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Theo's reinvention of the logic of conditional statements' proofs rooted in set-based reasoning

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ABSTRACT

This report documents how one undergraduate student used set-based reasoning to reinvent logical principles related to conditional statements and their proofs. This learning occurred in a teaching experiment intended to foster abstraction of these logical relationships by comparing the relationships between predicates within the conditional statements and inference structures among various proofs (in number theory and geometry). We document the progression of Theo's set-based emergent model (Gravemeijer, 1999) from a model-of the truth of statements to a model-for logical relationships. This constitutes some of the first evidence for how students can abstract such logical concepts in this way and provides evidence for the viability of the learning progression that guided the instructional design.

1. Introduction

Teaching logic for the purpose of supporting students' apprenticeship into mathematical proving imposes fundamental challenges regarding how the content-general relationships of logic can be operationalized within students' reasoning about particular mathematical concepts. Scholars affirm that this requires that logic be understood in both its syntactic and semantic aspects (Barrier, Durand-Guerrier, & Blossier, 2009; Durand-Guerrier, Boero, Douek, Epp & Tanguay, 2012). In other words, students must be able to reason about the form of statements and arguments as well as the way they refer to mathematical objects. Previous studies find that logic taught syntactically often does not foster understandings that are useful in context (e.g., Hawthorne and Rasmussen, 2015), and textbooks downplay the referential aspects of logic (Dawkins, Zazkis, & Cook, 2022). However, syntax also cannot be ignored. Teaching logic has become quite common in the United States (David and Zazkis, 2020) in introduction to proof courses because there is an acknowledged need for students to learn some syntactic skills before engaging with more complex proof content. Question 1: How then are students to abstract logical relationships that generalize across contexts and yet interface with their meanings for particular concepts? Question 2: How do such logical understandings become functional for comprehending mathematical proofs?

In our ongoing investigations of these questions (Dawkins, 2017, 2019; Dawkins & Roh, 2020), we have found that set-based reasoning can provide a unifying structure by which students abstract key logical relationships while reasoning about semantically meaningful statements and proofs. Set-based reasoning is propitious for student thinking and it provides a clear means by which students can interpret mathematical statements about very different topics as being in some sense the same (Hub & Dawkins, 2018).

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We claim the two questions above can be answered by guiding students to formulate logical understandings by comparing interpretations and generalizing their reasoning about mathematical texts in particular contexts.

In this report, we share a case study that illustrates one student's pathway to reinventing some basic logical principles of conditional statements: proof by universal generalization, converse independence, and contrapositive equivalence. Our use of constructivist teaching experiments (Steffe & Thompson, 2000) allows us to provide a detailed account of the student Theo's learning process, rooted in his meanings and activity (Piaget & Garcia, 1991; Thompson, 2013). This account of a student abstracting logical relationships is a novel contribution to the literature, especially regarding how proof frameworks (Selden & Selden, 1995) can develop conceptual understanding of logic and how diagrams can support such learning. Specifically, Theo's story of learning addresses the two questions we raised above by portraying how he abstracted logical structure from his reasoning about various specific mathematical statements and topics (Question 1) and how he used this emergent logical structure to understand how proofs do or do not justify conditional claims (Question 2). We analyze the series of learning episodes using the emergent models framework to document the emergence of a new mathematical reality (Gravemeijer, 1999), namely that of content-general logical structure rooted in set relationships. Consistent with our design-based project that simultaneously studied student learning while engineering the instructional activities that could support that learning, we answer our research questions with an "existence proof" by providing a deep account of one students' learning. We highlight Theo's learning pathway to demonstrate the viability of the learning progression, which closely matched what we intended in the instructional design. Naturally, this does not address questions of scale, but given that the literature lacks such qualitative accounts of the learning of logic, we offer this as a contribution to our knowledge of what is possible.

2. Conceptual analysis of the logic of conditionals

In our prior teaching experiments (Dawkins, 2017, 2018; Hub &), we guided students to reinvent logical facts (such as the truth conditions and negations of statements) by comparing their interpretations of mathematical statements of the same logical form. In this experiment, we extended this task sequence by asking students to read conditional statements each paired with 2–4 proofs. Their task was to determine whether each proof proved its associated theorem. We encouraged students to associate to each part of a statement with the set of objects that makes it true (*reasoning about predicates*, Dawkins, 2017). This allowed them to formulate generalizable truth-conditions for the statements and generalizable interpretations of the proofs. In this section, we shall present a conceptual analysis (Thompson, 2008) of these set-based understandings to clarify what we intended students to learn.

For these new proof tasks, we always presented true statements labeled "theorem to be proven." They all were universally quantified conditionals: "For any $[x \in S]$, if $[P(x)]$, then $[Q(x)]$." We use brackets since the statements/proofs that students saw always had particular objects and properties in these slots (e.g., "For every integer x, if x is a multiple of 6, then x is a multiple of 3" and "For all quadrilaterals ∎*ABCD*, if ∎*ABCD* is a rhombus, then it is a parallelogram"). Each proof was either a direct proof, a proof/disproof of the converse, a proof of the contrapositive, or a proof/disproof of the inverse (see Table 1). No proofs contained mathematical or logical errors (we told students this). All the proofs (as opposed to disproofs) used universal generalization. The principle of universal generalization (Copi, 1954) states that a proof regarding an arbitrary particular element of a set justifies the claim for the whole set of such objects. Choosing such an arbitrary particular is conventionally expressed using the imperative "let" and assigning a property to an object. The argument that follows must depend only on that property and thereby the argument will carry to all objects with the property (Alcock & Simpson, 2002). Such proofs that "if x has property P, then it has property Q" justify a subset relationship: the set of objects with P is a subset of the set of objects with Q. This is the truth-condition for such conditional statements, which we refer to as the *subset meaning* (Hub & Dawkins, 2018). Such statements are false when there is an object with property P and not property Q. To connect the subset meaning to proofs, students must relate chains of inference to the underlying sets of objects. Proofs establish implication relationships among properties, which are tantamount to containment relationships between the sets of objects with the properties: if the property P implies or entails the property Q, then all objects with property P are in the set of objects with property Q. Counterexamples show lack of set containment and also a lack of property implication.

Notice that the direct proof of a conditional and the (direct) proof of its converse (if both possible) deal with the same two sets of objects. They prove two facts about those sets: the set of objects with property Q contains the set of objects with property P, and vice versa. In this case the two truth sets are equal, meaning the exact same objects have the two properties. Since not every conditional involves two equal sets, these two proofs are taken as independent (the converse proof does not prove the original theorem). However, the contrapositive proof is understood to prove the theorem as the contrapositive statement is logically equivalent to the original theorem (arguments for this will appear in the results section). Since we expected students to abstract these structures from reading statements and proofs, we purposefully maintained a parallel structure to the proofs. The disproofs are somewhat oddly stated (since

Table 1
Forms of proof presented for comparison.

	Direct proof	Converse proof	Converse disproof	Contrapositive proof	Inverse disproof
Statement (directly) proven or disproven	"For any $x \in S$, if $P(x)$, then $Q(x)$."	"For any $x \in S$, if $Q(x)$, then $P(x)$."	"For any $x \in S$, if $Q(x)$, then $P(x)$."	"For any $x \in S$, if not $Q(x)$, then not $P(x)$."	"For any $x \in S$, if not $P(x)$, then not $Q(x)$."
Frame of proof text	Proof: Let x have property P. … Thus, x has property Q.	Proof: Let x have property Q. … Thus, x has property P.	Proof: Let x have property Q. x could be a. a does not have property P.	Proof: Let x not have property Q. … Thus, x does not have property P.	Proof: Let x not have property P. x could be a. a does not have property Q.

we usually do not write a disproof as a proof at all, but rather as an explanation for why we cannot prove something), but we intended for all proofs and disproofs to exhibit the same first-line/last-line structure to help students' associate the proofs with the theorem statements. Selden and Selden (1995) discussed this first-line/last-line structure of the proof as a *proof framework*, which they consider a purely procedural part of proof production (Selden, Selden, & Benkhalti, 2018). Our conceptual analysis portrays how we see proof frameworks as pertinent to conceptual understanding of logic and the reference structure of certain types of proofs.

3. Some insights into student thinking about conditionals

In this section, we review some relevant research findings from cognitive psychology and mathematics education related to people's reasoning about conditional statements. Rather than survey the rather large body of literature, we highlight the phenomena relevant to our findings.

3.1. Psychological studies

There exists a vast psychological literature on how adults reason about conditional statements in everyday and abstract settings. We will review two key findings that are relevant to this study. First, the most well-evidenced model of how people interpret conditionals is the Ramsey Test, which posits that people's degree of affirmation of a conditional statement "if p, then q" is based primarily on their assessed probability of q given p, sometimes notated $\Pr(q|p)$ (Evans & Over, 2004; Oaksford & Chater, 2020). In other words, people imagine p to be the case and consider how likely they judge q to be. This model suggests that people implicitly treat the conditional as an instruction to imagine that p is true. This model accords with logic inasmuch as cases where p is true and q is false are treated as counterexamples. This conflicts with standard logical practice that wants to assign a truth-value to the conditional even when p is false. However, the probabilistic model helps explain the finding that people will affirm a conditional in the presence of known counterexamples (Evans & Over, 2004; Over, Hadjichristidis, Evans, Handley, & Sloman, 2007), contrary to formal logic. The probabilistic account also calls into question whether the contrapositive claim is actually equivalent to the original conditional (i.e., they always share a truth value). This is because the conditional probability $\Pr(\neg p|\neg q)$ (associated with the contrapositive) can be quite different from the conditional probability $\Pr(q|p)$ (associated with the original conditional).

