

Quasilinear theory for inhomogeneous plasma

I. Y. Dodin[†]

Princeton Plasma Physics Laboratory, Princeton, NJ 08543, USA

Department of Astrophysical Sciences, Princeton University, Princeton, NJ 08544, USA

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This paper presents quasilinear theory (QLT) for classical plasma interacting with inhomogeneous turbulence. The particle Hamiltonian is kept general; for example, relativistic, electromagnetic, and gravitational effects are subsumed. A Fokker–Planck equation for the dressed ‘oscillation-center’ distribution is derived from the Klimontovich equation and captures quasilinear diffusion, interaction with the background fields, and ponderomotive effects simultaneously. The local diffusion coefficient is manifestly positive-semidefinite. Waves are allowed to be off-shell (i.e. not constrained by a dispersion relation), and a collision integral of the Balescu–Lenard type emerges in a form that is not restricted to any particular Hamiltonian. This operator conserves particles, momentum, and energy, and it also satisfies the H -theorem, as usual. As a spin-off, a general expression for the spectrum of microscopic fluctuations is derived. For on-shell waves, which satisfy a quasilinear wave-kinetic equation, the theory conserves the momentum and energy of the wave–plasma system. The action of nonresonant waves is also conserved, unlike in the standard version of QLT. Dewar’s oscillation-center QLT of electrostatic turbulence (1973, *Phys. Fluids* **16**, 1102) is proven formally as a particular case and given a concise formulation. Also discussed as examples are relativistic electromagnetic and gravitational interactions, and QLT for gravitational waves is proposed.

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[†] Email address for correspondence: idodin@princeton.edu

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1. Introduction

1.1. Background

Electromagnetic waves are present in plasmas naturally, and they are also launched into plasmas using external antennas, for example, for plasma heating and current drive (Stix 1992; Fisch 1987; Pinsker 2001). Nonlinear effects produced by these waves are often modeled within the quasilinear (QL) approximation, meaning that the nonlinearities are retained in the low-frequency (‘average’) dynamics but neglected in the high-frequency dynamics. Two separate paradigms exist within this approach.

In the *first paradigm*, commonly known as ‘the’ QL theory (QLT), the focus is made on resonant interactions. Nonresonant particles are considered as a background that is homogeneous in spatial (Vedenov *et al.* 1961; Drummond & Pines 1962; Kennel & Engelmann 1966; Rogister & Oberman 1968, 1969) or generalized coordinates (Kaufman 1972; Eriksson & Helander 1994; Catto *et al.* 2017); then the oscillating fields can be described in terms of global modes. This approach has the advantage of simplicity, but its applications are limited in that real plasmas are never actually homogeneous in any predefined variables (and, furthermore, tend to exhibit nonlinear instabilities in the presence of intense waves). The ‘ponderomotive’ dynamics determined by the gradients of the wave and plasma parameters is lost in this approach; then, spurious effects can emerge and have to be dealt with (Lee *et al.* 2018).

The *second paradigm* successfully captures the ponderomotive dynamics by introducing effective Hamiltonians for the particle average motion (Gaponov & Miller 1958; Motz & Watson 1967; Cary & Kaufman 1981; Kaufman 1987; Dodin 2014). But as usual in perturbation theory (Lichtenberg & Lieberman 1992), those Hamiltonians are by default singular for resonant interactions. Thus, such models have limited reach as well, and remarkable subtleties are still found even in basic QL problems. For example, it is still debated (Ochs & Fisch 2021a; Ochs 2021) to which extent the QL effects that remove resonant particles while capturing their energy (Fisch & Rax 1992) also remove charge along with the resonant particles thereby driving plasma rotation (Fetterman & Fisch 2008). This state of affairs means, arguably, that a clear comprehensive theory of QL wave–plasma interactions remains to be developed — a challenge that must be faced.

The first framework that subsumed both resonant and nonresonant interactions in inhomogeneous plasmas was proposed by Dewar (1973) for electrostatic turbulence in nonmagnetized plasma and is known as ‘oscillation-center’ (OC) QLT. It was later extended by McDonald *et al.* (1985) to nonrelativistic magnetized plasma. However,

both of these models are partly heuristic and limited in several respects. For example, they are bounded by the limitations of the variational approach used therein, and they separate resonant particles from nonresonant particles somewhat arbitrarily (see also (Ye & Kaufman 1992)). Both models also assume specific particle Hamiltonians and require that waves be governed by a QL wave-kinetic equation (WKE), i.e. be only *weakly* dissipative, or ‘on-shell’. (Somewhat similar formulations were also proposed, independently and without references to the OC formalism, in (Weibel 1981; Yasseen 1983; Yasseen & Vaclavik 1986).) This means that collisions and microscopic fluctuations are automatically excluded. Attempts to merge QLT and the WKE with theory of plasma collisions were made (Rogister & Oberman 1968; Schlickeiser & Yoon 2014; Yoon *et al.* 2016) but have not yielded a local theory applicable to inhomogeneous plasma. In particular, the existing models rely on global-mode decompositions and treat complex frequencies heuristically. Thus, the challenge stands.

Related problems are also of interest in the context of gravitostatic interactions (Chavanis 2012; Hamilton 2020; Magorrian 2021), where inhomogeneity of the background fields cannot be neglected in principle (Binney & Tremaine 2008). (To our knowledge, OC QLT analogs have not been considered in this field.) Similar challenges also arise in QLT of dispersive gravitational waves (Garg & Dodin 2021a, 2020). Hence, one cannot help but wonder whether a specific form of the particle Hamiltonian really matters for developing QLT or it is irrelevant and therefore should not be assumed. Since basic theory of linear waves is independent of Maxwell’s equations (Tracy *et al.* 2014; Dodin & Fisch 2012; Dodin *et al.* 2017), a general QLT might be possible too, and it might be easier to develop than a zoo of problem-specific models.

1.2. Outline

Here, we propose a general QLT that allows for plasma inhomogeneity and is not restricted to any particular Hamiltonian or interaction field. By starting with the Klimontovich equation, we derive a model that captures QL diffusion, interaction with background fields, and ponderomotive effects simultaneously. The local diffusion coefficient in this model is manifestly positive-semidefinite. Waves are allowed to be off-shell, and a collision integral of the Balescu–Lenard type emerges for general Hamiltonian interactions. This operator conserves particles, momentum, and energy, and it also satisfies the H -theorem, as usual. As a spin-off, a general expression for the spectrum of microscopic fluctuations of the interaction field is derived. For on-shell waves governed by the WKE, the theory conserves the momentum and energy of the wave–plasma system. The action of nonresonant waves is also conserved, unlike in the standard version of QLT.† Dewar’s OC QLT of electrostatic turbulence (Dewar 1973) is proven formally as a particular case and given a concise formulation. Also discussed as examples are relativistic electromagnetic and gravitational interactions, and QLT for gravitational waves is proposed. Overall, our formulation interconnects many known results that in the past were derived independently and reproduces them within a unifying framework.

This progress is made by giving up the traditional Fourier–Laplace approach. The author takes the stance that the global-mode language is unnatural for inhomogeneous-plasma problems (i.e. all real-plasma problems). A fundamental theory must be local. Likewise, the variational approach that is used sometimes in QL calculations is not universally advantageous, especially for describing dissipation. Instead of those methods, we use operator analysis and the Weyl symbol calculus, as has also been proven fruitful in

† The standard QLT (as in, for example, (Drummond & Pines 1962)) does not properly conserve energy–momentum either, even though it is formally conservative (see section 7.3.2).

other recent studies of ponderomotive effects and turbulence (Ruiz 2017; Ruiz & Dodin 2017b; Zhu & Dodin 2021) and linear-wave theory (Dodin *et al.* 2019). No logical leaps are made in this paper other than assuming the QL approximation *per se* and a certain ordering.‡ In a nutshell, we treat the commonly known QL-diffusion coefficient as a nonlocal operator, and we systematically approximate it using the Weyl symbol calculus. It is the nonlocality of this operator that gives rise to ponderomotive effects and ensures the proper conservation laws. The existing concept of ‘adiabatic diffusion’ (Galeev & Sagdeev 1985; Stix 1992) captures some of that, but systematic application of operator analysis yields a more general, more accurate, and more rigorous theory.

The author hopes not that this paper is an entertaining read. However, the paper was intended as self-contained, maximally structured, and easily searchable, so readers interested in specific questions could find and understand answers without having to read the whole paper. The text is organized as follows. In section 2, we present a primer on the Weyl symbol calculus and the associated notation. In section 3, we formulate our general model. In section 4, we introduce the necessary auxiliary theorems. In section 5, we derive a QL model for plasma interacting with prescribed waves. The waves may or may not be on-shell or self-consistent. (Their origin and dynamics are not addressed in section 5.) In section 6, we consider interactions with self-consistent waves. In particular, we separate out microscopic fluctuations, calculate their average distribution, and derive the corresponding collision operator. In section 7, we assume that the remaining macroscopic waves are on-shell, rederive the WKE, and show that our QL model combined with the WKE is conservative. In section 8, we discuss the general properties that our model predicts for plasmas in thermal equilibrium. In section 9, we show how to apply our theory to nonrelativistic electrostatic interactions, relativistic electromagnetic interactions, Newtonian gravity, and relativistic gravity. In section 10, we summarize our results. Auxiliary calculations are presented in appendices A–D, and appendix G summarizes our notation. This notation is extensive and may not be particularly intuitive. Thus, readers are encouraged to occasionally scout section 9 for examples even before fully absorbing the preceding sections.

An impatient reader can also skip calculations entirely and consult only the summaries of the individual sections (2.3, 3.4, 4.4, 5.6, 6.9, 7.6, 8.5; they are mostly self-contained) and then proceed to the examples in section 9. However, the main point of this work is not just the final results *per se* (surely, some readers will find them obvious) but also that they are derived with minimal assumptions and rigorously, which makes them reliable. A reader may also notice that we rederive some known results along the way, for example, basic linear-wave theory and the WKE. This is done for completeness and, more importantly, with the goal to present all pieces of the story within a unified notation.

2. A math primer

Here, we summarize the machinery to be used in the next sections. This machinery is not new, but a brief overview is in order at least to introduce the necessary notation. A more comprehensive summary, with proofs, can be found in (Dodin *et al.* 2019, supplementary material). For extended discussions, see (Tracy *et al.* 2014; Ruiz 2017; McDonald 1988; Littlejohn 1986).

‡ We treat the traditional QL approximation as a given mathematical model. We seek to push this model to its limits rather than to examine its validity, which is a separate issue. For discussions on the validity of the QL approximation, see (Besse *et al.* 2011; Escande *et al.* 2018; Crews & Shumlak 2022).

2.1. Weyl symbol calculus on spacetime

2.1.1. Basic notation

We denote the time variable as t , space coordinates as $\mathbf{x} \equiv (x^1, x^2, \dots, x^n)$, spacetime coordinates as $\mathbf{x} \equiv (x^0, x^1, \dots, x^n)$, where $x^0 \doteq t$ and $x^i \doteq x^i$. The symbol $\doteq$ denotes definitions, and Latin indices from the middle of the alphabet ($i, j, \dots$) range from 1 to n unless specified otherwise. We assume the spacetime-coordinate domain to be $\mathbb{R}^n$.[†] Functions on $\mathbf{x}$ form a Hilbert space $\mathcal{H}_x$ with an inner product that we define as

$$\langle \xi | \psi \rangle \doteq \int d\mathbf{x} \xi^*(\mathbf{x}) \psi(\mathbf{x}). \quad (2.1)$$

The symbol $*$ denotes complex conjugate,

$$d\mathbf{x} \doteq dx^0 dx^1 \dots dx^n = dt dx^1 \dots dx^n \quad (2.2)$$

(a similar convention is assumed also for other multi-dimensional variables used below), and integrals in this paper are taken over $(-\infty, \infty)$ unless specified otherwise. Operators on $\mathcal{H}_x$ will be denoted with carets, and we will use indexes H and A to denote their Hermitian and anti-Hermitian parts. For a given operator $\hat{A}$, one has $\hat{A} = \hat{A}_{\text{H}} + i\hat{A}_{\text{A}}$,

$$\hat{A}_{\text{H}} = \hat{A}_{\text{H}}^\dagger \doteq \frac{1}{2} (\hat{A} + \hat{A}^\dagger), \quad \hat{A}_{\text{A}} = \hat{A}_{\text{A}}^\dagger \doteq \frac{1}{2i} (\hat{A} - \hat{A}^\dagger), \quad (2.3)$$

where † denotes the Hermitian adjoint with respect to the inner product (2.1). The case of a more general inner product is detailed in (Dodin *et al.* 2019, supplementary material).

2.1.2. Vector fields

For multi-component fields $\boldsymbol{\psi} \equiv (\psi^1, \psi^2, \dots, \psi^M)^\tau$ (our τ denotes the matrix transpose), or ‘row vectors’ (actually, tuples), we *define* the dual ‘column vectors’ $\boldsymbol{\psi}^\dagger \equiv (\psi_1^*, \psi_2^*, \dots, \psi_M^*)$ via $\boldsymbol{\psi}^\dagger \doteq \mathbf{g} \boldsymbol{\psi}^*$. The matrix $\mathbf{g}$ is assumed to be real, diagonal, invertible, and constant; other than that, it can be chosen as suits a specific problem. (For example, a unit matrix may suffice.) This induces the standard rule of index manipulation

$$\psi_i = g_{ij} \psi^j, \quad \psi^i = g^{ij} \psi_j, \quad i, j = 1, 2, \dots, M, \quad (2.4)$$

where g_{ij} are elements of $\mathbf{g}$ and g^{ij} are elements of $\mathbf{g}^{-1}$. Summation over repeating indices is assumed. The rules of matrix multiplication apply to row and column vectors as usual. Then, for $\boldsymbol{\psi} \equiv (\psi^1, \psi^2, \dots, \psi^M)^\tau$ and $\boldsymbol{\xi} \equiv (\xi^1, \xi^2, \dots, \xi^M)^\tau$, the quantity $\boldsymbol{\psi} \boldsymbol{\xi}$ is a matrix with elements $\psi^i \xi^j$, $\boldsymbol{\psi} \boldsymbol{\xi}^\dagger$ is a matrix with elements $\psi^i \xi_j^*$ and $\boldsymbol{\xi}^\dagger \boldsymbol{\psi}$ is its scalar trace:[†]

$$\boldsymbol{\xi}^\dagger \boldsymbol{\psi} = \text{tr}(\boldsymbol{\psi} \boldsymbol{\xi}^\dagger) = \xi_i^* \psi^i = g_{ij} \xi^{i*} \psi^j, \quad i, j = 1, 2, \dots, M. \quad (2.5)$$

(Similarly, for $\boldsymbol{\chi} \equiv (\chi_1, \chi_2, \dots, \chi_M)$ and $\boldsymbol{\eta} \equiv (\eta_1, \eta_2, \dots, \eta_M)$, $\boldsymbol{\eta} \boldsymbol{\chi}$ is a matrix with elements $\eta_i \chi_j$.) We use the (2.5) to define a Hilbert space $\mathcal{H}_x^M$ of M -dimensional vector fields on $\mathbf{x}$, specifically, by adopting the inner product

$$\langle \boldsymbol{\xi} | \boldsymbol{\psi} \rangle \doteq \int d\mathbf{x} \boldsymbol{\xi}^\dagger(\mathbf{x}) \boldsymbol{\psi}(\mathbf{x}). \quad (2.6)$$

[†] This excludes periodic boundary conditions, albeit not entirely (section 3.1). Other than that, the spacetime metric can still be non-Euclidean, as illustrated by an application to relativistic gravity in section 9.4. See also the footnotes on pages 7 and 30.

[†] A common notation is $\boldsymbol{\xi}^\dagger \boldsymbol{\psi} = \boldsymbol{\xi} \cdot \boldsymbol{\psi}$, but we reserve the dot-product notation for a scalar product of different quantities (section 2.1.3).

Below, the distinction between $\mathcal{H}_x^M$ and $\mathcal{H}_x$ will be assumed but not emphasized. Also note that (2.5) yields

$$\boldsymbol{\xi}^\dagger(\boldsymbol{\psi}\boldsymbol{\psi}^\dagger)\boldsymbol{\xi} = (\boldsymbol{\xi}^\dagger\boldsymbol{\psi})(\boldsymbol{\psi}^\dagger\boldsymbol{\xi}) = |\boldsymbol{\xi}^\dagger\boldsymbol{\psi}|^2 \geq 0 \quad (2.7)$$

for any $\boldsymbol{\xi}$ and $\boldsymbol{\psi}$. Thus, dyadic matrices of the form $\boldsymbol{\psi}\boldsymbol{\psi}^\dagger$ are positive-semidefinite, even though $\boldsymbol{\psi}^\dagger\boldsymbol{\psi}$ may be negative (when $\mathbf{g}$ is not positive-definite).

For general matrices, the indices can be raised and lowered using $\mathbf{g}$ and $\mathbf{g}^{-1}$ as usual. The Hermitian adjoint $\mathbf{A}^\dagger$ for a given matrix $\mathbf{A}$ is defined such that $(\mathbf{A}^\dagger\boldsymbol{\xi})^\dagger\boldsymbol{\psi} = \boldsymbol{\xi}^\dagger(\mathbf{A}\boldsymbol{\psi})$ for any $\boldsymbol{\psi}$ and $\boldsymbol{\xi}$, which means

$$(\mathbf{A}^\dagger)_j^i = (\mathbf{A})^{i*}_j = \mathbf{A}^{i*}_{j^*}, \quad i, j = 1, 2, \dots, M. \quad (2.8)$$

The Hermitian and anti-Hermitian parts are defined as

$$\mathbf{A}_H = \mathbf{A}_H^\dagger \doteq \frac{1}{2}(\mathbf{A} + \mathbf{A}^\dagger), \quad \mathbf{A}_A = \mathbf{A}_A^\dagger \doteq \frac{1}{2i}(\mathbf{A} - \mathbf{A}^\dagger), \quad (2.9)$$

so $\mathbf{A} = \mathbf{A}_H + i\mathbf{A}_A$. For one-dimensional matrices (scalars), one has $\mathbf{A} = A$,

$$A_H = A_H^* = \text{re } A, \quad A_A = A_A^* = \text{im } A, \quad (2.10)$$

where re and im denote the real part and the imaginary part, respectively. We also define matrix operators $\hat{\mathbf{A}}$ as matrices of the corresponding operators $\hat{\mathbf{A}}^i_j$. Because $\mathbf{g}$ is constant, index manipulation applies as usual. Also as usual, one has

$$\hat{\mathbf{A}}_H = \hat{\mathbf{A}}_H^\dagger \doteq \frac{1}{2}(\hat{\mathbf{A}} + \hat{\mathbf{A}}^\dagger), \quad \hat{\mathbf{A}}_A = \hat{\mathbf{A}}_A^\dagger \doteq \frac{1}{2i}(\hat{\mathbf{A}} - \hat{\mathbf{A}}^\dagger), \quad (2.11)$$

and $\hat{\mathbf{A}} = \hat{\mathbf{A}}_H + i\hat{\mathbf{A}}_A$, where † is the Hermitian adjoint with respect to the inner product (2.6).

2.1.3. Bra-ket notation

Let us define the following operators that are Hermitian under the inner product (2.1):

$$\hat{x}^0 \equiv \hat{t} \doteq t, \quad \hat{x}^i \equiv \hat{x}^i \doteq x^i, \quad \hat{k}_0 \equiv -\hat{\omega} \doteq -i\partial_t, \quad \hat{k}_i \equiv -i\partial_i, \quad (2.12)$$

where $\partial_0 \equiv \partial_t \doteq \partial/\partial x^0$ and $\partial_i \doteq \partial/\partial x^i$. Accordingly,

$$\hat{\mathbf{x}} \equiv (\hat{x}^0, \hat{x}^1, \dots, \hat{x}^n) = (\hat{t}, \hat{\mathbf{x}}), \quad \hat{\mathbf{k}} \equiv (\hat{k}_0, \hat{k}_1, \dots, \hat{k}_n) = (-\hat{\omega}, \hat{\mathbf{k}}) \quad (2.13)$$

are understood as the spacetime-position operator and the corresponding wavevector operator, which will also be expressed as follows:

$$\hat{\mathbf{x}} = \mathbf{x}, \quad \hat{\mathbf{k}} = -i\partial_{\mathbf{x}}. \quad (2.14)$$

Also note the commutation property, where δ_i^j is the Kronecker symbol:[†]

$$[\hat{x}^i, \hat{k}_j] = i\delta_j^i, \quad i, j = 0, 1, \dots, n. \quad (2.15)$$

The eigenvectors of the operators (2.14) will be denoted as ‘kets’ $|\mathbf{x}\rangle$ and $|\mathbf{k}\rangle$:[‡]

$$\hat{\mathbf{x}}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle, \quad \hat{\mathbf{k}}|\mathbf{k}\rangle = \mathbf{k}|\mathbf{k}\rangle, \quad (2.16)$$

[†] Spaces with periodic boundary conditions require a different approach (Rigas *et al.* 2011), so they are not considered here (yet see section 3.1). That said, for a system that is large enough, the boundary conditions are unimportant; then the toolbox presented here is applicable as is.

[‡] More precisely, $|\mathbf{x}\rangle$ is the ket $|\mathbf{e}(\hat{\mathbf{x}}; \mathbf{x})\rangle$ that is an eigenvector of each $\hat{x}^i$ with the corresponding eigenvalues being x^i . A similar comment applies to $|\mathbf{k}\rangle$.

and we assume the usual normalization:

$$\langle \mathbf{x}_1 | \mathbf{x}_2 \rangle = \delta(\mathbf{x}_1 - \mathbf{x}_2), \quad \langle \mathbf{k}_1 | \mathbf{k}_2 \rangle = \delta(\mathbf{k}_1 - \mathbf{k}_2), \quad (2.17)$$

where δ is the Dirac delta function. Both sets $\{|\mathbf{x}\rangle, \mathbf{x} \in \mathbb{R}^n\}$ and $\{|\mathbf{k}\rangle, \mathbf{k} \in \mathbb{R}^n\}$, where $n \doteq n + 1$, form a complete basis on $\mathcal{H}_x$, and the eigenvalues of these operators form an extended *real* phase space $(\mathbf{x}, \mathbf{k})$, where

$$\mathbf{x} \equiv (t, \mathbf{x}), \quad \mathbf{k} \equiv (-\omega, \mathbf{k}). \quad (2.18)$$

The notation $\mathbf{k} \cdot \mathbf{s} \doteq -\omega\tau + \mathbf{k} \cdot \mathbf{s}$ will be assumed for any $\mathbf{s} \equiv (\tau, \mathbf{s})$, and $\mathbf{k} \cdot \mathbf{s} \doteq k_i s^i$. In particular, for any ψ and constant $\mathbf{s}$, one has

$$\exp(i\hat{\mathbf{k}} \cdot \mathbf{s})\psi(\mathbf{x}) = \exp(\mathbf{s} \cdot \partial_{\mathbf{x}})\psi(\mathbf{x}) = \psi(\mathbf{x} + \mathbf{s}), \quad (2.19)$$

as seen from comparing the Taylor expansions of the latter two expressions. (A generalization of this formula is discussed in section 4.1.) Also,

$$\langle \mathbf{x} | \mathbf{k} \rangle = \langle \mathbf{k} | \mathbf{x} \rangle^* = (2\pi)^{-n/2} \exp(i\mathbf{k} \cdot \mathbf{x}), \quad (2.20)$$

and

$$\int d\mathbf{x} |\mathbf{x}\rangle \langle \mathbf{x}| = \hat{1}, \quad \int d\mathbf{k} |\mathbf{k}\rangle \langle \mathbf{k}| = \hat{1}. \quad (2.21)$$

Here, ‘bra’ $\langle \mathbf{x}|$ is the one-form dual to $|\mathbf{x}\rangle$, $\langle \mathbf{k}|$ is the one-form dual to $|\mathbf{k}\rangle$, and $\hat{1}$ is the unit operator. Any field ψ on $\mathbf{x}$ can be viewed as the $\mathbf{x}$ representation (‘coordinate representation’) of $|\psi\rangle$, i.e. the projection of an abstract ket vector $|\psi\rangle \in \mathcal{H}_x$ on $|\mathbf{x}\rangle$:

$$\psi(\mathbf{x}) = \langle \mathbf{x} | \psi \rangle. \quad (2.22)$$

Similarly, $\langle \mathbf{k} | \psi \rangle$ is the $\mathbf{k}$ representation (‘spectral representation’) of $|\psi\rangle$, or the Fourier image of ψ :

$$\hat{\psi}(\mathbf{k}) \doteq \langle \mathbf{k} | \psi \rangle = \frac{1}{(2\pi)^{n/2}} \int d\mathbf{x} e^{-i\mathbf{k} \cdot \mathbf{x}} \psi(\mathbf{x}). \quad (2.23)$$

2.1.4. Wigner–Weyl transform

For a given operator $\hat{A}$ and a given field ψ , $\hat{A}\psi$ can be expressed in the integral form

$$\hat{A}\psi(\mathbf{x}) = \int d\mathbf{x}' \langle \mathbf{x} | \hat{A} | \mathbf{x}' \rangle \psi(\mathbf{x}'), \quad (2.24)$$

where $\langle \mathbf{x} | \hat{A} | \mathbf{x}' \rangle$ is a function of $(\mathbf{x}, \mathbf{x}')$. This is called the $\mathbf{x}$ representation (‘coordinate representation’) of $\hat{A}$. Equivalently, $\hat{A}$ can be given a phase-space, or Weyl, representation, i.e. expressed through a function of the phase-space coordinates, $A(\mathbf{x}, \mathbf{k})$:[†]

$$\hat{A} = \frac{1}{(2\pi)^n} \int d\mathbf{x} d\mathbf{k} d\mathbf{s} |\mathbf{x} + \mathbf{s}/2\rangle A(\mathbf{x}, \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{s}} \langle \mathbf{x} - \mathbf{s}/2| \equiv \text{oper}_x A. \quad (2.25)$$

The function $A(\mathbf{x}, \mathbf{k})$, called the Weyl symbol (or just ‘symbol’) of $\hat{A}$, is given by

$$A(\mathbf{x}, \mathbf{k}) \doteq \int d\mathbf{s} \langle \mathbf{x} + \mathbf{s}/2 | \hat{A} | \mathbf{x} - \mathbf{s}/2 \rangle e^{-i\mathbf{k} \cdot \mathbf{s}} \equiv \text{symb}_x \hat{A}. \quad (2.26)$$

The $\mathbf{x}$ and phase-space representations are connected by the Fourier transform:

$$\langle \mathbf{x} | \hat{A} | \mathbf{x}' \rangle = \frac{1}{(2\pi)^n} \int d\mathbf{k} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} A\left(\frac{\mathbf{x} + \mathbf{x}'}{2}, \mathbf{k}\right). \quad (2.27)$$

[†] Analytic continuation to complex arguments is possible, but by default, $\mathbf{x}$ and $\mathbf{k}$ are real.

This also leads to the following notable properties of Weyl symbols:

$$\langle \mathbf{x} | \hat{A} | \mathbf{x} \rangle = \frac{1}{(2\pi)^n} \int d\mathbf{k} A(\mathbf{x}, \mathbf{k}), \quad \langle \mathbf{k} | \hat{A} | \mathbf{k} \rangle = \frac{1}{(2\pi)^n} \int d\mathbf{x} A(\mathbf{x}, \mathbf{k}). \quad (2.28)$$

An operator unambiguously determines its symbol, and *vice versa*. We denote this isomorphism as $\hat{A} \leftrightarrow A$. The mapping $\hat{A} \mapsto A$ is called the Wigner transform, and $A \mapsto \hat{A}$ is called the Weyl transform. For uniformity, we call them the direct and inverse Wigner–Weyl transform. The isomorphism $\leftrightarrow$ is natural in that it has the following properties:

$$\hat{1} \leftrightarrow 1, \quad \hat{\mathbf{x}} \leftrightarrow \mathbf{x}, \quad \hat{\mathbf{k}} \leftrightarrow \mathbf{k}, \quad h(\hat{\mathbf{x}}) \leftrightarrow h(\mathbf{x}), \quad h(\hat{\mathbf{k}}) \leftrightarrow h(\mathbf{k}), \quad \hat{A}^\dagger \leftrightarrow A^*, \quad (2.29)$$

where h is any function and $\hat{A}$ is any operator. The product of two operators maps to the so-called Moyal product, or star product, of their symbols (Moyal 1949):

$$\hat{A}\hat{B} \leftrightarrow A(\mathbf{x}, \mathbf{k}) \star B(\mathbf{x}, \mathbf{k}) \doteq A(\mathbf{x}, \mathbf{k}) e^{i\hat{\mathcal{L}}_{\mathbf{x}}/2} B(\mathbf{x}, \mathbf{k}), \quad (2.30)$$

which is associative:

$$\hat{A}\hat{B}\hat{C} \leftrightarrow (A \star B) \star C = A \star (B \star C) \equiv A \star B \star C. \quad (2.31)$$

Here, $\hat{\mathcal{L}}_{\mathbf{x}} \doteq \overleftarrow{\partial}_{\mathbf{x}} \cdot \overrightarrow{\partial}_{\mathbf{k}} - \overleftarrow{\partial}_{\mathbf{k}} \cdot \overrightarrow{\partial}_{\mathbf{x}}$, and the arrows indicate the directions in which the derivatives act. For example, $A\hat{\mathcal{L}}_{\mathbf{x}}B$ is just the canonical Poisson bracket on $(\mathbf{x}, \mathbf{k})$:

$$A\hat{\mathcal{L}}_{\mathbf{x}}B = \{A, B\}_{\mathbf{x}} \doteq -\frac{\partial A}{\partial t} \frac{\partial B}{\partial \omega} + \frac{\partial A}{\partial \omega} \frac{\partial B}{\partial t} + \frac{\partial A}{\partial x^i} \frac{\partial B}{\partial k_i} - \frac{\partial A}{\partial k_i} \frac{\partial B}{\partial x^i}. \quad (2.32)$$

These formulas readily yield

$$h(\hat{\mathbf{x}})\hat{k}_\alpha \leftrightarrow k_\alpha h(\mathbf{x}) + \frac{i}{2} \partial_\alpha h(\mathbf{x}), \quad \hat{k}_\alpha h(\hat{\mathbf{x}}) \leftrightarrow k_\alpha h(\mathbf{x}) - \frac{i}{2} \partial_\alpha h(\mathbf{x}), \quad (2.33)$$

also $h(\hat{\mathbf{k}})e^{i\mathbf{k} \cdot \hat{\mathbf{x}}} \leftrightarrow h(\mathbf{k}) \star e^{i\mathbf{K} \cdot \mathbf{x}} = h(\mathbf{k} + \mathbf{K}/2)e^{i\mathbf{K} \cdot \mathbf{x}}$, etc. Another notable formula to be used below, which flows from (2.28) and (2.31), is

$$\langle \mathbf{x} | \hat{A}\hat{B}\hat{C} | \mathbf{x} \rangle = \frac{1}{(2\pi)^n} \int d\mathbf{k} (A \star B \star C)(\mathbf{x}, \mathbf{k}). \quad (2.34)$$

The Moyal product is particularly handy when $\partial_{\mathbf{x}}\partial_{\mathbf{k}} \sim \epsilon \ll 1$. Such ϵ is often called the geometrical-optics parameter. Since $\hat{\mathcal{L}}_{\mathbf{x}} = \mathcal{O}(\epsilon)$, one can express the Moyal product as an asymptotic series in powers of ϵ :

$$\star = \hat{1} + i\hat{\mathcal{L}}_{\mathbf{x}}/2 - \hat{\mathcal{L}}_{\mathbf{x}}^2/8 + \dots \quad (2.35)$$

2.1.5. Weyl expansion of operators

Operators can be approximated by approximating their symbols (Dodin *et al.* 2019; McDonald 1988). If $\hat{A}$ is approximately local in $\mathbf{x}$ (i.e. if $\hat{A}\psi(\mathbf{x})$ is determined by values $\psi(\mathbf{x} + \mathbf{s})$ only with small enough $\mathbf{s}$), its symbol can be adequately represented by the first few terms of the Taylor expansion in $\mathbf{k}$:

$$A(\mathbf{x}, \mathbf{k}) = A(\mathbf{x}, \mathbf{0}) + \Theta_0(\mathbf{x}) \cdot \mathbf{k} + \dots, \quad \Theta_0(\mathbf{x}) \doteq (\partial_{\mathbf{k}} A(\mathbf{x}, \mathbf{k}))_{\mathbf{k}=\mathbf{0}}. \quad (2.36)$$

Application of $\text{oper}_{\mathbf{x}}$ to this formula leads to

$$\hat{A} \approx A(\hat{\mathbf{x}}, \mathbf{0}) + \frac{1}{2} (\hat{\Theta}_0 \cdot \hat{\mathbf{k}} + \hat{\mathbf{k}} \cdot \hat{\Theta}_0) + \dots, \quad (2.37)$$

where $\hat{\Theta}_0 \doteq \Theta_0(\hat{\mathbf{x}})$. One can also rewrite (2.37) using the commutation property

$$[\hat{\mathbf{k}}, \hat{\Theta}_0] = -i(\partial_{\mathbf{x}} \cdot \Theta_0)(\hat{\mathbf{x}}). \quad (2.38)$$

In the $\mathbf{x}$ representation, this leads to

$$\hat{\mathbf{A}} = \mathbf{A}(\mathbf{x}, \mathbf{0}) - i\boldsymbol{\Theta}_0(\mathbf{x}) \cdot \partial_{\mathbf{x}} - \frac{i}{2} (\partial_{\mathbf{x}} \cdot \boldsymbol{\Theta}_0(\mathbf{x})) + \dots \quad (2.39)$$

The effect of a nonlocal operator on eikonal (monochromatic or quasimonochromatic) fields can be approximated similarly. Suppose $\psi = e^{i\theta}\check{\psi}$, where the dependence of $\bar{\mathbf{k}} \doteq \partial_{\mathbf{x}}\theta$ and $\check{\psi}$ on $\mathbf{x}$ is slower than that of θ by factor $\epsilon \ll 1$. Then, $\hat{\mathbf{A}}\psi = e^{i\theta}\hat{\mathbf{A}}'\check{\psi}$, where $\hat{\mathbf{A}}' \doteq e^{-i\theta(\tilde{\mathbf{x}})}\hat{\mathbf{A}}e^{i\theta(\tilde{\mathbf{x}})}$, and the symbol of $\hat{\mathbf{A}}'$ can be approximated as follows:

$$\mathbf{A}'(\mathbf{x}, \mathbf{k}) = \mathbf{A}(\mathbf{x}, \bar{\mathbf{k}}(\mathbf{x}) + \mathbf{k}) + \mathcal{O}(\epsilon^2). \quad (2.40)$$

By expanding this in $\mathbf{k}$ and applying $\text{oper}_{\mathbf{x}}$, one obtains

$$\hat{\mathbf{A}}' = \mathbf{A}(\mathbf{x}, \bar{\mathbf{k}}(\mathbf{x})) - i\boldsymbol{\Theta}(\mathbf{x}) \cdot \partial_{\mathbf{x}} - \frac{i}{2} (\partial_{\mathbf{x}} \cdot \boldsymbol{\Theta}(\mathbf{x})) + \mathcal{O}(\epsilon^2), \quad (2.41)$$

where $\boldsymbol{\Theta}(\mathbf{x}) \doteq (\partial_{\mathbf{k}}\mathbf{A}(\mathbf{x}, \mathbf{k}))_{\mathbf{k}=\bar{\mathbf{k}}(\mathbf{x})}$. Neglecting the $\mathcal{O}(\epsilon^2)$ corrections in this formula leads to what is commonly known as the geometrical-optics approximation ([Dodin *et al.* 2019](#)).

2.1.6. Wigner functions

Any ket $|\psi\rangle$ generates a dyadic $|\psi\rangle\langle\psi|$. In quantum mechanics, such dyadics are known as density operators (of pure states). For our purposes, though, it is more convenient to define the density operator in a slightly different form, namely, as

$$\hat{\mathbf{W}}_{\psi} \doteq (2\pi)^{-n} |\psi\rangle\langle\psi|. \quad (2.42)$$

The symbol of this operator, $\mathbf{W}_{\psi} = \text{symb}_{\mathbf{x}}\hat{\mathbf{W}}_{\psi}$, is a real function called the Wigner function. It is given by

$$\begin{aligned} \mathbf{W}_{\psi}(\mathbf{x}, \mathbf{k}) &= \frac{1}{(2\pi)^n} \int d\mathbf{s} \langle \mathbf{x} + \mathbf{s}/2 | \psi \rangle \langle \psi | \mathbf{x} - \mathbf{s}/2 \rangle e^{-i\mathbf{k} \cdot \mathbf{s}} \\ &= \frac{1}{(2\pi)^n} \int d\mathbf{s} \psi(\mathbf{x} + \mathbf{s}/2) \psi^*(\mathbf{x} - \mathbf{s}/2) e^{-i\mathbf{k} \cdot \mathbf{s}}, \end{aligned} \quad (2.43)$$

which is manifestly real and can be understood as the (inverse) Fourier image of

$$\mathbf{C}_{\psi}(\mathbf{x}, \mathbf{s}) \doteq \psi(\mathbf{x} + \mathbf{s}/2) \psi^*(\mathbf{x} - \mathbf{s}/2) = \int d\mathbf{k} \mathbf{W}_{\psi}(\mathbf{x}, \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{s}}. \quad (2.44)$$

Any function bilinear in ψ and ψ^* can be expressed through $\mathbf{W}_{\psi}$. Specifically, for any operators $\hat{\mathbf{L}}$ and $\hat{\mathbf{R}}$, one has

$$\begin{aligned} (\hat{\mathbf{L}}\psi(\mathbf{x}))(\hat{\mathbf{R}}\psi(\mathbf{x}))^* &= \langle \mathbf{x} | \hat{\mathbf{L}} | \psi \rangle \langle \psi | \hat{\mathbf{R}}^{\dagger} | \mathbf{x} \rangle \\ &= (2\pi)^n \langle \mathbf{x} | \hat{\mathbf{L}} \hat{\mathbf{W}}_{\psi} \hat{\mathbf{R}}^{\dagger} | \mathbf{x} \rangle \\ &= \int d\mathbf{k} \mathbf{L}(\mathbf{x}, \mathbf{k}) \star \mathbf{W}_{\psi}(\mathbf{x}, \mathbf{k}) \star \mathbf{R}^*(\mathbf{x}, \mathbf{k}), \end{aligned} \quad (2.45)$$

where $\mathbf{L}$ and $\mathbf{R}$ are the corresponding symbols and (2.28) was used along with (2.31). As a corollary, and as also seen from (2.28), one has

$$|\psi(\mathbf{x})|^2 = \int d\mathbf{k} \mathbf{W}_{\psi}(\mathbf{x}, \mathbf{k}), \quad |\check{\psi}(\mathbf{k})|^2 = \int d\mathbf{x} \mathbf{W}_{\psi}(\mathbf{x}, \mathbf{k}). \quad (2.46)$$

As a reminder, $\psi(\mathbf{x}) = \langle \mathbf{x} | \psi \rangle$ and $\check{\psi}(\mathbf{k}) \doteq \langle \mathbf{k} | \psi \rangle$ is the Fourier image of ψ (2.23), so $|\psi(\mathbf{x})|^2$ and $|\check{\psi}(\mathbf{k})|^2$ can be loosely understood as the densities of quanta (associated with the field ψ) in the $\mathbf{x}$ -space and the $\mathbf{k}$ -space, respectively. Because of (2.46), $\mathbf{W}_{\psi}$ is commonly attributed as a quasiprobability distribution of wave quanta in phase space.

(The prefix ‘quasi’ is added because W_ψ can be negative.) In case of real fields, which satisfy $\langle \mathbf{x} | \psi \rangle = \langle \psi | \mathbf{x} \rangle$, one also has

$$W_\psi(\mathbf{x}, \mathbf{k}) = W_\psi(\mathbf{x}, -\mathbf{k}). \quad (2.47)$$

Of particular importance are Wigner functions averaged over a sufficiently large phase-space volume $\Delta \mathbf{x} \Delta \mathbf{k} \gtrsim 1$. The average Wigner function $\overline{W}_\psi$ is a *local* property of the field (as opposed to, say, the field’s global Fourier spectrum) and satisfies (appendix A)

$$\overline{W}_\psi \geq 0. \quad (2.48)$$

2.1.7. Generalization to vector fields

In case of vector (tuple) fields $\boldsymbol{\psi} = (\psi^1, \psi^2, \dots, \psi^M)^\top$, kets are column vectors, $|\boldsymbol{\psi}\rangle = (|\psi^1\rangle, |\psi^2\rangle, \dots, |\psi^M\rangle)$, and bras are row vectors, $\langle \boldsymbol{\psi}| = (\langle \psi^1|, \langle \psi^2|, \dots, \langle \psi^M|)$. The operators acting on such kets and bras are matrices of operators. The Weyl symbol of a matrix operator is defined as the matrix of the corresponding symbols. As a result, the symbol of a Hermitian adjoint of a given operator is the Hermitian adjoint of the symbol of that operator:

$$\widehat{\mathbf{A}}^\dagger \leftrightarrow \mathbf{A}^\dagger, \quad (2.49)$$

and as a corollary, the symbol of a Hermitian matrix operator is a Hermitian matrix.

In particular, the density operator of a given vector field $\boldsymbol{\psi}$ is a matrix operator

$$\widehat{\mathbf{W}}_\psi \doteq (2\pi)^{-n} |\boldsymbol{\psi}\rangle \langle \boldsymbol{\psi}|. \quad (2.50)$$

The symbol of this operator, $\mathbf{W}_\psi = \text{symp}_x \widehat{\mathbf{W}}_\psi$, is a Hermitian matrix function[†]

$$\mathbf{W}_\psi(\mathbf{x}, \mathbf{k}) = \frac{1}{(2\pi)^n} \int d\mathbf{s} \, \boldsymbol{\psi}(\mathbf{x} + \mathbf{s}/2) \boldsymbol{\psi}^\dagger(\mathbf{x} - \mathbf{s}/2) e^{-i\mathbf{k} \cdot \mathbf{s}}, \quad (2.51)$$

called the Wigner matrix. (It is also called the ‘Wigner tensor’ when $\boldsymbol{\psi}$ is a true vector rather than a tuple.) It can be understood as the (inverse) Fourier image of

$$\mathbf{C}_\psi(\mathbf{x}, \mathbf{s}) \doteq \boldsymbol{\psi}(\mathbf{x} + \mathbf{s}/2) \boldsymbol{\psi}^\dagger(\mathbf{x} - \mathbf{s}/2) = \int d\mathbf{k} \, \mathbf{W}_\psi(\mathbf{x}, \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{s}}. \quad (2.52)$$

The analog of (2.45) is (appendix B.1)

$$(\widehat{\mathbf{L}}\boldsymbol{\psi})(\widehat{\mathbf{R}}\boldsymbol{\psi})^\dagger = \int d\mathbf{k} \, \mathbf{L}(\mathbf{x}, \mathbf{k}) \star \mathbf{W}_\psi(\mathbf{x}, \mathbf{k}) \star \mathbf{R}^\dagger(\mathbf{x}, \mathbf{k}). \quad (2.53a)$$

The Wigner matrix averaged over a sufficiently large phase-space volume $\Delta \mathbf{x} \Delta \mathbf{k} \gtrsim 1$ is a local property of the field, and it is positive-semidefinite (appendix A).

For real fields, one also has

$$\mathbf{W}_\psi(\mathbf{x}, \mathbf{k}) = \mathbf{W}_\psi^\top(\mathbf{x}, -\mathbf{k}) = \mathbf{W}_\psi^*(\mathbf{x}, -\mathbf{k}), \quad (2.53b)$$

and (2.53) yields the following corollary at $\epsilon \rightarrow 0$, when $\star$ becomes the usual product (appendix B.1):

$$(\widehat{\mathbf{L}}\boldsymbol{\psi})^\dagger \widehat{\mathbf{R}}\boldsymbol{\psi} = \int d\mathbf{k} \, \text{tr}(\mathbf{W}_\psi(\mathbf{L}^\dagger \mathbf{R})_{\mathbb{H}}). \quad (2.53c)$$

The generalizations of the other formulas from the previous sections are obvious.

[†] By construction, $\widehat{\mathbf{W}}_\psi$ is a matrix with mixed indices, $(\widehat{\mathbf{W}}_\psi)^i_j$. In sections 5.1 and 5.2, we also operate with a Wigner matrix that has two upper indices. Because the field of interest is real there, the dagger [†] in (2.51) is assumed omitted in that case.

2.2. Weyl symbol calculus on phase space

2.2.1. Notation

Consider a Hamiltonian system with coordinates $\mathbf{x} \equiv (x^1, x^2, \dots, x^n)$ and canonical momenta $\mathbf{p} \equiv (p_1, p_2, \dots, p_n)$. Together, these variables comprise the phase-space coordinates $\mathbf{z} \equiv (\mathbf{x}, \mathbf{p})$, i.e.

$$\mathbf{z} \equiv (z^1, \dots, z^{2n}) = (x^1, \dots, x^n, p_1, \dots, p_n). \quad (2.54)$$

Components of $\mathbf{z}$ will be denoted with Greek indices ranging from 1 to $2n$.[†]

Hamilton's equations for z^α can be written as $\dot{z}^\alpha = \{z^\alpha, H\}$, or equivalently, as

$$\dot{z}^\alpha = J^{\alpha\beta} \partial_\beta H. \quad (2.55)$$

Here, $H = H(t, \mathbf{z})$ is a Hamiltonian, $\partial_\beta \doteq \partial/\partial z^\beta$,

$$\{A, B\} \doteq J^{\alpha\beta} (\partial_\alpha A)(\partial_\beta B) \quad (2.56)$$

is the Poisson bracket on $\mathbf{z}$, $J^{\alpha\beta}$ is the canonical Poisson structure:

$$\mathbf{J} = -\mathbf{J}^\top = \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix}, \quad (2.57)$$

$\mathbf{0}_n$ is an n -dimensional zero matrix, and $\mathbf{1}_n$ is an n -dimensional unit matrix. The corresponding equation for the probability distribution $f(t, \mathbf{z})$ is

$$\partial_t f = \{H, f\}. \quad (2.58)$$

Solutions of (2.58) and other functions of the extended-phase-space coordinates $\mathbf{X} \equiv (t, \mathbf{z})$ can be considered as vectors in the Hilbert space $\mathcal{H}_X$ with the usual inner product[‡]

$$\langle \xi | \psi \rangle \doteq \int d\mathbf{X} \xi^*(\mathbf{X}) \psi(\mathbf{X}). \quad (2.59)$$

Assuming the notation $N \doteq \dim \mathbf{X} = 2n + 1$, one has

$$d\mathbf{X} \doteq dX^1 dX^2 \dots dX^N = dt dx^1 \dots dx^n dp_1, \dots, dp_n. \quad (2.60)$$

Let us introduce the position operator on $\mathbf{z}$,

$$\hat{\mathbf{z}} \doteq (\underbrace{x^1, \dots, x^n}_{\hat{\mathbf{x}}}, \underbrace{p_1, \dots, p_n}_{\hat{\mathbf{p}}}), \quad (2.61)$$

and the momentum operator on $\mathbf{z}$,

$$\hat{\mathbf{q}} \equiv (\underbrace{-i\partial_1, \dots, -i\partial_n}_{\hat{\mathbf{k}}}, \underbrace{-i\partial^1, \dots, -i\partial^n}_{\hat{\mathbf{r}}}), \quad (2.62)$$

where $\partial_i \doteq \partial/\partial x^i$ but $\partial^i \doteq \partial/\partial p_i$; that is, $\hat{\mathbf{z}} = (\hat{\mathbf{x}}, \hat{\mathbf{p}})$, $\hat{\mathbf{q}} = (\hat{\mathbf{k}}, \hat{\mathbf{r}})$, and

$$\hat{z}^\alpha \doteq z^\alpha, \quad \hat{q}_\alpha \doteq -i\partial_\alpha. \quad (2.63)$$

Then, much like in section 2.1, one can also introduce the position and momentum operators on the extended phase space $\mathbf{X}$:

$$\hat{\mathbf{X}} = (\hat{t}, \hat{\mathbf{z}}) = (\hat{t}, \hat{\mathbf{x}}, \hat{\mathbf{p}}), \quad \hat{\mathbf{K}} = (-\hat{\omega}, \hat{\mathbf{q}}) = (-\hat{\omega}, \hat{\mathbf{k}}, \hat{\mathbf{r}}). \quad (2.64)$$

[†] However, the index σ is reserved as a tag for individual particles and waves.

[‡] Note that the inner product (2.59) is different from (2.1). Still, we use the same notation assuming it will be clear from the context which inner product is used in each given case.

Assuming the convention that Latin indices from the beginning of the alphabet ($a, b, c, \dots$) range from 0 to $2n$, and $\partial_a \doteq \partial/\partial X^a$, one can compactly express this as

$$\hat{X}^a = X^a, \quad \hat{K}_a = -i\partial_a. \quad (2.65)$$

The eigenvectors of these operators will be denoted $|\mathbf{X}\rangle$ and $|\mathbf{K}\rangle$:

$$\hat{\mathbf{X}}|\mathbf{X}\rangle = \mathbf{X}|\mathbf{X}\rangle, \quad \hat{\mathbf{K}}|\mathbf{K}\rangle = \mathbf{K}|\mathbf{K}\rangle, \quad (2.66)$$

and we assume the usual normalization:

$$\langle \mathbf{X}_1 | \mathbf{X}_2 \rangle = \delta(\mathbf{X}_1 - \mathbf{X}_2), \quad \langle \mathbf{K}_1 | \mathbf{K}_2 \rangle = \delta(\mathbf{K}_1 - \mathbf{K}_2). \quad (2.67)$$

Both sets $\{|\mathbf{X}\rangle, \mathbf{X} \in \mathbb{R}^N\}$ and $\{|\mathbf{K}\rangle, \mathbf{K} \in \mathbb{R}^N\}$ form a complete basis on $\mathcal{H}_X$, and the eigenvalues of these operators form a real extended phase space $(\mathbf{X}, \mathbf{K})$, where

$$\mathbf{X} \equiv (t, \mathbf{z}), \quad \mathbf{K} \equiv (-\omega, \mathbf{q}). \quad (2.68)$$

Particularly note the following formula, which will be used below:

$$J^{\alpha\beta} \hat{q}_\alpha q_\beta = \begin{pmatrix} \hat{\mathbf{k}} & \hat{\mathbf{r}} \end{pmatrix} \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \begin{pmatrix} \mathbf{k} \\ \mathbf{r} \end{pmatrix} = \hat{\mathbf{k}} \cdot \mathbf{r} - \hat{\mathbf{r}} \cdot \mathbf{k}. \quad (2.69)$$

2.2.2. Wigner–Weyl transform

One can construct the Weyl symbol calculus on the extended phase space $\mathbf{X}$ just like it is done on spacetime $\mathbf{x}$ in section 2.1, with an obvious modification of the notation. The Wigner–Weyl transform is defined as

$$A(\mathbf{X}, \mathbf{K}) = \int d\mathbf{S} \langle \mathbf{X} + \mathbf{S}/2 | \hat{A} | \mathbf{X} - \mathbf{S}/2 \rangle e^{-i\mathbf{K} \cdot \mathbf{S}} \equiv \text{symb}_X \hat{A}, \quad (2.70)$$

$$\hat{A} = \frac{1}{(2\pi)^N} \int d\mathbf{X} d\mathbf{K} d\mathbf{S} |\mathbf{X} + \mathbf{S}/2\rangle A(\mathbf{X}, \mathbf{K}) \langle \mathbf{X} - \mathbf{S}/2| e^{i\mathbf{K} \cdot \mathbf{S}} \equiv \text{oper}_X A. \quad (2.71)$$

(Notice the change in the font and in the index compared to (2.26) and (2.25).) The corresponding Moyal product is denoted $\star$ (as opposed to $\star$ introduced earlier):

$$A \star B = A(\mathbf{X}, \mathbf{K}) e^{i\hat{\mathcal{L}}_X/2} B(\mathbf{X}, \mathbf{K}), \quad (2.72)$$

where $\hat{\mathcal{L}}_X \doteq \overleftarrow{\partial}_{\mathbf{X}} \cdot \overrightarrow{\partial}_{\mathbf{K}} - \overleftarrow{\partial}_{\mathbf{K}} \cdot \overrightarrow{\partial}_{\mathbf{X}}$ can be expressed as follows:

$$A \hat{\mathcal{L}}_X B \doteq -\frac{\partial A}{\partial t} \frac{\partial B}{\partial \omega} + \frac{\partial A}{\partial \omega} \frac{\partial B}{\partial t} + \frac{\partial A}{\partial x^i} \frac{\partial B}{\partial k_i} - \frac{\partial A}{\partial k_i} \frac{\partial B}{\partial x^i} + \frac{\partial A}{\partial p^i} \frac{\partial B}{\partial r_i} - \frac{\partial A}{\partial r_i} \frac{\partial B}{\partial p^i}. \quad (2.73)$$

If an operator $\hat{A}$ is local in $\mathbf{p}$, its $\mathbf{X}$ representation and $\mathbf{x}$ representation satisfy

$$\langle t, \mathbf{x}, \mathbf{p} | \hat{A} | t', \mathbf{x}', \mathbf{p}' \rangle = \langle t, \mathbf{x} | \hat{A} | t', \mathbf{x}' \rangle \delta(\mathbf{p} - \mathbf{p}'), \quad (2.74)$$

and therefore the Weyl symbol of $\hat{A}$ is the same irrespective of whether the operator is considered on $\mathcal{H}_X$ or on $\mathcal{H}_x$. In this case, we will use a unifying notation $\text{symb} \hat{A}$ instead of $\text{symb}_X \hat{A}$ and $\text{symb}_x \hat{A}$.

2.2.3. Wigner functions and Wigner matrices

The density operator of a given scalar field ψ is given by

$$\hat{W}_\psi \doteq (2\pi)^{-N} |\psi\rangle \langle \psi|. \quad (2.75)$$

The symbol of this operator, $W_\psi = \text{symb}_X \hat{W}_\psi$, is a real function called the Wigner function. It is given by

$$W_\psi(\mathbf{X}, \mathbf{K}) = \frac{1}{(2\pi)^N} \int d\mathbf{S} \psi(\mathbf{X} + \mathbf{S}/2) \psi^*(\mathbf{X} - \mathbf{S}/2) e^{-i\mathbf{K} \cdot \mathbf{S}}, \quad (2.76)$$

which can be understood as the (inverse) Fourier image of

$$C_\psi(\mathbf{X}, \mathbf{S}) \doteq \psi(\mathbf{X} + \mathbf{S}/2)\psi^*(\mathbf{X} - \mathbf{S}/2) = \int d\mathbf{K} W_\psi(\mathbf{X}, \mathbf{K}) e^{i\mathbf{K} \cdot \mathbf{S}}. \quad (2.77)$$

In particular, one has

$$\int d\mathbf{r} \text{ symb}_X \widehat{W}_\psi = \text{ symb}_x \widehat{W}_\psi(\mathbf{p}), \quad (2.78)$$

where the right-hand side is W_ψ given by (2.43), with $\mathbf{p}$ treated as a parameter. Also, for real fields,

$$W_\psi(\mathbf{X}, \mathbf{K}) = W_\psi(\mathbf{X}, -\mathbf{K}). \quad (2.79)$$

The density operator of a given vector field $\boldsymbol{\psi} = (\psi^1, \psi^2, \dots, \psi^M)$ is a matrix operator

$$\widehat{\mathbf{W}}_\psi \doteq (2\pi)^{-N} |\boldsymbol{\psi}\rangle\langle\boldsymbol{\psi}|. \quad (2.80)$$

The symbol of this operator, or the Wigner matrix, is a Hermitian matrix function

$$\mathbf{W}_\psi(\mathbf{X}, \mathbf{K}) = \frac{1}{(2\pi)^N} \int d\mathbf{S} \boldsymbol{\psi}(\mathbf{X} + \mathbf{S}/2) \boldsymbol{\psi}^\dagger(\mathbf{X} - \mathbf{S}/2) e^{-i\mathbf{K} \cdot \mathbf{S}}, \quad (2.81)$$

which can be understood as the (inverse) Fourier image of

$$\mathbf{C}_\psi(\mathbf{X}, \mathbf{S}) \doteq \boldsymbol{\psi}(\mathbf{X} + \mathbf{S}/2) \boldsymbol{\psi}^\dagger(\mathbf{X} - \mathbf{S}/2) = \int d\mathbf{K} \mathbf{W}_\psi(\mathbf{X}, \mathbf{K}) e^{i\mathbf{K} \cdot \mathbf{S}}. \quad (2.82)$$

In particular, one has

$$\int d\mathbf{r} \text{ symb}_X \widehat{\mathbf{W}}_\psi = \text{ symb}_x \widehat{\mathbf{W}}_\psi(\mathbf{p}), \quad (2.83)$$

where the right-hand side is $\mathbf{W}_\psi$ given by (2.51), with $\mathbf{p}$ treated as a parameter. Also, for real fields,

$$\mathbf{W}_\psi(\mathbf{X}, \mathbf{K}) = \mathbf{W}_\psi^\dagger(\mathbf{X}, -\mathbf{K}) = \mathbf{W}_\psi^*(\mathbf{X}, -\mathbf{K}). \quad (2.84)$$

Like those on $(\mathbf{x}, \mathbf{k})$, the Wigner matrices (Wigner functions) on $(\mathbf{X}, \mathbf{K})$ become positive-semidefinite (non-negative), and characterize local properties of the corresponding fields, when averaged over a sufficiently large phase-space volume $\Delta\mathbf{X} \Delta\mathbf{K} \gtrsim 1$.

2.3. Summary of section 2

In summary, we have introduced a generic n -dimensional physical space $\mathbf{x}$, the dual n -dimensional wavevector space $\mathbf{k}$, the corresponding n -dimensional ($n = n + 1$) spacetime $\mathbf{x} \equiv (t, \mathbf{x})$, and the dual n -dimensional wavevector space $\mathbf{k} \equiv (-\omega, \mathbf{k})$. We have also introduced an n -dimensional momentum space $\mathbf{p}$, the corresponding $2n$ -dimensional phase space $\mathbf{z} \equiv (\mathbf{x}, \mathbf{p})$, the N -dimensional ($N = 2n + 1$) extended space $\mathbf{X} \equiv (t, \mathbf{z}) \equiv (t, \mathbf{x}, \mathbf{p})$, and the dual N -dimensional wavevector space $\mathbf{K} \equiv (-\omega, \mathbf{q}) \equiv (-\omega, \mathbf{k}, \mathbf{r})$, where $\mathbf{r}$ is the n -dimensional wavevector space dual to $\mathbf{p}$. We have also introduced the $2N$ -dimensional phase space $(\mathbf{X}, \mathbf{K})$. Each of the said variables has a corresponding operator associated with it, which is denoted with a caret. For example, $\hat{\mathbf{x}}$ is the operator of position in the $\mathbf{x}$ space, and $\hat{\mathbf{k}} = -i\partial_{\mathbf{x}}$ is the corresponding wavevector operator.

Functions on $\mathbf{x}$ form a Hilbert space $\mathcal{H}_x$, and the corresponding bra-ket notation is introduced as usual. Any operator $\hat{\mathbf{A}}$ on $\mathcal{H}_x$ can be represented by its Weyl symbol $\mathbf{A}(\mathbf{x}, \mathbf{k})$. The correspondence between operators and their symbols, $\hat{\mathbf{A}} \leftrightarrow \mathbf{A}$, is determined by the Wigner–Weyl transform and is natural in the sense that (2.29) is satisfied. In

particular, $\hat{\mathbf{A}}\hat{\mathbf{B}} \leftrightarrow \mathbf{A} \star \mathbf{B}$, where $\star$ is the Moyal product on $(\mathbf{x}, \mathbf{k})$. When the geometrical-optics parameter is negligible ($\epsilon \rightarrow 0$), one has $\hat{\mathbf{A}} = \mathbf{A}(\hat{\mathbf{x}}, \hat{\mathbf{k}})$ and the Moyal product becomes the usual product. Similarly, functions on $\mathbf{X}$ form a Hilbert space $\mathcal{H}_X$, the corresponding bra-ket notation is also introduced as usual, any operator $\hat{A}$ on $\mathcal{H}_X$ can be represented by its Weyl symbol $A(\mathbf{X}, \mathbf{K})$, and $\hat{A}\hat{B} \leftrightarrow A \star B$. An operator that is local in $\mathbf{p}$ has the same symbol irrespective of whether it is considered on $\mathcal{H}_x$ or on $\mathcal{H}_X$.

Any given field ψ generates the corresponding density operator $(2\pi)^{-n} |\psi\rangle\langle\psi|$ and its symbol called the Wigner function (Wigner matrix, if the field is a vector). If the density operator is considered on $\mathcal{H}_x$, it is denoted $\hat{W}_\psi$ and the corresponding Wigner function is denoted $W_\psi(\mathbf{x}, \mathbf{k})$. If the density operator is considered on $\mathcal{H}_X$, it is denoted $\hat{W}_\psi$ and the corresponding Wigner function is denoted $W_\psi(\mathbf{X}, \mathbf{K})$. The two Wigner functions are related via $\int d\mathbf{r} W_\psi(t, \mathbf{x}, \mathbf{p}, \omega, \mathbf{k}, \mathbf{r}) = W_\psi(t, \mathbf{x}, \omega, \mathbf{k}; \mathbf{p})$, where $\mathbf{p}$ enters W_ψ as a parameter, if at all. If averaged over a sufficiently large phase-space volume, the Wigner functions (matrices) are non-negative (positive-semidefinite) and characterize local properties of the corresponding fields.

3. Model

Here, we introduce the general assumptions and the key ingredients of our theory.

3.1. Basic assumptions

3.1.1. Ordering

Let us consider particles governed by a Hamiltonian $H = \overline{H} + \tilde{H}$ such that

$$\tilde{H} = \mathcal{O}(\varepsilon) \ll \overline{H} = \mathcal{O}(1). \quad (3.1)$$

In other words, $\tilde{H}$ serves as a small perturbation to the leading-order Hamiltonian $\overline{H}$. The system will be described in canonical variables $\mathbf{z} \equiv (\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{2n}$. Let us also assume that the system is close to being homogeneous in $\mathbf{x}$. This includes two conditions. First, we require that the external fields are weak (yet see section 3.1.2), meaning

$$\partial_{\mathbf{x}} \overline{H} \sim \kappa_x \overline{H} = \mathcal{O}(\epsilon), \quad \partial_{\mathbf{p}} \overline{H} \sim \kappa_p \overline{H} = \mathcal{O}(1), \quad (3.2)$$

where $\epsilon \ll 1$ is a small parameter, κ_x and κ_p are the characteristic inverse scales in the $\mathbf{x}$ and $\mathbf{p}$ spaces, respectively, and the bar denotes local averaging.[†] Hence, the particle momenta $\mathbf{p}$ are close to being local invariants. Second, the *statistical* properties of $\tilde{H}$ are also assumed to vary in $\mathbf{x}$ slowly. These properties can be characterized using the density operator of the perturbation Hamiltonian,

$$\hat{W} \doteq (2\pi)^{-N} |\tilde{H}\rangle\langle\tilde{H}|, \quad (3.3)$$

and its symbol, the (real) Wigner function, as in (2.43):

$$W(\mathbf{X}, \mathbf{K}) = \frac{1}{(2\pi)^N} \int d\mathbf{S} \tilde{H}(\mathbf{X} + \mathbf{S}/2) \tilde{H}(\mathbf{X} - \mathbf{S}/2) e^{-i\mathbf{K} \cdot \mathbf{S}}. \quad (3.4)$$

Specifically, we will use the *average* Wigner function, $\overline{W}$, which represents the Fourier spectrum of the symmetrized autocorrelation function of $\tilde{H}$:

$$\overline{C}(\mathbf{X}, \mathbf{S}) \doteq \overline{\tilde{H}(\mathbf{X} + \mathbf{S}/2) \tilde{H}(\mathbf{X} - \mathbf{S}/2)} = \int d\mathbf{K} \overline{W}(\mathbf{X}, \mathbf{K}) e^{i\mathbf{K} \cdot \mathbf{S}}. \quad (3.5)$$

[†] An exception will be made for eikonal waves, specifically, for quantities evaluated on the local wavevector $\mathbf{k} \equiv (-\overline{\omega}, \mathbf{k})$.

The averaging is performed over sufficiently large volume of $\mathbf{x}$ to eliminate rapid oscillations and also over phase-space volumes $\Delta\mathbf{X} \Delta\mathbf{K} \gtrsim 1$, which guarantees $\overline{W}$ to be non-negative and local (section 2.2.3). The function $\overline{W}$ can be understood as a measure of the phase-space density of wave quanta when the latter is well defined (section 7).

We will assume[†]

$$\partial_t \overline{W} = \mathcal{O}(\epsilon), \quad \partial_{\mathbf{x}} \overline{W} = \mathcal{O}(\epsilon), \quad \partial_{\mathbf{p}} \overline{W} = \mathcal{O}(1). \quad (3.6)$$

That said, we will also allow (albeit not require) for oscillations to be constrained by a dispersion relation. In this case, $\overline{W} \propto \delta(\omega - \overline{\omega}(t, \mathbf{x}))$, so (3.6) *per se* is not satisfied; then we assume a similar ordering for $\int d\omega \overline{W}$ instead. Also note that in application to the standard QLT of homogeneous turbulence (Stix 1992, chapter 16), ϵ is understood as the geometrical-optics parameter characterizing the smallness of the linear-instability growth rates. (We discuss the ordering further in the end of section 3.3.)

3.1.2. Quasilinear approximation

The particle-motion equations can be written as

$$\dot{z}^\alpha = \{z^\alpha, \overline{H} + \tilde{H}\} = v^\alpha + u^\alpha, \quad (3.7)$$

where v^α and u^α are understood as the unperturbed phase-space velocity and the perturbation to the phase-space velocity, respectively:

$$v^\alpha \doteq J^{\alpha\beta} \partial_\beta \overline{H}, \quad u^\alpha \doteq J^{\alpha\beta} \partial_\beta \tilde{H}. \quad (3.8)$$

The notation v^i (with $i = 1, 2, \dots, n$) will also be used for the *spatial part* of the phase-space velocity v^α , i.e. for the true velocity *per se*. Likewise, $\mathbf{v}$ will be used to denote either the phase-space velocity vector or the spatial velocity vector depending on the context. Also note that a slightly different definition of $\mathbf{v}$ will be used starting from section 5.6.

The corresponding Klimontovich equation for the particle distribution $f(t, \mathbf{z})$ is

$$\partial_t f = \{\overline{H} + \tilde{H}, f\}. \quad (3.9)$$

(If collisions are not of interest, (3.9) can as well be understood as the Vlasov equation. Also, a small collision term can be included *ad hoc*; see the comment in the end of section 3.3.) Let us search for f in the form

$$f = \overline{f} + \tilde{f}, \quad \overline{\tilde{f}} = 0. \quad (3.10)$$

The equations for $\overline{f}$ and $\tilde{f}$ are obtained as the average and oscillating parts of (3.9), and we neglect the nonlinearity in the equation for $\tilde{f}$, following the standard QL approximation (Stix 1992, chapter 16). Then, one obtains

$$\partial_t \overline{f} = \{\overline{H}, \overline{f}\} + \overline{\{\tilde{H}, \tilde{f}\}}, \quad (3.11)$$

$$\partial_t \tilde{f} = \{\overline{H}, \tilde{f}\} + \{\tilde{H}, \overline{f}\}. \quad (3.12)$$

A comment is due here regarding plasmas in strong fields and magnetized plasmas in particular. Our formulation can be applied to such plasmas in canonical angle–action variables $(\phi, \mathbf{J})$. For fast angle variables, the ordering (3.2) is not satisfied and the Weyl symbol calculus is inapplicable as is (see the footnote on p. 7). Such systems can be accommodated by representing the distribution function as a Fourier series in ϕ and treating the individual-harmonic amplitudes separately as slow functions of the

[†] As a reminder, the notation $A = \mathcal{O}(\epsilon)$ does not rule out the possibility that A/ϵ is small. Also note that the terms ‘ $\sim$ ’ and ‘of order’ in this paper mean the same as ‘ $\mathcal{O}$ ’.

remaining coordinates. Then, our averaging procedure subsumes averaging over ϕ , so the averaged quantities are ϕ -independent and (3.2) is reinstated. In particular, magnetized plasmas can be described using guiding-center variables. Although not canonical by default (Littlejohn 1983), they can always be cast in a canonical form, at least in principle (Littlejohn 1979). Examples of canonical guiding-center variables are reviewed in (Cary & Brizard 2009). To make the connection with the homogeneous-plasma theory, one can also order the canonical pairs of guiding-center variables such that they would describe the gyromotion, the parallel motion, and the drifts separately (Wong 2000). This readily leads to results similar to those in (Catto *et al.* 2017). Further discussions on this topic are left to future papers.

3.2. Equation for $\tilde{f}$

Let us consider solutions of (3.12) as a subclass of solutions of the more general equation

$$\partial_\tau \tilde{f} = \hat{L} \tilde{f} + \mathcal{F}, \quad \mathcal{F}(\mathbf{X}) \doteq \{\tilde{H}, \bar{f}\}. \quad (3.13)$$

Here, we have introduced an auxiliary second ‘time’ τ , the operator

$$\hat{L} \doteq -\partial_t + \{\bar{H}, \blacksquare\} = -\partial_t + J^{\alpha\beta}(\partial_\alpha \bar{H}) \partial_\beta = -\partial_t - v^\lambda \partial_\lambda = -V^a \partial_a \quad (3.14)$$

(here and further, $\blacksquare$ denotes a placeholder), and $\mathbf{V}(\mathbf{X}) \equiv (1, \mathbf{v}(t, \mathbf{z}))$ is the unperturbed velocity in the $\mathbf{X}$ space. Note that

$$\partial_a V^a = \partial_\lambda v^\lambda = 0 \quad (3.15)$$

due to the incompressibility of the phase flow. Hence, $[\partial_a, V^a] = 0$, so $\hat{L}$ is anti-Hermitian.

Let us search for a solution of (3.13) in the form[†]

$$\tilde{f}(\tau, \mathbf{X}) = e^{\hat{L}\tau} \xi(\tau, \mathbf{X}). \quad (3.16)$$

Then, $\partial_\tau \tilde{f} = \hat{L} \tilde{f} + e^{\hat{L}\tau} \partial_\tau \xi$, so $\partial_\tau \xi = e^{-\hat{L}\tau} \mathcal{F}(\mathbf{X})$ and therefore

$$\xi(\tau, \mathbf{X}) = e^{-\hat{L}\tau_0} \xi_0(\mathbf{X}) + \int_{\tau_0}^{\tau} d\tau' e^{-\hat{L}\tau'} \mathcal{F}(\mathbf{X}), \quad (3.17)$$

where $\xi_0(\mathbf{X}) \doteq \tilde{f}(\tau_0, \mathbf{X})$. Hence, one obtains

$$\tilde{f}(\tau, \mathbf{X}) = e^{\hat{L}(\tau-\tau_0)} \xi_0(\mathbf{X}) + \int_{\tau_0}^{\tau} d\tau' e^{-\hat{L}(\tau'-\tau)} \mathcal{F}(\mathbf{X}), \quad (3.18)$$

or equivalently, using $\tau'' \doteq \tau - \tau'$,

$$\tilde{f}(\tau, \mathbf{X}) = g_0(\tau, \mathbf{X}) + \int_0^{\tau-\tau_0} d\tau'' \hat{T}_{\tau''} \mathcal{F}(\mathbf{X}). \quad (3.19)$$

Here, g_0 is a solution of $\partial_\tau g_0 = \hat{L} g_0$, specifically,

$$g_0(\tau, \mathbf{X}) \doteq \hat{T}_{\tau-\tau_0} \xi_0(\mathbf{X}), \quad g_0(\tau_0, \mathbf{X}) = \tilde{f}(\tau_0, \mathbf{X}), \quad (3.20)$$

and we have also introduced

$$\hat{T}_\tau \doteq e^{\hat{L}\tau} = e^{-\tau V^a \partial_a}. \quad (3.21)$$

Because $\hat{L}$ is anti-Hermitian, the operator $\hat{T}_\tau$ is unitary, and comparison with (2.19) shows that it can be recognized as a shift operator. For further details, see section 4.1.

[†] Using the auxiliary variable τ allows us to express the propagator as a regular exponential, rather than ordered exponential, even for t -dependent $\bar{H}$, because $\hat{L}$ is independent of τ .

Using $\hat{T}_\tau$, one can express (3.19) as

$$\tilde{f} = g_0 + \hat{\mathcal{G}}\mathcal{F}, \quad \hat{\mathcal{G}} \doteq \int_0^{\tau-\tau_0} d\tau' \hat{T}_{\tau'}, \quad (3.22)$$

where $\hat{\mathcal{G}}$ is the Green's operator understood as the right inverse of the operator $\partial_\tau - \hat{L}$, or on the space of τ -independent functions, $\partial_t - \{\bar{H}, \blacksquare\}$. Let us rewrite this operator as $\hat{\mathcal{G}} = \hat{\mathcal{G}}_< + \hat{\mathcal{G}}_>$, where

$$\hat{\mathcal{G}}_< = \int_0^{\tau-\tau_0} d\tau' e^{-\nu\tau'} \hat{T}_{\tau'}, \quad \hat{\mathcal{G}}_> = \int_0^{\tau-\tau_0} d\tau' (1 - e^{-\nu\tau'}) \hat{T}_{\tau'}, \quad (3.23)$$

and ν is a positive constant. Note that $\hat{\mathcal{G}}_<$ is well defined at $\tau_0 \rightarrow -\infty$, meaning that $\hat{\mathcal{G}}_<\mathcal{F}$ is well defined for any physical (bounded) field $\mathcal{F}$.[†] Thus, so is $g_0 + \hat{\mathcal{G}}_>\mathcal{F}$. Let us take $\tau_0 \rightarrow -\infty$ and then take $\nu \rightarrow 0+$. (Here, $0+$ denotes that ν must remain positive, i.e. the *upper* limit is taken.) Then, (3.22) can be expressed as

$$\tilde{f} = g + \hat{G}\mathcal{F}, \quad g \doteq \lim_{\nu \rightarrow 0+} \lim_{\tau_0 \rightarrow -\infty} (g_0 + \hat{\mathcal{G}}_>\mathcal{F}). \quad (3.24)$$

Here, we introduced an ‘effective’ Green's operator $\hat{G} \doteq \lim_{\nu \rightarrow 0+} \lim_{\tau_0 \rightarrow -\infty} \hat{\mathcal{G}}_<$, i.e.

$$\hat{G} \doteq \lim_{\nu \rightarrow 0+} \int_0^\infty d\tau e^{-\nu\tau} \hat{T}_\tau. \quad (3.25)$$

This operator will be discussed in section 4.2, and g will be discussed in section 4.3. Meanwhile, note that because τ is just an auxiliary variable, we will be interested in solutions independent of τ . In particular, this means that $\tilde{f}(\tau_0, \mathbf{X}) = \tilde{f}(\mathbf{X})$, so $\xi_0(\mathbf{X}) = \tilde{f}(\mathbf{X})$, so (3.20) leads to

$$g_0(\tau, \mathbf{X}) = \hat{T}_{\tau-\tau_0} \tilde{f}(\mathbf{X}). \quad (3.26)$$

3.3. Equation for $\bar{f}$

Using (3.22), one can rewrite (3.12) for $\bar{f}$ as follows:

$$\partial_t \bar{f} = \{\bar{H}, \bar{f}\} + \{\tilde{H}, g\} + \{\tilde{H}, \hat{G}\{\tilde{H}, \bar{f}\}\}. \quad (3.27)$$

Notice that

$$\{\tilde{H}, g\} = -\{g, \tilde{H}\} = -\partial_\alpha (J^{\alpha\beta} g \partial_\beta \tilde{H}) = -\partial_\alpha (u^\alpha g) \quad (3.28)$$

and also

$$\begin{aligned} \{\tilde{H}, \hat{G}\{\tilde{H}, \bar{f}\}\} &= \partial_\beta (J^{\alpha\beta} (\partial_\alpha \tilde{H}) \hat{G}\{\tilde{H}, \bar{f}\}) \\ &= \partial_\beta (J^{\alpha\beta} (\partial_\alpha \tilde{H}) \hat{G}(J^{\mu\nu} (\partial_\mu \tilde{H}) (\partial_\nu \bar{f}))) \\ &= \partial_\beta (u^\beta \hat{G}(u^\nu \partial_\nu \bar{f})). \end{aligned} \quad (3.29)$$

The field u^α enters here as a multiplication factor and can be considered as an operator:

$$\hat{u}^\alpha \psi(\mathbf{X}) \doteq u^\alpha(\mathbf{X}) \psi(\mathbf{X}). \quad (3.30)$$

Then, (3.29) can be compactly represented as

$$\{\tilde{H}, \hat{G}\{\tilde{H}, \bar{f}\}\} = \partial_\alpha (\hat{u}^\alpha \hat{G} \hat{u}^\beta \partial_\beta \bar{f}). \quad (3.31)$$

We will also use the notation

$$d_t \doteq \partial_t + v^\gamma \partial_\gamma = \partial_t - \{\bar{H}, \blacksquare\}. \quad (3.32)$$

[†] Unlike classic plasma-wave theory, this approach does not involve spectral decomposition, so there is no need to consider fields that are exponential in time on the *whole* interval $(-\infty, \infty)$.

This leads to the following equation for $\bar{f}$:

$$d_t \bar{f} = \partial_\alpha (\hat{D}^{\alpha\beta} \partial_\beta \bar{f}) + \Gamma, \quad (3.33)$$

where we introduced the following average quantities:

$$\hat{D}^{\alpha\beta} \doteq \overline{\hat{u}^\alpha \hat{G} \hat{u}^\beta}, \quad \Gamma \doteq -\partial_\alpha (\overline{u^\alpha g}), \quad (3.34)$$

Our goal is to derive explicit approximate expressions for the quantities (3.34) and to rewrite (3.33) in a more tractable form using the assumptions introduced in section 3.1. We will use[†]

$$\partial_t \bar{f} \sim \{\bar{H}, \bar{f}\} = \mathcal{O}(\epsilon), \quad d_t \bar{f} = \mathcal{O}(\epsilon^2), \quad (3.35)$$

and we will keep terms of order ϵ , ϵ^2 , and $\epsilon\epsilon^2$ in the equation for $\bar{f}$, while terms of order ϵ^4 , $\epsilon^2\epsilon^2$, and higher will be neglected. This implies the ordering

$$\epsilon^2 \ll \epsilon \ll \varepsilon \ll 1. \quad (3.36)$$

As a reminder, ε is a linear measure of the characteristic amplitude of oscillations, and ϵ is the geometrical-optics parameter, which is proportional to the inverse scale of the plasma inhomogeneity in spacetime. As usual then, linear dissipation is assumed to be of order ϵ . This model implies the assumption that collisionless dissipation is much stronger than collisional dissipation, which is to emerge as an effect quadratic in $\tilde{f}$ (section 6). Furthermore, the inverse plasma parameter[‡] will be assumed to be of order ϵ , so the collision operator for $\tilde{f}$ (section 6.8) will be of order $\epsilon\epsilon^2$. Within the assumed accuracy, this operator must be retained, while the dynamics of $\tilde{f}$ is considered linear and therefore collisionless. Alternatively, one can switch from the Klimontovich description to the Vlasov–Boltzmann description and introduce an *ad hoc* order- ϵ collision operator directly in (3.9). This will alter the Green’s operator, but the conceptual formulation would remain the same, so it will not be considered separately in detail.

3.4. Summary of section 3

Our QL model is defined as usual, except: (i) we allow for a general particle Hamiltonian H ; (ii) we use the Klimontovich equation rather than the Vlasov equation to retain collisions; (iii) we use *local* averaging (denoted with overbar) and allow for weak inhomogeneity of all averaged quantities; (iv) we retain the initial conditions g for the oscillating part of the distribution function (defined as in (3.24) but yet to be calculated explicitly). Then, the average part of the distribution function satisfies

$$\partial_t \bar{f} - \{\bar{H}, \bar{f}\} = \partial_\alpha (\hat{D}^{\alpha\beta} \partial_\beta \bar{f}) + \Gamma, \quad (3.37)$$

where $\hat{D}^{\alpha\beta} \doteq \overline{\hat{u}^\alpha \hat{G} \hat{u}^\beta}$, $\Gamma \doteq -\partial_\alpha (\overline{u^\alpha g})$, u^α is the wave-driven perturbation of the phase-space velocity (see (3.8)), $\hat{u}^\alpha$ is the same quantity considered as an operator on $\mathcal{H}_X$ (see (3.30)), and $\hat{G}$ is the ‘effective’ Green’s operator given by

$$\hat{G} \doteq \lim_{\nu \rightarrow 0+} \int_0^\infty d\tau e^{-\nu\tau - \tau V^\alpha \partial_\alpha}. \quad (3.38)$$

Also, $\partial_\alpha \equiv \partial/\partial z^\alpha$, and $\{\blacksquare, \blacksquare\}$ is the Poisson bracket on the particle phase space $\mathbf{z}$. The equation for $\bar{f}$ used in the standard QLT is recovered from (3.37) by neglecting Γ and the spatial gradients (in particular, the whole Poisson bracket) and also by approximating the operator $\hat{D}^{\alpha\beta}$ with a local function of $\mathbf{z}$.

[†] Starting with section 5.6, we will assume $d_t \bar{f} \sim \epsilon\epsilon^2 \bar{f}$ instead.

[‡] By the plasma parameter we mean the number of particles within the Debye sphere.

4. Preliminaries

Before we start calculating the functions in (3.37) explicitly, let us get some preliminaries out of the way. In this section, we discuss the shift operators $\hat{T}_\tau$ (section 4.1), approximate the operator $\hat{G}$ (section 4.2), and develop a model for the function g that encodes the initial conditions for $\tilde{f}$ (section 4.3).

4.1. Shift operator

Here, we derive some properties of the shift operator $\hat{T}_\tau$ introduced in section 3.2.

4.1.1. $\hat{T}_\tau$ as a shift

Here, we formally prove (an admittedly obvious fact) that

$$\hat{T}_\tau \psi(\mathbf{X}) = \psi(\mathbf{X} - \ell_\tau(\mathbf{X})), \quad \ell_\tau^a(\mathbf{X}) \doteq \int_0^\tau dt V^a(\mathbf{Y}(t, \mathbf{X})), \quad (4.1)$$

where the ‘characteristics’ Y^a solve[†]

$$\frac{dY^a}{d\tau} = -V^a(\mathbf{Y}), \quad Y^a(\tau = 0) = X^a, \quad (4.2)$$

and thus ℓ_τ^a can be Taylor-expanded in τ as

$$\ell_\tau^a(\mathbf{X}) = \tau V^a - \frac{1}{2} \tau^2 V^b \partial_b V^a + \dots, \quad V^a \equiv V^a(\mathbf{X}). \quad (4.3)$$

As the first step to proving (4.1), let us Taylor-expand V^a around a fixed point $\mathbf{X}_1$:

$$V^a = V_1^a + (\partial_b V_1^a) \delta X^b + \dots, \quad \delta X^a \doteq X^a - X_1^a, \quad (4.4)$$

where $V_1^a \equiv V^a(\mathbf{X}_1)$. If one neglects the *first* and higher derivatives of V^a , one obtains

$$\hat{T}_\tau \psi(\mathbf{X}) \approx e^{-\tau V_1^a \partial_a} \psi(\mathbf{X}) = \psi(\mathbf{X} - \tau \mathbf{V}_1). \quad (4.5)$$

By taking the limit $\mathbf{X}_1 \rightarrow \mathbf{X}$, which corresponds to $\mathbf{V}_1 \rightarrow \mathbf{V}$, one obtains

$$\hat{T}_\tau \psi(\mathbf{X}) = \psi(\mathbf{X} - \tau \mathbf{V}) + \mathcal{O}(\tau^2). \quad (4.6)$$

Similarly, if one neglects the *second* and higher derivatives of V^a , one obtains[‡]

$$\begin{aligned} \hat{T}_\tau \psi(\mathbf{X}) &= e^{-\tau(V_1^a + (\partial_b V_1^a) \delta X^b + \dots) \partial_a} \psi(\mathbf{X}) \\ &\approx e^{-\tau(\partial_b V_1^a) \delta X^b \partial_a} e^{-\tau V_1^a \partial_a} e^{-\frac{1}{2}[-\tau(\partial_b V_1^a) \delta X^b \partial_a, -\tau V_1^c \partial_c]} \psi(\mathbf{X}) \\ &\approx e^{-\tau(\partial_b V_1^a) \delta X^b \partial_a} e^{-\tau V_1^a \partial_a} e^{\frac{1}{2} \tau^2 V_1^c (\partial_b V_1^a) [\partial_c, \delta X^b \partial_a]} \psi(\mathbf{X}) \\ &\approx e^{-\tau(\partial_b V_1^a) \delta X^b \partial_a} e^{-\tau V_1^a \partial_a} e^{\frac{1}{2} \tau^2 V_1^b (\partial_b V_1^a) \partial_a} \psi(\mathbf{X}) \\ &\approx e^{-\tau(\partial_b V_1^a) \delta X^b \partial_a} e^{-\tau V_1^a \partial_a + \frac{1}{2} \tau^2 V_1^b (\partial_b V_1^a) \partial_a} \psi(\mathbf{X}) \\ &\approx e^{-\tau(\partial_b V_1^a) \delta X^b \partial_a} \psi(\mathbf{X} - \tau \mathbf{V}_1 + \frac{1}{2} \tau^2 V_1^b \partial_b \mathbf{V}_1). \end{aligned} \quad (4.7)$$

In the limit $\mathbf{X}_1 \rightarrow \mathbf{X}$, when $e^{-\tau(\partial_b V_1^a) \delta X^b \partial_a} \rightarrow 1$ and $\mathbf{V}_1 \rightarrow \mathbf{V}$, one obtains

$$\hat{T}_\tau \psi(\mathbf{X}) = \psi\left(\mathbf{X} - \tau \mathbf{V}(\mathbf{X}) + \frac{1}{2} \tau^2 (\mathbf{V} \cdot \partial_{\mathbf{X}}) \mathbf{V}\right) + \mathcal{O}(\tau^3). \quad (4.8)$$

In conjunction with (4.3), equations (4.6) and (4.8) show agreement with the sought

[†] In terms of $t' \doteq t - \tau$, (4.2) has a more recognizable form $dY^a/dt' = V^a(\mathbf{Y})$, with $Y^a(t' = t) = X^a$.

[‡] We use the Zassenhaus formula $e^{\hat{A} + \hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-[\hat{A}, \hat{B}]/2} e^{[\hat{B}, [\hat{A}, \hat{B}]]/3 + [\hat{A}, [\hat{A}, \hat{B}]]/6} \dots$

result (4.3) within the assumed accuracy. One can also retain m derivatives of $\mathbf{V}$ and derive the corresponding approximations similarly. Then the error will be $\mathcal{O}(\tau^{m+2})$.

For an order-one time interval τ , one can split this interval on $N_\tau \gg 1$ subintervals of small duration τ/N_τ and apply finite- m formulas (for example, (4.6) or (4.8)) to those. Then the total error scales as $\mathcal{O}(N_\tau^{-m-1})$ and the exact formula (4.1) is obtained at $N_\tau \rightarrow \infty$.

4.1.2. Symbol of $\hat{T}_\tau$

Using the bra-ket notation, (4.1) can be written as

$$\langle \mathbf{X} | \hat{T}_\tau | \psi \rangle = \langle \mathbf{X} - \ell_\tau(\mathbf{X}) | \psi \rangle. \quad (4.9)$$

Thus, $\langle \mathbf{X} | \hat{T}_\tau = \langle \mathbf{X} - \ell_\tau(\mathbf{X}) |$, so

$$\langle \mathbf{X}_1 | \hat{T}_\tau | \mathbf{X}_2 \rangle = \langle \mathbf{X}_1 - \ell_\tau(\mathbf{X}_1) | \mathbf{X}_2 \rangle = \delta(\mathbf{X}_1 - \mathbf{X}_2 - \ell_\tau(\mathbf{X}_1)). \quad (4.10)$$

Using (2.70), one obtains the Weyl symbol of $\hat{T}_\tau$ in the form

$$T_\tau(\mathbf{X}, \mathbf{K}) = \int d\mathbf{S} e^{-i\mathbf{K} \cdot \mathbf{S}} \delta(\mathbf{S} - \ell_\tau(\mathbf{X} + \mathbf{S}/2)). \quad (4.11)$$

From (4.3), one has

$$\begin{aligned} \ell_\tau^a(\mathbf{X} + \mathbf{S}/2) &= \tau V^a(\mathbf{X} + \mathbf{S}/2) - (\tau^2/2) V^b \partial_b V^a + \mathcal{O}(\epsilon^2) \\ &= \tau V^a + (\tau/2)(\partial_b V^a) S^b - (\tau^2/2) V^b \partial_b V^a + \mathcal{O}(\epsilon^2) \\ &= M^a_b V^b \tau + m^a_b S^b + \mathcal{O}(\epsilon^2), \end{aligned} \quad (4.12)$$

where we introduced a matrix $\mathbf{M} \doteq \mathbf{1} - \mathbf{m}$, or explicitly,

$$M^a_b \doteq \delta^a_b - m^a_b, \quad m^a_b \doteq (\tau/2)(\partial_b V^a). \quad (4.13)$$

Let us express the term $\mathcal{O}(\epsilon^2)$ in (4.12) as $-M^a_b \mu^b$. Then,

$$\begin{aligned} \delta(\mathbf{S} - \ell_\tau(\mathbf{X} + \mathbf{S}/2)) &= \delta(\mathbf{S} - \mathbf{M} \mathbf{V} \tau - \mathbf{m} \mathbf{S} + \mathbf{M} \boldsymbol{\mu}) \\ &= \delta(\mathbf{M}(\mathbf{S} - \mathbf{V} \tau + \boldsymbol{\mu})) \\ &= \delta(\mathbf{S} - \mathbf{V} \tau + \boldsymbol{\mu}) / |\det \mathbf{M}|. \end{aligned} \quad (4.14)$$

Because $\mathbf{m} = \mathcal{O}(\epsilon)$, the well-known formula yields $\det \mathbf{M} = 1 + \text{tr} \mathbf{m} + \mathcal{O}(\epsilon^2)$. But $\text{tr} \mathbf{m} = 0$ by (3.15), so

$$\delta(\mathbf{S} - \ell_\tau(\mathbf{X} + \mathbf{S}/2)) = \delta(\mathbf{S} - \mathbf{V} \tau + \boldsymbol{\mu}) + \mathcal{O}(\epsilon^2). \quad (4.15)$$

The last term $\mathcal{O}(\epsilon^2)$ is insignificant and can be neglected right away, so (4.11) leads to

$$T_\tau(\mathbf{X}, \mathbf{K}) \approx \exp(i\tau \Omega(\mathbf{X}, \mathbf{K}) + i\mathbf{K} \cdot \boldsymbol{\mu}), \quad (4.16)$$

where we have introduced the following notation:

$$\Omega(\mathbf{X}, \mathbf{K}) \doteq -\mathbf{K} \cdot \mathbf{V}(\mathbf{X}) = \omega - q_\alpha v^\alpha = \omega - \mathbf{k} \cdot \mathbf{v} + \mathcal{O}(\epsilon). \quad (4.17)$$

By definition, $\boldsymbol{\mu}$ is a polynomial of τ with coefficients that are of order ϵ^2 and therefore small. But because τ can be large, and because $\boldsymbol{\mu}$ is under the exponent, this makes T_τ potentially sensitive to this term, so we retain it (for now).