A second key finding from these psychological studies is that people quite frequently reason about negatively stated claims (e.g., "is not a bicycle") by focusing their attention on the condition being negated (e.g., "is a bicycle" or examples of bicycles). Tasks that include negative descriptions greatly increase the frequency at which people select responses that do not match what is described in the given statement. This has been called "matching bias" or the "matching heuristic" (Evans and Lynch, 1973) and has been shown to be a very persistent and robust effect in many psychological studies (Evans & Over, 2004). This phenomenon relates to our study because proofs of the contrapositive require negating claims.

3.2. Mathematics education studies

The Ramsey test and matching bias have been studied and observed in mathematical settings, though with slight modifications due to the unique nature of mathematical topics and language. Inglis's (2006) study of how mathematicians reasoned about mathematical conjectures found that their reasoning was also best modeled by the Ramsey Test, though with one caveat. In everyday settings, people will affirm a conditional "If p, then q" when $\Pr(q|p)$ is high, but not equal to 1. The mathematicians in Inglis' study may judge a conjecture to be likely, but they did not affirm it unless they judged q to be *necessary* given p (i.e., $\Pr(q|p) = 1$). To make such a judgment of certainty, these mathematicians sought to construct a proof (or to be convinced they could produce one). Consistent with the Ramsey Test, mathematicians reasoned about the conjectures by considering cases where p was true. Hoyles and Küchemann (2002) relatedly observed that 8th and 9th grade learners did not think conditionals were true or false in cases where p was false.

Inglis (2006) also observed evidence of the matching heuristic among these mathematicians. His study participants more often adopted indirect lines of reasoning (contradiction or contrapositive) when reasoning about the task "If n is abundant, then n is not of the form p^m for some natural m and prime p" as compared to the other statements that lacked negated claims. Dawkins (2017) noted that students often sought to replace negative descriptions with positive ones. This is logically viable in the case of replacing "is not even" with "is odd," but less so in the case of replacing "is not a rectangle" with "is a parallelogram." Dawkins (2019) summarized that for students, though, "negative categories are implicitly viewed as less valuable than positive ones because they do not clearly point to the things being referenced" (p. 21).

We note that the presence of negated properties poses two challenges for learning relevant to the goals of our study. First, if negatives change the way students reason about statements, then they may render different statements as having different logical structure. Part of logic is to see all statements of the same form as having the same structure, but students must construct that structure. Second, if we want students to understand why contrapositive statements are always equivalent to the original conditional, and relatedly why contrapositive proofs always prove a given conditional, then students must be able to work fluently with negations. For example, students must be able to negate negative claims to form the contrapositive of a statement such as, "Given any triangle, if it is equilateral, then it is not obtuse." Dawkins (2017) described one important way this can be taught, which we use in this teaching experiment. First, students must learn to *reason with predicates*, which means to associate to a mathematical property the set of all objects that have the property (e.g., "is a parallelogram" refers to the set of all parallelograms). Second, they need to understand the *negation-complement relation* which posits that a negative property is associated with the complement set of objects relative to the positive property (e.g., "is not a parallelogram" refers to the complement of the set of parallelograms). Constructing such complement sets relies on the presence of a universal set, since the complement of the set of parallelograms is different in the space of all quadrilaterals as opposed to the set of all polygons. We thus consistently identify a universal set in our experiment tasks.

4. Guided reinvention and emergent models

Our instructional sequence was inspired by the Realistic Mathematics Education design heuristics of guided reinvention and emergent models[1] (Freudenthal, 1973, 1991; Gravemeijer, 1999). Guided reinvention entails providing students with experientially real situations they can easily imagine and from which they might elaborate key mathematical ideas. The emergent models heuristic describes how students may first develop a *model-of* the situation that advances over time to become a *model-for* further mathematics. The initial activity within some imagined situation is called the *situational level of activity* (Gravemeijer, 1999). Instructors then present students with other situations to which the students' model may be adapted. Inasmuch as students see these new situations as expressing the same structure, Gravemeijer (1999) calls this the *referential level of activity*. It is referential in the sense that the model still refers to one or more of these situations to guide the student activity. As students elaborate the model by applying it to various situations, the model comes to constitute a new body of understanding apart from the situation(s) it interpreted. Once students can reason within the model itself apart from situation-specific imagery, they have begun the *general level of activity* (Gravemeijer, 1999). The model has at this point become a *model-for* reasoning about new problems and concepts. The model's elaboration for mathematical exploration constitutes the establishment of a *new mathematical reality* for the student. As students engage with their model-for asking new mathematical questions, they can move into the *formal level of activity* in which the model may exceed its situational origins to constitute new questions and ways of sense-making.

To apply these tools to teaching logic to undergraduates, we first wondered what kind of experientially real activity would lead students to perceive questions about logical structure (such as unform truth conditions for statements of a given type or general properties of proof frameworks). We found it useful to notice how logic generalizes across language, semantic contexts, and proofs. This led us to engage students in comparative reading of statements, that differed in their semantic context, and in reading proofs of parallel form. As noted above, we want the set-based structure (how statements and proofs refer to classes of objects) to unify the various statements and proofs. By helping students focus on set structure, students can develop a *model-of* how each statement refers to sets of objects (reasoning about predicates) and what it means for conditional statements to be true and false (the subset meaning and its negation). By considering how this set structure repeats across various statements and proof texts, students may extend their *model-for* reasoning about content-general logical relationships.

We can now specify the four levels of activity in our task sequence. The situations we invite students to reason about are the statements with their related semantic contexts, as well as the proof texts. We think of each statement as its own "situation" and the experientially real activity is determining whether a statement is true or false. Referential activity marks when students come to see statements as having similar underlying structure (for us, set-based structure), which is facilitated in our study by all statements being universally quantified conditionals.

Students can engage in referential activity in two primary ways in our tasks. The first way students may engage in referential activity is to connect conditional statements that have different semantic contexts because they share the conditional form (if… then…). If a student sees their activity reasoning about the converse or contrapositive statements as related to the task of interpreting "For every integer x, if x is a multiple of 6, then it is a multiple of 3," then they are constructing a model-of conditional statements more generally. Such a model can connect disparate semantic contexts. Second, when students read a statement and its converse or contrapositive, they may see all three statements as referring to the same underlying set relationships. For instance, when students read (original) "For every triangle, if the triangle is obtuse, then it is not equilateral" and (contrapositive) "For every triangle, if the triangle is equilateral, then it is not obtuse," they can treat these as somewhat separate tasks (as observed in Hub & Dawkins, 2018). On the other hand, they may treat them as expressing two different relationships (asking two different questions) about the same underlying arrangement of objects. In this case, the student is connecting the two "situations" via a model-of the semantic context of equilateral and obtuse triangles. This might be done using a spatial diagram such as an Euler diagram or a Venn diagram, which we encouraged in our experiments. The two types of referential activity are not independent. The formation of Euler diagrams to connect different statements in the same context may provide the means of connecting those statements to others in a different semantic context.

As students engage in the second type of referential activity, they may transition into general activity in which they can reason about the set-based model of conditional statements in ways that transcend the semantic contexts of any particular statement or mathematical domain (geometry or number theory). This constitutes the creation of a new mathematical reality, namely that of the mathematical conditional. While mathematical logic is taken to refer to many different ideas and relationships, a common feature is that logic bridges between semantic contexts. We thus see the general stage of activity as constituting students' construction of logical structure. However, engaging in general activity does not preclude students returning to referential activity in which they reason about more specific situations. In our case, students should alternate between focusing on semantic specifics (about triangles or integers) and generalizable structures in terms of sets (subset, complement) or syntax (converse, contrapositive, negation). This is necessary to relate proofs to statements since students must think about the overall form of the proof (direct, converse, or contrapositive) as well as the specific claims and warrants in each line of the proof. It is in this sense that our operationalization of logic using emergent models bridges syntax and semantics, as recommended by Durand-Guerrier et al. (2012). There is a key inversion from the referential level to

[1] Note that the term "model" in the *emergent models* framework refers to the student's mathematical model of a situation (or situations). The use of *model* in much radical constructivist research and the psychological work cited above instead uses the term to refer to researcher models.

the general level in the sense that students' focus shifts from using language to describe situations to studying mathematical language itself. The specific statements become examples of the general phenomenon of "conditional statements," and the focus of study becomes the general structure. We shall not focus on formal level activity in this study since our study was not designed to observe how students might use logic in further mathematical activity. Fig. 1 portrays our interpretation of the levels of emergent models to the construction of the logic of conditional statements rooted in reasoning about sets of objects.