4.2. Effective Green's operator

The effective Green's operator (3.25) can be understood as the right inverse of the operator (cf. section 3.2)

$$\hat{L}_{\text{eff}} \doteq \lim_{\nu \rightarrow 0+} (\partial_t - \{\overline{H}, \blacksquare\} + \nu), \quad (4.18)$$

so we denote it also as $\hat{G} = \hat{L}_{\text{eff}}^{-1}$ (which is admittedly abuse of notation). Because $\hat{L}_{\text{eff}}$ has real $\mathbf{X}$ representation by definition, the $\mathbf{X}$ representation of $\hat{G}$ is real too. In particular, $\langle \mathbf{X} + \mathbf{S}/2 | \hat{G} | \mathbf{X} - \mathbf{S}/2 \rangle$ is real, hence

$$G(\mathbf{X}, -\mathbf{K}) = G^*(\mathbf{X}, \mathbf{K}) \quad (4.19)$$

by definition of the Weyl symbol (2.70). As a corollary, the derivative of $G(\mathbf{X}, \mathbf{K})$ with respect to the a th component of the whole second argument, denoted $G^{[a]}$, satisfies

$$(G^{[a]}(\mathbf{X}, \mathbf{K}))^* = -G^{[a]}(\mathbf{X}, -\mathbf{K}). \quad (4.20)$$

Also note that $\hat{G}$ can be expressed as

$$\hat{G} = \lim_{\nu \rightarrow 0+} i(\hat{\omega} - \dot{x}^i \hat{k}_i - \dot{p}_i \hat{r}^i + i\nu)^{-1} \quad (4.21)$$

(the notation ' $\lim_{\nu \rightarrow 0+} A(\omega + i\nu)$ ' will also be shortened as ' $A(\omega + i0)$ '), whence

$$\frac{\partial G}{\partial r^i} = \mathcal{O}(\dot{p}_i) = \mathcal{O}(\epsilon). \quad (4.22)$$

Due to (4.16), the leading-order approximation of the symbol of the operator (3.25) is $G(\mathbf{X}, \mathbf{K}) = G_0(\Omega(\mathbf{X}, \mathbf{K}))$, where

$$G_0(\Omega) \doteq \lim_{\nu \rightarrow 0+} \int_0^\infty d\tau e^{-\nu\tau + i\Omega\tau} = \pi \delta(\Omega) + i \text{pv} \frac{1}{\Omega} \quad (4.23)$$

and the (standard) notation $\text{pv}(1/\Omega)$ is defined as follows:

$$\text{pv} \frac{1}{\Omega} \doteq \lim_{\nu \rightarrow 0+} \frac{\Omega}{\nu^2 + \Omega^2}. \quad (4.24)$$

This means, in particular, that for any A , one has

$$\begin{aligned} \mathcal{J}[A, G_0] &\doteq \int d\mathbf{K} A(\mathbf{X}, \mathbf{K}) G_0(\Omega(\mathbf{X}, \mathbf{K})) \\ &= \pi \int d\mathbf{K} A(\mathbf{X}, \mathbf{K}) \delta(\Omega(\mathbf{X}, \mathbf{K})) + i \oint d\mathbf{K} \frac{A(\mathbf{X}, \mathbf{K})}{\Omega(\mathbf{X}, \mathbf{K})}, \end{aligned} \quad (4.25)$$

where $\oint$ is a principal-value integral. Also usefully, $\overline{G_0} = G_0$ and

$$\begin{aligned} \partial_a \mathcal{J}[A, G_0] &= \int d\mathbf{K} A(\mathbf{X}, \mathbf{K}) G'_0(\Omega(\mathbf{X}, \mathbf{K})) \partial_a \Omega(\mathbf{X}, \mathbf{K}) \\ &= -(\partial_a V^b(\mathbf{X})) \int d\mathbf{K} K_b A(\mathbf{X}, \mathbf{K}) G'_0(\Omega(\mathbf{X}, \mathbf{K})) \\ &= -(\partial_a V^b(\mathbf{X})) \frac{\tilde{\partial}}{\partial \Omega} \int d\mathbf{K} K_b A(\mathbf{X}, \mathbf{K}) G_0(\Omega(\mathbf{X}, \mathbf{K})), \end{aligned} \quad (4.26)$$

where the notation $\tilde{\partial}/\partial \lambda \equiv \tilde{\partial}_\lambda$ is defined, for any λ and Q , as follows:

$$\frac{\tilde{\partial}}{\partial \lambda} \int Q(\lambda) \doteq \left(\frac{\partial}{\partial \vartheta} \int Q(\lambda + \vartheta) \right)_{\vartheta=0}. \quad (4.27)$$

Now let us reinstate the term μ in (4.16). It is readily seen (appendix B.2) that although μ may significantly affect T_τ *per se*, its effect on $\mathcal{J}[A, G]$ is small, namely,

$$\mathcal{J}[A, G] - \mathcal{J}[A, G_0] = \mathcal{O}(\epsilon^2). \quad (4.28)$$

Below, we apply this formulation to $A = \mathcal{O}(\epsilon^2)$, in which case (4.28) becomes $\mathcal{J}[A, G] - \mathcal{J}[A, G_0] = \mathcal{O}(\epsilon^2 \epsilon^2)$. Such corrections are negligible within our model, so from now on we adopt

$$G(\mathbf{X}, \mathbf{K}) \approx \bar{G}(\mathbf{X}, \mathbf{K}) \approx G_0(\Omega(\mathbf{X}, \mathbf{K})). \quad (4.29)$$

4.3. Initial conditions

Consider the function g from (3.24). Using (3.26), the latter can be written as follows:

$$g = \lim_{\nu \rightarrow 0+} \lim_{\tau_0 \rightarrow -\infty} \left(\hat{T}_{\tau-\tau_0} \tilde{f}(\mathbf{X}) + \int_0^{\tau-\tau_0} d\tau' (1 - e^{-\nu\tau'}) \hat{T}_{\tau'} \mathcal{F}(\mathbf{X}) \right). \quad (4.30)$$

Because $(1 - e^{-\nu\tau})$ is smooth and $\hat{T}_\tau \mathcal{F}$ is rapidly oscillating, the second term in the external parenthesis is an oscillatory function of τ_0 with the average negligible at $\nu \rightarrow 0$. But the whole expression in these parenthesis is independent of τ_0 at large τ_0 (section 3.2). Thus, it can be replaced with its own average over τ_0 , denoted $\langle \dots \rangle_{\tau_0}$. Because there is no ν -dependence left in this case, one can also omit $\lim_{\nu \rightarrow 0+}$. That gives

$$g = \lim_{\tau_0 \rightarrow -\infty} \langle \hat{T}_{\tau-\tau_0} \tilde{f}(\mathbf{X}) \rangle_{\tau_0}. \quad (4.31)$$

Using

$$f(\mathbf{X}) = \sum_{\sigma} \delta(\mathbf{z} - \mathbf{z}_{\sigma}(t)), \quad \mathfrak{f}(\mathbf{X}) \doteq \sum_{\sigma} \delta(\mathbf{z} - \bar{\mathbf{z}}_{\sigma}(t)), \quad (4.32)$$

where the sum is taken over individual particles, one can write[†]

$$f(\mathbf{X}) \approx \mathfrak{f}(\mathbf{X}) - \sum_{\sigma} \tilde{\mathbf{z}}_{\sigma}(\mathbf{X}) \partial_{\mathbf{z}} \delta(\mathbf{z} - \bar{\mathbf{z}}_{\sigma}(\mathbf{X})), \quad (4.33)$$

where $\mathbf{z}_{\sigma} \doteq \mathbf{z}_{\sigma} - \bar{\mathbf{z}}_{\sigma}$ are the $\tilde{H}$ -driven small deviations from the particle unperturbed trajectories $\bar{\mathbf{z}}_{\sigma}$. Then, $\bar{f} = \mathfrak{f}(\mathbf{X})$, and the linearized perturbation $\tilde{f} \doteq f - \bar{f}$ is given by

$$\tilde{f}(\mathbf{X}) = \underbrace{\mathfrak{f}(\mathbf{X}) - \bar{\mathfrak{f}}(\mathbf{X})}_{\tilde{\mathfrak{f}}} - \underbrace{\sum_{\sigma} \tilde{\mathbf{z}}_{\sigma}(\mathbf{X}) \partial_{\mathbf{z}} \delta(\mathbf{z} - \bar{\mathbf{z}}_{\sigma}(\mathbf{X}))}_{\tilde{\mathfrak{f}}}. \quad (4.34)$$

By definition, the unperturbed trajectories $\bar{\mathbf{z}}_{\sigma}$ satisfy $\hat{L}\delta(\mathbf{z} - \bar{\mathbf{z}}_{\sigma}(\mathbf{X})) = 0$, where $\hat{L}$ as in (3.14); thus,

$$\hat{T}_{\tau-\tau_0} \tilde{f} = e^{\hat{L}(\tau-\tau_0)} \tilde{f} = \tilde{\mathfrak{f}}. \quad (4.35)$$

Also, $\langle \hat{T}_{\tau-\tau_0} \tilde{\mathfrak{f}} \rangle_{\tau_0} = 0$, because $\tilde{\mathbf{z}}_{\sigma}$ are oscillatory functions of $\mathbf{X}$ that is slowly evolved by $\hat{T}_{\tau-\tau_0}$. Hence, g is the microscopic part of the unperturbed distribution function:

$$g = \tilde{\mathfrak{f}} = \mathfrak{f}(\mathbf{X}) - \bar{\mathfrak{f}}(\mathbf{X}). \quad (4.36)$$

This indicates that the term Γ defined in (3.34) is due to collisional effects. We postpone discussing these effects until section 6, so Γ will be ignored for now.

[†] Taylor-expanding delta functions is admittedly a questionable procedure, but here it is understood as a shorthand for Taylor-expanding integrals of f .

4.4. Summary of section 4

The main result of this section is that the Weyl symbol of the effective Green's operator $\hat{G}$ can be approximated within the assumed accuracy as follows:

$$G(\mathbf{X}, \mathbf{K}) \approx G_0(\Omega(\mathbf{X}, \mathbf{K})), \quad \Omega(\mathbf{X}, \mathbf{K}) \doteq -\mathbf{K} \cdot \mathbf{V}(\mathbf{X}). \quad (4.37)$$

Here, $\mathbf{V}$ is the unperturbed velocity in the $\mathbf{X}$ space, so $\Omega(\mathbf{X}, \mathbf{K}) = \omega - \mathbf{k} \cdot \mathbf{v} + \mathcal{O}(\epsilon)$, where $\mathbf{v}$ is the unperturbed velocity in the $\mathbf{x}$ space, and

$$G_0(\Omega) = \pi \delta(\Omega) + i \operatorname{pv} \frac{1}{\Omega}, \quad \operatorname{pv} \frac{1}{\Omega} \doteq \lim_{\nu \rightarrow 0+} \frac{\Omega}{\nu^2 + \Omega^2}. \quad (4.38)$$

We also show that the term Γ defined in (3.34) is due to collisional effects. We postpone discussing these effects until section 6, so Γ will be ignored for now.

5. Interaction with prescribed fields

In this section, we explore the effect of the diffusion operator $\hat{D}^{\alpha\beta}$. The oscillations will be described by $\bar{W}$ as a prescribed function, so they are allowed (yet not required) to be 'off-shell', i.e. do not have to be constrained by a dispersion relation. Examples of off-shell fluctuations include driven near-field oscillations, evanescent waves, and microscopic fluctuations (see also section 6). We will first derive the symbol of $\hat{D}^{\alpha\beta}$ and, using this symbol, approximate the diffusion operator with a differential operator (section 5.1). Then, we will calculate the coefficients in the approximate expression for $\hat{D}^{\alpha\beta}$ (sections 5.2 and 5.3). Finally, we will introduce the concept of the OC distribution (section 5.4) and summarize and simplify the resulting equations (section 5.6).

5.1. Expansion of the dispersion operator

The (effective) Green's operator can be represented through its symbol G using (2.71):

$$\hat{G} = \frac{1}{(2\pi)^N} \int d\mathbf{X} d\mathbf{K} d\mathbf{S} |\mathbf{X} + \mathbf{S}/2\rangle G(\mathbf{X}, \mathbf{K}) \langle \mathbf{X} - \mathbf{S}/2| e^{i\mathbf{K} \cdot \mathbf{S}}. \quad (5.1)$$

The corresponding representation of $\hat{u}^\alpha$ is even simpler, because the symbol of $\hat{u}^\alpha$ is independent of $\mathbf{K}$:[†]

$$\hat{u}^\alpha = \int d\mathbf{X} |\mathbf{X}\rangle u^\alpha(\mathbf{X}) \langle \mathbf{X}|. \quad (5.2)$$

Let us also introduce the Wigner matrix of u^α , denoted $W_{\mathbf{u}}^{\alpha\beta}$, and its inverse Fourier transform $C_{\mathbf{u}}^{\alpha\beta}$ as in section 2.2.3. Using these together with (2.67), one obtains

$$\begin{aligned} (2\pi)^N \hat{u}^\alpha \hat{G} \hat{u}^\beta &= \int d\mathbf{X}' d\mathbf{X}'' d\mathbf{X} d\mathbf{K} d\mathbf{S} u^\alpha(\mathbf{X}') u^\beta(\mathbf{X}'') G(\mathbf{X}, \mathbf{K}) e^{i\mathbf{K} \cdot \mathbf{S}} \\ &\quad \times |\mathbf{X}'\rangle \langle \mathbf{X}'| \mathbf{X} + \mathbf{S}/2 \rangle \langle \mathbf{X} - \mathbf{S}/2| \mathbf{X}'' \rangle \langle \mathbf{X}''| \\ &= \int d\mathbf{X} d\mathbf{K} d\mathbf{S} C_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}, \mathbf{S}) G(\mathbf{X}, \mathbf{K}) e^{i\mathbf{K} \cdot \mathbf{S}} |\mathbf{X} + \mathbf{S}/2\rangle \langle \mathbf{X} - \mathbf{S}/2| \\ &= \int d\mathbf{X}' d\mathbf{K}' d\mathbf{K}'' d\mathbf{S}' W_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}', \mathbf{K}') G(\mathbf{X}', \mathbf{K}'') e^{i(\mathbf{K}' + \mathbf{K}'') \cdot \mathbf{S}'} \\ &\quad \times |\mathbf{X}' + \mathbf{S}'/2\rangle \langle \mathbf{X}' - \mathbf{S}'/2|. \end{aligned} \quad (5.3)$$

[†] One can also derive (5.2) formally from (2.71).

Then, by taking symb_X of (5.3), one finds that the symbol of $\hat{D}^{\alpha\beta}$ is a convolution of $\overline{W}_u^{\alpha\beta}$ and G (appendix B.3):

$$D^{\alpha\beta}(\mathbf{X}, \mathbf{K}) = \int d\mathbf{K}' \overline{W}_u^{\alpha\beta}(\mathbf{X}, \mathbf{K}') G(\mathbf{X}, \mathbf{K} - \mathbf{K}'). \quad (5.4)$$

Let us Taylor-expand the symbol (5.4) in $\mathbf{K}$:

$$\begin{aligned} D^{\alpha\beta}(\mathbf{X}, \mathbf{K}) &\approx \int d\mathbf{K}' \overline{W}_u^{\alpha\beta}(\mathbf{X}, \mathbf{K}') G(\mathbf{X}, -\mathbf{K}') \\ &+ K_c \int d\mathbf{K}' \overline{W}_u^{\alpha\beta}(\mathbf{X}, \mathbf{K}') G^{[c]}(\mathbf{X}, -\mathbf{K}') + \mathcal{O}(K_a K_b G^{[ab]}). \end{aligned} \quad (5.5)$$

As a reminder, $G^{[a]}(\mathbf{X}, -\mathbf{K}) = -\partial^a G(\mathbf{X}, -\mathbf{K})$ denotes the derivative of G with respect to (the a th component of) the whole second argument, $-\mathbf{K}_a$, and

$$K_a \frac{\partial G}{\partial K_a} = \omega \frac{\partial G}{\partial \omega} + k_i \frac{\partial G}{\partial k_i} + r^i \frac{\partial G}{\partial r^i}. \quad (5.6)$$

Upon application of oper_X , ω gets replaced (roughly) with $i\partial_t = \mathcal{O}(\epsilon)$ and k_i gets replaced (also roughly) with $-i\partial_i = \mathcal{O}(\epsilon)$. By (4.22), the last term in (5.6) is of order ϵ too. This means that the contribution of the whole $K_a \partial^a G$ term to the equation for $\bar{f}$ is of order ϵ . The standard QLT neglects this contribution entirely, i.e. adopts $D^{\alpha\beta}(\mathbf{X}, \mathbf{K}) \approx D^{\alpha\beta}(\mathbf{X}, \mathbf{0})$, in which case the diffusion operator becomes just a local function of phase-space variables, $\hat{D}^{\alpha\beta} \approx D^{\alpha\beta}(\mathbf{X}, \mathbf{0})$. In this work, we retain corrections to the first order in $\mathbf{X}$, i.e. keep the second term in (5.5) as well, while neglecting the higher-order terms as usual.

Within this model, one can rewrite (5.5) as follows:

$$D^{\alpha\beta}(\mathbf{X}, \mathbf{K}) \approx D_0^{\alpha\beta}(\mathbf{X}) + K_c \Theta^{\alpha\beta c}(\mathbf{X}). \quad (5.7)$$

Here, we used (4.19) and introduced

$$D_0^{\alpha\beta}(\mathbf{X}) \doteq \int d\mathbf{K} \overline{W}_u^{\alpha\beta}(\mathbf{X}, \mathbf{K}) G^*(\mathbf{X}, \mathbf{K}), \quad (5.8)$$

$$\Theta^{\alpha\beta c}(\mathbf{X}) \doteq - \int d\mathbf{K} \overline{W}_u^{\alpha\beta}(\mathbf{X}, \mathbf{K}) (G^{[c]}(\mathbf{X}, \mathbf{K}))^*, \quad (5.9)$$

which satisfy (appendix B.4)

$$D_0^{\alpha\beta}(\mathbf{X}) = (D_0^{\alpha\beta}(\mathbf{X}))^*, \quad \Theta^{\alpha\beta c}(\mathbf{X}) = -(\Theta^{\alpha\beta c}(\mathbf{X}))^*. \quad (5.10)$$

The first-order Weyl expansion of $\hat{D}^{\alpha\beta}$ is obtained by applying oper_X to (5.7). Namely, for any ψ , one has (cf. section 2.1.5)

$$\hat{D}^{\alpha\beta} \psi \approx D_0^{\alpha\beta} \psi - i\Theta^{\alpha\beta c} \partial_c \psi - \frac{i}{2} (\partial_c \Theta^{\alpha\beta c}) \psi. \quad (5.11)$$

What remains now is to calculate the functions $D_0^{\alpha\beta}$ and $\Theta^{\alpha\beta c}$ explicitly.

5.2. Wigner matrix of the velocity oscillations

To express $\hat{D}^{\alpha\beta}$ through the Wigner function W of the perturbation Hamiltonian (section 3.1.1), we need to express $W_u^{\alpha\beta}$ through W . Recall that $W_u^{\alpha\beta}$ is the symbol of the density operator $\hat{W}_u^{\alpha\beta} \doteq (2\pi)^{-N} |u^\alpha\rangle \langle u^\beta|$ (section 2.2.3). By definition (3.8), one has $u^\alpha = iJ^{\alpha\mu} \hat{q}_\mu \hat{H}$, where $\hat{q}_\alpha \doteq -i\partial_\alpha$ (section 2.2.1). Then,

$$\hat{W}_u^{\alpha\beta} = (2\pi)^{-N} J^{\alpha\mu} \hat{q}_\mu |\tilde{H}\rangle \langle \tilde{H}| \hat{q}_\nu J^{\beta\nu} = J^{\alpha\mu} J^{\beta\nu} \hat{q}_\mu \hat{W} \hat{q}_\nu, \quad (5.12)$$

where $\widehat{W}$ is the density operator whose symbol is W . By applying symb_X , one obtains

$$W_{\mathbf{u}}^{\alpha\beta} = J^{\alpha\mu} J^{\beta\nu} (q_\mu \star W \star q_\nu), \quad (5.13)$$

where $\star$ is the Moyal product (2.72). Using formulas analogous to (2.33) in the $(\mathbf{X}, \mathbf{K})$ space, one obtains

$$\begin{aligned} q_\mu \star W \star q_\nu &= \left(q_\mu W - \frac{i}{2} \frac{\partial W}{\partial z^\mu} \right) \star q_\nu \\ &= q_\mu q_\nu W - \frac{i}{2} q_\nu \frac{\partial W}{\partial z^\mu} + \frac{i}{2} \frac{\partial}{\partial z^\nu} \left(q_\mu W - \frac{i}{2} \frac{\partial W}{\partial z^\mu} \right) \\ &= q_\mu q_\nu W_h + \frac{i}{2} \left(q_\mu \frac{\partial W}{\partial z^\nu} - q_\nu \frac{\partial W}{\partial z^\mu} \right) + \frac{1}{4} \frac{\partial^2 W}{\partial z^\mu \partial z^\nu}. \end{aligned} \quad (5.14)$$

Hence, $W_{\mathbf{u}}^{\alpha\beta}$ and W are connected via the following exact formula:

$$W_{\mathbf{u}}^{\alpha\beta} = J^{\alpha\mu} J^{\beta\nu} \left(q_\mu q_\nu W - \frac{i}{2} \left(q_\nu \frac{\partial W}{\partial z^\mu} - q_\mu \frac{\partial W}{\partial z^\nu} \right) + \frac{1}{4} \frac{\partial^2 W}{\partial z^\mu \partial z^\nu} \right). \quad (5.15)$$

5.3. Nonlinear potentials

Due to (5.10), one has $D_0^{\alpha\beta} = \text{re } D_0^{\alpha\beta}$. Using this together with (5.8), (5.15), (4.29), and (4.23), one obtains

$$\begin{aligned} D_0^{\alpha\beta} &= J^{\alpha\mu} J^{\beta\nu} \text{re} \int d\mathbf{K} \left(\pi \delta(\Omega) - i \text{pv} \frac{1}{\Omega} \right) \\ &\quad \times \left(q_\mu q_\nu \overline{W} - \frac{i}{2} \left(q_\nu \frac{\partial \overline{W}}{\partial z^\mu} - q_\mu \frac{\partial \overline{W}}{\partial z^\nu} \right) + \frac{1}{4} \frac{\partial^2 \overline{W}}{\partial z^\mu \partial z^\nu} \right), \end{aligned} \quad (5.16)$$

with notation as in (2.10). This can be written as $D_0^{\alpha\beta} = \mathcal{D}^{\alpha\beta} + \varrho^{\alpha\beta} + \varsigma^{\alpha\beta}$, where

$$\mathcal{D}^{\alpha\beta} \doteq J^{\alpha\mu} J^{\beta\nu} \int d\mathbf{K} \pi \delta(\Omega) q_\mu q_\nu \overline{W}, \quad (5.17)$$

and we also introduced

$$\varrho^{\alpha\beta} \doteq -\frac{1}{2} J^{\alpha\mu} J^{\beta\nu} \oint d\mathbf{K} \left(q_\nu \frac{\partial \overline{W}}{\partial z^\mu} - q_\mu \frac{\partial \overline{W}}{\partial z^\nu} \right) \frac{1}{\Omega}, \quad (5.18)$$

$$\varsigma^{\alpha\beta} \doteq \frac{1}{4} J^{\alpha\mu} J^{\beta\nu} \int d\mathbf{K} \pi \delta(\Omega) \frac{\partial^2 \overline{W}}{\partial z^\mu \partial z^\nu}. \quad (5.19)$$

As shown in appendix B.5, the contributions of these two functions to (3.33) are

$$\frac{\partial}{\partial z^\alpha} \left(\varrho^{\alpha\beta} \frac{\partial \overline{f}}{\partial z^\beta} \right) = \mathcal{O}(\epsilon \epsilon^2), \quad \frac{\partial}{\partial z^\alpha} \left(\varsigma^{\alpha\beta} \frac{\partial \overline{f}}{\partial z^\beta} \right) = \mathcal{O}(\epsilon^2 \epsilon^2). \quad (5.20)$$

Thus, $\varrho^{\alpha\beta}$ must be retained and $\varsigma^{\alpha\beta}$ must be neglected, which leads to

$$D_0^{\alpha\beta} \approx \mathcal{D}^{\alpha\beta} + \varrho^{\alpha\beta}. \quad (5.21)$$

The function $\Theta^{\alpha\beta c} = i \text{im } \Theta^{\alpha\beta c}$ can be written as follows:

$$\Theta^{\alpha\beta c}(\mathbf{X}) = i J^{\alpha\mu} J^{\beta\nu} \int d\mathbf{K} q_\mu q_\nu \overline{W}(\mathbf{X}, \mathbf{K}) \frac{\partial}{\partial K_c} \text{pv} \frac{1}{\Omega(\mathbf{X}, \mathbf{K})} = -i V^c(\mathbf{X}) \Theta^{\alpha\beta}(\mathbf{X}),$$

where we introduced

$$\Theta^{\alpha\beta} \doteq J^{\alpha\mu} J^{\beta\nu} \oint d\mathbf{K} \frac{q_\mu q_\nu \overline{W}}{\Omega} \quad (5.22)$$

and $\tilde{\partial}$ is defined as in (4.27). Then finally, one can rewrite (5.11) as follows:

$$\begin{aligned}\hat{D}^{\alpha\beta}\psi &\approx (\mathcal{D}^{\alpha\beta} + \varrho^{\alpha\beta})\psi - \Theta^{\alpha\beta} V^c \partial_c \psi - \frac{1}{2} V^c (\partial_c \Theta^{\alpha\beta}) \psi \\ &= (\mathcal{D}^{\alpha\beta} + \varrho^{\alpha\beta})\psi - \Theta^{\alpha\beta} (\partial_t + v^\lambda \partial_\lambda) \psi - \frac{1}{2} ((\partial_t + v^\lambda \partial_\lambda) \Theta^{\alpha\beta}) \psi,\end{aligned}\quad (5.23)$$

where we used (3.15). With some algebra (appendix B.6), and assuming the notation

$$\Phi = -J^{\mu\nu} \frac{\partial}{\partial z^\mu} \oint d\mathbf{K} \frac{q_\nu \overline{W}}{2\Omega}, \quad (5.24)$$

one finds that (5.23) leads to

$$\partial_\alpha (\hat{D}^{\alpha\beta} \partial_\beta \overline{f}) = \partial_\alpha (\mathcal{D}^{\alpha\beta} \partial_\beta \overline{f}) - \frac{1}{2} d_t \partial_\alpha (\Theta^{\alpha\beta} \partial_\beta \overline{f}) + \{\Phi, \overline{f}\}. \quad (5.25)$$

Hence, (3.33) becomes (to the extent that Γ is negligible; see section 6.7)

$$d_t \overline{f} + \frac{1}{2} d_t \partial_\alpha (\Theta^{\alpha\beta} \partial_\beta \overline{f}) - \{\Phi, \overline{f}\} = \partial_\alpha (\mathcal{D}^{\alpha\beta} \partial_\beta \overline{f}). \quad (5.26)$$

The functions $\Theta^{\alpha\beta}$, Φ , and $\mathcal{D}^{\alpha\beta}$ that determine the coefficients in this equation are fundamental and, for the lack of a better term, will be called nonlinear potentials.

5.4. Oscillation-center distribution

Let us introduce

$$F \doteq \overline{f} + \frac{1}{2} \partial_\alpha (\Theta^{\alpha\beta} \partial_\beta \overline{f}). \quad (5.27)$$

Then, using (5.25), one can rewrite (5.26) as[†]

$$\partial_t F - \{\mathcal{H}, F\} = \partial_\alpha (\mathcal{D}^{\alpha\beta} \partial_\beta F), \quad (5.28)$$

where corrections $\mathcal{O}(\varepsilon^4)$ have been neglected and we introduced $\mathcal{H} \doteq \overline{H} + \Phi$. As a reminder, the nonlinear potentials in (5.28) are as follows:

$$\mathcal{D}^{\alpha\beta} = J^{\alpha\mu} J^{\beta\nu} \int d\mathbf{K} \pi \delta(\Omega) q_\mu q_\nu \overline{W}, \quad (5.29)$$

$$\Theta^{\alpha\beta} = J^{\alpha\mu} J^{\beta\nu} \frac{\tilde{\partial}}{\partial \Omega} \oint d\mathbf{K} \frac{q_\mu q_\nu \overline{W}}{\Omega}, \quad (5.30)$$

$$\Phi = -J^{\mu\nu} \frac{\partial}{\partial z^\mu} \oint d\mathbf{K} \frac{q_\nu \overline{W}}{2\Omega}. \quad (5.31)$$

Equations (5.27)–(5.31) form a closed model that describes the evolution of the average distribution $\overline{f}$ in turbulence with prescribed $\overline{W}$. In particular, (5.28) can be interpreted as a Liouville-type equation for F as an effective, or ‘dressed’, distribution. The latter can be understood as the distribution of ‘dressed’ particles called OCs. Then, $\mathcal{H}$ serves as the OC Hamiltonian, $\mathcal{D}^{\alpha\beta}$ is the phase-space diffusion coefficient, Φ is the ponderomotive energy, $\Omega = \omega - q_\alpha v^\alpha$, and $v^\alpha \doteq J^{\alpha\beta} \partial_\beta \overline{H}$. Within the assumed accuracy, one can redefine v^α to be the OC velocity rather than the particle velocity; specifically,[‡]

$$v^\alpha \doteq J^{\alpha\beta} \partial_\beta \mathcal{H} = J^{\alpha\beta} \partial_\beta \overline{H} + \mathcal{O}(\varepsilon^2). \quad (5.32)$$

[†] The difference between F and $\overline{f}$ is related to the concept of so-called adiabatic diffusion (Galeev & Sagdeev 1985; Stix 1992), which captures some but not all adiabatic effects.

[‡] The advantage of the amended definition (5.32) is that it will lead to *exact* conservation laws of our theory, as to be discussed in section 7.5.

Then, the presence of $\delta(\Omega)$ in (5.29) signifies that OCs diffuse in phase space in response to waves they are resonant with. Below, we use the terms ‘OCs’ and ‘particles’ interchangeably except where specified otherwise.

That said, the interpretation of OCs as particle-like objects is limited. Single-OC motion equations are not introduced in our approach. (They would have been singular for resonant interactions.) Accordingly, the transformation (5.27) of the distribution function $\bar{f} \mapsto F$ is not *derived* from a coordinate transformation but rather is *fundamental*. As a result, particles and OCs live in the same phase space, but the ‘dynamics of OCs’ can be irreversible (section 5.5). This qualitatively distinguishes our approach from the traditional OC theory (Dewar 1973) and from the conceptually similar gyrokinetic theory (Littlejohn 1981; Cary & Brizard 2009), where coordinate transformations are central.

5.5. H -theorem

Because $\bar{W}$ is non-negative (section 2.1.6), $D^{\alpha\beta}$ is positive-semidefinite; that is,

$$D^{\alpha\beta} \psi_\alpha \psi_\beta = \int d\mathbf{K} \pi \delta(\Omega) a^2 \bar{W} \geq 0, \quad a \doteq J^{\alpha\mu} \psi_\alpha q_\mu \quad (5.33)$$

for any real ψ . This leads to the following theorem. Consider the OC entropy defined as

$$\mathcal{S} \doteq - \int d\mathbf{z} F(t, \mathbf{z}) \ln F(t, \mathbf{z}). \quad (5.34)$$

According to (5.28), $\mathcal{S}$ satisfies

$$\begin{aligned} \frac{d\mathcal{S}}{dt} &= - \int d\mathbf{z} \frac{d(F \ln F)}{dF} (\{\mathcal{H}, F\} + \partial_\alpha (D^{\alpha\beta} \partial_\beta F)) \\ &= - \int d\mathbf{z} \frac{d(F \ln F)}{dF} J^{\alpha\beta} (\partial_\alpha \mathcal{H}) (\partial_\beta F) - \int d\mathbf{z} (1 + \ln F) \partial_\alpha (D^{\alpha\beta} \partial_\beta F) \\ &= - \int d\mathbf{z} J^{\alpha\beta} (\partial_\alpha \mathcal{H}) \partial_\beta (F \ln F) - \int d\mathbf{z} \ln F \partial_\alpha (D^{\alpha\beta} \partial_\beta F) \\ &= \int d\mathbf{z} (J^{\alpha\beta} \partial_{\alpha\beta}^2 \mathcal{H}) F \ln F + \int d\mathbf{z} D^{\alpha\beta} (\partial_\alpha \ln F) (\partial_\beta \ln F) F. \end{aligned} \quad (5.35)$$

The first integral vanishes due to $J^{\alpha\beta} \partial_{\alpha\beta}^2 = 0$. The second integral is non-negative due to (5.33). Thus,

$$\frac{d\mathcal{S}}{dt} \geq 0, \quad (5.36)$$

which is recognized as the H -theorem (Lifshitz & Pitaevskii 1981, section 4) for QL OC dynamics.

5.6. Summary of section 5

From now on, we assume that the right-hand side of (5.28) scales not as $\mathcal{O}(\varepsilon^2)$ but as $\mathcal{O}(\varepsilon\varepsilon^2)$, either due to the scarcity of resonant particles or, for QL diffusion driven by microscopic fluctuations (section 6), due to the plasma parameter’s being large. Also, the spatial derivatives can be neglected within the assumed accuracy in the definition of F (5.27) and on the right-hand side of (5.28). Using this together with (2.69), and with (2.56) for the Poisson bracket, our results can be summarized as follows.

QL evolution of a particle distribution in a prescribed wave field is governed by[†]

$$\frac{\partial F}{\partial t} - \frac{\partial \mathcal{H}}{\partial \mathbf{x}} \cdot \frac{\partial F}{\partial \mathbf{p}} + \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \cdot \frac{\partial F}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{p}} \cdot \left(\mathbf{D} \frac{\partial F}{\partial \mathbf{p}} \right). \quad (5.37)$$

The OC distribution F is defined as

$$F = \bar{f} + \frac{1}{2} \frac{\partial}{\partial \mathbf{p}} \cdot \left(\mathbf{\Theta} \frac{\partial \bar{f}}{\partial \mathbf{p}} \right), \quad (5.38)$$

so the density of OCs is the same as the locally averages density of the true particles:

$$\mathcal{N} \doteq \int d\mathbf{p} F = \int d\mathbf{p} \bar{f}. \quad (5.39)$$

The function $\mathcal{H}$ is understood as the OC Hamiltonian. It is given by

$$\mathcal{H} \doteq \bar{H} + \Phi, \quad (5.40)$$

where $\bar{H}$ is the average Hamiltonian, which may include interaction with background fields, and Φ is the ponderomotive potential. The nonlinear potentials that enter (5.37) can be calculated to the zeroth order in ϵ and are given by[‡]

$$\mathbf{D} = \int d\omega d\mathbf{k} \pi \mathbf{k} \mathbf{k} \bar{W}(t, \mathbf{k} \cdot \mathbf{v}, \mathbf{k}; \mathbf{p}), \quad (5.41)$$

$$\mathbf{\Theta} = \frac{\partial}{\partial \vartheta} \oint d\omega d\mathbf{k} \frac{\mathbf{k} \mathbf{k} \bar{W}}{\omega - \mathbf{k} \cdot \mathbf{v} + \vartheta} \Big|_{\vartheta=0}, \quad (5.42)$$

$$\Phi = \frac{1}{2} \frac{\partial}{\partial \mathbf{p}} \cdot \oint d\omega d\mathbf{k} \frac{\mathbf{k} \bar{W}}{\omega - \mathbf{k} \cdot \mathbf{v}}, \quad (5.43)$$

where $\mathbf{k} \mathbf{k}$ is a dyadic matrix with two lower indices, and the same conventions apply as in section 2.1.2. Also, $\mathbf{v}$ is hereby redefined as the OC spatial velocity, namely,

$$\mathbf{v} \doteq \partial_{\mathbf{p}} \mathcal{H} = \partial_{\mathbf{p}} \bar{H} + \mathcal{O}(\epsilon^2). \quad (5.44)$$

The function $\bar{W}$ is defined as

$$\bar{W}(t, \mathbf{x}, \omega, \mathbf{k}; \mathbf{p}) \doteq \int d\mathbf{r} \bar{W}(t, \mathbf{x}, \mathbf{p}, \omega, \mathbf{k}, \mathbf{r}), \quad (5.45)$$

where $\bar{W}$ is the average Wigner function (3.4) of the perturbation Hamiltonian, i.e. the spectrum of its symmetrized autocorrelation function (3.5). Due to (2.78), it can be understood as the average of $W \doteq \text{sym}_x \hat{W}$ (where $\hat{W}$ is defined in (3.3)), i.e. as the Wigner function of the perturbation Hamiltonian with $\mathbf{p}$ treated as a parameter. As such, $\bar{W}$ is non-negative, so $\mathbf{D}$ is positive-semidefinite. This leads to an H -theorem (proven similarly to (5.36)) for the entropy density $\sigma \doteq - \int d\mathbf{p} F \ln F$:

$$\left(\frac{d\sigma}{dt} \right)_{\mathbf{D}} \geq 0, \quad \left(\frac{\partial \psi}{\partial t} \right)_{\mathbf{D}} \doteq \frac{\partial}{\partial \mathbf{p}} \cdot \left(\mathbf{D} \frac{\partial \psi}{\partial \mathbf{p}} \right). \quad (5.46)$$

Also note that for homogeneous turbulence in particular, where $\bar{W}$ is independent of $\mathbf{x}$,

[†] Remember that here we neglect Γ (3.34), which is a part of the collision operator to be reinstated in section 6.

[‡] See section 9 for examples and section 6.6 for the explanation on how Φ is related to Δ , which is yet to be introduced. Also note that in combination with (5.40), equation (5.43) generalizes the related results from (Kentwell 1987; Fraiman & Kostyukov 1995; Dodin & Fisch 2014).

(2.46) yields that

$$\int d\omega \overline{W}(t, \mathbf{x}, \omega, \mathbf{k}; \mathbf{p}) = \frac{1}{\mathcal{V}_n} \int d\omega d\mathbf{x} \overline{W}(t, \mathbf{x}, \omega, \mathbf{k}; \mathbf{p}) = \frac{1}{\mathcal{V}_n} \overline{|\hat{H}(t, \mathbf{k}, \mathbf{p})|^2}, \quad (5.47)$$

where $\mathcal{V}_n$ is the plasma volume (the index n denotes the number of spatial dimensions) and $\hat{H}$ is the spatial spectrum of $\tilde{H}$ as defined in (2.23).

Equation (5.37) can be used to calculate the ponderomotive force $\partial_t \int d\mathbf{p} \bar{f}$ that a given wave field imparts on a plasma. This potentially resolves the controversies mentioned in (Kentwell & Jones 1987). We will revisit this subject for on-shell waves in section 7.5.

6. Interaction with self-consistent fields

Here, we explain how to calculate the function $\overline{W}$ in the presence of microscopic fluctuations (nonzero g). In particular, we reinstate the term Γ that was omitted in section 5. We also show that a collision operator of the Balescu–Lenard type emerges from our theory within a general interaction model. This calculation can be considered as a generalization of that in (Rogister & Oberman 1968) for homogeneous plasmas. Another related calculation was proposed in (Chavanis 2012) in application to potential interactions in inhomogeneous systems using action–angle variables, with global averaging over the angles. (See also (Mynick 1988) for a related calculation in action–angle variables based on the Fokker–Planck approach.) In contrast, our model holds for any Hamiltonian interactions via any vector fields and allows for weak inhomogeneities in canonical coordinates.

6.1. Interaction model

Let us assume that particles interact via an M -component real field $\Psi \equiv (\psi^1, \psi^2, \dots, \psi^M)^\top$. It is treated below as a column vector; hence the index $\top$. (A complex field can be accommodated by considering its real and imaginary parts as separate components.) We split this field into the average part $\overline{\Psi}$ and the oscillating part $\tilde{\Psi}$. The former is considered given. For the latter, we assume the action integral of this field without plasma in the form

$$S_0 = \int d\mathbf{x} \mathcal{L}_0, \quad \mathcal{L}_0 = \frac{1}{2} \tilde{\Psi}^\dagger \hat{\Xi}_0 \tilde{\Psi} \quad (6.1)$$

(see section 9 for examples), where $\hat{\Xi}_0$ is a Hermitian operator[†] and $\tilde{\Psi}^\dagger = \tilde{\Psi}^\top$ is a row vector dual to $\tilde{\Psi}$. Plasma is allowed to consist of multiple species, henceforth denoted with index s . Because $\tilde{\Psi}$ is assumed small, the generic Hamiltonian for each species s can be Taylor-expanded in $\tilde{\Psi}$ and represented in a generic form

$$H_s(t, \mathbf{x}, \mathbf{p}) \approx H_{0s} + \hat{\alpha}_s^\dagger \tilde{\Psi} + \frac{1}{2} (\hat{L}_s \tilde{\Psi})^\dagger (\hat{R}_s \tilde{\Psi}) \quad (6.2)$$

(see section 9 for examples), which can be considered as a second-order Taylor expansion of the full Hamiltonian in $\tilde{\Psi}$. Here, $H_{0s} \equiv H_{0s}(t, \mathbf{x}, \mathbf{p})$ is independent of $\tilde{\Psi} \equiv \tilde{\Psi}(t, \mathbf{x})$, $\hat{\alpha}_s \equiv (\hat{\alpha}_{s,1}, \hat{\alpha}_{s,2}, \dots, \hat{\alpha}_{s,M})^\top$ is a column vector whose elements $\hat{\alpha}_{s,i}$ are linear operators on $\mathcal{H}_x$, and the dagger is added so that $\hat{\alpha}_s^\dagger$ could be understood as a row vector whose elements $\hat{\alpha}_{s,i}^\dagger$ act on the individual components of the field; i.e. $\hat{\alpha}_s^\dagger \tilde{\Psi} \equiv \hat{\alpha}_{s,i}^\dagger \tilde{\psi}^i$. We let $\hat{\alpha}_s$

[†] The field action often has the form $S_0 = \frac{1}{2} \int d\mathbf{x} \sqrt{g} (\mathbf{g} \tilde{\Psi}^*) \hat{\Xi}_0 \tilde{\Psi}$, where $\mathbf{g}(\mathbf{x})$ is a spacetime metric, $\mathbf{g} \doteq |\det \mathbf{g}|$, and $\hat{\Xi}_0$ is Hermitian with respect to the inner product $\langle \xi | \psi \rangle_{\mathbf{g}} \doteq \int d\mathbf{x} \sqrt{g} (\mathbf{g} \xi^*) \psi$. Using $\tilde{\Psi}' \doteq \mathbf{g}^{1/4} \tilde{\Psi}$ and $\hat{\Xi}'_0 \doteq \mathbf{g}^{1/4} \hat{\Xi}_0 \mathbf{g}^{-1/4}$, one can cast this action in the form (6.1), with $\hat{\Xi}'_0$ that is Hermitian with respect to the inner product (2.6).

be nonlocal in t and $\mathbf{x}$ (for example, $\hat{\alpha}_s$ can be a spacetime derivative or a spacetime integral), and we also let $\hat{\alpha}_s$ depend on $\mathbf{p}$ parametrically, so

$$\text{symb } \hat{\alpha}_s = \alpha_s(t, \mathbf{x}, \omega, \mathbf{k}; \mathbf{p}). \quad (6.3)$$

The matrix operators $\hat{\mathbf{L}}_s$ and $\hat{\mathbf{R}}_s$ and their symbols $\mathbf{L}_s$ and $\mathbf{R}_s$ are understood similarly.

The Lagrangian density of the oscillating-field–plasma system is

$$\mathfrak{L}_p = \frac{1}{2} \tilde{\Psi}^\dagger \hat{\Xi}_0 \tilde{\Psi} + \sum_s \sum_{\sigma_s} (\mathbf{p}_{\sigma_s} \cdot \dot{\mathbf{x}}_{\sigma_s} - H_s(t, \mathbf{x}_{\sigma_s}, \mathbf{p}_{\sigma_s})) \delta(\mathbf{x} - \mathbf{x}_{\sigma_s}(t)), \quad (6.4)$$

where the sum is taken over individual particles. Note that

$$\begin{aligned} \sum_{\sigma_s} H_s(t, \mathbf{x}_{\sigma_s}, \mathbf{p}_{\sigma_s}) \delta(\mathbf{x} - \mathbf{x}_{\sigma_s}(t)) &= \int d\mathbf{p} \sum_{\sigma_s} \delta(\mathbf{z} - \mathbf{z}_{\sigma_s}(t)) H_s(t, \mathbf{x}, \mathbf{p}) \\ &= \int d\mathbf{p} f_s(t, \mathbf{x}, \mathbf{p}) H_s(t, \mathbf{x}, \mathbf{p}), \end{aligned} \quad (6.5)$$

so the $\tilde{\Psi}$ -dependent part of the system action can be written as $S = \int d\mathbf{x} \mathfrak{L}$ with

$$\mathfrak{L} = \frac{1}{2} \tilde{\Psi}^\dagger \hat{\Xi}_p \tilde{\Psi} - \sum_s \int d\mathbf{p} \tilde{f}_s \hat{\alpha}_s^\dagger \tilde{\Psi}, \quad (6.6)$$

$$\hat{\Xi}_p \doteq \hat{\Xi}_0 - \sum_s \int d\mathbf{p} \hat{\mathbf{L}}_s^\dagger f_s \hat{\mathbf{R}}_s. \quad (6.7)$$

(The contribution of $\tilde{f}_s$ to the second term in (6.6) has been omitted because it averages to zero at integration over spacetime and thus does not contribute to S .) This ‘abridged’ action is not sufficient to describe the particle motion, but it is sufficient to describe the dynamics of $\tilde{\Psi}$ at given f_s , as discussed below. The operator $\hat{\Xi}_p$ can be considered Hermitian without loss of generality, because its anti-Hermitian part does not contribute to S . Also, we assume that unless either of $\hat{\mathbf{L}}$ and $\hat{\mathbf{R}}$ is zero, the high-frequency field has no three-wave resonances, so terms cubic in $\tilde{\Psi}$ can be neglected in S ;† then,

$$\hat{\Xi}_p \approx \hat{\Xi}_0 - \sum_s \int d\mathbf{p} (\hat{\mathbf{L}}_s^\dagger F_s \hat{\mathbf{R}}_s)_H. \quad (6.8)$$

Using the same assumption, one can also adopt

$$\overline{H}_s = H_{0s} + \frac{1}{2} \overline{(\hat{\mathbf{L}}_s \tilde{\Psi})^\dagger (\hat{\mathbf{R}}_s \tilde{\Psi})}, \quad \tilde{H}_s \approx \hat{\alpha}_s^\dagger \tilde{\Psi}, \quad (6.9)$$

because in the absence of three-wave resonances, the *oscillating* part of $(\hat{\mathbf{L}}_s \tilde{\Psi})^\dagger (\hat{\mathbf{R}}_s \tilde{\Psi})$ contributes only $\mathcal{O}(\varepsilon^4)$ terms to the equation for F_s .

6.2. Field equations

The Euler–Lagrange equation for $\tilde{\Psi}$ derived from (6.6) is

$$\hat{\Xi}_p \tilde{\Psi} = \sum_s \int d\mathbf{p} \hat{\alpha}_s \tilde{f}_s. \quad (6.10)$$

† This is tacitly assumed already in (6.2), where cubic terms are neglected. Also note that three-wave interactions that involve resonances between low-frequency oscillations of F_s and two high-frequency waves, like Raman scattering (Balakin *et al.* 2016), are still allowed.

Then, to the extent that the linear approximation for $\tilde{f}_s$ is sufficient (see below), one finds that the oscillating part of the field satisfies

$$\hat{\mathbf{E}}_p \tilde{\Psi} - \sum_s \int d\mathbf{p} \hat{\alpha}_s \hat{G}_s \{\hat{\alpha}_s^\dagger \tilde{\Psi}, \bar{f}_s\} = \sum_s \int d\mathbf{p} \hat{\alpha}_s g_s, \quad (6.11)$$

where we used (3.24). Note that the right-hand side of (6.11) is determined by microscopic fluctuations $g_s(t, \mathbf{x}, \mathbf{p})$ (section 4.3). Equation (6.11) can also be expressed as

$$\hat{\mathbf{E}} \tilde{\Psi} = \sum_s \int d\mathbf{p} \hat{\alpha}_s g_s, \quad (6.12)$$

where $\hat{\mathbf{E}}$ is understood as the plasma dispersion operator and is given by

$$\hat{\mathbf{E}} \doteq \hat{\mathbf{E}}_p - \sum_s \int d\mathbf{p} \hat{\alpha}_s \hat{G}_s \{\hat{\alpha}_s^\dagger \blacksquare, \bar{f}_s\}, \quad (6.13)$$

where $\blacksquare$ is a placeholder. The general solution of (6.12) can be written as

$$\tilde{\Psi} = \tilde{\underline{\Psi}} + \tilde{\underline{\Psi}}, \quad \tilde{\underline{\Psi}} = \sum_s \int d\mathbf{p} \hat{\mathbf{E}}^{-1} \hat{\alpha}_s g_s. \quad (6.14)$$

Here, $\hat{\mathbf{E}}^{-1}$ is the right inverse of $\hat{\mathbf{E}}$ (meaning $\hat{\mathbf{E}} \hat{\mathbf{E}}^{-1} = \hat{\mathbf{1}}$ yet $\hat{\mathbf{E}}^{-1} \hat{\mathbf{E}} \neq \hat{\mathbf{1}}$) such that $\tilde{\underline{\Psi}}$ vanishes at zero g .[†] The rest of the solution, $\tilde{\underline{\Psi}}$, is the macroscopic field that satisfies

$$\hat{\mathbf{E}} \tilde{\underline{\Psi}} = 0. \quad (6.15)$$

In the special case when the dispersion operator is Hermitian ($\hat{\mathbf{E}} = \hat{\mathbf{E}}_H$), (6.15) also flows from the ‘adiabatic’ macroscopic part of the action S , namely,

$$S_{\text{ad}} \doteq \frac{1}{2} \int d\mathbf{x} \tilde{\underline{\Psi}}^\dagger \hat{\mathbf{E}}_H \tilde{\underline{\Psi}}. \quad (6.16)$$

Because we have assumed a linear model for f_s in (6.11), $\tilde{\underline{\Psi}}$ is decoupled from $\tilde{\underline{\Psi}}$, and hence the dynamics of $\tilde{\underline{\Psi}}$ turns out to be collisionless. This is justified, because collisional dissipation is assumed to be much slower than collisionless dissipation (section 3.3). One can reinstate collisions in (6.15) by modifying $\hat{G}_s$ *ad hoc*, if necessary. Alternatively, one can avoid separating $\tilde{\underline{\Psi}}$ and $\tilde{\underline{\Psi}}$ and, instead, derive an equation for the average Wigner matrix of the whole $\tilde{\Psi}$ (McDonald 1991). However, this approach is beyond QLT, so it is not considered in this paper.

6.3. Dispersion matrix

As readily seen from the definition (6.13), the operator $\hat{\mathbf{E}}$ can be expressed as

$$\hat{\mathbf{E}} = \hat{\mathbf{E}}_p - i k_j \sum_s \int d\mathbf{p} \hat{\alpha}_s \hat{G}_s \hat{\alpha}_s^\dagger \frac{\partial F_s}{\partial p_j} + \mathcal{O}(\epsilon, \varepsilon^2). \quad (6.17)$$

The corrections caused by nonzero ϵ and ε in this formula will be insignificant for our purposes, so they will be neglected. In particular, this means that $G_s \doteq \text{symb}_X \hat{G}_s$ can be adopted in the form independent of $\mathbf{r}$ (section 4.2):

$$G_s \approx \frac{i}{\omega - \mathbf{k} \cdot \mathbf{v}_s + i0} = \pi \delta(\omega - \mathbf{k} \cdot \mathbf{v}_s) + i \text{pv} \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_s}. \quad (6.18)$$

[†] Most generally, the problem of finding $\hat{\mathbf{E}}^{-1}$ is the standard problem of calculating the field produced by a given radiation source.

Then, $\hat{G}_s$ can be considered as an operator on $\mathcal{H}_x$ with $\mathbf{p}$ as a parameter, and $\text{symb } \hat{G}_s = G_s(t, \mathbf{x}, \omega, \mathbf{k}; \mathbf{p})$. Also,

$$\text{symb}(\hat{\alpha}_s \hat{G}_s \hat{\alpha}_s^\dagger) = \alpha_s \star G_s \star \alpha_s^\dagger \approx \alpha_s G_s \alpha_s^\dagger. \quad (6.19)$$

This readily yields the ‘dispersion matrix’ $\Xi \doteq \text{symb}_x \hat{\Xi}$:

$$\Xi(\omega, \mathbf{k}) \approx \Xi_p(\omega, \mathbf{k}) + \sum_s \int d\mathbf{p} \frac{\alpha_s(\omega, \mathbf{k}; \mathbf{p}) \alpha_s^\dagger(\omega, \mathbf{k}; \mathbf{p})}{\omega - \mathbf{k} \cdot \mathbf{v}_s + i0} \mathbf{k} \cdot \frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}}, \quad (6.20)$$

$$\Xi_p(\omega, \mathbf{k}) \approx \Xi_0(\omega, \mathbf{k}) - \sum_s \int d\mathbf{p} \wp_s(\omega, \mathbf{k}; \mathbf{p}) F_s(\mathbf{p}) \quad (6.21)$$

(see section 9 for examples). Here, $\alpha_s \alpha_s^\dagger$ is a dyadic matrix, and the arguments t and $\mathbf{x}$ are henceforth omitted for brevity. Also, we introduced the operators $\hat{\wp}_s = \hat{\wp}_s^\dagger$ and their symbols $\wp_s = \wp_s^\dagger$ as

$$\hat{\wp}_s \doteq (\hat{L}_s^\dagger \hat{R})_H, \quad \wp_s \doteq \text{symb } \hat{\wp}_s \approx (L_s^\dagger R)_H. \quad (6.22)$$

The appearance of $+i0$ in the denominator in (6.20) is related to the Landau rule. (Remember that as arguments of Weyl symbols, ω and $\mathbf{k}$ are real by definition.) The Hermitian and anti-Hermitian parts of the dispersion matrix are

$$\Xi_H(\omega, \mathbf{k}) \approx \Xi_p(\omega, \mathbf{k}) + \sum_s \oint d\mathbf{p} \frac{\alpha_s(\omega, \mathbf{k}; \mathbf{p}) \alpha_s^\dagger(\omega, \mathbf{k}; \mathbf{p})}{\omega - \mathbf{k} \cdot \mathbf{v}_s} \mathbf{k} \cdot \frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}}, \quad (6.23)$$

$$\Xi_A(\omega, \mathbf{k}) \approx -\pi \sum_s \int d\mathbf{p} \alpha_s(\omega, \mathbf{k}; \mathbf{p}) \alpha_s^\dagger(\omega, \mathbf{k}; \mathbf{p}) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_s) \mathbf{k} \cdot \frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}}. \quad (6.24)$$

Assuming the notation $\Xi^{-\dagger} \doteq (\Xi^\dagger)^{-1}$, the inverse dispersion matrix can be expressed as

$$\Xi^{-1} = \Xi^{-1} \Xi^\dagger \Xi^{-\dagger} = \Xi^{-1} \Xi_H \Xi^{-\dagger} - i \Xi^{-1} \Xi_A \Xi^{-\dagger}. \quad (6.25)$$

Because $\Xi^{-\dagger} = (\Xi^{-1})^\dagger$, this leads to the following formulas, which we will need later:

$$(\Xi^{-1})_H = \Xi^{-1} \Xi_H \Xi^{-\dagger}, \quad (\Xi^{-1})_A = -\Xi^{-1} \Xi_A \Xi^{-\dagger}. \quad (6.26)$$

6.4. Spectrum of microscopic fluctuations

Other objects to be used below are the density operators of the oscillating fields:

$$\widehat{\mathbf{W}}_{\underline{\Psi}} \doteq (2\pi)^{-n} |\underline{\Psi}\rangle \langle \underline{\Psi}|, \quad \widehat{\mathbf{W}}_{\underline{\tilde{\Psi}}} \doteq (2\pi)^{-n} |\underline{\tilde{\Psi}}\rangle \langle \underline{\tilde{\Psi}}|, \quad (6.27)$$

and the corresponding average Wigner matrices on $(\mathbf{x}, \mathbf{k})$. The former, $\mathbf{U} \doteq \overline{\mathbf{W}}_{\underline{\Psi}}$, is readily found by definition (2.51), and the latter, $\mathbf{W} \doteq \overline{\mathbf{W}}_{\underline{\tilde{\Psi}}}$, is calculated as follows. Let us consider $g_s(t, \mathbf{x}, \mathbf{p})$ as a ket in $\mathcal{H}_x$, with $\mathbf{p}$ as a parameter. Then, (6.14) readily yields

$$\widehat{\mathbf{W}}_{\underline{\tilde{\Psi}}} = \frac{1}{(2\pi)^n} \sum_{s, s'} \int d\mathbf{p} d\mathbf{p}' \hat{\Xi}^{-1} \hat{\alpha}_s(\mathbf{p}) |g_s(\mathbf{p})\rangle \langle g_{s'}(\mathbf{p}')| \hat{\alpha}_{s'}^\dagger(\mathbf{p}') \hat{\Xi}^{-\dagger}. \quad (6.28)$$

By applying symb_x to this, one obtains

$$\mathbf{W}(\omega, \mathbf{k}) = \sum_{s, s'} \int d\mathbf{p} d\mathbf{p}' \Xi^{-1} \star \alpha_s(\mathbf{p}) \star \wp_{ss'}(\mathbf{p}, \mathbf{p}') \star \alpha_{s'}^\dagger(\mathbf{p}') \star \Xi^{-\dagger}, \quad (6.29)$$

where most arguments are omitted for brevity and (appendix B.7)

$$\wp_{ss'}(\mathbf{p}, \mathbf{p}') \doteq \frac{1}{(2\pi)^n} \int d\mathbf{s} e^{-i\mathbf{k} \cdot \mathbf{s}} \overline{g_s(\mathbf{x} + \mathbf{s}/2, \mathbf{p})} g_{s'}(\mathbf{x} - \mathbf{s}/2, \mathbf{p}')$$

$$\approx \frac{1}{(2\pi)^n} \delta_{ss'} \delta(\mathbf{p} - \mathbf{p}') \delta(\omega - \mathbf{k} \cdot \mathbf{v}_s) F_s(\mathbf{p}), \quad (6.30)$$

assuming corrections due to inter-particle correlations are negligible. Then, (6.29) gives

$$\begin{aligned} \mathfrak{W}(\omega, \mathbf{k}) &= \frac{1}{(2\pi)^n} \sum_{s'} \int d\mathbf{p}' \delta(\omega - \mathbf{k} \cdot \mathbf{v}'_{s'}) F_{s'}(\mathbf{p}') \\ &\quad \times \Xi^{-1}(\omega, \mathbf{k})(\boldsymbol{\alpha}_{s'} \boldsymbol{\alpha}_{s'}^\dagger)(\omega, \mathbf{k}; \mathbf{p}') \Xi^{-\dagger}(\omega, \mathbf{k}), \end{aligned} \quad (6.31)$$

where $\mathbf{v}'_{s'} \doteq \mathbf{v}_{s'}(t, \mathbf{x}, \mathbf{p}')$. It is readily seen from (6.31) that $\mathfrak{W}$ is positive-semidefinite. One can also recognize (6.31) as a manifestation of the dressed-particle superposition principle (Rostoker 1964). Specifically, (6.31) shows that the contributions of individual particles to $\mathfrak{W}$ are additive and affected by the plasma collective response, i.e. by the difference between Ξ and the vacuum dispersion matrix Ξ_0 .

Using (6.31), one can also find other averages quadratic in the field via (cf. (2.53a))

$$\overline{(\hat{\mathbf{L}} \tilde{\Psi})(\hat{\mathbf{R}} \tilde{\Psi})^\dagger} \approx \int d\omega d\mathbf{k} (\mathbf{L} \mathfrak{W} \mathbf{R}^\dagger)(\omega, \mathbf{k}), \quad (6.32)$$

where $\hat{\mathbf{L}}$ and $\hat{\mathbf{R}}$ are any linear operators and $\mathbf{L}$ and $\mathbf{R}$ are their symbols; for example,

$$\overline{\tilde{\Psi}(t, \mathbf{x}) \tilde{\Psi}^\dagger(t, \mathbf{x})} \approx \int d\omega d\mathbf{k} \mathfrak{W}(\omega, \mathbf{k}). \quad (6.33)$$

Because of this, we loosely attribute $\mathfrak{W}$ as the spectrum of microscopic oscillations, but see also section 8.2, where an alternative notation is introduced and a fluctuation–dissipation theorem is derived from (6.31) for plasma in thermal equilibrium. See also section 9 for specific examples.

6.5. Nonlinear potentials

From (6.14), the oscillating part of the Hamiltonian (6.9) can be split into the macroscopic part and the microscopic part as $\tilde{H}_s = \underline{\tilde{H}}_s + \tilde{H}_s$, $\underline{\tilde{H}}_s = \hat{\boldsymbol{\alpha}}_s^\dagger \tilde{\Psi}$, and

$$\tilde{H}_s(\mathbf{p}) = \sum_{s'} \int d\mathbf{p}' \hat{\mathcal{X}}_{ss'}(\mathbf{p}, \mathbf{p}') g_{s'}(\mathbf{p}'). \quad (6.34)$$

Here, $\hat{\mathcal{X}}_{ss'}$ is an operator on $\mathcal{H}_x$ given by

$$\hat{\mathcal{X}}_{ss'}(\mathbf{p}, \mathbf{p}') \doteq \hat{\boldsymbol{\alpha}}_s^\dagger(\mathbf{p}) \hat{\Xi}^{-1} \hat{\boldsymbol{\alpha}}_{s'}(\mathbf{p}'), \quad (6.35)$$

with the symbol

$$\mathcal{X}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}') \approx \boldsymbol{\alpha}_s^\dagger(\omega, \mathbf{k}; \mathbf{p}) \Xi^{-1}(\omega, \mathbf{k}) \boldsymbol{\alpha}_{s'}(\omega, \mathbf{k}; \mathbf{p}') \quad (6.36)$$

(see section 9 for examples). The corresponding average Wigner functions on $(\mathbf{x}, \mathbf{k})$ are $\overline{W}_s = \overline{W}_s^{(m)} + \overline{W}_s^{(\mu)}$, where the index ‘m’ stands for ‘macroscopic’ and the index ‘μ’ stands for ‘microscopic’. Because the dependence on t and $\mathbf{x}$ is slow, one can approximate them as follows:

$$\overline{W}_s^{(m)} \approx \boldsymbol{\alpha}_s^\dagger(\omega, \mathbf{k}; \mathbf{p}) \mathbf{U}(\omega, \mathbf{k}) \boldsymbol{\alpha}_s(\omega, \mathbf{k}; \mathbf{p}), \quad (6.37a)$$

$$\overline{W}_s^{(\mu)} \approx \boldsymbol{\alpha}_s^\dagger(\omega, \mathbf{k}; \mathbf{p}) \mathfrak{W}(\omega, \mathbf{k}) \boldsymbol{\alpha}_s(\omega, \mathbf{k}; \mathbf{p}). \quad (6.37b)$$

The matrix $\mathbf{U}$ is positive-semidefinite as an average Wigner tensor (section 2.1.7), and so is $\mathfrak{W}$ (section 6.4). Hence, both $\overline{W}_s^{(m)}$ and $\overline{W}_s^{(\mu)}$ are non-negative. Using (6.31), one can also rewrite the Wigner function of $\tilde{H}_s$ more compactly as

$$\overline{W}_s^{(\mu)}(\omega, \mathbf{k}; \mathbf{p}) = (2\pi)^{-n} \sum_{s'} \int d\mathbf{p}' \delta(\omega - \mathbf{k} \cdot \mathbf{v}'_{s'}) |\mathcal{X}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2 F_{s'}(\mathbf{p}'). \quad (6.38)$$

Now we can represent the nonlinear potentials (5.41)–(5.43) as

$$\mathbf{D}_s = \mathbf{D}_s^{(m)} + \mathbf{D}_s^{(\mu)}, \quad \Theta_s = \Theta_s^{(m)} + \Theta_s^{(\mu)}, \quad \Phi_s = \Phi_s^{(m)} + \Phi_s^{(\mu)}. \quad (6.39)$$

Here, the index $^{(m)}$ denotes contributions from $\overline{W}_s^{(m)}$ and the index $^{(\mu)}$ denotes contributions from $\overline{W}_s^{(\mu)}$. Specifically,

$$\mathbf{D}_s^{(m)} = \int d\mathbf{k} \pi \mathbf{k} \overline{W}_s^{(m)}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}), \quad (6.40)$$

$$\Theta_s^{(m)} = \frac{\partial}{\partial \vartheta} \oint d\omega d\mathbf{k} \frac{\mathbf{k} \overline{W}_s^{(m)}(\omega, \mathbf{k}; \mathbf{p})}{\omega - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \Big|_{\vartheta=0}, \quad (6.41)$$

$$\Phi_s^{(m)} = \frac{1}{2} \frac{\partial}{\partial \mathbf{p}} \cdot \oint d\omega d\mathbf{k} \frac{\mathbf{k} \overline{W}_s^{(m)}(\omega, \mathbf{k}; \mathbf{p})}{\omega - \mathbf{k} \cdot \mathbf{v}_s}. \quad (6.42)$$

Here, $\overline{W}_s^{(m)}$ is a non-negative function (6.37a), so $\mathbf{D}_s^{(m)}$ is positive-semidefinite and leads to an H -theorem similar to (5.46). One also has

$$\mathbf{D}_s^{(\mu)} = \sum_{s'} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' \pi \mathbf{k} \delta(\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}_{s'}) |\chi_{ss'}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2 F_{s'}(\mathbf{p}'), \quad (6.43)$$

$$\Theta_s^{(\mu)} = \sum_{s'} \frac{\partial}{\partial \vartheta} \oint \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' \frac{\mathbf{k} F_{s'}(\mathbf{p}')}{\mathbf{k} \cdot (\mathbf{v}_{s'} - \mathbf{v}_s) + \vartheta} |\chi_{ss'}(\mathbf{k} \cdot \mathbf{v}_{s'}, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2 \Big|_{\vartheta=0}, \quad (6.44)$$

$$\Phi_s^{(\mu)} = \sum_{s'} \frac{\partial}{\partial \mathbf{p}} \cdot \oint \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' \frac{\mathbf{k} F_{s'}(\mathbf{p}')}{2\mathbf{k} \cdot (\mathbf{v}_{s'} - \mathbf{v}_s)} |\chi_{ss'}(\mathbf{k} \cdot \mathbf{v}_{s'}, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2. \quad (6.45)$$

The functions $\Theta_s^{(\mu)}$ and $\Phi_s^{(\mu)}$ scale as $\overline{W}_s^{(\mu)}$, i.e. as $\epsilon \varepsilon^2$ (section 3.3). Their contribution to (5.37) is of order $\epsilon \Theta_s^{(\mu)}$ and $\epsilon \Phi_s^{(\mu)}$, respectively, so it scales as $\epsilon^2 \varepsilon^2$ and therefore is negligible within our model. In contrast, $\mathbf{D}_s^{(\mu)}$ must be retained alongside with $\mathbf{D}_s^{(m)}$. This is because although weak, macroscopic fluctuations can resonate with particles from the bulk distribution, while the stronger macroscopic fluctuations are assumed to resonate only with particles from the tail distribution, which are few.

6.6. Oscillation-center Hamiltonian

Within the assumed accuracy, the OC Hamiltonian is $\mathcal{H}_s = \overline{H}_s + \Phi_s^{(m)}$, and $\overline{H}_s$ is given by (6.9). Combined with the general theorem (2.53c), the latter readily yields $\overline{H}_s = H_{0s} + \phi_s$, where

$$\phi_s \approx \frac{1}{2} \int d\omega d\mathbf{k} \operatorname{tr}(\mathbf{U} \boldsymbol{\wp}_s) \quad (6.46)$$

and the contribution of $\mathfrak{W}$ has been neglected. Because both $\Phi_s^{(m)}$ and ϕ_s are quadratic in $\tilde{\Psi}$ and enter $\mathcal{H}_s$ only in the combination $\Delta_s \doteq \Phi_s^{(m)} + \phi_s$, it is convenient to attribute the latter as the ‘total’ ponderomotive energy. Using (6.42) in combination with (6.37a), one can express it as follows:

$$\Delta_s = \frac{1}{2} \frac{\partial}{\partial \mathbf{p}} \cdot \oint d\omega d\mathbf{k} \mathbf{k} \frac{\boldsymbol{\alpha}_s^\dagger \mathbf{U} \boldsymbol{\alpha}_s}{\omega - \mathbf{k} \cdot \mathbf{v}_s} + \frac{1}{2} \int d\omega d\mathbf{k} \operatorname{tr}(\mathbf{U} \boldsymbol{\wp}_s) \quad (6.47)$$

(see section 9 for examples). Notably,

$$\Delta_s = -\frac{1}{2} \frac{\delta}{\delta F_s} \oint d\omega d\mathbf{k} \operatorname{tr}(\boldsymbol{\Xi}_H \mathbf{U}) = -\frac{\delta S_{\text{ad}}}{\delta F_s}, \quad (6.48)$$

where $\delta/\delta F_s$ denotes a functional derivative and S_{ad} is the adiabatic action defined in (6.16). Equation (6.48) is a generalization of the well-known ‘ K – χ theorem’ (Kaufman

& Holm 1984; Kaufman 1987). Loosely speaking, it says that the coefficient connecting Δ_s with $\mathbf{U}$ is proportional to the linear polarizability of an individual particle of type s (Dodin *et al.* 2017; Dodin & Fisch 2010a). (‘ K ’ in the name of this theorem is the same as our Δ_s , and ‘ χ ’ is the linear susceptibility.) Also, the OC Hamiltonian and the OC velocity can be expressed as

$$\mathcal{H}_s = H_{0s} + \Delta_s, \quad \mathbf{v} = \partial_{\mathbf{p}} H_{0s} + \partial_{\mathbf{p}} \Delta_s. \quad (6.49)$$

6.7. Polarization drag

Within the assumed accuracy, the OC distribution can be expressed as

$$F_s = \bar{f}_s + \frac{1}{2} \frac{\partial}{\partial \mathbf{p}} \cdot \left(\boldsymbol{\Theta}_s^{(m)} \frac{\partial \bar{f}_s}{\partial \mathbf{p}} \right), \quad (6.50)$$

and (5.37) becomes

$$\frac{\partial F_s}{\partial t} - \frac{\partial \mathcal{H}_s}{\partial \mathbf{x}} \cdot \frac{\partial F_s}{\partial \mathbf{p}} + \frac{\partial \mathcal{H}_s}{\partial \mathbf{p}} \cdot \frac{\partial F_s}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{p}} \cdot \left(\mathbf{D}_s^{(m)} \frac{\partial F_s}{\partial \mathbf{p}} \right) + \mathcal{C}_s, \quad (6.51)$$

$$\mathcal{C}_s \doteq \frac{\partial}{\partial \mathbf{p}} \cdot \left(\mathbf{D}_s^{(\mu)} \frac{\partial F_s}{\partial \mathbf{p}} \right) + \Gamma_s, \quad (6.52)$$

where we have reinstated the term Γ_s introduced in section 3.3. As a collisional term, Γ_s is needed only to the zeroth order in ϵ , so

$$\Gamma_s = \overline{\{\tilde{H}_s, g_s\}} \approx \partial_{\mathbf{p}} \cdot \overline{(g_s \partial_{\mathbf{x}} \tilde{H}_s)} \equiv \partial_{\mathbf{p}} \cdot \boldsymbol{\zeta}_s, \quad \boldsymbol{\zeta}_s = \overline{\mathbf{i} \langle \mathbf{x} | \hat{\mathbf{k}} \tilde{H}_s \rangle \langle \mathbf{x} | g_s \rangle}. \quad (6.53)$$

Correlating with g_s is only the microscopic part of $\tilde{H}_s$, so using (6.34) one obtains

$$\boldsymbol{\zeta}_s = \mathbf{i} \sum_{s'} \int d\mathbf{p}' \langle \mathbf{x} | \hat{\mathbf{k}} \hat{\mathcal{X}}_{ss'}(\mathbf{p}, \mathbf{p}') | g_{s'}(\mathbf{p}') \rangle \overline{\langle g_s(\mathbf{p}) | \mathbf{x} \rangle}. \quad (6.54)$$

Next, let us use (2.28) and $\boldsymbol{\zeta} = \text{re } \boldsymbol{\zeta}$ to express this result as follows:

$$\begin{aligned} \boldsymbol{\zeta}_s &= \mathbf{i} \sum_{s'} \int d\omega d\mathbf{k} d\mathbf{p}' \mathbf{k} \star \mathcal{X}_{ss'}(\omega, \mathbf{k}, \mathbf{p}, \mathbf{p}') \star \mathfrak{F}_{s's}(\omega, \mathbf{k}, \mathbf{p}', \mathbf{p}) \\ &\approx \mathbf{i} \int d\omega d\mathbf{k} d\mathbf{p}' \mathbf{k} \mathcal{X}_{ss'}(\omega, \mathbf{k}, \mathbf{p}, \mathbf{p}') (2\pi)^{-n} \delta(\mathbf{p} - \mathbf{p}') \delta(\omega - \mathbf{k} \cdot \mathbf{v}_s) F_s(\mathbf{p}) \\ &= - \int \frac{d\mathbf{k}}{(2\pi)^n} \mathbf{k} \text{im } \mathcal{X}_{ss'}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}, \mathbf{p}, \mathbf{p}) F_s(\mathbf{p}), \end{aligned} \quad (6.55)$$

where we have approximated $\star$ with the usual product and substituted (6.30). Hence,

$$\Gamma_s \approx -\partial_{\mathbf{p}} \cdot (\boldsymbol{\mathfrak{F}}_s F_s), \quad (6.56)$$

where $\boldsymbol{\mathfrak{F}}_s$ can be interpreted as the polarization drag (i.e. the average force that is imposed on an OC by its dress) and is given by

$$\boldsymbol{\mathfrak{F}}_s = \int \frac{d\mathbf{k}}{(2\pi)^n} \mathbf{k} \text{im } \mathcal{X}_{ss'}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}, \mathbf{p}, \mathbf{p}). \quad (6.57)$$

Using (6.36), one also rewrite this as follows:

$$\boldsymbol{\mathfrak{F}}_s \approx \int \frac{d\mathbf{k}}{(2\pi)^n} \mathbf{k} (\boldsymbol{\alpha}_s^\dagger (\boldsymbol{\Xi}^{-1})_{\text{A}} \boldsymbol{\alpha}_s)(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}) \quad (6.58a)$$

$$\approx - \int \frac{d\mathbf{k}}{(2\pi)^n} \mathbf{k} (\boldsymbol{\alpha}_s^\dagger \boldsymbol{\Xi}^{-1} \boldsymbol{\Xi}_{\text{A}} \boldsymbol{\Xi}^{-\dagger} \boldsymbol{\alpha}_s)(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}), \quad (6.58b)$$

where we have substituted (6.26) for $(\Xi^{-1})_A$. With (6.24) for Ξ_A , this yields

$$\begin{aligned} \mathfrak{F}_s \approx \sum_{s'} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' \pi \delta(\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}'_{s'}) \mathbf{k} \mathbf{k} \cdot \frac{\partial F_{s'}(\mathbf{p}')}{\partial \mathbf{p}'} \\ \times \alpha_s^\dagger(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}) \Xi^{-1}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}) \alpha_{s'}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}') \\ \times \alpha_{s'}^\dagger(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}') \Xi^{-\dagger}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}) \alpha_s(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}). \end{aligned} \quad (6.59)$$

The product of the last two lines equals $|\mathcal{X}_{ss'}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2$. Hence,

$$\mathfrak{F}_s = \sum_{s'} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' \pi \delta(\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}'_{s'}) |\mathcal{X}_{ss'}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2 \mathbf{k} \mathbf{k} \cdot \frac{\partial F_{s'}(\mathbf{p}')}{\partial \mathbf{p}'}. \quad (6.60)$$

6.8. Collision operator

By combining (6.56) for Γ_s with (6.43) for $\mathbf{D}_s^{(\mu)}$, one can express $\mathcal{C}_s$ as

$$\begin{aligned} \mathcal{C}_s = \frac{\partial}{\partial \mathbf{p}} \cdot \sum_{s'} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' \pi \delta(\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}'_{s'}) |\mathcal{X}_{ss'}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2 \\ \times \mathbf{k} \mathbf{k} \cdot \left(\frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}} F_{s'}(\mathbf{p}') - F_s(\mathbf{p}) \frac{\partial F_{s'}(\mathbf{p}')}{\partial \mathbf{p}'} \right), \end{aligned} \quad (6.61)$$

where $\mathcal{X}_{ss'}$ is given by (6.36). One can recognize this as a generalization of the Balescu–Lenard collision operator (Krall & Trivelpiece 1973, section 11.11) to interactions via a general multi-component field Ψ . Specific examples can be found in section 9.

It is readily seen that $\mathcal{C}_s$ conserves particles, i.e.

$$\int d\mathbf{p} \mathcal{C}_s = 0, \quad (6.62)$$

and vanishes in thermal equilibrium (section 8.1). Other properties of $\mathcal{C}_s$ are determined by the properties of the coupling coefficient $\mathcal{X}_{ss'}$, which are as follows. Note that

$$|\mathcal{X}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2 = \mathcal{Q}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}') + \mathcal{R}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}')/2, \quad (6.63)$$

where we introduced

$$\mathcal{Q}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}') \doteq (|\mathcal{X}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2 + |\mathcal{X}_{s's}(\omega, \mathbf{k}; \mathbf{p}', \mathbf{p})|^2)/2, \quad (6.64)$$

$$\mathcal{R}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}') \doteq |\mathcal{X}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2 - |\mathcal{X}_{s's}(\omega, \mathbf{k}; \mathbf{p}', \mathbf{p})|^2. \quad (6.65)$$

To calculate $\mathcal{R}_{ss'}$, note that (6.36) yields

$$\begin{aligned} |\mathcal{X}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2 &\approx |\alpha_s^\dagger(\omega, \mathbf{k}; \mathbf{p}) \Xi^{-1}(\omega, \mathbf{k}) \alpha_{s'}(\omega, \mathbf{k}; \mathbf{p}')|^2, \\ |\mathcal{X}_{s's}(\omega, \mathbf{k}; \mathbf{p}', \mathbf{p})|^2 &\approx |\alpha_s^\dagger(\omega, \mathbf{k}; \mathbf{p}) \Xi^{-\dagger}(\omega, \mathbf{k}) \alpha_{s'}(\omega, \mathbf{k}; \mathbf{p}')|^2, \end{aligned} \quad (6.66)$$

whence one obtains

$$\begin{aligned} \mathcal{R}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}') &\approx 4 \operatorname{im} (\alpha_s^\dagger(\omega, \mathbf{k}; \mathbf{p}) (\Xi^{-1})_H(\omega, \mathbf{k}) \alpha_{s'}(\omega, \mathbf{k}; \mathbf{p}') \\ &\quad \alpha_{s'}^\dagger(\omega, \mathbf{k}; \mathbf{p}') (\Xi^{-1})_A(\omega, \mathbf{k}) \alpha_s(\omega, \mathbf{k}; \mathbf{p})). \end{aligned} \quad (6.67)$$

The operators $(\Xi^{-1})_H$, $(\Xi^{-1})_A$, and $\hat{\alpha}_s$ (for all s) have been introduced for real fields, so their matrix elements in the coordinate representation are real. Then, the corresponding symbols satisfy $\mathbf{A}(-\omega, -\mathbf{k}) = \mathbf{A}^*(\omega, \mathbf{k})$, where $\mathbf{A}$ is any of the three symbols. This gives

$$\mathcal{R}_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') \doteq \mathcal{R}_{ss'}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}, \mathbf{p}') = -\mathcal{R}_{ss'}(-\mathbf{k}; \mathbf{p}, \mathbf{p}'). \quad (6.68)$$

Because the rest of the integrand in (6.61) is even in $\mathbf{k}$, (6.68) signifies that $\mathcal{R}_{ss'}$ does

not contribute to $\mathcal{C}_s$. Thus, $\mathcal{X}_{ss'}$ in (6.61) can as well be replaced with $\mathcal{Q}_{ss'}$:

$$\begin{aligned} \mathcal{C}_s = \frac{\partial}{\partial \mathbf{p}} \cdot \sum_{s'} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' \pi \delta(\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}_{s'}) \mathcal{Q}_{ss'}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}, \mathbf{p}') \\ \times \mathbf{k} \mathbf{k} \cdot \left(\frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}} F_{s'}(\mathbf{p}') - F_s(\mathbf{p}) \frac{\partial F_{s'}(\mathbf{p}')}{\partial \mathbf{p}'} \right). \end{aligned} \quad (6.69)$$

In this representation, the coupling coefficient in $\mathcal{C}_s$ is manifestly symmetric,

$$\mathcal{Q}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}') = \mathcal{Q}_{s's}(\omega, \mathbf{k}; \mathbf{p}', \mathbf{p}), \quad (6.70)$$

which readily leads to momentum and energy conservation (appendix C):[†]

$$\sum_s \int d\mathbf{p} \mathbf{p} \mathcal{C}_s = 0, \quad \sum_s \int d\mathbf{p} \mathcal{H}_s \mathcal{C}_s = 0. \quad (6.71)$$

The collision operator $\mathcal{C}_s$ also satisfies the H -theorem (appendix C.3):

$$\left(\frac{d\sigma}{dt} \right)_{\text{coll}} \geq 0, \quad (6.72)$$

where the entropy density σ is defined as

$$\sigma \doteq - \sum_s \int d\mathbf{p} F_s(\mathbf{p}) \ln F_s(\mathbf{p}) \quad (6.73)$$

and $(\partial_t F_s)_{\text{coll}} \doteq \mathcal{C}_s$. Note that these properties are not restricted to any particular $\mathcal{H}_s$. Also note that if applied in proper variables (section 3.1.2), our formula (6.69) can describe collisions in strong background fields. This topic, including comparison with the relevant literature, is left to future work.

6.9. Summary of section 6

Let us summarize the above general results (for examples, see section 9). We consider species s governed by a Hamiltonian of the form

$$H_s = H_{0s} + \hat{\alpha}_s^\dagger(\mathbf{p}) \tilde{\Psi} + \frac{1}{2} (\hat{\mathbf{L}}_s \tilde{\Psi})^\dagger (\hat{\mathbf{R}}_s \tilde{\Psi}), \quad (6.74)$$

where $\tilde{\Psi}$ is a real oscillating field (of any dimension M), which generally consists of a macroscopic part $\tilde{\underline{\Psi}}$ and a microscopic part $\tilde{\underline{\Psi}}$. The term H_{0s} is independent of $\tilde{\Psi}$, and the operators $\hat{\alpha}_s^\dagger$, $\hat{\mathbf{L}}_s$, and $\hat{\mathbf{R}}_s$ may be nonlocal in t and $\mathbf{x}$ and may depend on the momentum $\mathbf{p}$ parametrically. The dynamics of this system averaged over the fast oscillations can be described in terms of the OC distribution function

$$F_s = \bar{f}_s + \frac{1}{2} \frac{\partial}{\partial \mathbf{p}} \cdot \left(\mathbf{\Theta}_s \frac{\partial \bar{f}_s}{\partial \mathbf{p}} \right) \quad (6.75)$$

(the index ^(m) is henceforth omitted for brevity), which is governed by the following equation of the Fokker–Planck type:

$$\frac{\partial F_s}{\partial t} - \frac{\partial \mathcal{H}_s}{\partial \mathbf{x}} \cdot \frac{\partial F_s}{\partial \mathbf{p}} + \frac{\partial \mathcal{H}_s}{\partial \mathbf{p}} \cdot \frac{\partial F_s}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{p}} \cdot \left(\mathbf{D}_s \frac{\partial F_s}{\partial \mathbf{p}} \right) + \mathcal{C}_s. \quad (6.76)$$

[†] Remember that $\mathbf{v}_s$ is defined as the OC velocity in the above formulas (section 5.6). If $\mathbf{v}_s$ is treated as the particle velocity instead, then $\mathcal{H}_s$ in (6.71) should be replaced with $\bar{H}_s$. Both options are admissible within the assumed accuracy, but the former option is preferable because it leads to other conservation laws that are exact within our model (section 7.5).

Here, $\mathcal{H}_s = H_{0s} + \Delta_s$ is the OC Hamiltonian, Θ_s is the dressing function, and Δ_s is the total ponderomotive energy (i.e. the part of the OC Hamiltonian that is quadratic in $\tilde{\Psi}$), so $\mathbf{v}_s(t, \mathbf{x}, \mathbf{p}) \doteq \partial_{\mathbf{p}} \mathcal{H}_s$ is the OC velocity. Specifically,

$$\mathbf{D}_s = \int d\mathbf{k} \pi \mathbf{k} \mathbf{k} \overline{\mathbf{W}}_s(t, \mathbf{x}, \mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}), \quad (6.77a)$$

$$\Theta_s = \frac{\partial}{\partial \vartheta} \oint d\omega d\mathbf{k} \mathbf{k} \mathbf{k} \frac{\overline{\mathbf{W}}_s(t, \mathbf{x}, \omega, \mathbf{k}; \mathbf{p})}{\omega - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \Big|_{\vartheta=0}, \quad (6.77b)$$

$$\begin{aligned} \Delta_s &= \frac{1}{2} \frac{\partial}{\partial \mathbf{p}} \cdot \oint d\omega d\mathbf{k} \mathbf{k} \frac{\overline{\mathbf{W}}_s(t, \mathbf{x}, \omega, \mathbf{k}; \mathbf{p})}{\omega - \mathbf{k} \cdot \mathbf{v}_s} \\ &\quad + \frac{1}{2} \int d\omega d\mathbf{k} \text{tr}(\mathbf{U} \boldsymbol{\wp}_s)(t, \mathbf{x}, \omega, \mathbf{k}; \mathbf{p}). \end{aligned} \quad (6.77c)$$

Here, $\overline{\mathbf{W}}_s = \boldsymbol{\alpha}_s^\dagger \mathbf{U} \boldsymbol{\alpha}_s$ is a scalar function, the average Wigner matrix $\mathbf{U}$ is understood as the Fourier spectrum of the symmetrized autocorrelation matrix of the macroscopic oscillations:

$$\mathbf{U}(t, \mathbf{x}, \omega, \mathbf{k}) = \int \frac{d\tau}{2\pi} \frac{d\mathbf{s}}{(2\pi)^n} \overline{\underline{\tilde{\Psi}}(t + \tau/2, \mathbf{x} + \mathbf{s}/2) \underline{\tilde{\Psi}}^\dagger(t - \tau/2, \mathbf{x} - \mathbf{s}/2)} e^{i\omega\tau - i\mathbf{k} \cdot \mathbf{s}}, \quad (6.78)$$

with $n \doteq \dim \mathbf{x}$. Also, the vector $\boldsymbol{\alpha}_s(t, \mathbf{x}, \omega, \mathbf{k}; \mathbf{p})$ is the Weyl symbol of $\hat{\boldsymbol{\alpha}}_s$ as defined in (2.26), $\boldsymbol{\wp}_s(t, \mathbf{x}, \omega, \mathbf{k}; \mathbf{p}) \approx (\mathbf{L}_s^\dagger \mathbf{R}_s)_\text{H}$, $\mathbf{L}_s$ and $\mathbf{R}_s$ are the Weyl symbols of $\hat{\mathbf{L}}_s$ and $\hat{\mathbf{R}}_s$, respectively, and H denotes the Hermitian part. The matrix $\mathbf{D}_s$ is positive-semidefinite and satisfies an H -theorem of the form (5.46). Also, Δ_s satisfies the ‘ K - χ theorem’:

$$\Delta_s = -\frac{1}{2} \frac{\delta}{\delta F_s} \oint d\omega d\mathbf{k} \text{tr}(\boldsymbol{\Xi}_\text{H} \mathbf{U}). \quad (6.79)$$

The matrix $\boldsymbol{\Xi}$ characterizes the collective plasma response to the field $\tilde{\Psi}$ and is given by

$$\boldsymbol{\Xi} \approx \boldsymbol{\Xi}_0 + \sum_s \int d\mathbf{p} \left(\frac{\boldsymbol{\alpha}_s(\mathbf{p}) \boldsymbol{\alpha}_s^\dagger(\mathbf{p})}{\omega - \mathbf{k} \cdot \mathbf{v}_s(\mathbf{p}) + i0} \mathbf{k} \cdot \frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}} - \boldsymbol{\wp}_s(\mathbf{p}) F_s(\mathbf{p}) \right). \quad (6.80)$$

Here, the arguments $(t, \mathbf{x}, \omega, \mathbf{k})$ are omitted for brevity, $\boldsymbol{\alpha}_s \boldsymbol{\alpha}_s^\dagger$ is a dyadic matrix, and $\boldsymbol{\Xi}_0$ is the symbol of the Hermitian dispersion operator $\hat{\boldsymbol{\Xi}}_0$ that governs the field $\tilde{\Psi}$ in the absence of plasma. Specifically, $\hat{\boldsymbol{\Xi}}_0$ is defined such that the field Lagrangian density without plasma is $\mathfrak{L}_0 = \tilde{\Psi}^\dagger \hat{\boldsymbol{\Xi}}_0 \tilde{\Psi} / 2$.

The spectrum of microscopic fluctuations (specifically, the spectrum of the symmetrized autocorrelation function of the microscopic field $\tilde{\Psi}$) is a positive-semidefinite matrix function and given by

$$\begin{aligned} \mathfrak{W}(\omega, \mathbf{k}) &= \frac{1}{(2\pi)^n} \sum_{s'} \int d\mathbf{p}' \delta(\omega - \mathbf{k} \cdot \mathbf{v}'_{s'}) F_{s'}(\mathbf{p}') \\ &\quad \times \boldsymbol{\Xi}^{-1}(\omega, \mathbf{k}) (\boldsymbol{\alpha}_{s'} \boldsymbol{\alpha}_{s'}^\dagger)(\omega, \mathbf{k}; \mathbf{p}') \boldsymbol{\Xi}^{-\dagger}(\omega, \mathbf{k}), \end{aligned} \quad (6.81)$$

where $\mathbf{v}'_{s'} \doteq \mathbf{v}_{s'}(\mathbf{p}')$. (The dependence on t and $\mathbf{x}$ is assumed too but not emphasized.) The microscopic fluctuations give rise to a collision operator of the Balescu–Lenard type:

$$\begin{aligned} \mathcal{C}_s &= \frac{\partial}{\partial \mathbf{p}} \cdot \sum_{s'} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' \pi \delta(\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}'_{s'}) \mathcal{Q}_{ss'}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}, \mathbf{p}') \\ &\quad \times \mathbf{k} \mathbf{k} \cdot \left(\frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}} F_{s'}(\mathbf{p}') - F_s(\mathbf{p}) \frac{\partial F_{s'}(\mathbf{p}')}{\partial \mathbf{p}'} \right), \end{aligned} \quad (6.82)$$

where the coupling coefficient $\mathcal{Q}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}') = \mathcal{Q}_{s's}(\omega, \mathbf{k}; \mathbf{p}', \mathbf{p})$ is given by

$$\mathcal{Q}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}') = (|\mathcal{X}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2 + |\mathcal{X}_{s's}(\omega, \mathbf{k}; \mathbf{p}', \mathbf{p})|^2)/2, \quad (6.83)$$

$$\mathcal{X}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}') \approx \alpha_s^\dagger(\omega, \mathbf{k}; \mathbf{p}) \Xi^{-1}(\omega, \mathbf{k}) \alpha_{s'}(\omega, \mathbf{k}; \mathbf{p}'). \quad (6.84)$$

The operator $\mathcal{C}_s$ satisfies the H -theorem and conserves particles, momentum, and energy:

$$\int d\mathbf{p} \mathcal{C}_s = 0, \quad \sum_s \int d\mathbf{p} \mathbf{p} \mathcal{C}_s = 0, \quad \sum_s \int d\mathbf{p} \mathcal{H}_s \mathcal{C}_s = 0.$$

7. Interaction with on-shell waves

Here, we discuss QL interaction of plasma with ‘on-shell’ waves, i.e. waves constrained by dispersion relations. To motivate the assumptions that will be adopted, and also to systematically introduce our notation, we start with briefly overviewing theory of linear waves in dispersive media (Tracy *et al.* 2014; Whitham 1974), including monochromatic waves (section 7.1), conservative eikonal waves (section 7.2), general eikonal waves (section 7.3), and general broadband waves described by the WKE (section 7.4). After that, we derive conservation laws for the total momentum and energy, which are exact within our model (section 7.5). All waves in this section are considered macroscopic, so we adopt a simplified notation $\tilde{\underline{\Psi}} \equiv \tilde{\underline{\Psi}}$ and the index ^(m) will be omitted.