5. Methods

As part of a grant project developing constructivist models of students' learning of logical principles through guided reinvention, we conducted 8–12 session teaching experiments with pairs of undergraduate students recruited from Calculus 3 classes at two large public universities in the United States. Students volunteered to participate and completed a screening survey to verify that they did not already know the target concepts to be taught (see Roh & Lee, 2018). Given that this study occurred during the Covid pandemic, the number of volunteers was quite low and Theo and Phil were chosen as the only pair of students meeting our selection criteria in terms of schedule availability and responses on the screening survey. Specifically, they had not taken any proof-based courses at university and did not answer questions about conditional statements in a manner consistent with the truth-table definition. The experiment featured in this paper was conducted remotely once per week over Zoom using OneNote as a shared space for reading and

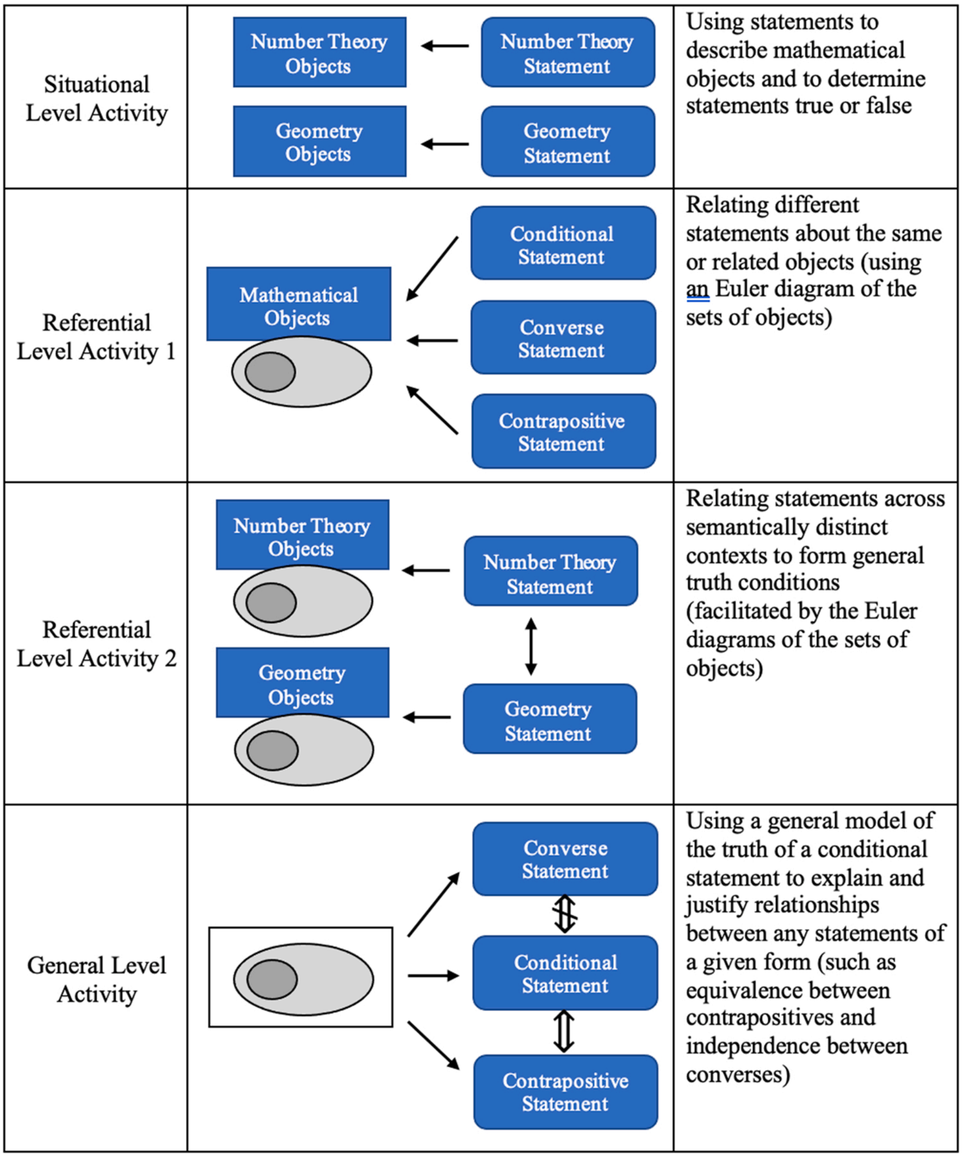


Fig. 1. The levels of emergent models for the logic of conditional statements.

writing. The two participants chose the pseudonyms Theorem (which we abbreviate as "Theo" for clarity) and Phil. The lead author served as the teacher/researcher and the other two authors served as witnesses (Steffe & Thompson, 2000). Each session lasted between 60 and 90 min and participants were compensated monetarily for their time.

This experiment consisted of an intake interview with a pre-test, nine instructional sessions, and an exit interview with a post-test. During the exit interview, we asked students to choose how they wanted to be identified in terms of their ethnic and gender identities and how those identities were significant for their mathematics learning at university. Theo identified himself as a white, non-Hispanic male. At the time of the study, he was in his first year of university as a finance and mathematics double major. He described himself as "passionate" about mathematics. Phil, an engineering technology major, identified himself as a Hispanic male. The two students worked productively and respectfully together, though they generally operated in parallel rather than interactively. We focus on Theo in this report because of the clear evidence of his progression toward our learning goals, helping us learn about the processes of learning along our intended trajectory. However, Theo's story of learning depends upon his interactions with Phil. In this paper, we attend to Phil's contributions and reasoning inasmuch as they are necessary for telling the story of Theo's learning. Phil's learning will be featured in other reports on the project overall.

The research team met once or twice between sessions to analyze and plan for subsequent sessions. At these meetings, the research team continuously made conjectures about the two students' understanding and tested those conjectures through questioning and iterative task design, consistent with teaching experiment methodology (Steffe & Thompson, 2000). All sessions were recorded on at least two or three screens: the interviewer screen that moved between pages in OneNote and two screens dedicated to capturing Theo and Phil's pages respectively. All main study sessions were transcribed. Our retrospective analysis drew upon field notes, transcripts, and compiled video.

In the first two instructional sessions of the experiment, Theo and Phil read sequences of universally quantified conditional statements and considered the relationships between the sets of objects that made the if-part true and objects that made the then-part true (hereafter the "if-set" and "then-set"). We intended them to formulate the *subset meaning* (see Conceptual Analysis) for such statements and the conditions for a counterexample. In the next three sessions, they read theorems and proofs as shown in Fig. 2. Theorems 1–4 were chosen to intentionally vary the relationships between the underlying sets (proper subset in 1 and 3; set equality in 2 and 4) and to vary the mathematical context (number theory in 1 and 2; geometry in 3 and 4). In the sixth session, in which Phil was absent, Theo reviewed all of the theorems and proofs and his decisions about which proved the associated theorems. We call this the Comparison Task. We sought for him to systematize the relationship between the logical form of the proof and whether it proved the given theorem (evidence of a model-for logical reasoning that generalizes across context). In the seventh session we invited Theo and Phil to anticipate how they would form various kinds of proofs for some new statements, extending their claims from the Comparison Task. Finally, in sessions eight and nine we invited Theo and Phil to abstract their understandings to answer some questions about two mystery properties P and Q.

6. Results

In this section we shall present a chronological narrative of Theo's reasoning as it developed over the course of the experiment. Our main goal is to trace Theo's learning through the lens of emergent models to argue how he reached the general level of activity within his set-based model-for reasoning about the logic of conditional statements. To do this, we will begin with a rather detailed account of the emergence of how Theo reasoned about the sets of objects referred to by the statements and proofs. Our intent is to justify that Theo did not begin the experiment with these powerful modes of reasoning, but his emergent model of the set-based meaning of conditional statements was sufficient for him to reinvent some key logical principles, to explain and justify them in powerful ways, and to abstract them to a general setting apart from the specific situations from which they arose.

6.1. Developing set-based meanings (Intake interview and day 1)

In the intake interview (prior to the first teaching session), Theo read a direct proof, inverse proof, converse proof, and contrapositive proof of the claim "For any integer x, if x is not a multiple of 3, then $x^2 - 1$ is a multiple of 3." He affirmed the direct proof proved the theorem and denied that the other three did. His rejection of the converse proof was not based on its reverse order from the theorem. Productively, he showed early evidence of associating an equation such as $x = 3k + 1$ to a set of values (***reasoning with predicates***; Dawkins, 2017). We note this to provide evidence that he did not already know the rules about proofs of conditionals prior to the experiment.

6.1.1. Identifying counterexamples using list-and-test – situational level activity

During the first session, Theo and Phil began by determining whether a provided list of universally quantified conditional statements was true or false. Once they had done so, the interviewer asked them whether they saw any pattern among the statements, specifically the statements that were false.

Phil: I mean kind of but not really at the same time.

Theo: I feel for the false ones, they were kind of obvious. You could test it pretty quickly if they were false.

Int: Yeah. And so what did you test? What is it that you found to identify that it was false?

Theo: I mean, just straight away for [statement] four, it says "if x is a multiple of 3, and then it's a multiple of 5," you can just barely use 3 or 6 and see that it's not a multiple of 5.

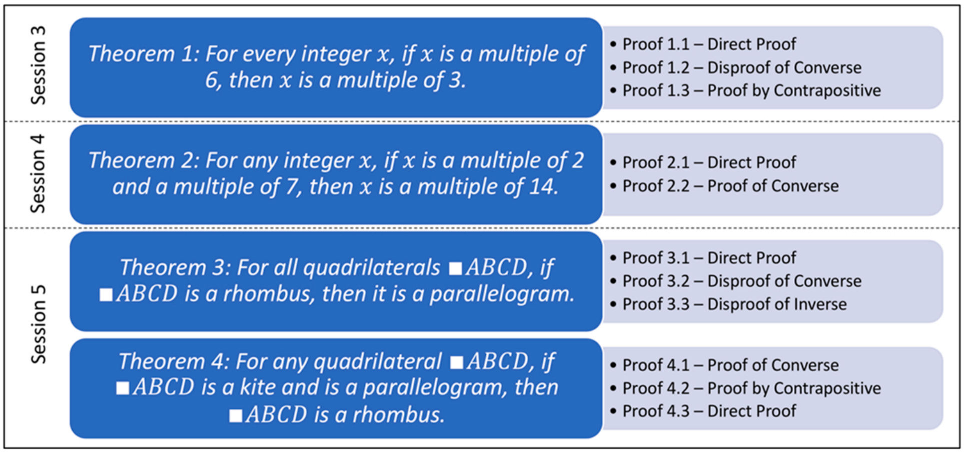


Fig. 2. Sequence of proof reading tasks.

Int: Okay, and looking at Phil's page. Phil, I noticed for [statement 2], you listed out a few multiples of 3, and then drew some slashes (see Fig. 3). Can you tell me about what you meant by that?

Phil: […] The slashes were just instances where it's a multiple of 3, but it is not a multiple of 6, so it kind of negated their premise of [statement] two, I just did slashes for those because those are the values that I've found that aren't supported by the claim.

Fig. 3 presents Phil's work by which he decided whether the first few statements were true or false. We note here that while both Phil and Theo seemed to identify counterexamples in a normative way, they explained why these numbers were counterexamples in a rather limited way. Regarding 3 and 6 as counterexamples for Statement 4, Theo only explained that they are "not multiples of 5." Phil in a similar manner listed the numbers that make the premise true and then slashes through the numbers that fail to satisfy the conclusion. The interviewer wanted them to reason more explicitly about their conditions for a counterexample.

Theo: There's one example [of] the way it doesn't work.

Int: Yeah. And can you tell me more about what you mean by "it doesn't work?" […][2].

Theo: You just need to prove a single instance where the statement proves false.

Int: Phil, you want to add to that? What do you mean by "statement proves false?".

Phil: If x is a multiple of 3, you're proving you can get an assembly of numbers like I did [statement] number two where I did the multiples of 3: 3, 6, 9, 12, 15, etc. Then you have to prove that in one of these cases or several of these cases the x-values that you produce are not a multiple of 5 or is not a multiple of 5 proving wrong then or disproving the then[-part]. As far as the if/then structure you're trying to disprove, or we are trying to. I was trying to disprove the then[-part] based on the cause of if it is this.

Int: Okay. So just apply that for me back in the case that we have in the shared [OneNote] space about if parallelogram, then a rhombus. So, what am I looking for to make that false?

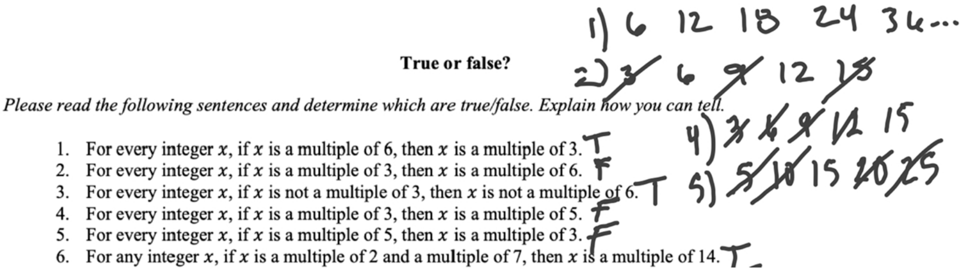
Theo: Just an instance where a parallelogram is not considered rhombus, which I mean would just happen when any of the opposing. the non-opposing sides are just greater in them.

After prompting, Phil and Theo thus identified that a counterexample was an element of the if-set that made the then-property false. We state it this way to point that we observe an asymmetry in the ways that Phil and Theo interpreted the two conditions. The if-condition determined the set of objects being considered and the then-condition was a test applied to those numbers. Much like the Ramsey Test, they took the conditional as instruction to consider the space in which the premise is true. Unlike the Ramsey Test the students held that one counterexample was sufficient to render a claim false. We refer to this interpretation of a conditional claim as the *list-and-test meaning*.

Once Theo and Phil had agreed what made a statement false, the interviewer turned their attention to the next prompt at the bottom of their list of statements. The prompt invited them to imagine forming all of the things that made each part of the sentence true. It then asked, "For the true statements, what is the relationship between the two sets? For the false statements, what is the relationship between the two sets?".

Fig. 4 presents Theo's initial set diagrams, which labeled the if-set as 1st and the then-set as 2nd. It is unclear what Theo meant by the arrow on the right, but the left diagram conflicts with the normative interpretation in which the if-set is a subset of the then-set. However, Theo quickly reversed this interpretation when the interviewer invited him to consider a specific true statement (Statement

[2] We include an ellipsis in brackets to mark when we have deleted some dialogue for clarity or brevity. Ellipses without brackets mark a pause in the discussion.


True or false?

Please read the following sentences and determine which are true/false. Explain how you can tell.

1. For every integer x, if x is a multiple of 6, then x is a multiple of 3. T
2. For every integer x, if x is a multiple of 3, then x is a multiple of 6. F
3. For every integer x, if x is not a multiple of 3, then x is not a multiple of 6. T
4. For every integer x, if x is a multiple of 3, then x is a multiple of 5. F
5. For every integer x, if x is a multiple of 5, then x is a multiple of 3. F
6. For any integer x, if x is a multiple of 2 and a multiple of 7, then x is a multiple of 14. T

Fig. 3. Phil's lists of numbers for analyzing statements 1–5.

Fig. 4. Theo's initial set diagrams.

1).

Phil: Yeah. Okay. I guess you're saying in the multiples of 6.

Int: So, I'm trying to follow, so I see that in the outer circle you wrote first-

Theo: And then I guess in this case it would be the other way around.

Int: Okay, tell me just more about how you're thinking about that. So why did you write first on the outside and second on the inside [see Fig. 4]? What were you thinking about?

Theo: I guess in that case, I was just generally thinking of it, but I mean we. And this one we could say we could use first can be whatever, we consider a broader definition. So, multiples of 3, you're going to have more than multiples of 6. I guess finitely speaking, if you did one to 100, and then encompassed in that multiples, we could instead list it out like 3, 6, 9, 12. We'd have a subset of multiples of 6.

Theo then explained that he could adapt this type of diagram to consider other statements, such as Statement 2. He said the following after drawing the new diagram in Fig. 5: "And then we can say for [Statement] 2, because it's saying 'x is a multiple of 3,' so, it's that ginormous area and x is a multiple of 6 that it could instead go into the second circle, which is the non-6 multiples. So that's why it's false."

Theo's revised diagram follows the *list-and-test meaning* inasmuch as he populated the regions with elements of the if-set and then applied the then-condition as a test to those numbers. The space outside of the regions did not appear to bear any significance for him. We thus call these *regions that gather* because they group objects that share a property but do not inherently partition those inside the region from those outside. This is why Theo introduced a new region to represent the non-multiples of 6, rather than the outside of the other region already signifying this. In classical logical diagrams, each region partitions the objects inside from those outside. We refer to the normative diagram use as *regions that partition*. Enclosing the non-multiples of 6 was the first time we observed him formulating a

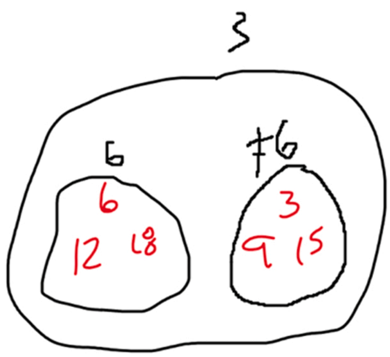


Fig. 5. Theo's revised set diagram.

complement set, though this set does not cover all non-multiples of 6, but only those within the multiples of 3. This is also the first time we observed Theo repurposing the same representation to interpret two different statements, suggesting movement toward the referential level of activity.

6.1.2. Addressing complement sets in the contrapositive statement – situational level activity

The interviewer was curious how Theo and Phil would deal with sets defined negatively (by lack of some property), given the challenges they had posed to students in previous experiments (Dawkins, 2017). He thus asked Theo and Phil to draw similar diagrams for Statement 3 (the contrapositive of Statement 1). The students did indeed struggle to construct such diagrams, largely returning to their previous strategy of listing numbers in the if-set and applying the then-condition test.

Theo: I guess you can look at it there's different. Like a Venn diagram within a crossover and the circles could be different odd numbers. So, you would have another Venn with 5, another one with 7, another one with 9. I just continuously go up and none of these circles are intermingling.