7.1. Monochromatic waves

Conservative (nondissipative) waves can be described using the least-action principle $\delta S = 0$. Assuming the notation as in section 6.2, the action integral can be expressed as $S = \int d\mathbf{x} \mathcal{L}$ with the Lagrangian density given by

$$\mathcal{L} = \frac{1}{2} \tilde{\Psi}^\dagger \hat{\Xi}_H \tilde{\Psi}. \quad (7.1)$$

First, let us assume a homogeneous stationary medium, so $\Xi_H(t, \mathbf{x}, \omega, \mathbf{k}) = \Xi_H(\omega, \mathbf{k})$. Because we assume real fields, $\dagger \langle t_1, \mathbf{x}_1 | \hat{\Xi} | t_2, \mathbf{x}_2 \rangle$ is real for all $(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2)$, one also has

$$\Xi_H(-\omega, -\mathbf{k}) = \Xi_H^*(\omega, \mathbf{k}) = \Xi_H^\top(\omega, \mathbf{k}), \quad (7.2)$$

where the latter equality is due to $\Xi_H^\dagger(\omega, \mathbf{k}) = \Xi_H(\omega, \mathbf{k})$.

Because $\Xi_H(\omega, \mathbf{k})$ is Hermitian, it has $M \doteq \dim \Xi_H$ orthonormal eigenvectors $\boldsymbol{\eta}_b$:

$$\Xi_H(\omega, \mathbf{k}) \boldsymbol{\eta}_b(\omega, \mathbf{k}) = \Lambda_b(\omega, \mathbf{k}) \boldsymbol{\eta}_b(\omega, \mathbf{k}), \quad \boldsymbol{\eta}_b^\dagger(\omega, \mathbf{k}) \boldsymbol{\eta}_{b'}(\omega, \mathbf{k}) = \delta_{b,b'}. \quad (7.3)$$

Here Λ_b are the corresponding eigenvalues, which are real and satisfy

$$\Lambda_b(\omega, \mathbf{k}) = \boldsymbol{\eta}_b^\dagger(\omega, \mathbf{k}) \Xi_H(\omega, \mathbf{k}) \boldsymbol{\eta}_b(\omega, \mathbf{k}). \quad (7.4)$$

Due to (7.2), one has

$$\Lambda_b(-\omega, -\mathbf{k}) = \text{eigv}_b(\Xi_H(\omega, \mathbf{k}))^\top = \text{eigv}_b \Xi_H(\omega, \mathbf{k}) = \Lambda_b(\omega, \mathbf{k}), \quad (7.5)$$

where eigv_b stands for the b th eigenvalue. Using this together with (7.2), one obtains from (7.3) that

$$\Xi_H^*(\omega, \mathbf{k}) \boldsymbol{\eta}_b(-\omega, -\mathbf{k}) = \Lambda_b(\omega, \mathbf{k}) \boldsymbol{\eta}_b(-\omega, -\mathbf{k}), \quad (7.6)$$

whence

$$\boldsymbol{\eta}_b(-\omega, -\mathbf{k}) = \boldsymbol{\eta}_b^*(\omega, \mathbf{k}). \quad (7.7)$$

† A complex field can be accommodated by considering its real and imaginary parts as separate components.

Let us consider a monochromatic wave of the form

$$\tilde{\Psi}(t, \mathbf{x}) = \text{re}(e^{-i\bar{\omega}t + i\bar{\mathbf{k}} \cdot \mathbf{x}} \check{\Psi}), \quad (7.8)$$

with real frequency $\bar{\omega}$, real wavevector $\bar{\mathbf{k}}$, and complex amplitude $\check{\Psi}$. For such a wave, the action integral can be expressed as $S = \int d\mathbf{x} \bar{\mathcal{L}}$, where the average Lagrangian density $\bar{\mathcal{L}}$ is given by[†]

$$\bar{\mathcal{L}} = \frac{1}{2} \overline{\tilde{\Psi}^\dagger \hat{\Xi}_H \tilde{\Psi}} = \frac{1}{4} \text{re}(\check{\Psi}^\dagger \Xi_H(\bar{\omega}, \bar{\mathbf{k}}) \check{\Psi}) = \frac{1}{4} \check{\Psi}^\dagger \Xi_H(\bar{\omega}, \bar{\mathbf{k}}) \check{\Psi}. \quad (7.9)$$

Let us decompose $\check{\Psi}$ in the basis formed by the eigenvectors $\boldsymbol{\eta}_b$, that is, as

$$\check{\Psi} = \sum_b \boldsymbol{\eta}_b \check{a}^b. \quad (7.10)$$

Then, (7.9) becomes

$$\bar{\mathcal{L}} = \frac{1}{4} \sum_b A_b(\bar{\omega}, \bar{\mathbf{k}}) |\check{a}^b|^2. \quad (7.11)$$

The real and imaginary parts of the amplitudes $\check{a}^b$ can be treated as independent variables. This is equivalent to treating $\check{a}^{b*}$ and $\check{a}^b$ as independent variables, so one arrives at the following Euler–Lagrange equations:

$$0 = \frac{\delta S[\check{\mathbf{a}}^*, \check{\mathbf{a}}]}{\delta \check{a}^{b*}} = \frac{1}{4} A_b(\bar{\omega}, \bar{\mathbf{k}}) \check{a}^b, \quad 0 = \frac{\delta S[\check{\mathbf{a}}^*, \check{\mathbf{a}}]}{\delta \check{a}^b} = \frac{1}{4} \check{a}^{b*} A_b(\bar{\omega}, \bar{\mathbf{k}}). \quad (7.12)$$

Hence the b th mode with a nonzero amplitude $\check{a}^b$ satisfies the dispersion relation

$$0 = A_b(\bar{\omega}, \bar{\mathbf{k}}) = A_b(-\bar{\omega}, -\bar{\mathbf{k}}). \quad (7.13)$$

Equation (7.13) determines a dispersion surface in the $\mathbf{k}$ space where the waves can have nonzero amplitude. This surface is sometimes called a shell, so waves constrained by a dispersion relation are called on-shell. Also note that combining (7.13) with (7.3) yields that on-shell waves satisfy

$$\Xi_H(\bar{\omega}, \bar{\mathbf{k}}) \boldsymbol{\eta}_b(\bar{\omega}, \bar{\mathbf{k}}) = \mathbf{0}, \quad \boldsymbol{\eta}_b^\dagger(\bar{\omega}, \bar{\mathbf{k}}) \Xi_H(\bar{\omega}, \bar{\mathbf{k}}) = \mathbf{0}, \quad (7.14)$$

which are two mutually adjoint representations of the same equation.

Below, we consider the case when (7.13) is satisfied only for one mode at a time, so summation over b and the index b itself can be omitted. (A more general case is discussed, for example, in (Dodin *et al.* 2019).) Then, $\check{\Psi} = \boldsymbol{\eta}(\bar{\omega}, \bar{\mathbf{k}}) \check{a}$,

$$\bar{\mathcal{L}} = \frac{1}{4} A(\bar{\omega}, \bar{\mathbf{k}}) |\check{a}|^2, \quad (7.15)$$

and $\bar{\omega}$ is connected with $\bar{\mathbf{k}}$ via $\bar{\omega} = w(\bar{\mathbf{k}})$, where $w(\mathbf{k}) = -w(-\mathbf{k})$ is the function that solves $A(w(\mathbf{k}), \mathbf{k}) = 0$. Also importantly, (7.14) ensures that

$$\begin{aligned} \partial_\bullet A(\bar{\omega}, \bar{\mathbf{k}}) &= ((\partial_\bullet \boldsymbol{\eta}^\dagger) \Xi_H \boldsymbol{\eta} + \boldsymbol{\eta}^\dagger (\partial_\bullet \Xi_H) \boldsymbol{\eta} + \boldsymbol{\eta}^\dagger \Xi_H (\partial_\bullet \boldsymbol{\eta})) \Big|_{(\omega, \mathbf{k})=(\bar{\omega}, \bar{\mathbf{k}})} \\ &= (\boldsymbol{\eta}^\dagger (\partial_\bullet \Xi_H) \boldsymbol{\eta}) \Big|_{(\omega, \mathbf{k})=(\bar{\omega}, \bar{\mathbf{k}})}, \end{aligned} \quad (7.16)$$

where $\bullet$ can be replaced with any variable.

[†] Here we use that for any oscillating $a = \text{re}(e^{i\theta} \check{a})$ and $b = \text{re}(e^{i\theta} \check{b})$, one has $\overline{ab} = \text{re}(\check{a}^* \check{b})/2$ and that $\check{\Psi}^\dagger \Xi_H(\bar{\omega}, \bar{\mathbf{k}}) \check{\Psi}$ is real because $\Xi_H(\bar{\omega}, \bar{\mathbf{k}})$ is Hermitian.

7.2. Conservative eikonal waves

7.2.1. Basic properties

In case of a quasimonochromatic eikonal wave and, possibly, inhomogeneous non-stationary plasma, one can apply the same arguments as in section 7.1 except the above equalities are now satisfied up to $\mathcal{O}(\epsilon)$. For a single-mode wave, one has

$$\tilde{\Psi}(t, \mathbf{x}) = \text{re}(\tilde{\Psi}_c(t, \mathbf{x})) + \mathcal{O}(\epsilon), \quad \tilde{\Psi}_c = e^{i\theta(t, \mathbf{x})} \boldsymbol{\eta}(t, \mathbf{x}) \check{a}(t, \mathbf{x}), \quad (7.17)$$

where the local frequency and the wavevector,

$$\overline{\omega} \doteq -\partial_t \theta, \quad \overline{\mathbf{k}} \doteq \partial_{\mathbf{x}} \theta \quad (7.18)$$

are slow functions of $(t, \mathbf{x})$, and so is $\boldsymbol{\eta}(t, \mathbf{x}) \doteq \boldsymbol{\eta}(t, \mathbf{x}, \overline{\omega}(t, \mathbf{x}), \overline{\mathbf{k}}(t, \mathbf{x}))$, which satisfies (7.3). Then,

$$\overline{\mathcal{L}} = \frac{1}{4} A(t, \mathbf{x}, \overline{\omega}, \overline{\mathbf{k}}) |\check{a}(t, \mathbf{x})|^2 + \mathcal{O}(\epsilon). \quad (7.19)$$

Within the leading-order theory, the term $\mathcal{O}(\epsilon)$ is neglected.[†] Then, the least action principle

$$0 = \frac{\delta S[\theta, \check{a}^*, \check{a}]}{\delta \check{a}^{b*}} \approx \frac{1}{4} A_b(\overline{\omega}, \overline{\mathbf{k}}) \check{a}^b, \quad 0 = \frac{\delta S[\theta, \check{a}^*, \check{a}]}{\delta \check{a}^b} \approx \frac{1}{4} \check{a}^{b*} A_b(\overline{\omega}, \overline{\mathbf{k}}) \quad (7.20)$$

leads to the same (but now local) dispersion relation as for monochromatic waves, $A(t, \mathbf{x}, \overline{\omega}, \overline{\mathbf{k}}) = 0$. This shows that quasimonochromatic waves are also on-shell, and thus they satisfy (7.16) as well. Also notice that the dispersion relation can now be understood as a Hamilton–Jacobi equation for the eikonal phase θ :

$$A(t, \mathbf{x}, -\partial_t \theta, \partial_{\mathbf{x}} \theta) = 0. \quad (7.21)$$

Like in the previous section, let us introduce the function w that solves

$$A(t, \mathbf{x}, w(t, \mathbf{x}, \mathbf{k}), \mathbf{k}) = 0 \quad (7.22)$$

and therefore satisfies

$$w(t, \mathbf{x}, \mathbf{k}) = -w(t, \mathbf{x}, -\mathbf{k}). \quad (7.23)$$

Differentiating (7.22) with respect to t , $\mathbf{x}$, and $\mathbf{k}$ leads to

$$\partial_t A + (\partial_{\omega} A) \partial_t w = 0, \quad (7.24a)$$

$$\partial_{\mathbf{x}} A + (\partial_{\omega} A) \partial_{\mathbf{x}} w = 0, \quad (7.24b)$$

$$\partial_{\mathbf{k}} A + (\partial_{\omega} A) \partial_{\mathbf{k}} w = 0, \quad (7.24c)$$

where the derivatives of A are evaluated at $(t, \mathbf{x}, w(t, \mathbf{x}, \mathbf{k}), \mathbf{k})$. In particular, (7.24c) gives

$$\mathbf{v}_g \doteq \frac{\partial w}{\partial \mathbf{k}} = -\frac{\partial_{\mathbf{k}} A}{\partial_{\omega} A}, \quad (7.25)$$

for the group velocity $\mathbf{v}_g$, whose physical meaning is to be specified shortly.

Because θ is now an additional dynamical variable, one also obtains an additional Euler–Lagrange equation:

$$0 = \delta_{\theta} S[\theta, \check{a}^*, \check{a}] = \partial_t \mathcal{I} + \partial_{\mathbf{x}} \cdot \mathcal{J}, \quad (7.26)$$

[†] Corrections to the lowest-order dispersion relation produce the so-called spin Hall effect; see (Dodin *et al.* 2019; Ruiz & Dodin 2017a) for an overview and (Bliokh *et al.* 2015; Ruiz & Dodin 2015a; Oancea *et al.* 2020; Andersson *et al.* 2021) for examples. These corrections are beyond the accuracy of the model considered, so they will be ignored.

where $\mathcal{I}$ is called the action density and $\mathcal{J}$ is the action flux density:

$$\mathcal{I} \doteq \frac{\partial \mathcal{L}}{\partial \omega} = \frac{|\check{a}|^2}{4} \frac{\partial \Lambda}{\partial \omega} = \frac{|\check{a}|^2}{4} \boldsymbol{\eta}^\dagger \frac{\partial \boldsymbol{\Xi}_H}{\partial \omega} \boldsymbol{\eta}, \quad (7.27)$$

$$\mathcal{J}^i \doteq -\frac{\partial \mathcal{L}}{\partial k_i} = -\frac{|\check{a}|^2}{4} \frac{\partial \Lambda}{\partial k_i} = -\frac{|\check{a}|^2}{4} \boldsymbol{\eta}^\dagger \frac{\partial \boldsymbol{\Xi}_H}{\partial k_i} \boldsymbol{\eta}, \quad (7.28)$$

where we used (7.16) and the derivatives are evaluated on $(t, \mathbf{x}, w(t, \mathbf{x}, \bar{\mathbf{k}}(t, \mathbf{x})), \bar{\mathbf{k}}(t, \mathbf{x}))$. Using (7.25), one can also rewrite (7.28) as

$$\mathcal{J} = \bar{\mathbf{v}}_g \mathcal{I}, \quad \bar{\mathbf{v}}_g(t, \mathbf{x}) \doteq \mathbf{v}_g(t, \mathbf{x}, \bar{\mathbf{k}}(t, \mathbf{x})). \quad (7.29)$$

(The arguments $(t, \mathbf{x})$ will be omitted from now on for brevity. We will also use $(\mathbf{k})$ as a shorthand for $(w(\mathbf{k}), \mathbf{k})$ where applicable.) Then, (7.26) becomes

$$\partial_t \mathcal{I} + \partial_{\mathbf{x}} \cdot (\bar{\mathbf{v}}_g \mathcal{I}) = 0, \quad (7.30)$$

which can be understood as a continuity equation for quasiparticles (‘photons’ or, more generally, ‘wave quanta’) with density $\mathcal{I}$ and fluid velocity $\bar{\mathbf{v}}_g$ (see also section 7.2.2). Thus, if an eikonal wave satisfies the least-action principle, its total action $\int d\mathbf{x} \mathcal{I}$ (‘number of quanta’) is an invariant. This conservation law can be attributed to the fact that the wave Lagrangian density $\mathcal{L}$ depends on derivatives of θ but not on θ *per se*.

Also notice the following. By expanding (7.19) in $\partial_t \theta$ around $\partial_t \theta = -w(t, \mathbf{x}, \partial_{\mathbf{x}} \theta)$, which is satisfied on any solution, one obtains

$$\bar{\mathcal{L}} \approx -\frac{1}{4} (\partial_t \theta + w(t, \mathbf{x}, \partial_{\mathbf{x}} \theta)) \partial_{\omega} \Lambda |\check{a}|^2 = -(\partial_t \theta + w(t, \mathbf{x}, \partial_{\mathbf{x}} \theta)) \mathcal{I}, \quad (7.31)$$

where we used that $\bar{\mathcal{L}}(t, \mathbf{x}, -\partial_t \theta, \partial_{\mathbf{x}} \theta) = 0$ due to (7.21). Then, one arrives at the canonical form of the action integral (Hayes 1973)

$$S[\mathcal{I}, \theta] = - \int dt d\mathbf{x} (\partial_t \theta + w(t, \mathbf{x}, \mathbf{k})) \mathcal{I}. \quad (7.32)$$

From here, $\delta_{\mathcal{I}} S = 0$ yields the dispersion relation in the Hamilton–Jacobi form $\partial_t \theta + w(t, \mathbf{x}, \mathbf{k}) = 0$, and $\delta_{\theta} S = 0$ yields the action conservation (7.30).

7.2.2. Ray equations

By (7.18), one has the so-called consistency relations:

$$\partial_t \bar{k}_i + \partial_i \bar{\omega} = 0, \quad \partial_i \bar{k}_j = \partial_j \bar{k}_i. \quad (7.33)$$

These lead to

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \bar{\mathbf{v}}_g \cdot \frac{\partial}{\partial \mathbf{x}} \right) \bar{k}_i(t, \mathbf{x}) &= -\frac{\partial w(t, \mathbf{x}, \bar{\mathbf{k}}(t, \mathbf{x}))}{\partial x^i} + \bar{\mathbf{v}}_g \cdot \frac{\partial \bar{k}_i(t, \mathbf{x})}{\partial \mathbf{x}} \\ &= -\left(\frac{\partial w(t, \mathbf{x}, \mathbf{k})}{\partial x^i} \right)_{\mathbf{k}=\bar{\mathbf{k}}(t, \mathbf{x})} - \bar{v}_g^j \frac{\partial \bar{k}_j(t, \mathbf{x})}{\partial x^i} + \bar{v}_g^j \frac{\partial \bar{k}_i(t, \mathbf{x})}{\partial x^j} \\ &= -\left(\frac{\partial w(t, \mathbf{x}, \mathbf{k})}{\partial x^i} \right)_{\mathbf{k}=\bar{\mathbf{k}}(t, \mathbf{x})}, \end{aligned} \quad (7.34)$$

and similarly,

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \bar{\mathbf{v}}_g \cdot \frac{\partial}{\partial \mathbf{x}} \right) \bar{\omega}(t, \mathbf{x}) &= \left(\frac{\partial}{\partial t} + \bar{\mathbf{v}}_g \cdot \frac{\partial}{\partial \mathbf{x}} \right) w(t, \mathbf{x}, \bar{\mathbf{k}}(t, \mathbf{x})) \\ &= \left(\frac{\partial w(t, \mathbf{x}, \mathbf{k})}{\partial t} + \bar{v}_g^i \frac{\partial w(t, \mathbf{x}, \mathbf{k})}{\partial x^i} \right)_{\mathbf{k}=\bar{\mathbf{k}}(t, \mathbf{x})} + \bar{v}_g^i \left(\frac{\partial}{\partial t} + \bar{\mathbf{v}}_g \cdot \frac{\partial}{\partial \mathbf{x}} \right) \bar{k}_i(t, \mathbf{x}) \end{aligned}$$

$$= \left(\frac{\partial w(t, \mathbf{x}, \mathbf{k})}{\partial t} \right)_{\mathbf{k}=\bar{\mathbf{k}}(t, \mathbf{x})}, \quad (7.35)$$

where we used (7.34). Using the convective derivative associated with the group velocity,

$$d/dt \equiv d_t \doteq \partial_t + (\bar{\mathbf{v}}_g \cdot \partial_{\mathbf{x}}), \quad (7.36)$$

one can rewrite these compactly as

$$\frac{d\bar{k}_i(t, \mathbf{x})}{dt} = - \left(\frac{\partial w(t, \mathbf{x}, \mathbf{k})}{\partial x^i} \right)_{\mathbf{k}=\bar{\mathbf{k}}(t, \mathbf{x})}, \quad \frac{d\bar{\omega}(t, \mathbf{x})}{dt} = \left(\frac{\partial w(t, \mathbf{x}, \mathbf{k})}{\partial t} \right)_{\mathbf{k}=\bar{\mathbf{k}}(t, \mathbf{x})}. \quad (7.37)$$

One can also represent (7.37) as *ordinary* differential equations for $\bar{\mathbf{k}}(t) \doteq \bar{\mathbf{k}}(t, \bar{\mathbf{x}}(t))$ and $\bar{\omega}(t) \doteq \bar{\omega}(t, \bar{\mathbf{x}}(t))$, where $\bar{\mathbf{x}}(t)$ are the ‘ray trajectories’ governed by

$$\frac{d\bar{x}^i(t)}{dt} = v_g^i(t, \bar{\mathbf{x}}(t), \bar{\mathbf{k}}(t)). \quad (7.38)$$

Specifically, together with (7.38), equations (7.37) become Hamilton’s equations also known as the ray equations:

$$\frac{d\bar{x}^i}{dt} = \frac{\partial w(t, \bar{\mathbf{x}}, \bar{\mathbf{k}})}{\partial \bar{k}_i}, \quad \frac{d\bar{k}_i}{dt} = - \frac{\partial w(t, \bar{\mathbf{x}}, \bar{\mathbf{k}})}{\partial \bar{x}^i}, \quad \frac{d\bar{\omega}}{dt} = \frac{\partial w(t, \bar{\mathbf{x}}, \bar{\mathbf{k}})}{\partial t}, \quad (7.39)$$

where $\bar{\mathbf{x}}$ is the coordinate, $\hbar\bar{\mathbf{k}}$ is the momentum, $\hbar\bar{\omega}$ is the energy, $\hbar w$ is the Hamiltonian, and the constant factor $\hbar$ can be anything. If $\hbar$ is chosen to be the Planck constant, then (7.39) can be interpreted as the motion equations of individual wave quanta, for example, photons. Hamilton’s equations for ‘true’ particles, such as electrons and ions, are also subsumed under (7.39) in that they can be understood as the ray equations of the particles considered as quantum-matter waves in the semiclassical limit.

Also notably, (7.39) can be obtained by considering the point-particle limit of (7.32) (Ruiz & Dodin 2015b). Specifically, adopting $\mathcal{I}(t, \mathbf{x}) \propto \delta(\mathbf{x} - \bar{\mathbf{x}}(t))$ and taking the integral in (7.32) by parts leads to a canonical action $S \propto \int dt (\bar{\mathbf{k}} \cdot \bar{\mathbf{x}} - w(t, \bar{\mathbf{x}}, \bar{\mathbf{k}}))$, whence Hamilton’s equations follow as usual.

7.2.3. Wave momentum and energy

Using (7.30) and (7.36), one arrives at the following equality for any given field $\mathbf{X}$:

$$\begin{aligned} \partial_t(\mathbf{X}\mathcal{I}) + \partial_{\mathbf{x}} \cdot (\mathbf{X}\mathcal{I}\bar{\mathbf{v}}_g) &= (\partial_t\mathbf{X})\mathcal{I} + \mathbf{X}(\partial_t\mathcal{I}) + [\partial_{\mathbf{x}} \cdot (\mathcal{I}\bar{\mathbf{v}}_g)]\mathbf{X} + \mathcal{I}(\bar{\mathbf{v}}_g \cdot \partial_{\mathbf{x}})\mathbf{X} \\ &= \mathcal{I}[\partial_t + (\bar{\mathbf{v}}_g \cdot \partial_{\mathbf{x}})]\mathbf{X} + \mathbf{X}[\partial_t\mathcal{I} + \partial_{\mathbf{x}} \cdot (\mathcal{I}\bar{\mathbf{v}}_g)] \\ &= \mathcal{I} d_t \mathbf{X}. \end{aligned} \quad (7.40)$$

For $\mathbf{X} = \bar{k}_i$ and $\mathbf{X} = \bar{\omega}$, (7.40) yields, respectively,

$$\partial_t P_{w,i} + \partial_{\mathbf{x}} \cdot (\bar{\mathbf{v}}_g P_{w,i}) = -\mathcal{I} \partial_i w, \quad (7.41a)$$

$$\partial_t \mathcal{E}_w + \partial_{\mathbf{x}} \cdot (\bar{\mathbf{v}}_g \mathcal{E}_w) = \mathcal{I} \partial_t w, \quad (7.41b)$$

where we used (7.37) and introduced the following notation:

$$\mathbf{P}_w \doteq \bar{\mathbf{k}}\mathcal{I}, \quad \mathcal{E}_w \doteq \bar{\omega}\mathcal{I}. \quad (7.42)$$

When a medium is homogeneous along x^i , (7.41a) yields $\int d\mathbf{x} P_{w,i} = \text{const.}$ Likewise, when a medium is stationary, (7.41b) yields $\int d\mathbf{x} \mathcal{E}_w = \text{const.}$ Hence, by definition, $\mathbf{P}_w$ and $\mathcal{E}_w$ are the densities of the wave canonical momentum and energy, at least up to a

constant factor κ .[†] A proof that $\kappa = 1$ can be found, for example, in (Dodin & Fisch 2012). In section 7.5, we will show this using different arguments.

7.3. Non-conservative eikonal waves

In a medium with nonzero Ξ_A , where waves are non-conservative, the wave properties are defined as in the previous section but the wave action evolves differently. The variational principle is not easy to apply in this case (however, see (Dodin *et al.* 2017)), so a different approach will be used to derive the action equation. A more straightforward but less intuitive approach can be found in (Dodin *et al.* 2019; McDonald 1988).

7.3.1. Monochromatic waves

First, consider a homogeneous stationary medium and a ‘monochromatic’ (exponentially growing at a constant rate) wave field in the form

$$\tilde{\Psi}(t, \mathbf{x}) = \text{re}(e^{-i\bar{\omega}t + i\bar{\mathbf{k}} \cdot \mathbf{x}} \check{\Psi}_c), \quad \check{\Psi}_c = e^{\bar{\gamma}t} \times \text{const}, \quad (7.43)$$

where the constants $\bar{\omega}$ and $\bar{\mathbf{k}}$ are, as usual, the real frequency and wavenumber, and $\bar{\gamma}$ is the linear growth rate, which can have either sign. Then, (6.15) becomes

$$0 = \Xi(\bar{\omega} + i\bar{\gamma}, \bar{\mathbf{k}})\check{\Psi}_c = \Xi_H(\bar{\omega}, \bar{\mathbf{k}})\check{\Psi}_c + i(\bar{\gamma}\partial_\omega \Xi_H(\bar{\omega}, \bar{\mathbf{k}}) + \Xi_A(\bar{\omega}, \bar{\mathbf{k}}))\check{\Psi}_c + \mathcal{O}(\epsilon^2), \quad (7.44)$$

where we assume that Ξ is a smooth function of ω and also that both Ξ_A and $\bar{\gamma}$ are $\mathcal{O}(\epsilon)$. Like in section 7.2.1, we adopt $\check{\Psi}_c = \boldsymbol{\eta}\check{a} + \mathcal{O}(\epsilon)$, where the polarization vector $\boldsymbol{\eta}$ is the relevant eigenvector of Ξ_H . Then, by projecting (7.44) on $\boldsymbol{\eta}$, one obtains

$$0 = A(\bar{\omega}, \bar{\mathbf{k}})\check{a} + i(\bar{\gamma}\partial_\omega A + \boldsymbol{\eta}^\dagger \Xi_A \boldsymbol{\eta})|_{(\omega, \mathbf{k})=(\bar{\omega}, \bar{\mathbf{k}})}\check{a} + \mathcal{O}(\epsilon^2), \quad (7.45)$$

where $A = \boldsymbol{\eta}^\dagger \Xi_H \boldsymbol{\eta}$ is the corresponding eigenvalue of Ξ_H and we used (7.16). Let us neglect $\mathcal{O}(\epsilon^2)$, divide (7.45) by $\check{a}$, and consider the real and imaginary parts of the resulting equation separately:

$$A(\bar{\omega}, \bar{\mathbf{k}}) = 0, \quad (\bar{\gamma}\partial_\omega A + \boldsymbol{\eta}^\dagger \Xi_A \boldsymbol{\eta})|_{(\omega, \mathbf{k})=(\bar{\omega}, \bar{\mathbf{k}})} = 0. \quad (7.46)$$

The former is the same dispersion relation for $\bar{\omega}$ as for conservative waves, and the latter yields $\bar{\gamma} = \gamma(\bar{\mathbf{k}})$, where

$$\gamma(\mathbf{k}) \doteq -\frac{\boldsymbol{\eta}^\dagger \Xi_A \boldsymbol{\eta}}{\partial_\omega A}. \quad (7.47)$$

Because $|\check{a}| \propto e^{\bar{\gamma}t}$, one can write the amplitude equation as

$$\partial_t |\check{a}|^2 = 2\bar{\gamma} |\check{a}|^2. \quad (7.48)$$

One can also define the action density $\mathcal{I}$ as in section 7.2.1 and rewrite (7.48) in terms of that. Because $\mathcal{I} = |\check{a}|^2 \times \text{const}$, one obtains

$$\partial_t \mathcal{I} = 2\bar{\gamma} \mathcal{I}. \quad (7.49)$$

7.3.2. Non-monochromatic waves

When weak inhomogeneity and weak dissipation coexist, their effect on the action density is additive, so (7.30) and (7.49) merge into a general equation

$$\partial_t \mathcal{I} + \partial_{\mathbf{x}}(\bar{\mathbf{v}}_g \mathcal{I}) = 2\bar{\gamma} \mathcal{I}. \quad (7.50)$$

[†] Therefore, in a zero-dimensional wave, where $\int d\mathbf{x}$ can be omitted, conservation of the total action $\mathcal{I}$ implies conservation of $\mathcal{E}_w/\omega$, which is a well-known adiabatic invariant of a discrete harmonic oscillator with a slowly varying frequency (Landau & Lifshitz 1976, section 49).

(A formal derivation of (7.50), which uses the Weyl expansion (2.41) and projection of the field equation on the polarization vector, can be found in (Dodin *et al.* 2019).) Then, (7.40) is modified as follows:

$$\partial_t(\mathbf{X}\mathcal{I}) + \partial_{\mathbf{x}} \cdot (\mathbf{X}\mathcal{I}\bar{\mathbf{v}}_g) = \mathcal{I} d_t \mathbf{X} + 2\bar{\gamma}\mathbf{X}\mathcal{I}, \quad (7.51)$$

and the equations (7.41) for the wave momentum and energy (7.42) become

$$\partial_t P_{w,i} + \partial_{\mathbf{x}} \cdot (\bar{\mathbf{v}}_g P_{w,i}) = 2\bar{\gamma} P_{w,i} - \mathcal{I} \partial_i w, \quad (7.52a)$$

$$\partial_t \mathcal{E}_w + \partial_{\mathbf{x}} \cdot (\bar{\mathbf{v}}_g \mathcal{E}_w) = 2\bar{\gamma} \mathcal{E}_w + \mathcal{I} \partial_t w. \quad (7.52b)$$

A comment is due here regarding the relation between (7.50) and the amplitude equation (7.48) that is commonly used in the standard QLT for homogeneous plasma (for example, see (2.21) in (Drummond & Pines 1962)). In a nutshell, the latter is incorrect, even when $\partial_{\mathbf{x}} = 0$. Because $\bar{f}$ is time-dependent, waves do not grow or decay exponentially. Rather, they can be considered as geometrical-optics (WKB) waves, and unlike in section 7.3.1, the ratio $|\check{a}|^2/\mathcal{I}$ generally evolves at a rate comparable to $\bar{\gamma}$. The standard QLT remains conservative only because it also incorrectly replaces (3.19) with its stationary-plasma limit ($\epsilon = 0$) and the two errors cancel each other. These issues are less of a problem for waves in not-too-hot plasmas (e.g. Langmuir waves), because in such plasmas, changing the distribution functions does not significantly affect the dispersion relations and thus $|\check{a}|^2/\mathcal{I}$ does in fact approximately remain constant. See also the discussion in section 9.1.4.

7.4. General waves

Let us now discuss a more general case that includes broadband waves. The evolution of such waves can be described statistically in terms of their average Wigner matrix $\mathbf{U}$. This matrix also determines the function $\bar{\mathbf{W}}_s$ that is given by (6.37a) and enters the nonlinear potentials (6.77). Below, we derive the general form of $\mathbf{U}$ in terms of the phase-space action density J and the governing equation for J (sections 7.4.1–7.4.3). Then, we also express the function $\bar{\mathbf{W}}_s$ through J (section 7.4.4). Related calculations can also be found in (McDonald & Kaufman 1985; Ruiz 2017).

7.4.1. Average Wigner matrix of an eikonal wave field

Let us start with calculating the average Wigner matrix of an eikonal field $\tilde{\Psi}$ of the form (7.17) (see also appendix A.2). Using $\tilde{\Psi} = (\tilde{\Psi}_c + \tilde{\Psi}_c^*)/2$, it can be readily expressed through the average Wigner functions of the complexified field[†] $\tilde{\Psi}_c$ and of its complex conjugate:

$$\mathbf{U} \approx (\bar{\mathbf{W}}_{\tilde{\Psi}_c} + \bar{\mathbf{W}}_{\tilde{\Psi}_c^*})/4 \equiv (\mathbf{U}_{c+} + \mathbf{U}_{c-})/4. \quad (7.53)$$

For $\tilde{\Psi}_c = \check{a}\boldsymbol{\eta}(\bar{\omega}, \bar{\mathbf{k}})e^{i\theta}$, where the arguments $(t, \mathbf{x})$ are omitted for brevity, one has

$$\mathbf{U}_c \equiv \mathbf{U}_{c+} \approx (\boldsymbol{\eta}\boldsymbol{\eta}^\dagger)(\bar{\omega}, \bar{\mathbf{k}})|\check{a}|^2 \int \frac{d\tau d\mathbf{s}}{(2\pi)^n} e^{i\theta(t+\tau/2, \mathbf{x}+\mathbf{s}/2)} e^{-i\theta(t-\tau/2, \mathbf{x}-\mathbf{s}/2)} e^{i\omega\tau - i\mathbf{k}\cdot\mathbf{s}}, \quad (7.54)$$

where we neglected the dependence of $\check{a}$ and $\boldsymbol{\eta}$ on $(t, \mathbf{x})$ because it is weak compared to that of $e^{\pm i\theta}$. By Taylor-expanding θ , one obtains

$$\mathbf{U}_c \approx (\boldsymbol{\eta}\boldsymbol{\eta}^\dagger)(\bar{\omega}, \bar{\mathbf{k}})|\check{a}|^2 \int \frac{d\tau d\mathbf{s}}{(2\pi)^n} e^{i(\omega-\bar{\omega})\tau - i(\mathbf{k}-\bar{\mathbf{k}})\cdot\mathbf{s}} = (\boldsymbol{\eta}\boldsymbol{\eta}^\dagger)(\bar{\omega}, \bar{\mathbf{k}})|\check{a}|^2 \delta(\omega - \bar{\omega}) \delta(\mathbf{k} - \bar{\mathbf{k}}).$$

[†] Field complexification is discussed, for example, in (Brizard *et al.* 1993).

For $\tilde{\Psi}_c^* = \check{a}^* \boldsymbol{\eta}^*(\bar{\omega}, \bar{\mathbf{k}}) e^{-i\theta}$, which can also be written as $\tilde{\Psi}_c^* = \check{a}^* \boldsymbol{\eta}(-\bar{\omega}, -\bar{\mathbf{k}}) e^{-i\theta}$ due to (7.7), the result is the same up to replacing $\bar{\omega} \rightarrow -\bar{\omega}$ and $\bar{\mathbf{k}} \rightarrow -\bar{\mathbf{k}}$. Also notice that

$$\begin{aligned} \delta(\omega \mp \bar{\omega}) \delta(\mathbf{k} \mp \bar{\mathbf{k}}) &= \delta(\omega \mp w(\bar{\mathbf{k}})) \delta(\mathbf{k} \mp \bar{\mathbf{k}}) \\ &= \delta(\omega \mp w(\pm \mathbf{k})) \delta(\mathbf{k} \mp \bar{\mathbf{k}}) \\ &= \delta(\omega - w(\mathbf{k})) \delta(\mathbf{k} \mp \bar{\mathbf{k}}), \end{aligned} \quad (7.55)$$

so one can rewrite $\mathbf{U}_{c\pm}$ as follows:

$$\mathbf{U}_{c\pm} = \boldsymbol{\eta}(\mathbf{k}) \boldsymbol{\eta}^\dagger(\mathbf{k}) |\check{a}|^2 \delta(\omega - w(\mathbf{k})) \delta(\mathbf{k} \mp \bar{\mathbf{k}}), \quad (7.56)$$

where $(\mathbf{k}) \equiv (w(\mathbf{k}), \mathbf{k})$. Thus finally,

$$\mathbf{U}(\omega, \mathbf{k}) \approx \boldsymbol{\eta}(\mathbf{k}) \boldsymbol{\eta}^\dagger(\mathbf{k}) |\check{a}|^2 (\delta(\mathbf{k} - \bar{\mathbf{k}}) + \delta(\mathbf{k} + \bar{\mathbf{k}})) \delta(\omega - w(\mathbf{k}))/4. \quad (7.57)$$

7.4.2. Average Wigner matrix of a general wave

Assuming the background medium is sufficiently smooth, a general wave field can be represented as a superposition of eikonal fields:

$$\tilde{\Psi} = \text{re } \tilde{\Psi}_c, \quad \tilde{\Psi}_c = \sum_\sigma \tilde{\Psi}_{\sigma,c}, \quad \tilde{\Psi}_{\sigma,c} = \check{a}_\sigma e^{i\theta_\sigma}. \quad (7.58)$$

As a quadratic functional, its average Wigner matrix $\mathbf{U}$ equals the sum of the average Wigner matrices $\mathbf{U}_\sigma$ of the individual eikonal waves:

$$\mathbf{U} = \sum_\sigma \mathbf{U}_\sigma = \sum_\sigma (\mathbf{U}_{\sigma,c+} + \mathbf{U}_{\sigma,c-})/4, \quad (7.59)$$

where $\mathbf{U}_{\sigma,c+} \equiv \mathbf{U}_{\sigma,c}$ and $\mathbf{U}_{\sigma,c-}$ are the average Wigner matrices of $\tilde{\Psi}_{\sigma,c}$ and $\tilde{\Psi}_{\sigma,c}^*$, respectively:

$$\mathbf{U}_{\sigma,c\pm} = \boldsymbol{\eta}(\mathbf{k}) \boldsymbol{\eta}^\dagger(\mathbf{k}) |\check{a}_\sigma|^2 \delta(\mathbf{k} \mp \bar{\mathbf{k}}_\sigma) \delta(\omega - w(\mathbf{k})). \quad (7.60)$$

Equation (7.59) can also be expressed as

$$\mathbf{U} = (\mathbf{U}_{c+} + \mathbf{U}_{c-})/4, \quad \mathbf{U}_{c\pm} = \sum_\sigma \mathbf{U}_{\sigma,c\pm}, \quad (7.61)$$

where $\mathbf{U}_{c\pm}$ are the average Wigner matrices of $\tilde{\Psi}_c$ and $\tilde{\Psi}_c^*$, respectively:

$$\mathbf{U}_{c\pm} = \boldsymbol{\eta}(\mathbf{k}) \boldsymbol{\eta}^\dagger(\mathbf{k}) h_{c\pm}(\mathbf{k}) \delta(\omega - w(\mathbf{k})), \quad h_{c\pm}(\mathbf{k}) \doteq \sum_\sigma |\check{a}_\sigma|^2 \delta(\mathbf{k} \mp \bar{\mathbf{k}}_\sigma). \quad (7.62)$$

Because $h_{c-}(\mathbf{k}) = h_{c+}(-\mathbf{k}) \equiv h_c(-\mathbf{k})$, the matrix $\mathbf{U}$ can also be written as follows:

$$\mathbf{U}(\omega, \mathbf{k}) = (\boldsymbol{\eta} \boldsymbol{\eta}^\dagger)(\mathbf{k}) (h(\mathbf{k}) + h(-\mathbf{k})) \delta(\omega - w(\mathbf{k})), \quad (7.63)$$

where $h(\mathbf{k}) \doteq h_c(\mathbf{k})/4$ is given by

$$h(\mathbf{k}) = \frac{1}{4} \sum_\sigma |\check{a}_\sigma|^2 \delta(\mathbf{k} - \bar{\mathbf{k}}_\sigma) \geq 0. \quad (7.64)$$

This shows that for broadband waves comprised of eikonal waves, $\mathbf{U}$ has the same form as for an eikonal wave except $h(\mathbf{k})$ is not necessarily delta-shaped.