Phil: Yeah. I mean, for this one, I pretty much just chose any value that wasn't a multiple of 3, like one. I was like, "Okay, if it's not a multiple." I know these values that I chose are not multiples of 3. Let me see if they're divisible by 6 anyway. And if they weren't thereby concluded that it was true. I'm trying to put it into diagram form, but it's not.

It seems that Theo and Phil struggled to think of the negative properties as defining a set in the same way the positive properties did. Theo initially imagined putting each odd number in a separate circle, since they were not unified by some property. We further see evidence that the outside of the circles in Theo's diagram did not represent the complement set of objects, as is normatively understood in logical diagrams.

Theo then introduced a new idea that allowed him to bypass these challenges and produce a diagram for Statement 3. Specifically, he recognized that any integer could be represented by one of the equations $x = 3k$, $y = 3k - 1$, or $z = 3k - 2$. He drew a new diagram (see Fig. 6) to show how this organized the integers, which allowed both he and Phil to produce justifications of statement 3.

Theo: So it's like three different circles and where the points don't crossover (see Fig. 6).

Phil: Where they don't crossover?

Theo: Yeah. Because there's nothing that… They're independently like say subsets. So, like the one is 3, 6, 9. The other one would be 2, 5, 8, 11, and then the first one would be 1, 4, 7.

Phil: Would y and z be shared? As these are values that are not multiples of 3. Would they have a shared little bubble?

Theo: Yeah. I guess you could share the bubble and, and make that bigger set called non-3 multiples. Oh, actually, yeah. That might work because. And then, because the 6 one is inside the x. We can see that what proved true for either one. Because it's saying, "if it's not for 3" and since the 6 one exists in 3, then it's. I can't remember if you asked for true or false.

Phil: If x is not a multiple of 3, then x is not a multiple of 6.

Theo: Yeah. So then we can prove it's true because it's in a whole 'nother group, so 6 is inside.

Phil: In order to disprove it we need to find a value inside of the y and z equations that are multiple, that produce a multiple of 6. Is that how we would disprove it? Or I am just trying to follow what you're saying.

Theo: So yeah. I guess if you wanted to. I mean, we know that 6 is a multiple of 3, so therefore it exists in x as a subset of x, but x has no. I don't know what to call it, an intersections with y or z. And since we're asking whether the little subset 6, has a little crossover point until y or z, but it doesn't because it exists purely in x.

Because Theo was able to substitute the negative condition (not a multiple of 3) with two positive conditions ($y = 3k - 1$ or $z = 3k - 2$), he now could reason about the if-set for Statement 3. Phil noted that grouping these sets y and z together formed the set of non-multiples of 3. Theo understood that each of these equations/regions represented a whole class of integers that each were disjoint from the others and that together exhausted all integers (the two properties necessary for a complement set). Building on their understanding that the multiples of 6 were contained in the multiples of 3, both students then argued why this statement must be true. Phil's

Fig. 6. Theo's complement set diagram.

argument was akin to Yopp's (2017) idea of *eliminating counterexamples* since he argued that you could not find a multiple of 6 in either the *y* or *z* circle. Theo's argument is akin to what Hub and Dawkins (2018) called the *empty intersection* meaning. Rather than justifying statement three by saying all non-multiples of 3 are non-multiples of 6 (subset meaning), he argued that the set of non-multiples of 3 and the set of multiples of 6 were disjoint.

The interviewer wanted to confirm that the students were attending to the complement sets, so he asked them what region of the diagram represented the non-multiples of 6. Theo explained, "So I think basically so the bigger one where it has 6 and non-6, So, everything that's not 6, is the rest of x the other half of x and then y and z." The interviewer asked, "Which is the bigger group of numbers?" Theo said, "not in multiples of 6." However, when the interviewer tried to draw a simpler subset diagram labeling the inner circle "non 3" and the outer circle as "non 6," this diagram revealed some key aspects of their reasoning. When the interviewer asked them to list numbers in each region, they identified 1, 2, 5, and 4 as numbers that were in the inner circle. With some effort, Theo listed 9 and 15 as numbers in the larger region, but outside the smaller. When the interviewer asked what was outside the larger circle, Phil replied "non-multiples of either" while Theo cited some universal sets such as the reals or integers. Only when the interviewer pointed out that the non-multiples were in the center was Theo able to list 6, 12, and 18 as numbers outside of both circles.

Why did Theo and Phil have trouble reasoning about the outsides of the circles in the regions the interviewer drew, though they had just constructed the same sets of numbers and their complements in Theo's diagram (Fig. 6)? We provide two explanations. First, we think Theo and Phil's construction of the negative categories as properties that had truth sets was still quite tenuous. The interviewer expected them to treat these as categories of numbers, but for the students these negative categories still needed to be constructed from their positive counterparts (consistent with the idea of the matching heuristic, Evans & Lynch, 1973). Second, we note that Theo used *regions that gather* in which the outside of the regions did not refer to anything. The interviewer's question about the outside of the circles drew on normative conventions for diagrams that Theo and Phil were not using. These descriptions (*list-and-test* as represented by *regions that gather*) of Theo's activity express his set-based model-of these conditional statements, which constitute his initial activity at the situational level.

6.2. Generalizing the subset truth condition (Day 2) – referential level activity

Based on analysis of Day 1, the Day 2 activities presented Phil and Theo with seven conditional statements to represent using the set diagrams. We shall focus on the first six statements, which were paired to discuss the same sets of objects (or complements). Specifically, Statements 1 and 2 were converses with only one of them true while Statements 5 and 6 were converses that were both true. We provided rectangles labeled as the universal set to guide them to use the outside of the regions to represent complements. Neither Phil nor Theo used the diagrams in this way. Theo specifically began by drawing the if-set and then-set regions next to one another before drawing an arrow from one to another (to show a subset relation). Later on, he began to draw the diagrams with one region inside the other or coinciding with the other (when the sets were equal). Theo and Phil's reasoning on this second day largely confirmed our models of their reasoning from Day 1, as described above. They again struggled to accurately represent the relationship between two negatively defined sets, specifically in the statement "For any quadrilateral, if it is not a parallelogram, then it is not a rhombus." Though they correctly affirmed this statement as true, they drew the non-rhombus set as a subset of the non-parallelogram set, which conflicts with the normative relationship between these sets. We conjecture they thought of non-rhombi as contained in non-parallelograms because they knew rhombi are a subset of parallelograms and were trying to reason about the negative properties by thinking about the positive ones.

For the sake of space, we shall focus on the discussion toward the end of the session when the students reflected on their work to generalize conditions under which a conditional statement is true. Note that we had asked them to color the diagrams where the if-set was blue and the then-set was yellow.

Int: In these cases, what do you actually notice as the relationship between the things that make the if-part true and the things that make them then-part true? The blue [if-set] and the yellow [then-set].
Phil: The then-part is more, encompassing more shapes exist within the then-part than within the if-part.
Int: Okay. Theo, how would you describe it? What do you notice about the relationship between the blue and the yellow in these cases?
Theo: Yeah. I would have to say the same that the then[-part] is more broad or has more options inside it than just the if.
Phil: The if[-part] provides a condition where it limits the amount of things you'd have inside of it.

In this interchange, Theo and Phil provided a general statement of a truth condition consistent with the *subset meaning*. Consistent with their *list-and-test meaning*, they tended to articulate the subset meaning as though the second condition acts upon the first. However, we see evidence they had moved beyond the list-and-test meaning because both conditions constituted a set of objects. They reasoned about the whole then-set, not just within the universe of the if-set. Notice that Theo and Phil's language attributed breadth ("more broad") and quantity ("more options") to the conditions in the statements, showing their engagement in *reasoning with predicates* as represented by the spatial regions in their diagrams.

The interviewer then drew Theo and Phil's attention to Statements 5 and 6 ("For each integer x, if x is a multiple of 2 and a multiple of 7, then x is a multiple of 14" and converse), where the students seemed aware that the exact same objects were in both sets. This led the students to formulate a more nuanced truth condition that acknowledged the two possible arrangements. Theo further conjectured how the different set relations corresponded to a pattern in the truth-values of converse statements.

Int: Okay. Do you all see now that one way we can think about all these different types of statements is when we say, “if this is true, then that is true,” then we’re saying the then-set at least completely includes or covers the if-set. In case we haven’t made this explicit, what is the relationship between Sentence 1 and 2?

Theo: They change the if and the then.

Int: Can you elaborate on that a little more, Theo?

Theo: I guess a better way to be putting it would be changing the big circle and the little circle, switching places. And it would be, changes to false. It switches results if they’re not the same size.

Int: Okay. And what’s the other alternative? Why do you make that caveat of it? It switches the results. I think you’re meaning, right, one is true and one is false. It switches the result if they’re not the same size. What’s the alternative?

Theo: That if they’re the same size, it stays the same.

Phil: Yeah. Then nothing changes, they are both true.

Thus, Theo claimed that when one set is a proper subset of the other, the converses would have opposite truth values and when the sets are the same size, the converses will both be true.

Interestingly, Theo perceived a close link between the contrapositive statements (Statement 3 “For any quadrilateral, if it is not a parallelogram, then it is not a rhombus” and Statement 4 its contrapositive), though he could not fully explicate it.

Int: How are Statements 3 and 4 related?

Theo: Is it asking the same thing?

Int: What do you mean?

Theo: Three is like saying okay, this is not a parallelogram then it’s not a rhombus. Then the other one’s saying that okay, if it is a rhombus then it is a parallelogram. [...] It’s the opposites of each other. It’s basically saying that it’s like testing the other parts that maybe not the other parts, maybe the opposite relation, I guess. I don’t know what’s the word. If it is a rhombus instead of if it’s not a rhombus.

The interviewer took this opportunity to introduce the term “complement” to express what Theo was recognizing as “opposites.”

We see here some key features of Theo’s reasoning that mark his movement into referential level activity and the development of his model-of conditional statements. First, he saw how his truth condition linked all the statements and he treated all conditionals as instances of the same thing. This model bridged between the different statements both at the level of syntactic manipulations in the same context (converses and contrapositives) and across contexts (number theory and geometry). Further, he recognized how this truth condition can both connect and distinguish two types of situations, proper subset situations and equal set situations. The truth-values of converse statements indicate which situation is being described.

6.3. Interpreting proof by contrapositive (Day 3) – beginning general level activity

On Day 3, we moved from comparing statements and their truth values to reading proofs and considering whether they proved a given conditional. At the beginning of the session, the interviewer asked Theo to summarize what he had learned the previous two days. He reported:

> It’s true if the statement, if [the if-set] exists inside then[-set] or is the same size as then[-set]. [...] If the [if-]condition exists outside of the parameters of the then statement, like if it goes beyond the bubbles or diagrams that we created, if it extends beyond it then that’s when it’s not true.

Again, we note the way Theo’s explanation associated the conditions in the statement as regions in space. Also, his ability to provide a unified explanation for the various statements shows he is operating at the referential level where he has a model-of the various statements.

During the third and fourth teaching sessions, Theo adopted normative answers as to whether each proof proved the associated theorem based on his set-based reasoning developed in the first two days. The main questions the students had to answer for each proof were: *Does this proof prove the given theorem?* and *If not, what statement does it prove (or disprove)?* Theo affirmed that Proof 1.1 (direct) proved Theorem 1. He did so by focusing on the steps within the proof, not the order from first to last line. Phil denied that Proof 1.2 (disproof of converse) proved the theorem, saying:

> I don’t agree with this theorem [sic] because we’re trying to say that “if it’s a multiple of 6 then it’s a multiple of 3,” not “if it’s a multiple of 3 then it’s a multiple of 6.” It kind of goes into what we were saying last week, if the condition falls outside of the realm of all possibilities and the then statement, then it doesn’t hold up, it’s not true.

Theo added, “But [Proof 1.2] also doesn’t disprove the theorem.” In the first part of the quote, Phil restated Theorem 1 as what “we’re trying to say” and contrasted it with what Proof 1.2 is addressing, which he articulated as the converse conditional. He thus attended to the order of the theorem and the first/last-lines of the proof in order to distinguished the meaning of the theorem from what the proof accomplished and to show conflict between the two. He then elaborated what the proof (which presents the counterexample 15) proved: that the if-condition for the converse “falls outside the realm” of the then-statement. He thus shifted back into the language of sets of objects as spatial regions. Theo explained later that Proof 1.2 disproved the converse of Theorem 1, saying, “It’s kind of like, if it’s false one way, it’s usually true the other way, kind of. It’s false because the if is larger than the then, which means that if you switch them around, the if is going to be smaller than the then or exist inside the then.”

Both Theo and Phil agreed that Proof 1.3 (a proof by contrapositive) proved Theorem 1. Theo began by listing the first 12 integers and drawing arrows from the various equations in the proof to the numbers they described. This matched closely their list-and-test meaning. The interviewer invited Theo to draw a diagram for how he understood the proof.

Int: Okay. Can y’all try to use the diagrams that we were using the last two times we met? We have this kind of meaning for what the theorem says in terms of the group of multiples of 3 covering the group of multiples of 6. Can y’all try to explain to me how is it that Proof 1.3 proves it using that idea?

Theo: I think you got to look at, it would be the pattern of all the non-multiples of 3 and you could be like, 1, 2, 4, 5, 7, 8. And you have that subset of numbers, and then you have the other subset that’s 3 and obviously they’re not in each other. However, the multiples of 6 does not exist inside the non-multiples of 3. It only lives inside the multiples of 3. […] It’s talking about the subspace when x is not a multiple of 3 [see Fig. 7], which is going to be this whole range of numbers on the left side. And basically, it proves that there exists no of this smaller subset that’s on the right side, the blue circle that exists in the non-multiples of three, not even like a cross over even.

This argument closely mirrored Theo’s argument on Day 1, stating that the set of non-multiples of 3 and the set of multiples of 6 are disjoint (*empty intersection meaning*). This interpretation renders the condition “not a multiple of 6” as saying “x is not an element of the multiples of 6” rather than “x is an element of the non-multiples of 6.” This is a common move we observe that allows students to avoid forming two different complement sets (Hub & Dawkins, 2018). Unlike Day 1, Theo now treated the non-multiples of 3 as a single set, even though the proof represented this set using two equations as he previously had. Theo later summarized and connected the two proofs of Theorem 1 using his diagram, saying:

> I think a good way to put it would be [Proof] 1.3 shows that the [multiple of 6] circle doesn’t exist in the circle of non-multiples of 3, while Proof 1.1 would show it exists in the circle with multiples of 3.

Theo’s argument shows how he perceived symmetry between the ways that the direct proof (1.1) and the contrapositive proof (1.3) justified Theorem 1. This way of reasoning shows a marked shift from the original list-and-test meaning because that would suggest that the two proofs were about completely separate sets of numbers suggesting they are independent of each other. Instead, Theo’s set-based model supported him in linking across the proofs and in producing rich logical explanations and justifications. This to some extent marks a shift toward general level activity since it is fostering the emergence of a new mathematical reality. His set-based thinking is allowing him to ask new questions in the set-based model that would not have made sense in the original situation of reading individual conditional statements.

6.4. Interpreting converse proofs (Day 4)

During Day 4, Theo affirmed Proof 2.1 (direct) and denied Proof 2.2 (converse proof) as proofs of Theorem 2. He did so by using an analogy to the other statements they had read. He explained:

> In this case, the if and the then are the same set. But, if you switch them around in a set where they’re not the same, then it doesn’t necessarily work out that way. In this example, it works out, but switching the if and then doesn’t necessarily mean it will work out every time. […] Earlier, we were doing the circles of sets. And, in this case, the circle of the if exists inside the then, but it encompasses the whole then. They’re the same set. So, in 2.2, when we switch them around, the if and the then are still the same sets, but, if we had an example where the if is a subspace of the space, and then you switch it around, it’s not necessarily true. In this case, it is, but, in general, if you switch them, it might not work.

This argument marks a key development in Theo’s thinking because his model-of the set structure allows him to make an analogy between the different conditional statements that determines how the proofs do or do not support the theorems. It shows how his set-theoretic model had become a model-for reasoning about more abstract relationships between theorems and proofs. However, we learned in the next sessions that his model still carried some contextual dependence.

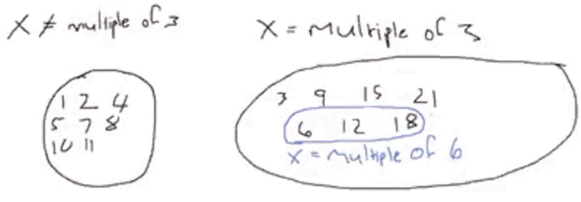


Fig. 7. Theo’s diagram for Proof 1.3.

6.5. Different interpretations emerge for a more complex geometry statement (Day 5)

Recall that our operating definition for a student's understanding as being logical is that it generalizes across semantic content. Theo's use of his set-theoretic model showed that to some extent he was attending to logical structure on the number theory tasks. In contrast with his prior reasoning (from our perspective), on Day 5 Theo affirmed that Proof 4.1 (converse) proved the theorem and he denied that Proof 4.2 (contrapositive) did so. Initially Theo and Phil judged that Proof 4.2 was irrelevant to Theorem 4, as we might expect from the Ramsey test reasoning that they are about different sets of objects. Theo articulated what Proof 4.2 proved as, "if ABCD is not a rhombus, then it may have properties of a kite or have properties of a parallelogram, but not have both." Though Phil later developed an indirect argument for why Proof 4.2 supported Theorem 4, neither judged that the proof proved the theorem.

Theo also had trouble applying his subset meaning to this theorem because he represented the if-set (kite and parallelogram) using two overlapping circles to show what is shared among kites and parallelograms (see Fig. 8). He identified that the conclusion (rhombi) existed in the overlap, which led him to imagine the then-set as nested within the if-set. In other words, he saw the then-set as a subset of the if-set, reversing the previous pattern. We hypothesize that the three-set structure and the more complex nature of negating the hypothesis kept Theo from structuring these theorem/proof pairs in a manner consistent with his prior activity. To be clear, we cannot distinguish the extent to which the geometry context or the three-set structure influenced Theo to reason differently about this contrapositive proof. While the three-set structure clearly seemed to influence his reasoning, the geometry context may have influenced Theo to think of "is a kite and is a parallelogram" as the combination of two properties rather than as a single property. In any case, these geometry tasks revealed that Theo was still operating referentially with regard to some conditional statements, specifically when he did not see them fitting into his subset model-of conditional statements. He still needed more work to see all of these conditional statements as instances of a general structure and to operate within that structure consistently across semantic contexts.