7.4.3. Phase-space action density and the wave-kinetic equation

The wave equation for the complexified field $\tilde{\Psi}_c$ can be written in the invariant form as $\hat{\Xi} |\tilde{\Psi}_c\rangle = |0\rangle$. Multiplying it by $\langle \tilde{\Psi}_c |$ from the right leads to

$$\hat{\Xi} \hat{\mathbf{U}}_c = \hat{0}, \quad \hat{\mathbf{U}}_c \doteq (2\pi)^{-n} |\tilde{\Psi}_c\rangle \langle \tilde{\Psi}_c|. \quad (7.65)$$

This readily yields an equation for the Wigner matrix: $\boldsymbol{\Xi} \star \mathbf{U}_c = \mathbf{0}$. Let us integrate this equation over ω to make the left-hand side a smooth function of $(t, \mathbf{x}, \mathbf{k})$. Let us also take

the trace of the resulting equation to put it in a scalar form:

$$\text{tr} \int d\omega \, \Xi \star \mathbf{U}_c = 0. \quad (7.66)$$

As usual, we assume $\Xi = \Xi_H + i\Xi_A$ with $\Xi_A = \mathcal{O}(\epsilon) \ll \Xi_H = \mathcal{O}(1)$ for generic $(\mathbf{x}, \mathbf{k})$. The integrand in (7.66) can be written as $\Xi \star \mathbf{U}_c = \Xi e^{i\hat{\mathcal{L}}_x/2} \mathbf{U}_c$, and its expansion in the differential operator $\hat{\mathcal{L}}_x$ (2.32) contains derivatives of all orders. High-order derivatives on $\mathbf{U}_c$ are not negligible *per se*, because for on-shell waves this function is delta-shaped. However, using integration by parts, one can reapply all derivatives with respect to ω to Ξ and take the remaining derivatives (with respect to t , $\mathbf{x}$, and $\mathbf{k}$) outside the integral. Then it is seen that each power m of $\hat{\mathcal{L}}_x$ in the expansion of $\Xi e^{i\hat{\mathcal{L}}_x/2} \mathbf{U}_c$ contributes $\mathcal{O}(\epsilon^m)$ to the integral. Let us neglect terms with $m \geq 2$ and use (7.62). Hence, one obtains[†]

$$\begin{aligned} 0 &\approx \text{tr} \int d\omega \left(\Xi_H \mathbf{U}_c + i\Xi_A \mathbf{U}_c + \frac{i}{2} \{ \Xi_H, \mathbf{U}_c \}_x \right) \\ &\approx (\boldsymbol{\eta}^\dagger \Xi_H \boldsymbol{\eta} + i\boldsymbol{\eta}^\dagger \Xi_A \boldsymbol{\eta}) h_c + \frac{i}{2} \text{tr} \int d\omega \left(\frac{\partial \Xi_H}{\partial x^i} \frac{\partial \mathbf{U}_c}{\partial k_i} - \frac{\partial \Xi_H}{\partial k_i} \frac{\partial \mathbf{U}_c}{\partial x^i} \right). \end{aligned} \quad (7.67)$$

Let us also re-express this as follows, using (7.4) and (7.47):

$$\begin{aligned} 0 &\approx \left(A - i\gamma \frac{\partial A}{\partial \omega} \right) h_c - \frac{i}{2} \text{tr} \int d\omega \frac{\partial}{\partial \omega} \left(\frac{\partial \Xi_H}{\partial t} \mathbf{U}_c \right) + \frac{i}{2} \frac{\partial}{\partial t} \text{tr} \int d\omega \frac{\partial \Xi_H}{\partial \omega} \mathbf{U}_c \\ &\quad + \frac{i}{2} \frac{\partial}{\partial k_i} \text{tr} \int d\omega \frac{\partial \Xi_H}{\partial x^i} \mathbf{U}_c - \frac{i}{2} \frac{\partial}{\partial x^i} \text{tr} \int d\omega \frac{\partial \Xi_H}{\partial k_i} \mathbf{U}_c. \end{aligned} \quad (7.68)$$

Clearly,

$$\int d\omega \frac{\partial}{\partial \omega} \left(\frac{\partial \Xi_H}{\partial t} \mathbf{U}_c \right) = 0. \quad (7.69)$$

To simplify the remaining terms, we proceed as follows. As a Hermitian matrix, Ξ_H can be represented in terms of its eigenvalues Λ_b and eigenvectors $\boldsymbol{\eta}_b$ as $\Xi_H = \Lambda_b \boldsymbol{\eta}_b \boldsymbol{\eta}_b^\dagger$. For $\mathbf{U}_c$, let us use (7.62) again, where $\boldsymbol{\eta}$ is one of the vectors $\boldsymbol{\eta}_b$, say, $\boldsymbol{\eta} \equiv \boldsymbol{\eta}_0$. (Accordingly, $A \equiv \Lambda_0$.) Then, for any $\blacksquare \in \{\omega, x^i, k_i\}$, one has

$$\begin{aligned} \text{tr} \int d\omega \frac{\partial \Xi_H}{\partial \blacksquare} \mathbf{U}_c &= \frac{\partial \Lambda_b}{\partial \blacksquare} |\boldsymbol{\eta}_b^\dagger \boldsymbol{\eta}|^2 h_c + \Lambda_b \left(\boldsymbol{\eta}^\dagger \frac{\partial \boldsymbol{\eta}_b}{\partial \blacksquare} \right) (\boldsymbol{\eta}_b^\dagger \boldsymbol{\eta}) h_c + \Lambda_b (\boldsymbol{\eta}^\dagger \boldsymbol{\eta}_b) \left(\frac{\partial \boldsymbol{\eta}_b^\dagger}{\partial \blacksquare} \boldsymbol{\eta} \right) h_c \\ &= \frac{\partial \Lambda_b}{\partial \blacksquare} (\delta_{b,0})^2 h_c + \Lambda_b \left(\boldsymbol{\eta}^\dagger \frac{\partial \boldsymbol{\eta}_b}{\partial \blacksquare} \right) \delta_{b,0} h_c + \Lambda_b \delta_{b,0} \left(\frac{\partial \boldsymbol{\eta}_b^\dagger}{\partial \blacksquare} \boldsymbol{\eta} \right) h_c \\ &= \frac{\partial \Lambda}{\partial \blacksquare} h_c + \left(\boldsymbol{\eta}^\dagger \frac{\partial \boldsymbol{\eta}}{\partial \blacksquare} + \frac{\partial \boldsymbol{\eta}^\dagger}{\partial \blacksquare} \boldsymbol{\eta} \right) \Lambda h_c \\ &= \frac{\partial \Lambda}{\partial \blacksquare} h_c + \frac{\partial (\boldsymbol{\eta}^\dagger \boldsymbol{\eta})}{\partial \blacksquare} \Lambda h_c \\ &= \frac{\partial \Lambda}{\partial \blacksquare} h_c, \end{aligned} \quad (7.70)$$

where we used $\boldsymbol{\eta}_b^\dagger \boldsymbol{\eta} = \delta_{b,0}$ and, in particular, $\boldsymbol{\eta}^\dagger \boldsymbol{\eta} = 1$. Then, (7.68) can be written as

$$\Lambda h_c - 2i\mathcal{E} = 0, \quad (7.71)$$

[†] McDonald & Kaufman (1985) first Taylor-expand $\Xi \star \mathbf{U}_c$ and then integrate over ω . Strictly speaking, that is incorrect (because $\Xi \star \mathbf{U}_c$ is not smooth), but the final result is the same.

where

$$\mathcal{E} = 2\gamma \left(\frac{\partial \Lambda}{\partial \omega} h \right) - \frac{\partial}{\partial t} \left(\frac{\partial \Lambda}{\partial \omega} h \right) - \frac{\partial}{\partial k_i} \left(\frac{\partial \Lambda}{\partial x^i} h \right) + \frac{\partial}{\partial x^i} \left(\frac{\partial \Lambda}{\partial k_i} h \right). \quad (7.72)$$

The real part of (7.71) gives $\Lambda = 0$, which is the dispersion relation. The imaginary part of (7.71) gives $\mathcal{E} = 0$. To understand this equation, let us rewrite $\mathcal{E}$ as

$$\mathcal{E} = 2\gamma J - \frac{\partial J}{\partial t} + \frac{\partial}{\partial k_i} \left(\frac{\partial w}{\partial x^i} J \right) - \frac{\partial}{\partial x^i} \left(\frac{\partial w}{\partial k_i} J \right). \quad (7.73)$$

Here, we introduced

$$J(\mathbf{k}) \doteq h(\mathbf{k}) \partial_\omega \Lambda(\mathbf{k}), \quad \Lambda(\mathbf{k}) \doteq \Lambda(w(\mathbf{k}), \mathbf{k}), \quad (7.74)$$

which, according to (7.24), satisfy

$$J \partial_t w(\mathbf{k}) = -h(\partial_t \Lambda)(\mathbf{k}), \quad (7.75a)$$

$$J \partial_{\mathbf{x}} w(\mathbf{k}) = -h(\partial_{\mathbf{x}} \Lambda)(\mathbf{k}), \quad (7.75b)$$

$$J \partial_{\mathbf{k}} w(\mathbf{k}) = -h(\partial_{\mathbf{k}} \Lambda)(\mathbf{k}). \quad (7.75c)$$

Note that using (7.64), one can also express J as

$$J = \sum_\sigma \left(\frac{1}{4} |\check{a}_\sigma|^2 \partial_\omega \Lambda(\mathbf{k}_\sigma) \right) \delta(\mathbf{k} - \bar{\mathbf{k}}_\sigma) = \sum_\sigma \mathcal{I}_\sigma \delta(\mathbf{k} - \bar{\mathbf{k}}_\sigma), \quad (7.76)$$

where $\mathcal{I}_\sigma$ are the action densities (7.27) of the individual eikonal waves that comprise the total wave field (section 7.4.2). In particular, $\int d\mathbf{k} J = \sum_\sigma \mathcal{I}_\sigma$, which is the total action density. Therefore, the function J can be interpreted as the *phase-space* action density. In terms of J , the equation $\mathcal{E} = 0$ can be written as

$$\frac{\partial J}{\partial t} + \frac{\partial w}{\partial k_i} \frac{\partial J}{\partial x^i} - \frac{\partial w}{\partial x^i} \frac{\partial J}{\partial k_i} = 2\gamma J. \quad (7.77)$$

This equation, called the WKE, serves the same role in QL wave-kinetic theory as the Vlasov equation serves in plasma kinetic theory.[†] Unlike the field equation used in the standard QLT (Drummond & Pines 1962), (7.77) exactly conserves the action of nonresonant waves, i.e. those with $\gamma = 0$. Also note that (7.50) for eikonal waves can be deduced from (7.77) as a particular case by assuming the ansatz

$$J(t, \mathbf{x}, \mathbf{k}) = \mathcal{I}(t, \mathbf{x}) \delta(\mathbf{k} - \bar{\mathbf{k}}(t, \mathbf{x})) \quad (7.78)$$

and integrating over $\mathbf{k}$. In other words, eikonal-wave theory can be understood as the ‘cold-fluid’ limit of wave-kinetic theory.

7.4.4. Function $\overline{W}_s$ in terms of J

Here we explicitly calculate the function (6.37a) that determines the nonlinear potentials (6.77). Using (7.63), one obtains

$$\overline{W}_s(\omega, \mathbf{k}; \mathbf{p}) = |\alpha_s^\dagger \eta|^2(\mathbf{k}; \mathbf{p}) (h(\mathbf{k}) + h(-\mathbf{k})) \delta(\omega - w(\mathbf{k})) \geq 0, \quad (7.79)$$

where $(\mathbf{k}; \mathbf{p}) \equiv (w(\mathbf{k}), \mathbf{k}; \mathbf{p})$. By definition of $\hat{\alpha}_s$, the function $\langle t_1, \mathbf{x}_1 | \hat{\alpha}_s | t_2, \mathbf{x}_2 \rangle$ is real for all $(t_1, \mathbf{x}_1)$ and $(t_2, \mathbf{x}_2)$, so $\alpha_s(-\omega, -\mathbf{k}) = \alpha_s^*(\omega, \mathbf{k})$ by definition of the Weyl symbol

[†] The term ‘WKE’ is also used for the equation that describes nonlinear interactions of waves in statistically homogeneous media, or ‘wave–wave collisions’ (Zakharov *et al.* 1992). That is not what we consider here. Inhomogeneities are essential in our formulation, and the QL WKE is linear (in J) by definition of the QL approximation. That said, the Weyl symbol calculus that we use can facilitate derivations of wave–wave collision operators as well (Ruiz *et al.* 2019).

(2.26). Together with (7.7), this gives $|\alpha_s^\dagger \eta|^2(\omega, \mathbf{k}; \mathbf{p}) = |\alpha_s^\dagger \eta|^2(-\omega, -\mathbf{k}; \mathbf{p})$, so

$$|\alpha_s^\dagger \eta|^2 \equiv |\alpha_s^\dagger \eta|^2(\mathbf{k}; \mathbf{p}) = |\alpha_s^\dagger \eta|^2(-\mathbf{k}; \mathbf{p}), \quad (7.80a)$$

and similarly,

$$|\eta^\dagger \wp_s \eta|^2 \equiv |\eta^\dagger \wp_s \eta|^2(\mathbf{k}; \mathbf{p}) = |\eta^\dagger \wp_s \eta|^2(-\mathbf{k}; \mathbf{p}). \quad (7.80b)$$

This also means that $\overline{W}_s(\omega, \mathbf{k}; \mathbf{p}) = \overline{W}_s(-\omega, -\mathbf{k}; \mathbf{p})$. Then finally, using (7.74), one can express this function through the phase-space action density:

$$\overline{W}_s(\omega, \mathbf{k}; \mathbf{p}) = |\alpha_s^\dagger \eta|^2 (\varsigma_{\mathbf{k}} J(\mathbf{k}) + \varsigma_{-\mathbf{k}} J(-\mathbf{k})) \delta(\Lambda(\omega, \mathbf{k})), \quad (7.81)$$

$$\varsigma_{\mathbf{k}} \doteq \text{sgn } \partial_\omega \Lambda(\mathbf{k}) = \text{sgn}(J(\mathbf{k})/h(\mathbf{k})) = \text{sgn } J(\mathbf{k}). \quad (7.82)$$

7.5. Conservation laws

Let us rewrite (7.77) together with (6.76) in the ‘divergence’ form:

$$\frac{\partial J}{\partial t} + \frac{\partial(v_g^i J)}{\partial x^i} - \frac{\partial}{\partial k_i} \left(\frac{\partial w}{\partial x^i} J \right) = 2\gamma J, \quad (7.83)$$

$$\frac{\partial F_s}{\partial t} + \frac{\partial(v_s^i F_s)}{\partial x^i} - \frac{\partial}{\partial p_i} \left(\frac{\partial \mathcal{H}_s}{\partial x^i} F_s \right) = \frac{\partial}{\partial p_i} \left(D_{s,ij} \frac{\partial F_s}{\partial p_j} \right) + C_s. \quad (7.84)$$

Using (7.62), the diffusion matrix $D_{s,ij}$ can be represented as follows:

$$D_{s,ij} = 2\pi \int d\mathbf{k} k_i k_j |\alpha_s^\dagger \eta|^2 \frac{J(\mathbf{k})}{\partial_\omega \Lambda(\mathbf{k})} \delta(\mathbf{k} \cdot \mathbf{v}_s - w(\mathbf{k})). \quad (7.85)$$

Also, by substituting (6.24) into (7.47), one finds

$$\gamma = \pi \sum_s \int d\mathbf{p} \frac{|\alpha_s^\dagger \eta|^2}{\partial_\omega \Lambda(\mathbf{k})} \delta(w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s(\mathbf{p})) \mathbf{k} \cdot \frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}}. \quad (7.86)$$

Together with (7.75), these yield the following notable corollaries. First of all, if Ξ_0 , $|\alpha_s^\dagger \eta|^2$, and $\eta^\dagger \wp_s \eta$ are independent of $\mathbf{x}, \dagger$ one has for each l that (appendix D.1)

$$\begin{aligned} \frac{\partial}{\partial t} \left(\sum_s \int d\mathbf{p} p_l F_s + \int d\mathbf{k} k_l J \right) + \frac{\partial}{\partial x^i} \left(\sum_s \int d\mathbf{p} p_l v_s^i F_s + \int d\mathbf{k} k_l v_g^i J \right) \\ + \frac{\partial}{\partial x^l} \sum_s \int d\mathbf{p} \Delta_s F_s = - \sum_s \int d\mathbf{p} \frac{\partial H_{0s}}{\partial x^l} F_s. \end{aligned} \quad (7.87)$$

This can be viewed as a momentum-conservation theorem, because at $\partial_l H_{0s} = 0$, one has

$$\sum_s \int d\mathbf{x} d\mathbf{p} p_l F_s + \int d\mathbf{x} d\mathbf{k} k_l J = \text{const.} \quad (7.88)$$

Also, the ponderomotive force on a plasma is readily found from (7.87) as the sum of the terms quadratic in the wave amplitude (after F_s has been expressed through $\overline{f}_s$). Similarly, if Ξ_0 , $|\alpha_s^\dagger \eta|^2$, and $\eta^\dagger \wp_s \eta$ are independent of t , one has (appendix D.2)

$$\frac{\partial}{\partial t} \left(\sum_s \int d\mathbf{p} H_{0s} F_s + \int d\mathbf{k} w J \right) + \frac{\partial}{\partial x^i} \left(\sum_s \int d\mathbf{p} H_{0s} v_s^i F_s + \int d\mathbf{k} w v_g^i J \right)$$

$\dagger$ Having $\mathbf{x}$ -dependence in Ξ_0 , $|\alpha_s^\dagger \eta|^2$, or $\eta^\dagger \wp_s \eta$ would signify interaction with external fields not treated self-consistently. Such fields could exchange momentum with the wave–plasma system, so the momentum of the latter would not be conserved. A similar argument applies to the temporal dependence of these coefficients vs. energy conservation considered below.

Quantity	Notation	Interpretation
$\int d\mathbf{p} \mathbf{p} F_s$	$\mathbf{P}_s$	OC momentum density
$\int d\mathbf{p} H_{0s} F_s$	$\mathcal{E}_s$	OC energy density
$\int d\mathbf{p} (\mathbf{p} v_s + \Delta_s \mathbf{1}) F_s$	$\mathbf{\Pi}_s$	OC momentum flux density
$\int d\mathbf{p} (H_{0s} + \Delta_s) v_s F_s$	$\mathbf{Q}_s$	OC energy flux density
$\int d\mathbf{k} \mathbf{k} J$	$\mathbf{P}_w$	wave momentum density
$\int d\mathbf{k} w J$	$\mathcal{E}_w$	wave energy density
$\int d\mathbf{k} \mathbf{k} v_g J$	$\mathbf{\Pi}_w$	wave momentum flux density
$\int d\mathbf{k} w v_g J$	$\mathbf{Q}_w$	wave energy flux density

TABLE 1. Interpretation of the individual terms in (7.87) and (7.89). The wave energy–momentum is understood as the canonical (‘Minkowski’) energy–momentum, which must not be confused with the kinetic (‘Abraham’) energy–momentum (Dodin & Fisch 2012; Dewar 1977). Whether the terms with $\Delta_s F_s$ should be attributed to OCs or to the wave is a matter of convention, because $\Delta_s F_s$ scales linearly both with F_s and with J . In contrast, the wave energy density is defined unambiguously as $\mathcal{E}_s \doteq \int d\mathbf{p} H_{0s} F_s$ and does not contain Δ_s . This is because $\int d\mathbf{p} \Delta_s F_s$ is a part of the wave energy density $\mathcal{E}_w$ (Dodin & Fisch 2010a). Similarly, $\int d\mathbf{p} (\partial_{v_s} \Delta_s) F_s$ is a part of the wave momentum density (Dodin & Fisch 2012).

$$+ \frac{\partial}{\partial x^i} \sum_s \int d\mathbf{p} \Delta_s v_s^i F_s = \sum_s \int d\mathbf{p} \frac{\partial H_{0s}}{\partial t} F_s. \quad (7.89)$$

This can be viewed as an energy-conservation theorem, because at $\partial_t H_{0s} = 0$, one has

$$\sum_s \int d\mathbf{x} d\mathbf{p} H_{0s} F_s + \int d\mathbf{x} d\mathbf{k} w J = \text{const}. \quad (7.90)$$

Related equations are also discussed in (Dodin & Fisch 2012; Dewar 1977).

The individual terms in (7.87) and (7.89) can be interpreted as described in table 1. The results of section 7.2.3 are reproduced as a particular case for the eikonal-wave ansatz (7.78).[†] In particular, note that electrostatic *waves* carry nonzero momentum density $\int d\mathbf{k} \mathbf{k} J$ just like any other waves, even though the electrostatic *field* of these waves carries no momentum. The momentum is stored in the particle motion in this case (section 9.1.3), and it is pumped there via either temporal dependence (Liu & Dodin 2015, section II.2) or spatial dependence (Ochs & Fisch 2021b, 2022) of the wave amplitude. This shows that homogeneous-plasma models that ignore ponderomotive effects cannot adequately describe the energy–momentum transfer between waves and plasma even when resonant absorption *per se* occurs in a homogeneous-plasma region. The OC formalism presented here provides means to describe such processes rigorously, generally, and without cumbersome calculations.

7.6. Summary of section 7

In summary, we have considered plasma interaction with general broadband single-mode on-shell waves (for examples, see section 9). Assuming a general response matrix $\mathbf{\Xi}$, these waves have a dispersion function $\Lambda(t, \mathbf{x}, \omega, \mathbf{k})$ and polarization $\boldsymbol{\eta}(t, \mathbf{x}, \omega, \mathbf{k})$

[†] There is no ambiguity in the definition of the wave momentum and energy in this case (i.e. $\kappa = 1$), because (7.88) and (7.90) connect those with the momentum and energy of particles (OCs), which are defined unambiguously.

determined by

$$\Xi_H \eta = \Lambda \eta, \quad \Lambda = \eta^\dagger \Xi_H \eta, \quad (7.91)$$

where the normalization $\eta^\dagger \eta = 1$ is assumed. Specifically for Ξ given by (6.80), one has

$$\begin{aligned} \Lambda(t, \mathbf{x}, \omega, \mathbf{k}) &= \eta^\dagger \Xi_0 \eta - \sum_s \int d\mathbf{p} \eta^\dagger \wp_s(\mathbf{p}) \eta F_s(\mathbf{p}) \\ &\quad + \sum_s \oint d\mathbf{p} \frac{|\alpha_s^\dagger \eta|^2(\mathbf{p})}{\omega - \mathbf{k} \cdot \mathbf{v}_s(\mathbf{p})} \mathbf{k} \cdot \frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}}, \end{aligned} \quad (7.92)$$

where the arguments $(t, \mathbf{x}, \omega, \mathbf{k})$ are omitted for brevity. (Some notation is summarized in section 6.9.) The wave frequency $\omega = w(t, \mathbf{x}, \mathbf{k})$ satisfies

$$\Lambda(t, \mathbf{x}, w(t, \mathbf{x}, \mathbf{k}), \mathbf{k}) = 0 \quad (7.93)$$

and $w(t, \mathbf{x}, -\mathbf{k}) = -w(t, \mathbf{x}, \mathbf{k})$, where w is a real function at real arguments. The wave local linear growth rate γ , which is assumed to be small in this section, is

$$\gamma(t, \mathbf{x}, \mathbf{k}) = - \left(\frac{\eta^\dagger \Xi_A \eta}{\partial_\omega \Lambda} \right)_{(t, \mathbf{x}, w(t, \mathbf{x}, \mathbf{k}), \mathbf{k})}, \quad (7.94)$$

or explicitly,

$$\gamma(t, \mathbf{x}, \mathbf{k}) = \pi \sum_s \int d\mathbf{p} \frac{|\alpha_s^\dagger \eta|^2}{\partial_\omega \Lambda(t, \mathbf{x}, w, \mathbf{k})} \delta(w - \mathbf{k} \cdot \mathbf{v}_s(t, \mathbf{x}, \mathbf{p})) \mathbf{k} \cdot \frac{\partial F_s(t, \mathbf{x}, \mathbf{p})}{\partial \mathbf{p}},$$

where $w \equiv w(t, \mathbf{x}, \mathbf{k})$ and $|\alpha_s^\dagger \eta|^2 \equiv |\alpha_s^\dagger \eta|^2(t, \mathbf{x}, w, \mathbf{k}; \mathbf{p})$. The nonlinear potentials (6.77) are expressed through the scalar function

$$\overline{W}_s(t, \mathbf{x}, \omega, \mathbf{k}; \mathbf{p}) = |\alpha_s^\dagger \eta|^2 (\varsigma_{\mathbf{k}} J(t, \mathbf{x}, \mathbf{k}) + \varsigma_{-\mathbf{k}} J(t, \mathbf{x}, -\mathbf{k})) \delta(\Lambda(t, \mathbf{x}, \omega, \mathbf{k})), \quad (7.95)$$

where $\varsigma_{\mathbf{k}} \doteq \text{sgn}(\partial_\omega \Lambda(t, \mathbf{x}, \omega, \mathbf{k}))$ is evaluated at $\omega = w(t, \mathbf{x}, \mathbf{k})$; see also (7.82). The function J is the phase-space action density governed by the WKE:

$$\frac{\partial J}{\partial t} - \frac{\partial w}{\partial \mathbf{x}} \cdot \frac{\partial J}{\partial \mathbf{k}} + \frac{\partial w}{\partial \mathbf{k}} \cdot \frac{\partial J}{\partial \mathbf{x}} = 2\gamma J, \quad (7.96)$$

where $\partial_{\mathbf{k}} w = \mathbf{v}_g$ is the group velocity. Collisional dissipation is assumed small compared to collisionless dissipation, so it is neglected in (7.96) but can be reintroduced by an *ad hoc* modification of γ (section 6.2). Unlike the field equation used in the standard QLT, (7.96) exactly conserves the action of nonresonant waves, i.e. those with $\gamma = 0$. The WKE must be solved together with the QL equation for the OC distribution F_s ,

$$\frac{\partial F_s}{\partial t} - \frac{\partial \mathcal{H}_s}{\partial \mathbf{x}} \cdot \frac{\partial F_s}{\partial \mathbf{p}} + \frac{\partial \mathcal{H}_s}{\partial \mathbf{p}} \cdot \frac{\partial F_s}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{p}} \cdot \left(\mathbf{D}_s \frac{\partial F_s}{\partial \mathbf{p}} \right) + \mathcal{C}_s, \quad (7.97)$$

because F_s determines the coefficients in (7.96) and J determines the coefficients in (7.97). When Ξ_0 and $|\alpha_s^\dagger \eta|^2$ are independent of t and $\mathbf{x}$, (7.96) and (7.97) conserve the total momentum and energy of the system; specifically,

$$\partial_t (\sum_s P_{s,i} + P_{w,i}) + \partial_j (\sum_s \Pi_{s,i}^j + \Pi_{w,i}^j) = - \sum_s \int d\mathbf{p} F_s \partial_i H_{0s}, \quad (7.98)$$

$$\partial_t (\sum_s \mathcal{E}_s + \mathcal{E}_w) + \partial_j (\sum_s Q_s^j + Q_w^j) = \sum_s \int d\mathbf{p} F_s \partial_t H_{0s}. \quad (7.99)$$

Here, the notation is as in table 1, or see (7.87) and (7.89) instead.

8. Thermal equilibrium

In this section, we discuss, for completeness, the properties of plasmas in thermal equilibrium.

8.1. Boltzmann–Gibbs distribution

As discussed in section 6.8, collisions conserve the density of each species, the total momentum density, and the total energy density, while the plasma total entropy density σ either grows or remains constant. Let us search for equilibrium states in particular. At least one of the states in which σ remains constant is the one that maximizes the entropy density at fixed $\int d\mathbf{p} F_s$, $\sum_s \int d\mathbf{p} \mathbf{p} F_s$, and $\sum_s \int d\mathbf{p} \mathcal{H}_s F_s$. This ‘state of thermal equilibrium’ can be found as an extremizer of

$$\sigma' \doteq \sigma - \sum_s \lambda_s^{(\mathcal{N})} \int d\mathbf{p} F_s - \boldsymbol{\lambda}^{(\mathbf{P})} \cdot \sum_s \int d\mathbf{p} \mathbf{p} F_s - \lambda^{(\mathcal{H})} \sum_s \int d\mathbf{p} \mathcal{H}_s F_s \quad (8.1)$$

considered as a functional of all F_s , where $\lambda_s^{(\mathcal{N})}$, $\boldsymbol{\lambda}^{(\mathbf{P})}$, and $\lambda^{(\mathcal{H})}$ are Lagrange multipliers. Using (6.73), one finds that extremizers of σ' satisfy

$$0 = \frac{\delta \sigma'}{\delta F_s} = -\ln F_s - 1 - \lambda_s^{(\mathcal{N})} - \boldsymbol{\lambda}^{(\mathbf{P})} \cdot \mathbf{p} - \lambda^{(\mathcal{H})} \mathcal{H}_s, \quad (8.2)$$

whence

$$F_s = \text{const}_s \times \exp(-\boldsymbol{\lambda}^{(\mathbf{P})} \cdot \mathbf{p} - \lambda^{(\mathcal{H})} \mathcal{H}_s). \quad (8.3)$$

The pre-exponential constant is determined by the given density of species s , while $\boldsymbol{\lambda}^{(\mathbf{P})}$ and $\lambda^{(\mathcal{H})}$ can be expressed through the densities of the plasma momentum and energy stored in the whole distribution. Because

$$\frac{\delta^2 \sigma'}{\delta F_s \delta F_s} = -\frac{1}{F_s} < 0, \quad \frac{\delta^2 \sigma'}{\delta F_s \delta F_{s' \neq s}} = 0, \quad (8.4)$$

the matrix $\delta^2 \sigma' / \delta F_s \delta F_{s'}$ is positive-definite, so the entropy is maximal (as opposed to minimal) at the extremizer (8.3).

The distribution (8.3) is known as the Boltzmann–Gibbs distribution, with $T \doteq 1/\lambda^{(\mathcal{H})}$ being the temperature (common for all species). Also, let us introduce a new, rescaled Lagrange multiplier $\mathbf{u}$ via $\boldsymbol{\lambda}^{(\mathbf{P})} = -\mathbf{u}/T$. Then,

$$F_s(\mathbf{p}) = F_s^{(0)} \exp\left(-\frac{\mathcal{H}_s(\mathbf{p}) - \mathbf{u} \cdot \mathbf{p}}{T}\right), \quad (8.5)$$

where $F_s^{(0)}$ is independent of $\mathbf{p}$. Correspondingly,

$$\frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}} = -F_s(\mathbf{p}) \frac{\mathbf{v}_s - \mathbf{u}}{T}, \quad (8.6)$$

where we used (5.44). From (8.6), one obtains

$$\begin{aligned} & \delta(\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}'_{s'}) \mathbf{k} \cdot \left(\frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}} F_{s'}(\mathbf{p}') - F_s(\mathbf{p}) \frac{\partial F_{s'}(\mathbf{p}')}{\partial \mathbf{p}'} \right) \\ &= -\frac{1}{T} \delta(\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}'_{s'}) (\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}'_{s'}) F_s(\mathbf{p}) F_{s'}(\mathbf{p}') \\ &= 0, \end{aligned} \quad (8.7)$$

where $\mathcal{H}'_{s'} \doteq \mathcal{H}_{s'}(\mathbf{p}')$. Then, (6.61) yields that the collision operator vanishes on the Boltzmann–Gibbs distribution, and thus, expectedly, $(d\sigma/dt)_{\text{coll}} = 0$. One can also show

that the Boltzmann–Gibbs distribution is *the only* distribution (strictly speaking, a class of distributions parameterized by T and $\mathbf{u}$) for which the entropy density is conserved (appendix E).

The property (8.6) of the thermal-equilibrium state also leads to other notable results that we derive below. In doing so, we will assume the reference frame where $\mathbf{u} = 0$, so the Boltzmann–Gibbs distribution has a better known form

$$F_s(\mathbf{p}) = F_s^{(0)} \exp\left(-\frac{\mathcal{H}_s(\mathbf{p})}{T}\right), \quad \frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}} = -F_s(\mathbf{p}) \frac{\mathbf{v}_s}{T}. \quad (8.8)$$

(For $\mathcal{H}_s$ isotropic in $\mathbf{p}$, this is the frame where the plasma total momentum density $\sum_s \int d\mathbf{p} \mathbf{p} F_s$ is zero.) The generalizations to arbitrary $\mathbf{u}$ are straightforward.

8.2. Fluctuation–dissipation theorem

Let us describe microscopic fluctuations in equilibrium plasmas in terms of $\mathbf{S}(\omega, \mathbf{k}) \doteq (2\pi)^n \mathfrak{M}(\omega, \mathbf{k})$, i.e.

$$\mathbf{S}(\omega, \mathbf{k}) \doteq \int d\tau d\mathbf{s} \overline{\tilde{\Psi}(t + \tau/2, \mathbf{x} + \mathbf{s}/2) \tilde{\Psi}^\dagger(t - \tau/2, \mathbf{x} - \mathbf{s}/2)} e^{i\omega\tau - i\mathbf{k}\cdot\mathbf{s}}, \quad (8.9)$$

which can also be represented in terms of the Fourier image $\dot{\tilde{\Psi}}(\omega, \mathbf{k})$ of the microscopic field $\tilde{\Psi}(t, \mathbf{x})$:

$$\mathbf{S}(\omega, \mathbf{k}) = \frac{\overline{\dot{\tilde{\Psi}}(\omega, \mathbf{k}) \dot{\tilde{\Psi}}^\dagger(\omega, \mathbf{k})}}{\mathcal{T} \mathcal{V}_n}. \quad (8.10)$$

For statistically homogeneous fields that persist on time $\mathcal{T} \rightarrow \infty$ within volume $\mathcal{V}_n \rightarrow \infty$, the Fourier transform is formally divergent; hence the appearance of the factors $\mathcal{T}$ and $\mathcal{V}_n$ in (8.10).[†] Also, as seen from (6.32), any quadratic function of the microscopic field can be expressed through $\mathbf{S}$ via

$$\overline{(\hat{\mathbf{L}}\psi)(\hat{\mathbf{R}}\psi)^\dagger} \approx \int \frac{d\omega}{2\pi} \frac{d\mathbf{k}}{(2\pi)^n} (\mathbf{L}\mathbf{S}\mathbf{R}^\dagger)(\omega, \mathbf{k}), \quad (8.11)$$

where $\hat{\mathbf{L}}$ and $\hat{\mathbf{R}}$ are any linear operators and $\mathbf{L}$ and $\mathbf{R}$ are their symbols.

From (6.31), one finds that, in general,

$$\mathbf{S}(\omega, \mathbf{k}) = 2\pi \sum_{s'} \int d\mathbf{p}' \delta(\omega - \mathbf{k} \cdot \mathbf{v}'_{s'}) F_{s'}(\mathbf{p}') \Xi^{-1}(\omega, \mathbf{k}) (\alpha_{s'} \alpha_{s'}^\dagger)(\omega, \mathbf{k}; \mathbf{p}') \Xi^{-\dagger}(\omega, \mathbf{k}). \quad (8.12)$$

For a thermal distribution in particular, which satisfies (8.8), one can rewrite (6.24) as follows:

$$\begin{aligned} \Xi_A(\omega, \mathbf{k}) &\approx \frac{\pi}{T} \sum_s \int d\mathbf{p} \alpha_s(\omega, \mathbf{k}; \mathbf{p}) \alpha_s^\dagger(\omega, \mathbf{k}; \mathbf{p}) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_s) (\mathbf{k} \cdot \mathbf{v}_s) F_s(\mathbf{p}) \\ &= \frac{\pi\omega}{T} \sum_s \int d\mathbf{p} \alpha_s(\omega, \mathbf{k}; \mathbf{p}) \alpha_s^\dagger(\omega, \mathbf{k}; \mathbf{p}) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_s) F_s(\mathbf{p}). \end{aligned} \quad (8.13)$$

By comparing this with (8.12), one also finds that

$$\mathbf{S}(\omega, \mathbf{k}) = \frac{2T}{\omega} (\Xi^{-1} \Xi_A \Xi^{-\dagger})(\omega, \mathbf{k}). \quad (8.14)$$

[†] To make (8.10) look more physical (local), one can absorb the global factors $\mathcal{T}$ and $\mathcal{V}_n$ in the definition of the Fourier transform; cf. section 9.1.5.

Due to (6.26), this leads to the fluctuation–dissipation theorem in the following form:

$$\mathbf{S}(\omega, \mathbf{k}) = -\frac{2T}{\omega} (\boldsymbol{\Xi}^{-1})_{\text{A}}(\omega, \mathbf{k}). \quad (8.15)$$

For examples of $\boldsymbol{\Xi}$ for specific systems, see section 9.

8.3. Kirchhoff's law

Consider the power deposition via polarization drag:

$$\mathfrak{P} = \sum_s \int d\mathbf{p} (\mathbf{v}_s \cdot \boldsymbol{\mathfrak{F}}_s) F_s(\mathbf{p}). \quad (8.16)$$

Using (6.58a) for $\boldsymbol{\mathfrak{F}}_s$, (8.13) for $\boldsymbol{\Xi}_{\text{A}}$, and (8.15) for $\mathbf{S}$, this can also be expressed as follows:

$$\begin{aligned} \mathfrak{P} &\approx \sum_s \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p} (\mathbf{k} \cdot \mathbf{v}_s) (\boldsymbol{\alpha}_s^\dagger (\boldsymbol{\Xi}^{-1})_{\text{A}} \boldsymbol{\alpha}_s)(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}) F_s(\mathbf{p}) \\ &= \sum_s \int \frac{d\omega}{2\pi} \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p} 2\pi\omega \delta(\omega - \mathbf{k} \cdot \mathbf{v}_s) (\boldsymbol{\alpha}_s^\dagger (\boldsymbol{\Xi}^{-1})_{\text{A}} \boldsymbol{\alpha}_s)(\omega, \mathbf{k}; \mathbf{p}) F_s(\mathbf{p}) \\ &= 2T \int \frac{d\omega}{2\pi} \frac{d\mathbf{k}}{(2\pi)^n} \text{tr} \left((\boldsymbol{\Xi}^{-1})_{\text{A}} \frac{\pi\omega}{T} \sum_s \int d\mathbf{p} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_s) (\boldsymbol{\alpha}_s \boldsymbol{\alpha}_s^\dagger)(\omega, \mathbf{k}; \mathbf{p}) F_s(\mathbf{p}) \right) \\ &= - \int \frac{d\omega}{2\pi} \frac{d\mathbf{k}}{(2\pi)^n} \omega \text{tr}(\mathbf{S} \boldsymbol{\Xi}_{\text{A}})(\omega, \mathbf{k}). \end{aligned} \quad (8.17)$$

Thus, the spectral density of the power deposition via polarization drag is given by

$$\mathfrak{P}_{\omega, \mathbf{k}} = -\omega \text{tr}(\mathbf{S} \boldsymbol{\Xi}_{\text{A}}), \quad (8.18)$$

which is a restatement of Kirchhoff's law (Krall & Trivelpiece 1973, section 11.4). For examples of $\boldsymbol{\Xi}$ for specific systems, see section 9.

8.4. Equipartition theorem

As flows from section 7.5, the energy of on-shell waves of a field $\tilde{\Psi}$ in a homogeneous n -dimensional plasma of a given volume $\mathcal{V}_n$ can be written as

$$\begin{aligned} \mathcal{V}_n \mathcal{E}_{\text{w}} &= \int d\mathbf{k} \mathcal{V}_n w(\mathbf{k}) J(\mathbf{k}) \\ &= (2\pi)^n \int \frac{\mathcal{V}_n d\mathbf{k}}{(2\pi)^n} \int_0^\infty d\omega \omega \partial_\omega \Lambda(\omega, \mathbf{k}) h(\mathbf{k}) \delta(\omega - w(\mathbf{k})) \\ &= (2\pi)^n \sum_{\mathbf{k}} \int_0^\infty d\omega \omega \partial_\omega \Lambda(\omega, \mathbf{k}) (\boldsymbol{\eta}^\dagger \mathbf{U} \boldsymbol{\eta})(\omega, \mathbf{k}). \end{aligned} \quad (8.19)$$

To apply this to microscopic fluctuations, one can replace $\mathbf{U}$ with $\mathfrak{U}$ and substitute $\mathfrak{U} = (2\pi)^{-(n+1)} \mathbf{S}$. Then, the total energy of a mode with given wavevector $\mathbf{k}$ and polarization $\boldsymbol{\eta}$ can be expressed as

$$\mathcal{E}_{\mathbf{k}, \boldsymbol{\eta}} = \frac{1}{2\pi} \int_0^\infty d\omega \omega (\partial_\omega \Lambda) \boldsymbol{\eta}^\dagger \mathbf{S} \boldsymbol{\eta}, \quad (8.20)$$

where the arguments $(\omega, \mathbf{k})$ are omitted for brevity. For thermal equilibrium, one can substitute (8.15) for $\mathbf{S}$; then,

$$\mathcal{E}_{\mathbf{k}, \boldsymbol{\eta}} = -\frac{T}{\pi} \text{im} \int_0^\infty d\omega (\partial_\omega \Lambda) \boldsymbol{\eta}^\dagger \boldsymbol{\Xi}^{-1} \boldsymbol{\eta}. \quad (8.21)$$

The integrand peaks at $\omega = w(\mathbf{k})$, where the mode eigenvalue Λ is small. Due to damping, the actual zero of Λ is slightly below the real axis in the complex-frequency space. Then, at infinitesimally small damping, $\boldsymbol{\eta}^\dagger \boldsymbol{\Xi}^{-1} \boldsymbol{\eta}$ can be approximated near $\omega = w(\mathbf{k})$ as

$$\boldsymbol{\eta}^\dagger \boldsymbol{\Xi}^{-1} \boldsymbol{\eta} \approx \frac{1}{\Lambda} \approx \frac{1}{\partial_\omega \Lambda(\omega, \mathbf{k})} \left(\text{pv} \frac{1}{\omega - w(\mathbf{k})} - i\pi\delta(\omega - w(\mathbf{k})) \right). \quad (8.22)$$

This leads to the well-known equipartition theorem:

$$\mathcal{E}_{\mathbf{k}} = T. \quad (8.23)$$

Note that according to (8.23), the sum $\mathcal{V}_n \mathcal{E}_w = \sum_{\mathbf{k}, \boldsymbol{\eta}} \mathcal{E}_{\mathbf{k}, \boldsymbol{\eta}}$ is divergent. This indicates that not all modes can be classical and on-shell (weakly damped) simultaneously.

8.5. Summary of section 8

In thermal equilibrium, when all species have Boltzmann–Gibbs distributions with common temperature T , the collision operator vanishes, the entropy is conserved, and the spectrum of microscopic fluctuations (8.9) satisfies the fluctuation–dissipation theorem:

$$\mathbf{S}(\omega, \mathbf{k}) = -\frac{2T}{\omega} (\boldsymbol{\Xi}^{-1})_{\text{A}}(\omega, \mathbf{k}), \quad (8.24)$$

where $\boldsymbol{\Xi}$ is the dispersion matrix (6.80) and $_{\text{A}}$ denotes the anti-Hermitian part (or the imaginary part in case of scalar fields). From here, it is shown that the spectral density of the power deposition via polarization drag is given by $\mathfrak{P}_{\omega, \mathbf{k}} = -\omega \text{tr}(\mathbf{S} \boldsymbol{\Xi}_{\text{A}})$, which is a restatement of Kirchhoff’s law. For on-shell waves, (8.24) reduces to the equipartition theorem, which says that the energy per mode equals T . Applications to specific systems are discussed in section 9.

9. Examples

In this section, we show how to apply our general formulation to nonrelativistic electrostatic interactions (section 9.1), relativistic electromagnetic interactions (section 9.2), Newtonian gravity (section 9.3), and relativistic gravity, including gravitational waves (section 9.4).

9.1. Nonrelativistic electrostatic interactions

9.1.1. Main equations

Let us show how our general formulation reproduces (and generalizes) the well-known results for electrostatic turbulence in nonmagnetized nonrelativistic plasma. In this case,

$$H_s = \frac{p^2}{2m_s} + e_s \bar{\varphi} + e_s \tilde{\varphi}, \quad (9.1)$$

where e_s is the electric charge, φ is the electrostatic potential, and $\bar{\varphi}$ and $\tilde{\varphi}$ are its average and oscillating parts, respectively. Then, $H_{0s} = \bar{H}_s = p^2/(2m_s) + e_s \bar{\varphi}$, $\tilde{H}_s = e_s \tilde{\varphi}$, $\hat{\alpha}_s = e_s$, and $\hat{\mathbf{L}}_s = \hat{\mathbf{R}}_s = \hat{\mathbf{0}}$, so $\boldsymbol{\wp}_s = \mathbf{0}$. The matrix (6.78) is a scalar (Wigner function) given by

$$\mathcal{U}(t, \mathbf{x}, \omega, \mathbf{k}) = \int \frac{d\tau}{2\pi} \frac{d\mathbf{s}}{(2\pi)^n} \overline{\tilde{\varphi}(t + \tau/2, \mathbf{x} + \mathbf{s}/2)} \tilde{\varphi}(t - \tau/2, \mathbf{x} - \mathbf{s}/2) e^{i\omega\tau - i\mathbf{k} \cdot \mathbf{s}}. \quad (9.2)$$

(Underlining denotes the macroscopic part, $n \doteq \dim \mathbf{x}$, and the arguments $(t, \mathbf{x})$ will be omitted from now on.) Correspondingly,

$$\mathbf{D}_s \approx e_s^2 \int d\mathbf{k} \pi \mathbf{k} \mathbf{k} \mathbf{U}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}), \quad (9.3)$$

$$\Theta_s = e_s^2 \frac{\partial}{\partial \vartheta} \oint d\omega d\mathbf{k} \frac{\mathbf{k} \mathbf{k} \mathbf{U}(\omega, \mathbf{k})}{\omega - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \Big|_{\vartheta=0}, \quad (9.4)$$

$$\Delta_s = \Phi_s = \frac{e_s^2}{2} \frac{\partial}{\partial \mathbf{p}} \cdot \oint d\omega d\mathbf{k} \frac{\mathbf{k} \mathbf{U}(\omega, \mathbf{k})}{\omega - \mathbf{k} \cdot \mathbf{v}_s}, \quad (9.5)$$

and also

$$\mathcal{H}_s = \frac{p^2}{2m_s} + e_s \bar{\varphi} + \Delta_s, \quad \mathbf{v}_s = \frac{\mathbf{p}}{m_s} + \frac{\partial \Delta_s}{\partial \mathbf{p}}. \quad (9.6)$$

The Lagrangian density of a free electrostatic field is

$$\mathfrak{L}_0 = \frac{1}{8\pi} \delta^{ij} (\partial_i \tilde{\varphi})(\partial_j \tilde{\varphi}) = \frac{\partial}{\partial x^i} \left(\frac{1}{8\pi} \delta^{ij} \tilde{\varphi} (\partial_j \tilde{\varphi}) \right) + \frac{1}{2} \tilde{\varphi} \left(-\frac{\delta^{ij} \partial_i \partial_j}{4\pi} \right) \tilde{\varphi}. \quad (9.7)$$

The first term on the right-hand side does not contribute to the field action S_0 and thus can be ignored. The second term is of the form (6.1) with $M = 1$, $\mathbf{g} = 1$ (section 2.1.2), and $\hat{\Xi}_0 = \hat{k}^2/(4\pi)$, so $\Xi_0(\omega, \mathbf{k}) = k^2/(4\pi)$, where $k^2 \equiv \mathbf{k}^2 \equiv \delta^{ij} k_i k_j$. Then, (6.20) gives

$$\Xi(\omega, \mathbf{k}) = \Xi(\omega, \mathbf{k}) = \frac{k^2 \epsilon_{\parallel}(\omega, \mathbf{k})}{4\pi}, \quad (9.8)$$

where the arguments t and $\mathbf{x}$ are omitted for brevity and $\epsilon_{\parallel}$ is the parallel permittivity:

$$\epsilon_{\parallel}(\omega, \mathbf{k}) = 1 + \sum_s \frac{4\pi e_s^2}{k^2} \int d\mathbf{p} \frac{\mathbf{k}}{\omega - \mathbf{k} \cdot \mathbf{v}_s + i0} \cdot \frac{\partial F_s}{\partial \mathbf{p}}. \quad (9.9)$$

9.1.2. Collisions and fluctuations

By (8.12), the spectrum of microscopic oscillations of $\tilde{\varphi}$ is a scalar given by

$$S(\omega, \mathbf{k}) = 2\pi \sum_s \left(\frac{4\pi e_s}{k^2 |\epsilon_{\parallel}(\omega, \mathbf{k})|} \right)^2 \int d\mathbf{p} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_s) F_s(\mathbf{p}), \quad (9.10)$$

where we substituted $n = 3$ for three-dimensional plasma. For thermal equilibrium, (8.15) leads to the well-known formula (Lifshitz & Pitaevskii 1981, section 51)

$$S(\omega, \mathbf{k}) = -\frac{2T}{\omega} \operatorname{im} \left(\frac{1}{\Xi(\omega, \mathbf{k})} \right) = -\frac{8\pi T}{\omega k^2} \operatorname{im} \left(\frac{1}{\epsilon_{\parallel}(\omega, \mathbf{k})} \right) = \frac{8\pi T}{\omega k^2} \frac{\operatorname{im} \epsilon_{\parallel}(\omega, \mathbf{k})}{|\epsilon_{\parallel}(\omega, \mathbf{k})|^2}. \quad (9.11)$$

The spectrum S_{ρ} of charge-density fluctuations is found using Poisson's equation $\tilde{\rho} = \hat{k}^2 \tilde{\varphi}/4\pi$, whence $S_{\rho} \approx (k^2/4\pi)^2 S$. Fluctuations of other fields are found similarly. Also, (6.36) leads to

$$|\mathcal{X}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2 = \left(\frac{4\pi e_s e_{s'}}{k^2 |\epsilon_{\parallel}(\omega, \mathbf{k})|} \right)^2. \quad (9.12)$$

Then, (6.61) yields the standard Balescu–Lenard collision operator:

$$\mathcal{C}_s = \frac{\partial}{\partial \mathbf{p}} \cdot \sum_{s'} \int \frac{d\mathbf{k}}{(2\pi)^3} d\mathbf{p}' \frac{\pi \mathbf{k} \mathbf{k}}{|\epsilon_{\parallel}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k})|^2} \left(\frac{4\pi e_s e_{s'}}{k^2} \right)^2 \delta(\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}'_{s'})$$

$$\cdot \left(\frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}} F_{s'}(\mathbf{p}') - F_s(\mathbf{p}) \frac{\partial F_{s'}(\mathbf{p}')}{\partial \mathbf{p}'} \right). \quad (9.13)$$

(As a reminder, the distribution functions are normalized such that $\int d\mathbf{p} F_s$ is the local average density of species s (5.39).)