6.6. Comparison task (Day 6) – general level activity

On Day 6, we presented Theo with all four theorems and their proofs from the previous three sessions along with his decision about whether each proof proved the theorem (recall Phil was absent from this session). We asked him to look for patterns among the proofs, his decisions, and how the proofs did or did not prove the associated theorem. He began by grouping all the direct proofs and affirming they all proved their associated theorem: "You start by saying like, okay, we have this if, it meets such criteria and then like, it continues on to conclude, okay, this criteria can be put inside this larger space." We take this as a justification for the principle of universal generalization (Copi, 1954) arguing that a proof that an object meeting the if-condition must also have the then-condition proves that the if-set is a subset of the then-set. He initially placed Proof 4.1 (converse) in this group, and later removed it since it started with "the then."

Theo then grouped the first three (dis)proofs of converse (1.2, 2.2, 3.2), omitting Proof 4.1. Later he decided to move Proof 4.1 into that group. When asked about his claim that Proof 4.1 proved the theorem, he decided to change his decision "to be consistent." He now reiterated his prior line of thinking to include Proof 4.1:

> I think it's because it's proving the converse, and just like, in this situation, it happens to be the same space. But I think in terms of consistency, gotta not allow that. Again, Proof 1 and Proof 3. So, either both 2.2 and 4.1 are no's or either they're both yeses. That's how I feel.

He again connects situations where the two sets are the same ("happens to be the same space") to situations where one set is nested in the other ("Proof 1 and Proof 3", which we take to refer to Theorem 1 and Theorem 3 and the disproofs of their converses) as two instances of the same general relationship. So while his initial reading of the geometry proofs seemed inconsistent with his reasoning about the number theory proofs (both to us and to him), when invited to directly compare Theo decided to adopt a uniform and general interpretation of the proofs.

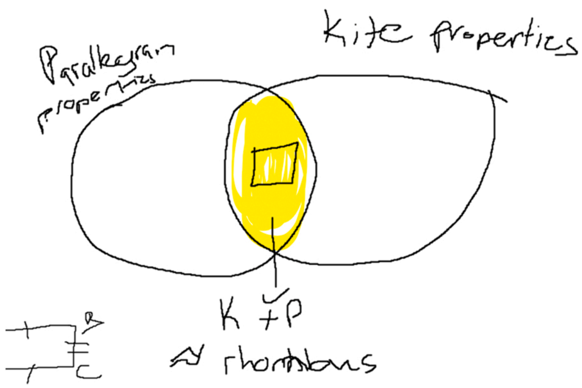


Fig. 8. Theo's diagram for Theorem 4.

Theo grouped all the rest of the proofs (inverse and contrapositive) together, saying, "I guess we could make a third group that kind of not like either, but I guess by not either, I'm talking about what it starts with. [...] So 1.3 and group that with maybe 4.2, maybe 3.3." We explain this deviation from the normative logical distinction between contrapositive and inverse in terms of Theo's set-based model, which operated with the structure of if-sets, then-sets, and everything else. Since his set model did not include the complement of the if-set, he did not distinguish the order of inverse and contrapositive. Theo's groupings demonstrate how the structure of the proofs that unified them reflected the structure of the sets they referred to, not the statements per se.

The final point at which Theo's reasoning was not yet aligning to our learning goals regarded contrapositive equivalence. He affirmed that Proof 1.3 proved Theorem 1 (number theory), but denied that Proof 4.2 proved Theorem 4 (geometry). We wanted to see if he could adapt his argument from the former to the latter. The interviewer asked Theo to explain again his argument for why Proof 1.3 proved Theorem 1, which he did in terms of his *empty intersection meaning*.

> It's discussing I think this whole subspace that, is this the subspace that is not multiples of 3, so 1, 2, 4, 5, and it's proving that, okay, so let's look at the sub space and do multiples of 6 possibly exist in this sub space. And it proves that there's no multiples of 6 that exist in this space. So, it has to exist in this other space that is multiples of 3. Doesn't specifically say that like, it's the same or it's a smaller group, but it says that it is in that space, which is what we're trying to prove.

The interviewer then asked him to apply the same argument to Proof 4.2. Upon considering he decided Proof 4.2 proved, explaining:

> I think looking at 4.2, it's, starting with, okay, let it be a non-rhombi. So once again, we're dealing with this different space, and it's saying that... Okay, yeah. It's saying, if it's not a rhombi, so it doesn't meet the certain criteria, then it's either not a kite, or it's either not a parallelogram. So, it's either in the spaces of one of those two, but not on both at the same time, however, a theorem to prove if it's in both of these spaces, then it's rhombi.

Theo thereby began to affirm that Proof 4.2 did prove Theorem 4.

We see a few key aspects of Theo's explanations of both Proof 1.3 and Proof 4.2 that suggest he was operating at a general level of activity. First, he rendered the argument of Proof 1.3 to capture only the relationship between the sets of objects, backgrounding his focus on the specific objects and the specific properties and inferences in the proof. This allowed him to see this proof about numbers as bearing the same structure as the proof about shapes. He pointed out that this proof does not show that multiples of 6 is a smaller group than multiples of 3, but rather shows that "it is in that space." Here he reasoned about the sets of objects and their relations, attending to his and Phil's set-based meaning for the theorem ("what we're trying to prove"). Despite Theorem 4.2 having a more complex structure due to the conjunction, Theo was able to adapt his set-based argument to show that "if it's in both of these spaces, then it's in rhombi." He was implicitly able to connect the condition "not a kite or not a parallelogram" to the negation of "is a kite and is a parallelogram," which we had not explicitly discussed in the experiment. We see his ability to adapt his Proof 1.3 argument to the more complex theorem as strong evidence of actively abstracting his set-based model to construct a logical justification. It is in the sense of affording general justification that we claim he was acting in the general level of activity since his set-based model was beginning to constitute a new mathematical reality for reasoning about logical structure (across the contexts) and forming logical explanations and justifications.

6.7. Reasoning about abstract properties in statements (Days 8–9) – general level activity

In this final section of results, we shall provide evidence that Theo was operating fully in the general stage of activity in which he could reason about the logic of properties, truth sets, conditional statements, and proofs apart from any specific situation (context). We

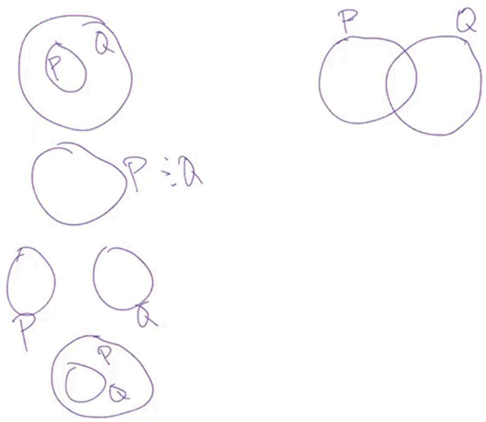


Fig. 9. Five possible relationships between the truth sets of two properties.

introduced this transition by asking the students to imagine two mystery properties *P* and *Q* both defined on the same universal set *S*. We asked them to reason about the statement "Given any *x* in *S*, if *x* has property *P*, then *x* has property *Q*." They were to identify the possible relationships between the truth sets of *P* and *Q* and, in each arrangement, identify some conditional statements that were true and some conditional statements that were false.

Largely led by Theo, the pair identified five possible relationships between the truth sets, as shown in Fig. 9. For the first arrangement, Theo quickly suggested that both the original conditional (provided in the task) and the contrapositive conditional ("For all *x*, if it doesn't have the property of *Q*, then it doesn't have the property *P*.") would be true. In the equal sets arrangement, Phil and Theo proposed four statements as true: original, inverse, converse, and contrapositive.

When the interviewer assigned the pair to work on the other arrangements independently, Theo typed his answers as shown in Fig. 10.

We notice a few interesting things about Theo's responses. First, his reasoning was internally consistent showing that he was able to reason coherently in his set-based model even without specific contexts or meanings for *P* and *Q*. He could identify contexts that matched the different arrangements, but did not rely on them. We provide this as evidence that he was operating at the general stage of activity. Second, inferring from his four statements for the "*P* unrelated to *Q*" (i.e., disjoint) arrangement, he interpreted that diagram to mean *P* and *Q*'s truth sets were complement sets (there were no objects that lacked both properties). This is consistent with his previous use of *regions that gather* instead of *regions that partition*. His representation system without a universal set did not allow him to distinguish the arrangement of complements from that of disjoint sets that do not exhaust the universal set. Third, there are not any normal conditional statements with *P*, *Q*, and their negations alone that are true in the final arrangement ("*P* and *Q* have a crossover"). In two of these cases, Theo improvised a new statement structure of the form "if, then might." This is not consistent with standard mathematical language in which conditionals are always universals (Hammack, 2013; Dawkins & Norton, 2022). However, it shows an appropriate way to improvise language to express a relationship he saw in the truth sets.