9.1.3. On-shell waves

For on-shell waves, (7.63) gives $U(\omega, \mathbf{k}) = (h(\mathbf{k}) + h(-\mathbf{k}))\delta(\omega - w(\mathbf{k}))$, where $w(\mathbf{k})$ is determined by the dispersion relation

$$\epsilon_{\parallel H}(w(\mathbf{k}), \mathbf{k}) = 0, \quad (9.14)$$

and $\epsilon_{\parallel H} \equiv \text{re } \epsilon_{\parallel}$ is given by

$$\epsilon_{\parallel H}(\omega, \mathbf{k}) = 1 + \sum_s \frac{4\pi e_s^2}{k^2} \oint d\mathbf{p} \frac{\mathbf{k}}{\omega - \mathbf{k} \cdot \mathbf{v}_s} \cdot \frac{\partial F_s}{\partial \mathbf{p}}. \quad (9.15)$$

The phase-space density of the wave action, defined in (7.74), is

$$J(\mathbf{k}) = h(\mathbf{k}) \frac{\partial \Xi_H(\mathbf{k})}{\partial \omega} = h(\mathbf{k}) \frac{k^2}{4\pi} \frac{\partial \epsilon_{\parallel H}(w(\mathbf{k}), \mathbf{k})}{\partial \omega}, \quad (9.16)$$

and the dressing function (9.4) is given by

$$\begin{aligned} \Theta_s &= e_s^2 \frac{\partial}{\partial \vartheta} \int d\mathbf{k} (h(\mathbf{k}) + h(-\mathbf{k})) \frac{\mathbf{k}\mathbf{k}}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \Big|_{\vartheta=0} \\ &= 2e_s^2 \frac{\partial}{\partial \vartheta} \int d\mathbf{k} h(\mathbf{k}) \frac{\mathbf{k}\mathbf{k}}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \Big|_{\vartheta=0}. \end{aligned} \quad (9.17)$$

Using these, one obtains (appendix F.1.1)

$$\sum_s \int d\mathbf{p} \mathbf{p} F_s + \int d\mathbf{k} \mathbf{k} J = \sum_s \int d\mathbf{p} \mathbf{p} \bar{f}_s, \quad (9.18)$$

so the conserved quantity (7.88) is the average momentum of the plasma (while the electrostatic field carries no momentum, naturally). Also (appendix F.1.2),

$$\sum_s \int d\mathbf{p} H_{0s} F_s + \int d\mathbf{k} w J = \sum_s \int d\mathbf{p} H_{0s} \bar{f}_s + \frac{1}{8\pi} \widetilde{\mathbf{E}^\dagger} \widetilde{\mathbf{E}}, \quad (9.19)$$

so, expectedly, the conserved quantity (7.90) is the average particle energy plus the energy of the electrostatic field. In combination with our equations for F_s and J (section 7.6), these results can be considered as a generalization and concise restatement of the OC QLT by Dewar (1973), which is rigorously reproduced from our general formulation as a particular case.

9.1.4. Eikonal waves

As a particular case, let us consider an eikonal wave

$$\tilde{\varphi} \approx \text{re}(e^{i\theta} \check{\varphi}), \quad \bar{\omega} \doteq -\partial_t \theta, \quad \bar{\mathbf{k}} \doteq \partial_{\mathbf{x}} \theta, \quad (9.20)$$

which may or may not be on-shell. As seen from section 7.4.1,

$$U \approx \frac{|\check{\varphi}|^2}{4} \sum_{\pm} \delta(\omega \pm \bar{\omega}) \delta(\mathbf{k} \pm \bar{\mathbf{k}}). \quad (9.21)$$

For nonresonant particles, the dressing function is well defined is found as follows:

$$\begin{aligned}\Theta_s &\approx - \int d\omega d\mathbf{k} \frac{e_s^2 \mathbf{k} \mathbf{k} |\check{\varphi}|^2}{4(\omega - \mathbf{k} \cdot \mathbf{v}_s)^2} \sum_{\pm} \delta(\omega \pm \bar{\omega}) \delta(\mathbf{k} \pm \bar{\mathbf{k}}) \\ &= - \frac{e_s^2 \bar{\mathbf{k}} \bar{\mathbf{k}} |\check{\varphi}|^2}{2(\bar{\omega} - \bar{\mathbf{k}} \cdot \mathbf{v}_s)^2}.\end{aligned}\quad (9.22)$$

Similarly, the ponderomotive energy for nonresonant particles is

$$\begin{aligned}\Delta_s &\approx \frac{e_s^2 |\check{\varphi}|^2}{8m_s} \frac{\partial}{\partial \mathbf{v}_s} \cdot \int d\omega d\mathbf{k} \frac{\mathbf{k}}{\omega - \mathbf{k} \cdot \mathbf{v}_s} \sum_{\pm} \delta(\omega \pm \bar{\omega}) \delta(\mathbf{k} \pm \bar{\mathbf{k}}) \\ &= \frac{e_s^2 |\check{\varphi}|^2}{8m_s} \int d\omega d\mathbf{k} \frac{k^2}{(\omega - \mathbf{k} \cdot \mathbf{v}_s)^2} \sum_{\pm} \delta(\omega \pm \bar{\omega}) \delta(\mathbf{k} \pm \bar{\mathbf{k}}) \\ &= \frac{e_s^2 \bar{k}^2 |\check{\varphi}|^2}{4m_s (\bar{\omega} - \bar{\mathbf{k}} \cdot \mathbf{v}_s)^2},\end{aligned}\quad (9.23)$$

in agreement with (Dewar 1972; Cary & Kaufman 1977). One can also express these functions in terms of the electric-field envelope $\check{\mathbf{E}} \approx -i\bar{\mathbf{k}}\check{\varphi}$:

$$\Theta_s \approx - \frac{e_s^2 \check{\mathbf{E}} \check{\mathbf{E}}^\dagger}{2(\bar{\omega} - \bar{\mathbf{k}} \cdot \mathbf{v}_s)^2}, \quad \Delta_s \approx \frac{e_s^2 |\check{\mathbf{E}}|^2}{4m_s (\bar{\omega} - \bar{\mathbf{k}} \cdot \mathbf{v}_s)^2}.\quad (9.24)$$

For on-shell in particular, one can use (9.16) together with $h(\mathbf{k}) = \frac{1}{4}|\check{\varphi}|^2 \delta(\mathbf{k} - \bar{\mathbf{k}})$ (cf. (7.64)) to obtain the well-known expression for the wave action density $\mathcal{I} \doteq \int d\mathbf{k} J$:

$$\mathcal{I} = \frac{|\check{\mathbf{E}}|^2}{16\pi} \frac{\partial \epsilon_{\parallel\text{H}}(\omega, \bar{\mathbf{k}})}{\partial \omega} \Big|_{\omega=w(\bar{\mathbf{k}})}.\quad (9.25)$$

For non-too-hot plasma, one has $\epsilon_{\parallel\text{H}}(\omega, \mathbf{k}) \approx 1 - \omega_p^2/\omega^2$, where $\omega_p \doteq \sum_s 4\pi N_s e_s^2/m_s$ is the plasma frequency. The corresponding waves are Langmuir waves. Their dispersion relation is $w(\bar{\mathbf{k}}) \approx \pm \omega_p$, so $\mathcal{I} \approx \pm |\check{\mathbf{E}}|^2/(8\pi\omega_p)$ (and accordingly, the wave energy density is $\mathcal{E}_w = w\mathcal{I} \geq 0$ for either sign). Remember, though, that this expression is only approximate. Using it instead of (9.25) can result in violation of the exact conservation laws of QLT. Conservation of the Langmuir-wave action in non-stationary plasmas beyond the cold-plasma approximation is also discussed in (Dodin *et al.* 2009; Dodin & Fisch 2010b; Schmit *et al.* 2010).

9.1.5. Homogeneous plasma

In homogeneous n -dimensional plasma of a given volume $\mathcal{V}_n$, the Wigner function (9.2) has the form $\mathcal{U} = \mathcal{U}(t, \mathbf{k}) \delta(\omega - w(t, \mathbf{k}))$. The function $\mathcal{U}$ is readily found using (5.47):

$$\mathcal{U}(t, \mathbf{k}) = \frac{1}{\mathcal{V}_n} \int d\mathbf{x} d\omega U = \frac{1}{\mathcal{V}_n} \overline{|\check{\varphi}(t, \mathbf{k})|^2} = \frac{1}{\mathcal{V}_n} |\check{\varphi}(t, \mathbf{k})|^2.\quad (9.26)$$

Then,

$$\mathbf{D}_s \approx \frac{\pi e_s^2}{\mathcal{V}_n} \int d\mathbf{k} \mathbf{k} \mathbf{k} |\check{\varphi}(t, \mathbf{k})|^2 \delta(w(t, \mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s).\quad (9.27)$$

This coincides with the well-known formula for the QL-diffusion coefficient in homogeneous electrostatic plasma.[†] The functions Θ_s and Δ_s are also important in homogeneous

[†] See, for example, equation (16.17) in (Stix 1992). The extra mass factor appears there because QL diffusion is considered in the velocity space instead of the momentum space.

turbulence in that they ensure the proper energy–momentum conservation; for example, see (Stix 1992, section 16.3) and (Liu & Dodin 2015, section II.2). These functions can be expressed through $\tilde{\varphi}$ too. However, they have a simpler representation in terms of the Wigner function $\mathbf{U}$, as in (9.4) and (9.5), respectively. This is because $\mathbf{U}$ is a *local* property of the field, which makes it more fundamental than the amplitudes of global Fourier harmonics commonly used in the literature.

9.2. Relativistic electromagnetic interactions

9.2.1. Main equations

Let us extend the above results to relativistic electromagnetic interactions. In this case,

$$H_s = \sqrt{m_s^2 c^4 + (\mathbf{p}c - e_s \mathbf{A})^2} + e_s \varphi, \quad (9.28)$$

where c is the speed of light and $\mathbf{A}$ is the vector potential. Let us adopt the Weyl gauge for the oscillating part of the electromagnetic field ($\tilde{\varphi} = 0$) and Taylor-expand H_s to the second order in $\mathbf{A}$. This leads to

$$H_s \approx H_{0s} - e_s \beta_s^\dagger \tilde{\mathbf{A}} + \frac{e_s^2}{2c^2} \tilde{\mathbf{A}}^\dagger \boldsymbol{\mu}_s^{-1} \tilde{\mathbf{A}}, \quad (9.29)$$

$$H_{0s} = \sqrt{m_s^2 c^4 + (\mathbf{p}c - e_s \overline{\mathbf{A}})^2} + e_s \overline{\varphi} \quad (9.30)$$

(although plasma is assumed nonmagnetized, a *weak* average magnetic field $\overline{\mathbf{B}} = \nabla \times \overline{\mathbf{A}}$ is allowed, so $\overline{\mathbf{A}}$ can be order-one and thus generally must be retained), where

$$\boldsymbol{\beta}_s = \frac{1}{m_s c \gamma_s} \left(\mathbf{p} - \frac{e_s}{c} \overline{\mathbf{A}} \right), \quad \boldsymbol{\mu}_s^{-1} = \frac{1 - \boldsymbol{\beta}_s \boldsymbol{\beta}_s^\dagger}{m_s \gamma_s}, \quad (9.31)$$

and $\gamma_s \doteq (1 - \beta_s^2)^{-1/2}$. In the equations presented below, $\boldsymbol{\beta}_s = \mathbf{v}_s/c$ (where $\mathbf{v}_s$ is the OC velocity) is a sufficiently accurate approximation. Also, $\boldsymbol{\mu}_s \equiv (\partial_{\mathbf{p}\mathbf{p}}^2 H_{0s})^{-1}$ can be interpreted as the relativistic-mass tensor.

Let us choose the field $\tilde{\Psi}$ of our general theory to be the oscillating electric field $\tilde{\mathbf{E}} = i\hat{\omega} \tilde{\mathbf{A}}/c$; then (cf. (6.2)),

$$\hat{\boldsymbol{\alpha}}_s = i e_s \mathbf{v}_s \hat{\omega}^{-1}, \quad \hat{\mathbf{L}}_s = e_s^2 \hat{\omega}^{-1}, \quad \hat{\mathbf{R}}_s = \boldsymbol{\mu}_s^{-1} \hat{\omega}^{-1}. \quad (9.32)$$

(Other ways to identify $\hat{\mathbf{L}}_s$ and $\hat{\mathbf{R}}_s$ are also possible and lead to the same results.) Then,

$$\boldsymbol{\alpha}_s = \frac{i e_s \mathbf{v}_s}{\omega}, \quad \boldsymbol{\wp}_s = \frac{e_s^2}{\omega^2} \boldsymbol{\mu}_s^{-1}. \quad (9.33)$$

The average Wigner matrix of $\tilde{\mathbf{E}}$ is

$$\mathbf{U}(t, \mathbf{x}, \omega, \mathbf{k}) = \int \frac{d\tau}{2\pi} \frac{d\mathbf{s}}{(2\pi)^3} \overline{\tilde{\mathbf{E}}(t + \tau/2, \mathbf{x} + \mathbf{s}/2)} \tilde{\mathbf{E}}^\dagger(t - \tau/2, \mathbf{x} - \mathbf{s}/2) e^{i\omega\tau - i\mathbf{k} \cdot \mathbf{s}} \quad (9.34)$$

(the arguments t and $\mathbf{x}$ are henceforth omitted), and the nonlinear potentials are

$$\mathbf{D}_s = \pi e_s^2 \int d\mathbf{k} \, \mathbf{k} \mathbf{k} \frac{\mathbf{v}_s^\dagger \mathbf{U}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}) \mathbf{v}_s}{(\mathbf{k} \cdot \mathbf{v}_s)^2}, \quad (9.35)$$

$$\boldsymbol{\Theta}_s = e_s^2 \frac{\partial}{\partial \vartheta} \oint d\omega d\mathbf{k} \frac{\mathbf{k} \mathbf{k}}{\omega^2} \frac{(\mathbf{v}_s^\dagger \mathbf{U} \mathbf{v}_s)}{\omega - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \Big|_{\vartheta=0}, \quad (9.36)$$

$$\Delta_s = \frac{e_s^2}{2} \frac{\partial}{\partial \mathbf{p}} \cdot \oint d\omega d\mathbf{k} \frac{\mathbf{k}}{\omega^2} \frac{(\mathbf{v}_s^\dagger \mathbf{U} \mathbf{v}_s)}{\omega - \mathbf{k} \cdot \mathbf{v}_s} + \frac{e_s^2}{2} \int d\omega d\mathbf{k} \frac{\text{tr}(\mathbf{U} \boldsymbol{\mu}_s^{-1})}{\omega^2}. \quad (9.37)$$

When plasma is nonrelativistic and the field is electrostatic (so $\mathbf{U} = \mathbf{k} \mathbf{k}^\dagger \mathbf{U}_\varphi$, where $\mathbf{U}_\varphi$ is

scalar), (9.35) gives the same $\mathbf{D}_s$ as (9.3) and (9.37) gives the same Δ_s as (9.5). For Θ_s , the equivalence between (9.36) and (9.4) should not be expected because Θ_s is a part of a distribution function, which is not gauge-invariant. (Canonical momenta in the Weyl gauge are different from those in the electrostatic gauge.) But it is precisely the dressing function (9.36) that leads to the correct expressions for the momentum and energy stored in the OC distribution (section 9.2.3).

The Lagrangian density of a free electromagnetic field is

$$\mathfrak{L}_0 = \frac{\tilde{\mathbf{E}}^\dagger \tilde{\mathbf{E}} - \tilde{\mathbf{B}}^\dagger \tilde{\mathbf{B}}}{8\pi}. \quad (9.38)$$

From Faraday's law, one has $\tilde{\mathbf{B}} = \hat{\omega}^{-1} c (\hat{\mathbf{k}} \times \mathbf{E})$.[†] Then, $-\tilde{\mathbf{B}}^\dagger \tilde{\mathbf{B}}/c^2$ can be represented as follows (up to a divergence, which is insignificant):

$$\begin{aligned} (\tilde{\mathbf{E}} \times \hat{\omega}^{-1} \hat{\mathbf{k}}) \cdot (\hat{\omega}^{-1} \hat{\mathbf{k}} \times \tilde{\mathbf{E}}) &= \tilde{\mathbf{E}} \cdot \hat{\omega}^{-2} (\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \tilde{\mathbf{E}})) \\ &= \tilde{\mathbf{E}}^\dagger \hat{\omega}^{-2} (\hat{\mathbf{k}} (\hat{\mathbf{k}} \cdot \tilde{\mathbf{E}}) - \tilde{\mathbf{E}} \hat{k}^2) \\ &= \tilde{\mathbf{E}}^\dagger \hat{\omega}^{-2} (\hat{\mathbf{k}} \hat{\mathbf{k}}^\dagger - \mathbf{1} \hat{k}^2) \tilde{\mathbf{E}}. \end{aligned} \quad (9.39)$$

Then, the vacuum dispersion operator can be written as (cf. (6.1))

$$\hat{\Xi}_0(\omega, \mathbf{k}) = \frac{1}{4\pi} \left(\mathbf{1} + c^2 \hat{\omega}^{-2} (\hat{\mathbf{k}} \hat{\mathbf{k}}^\dagger - \mathbf{1} \hat{k}^2) \right). \quad (9.40)$$

The total dispersion matrix is readily found to be

$$\Xi(\omega, \mathbf{k}) = \frac{1}{4\pi} \left(\epsilon(\omega, \mathbf{k}) + \frac{c^2}{\omega^2} (\mathbf{k} \mathbf{k}^\dagger - \mathbf{1} k^2) \right), \quad (9.41)$$

where ϵ (not to be confused with the small parameter ϵ that we introduced earlier) is the dielectric tensor:

$$\epsilon(\omega, \mathbf{k}) = \mathbf{1} - \frac{\mathfrak{w}_p}{\omega^2} + \sum_s \frac{4\pi e_s^2}{\omega^2} \int d\mathbf{p} \frac{\mathbf{v}_s \mathbf{v}_s^\dagger}{\omega - \mathbf{k} \cdot \mathbf{v}_s + i0} \mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{p}}. \quad (9.42)$$

Here, $\mathfrak{w}_p$ is the squared relativistic plasma frequency, which is a matrix, because the 'masses' μ_s are matrices:

$$\mathfrak{w}_p \doteq \sum_s 4\pi e_s^2 \int d\mathbf{p} F_s \mu_s^{-1}. \quad (9.43)$$

9.2.2. Collisions and fluctuations

By (8.12), the spectrum of microscopic oscillations of $\tilde{\mathbf{E}}$ is a matrix given by

$$\mathbf{S}(\omega, \mathbf{k}) = 2\pi \sum_s \left(\frac{4\pi e_s}{\omega} \right)^2 \int d\mathbf{p} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_s) F_s(\mathbf{p}) \epsilon^{-1}(\omega, \mathbf{k}) \mathbf{v}_s \mathbf{v}_s^\dagger \epsilon^{-\dagger}(\omega, \mathbf{k}). \quad (9.44)$$

In the electrostatic limit, one can replace ϵ^{-1} with $\epsilon_{\parallel}^{-1} \mathbf{k} \mathbf{k}^\dagger / k^2$, where $\epsilon_{\parallel}$ is the relativistic generalization of (9.9); then (9.44) leads to (9.10) as a particular case. For thermal equilibrium, one can also use (8.15) and the following form of ϵ^{-1} for isotropic plasma:

$$\epsilon^{-1} = \frac{1}{\epsilon_{\perp}} \left(\mathbf{1} - \frac{\mathbf{k} \mathbf{k}^\dagger}{k^2} \right) + \frac{1}{\epsilon_{\parallel}} \frac{\mathbf{k} \mathbf{k}^\dagger}{k^2}, \quad (9.45)$$

[†] Here, the oscillating field $\tilde{\Psi} = \tilde{\mathbf{E}}$ has the same dimension as $\mathbf{x}$, so the standard vector notation (including the dot product and the cross product) is naturally extended to $\tilde{\Psi}$.

where $\epsilon_\perp$ is the (scalar) transverse permittivity. Also, (6.36) leads to

$$|\mathcal{X}_{ss'}(\omega, \mathbf{k}; \mathbf{p}, \mathbf{p}')|^2 \approx \left(\frac{4\pi e_s e_{s'}}{\omega^2} \right)^2 |\mathbf{v}_s^\dagger \epsilon^{-1}(\omega, \mathbf{k}) \mathbf{v}_{s'}'|^2. \quad (9.46)$$

Then the collision operator (6.61) is obtained in the form

$$\begin{aligned} \mathcal{C}_s = \frac{\partial}{\partial \mathbf{p}} \cdot \sum_{s'} 2e_s^2 e_{s'}^2 \int \frac{d\mathbf{k}}{(2\pi)^3} d\mathbf{p}' \frac{|\mathbf{v}_s^\dagger \epsilon^{-1}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}) \mathbf{v}_{s'}'|^2}{(\mathbf{k} \cdot \mathbf{v}_s)^4} \delta(\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}_{s'}) \\ \times \mathbf{k} \mathbf{k} \cdot \left(\frac{\partial F_s(\mathbf{p})}{\partial \mathbf{p}} F_{s'}(\mathbf{p}') - F_s(\mathbf{p}) \frac{\partial F_{s'}(\mathbf{p}')}{\partial \mathbf{p}'} \right), \end{aligned} \quad (9.47)$$

which is in agreement with (Hizanidis *et al.* 1983; Silin 1961). Replacing ϵ^{-1} with $\epsilon_\parallel^{-1} \mathbf{k} \mathbf{k}^\dagger / k^2$ leads to the standard Balescu–Lenard operator (9.13) as a particular case.

9.2.3. On-shell waves

Electromagnetic on-shell waves satisfy

$$\left(\epsilon_H(w(\mathbf{k}), \mathbf{k}) + \frac{c^2}{w(\mathbf{k})^2} (\mathbf{k} \mathbf{k}^\dagger - 1k^2) \right) \check{\mathbf{E}} = 0, \quad (9.48)$$

where $\check{\mathbf{E}}$ is the complex envelope vector parallel to the polarization vector $\boldsymbol{\eta}$; also,

$$\epsilon_H(\omega, \mathbf{k}) = 1 - \frac{\mathbf{w}_p}{\omega^2} + \sum_s \frac{4\pi e_s^2}{\omega^2} \oint d\mathbf{p} \frac{\mathbf{v}_s \mathbf{v}_s^\dagger}{\omega - \mathbf{k} \cdot \mathbf{v}_s} \mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{p}}. \quad (9.49)$$

This yields (see (9.39))

$$\check{\mathbf{E}}^\dagger \epsilon_H(w(\mathbf{k}), \mathbf{k}) \check{\mathbf{E}} = -\frac{c^2}{w(\mathbf{k})^2} \check{\mathbf{E}}^\dagger (\mathbf{k} \mathbf{k}^\dagger - 1k^2) \check{\mathbf{E}} = \check{\mathbf{B}}^\dagger \check{\mathbf{B}}. \quad (9.50)$$

Then, the phase-space density of the wave action (7.74) can be cast in the form

$$J(\mathbf{k}) = h(\mathbf{k}) \boldsymbol{\eta}^\dagger \frac{\partial \Xi_H(\omega, \mathbf{k})}{\partial \omega} \boldsymbol{\eta} \Big|_{\omega=w(\bar{\mathbf{k}})} = \frac{h(\mathbf{k})}{4\pi\omega^2} \boldsymbol{\eta}^\dagger \frac{\partial (\omega^2 \epsilon_H(\omega, \mathbf{k}))}{\partial \omega} \boldsymbol{\eta} \Big|_{\omega=w(\bar{\mathbf{k}})} \quad (9.51)$$

(cf. (Dodin *et al.* 2019)), and the dressing function (9.36) is given by

$$\begin{aligned} \Theta_s = e_s^2 \frac{\partial}{\partial \vartheta} \oint d\mathbf{k} (h(\mathbf{k}) + h(-\mathbf{k})) \frac{\mathbf{k} \mathbf{k}}{w^2(\mathbf{k})} \frac{(\mathbf{v}_s^\dagger \boldsymbol{\eta} \boldsymbol{\eta}^\dagger \mathbf{v}_s)}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \Big|_{\vartheta=0} \\ = 2e_s^2 \frac{\partial}{\partial \vartheta} \oint d\mathbf{k} h(\mathbf{k}) \frac{\mathbf{k} \mathbf{k}}{w^2(\mathbf{k})} \frac{(\boldsymbol{\eta}^\dagger \mathbf{v}_s \mathbf{v}_s^\dagger \boldsymbol{\eta})}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \Big|_{\vartheta=0}. \end{aligned} \quad (9.52)$$

Using these, one obtains (appendix F.2.1)

$$\sum_s \int d\mathbf{p} \mathbf{p} F_s + \int d\mathbf{k} \mathbf{k} J = \mathcal{P}^{(\text{kin})} + \frac{\overline{\check{\mathbf{E}} \times \check{\mathbf{B}}}}{4\pi c}, \quad (9.53)$$

where $\mathcal{P}^{(\text{kin})}$ is the average density of the plasma *kinetic* (up to $\overline{\mathbf{A}}$) momentum,

$$\mathcal{P}^{(\text{kin})} \doteq \sum_s \int d\mathbf{p} \overline{(\mathbf{p} - e_s \tilde{\mathbf{A}}/c) f_s} = \sum_s \int d\mathbf{p} \mathbf{p} \overline{f_s^{(\text{kin})}}, \quad (9.54)$$

the functions $f_s^{(\text{kin})}(\mathbf{p}) \doteq f_s(\mathbf{p} + e_s \tilde{\mathbf{A}}/c)$ are the distributions of kinetic (up to $\overline{\mathbf{A}}$) momenta, and the second term in (9.53) is the well-known average momentum of electro-

magnetic field. Similarly (appendix F.2.2),

$$\sum_s \int d\mathbf{p} H_{0s} F_s + \int d\mathbf{k} w J = \mathcal{K}^{(\text{kin})} + \frac{1}{8\pi} \overline{(\tilde{\mathbf{E}}^\dagger \tilde{\mathbf{E}} + \tilde{\mathbf{B}}^\dagger \tilde{\mathbf{B}})}, \quad (9.55)$$

where $\mathcal{K}^{(\text{kin})}$ is given by

$$\mathcal{K}^{(\text{kin})} \doteq \sum_s \int d\mathbf{p} H_{0s} \overline{f_s^{(\text{kin})}}. \quad (9.56)$$

In other words, the total momentum and energy of the system in the OC-wave representation are the same as those in the original particle-field variables.

9.2.4. Eikonal waves

As a particular case, let us consider an eikonal wave

$$\tilde{\mathbf{E}} \approx \text{re}(e^{i\theta} \check{\mathbf{E}}), \quad \bar{\omega} \doteq -\partial_t \theta, \quad \bar{\mathbf{k}} \doteq \partial_{\mathbf{x}} \theta, \quad (9.57)$$

which may or may not be on-shell. Then, (9.36) and (9.37) lead to (cf. section 9.1.4)

$$\Theta_s = -\frac{\bar{\mathbf{k}} \bar{\mathbf{k}}}{\bar{\omega}^2} \frac{e_s^2 |\mathbf{v}_s^\dagger \check{\mathbf{E}}|^2}{2(\bar{\omega} - \bar{\mathbf{k}} \cdot \mathbf{v}_s)^2}, \quad (9.58)$$

$$\Delta_s = \frac{e_s^2 \check{\mathbf{E}}^\dagger \mu_s^{-1} \check{\mathbf{E}}}{4\bar{\omega}^2} + \frac{e_s^2 \bar{\mathbf{k}}}{4\bar{\omega}^2} \cdot \frac{\partial}{\partial \mathbf{p}} \left(\frac{|\mathbf{v}_s^\dagger \check{\mathbf{E}}|^2}{\bar{\omega} - \bar{\mathbf{k}} \cdot \mathbf{v}_s} \right). \quad (9.59)$$

For on-shell waves in particular, one also obtains the action density in the form

$$\begin{aligned} \mathcal{I} &= \frac{1}{16\pi\omega^2} \check{\mathbf{E}}^\dagger \frac{\partial(\omega^2 \epsilon_H(\omega, \mathbf{k}))}{\partial \omega} \check{\mathbf{E}} \Big|_{\omega=w(\bar{\mathbf{k}})} \\ &= \frac{1}{16\pi\omega} \left(\check{\mathbf{E}}^\dagger \frac{\partial(\omega \epsilon_H(\omega, \mathbf{k}))}{\partial \omega} \check{\mathbf{E}} + \check{\mathbf{B}}^\dagger \check{\mathbf{B}} \right) \Big|_{\omega=w(\bar{\mathbf{k}})}, \end{aligned} \quad (9.60)$$

where we used (9.50).

9.3. Newtonian gravity

For Newtonian interactions governed by a gravitostatic potential φ_g , one has

$$H_s = \frac{p^2}{2m_s} + m_s \bar{\varphi}_g + m_s \tilde{\varphi}_g, \quad \mathfrak{L}_0 = -\frac{(\nabla \tilde{\varphi}_g)^2}{8\pi\mathbf{G}}, \quad (9.61)$$

where $\mathbf{G}$ is the gravitational constant. This system is identical to that considered in section 9.1 for nonrelativistic electrostatic interactions up to coefficients. Specifically, e_s are replaced with m_s , a factor $-\mathbf{G}^{-1}$ appears in Ξ_0 , and the dispersion matrix becomes

$$\Xi(\omega, \mathbf{k}) = \Xi(\omega, \mathbf{k}) = -\frac{k^2 \epsilon_g(\omega, \mathbf{k})}{4\pi\mathbf{G}}. \quad (9.62)$$

Thus, $\epsilon_{||}$ is replaced with $-\epsilon_g/\mathbf{G}$, where ϵ_g is the gravitostatic permittivity given by

$$\epsilon_g(\omega, \mathbf{k}) = 1 - \sum_s \frac{4\pi\mathbf{G}m_s^2}{k^2} \int d\mathbf{p} \frac{\mathbf{k}}{\omega - \mathbf{k} \cdot \mathbf{v}_s + i0} \cdot \frac{\partial F_s}{\partial \mathbf{p}}. \quad (9.63)$$

This readily yields, for example, kinetic theory of the Jeans instability (Trigger *et al.* 2004), whose dispersion relation is given by $\epsilon_g(\omega, \mathbf{k}) = 0$ (modulo the usual analytic continuation of the permittivity to modes with $\text{im } \omega < 0$).

9.4. Relativistic gravity

9.4.1. Main equations

The dynamics of a relativistic neutral particle with mass m in a spacetime metric $g_{\alpha\beta}$ with signature $(-+++)$ is governed by a covariant Hamiltonian (see, for example, (Garg & Dodin 2020))

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m} (m^2 + g^{\alpha\beta}(\mathbf{x})p_\alpha p_\beta) \equiv H(\mathbf{g}, \mathbf{p}). \quad (9.64)$$

Here, $\mathbf{x} \equiv (x^0, \mathbf{x})$, and $x^0 = t$, as usual. Also, $\mathbf{p} \equiv (p_0, \mathbf{p})$ is the index-free notation for the four-momentum p_α , $g^{\alpha\beta}$ is the inverse metric, $\mathbf{g}$ is the index-free notation for $g^{\alpha\beta}$, the units are such that $c = 1$, and the species index is omitted.[†] The corresponding Hamilton's equations, with τ the proper time, are

$$\frac{dx^\alpha}{d\tau} = \frac{\partial H}{\partial p_\alpha}, \quad \frac{dp_\alpha}{d\tau} = -\frac{\partial H}{\partial x^\alpha}. \quad (9.65)$$

This dynamics is constrained to the shell $p_0 = P_0(t, \mathbf{x}, \mathbf{p})$, where P_0 is the (negative) solution of

$$H(\mathbf{g}, P_0(t, \mathbf{x}, \mathbf{p}), \mathbf{p}) = 0. \quad (9.66)$$

This means that the particle distribution in the $(\mathbf{x}, \mathbf{p})$ space is delta-shaped and thus does not satisfy (3.35). Hence, we will consider particles in the six-dimensional space $(\mathbf{x}, \mathbf{p})$ instead. The corresponding dynamics is governed by the Hamiltonian

$$H = -P_0(t, \mathbf{x}, \mathbf{p}). \quad (9.67)$$

This is seen from the fact that

$$\frac{\partial H}{\partial \blacksquare} = -\frac{\partial P_0}{\partial \blacksquare} = \frac{\partial H / \partial \blacksquare}{\partial H / \partial p_0}, \quad (9.68)$$

where $\blacksquare \in \{t, \mathbf{x}, \mathbf{p}\}$, so Hamilton's equations obtained from (9.67) are equivalent to (9.65):

$$\frac{dx^\alpha}{dt} = \frac{\partial H}{\partial p_\alpha} = \frac{\partial H / \partial p_\alpha}{\partial H / \partial p_0}, \quad \frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial x^\alpha} = \frac{\partial H / \partial x^\alpha}{\partial H / \partial p_0}. \quad (9.69)$$

Let us decompose the metric into the average part and oscillations, $g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}$, and approximate the inverse metric to the second order in $\tilde{\mathbf{g}}$:

$$g^{\alpha\beta} \approx \bar{g}^{\alpha\beta} - \tilde{g}^{\alpha\beta} + \tilde{g}^{\alpha\gamma} \bar{g}_{\gamma\delta} \tilde{g}^{\delta\beta}, \quad (9.70)$$

where the indices of $\tilde{\mathbf{g}}$ are manipulated using the background metric $\bar{g}_{\alpha\beta}$. This gives

$$H = \frac{1}{2m} (m^2 + \bar{g}^{\alpha\beta} p_\alpha p_\beta - \tilde{g}^{\alpha\beta} p_\alpha p_\beta + \tilde{g}^{\alpha\beta} \bar{g}_{\beta\gamma} \tilde{g}^{\gamma\delta} p_\alpha p_\delta). \quad (9.71)$$

The Hamiltonian (9.67) is expanded in $\tilde{\mathbf{g}}$ as follows:

$$H(\mathbf{g}, \mathbf{p}) \approx -P_0 - \frac{\partial P_0}{\partial \tilde{g}^{\alpha\beta}} \tilde{g}^{\alpha\beta} - \frac{1}{2} \frac{\partial^2 P_0}{\partial \tilde{g}^{\alpha\beta} \partial \tilde{g}^{\gamma\delta}} \tilde{g}^{\alpha\beta} \tilde{g}^{\gamma\delta}, \quad (9.72)$$

where P_0 and the derivatives on the right-hand side are evaluated on $(\bar{\mathbf{g}}, \mathbf{p})$. To calculate these derivatives, let us differentiate (9.66) and use (9.71) for H . This gives

$$0 = \frac{\partial H}{\partial \tilde{g}^{\alpha\beta}} + \frac{\partial H}{\partial p_0} \frac{\partial P_0}{\partial \tilde{g}^{\alpha\beta}} = \frac{1}{2m} \left(-p_\alpha p_\beta + 2P_0 \frac{\partial P_0}{\partial \tilde{g}^{\alpha\beta}} \right), \quad (9.73)$$

[†] This section uses notation different from that used in the rest of the paper. In particular, the Greek indices span from 0 to 3, and the standard rules of index manipulations apply.

where the derivatives with respect to the oscillating metric are taken at fixed p_α and at $\tilde{g} \rightarrow 0$, and $P^0 \equiv P^0(\mathbf{g}, \mathbf{p}) = \bar{g}^{0\alpha} p_\alpha$; thus,

$$\frac{\partial P_0}{\partial \tilde{g}^{\alpha\beta}} = \frac{p_\alpha p_\beta}{2P^0}. \quad (9.74)$$

Similarly, differentiating (9.66) twice gives

$$\begin{aligned} 0 &= \frac{\partial^2 H}{\partial \tilde{g}^{\alpha\beta} \partial \tilde{g}^{\gamma\delta}} + \frac{\partial H}{\partial p_0} \frac{\partial^2 P_0}{\partial \tilde{g}^{\alpha\beta} \partial \tilde{g}^{\gamma\delta}} + \frac{\partial P_0}{\partial \tilde{g}^{\alpha\beta}} \frac{\partial}{\partial p_0} \frac{\partial H}{\partial \tilde{g}^{\gamma\delta}} + \frac{\partial}{\partial p_0} \left(\frac{\partial H}{\partial \tilde{g}^{\alpha\beta}} + \frac{\partial H}{\partial p_0} \frac{\partial P_0}{\partial \tilde{g}^{\alpha\beta}} \right) \frac{\partial P_0}{\partial \tilde{g}^{\gamma\delta}} \\ &= \frac{\partial^2 H}{\partial \tilde{g}^{\alpha\beta} \partial \tilde{g}^{\gamma\delta}} + \frac{\partial H}{\partial p_0} \frac{\partial^2 P_0}{\partial \tilde{g}^{\alpha\beta} \partial \tilde{g}^{\gamma\delta}} + \frac{\partial P_0}{\partial \tilde{g}^{\alpha\beta}} \frac{\partial}{\partial p_0} \frac{\partial H}{\partial \tilde{g}^{\gamma\delta}} + \frac{\partial P_0}{\partial \tilde{g}^{\gamma\delta}} \frac{\partial}{\partial p_0} \frac{\partial H}{\partial \tilde{g}^{\alpha\beta}} + \frac{\partial^2 H}{\partial p_0 \partial p_0} \frac{\partial P_0}{\partial \tilde{g}^{\alpha\beta}} \frac{\partial P_0}{\partial \tilde{g}^{\gamma\delta}} \\ &= \frac{1}{2m} \left(\bar{g}_{\beta\gamma} p_\alpha p_\delta + \bar{g}_{\delta\alpha} p_\gamma p_\beta + 2P^0 \frac{\partial^2 P_0}{\partial \tilde{g}^{\alpha\beta} \partial \tilde{g}^{\gamma\delta}} - \frac{1}{2P^0} \frac{\partial(p_\alpha p_\beta p_\gamma p_\delta)}{\partial p_0} + \bar{g}^{00} \frac{p_\alpha p_\beta p_\gamma p_\delta}{2(P^0)^2} \right), \end{aligned}$$

whence

$$\frac{\partial^2 P_0}{\partial \tilde{g}^{\alpha\beta} \partial \tilde{g}^{\gamma\delta}} = -\frac{1}{2P^0} (\bar{g}_{\beta\gamma} p_\alpha p_\delta + \bar{g}_{\delta\alpha} p_\gamma p_\beta) + \frac{1}{4(P^0)^2} \frac{\partial(p_\alpha p_\beta p_\gamma p_\delta)}{\partial p_0} - \bar{g}^{00} \frac{p_\alpha p_\beta p_\gamma p_\delta}{4(P^0)^3}.$$

Then, (9.72) yields

$$H \approx H_0 + \alpha_{\alpha\beta} \tilde{g}^{\alpha\beta} + \frac{1}{2} \tilde{g}_{\alpha\beta} \wp^{\alpha\beta}{}_{\gamma\delta} \tilde{g}^{\gamma\delta}, \quad (9.75)$$

where we introduced $H_0 = -P_0$, $\alpha^{\alpha\beta} = p^\alpha p^\beta / (2P^0)$, and

$$\wp^{\alpha\beta}{}_{\gamma\delta} = \frac{\delta_\gamma^\beta p^\alpha p_\delta + \delta_\delta^\alpha p^\beta p_\gamma}{2P^0} - \frac{1}{4(P^0)^2} \frac{\partial(p^\alpha p^\beta p_\gamma p_\delta)}{\partial p_0} + \bar{g}^{00} \frac{p^\alpha p^\beta p_\gamma p_\delta}{4(P^0)^3}. \quad (9.76)$$

9.4.2. Nonlinear potentials

Let us treat $\tilde{\mathbf{g}}$ as a 16-dimensional vector (Garg & Dodin 2021b), so $\alpha_{\alpha\beta}$ serves as $\boldsymbol{\alpha}^\dagger$ and $\wp^{\alpha\beta}{}_{\gamma\delta}$ serves as $\boldsymbol{\wp}$. (Because these operators happen to be local in the $\mathbf{x}$ representation, here we do not distinguish them from their symbols.) Let us also introduce

$$\mathfrak{E} \doteq p_\alpha p_\beta p_\gamma p_\delta \mathcal{U}^{\alpha\beta\gamma\delta} \quad (9.77)$$

and notice that $v^i \approx \dot{x}^i = p^i / p^0$ (see (9.69)), so $\omega - \mathbf{k} \cdot \mathbf{v} = -k_\rho p^\rho / P^0$ and $\delta(\omega - \mathbf{k} \cdot \mathbf{v}) = P^0 \delta(k^\rho p_\rho)$. Then, one finds from (6.77) that (appendix B.8)

$$\mathbf{D} = \frac{\pi}{4P^0} \int d\mathbf{k} \mathbf{k} \mathbf{k} \mathfrak{E} \delta(k^\rho p_\rho), \quad (9.78)$$

$$\boldsymbol{\Theta} = \frac{1}{4P^0} \frac{\partial}{\partial \vartheta} \oint d\mathbf{k} \frac{\mathbf{k} \mathbf{k} \mathfrak{E}}{\vartheta P^0 - k^\rho p_\rho} \Big|_{\vartheta=0}, \quad (9.79)$$

$$\Delta = \frac{p_\alpha p_\beta}{2P^0} \int d\mathbf{k} \mathcal{U}^{\alpha\gamma\beta}{}_{\gamma} - \frac{1}{8P^0} \frac{\partial}{\partial p_\lambda} \oint d\mathbf{k} \frac{k_\lambda \mathfrak{E}}{k^\rho p_\rho}. \quad (9.80)$$

Equation (9.80) (where one takes $p_0 = P_0$ after the differentiation) is in agreement with the result that was obtained for quasimonochromatic waves in (Garg & Dodin 2020). The derivation of the dispersion matrix $\boldsymbol{\Xi}$ for relativistic gravitational interactions in matter is cumbersome, so it is not presented here, but see (Garg & Dodin 2022). The collision integral and fluctuations for relativistic gravitational interactions are straightforward to obtain from the general formulas presented in sections 6.9 and 8. This can be used to

describe QL interactions of gravitational waves, including not only the usual vacuum modes[†] but also waves coupled with matter, for example, the relativistic Jeans mode.

Also notice that the OC Hamiltonian $\mathcal{H} = H_0 + \Delta$ can be put in a covariant form as follows. Like in the original system (section 9.4.1), $\mathcal{H}$ determines the ponderomotively modified shell $p_0 = \mathcal{P}_0(t, \mathbf{x}, \mathbf{p})$ via $\mathcal{H} = -\mathcal{P}_0$. On one hand, the covariant OC Hamiltonian $\mathcal{H}'$ vanishes on this shell,[‡] so it can be Taylor-expanded as follows:

$$\mathcal{H}' \approx (p_0 - \mathcal{P}_0) \left. \frac{\partial \mathcal{H}}{\partial p_0} \right|_{p=\mathcal{P}_0} \approx (p_0 - P_0 + \Delta) \lambda, \quad \lambda \doteq \left. \frac{\partial H}{\partial p_0} \right|_{p=P_0}. \quad (9.81)$$

On the other hand, it can also be represented as $\mathcal{H}' = H(\bar{\mathbf{g}}, \mathbf{p}) + \Delta'$ (here Δ' is the ponderomotive term yet to be found) and Taylor-expanded around the unperturbed shell $p_0 = P_0(t, \mathbf{x}, \mathbf{p})$ as

$$\mathcal{H}' \approx \Delta' + (p_0 - P_0) \lambda = (p_0 - P_0 + \Delta'/\lambda) \lambda. \quad (9.82)$$

By comparing (9.81) with (9.82), one finds that $\Delta' = \lambda \Delta$. Because $\lambda = P^0/m$, this leads to the following covariant Hamiltonian for OCs:

$$\mathcal{H}' = \frac{1}{2m} \left(m^2 + g_{\text{eff}}^{\alpha\beta} p_\alpha p_\beta - \frac{1}{4} \frac{\partial}{\partial p_\lambda} \oint d\mathbf{k} \frac{k_\lambda \mathfrak{E}}{k^\rho p_\rho} \right), \quad (9.83)$$

$$g_{\text{eff}}^{\alpha\beta} \doteq \bar{g}^{\alpha\beta} + \int d\mathbf{k} \mathbf{U}^{\alpha\gamma}{}_\gamma{}^\beta. \quad (9.84)$$

9.4.3. Gauge invariance

As shown in (Garg & Dodin 2021a,b) adiabatic QL interactions via gravitational waves (i.e. those determined by Θ and Δ) can be formulated in a form invariant with respect to gauge transformations

$$\tilde{g}^{\alpha\beta} \rightarrow \tilde{g}'^{\alpha\beta} = \tilde{g}^{\alpha\beta} + \nabla^{(\alpha} \tilde{\xi}^{\beta)}, \quad (9.85)$$

where ∇ is the covariant derivative associated with the background metric $\bar{\mathbf{g}}$, $\tilde{\xi}^\mu$ is an arbitrary vector field, and $\psi^{(\alpha}\eta^{\beta)} \equiv (\psi^\alpha\eta^\beta + \psi^\beta\eta^\alpha)/2$. Let us show that this also extends to resonant interactions. Recall that within the assumed accuracy the nonlinear potentials are supposed to be calculated only to the zeroth order in the geometrical-optics parameter. Then, the modification of the average Wigner matrix of the metric oscillations under the transformation (9.85) can be written as

$$\begin{aligned} \mathbf{U}'^{\alpha\beta\gamma\delta} - \mathbf{U}^{\alpha\beta\gamma\delta} &= \text{symb}_x \left(i \hat{k}^{(\alpha} \overline{|\tilde{\xi}^{\beta)}\rangle} \langle \tilde{g}^{\gamma\delta} | - i \overline{|\tilde{g}^{\alpha\beta}\rangle} \langle \tilde{\xi}^{(\gamma} | \hat{k}^{\delta)} + \hat{k}^{(\alpha} \overline{|\tilde{\xi}^{\beta)}\rangle} \langle \tilde{\xi}^{(\gamma} | \hat{k}^{\delta)} \right) \\ &= i k^{(\alpha} \mathcal{W}^{\beta)\gamma\delta} - i k^{(\delta} \mathcal{W}^{\gamma)\alpha\beta*} + k^{(\alpha} \mathcal{W}_\xi^{\beta)(\gamma} k^{\delta)}, \end{aligned} \quad (9.86)$$

where $\mathcal{W}^{\beta\gamma\delta} \doteq \overline{\text{symb}_x |\tilde{\xi}^{\beta)}\rangle \langle \tilde{g}^{\gamma\delta} |}$ and $\mathcal{W}_\xi^{\beta\gamma}$ is the average Wigner matrix of $\tilde{\xi}^\alpha$. The corresponding change of $\mathfrak{E}$ is

$$\mathfrak{E}' - \mathfrak{E} = (k^\rho p_\rho) \left(i p_\beta p_\gamma p_\delta \mathcal{W}^{\beta\gamma\delta} - i p_\alpha p_\beta p_\gamma \mathcal{W}^{\gamma\alpha\beta*} + k^\lambda p_\lambda p_\beta p_\gamma \mathcal{W}_\xi^{\beta\gamma} \right) \equiv (k^\rho p_\rho) A.$$

[†] Vacuum gravitational waves satisfy $\omega^2 = |\mathbf{k}|^2$. Hence, satisfying the resonance condition $k^\rho p_\rho = 0$ requires $|\mathbf{k} \cdot \mathbf{v}| = |\mathbf{k}|$, which requires particle speeds not smaller than the speed of light (remember that $c = 1$ in our units). For massive particles, this cannot be satisfied, so $\mathbf{D}$ vanishes for vacuum gravitational waves. However, such waves can still produce adiabatic ponderomotive effects determined by Δ (Garg & Dodin 2020).