We provide one final detail of Theo's reasoning that highlights the complexity of constructing these syntactic logical relationships for students. When explaining how he produced the statements that were true in the "*P* unrelated to *Q*" arrangement, Theo called the second statement the converse. This is normatively the contrapositive, but Theo interpreted it as the converse because he viewed "not" as part of the statement structure, rather than as part of the second predicate. This can be viewed as treating "if, then not" as the syntax of the statement and *P* and *Q* as the semantic referents. This is not the normative way of analyzing the logical form of the statement, since it would change contrapositive equivalence in the presence of certain negations. We recognize it is quite a challenge for students to anticipate such consequences. When the interviewer suggested the standard interpretation and asked Theo why we "move the 'not'," Theo answered, "That's like the full description of that set. The set that does not have this property *Q*." We thus see that while negations continue to be a complicating factor in the construction of logic, Theo was comfortable by this point treating "not *Q*" as a property much as he would the mystery placeholder *Q*.

7. Discussion and conclusions

We identify three primary contributions this paper makes to the existing literature. First, we offer this study as the (to our knowledge) first qualitatively detailed account of how a student learned key concepts of mathematical logic. Our analysis of Theo's learning provides some evidential support for the set-based learning trajectory that underlies the experiment, though ongoing work is necessary to understand its viability at scale. Second, we identified some particular meanings that Theo and Phil employed over time that can be used to model and explain other students' reasoning about logic. Specifically, the *list-and-test meaning* that Theo and Phil used early in the experiment and the distinction between *regions that gather* and *regions that partition* regarding the way Theo

P unrelated to Q
1 If x has property P then x does not have property Q
2 If x has property Q then x does not have property P
3 If x does not property Q then x has property P
4 If x does not have property P then x has property Q

Q in P
1 If x has property P then x might have property Q
2 If x has property Q then x has property P

P and Q have a crossover
1 If x has property P then it might have property Q
2 If x has property Q then it might have property P

Fig. 10. Theo's proposed true statements for the three set diagrams in which P is not a subset of Q (see Fig. 9).

interpreted the spatial diagrams that are often used in logic instruction. Third, we analyzed Theo's learning through the lens of emergent models to portray how he advanced his activity from the situations of specific statements to the general stage of reasoning about mystery properties (logical variables). We think this story of logical abstraction building on the tools of guided reinvention contribute to the already rich literature on Realistic Mathematics Education design work at the undergraduate level.

7.1. Student construction of logic

We proffer this account of Theo's learning as an account of how logical understandings can emerge from set-based reasoning about the structure of conditional statements and their proofs. Naturally, teaching experiment data cannot provide any evidence comparing the learning of logic using sets to the learning of logic using more conventional approaches. What our experiment can show is the richness that set-based reasoning had for supporting Theo in constructing logic from his semantic reasoning and in abstracting his understanding. We argue that Theo's ability to see necessity in logical relationships (e.g., Theo concluding that converse proofs cannot prove, in order to be consistent) and to generalize logical arguments (e.g., adapting his empty intersection argument from Proof 1.3 to Proof 4.2) as evidence that his set-theoretic model constituted a new mathematical reality for *reasoning about logic* (Dawkins & Cook, 2017). His ability to identify valid logical relationships about the mystery properties showed that he had reached a powerful level of generality in his understanding of the logical relationships at play. Theo's pathway to logical abstraction also demonstrates how proof frames relate to conceptual understanding of logic, contrary to Selden, Selden, and Benkhalti's (2018) claim that they are purely procedural. In particular, Theo linked proofs and statements based syntactically on the first and last line of the proof, which for him was not a procedure, but rather connected to the semantic matter of subset relationships between sets of objects.

To further illustrate what was involved in Theo's learning, we highlight some shifts in Theo's ways of talking about the statements and categories in the statements. First, he became comfortable talking about negative categories such as non-rhombus. Second, he shifted rather fluidly between using a) set language interpreted as spatial regions such as "smallest subspace," b) property language such as "meets such criteria," and c) syntactic/temporal order language of "if," "then," and "start." In this way, Theo coordinated quantification, property relations, and statement syntax to give meaning to these complex proof texts. What is more, these understandings allowed him to perceive theorems/proofs about number theory categories and geometric categories as the same, since they all shared set-theoretic structure. We conjecture that developing negative categories and exploring how properties stand for whole classes of objects are essential parts of his construction of a logic of conditional statements and proofs.

7.2. Specific student meanings

While we cannot report on all of the details here, the experiment with Phil and Theo is one of six such teaching experiments we have run with this sequence of tasks. After the teaching experiment reported here, we have used the task sequence once in a whole class setting three times. We recognize the limitations of generalizing from a few students. However, we can say that our ongoing research and teaching suggests that several aspects of Theo and Phil's reasoning have recurred in our (limited but growing) body of experience teaching logic in this manner with undergraduates. Students often take a conditional to be only about the truth set of the antecedent, consistent with the *list-and-test meaning.* What this particular image of student interpretation adds is that the antecedent and consequent parts of a conditional play very different roles in their untrained interpretation. For students to construct the logical relationships we targeted in this experiment, they need to come to reason about them more symmetrically, which for us means imagining both as properties that have truth sets. The *list-and-test meaning* can be understood as a qualitative specification to mathematics of the general psychological model of the Ramsey Test (Evans & Over, 2004). It helps explain some previous findings, such as the claim that students think conditionals are irrelevant when the antecedent is false (Hoyles and Küchemann, 2002).

Very little research on student learning of logic has examined how students make sense of classical diagrams such as Venn or Euler diagrams. This study was not set up as a direct investigation of diagram usage, but we see that even a rather precocious student such as Theo may use these diagrams in powerful ways while still remaining at variance with standard mathematical practice. In particular, Theo's initial use of the diagrams did not entail complement sets at all. We described this using the distinction between *regions that gather* and *regions that partition*. Theo struggled to define negatively defined sets unless he could identity a positive description of the property, which may help explain why his construction of a truth set did not implicitly also create the complement set. Over time, he came to a three-set structure that partitioned the examples into the truth set of P, the truth set of Q, and the truth set of not Q. The *empty intersection meaning* allowed him to reason about the contrapositive without actually constructing two complement sets, which we think is part of its (implicit) appeal. What we take from Theo's story is the necessity of encouraging students to interpret the same diagram in multiple ways. This is exemplified in the final activity of using a diagram to identify multiple conditional statements that are true and false. Further, we encourage students to construct the complements of each set, which necessitates the inclusion of a universal set (which Theo never adopted during this experiment).

7.3. Realistic Mathematics Education

We adopt the emergent models framework to describe Theo's learning both because we think it provides an helpful portrayal of the key developments in his activity and to acknowledge the intellectual contributions that Realistic Mathematics Education (Freudenthal, 1973, 1991; Gravemeijer, 1999) provided us in designing the teaching sequence and running the experiment. Indeed, the guided reinvention heuristic guided how we designed the task sequence and the emergent models framework helped us think about how to scaffold Theo from specific contexts, to comparing contexts, to reasoning about mystery properties (transcending context). These tools for instructional design that builds key mathematical ideas upon student ideas and student activity continue to provide fruitful tools for advancing undergraduate instruction.

As with any guided reinvention instruction, the instructor must choose how to strike a balance between providing guidance and following student reinvention. In these laboratory experiments, we often err on the side of following student reinvention to ensure that we gather rich data about student learning and that we avoid students seeking to identify the answers that the instructor intended. In the classroom, we shift further toward guidance since we do not bear the same burden of gaining evidence of student thinking and learning and because we must manage the progress of the class more tightly. For example, we allowed Theo to develop his images of sets while reasoning about statements and gave ample time for Theo to revise his judgments about proofs as he compared his reasoning across proofs (in a later session). In the classroom, we have introduced lessons on sets (all defined by a unifying property or properties) before we introduce statements. We intentionally focus on students reasoning about negatively defined sets and conjunctively defined sets. In this way, we guide students to address these challenges more directly before they come up in the context of reasoning about proofs. We mention these examples to acknowledge the myriad choices we made in guiding Theo and Phil in the study and to make clear how those choices were modulated by the necessities of research as well as the goals of instruction. We have tried to acknowledge these choices as much as space and clarity allowed, though we are aware much more could be said in this regard.

7.4. Summary

We began with questions about how students' understanding of logical relationships can interact with their content-specific reasoning (Question 1) and how that understanding could apply to reasoning about proofs (Question 2). We claim that Theo's learning progression provides an actionable answer to these questions, even if only through an existence proof. Specifically, logical concepts can be reinvented in context via the emergence of set-based models for the truth and falsehood conditions and the structure of mathematical proofs. We see the richness of Theo's explanations as an endorsement of the potential of set-based reasoning for student learning of logic. While students may also be able to learn logic using truth-tables, we contend that set-based reasoning may be more cognitively amenable. Ongoing work is seeking to understand other students' pathway to these abstractions to create generalizable learning sequences for undergraduate students' introduction to mathematical proving.

Competing interests statement

We have no competing interests to declare regarding this submission.

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