[‡] The covariant Hamiltonian is the dispersion function of particles as quantum waves in the semiclassical limit (Garg & Dodin 2020).

Then, the difference in the diffusion coefficients (9.78) is

$$\mathbf{D}' - \mathbf{D} = \frac{\pi}{4P_0} \int d\mathbf{k} \mathbf{k} \mathbf{k} \delta(k^\rho p_\rho) (k^\rho p_\rho) A = 0, \quad (9.87)$$

because $\delta(k^\rho p_\rho) (k^\rho p_\rho) = 0$. In particular, this rules out QL diffusion via ‘coordinate waves’.

9.4.4. Lorenz gauge and effective metric

Let us consider gravitational waves in the Lorenz gauge, $\nabla_\alpha \tilde{g}^{\alpha\beta} = 0$. In this case,

$$k_\alpha U^{\alpha\beta\gamma\delta} = k_\beta U^{\alpha\beta\gamma\delta} = k_\gamma U^{\alpha\beta\gamma\delta} = k_\delta U^{\alpha\beta\gamma\delta} = 0, \quad (9.88)$$

and thus $k_\lambda \partial \mathfrak{E} / \partial p^\lambda = 0$. Then,

$$\frac{\partial}{\partial p_\lambda} \oint d\mathbf{k} \frac{k_\lambda \mathfrak{E}}{k^\rho p_\rho} = \frac{\partial}{\partial \vartheta} \oint d\mathbf{k} \frac{(k_\lambda k^\lambda) \mathfrak{E}}{k^\rho p_\rho + \vartheta} \Big|_{\vartheta=0}. \quad (9.89)$$

This simplifies the expression (9.80) for Δ and (9.83) for $\mathcal{H}'$. Furthermore, if the waves are not significantly affected by matter, so the vacuum dispersion $k_\lambda k^\lambda = 0$ can be assumed, the term (9.89) vanishes completely. Then, (9.83) becomes

$$\mathcal{H}' = \frac{1}{2m} \left(m^2 + g_{\text{eff}}^{\alpha\beta} p_\alpha p_\beta \right) \quad (9.90)$$

and QL diffusion disappears, because particles cannot resonate with waves. This shows that the only average QL effect of vacuum gravitational waves on particles is the modification of the spacetime metric by $\int d\mathbf{k} U_{\alpha\gamma}{}^{\gamma\beta} = \mathcal{O}(\varepsilon^2)$. For quasimonochromatic waves, this effect is discussed in further detail in (Garg & Dodin 2020).

10. Summary

In summary, we have presented quasilinear theory for classical plasma interacting with inhomogeneous turbulence in the presence of background fields. Because we use the Weyl symbol calculus, global-mode decomposition is not invoked, so our formulation is local and avoids the usual issues with complex-frequency modes. Also, the particle Hamiltonian is kept general, so the results are equally applicable to relativistic, electromagnetic, and even non-electromagnetic (for example, gravitational) interactions. Because our approach is not bounded by the limitations of variational analysis either, effects caused by collisional and collisionless dissipation are also included naturally.

Our main results are summarized in sections 5.6, 6.9, 7.6, 8.5 and are as follows. Starting from the Klimontovich equation, we derive a Fokker–Planck model for the dressed oscillation-center distribution. This model captures quasilinear diffusion, interaction with the background fields, and ponderomotive effects simultaneously. The local diffusion coefficient is manifestly positive-semidefinite. Waves are allowed to be off-shell (not constrained by a dispersion relation), and a collision integral of the Balescu–Lenard type emerges in a form that is not restricted to any particular Hamiltonian. This operator conserves particles, momentum, and energy, and it also satisfies the H -theorem, as usual. As a spin-off, a general expression for the spectrum (average Wigner matrix) of microscopic fluctuations of the interaction field is derived. For on-shell waves, which satisfy a quasilinear wave-kinetic equation, our theory conserves the momentum and energy of the wave–plasma system. Dewar’s oscillation-center quasilinear theory of electrostatic turbulence (Dewar 1973) is proven formally as a particular case and given

a concise formulation. Also discussed as examples are relativistic electromagnetic and gravitational interactions, and quasilinear theory for gravitational waves is proposed.

Aside from having the aesthetic appeal of a rigorous local theory, our formulation can help, for example, better understand and model quasilinear plasma heating and current drive. First of all, it systematically accounts for the wave-driven evolution of the nonresonant-particle distribution and for the ponderomotive effects caused by plasma inhomogeneity in both time and space. As discussed above (section 7.5), this is generally important for adequately calculating the energy–momentum transfer between waves and plasma even when resonant absorption *per se* occurs in a homogeneous-plasma region. Second, our formulation provides general formulas that equally hold in any canonical variables and for any Hamiltonians that satisfy our basic assumptions (section 3.1). Therefore, our results can be applied to various plasma models immediately. This eliminates the need for *ad hoc* calculations, which can be especially cumbersome beyond the homogeneous-plasma approximation. Discussing specific models of applied interest, however exciting, is beyond the scope of this paper and is left to future work.

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Declaration of interests

The author reports no conflict of interest.

Appendix A. Average Wigner matrices

A.1. Positive semidefiniteness

As known from (Cartwright 1976), the average Wigner function of any scalar field on the real axis is non-negative if the averaging is done over a sufficiently large phase-space volume. Here, we extend this theorem to average Wigner matrices of vector fields in a multi-dimensional space, $\psi(\mathbf{x})$, and show that such matrices are positive-semidefinite.

For any given function $h(\mathbf{z}) \equiv h(\mathbf{x}, \mathbf{k})$, we define its local phase-space average as the following convolution integral:[†]

$$\bar{h}(\mathbf{z}) \doteq \int d\mathbf{z}' \mathcal{G}(\mathbf{z} - \mathbf{z}') h(\mathbf{z}') \equiv \int d\mathbf{x}' d\mathbf{k}' \mathcal{G}(\mathbf{x} - \mathbf{x}', \mathbf{k} - \mathbf{k}') h(\mathbf{x}', \mathbf{k}') \quad (\text{A } 1)$$

with a Gaussian window function

$$\mathcal{G}(\mathbf{x}, \mathbf{k}) \doteq \frac{1}{(2\pi\sigma_{\mathbf{x}}\sigma_{\mathbf{k}})^n} \exp\left(-\frac{|\mathbf{x}|^2}{2\sigma_{\mathbf{x}}^2} - \frac{|\mathbf{k}|^2}{2\sigma_{\mathbf{k}}^2}\right) \quad (\text{A } 2)$$

and positive constants $\sigma_{\mathbf{x}}$ and $\sigma_{\mathbf{k}}$ yet to be specified. Unlike in section 2.1.1, the following notation will be assumed for the ‘scalar product’ for variables with upper, lower, and mixed indices:

$$\mathbf{x}' \cdot \mathbf{x}'' \doteq \delta_{ij} x'^i x''^j, \quad \mathbf{k}' \cdot \mathbf{k}'' \doteq \delta^{ij} k'_i k''_j, \quad \mathbf{k} \cdot \mathbf{x} \doteq k_i x^i. \quad (\text{A } 3)$$

(The Latin indices in this appendix range from 0 to n , δ_{ij} and δ^{ij} are unit matrices, and

[†] This ensures that $\partial_{\mathbf{z}} \bar{h} = \overline{\partial_{\mathbf{z}} h}$, as readily seen from (A 1) using integration by parts.

summation over repeating indices is assumed.) In particular, note that $|\mathbf{x}|^2 \doteq \mathbf{x} \cdot \mathbf{x} \geq 0$ and must not be confused with the squared spacetime interval, which can have either sign. Likewise, $|\mathbf{k}|^2 \doteq \mathbf{k} \cdot \mathbf{k} \geq 0$ must not be confused with $k_i k^i = -\omega^2 + \mathbf{k}^2$.

The average Wigner matrix of any given vector field $\boldsymbol{\psi}$ is given by

$$\overline{\mathbf{W}}_{\boldsymbol{\psi}}(\mathbf{x}, \mathbf{k}) = \frac{1}{(2\pi)^n} \frac{1}{(2\pi\sigma_x\sigma_k)^n} \int d\mathbf{s} d\mathbf{x}' d\mathbf{k}' \boldsymbol{\psi}(\mathbf{x}' + \mathbf{s}/2) \boldsymbol{\psi}^\dagger(\mathbf{x}' - \mathbf{s}/2) \times \exp\left(-\frac{|\mathbf{x} - \mathbf{x}'|^2}{2\sigma_x^2} - \frac{|\mathbf{k} - \mathbf{k}'|^2}{2\sigma_k^2} - i\mathbf{k}' \cdot \mathbf{s}\right). \quad (\text{A } 4)$$

The integral over $\mathbf{k}'$ can be readily taken:

$$\int d\mathbf{k}' \exp\left(-\frac{|\mathbf{k} - \mathbf{k}'|^2}{2\sigma_k^2} - i\mathbf{k}' \cdot \mathbf{s}\right) = (2\pi)^{n/2} \sigma_k^n \exp\left(-\frac{\sigma_k^2 |\mathbf{s}|^2}{2} - i\mathbf{k} \cdot \mathbf{s}\right). \quad (\text{A } 5)$$

Then, using the variables $\mathbf{x}_{1,2} \doteq \mathbf{x}' \pm \mathbf{s}/2$, one can rewrite (A 4) as follows:

$$\overline{\mathbf{W}}_{\boldsymbol{\psi}}(\mathbf{x}, \mathbf{k}) = \frac{1}{(2\pi)^{3n/2} \sigma_x^n} \int d\mathbf{x}_1 d\mathbf{x}_2 \boldsymbol{\psi}(\mathbf{x}_1) \boldsymbol{\psi}^\dagger(\mathbf{x}_2) e^{-\phi}, \quad (\text{A } 6)$$

$$\phi = \frac{|\mathbf{x} - (\mathbf{x}_1 + \mathbf{x}_2)/2|^2}{2\sigma_x^2} + \frac{\sigma_k^2 |\mathbf{x}_1 - \mathbf{x}_2|^2}{2} + i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2). \quad (\text{A } 7)$$

The function ϕ can also be expressed as $\phi = |\mathbf{x}|^2/(2\sigma_x^2) + \phi(\mathbf{x}_1) + \phi^*(\mathbf{x}_2) - \lambda \mathbf{x}_1 \cdot \mathbf{x}_2$, where

$$\phi(\mathbf{y}) \doteq \frac{|\mathbf{y}|^2}{2} \left(\frac{1}{4\sigma_x^2} + \sigma_k^2 \right) - \frac{\mathbf{x} \cdot \mathbf{y}}{2\sigma_x^2} + i\mathbf{k} \cdot \mathbf{y} \quad (\text{A } 8)$$

and $\lambda \doteq \sigma_k^2 - (4\sigma_x^2)^{-1}$. Then, using $\boldsymbol{\xi}(\mathbf{y}) \doteq \boldsymbol{\psi}(\mathbf{y})e^{-\phi(\mathbf{y})}$, one obtains from (A 6) that

$$\overline{\mathbf{W}}_{\boldsymbol{\psi}}(\mathbf{x}, \mathbf{k}) = \frac{e^{-|\mathbf{x}|^2/(2\sigma_x^2)}}{(2\pi)^{3n/2} \sigma_x^n} \int d\mathbf{x}_1 d\mathbf{x}_2 \boldsymbol{\xi}(\mathbf{x}_1) \boldsymbol{\xi}^\dagger(\mathbf{x}_2) e^{\lambda \mathbf{x}_1 \cdot \mathbf{x}_2}. \quad (\text{A } 9)$$

By Taylor-expanding $e^{\lambda \mathbf{x}_1 \cdot \mathbf{x}_2}$, one obtains

$$\overline{\mathbf{W}}_{\boldsymbol{\psi}}(\mathbf{x}, \mathbf{k}) = \frac{e^{-|\mathbf{x}|^2/(2\sigma_x^2)}}{(2\pi)^{3n/2} \sigma_x^n} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \mathbf{J}_m, \quad (\text{A } 10)$$

where $\mathbf{J}_m \doteq \int d\mathbf{x}_1 d\mathbf{x}_2 (\mathbf{x}_1 \cdot \mathbf{x}_2)^m \boldsymbol{\xi}(\mathbf{x}_1) \boldsymbol{\xi}^\dagger(\mathbf{x}_2)$. Note that

$$(\mathbf{x}_1 \cdot \mathbf{x}_2)^m = \sum_{\boldsymbol{\mu}(m)} \prod_{i=1}^n (x_1^i x_2^i)^{m_i}, \quad (\text{A } 11)$$

where the summation is performed over all combinations $\boldsymbol{\mu}(m) \equiv \{m_1, m_2, \dots, m_n\}$ of integers $m_i \geq 0$ such that $\sum_i m_i = m$. Thus,

$$\mathbf{J}_m = \sum_{\boldsymbol{\mu}(m)} \mathcal{J}_{\boldsymbol{\mu}} \mathcal{J}_{\boldsymbol{\mu}}^\dagger, \quad \mathcal{J}_{\boldsymbol{\mu}} = \int d\mathbf{y} \boldsymbol{\xi}(\mathbf{y}) \prod_{i=1}^n (y^i)^{m_i}. \quad (\text{A } 12)$$

Because each $\mathbf{J}_m$ is positive-semidefinite, the Wigner matrix $\overline{\mathbf{W}}_{\boldsymbol{\psi}}$ is positive-semidefinite when $\lambda \geq 0$, or equivalently, when $\sigma_x\sigma_k > 1/2$. This condition is assumed to be satisfied for the phase-space averaging of $\mathbf{W}_{\boldsymbol{\psi}}$ used in the main text. Loosely, this means that the averaging is done over the phase-space volume $\Delta\mathbf{x} \Delta\mathbf{k} \sim (\sigma_x\sigma_k)^n \gtrsim 1$.

A.2. Invariant limit for eikonal fields

For eikonal fields (7.17), one has

$$\boldsymbol{\psi}(\mathbf{x} + \mathbf{s}/2) \boldsymbol{\psi}^\dagger(\mathbf{x} - \mathbf{s}/2) \approx (\mathbf{A}(\mathbf{x}) e^{i\bar{\mathbf{k}}(\mathbf{x}) \cdot \mathbf{s}} + \text{c.c.}) + (\mathbf{B}(\mathbf{x}) e^{2i\theta(\mathbf{x})} + \text{c.c.}). \quad (\text{A } 13)$$

Here, $\mathbf{A} \doteq \boldsymbol{\eta}\boldsymbol{\eta}^\dagger|\check{a}|^2/4$, $\mathbf{B} \doteq \boldsymbol{\eta}\boldsymbol{\eta}^\dagger\check{a}^2/4$, ‘c.c.’ stands for complex conjugate, we used the linear approximation $\theta(\mathbf{x} \pm \mathbf{s}/2) \approx \theta(\mathbf{x}) \pm \bar{\mathbf{k}}(\mathbf{x}) \cdot \mathbf{s}/2$, with $\bar{\mathbf{k}} \equiv (-\bar{\omega}, \bar{\mathbf{k}})$. Then,

$$\mathbf{W}_\psi(\mathbf{x}, \mathbf{k}) \approx \mathbf{A}(\mathbf{x}) \delta(\mathbf{k} - \bar{\mathbf{k}}(\mathbf{x})) + \mathbf{A}^*(\mathbf{x}) \delta(\mathbf{k} + \bar{\mathbf{k}}(\mathbf{x})) + 2 \operatorname{re}(\mathbf{B}(\mathbf{x}) e^{2i\theta(\mathbf{x})} \delta(\mathbf{k})). \quad (\text{A } 14)$$

Let us adopt $\sigma_x \ll l_c$, where l_c is the least characteristic scale of $\check{a}$, $\boldsymbol{\eta}$, and $\bar{\mathbf{k}}$. Then,

$$\bar{\mathbf{W}}_\psi(\mathbf{x}, \mathbf{k}) \approx \mathbf{A}(\mathbf{x}) \mathcal{G}_k(\mathbf{k} - \bar{\mathbf{k}}) + \mathbf{A}^*(\mathbf{x}) \mathcal{G}_k(\mathbf{k} + \bar{\mathbf{k}}) + 2 \operatorname{re}(\mathbf{B}(\mathbf{x}) \mathcal{G}_k(\mathbf{k}) \zeta e^{2i\theta(\mathbf{x})}). \quad (\text{A } 15)$$

Here, $\mathcal{G}_k(\mathbf{k})$ are normalized Gaussians that can be replaced with delta functions if σ_k is small compared to any scale of interest in the $\mathbf{k}$ space:

$$\mathcal{G}_k(\mathbf{k}) \doteq \frac{1}{(\sqrt{2\pi}\sigma_k)^n} \exp\left(-\frac{|\mathbf{k}|^2}{2\sigma_k^2}\right) \rightarrow \delta(\mathbf{k}). \quad (\text{A } 16)$$

Also, the function

$$\zeta \approx \frac{1}{(\sqrt{2\pi}\sigma_x)^n} \int d\mathbf{x}' \exp\left(-\frac{|\mathbf{x}' - \mathbf{x}|^2}{2\sigma_x^2} + 2i\bar{\mathbf{k}}(\mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x})\right) = e^{-2|\bar{\mathbf{k}}(\mathbf{x})|^2 \sigma_x^2} \quad (\text{A } 17)$$

can be made exponentially small by adopting $\sigma_x \gg |\bar{\mathbf{k}}|^{-1}$.[†] In this limit, the average Wigner matrix of an eikonal field is independent of σ_x and σ_k :

$$\bar{\mathbf{W}}_\psi(\mathbf{x}, \mathbf{k}) \approx \mathbf{A}(\mathbf{x}) \delta(\mathbf{k} - \bar{\mathbf{k}}(\mathbf{x})) + \mathbf{A}^*(\mathbf{x}) \delta(\mathbf{k} + \bar{\mathbf{k}}(\mathbf{x})). \quad (\text{A } 18)$$

This $\bar{\mathbf{W}}_\psi$ is also Hermitian and positive-semidefinite (in agreement with the general theory from section A.1), because so are $\mathbf{A}$ and $\mathbf{A}^*$. The same properties pertain to the Wigner matrix of an ensemble of randomly phased eikonal fields, because it equals the sum of the Wigner matrices of the individual components (see also section 7.4).

Appendix B. Auxiliary proofs

B.1. Proof of (2.53)

Like in the case of (2.45), one finds that

$$\begin{aligned} (\hat{\mathbf{L}}\psi(\mathbf{x}))^i (\hat{\mathbf{R}}\psi(\mathbf{x}))^{*j} &= \langle \mathbf{x} | \hat{\mathbf{L}}_{i'}^i | \psi^{i'} \rangle \langle \psi^{j'} | (\hat{\mathbf{R}}^j_{j'})^\dagger | \mathbf{x} \rangle \\ &= (2\pi)^n \langle \mathbf{x} | \hat{\mathbf{L}}_{i'}^i \hat{\mathbf{W}}_\psi^{i'j'} (\hat{\mathbf{R}}^\dagger)_{j'^j} | \mathbf{x} \rangle \\ &= (2\pi)^n \langle \mathbf{x} | (\hat{\mathbf{L}} \hat{\mathbf{W}}_\psi \hat{\mathbf{R}}^\dagger)^{ij} | \mathbf{x} \rangle \\ &= \int d\mathbf{k} (\mathbf{L}(\mathbf{x}, \mathbf{k}) \star \mathbf{W}_\psi(\mathbf{x}, \mathbf{k}) \star \mathbf{R}^\dagger(\mathbf{x}, \mathbf{k}))^{ij}. \end{aligned} \quad (\text{B } 1)$$

This proves (2.53a). At $\epsilon \rightarrow 0$, when $\star$ becomes the usual product, (B 1) gives

$$(\hat{\mathbf{L}}\psi)(\hat{\mathbf{R}}\psi)^\dagger = \int d\mathbf{k} \mathbf{L} \mathbf{W}_\psi \mathbf{R}^\dagger, \quad (\text{B } 2)$$

and in particular, taking the trace of (B 2) yields

$$(\hat{\mathbf{R}}\psi)^\dagger (\hat{\mathbf{L}}\psi) = \operatorname{tr}((\hat{\mathbf{L}}\psi)(\hat{\mathbf{R}}\psi)^\dagger) = \int d\mathbf{k} \operatorname{tr}(\mathbf{L} \mathbf{W}_\psi \mathbf{R}^\dagger) = \int d\mathbf{k} \operatorname{tr}(\mathbf{W}_\psi \mathbf{Q}). \quad (\text{B } 3)$$

Here, $\mathbf{Q} \doteq \mathbf{R}^\dagger \mathbf{L}$, and we used that $\operatorname{tr}(\mathbf{A}\mathbf{B}) = \operatorname{tr}(\mathbf{B}\mathbf{A})$ for any matrices $\mathbf{A}$ and $\mathbf{B}$.

For real fields, one can also replace the integrand with

$$\operatorname{tr}(\mathbf{W}_\psi^* \mathbf{Q}^*) = \operatorname{tr}(\mathbf{Q}^\dagger \mathbf{W}_\psi^\dagger) = \operatorname{tr}(\mathbf{Q}^\dagger \mathbf{W}_\psi) = \operatorname{tr}(\mathbf{W}_\psi \mathbf{Q}^\dagger), \quad (\text{B } 4)$$

[†] Even though σ_x has been assumed small compared to l_c , the smallness of the geometrical-optics parameter $\epsilon \doteq (|\bar{\mathbf{k}}|/l_c)^{-1} \ll 1$ allows choosing σ_x in the interval $|\bar{\mathbf{k}}|^{-1} \ll \sigma_x \ll l_c$.

where we used $\text{tr } \mathbf{A}^\top = \text{tr } \mathbf{A}$, $(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$, $\mathbf{W}_\psi^\dagger = \mathbf{W}_\psi$, and, again, $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$, respectively. In summary then,

$$(\hat{\mathbf{R}}\psi)^\dagger(\hat{\mathbf{L}}\psi) = \int d\mathbf{k} \text{tr}(\mathbf{W}_\psi \mathbf{Q}) = \int d\mathbf{k} \text{tr}(\mathbf{W}_\psi \mathbf{Q}^\dagger), \quad (\text{B } 5)$$

so the anti-Hermitian part of $\mathbf{Q}$ does not contribute to the integrals. Thus,

$$(\hat{\mathbf{R}}\psi)^\dagger(\hat{\mathbf{L}}\psi) = \int d\mathbf{k} \text{tr}(\mathbf{W}_\psi(\mathbf{R}^\dagger \mathbf{L})_{\text{H}}). \quad (\text{B } 6)$$

Because $\hat{\mathbf{L}}$ and $\hat{\mathbf{R}}$ are arbitrary, they can as well be swapped; then one obtains (2.53c).

B.2. Proof of (4.28)

Suppose the dominant term in $\boldsymbol{\mu}$ in (4.16) has the form $\boldsymbol{\mu}_h \tau^h$, where h is a natural number and $\mu_h = \mathcal{O}(\epsilon^2)$. (Here h is a power index, so no summation over h is assumed.) Let us Taylor-expand $\mathcal{J}[A, G]$ in $\boldsymbol{\mu}_h$:

$$\begin{aligned} & \mathcal{J}[A, G] - \mathcal{J}[A, G_0] - \mathcal{O}(\epsilon^2) \\ & \approx \boldsymbol{\mu}_h \cdot \frac{\partial}{\partial \boldsymbol{\mu}_h} \left(\int d\mathbf{K} A(\mathbf{X}, \mathbf{K}) \lim_{\nu \rightarrow 0^+} \int_0^\infty d\tau e^{-\nu\tau + i\Omega\tau + i\mathbf{K} \cdot \boldsymbol{\mu}_h \tau^h} \right)_{\boldsymbol{\mu}_h=0} \\ & \approx \boldsymbol{\mu}_h \cdot \int d\mathbf{K} A(\mathbf{X}, \mathbf{K}) \lim_{\nu \rightarrow 0^+} \frac{\partial}{\partial \boldsymbol{\mu}_h} \left(\int_0^\infty d\tau e^{-\nu\tau + i\Omega\tau + i\mathbf{K} \cdot \boldsymbol{\mu}_h \tau^h} \right)_{\boldsymbol{\mu}_h=0} \\ & \approx i\boldsymbol{\mu}_h \cdot \int d\mathbf{K} \mathbf{K} A(\mathbf{X}, \mathbf{K}) \lim_{\nu \rightarrow 0^+} \int_0^\infty d\tau \tau^h e^{-\nu\tau + i\Omega\tau} \\ & \approx i^{1-h} \boldsymbol{\mu}_h \cdot \int d\mathbf{K} \mathbf{K} A(\mathbf{X}, \mathbf{K}) \frac{d^h G_0(\Omega(\mathbf{X}, \mathbf{K}))}{d\Omega^h} \\ & \approx i^{1-h} \boldsymbol{\mu}_h \cdot \frac{\partial^h}{\partial \Omega^h} \int d\mathbf{K} \mathbf{K} A(\mathbf{X}, \mathbf{K}) G_0(\Omega(\mathbf{X}, \mathbf{K})). \end{aligned} \quad (\text{B } 7)$$

Provided that A is sufficiently smooth and well behaved, the overall coefficient here is $\mathcal{O}(1)$, so $\mathcal{J}[A, G] - \mathcal{J}[A, G_0] = \mathcal{O}(\mu_h) + \mathcal{O}(\epsilon^2)$. Because $\mu_h = \mathcal{O}(\epsilon^2)$, this proves (4.28).

B.3. Proof of (5.4)

Here, we show that

$$\begin{aligned} & \text{symb}_x(\hat{u}^\alpha \hat{G} \hat{u}^\beta) \\ & = \frac{1}{(2\pi)^N} \int d\mathbf{S} e^{-i\mathbf{K} \cdot \mathbf{S}} \langle \mathbf{X} + \mathbf{S}/2 | \hat{u}^\alpha \hat{G} \hat{u}^\beta | \mathbf{X} - \mathbf{S}/2 \rangle \\ & = \frac{1}{(2\pi)^N} \int d\mathbf{X}' d\mathbf{K}'' d\mathbf{K}' d\mathbf{S} d\mathbf{S}' W_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}', \mathbf{K}') G(\mathbf{X}', \mathbf{K}'') e^{-i\mathbf{K} \cdot \mathbf{S} + i(\mathbf{K}' + \mathbf{K}'') \cdot \mathbf{S}'} \\ & \quad \times \delta(\mathbf{X} + \mathbf{S}/2 - \mathbf{X}' - \mathbf{S}'/2) \delta(\mathbf{X} - \mathbf{S}/2 - \mathbf{X}' + \mathbf{S}'/2) \\ & = \frac{1}{(2\pi)^N} \int d\mathbf{X}' d\mathbf{K}' d\mathbf{K}'' d\mathbf{S} d\mathbf{S}' W_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}', \mathbf{K}') G(\mathbf{X}', \mathbf{K}'') e^{-i\mathbf{K} \cdot \mathbf{S} + i(\mathbf{K}' + \mathbf{K}'') \cdot \mathbf{S}'} \\ & \quad \times \delta(\mathbf{S} - \mathbf{S}') \delta(\mathbf{X} - \mathbf{S}/2 - \mathbf{X}' + \mathbf{S}'/2) \\ & = \frac{1}{(2\pi)^N} \int d\mathbf{X}' d\mathbf{K}' d\mathbf{K}'' d\mathbf{S} W_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}', \mathbf{K}') G(\mathbf{X}', \mathbf{K}'') e^{i(\mathbf{K}' + \mathbf{K}'' - \mathbf{K}) \cdot \mathbf{S}} \delta(\mathbf{X} - \mathbf{X}') \\ & = \frac{1}{(2\pi)^N} \int d\mathbf{K}' d\mathbf{K}'' d\mathbf{S} W_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}, \mathbf{K}') G(\mathbf{X}, \mathbf{K}'') e^{i(\mathbf{K}' + \mathbf{K}'' - \mathbf{K}) \cdot \mathbf{S}} \\ & = \int d\mathbf{K}' W_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}, \mathbf{K}') G(\mathbf{X}, \mathbf{K} - \mathbf{K}'). \end{aligned} \quad (\text{B } 8)$$

B.4. *Proof of (5.10)*

Using (4.19), (4.20), and (2.84) in application to $W_{\mathbf{u}}^{\alpha\beta}$, one finds that

$$\begin{aligned}
D_0^{\alpha\beta}(\mathbf{X}) &\doteq \int d\mathbf{K} \overline{W}_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}, \mathbf{K}) G^*(\mathbf{X}, \mathbf{K}) \\
&= \int d\mathbf{K} \overline{W}_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}, -\mathbf{K}) G^*(\mathbf{X}, -\mathbf{K}) \\
&= \int d\mathbf{K} \overline{W}_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}, -\mathbf{K}) G(\mathbf{X}, \mathbf{K}) \\
&= \int d\mathbf{K} \overline{W}_{\mathbf{u}}^{\alpha\beta*}(\mathbf{X}, \mathbf{K}) G(\mathbf{X}, \mathbf{K}) \\
&= (D_0^{\alpha\beta}(\mathbf{X}))^*
\end{aligned} \tag{B 9}$$

and also

$$\begin{aligned}
\Theta^{\alpha\beta c}(\mathbf{X}) &\doteq - \int d\mathbf{K} \overline{W}_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}, \mathbf{K}) (G^{|c}(\mathbf{X}, \mathbf{K}))^* \\
&= \int d\mathbf{K} \overline{W}_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}, \mathbf{K}) G^{|c}(\mathbf{X}, -\mathbf{K}) \\
&= \int d\mathbf{K} \overline{W}_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}, -\mathbf{K}) G^{|c}(\mathbf{X}, \mathbf{K}) \\
&= \int d\mathbf{K} (\overline{W}_{\mathbf{u}}^{\alpha\beta}(\mathbf{X}, \mathbf{K}))^* G^{|c}(\mathbf{X}, \mathbf{K}) \\
&= -(\Theta^{\alpha\beta c}(\mathbf{X}))^*.
\end{aligned} \tag{B 10}$$

B.5. *Proof of (5.20)*

Let us estimate

$$\mathcal{L}^{(1)} \overline{f} \doteq \frac{\partial}{\partial z^\alpha} \left(J^{\alpha\mu} J^{\beta\nu} \mathcal{P}_{\mu\nu}^{(1)} \frac{\partial \overline{f}}{\partial z^\beta} \right), \tag{B 11}$$

where $\mathcal{P}_{\mu\nu}^{(1)}$ has the form

$$\mathcal{P}_{\mu\nu}^{(1)} \doteq \int d\mathbf{K} q_\nu \frac{\partial \overline{W}(\mathbf{X}, \mathbf{K})}{\partial z^\mu} G(\Omega(\mathbf{X}, \mathbf{K})). \tag{B 12}$$

First, notice that

$$\begin{aligned}
\mathcal{P}_{\mu\nu}^{(1)} &= \frac{\partial}{\partial z^\mu} \int d\mathbf{K} q_\nu \overline{W} G - \int d\mathbf{K} q_\nu \overline{W} \frac{\partial G}{\partial z^\mu} \\
&= \frac{\partial}{\partial z^\mu} \int d\mathbf{K} q_\nu \overline{W} G + \frac{\partial v^\lambda}{\partial z^\mu} \int d\mathbf{K} q_\nu q_\lambda \overline{W} G' \\
&= \frac{\partial}{\partial z^\mu} \int d\mathbf{K} q_\nu \overline{W} G + \frac{\partial v^\lambda}{\partial z^\mu} \frac{\partial}{\partial \Omega} \int d\mathbf{K} q_\nu q_\lambda \overline{W} G \\
&\equiv \frac{\partial \mathcal{Q}_\nu^{(1)}}{\partial z^\mu} + \frac{\partial v^\lambda}{\partial z^\mu} \mathcal{R}_{\lambda\nu}^{(1)}.
\end{aligned} \tag{B 13}$$

Because $\mathcal{Q}_\nu^{(1)}$ and $\mathcal{R}_{\lambda\nu}^{(1)}$ are $\mathcal{O}(\varepsilon^2)$, one has $\mathcal{P}_{\mu\nu}^{(1)} \sim \kappa_\mu \varepsilon^2$, where κ_μ is the characteristic inverse scale along the μ th phase-space axis. Thus,

$$\mathcal{L}^{(1)} \overline{f} \sim (J^{\alpha\mu} \kappa_\alpha \kappa_\mu) J^{\beta\nu} \kappa_\beta \varepsilon^2 \overline{f} = \mathcal{O}(\varepsilon^2), \tag{B 14}$$

where we used (see (2.69) and (3.2))

$$J^{\alpha\mu}\kappa_\alpha\kappa_\mu \sim \kappa_x\kappa_p = \mathcal{O}(\epsilon). \quad (\text{B } 15)$$

The first part of (5.20) is obtained by considering $\text{im } \mathcal{L}^{(1)}\bar{f}$ and using (B 14).

Let us also estimate

$$\mathcal{L}^{(2)}\bar{f} \doteq \frac{\partial}{\partial z^\alpha} \left(J^{\alpha\mu} J^{\beta\nu} \mathcal{P}_{\mu\nu}^{(2)} \frac{\partial \bar{f}}{\partial z^\beta} \right), \quad (\text{B } 16)$$

where $\mathcal{P}_{\mu\nu}^{(2)}$ has the form

$$\mathcal{P}_{\mu\nu}^{(2)} \doteq \int d\mathbf{K} \frac{\partial^2 \bar{W}}{\partial z^\mu \partial z^\nu} G(\Omega(\mathbf{X}, \mathbf{K})). \quad (\text{B } 17)$$

First, note that

$$\begin{aligned} \mathcal{P}_{\mu\nu}^{(2)} &= \int d\mathbf{K} \frac{\partial^2 \bar{W}}{\partial z^\mu \partial z^\nu} G \\ &= \frac{\partial}{\partial z^\mu} \int d\mathbf{K} \frac{\partial \bar{W}}{\partial z^\nu} G - \int d\mathbf{K} \frac{\partial \bar{W}}{\partial z^\nu} \frac{\partial G}{\partial z^\mu} \\ &= \frac{\partial^2}{\partial z^\mu \partial z^\nu} \int d\mathbf{K} \bar{W} G - \frac{\partial}{\partial z^\mu} \int d\mathbf{K} \bar{W} \frac{\partial G}{\partial z^\nu} - \frac{\partial}{\partial z^\nu} \int d\mathbf{K} \bar{W} \frac{\partial G}{\partial z^\mu} + \int d\mathbf{K} \bar{W} \frac{\partial^2 G}{\partial z^\mu \partial z^\nu} \\ &= \frac{\partial^2}{\partial z^\mu \partial z^\nu} \int d\mathbf{K} \bar{W} G - \frac{\partial}{\partial z^\mu} \int d\mathbf{K} \bar{W} G' \left(-q_\lambda \frac{\partial v^\lambda}{\partial z^\nu} \right) \\ &\quad - \frac{\partial}{\partial z^\nu} \int d\mathbf{K} \bar{W} G' \left(-q_\lambda \frac{\partial v^\lambda}{\partial z^\mu} \right) + \int d\mathbf{K} \bar{W} \frac{\partial^2 G}{\partial z^\mu \partial z^\nu} \\ &= \frac{\partial^2}{\partial z^\mu \partial z^\nu} \int d\mathbf{K} \bar{W} G + \frac{\partial}{\partial z^\mu} \left(\frac{\partial v^\lambda}{\partial z^\nu} \frac{\partial}{\partial \Omega} \int d\mathbf{K} q_\lambda \bar{W} G \right) \\ &\quad + \frac{\partial}{\partial z^\nu} \left(\frac{\partial v^\lambda}{\partial z^\mu} \frac{\partial}{\partial \Omega} \int d\mathbf{K} q_\lambda \bar{W} G \right) + \int d\mathbf{K} \bar{W} \frac{\partial^2 G}{\partial z^\mu \partial z^\nu}. \end{aligned} \quad (\text{B } 18)$$

Next, note that

$$\begin{aligned} \int d\mathbf{K} \bar{W} \frac{\partial^2 G}{\partial z^\mu \partial z^\nu} &= \int d\mathbf{K} \bar{W} \frac{\partial}{\partial z^\mu} \left(-q_\lambda G' \frac{\partial v^\lambda}{\partial z^\nu} \right) \\ &= -\frac{\partial v^\lambda}{\partial z^\nu} \int d\mathbf{K} q_\lambda \bar{W} \frac{\partial G'}{\partial z^\mu} - \frac{\partial^2 v^\lambda}{\partial z^\mu \partial z^\nu} \int d\mathbf{K} q_\lambda \bar{W} G' \\ &= \frac{\partial v^\lambda}{\partial z^\nu} \frac{\partial v^\delta}{\partial z^\mu} \int d\mathbf{K} q_\lambda q_\delta \bar{W} G'' - \frac{\partial^2 v^\lambda}{\partial z^\mu \partial z^\nu} \int d\mathbf{K} q_\lambda \bar{W} G' \\ &= \frac{\partial v^\lambda}{\partial z^\nu} \frac{\partial v^\delta}{\partial z^\mu} \frac{\partial^2}{\partial \Omega^2} \int d\mathbf{K} q_\lambda q_\delta \bar{W} G - \frac{\partial^2 v^\lambda}{\partial z^\mu \partial z^\nu} \frac{\partial}{\partial \Omega} \int d\mathbf{K} q_\lambda \bar{W} G. \end{aligned} \quad (\text{B } 19)$$

Assuming the notation

$$\begin{aligned} \mathcal{S}^{(2)} &\doteq \int d\mathbf{K} \bar{W} G = \mathcal{O}(\varepsilon^2), \\ \mathcal{Q}_\lambda^{(2)} &\doteq \frac{\partial}{\partial \Omega} \int d\mathbf{K} q_\lambda \bar{W} G = \mathcal{O}(\varepsilon^2), \\ \mathcal{R}_{\lambda\delta}^{(2)} &\doteq \frac{\partial^2}{\partial \Omega^2} \int d\mathbf{K} q_\lambda q_\delta \bar{W} G = \mathcal{O}(\varepsilon^2), \end{aligned} \quad (\text{B } 20)$$

one can then rewrite $\mathcal{P}_{\mu\nu}^{(2)}$ as follows:

$$\begin{aligned}\mathcal{P}_{\mu\nu}^{(2)} &= \frac{\partial^2 \mathcal{S}^{(2)}}{\partial z^\mu \partial z^\nu} + \frac{\partial}{\partial z^\mu} \left(\frac{\partial v^\lambda}{\partial z^\nu} \mathcal{Q}_\lambda^{(2)} \right) + \frac{\partial}{\partial z^\nu} \left(\frac{\partial v^\lambda}{\partial z^\mu} \mathcal{Q}_\lambda^{(2)} \right) - \frac{\partial^2 v^\lambda}{\partial z^\mu \partial z^\nu} \mathcal{Q}_\lambda^{(2)} + \frac{\partial v^\lambda}{\partial z^\nu} \frac{\partial v^\delta}{\partial z^\mu} \mathcal{R}_{\lambda\delta}^{(2)} \\ &= \frac{\partial^2 \mathcal{S}^{(2)}}{\partial z^\mu \partial z^\nu} + \frac{\partial^2 v^\lambda}{\partial z^\mu \partial z^\nu} \mathcal{Q}_\lambda^{(2)} + \frac{\partial v^\lambda}{\partial z^\nu} \frac{\partial \mathcal{Q}_\lambda^{(2)}}{\partial z^\mu} + \frac{\partial v^\lambda}{\partial z^\mu} \frac{\partial \mathcal{Q}_\lambda^{(2)}}{\partial z^\nu} + \frac{\partial^2 v^\lambda}{\partial z^\mu \partial z^\nu} \mathcal{Q}_\lambda^{(2)} + \frac{\partial v^\lambda}{\partial z^\nu} \frac{\partial v^\delta}{\partial z^\mu} \mathcal{R}_{\lambda\delta}^{(2)}.\end{aligned}$$

Each term on the right-hand side of this equation scales as $\varepsilon^2 \kappa_\mu \kappa_\nu$, so

$$\mathcal{L}^{(2)} \bar{f} \sim (J^{\alpha\mu} \kappa_\alpha \kappa_\mu) (J^{\beta\nu} \kappa_\beta \kappa_\nu) \varepsilon^2 \bar{f} \sim \varepsilon^2 \varepsilon^2 \bar{f}, \quad (\text{B } 21)$$

where we again used (B 15). The second part of (5.20) is obtained by considering $\text{re } \mathcal{L}^{(2)} \bar{f}$ and using (B 21).

B.6. Proof of (5.25)

Using (5.23) and assuming the notation $\text{d}_t \doteq \partial_t + v^\gamma \partial_\gamma$, one finds that

$$\begin{aligned}& \partial_\alpha (\hat{D}^{\alpha\beta} \partial_\beta \bar{f}) - \partial_\alpha ((\mathcal{D}^{\alpha\beta} + \varrho^{\alpha\beta}) \partial_\beta \bar{f}) \\ &= -\partial_\alpha \left(\Theta^{\alpha\beta} \text{d}_t \partial_\beta \bar{f} + \frac{1}{2} (\text{d}_t \Theta^{\alpha\beta}) \partial_\beta \bar{f} \right) \\ &= -\partial_\alpha \left(\frac{1}{2} \Theta^{\alpha\beta} \text{d}_t \partial_\beta \bar{f} + \frac{1}{2} \text{d}_t (\Theta^{\alpha\beta} \partial_\beta \bar{f}) \right) \\ &= -\partial_\alpha \left(\frac{1}{2} \Theta^{\alpha\beta} \partial_\beta \text{d}_t \bar{f} - \frac{1}{2} \Theta^{\alpha\beta} (\partial_\beta v^\gamma) \partial_\gamma \bar{f} \right) - \partial_\alpha \left(\frac{1}{2} \text{d}_t (\Theta^{\alpha\beta} \partial_\beta \bar{f}) \right) \\ &= -\partial_\alpha \left(\frac{1}{2} \Theta^{\alpha\beta} \partial_\beta \text{d}_t \bar{f} \right) + \partial_\alpha \left(\frac{1}{2} \Theta^{\alpha\beta} (\partial_\beta v^\gamma) \partial_\gamma \bar{f} \right) \\ &\quad - \text{d}_t \left(\frac{1}{2} \partial_\alpha (\Theta^{\alpha\beta} \partial_\beta \bar{f}) \right) - (\partial_\alpha v^\gamma) \partial_\gamma \left(\frac{1}{2} \Theta^{\alpha\beta} \partial_\beta \bar{f} \right).\end{aligned} \quad (\text{B } 22)$$

Because $\Theta^{\alpha\beta} = \mathcal{O}(\varepsilon^2)$ and $\text{d}_t \bar{f} = \mathcal{O}(\varepsilon^2)$, the first term on the right-hand side of (B 22) is negligible. Also note that due to (3.15), the factor $\partial_\alpha v^\gamma$ in the last term on the right-hand side of (B 22) commutes with ∂_γ . Hence, one obtains

$$\begin{aligned}& \partial_\alpha (\hat{D}^{\alpha\beta} \partial_\beta \bar{f}) - \partial_\alpha (\mathcal{D}^{\alpha\beta} \partial_\beta \bar{f}) + \text{d}_t \left(\frac{1}{2} \partial_\alpha (\Theta^{\alpha\beta} \partial_\beta \bar{f}) \right) \\ &= \partial_\alpha \left(\varrho^{\alpha\beta} \partial_\beta \bar{f} + \frac{1}{2} \Theta^{\alpha\beta} (\partial_\beta v^\gamma) \partial_\gamma \bar{f} \right) - \partial_\gamma \left(\frac{1}{2} \Theta^{\alpha\beta} (\partial_\alpha v^\gamma) \partial_\beta \bar{f} \right) \\ &= \partial_\alpha \left(\varrho^{\alpha\beta} \partial_\beta \bar{f} + \frac{1}{2} (\Theta^{\alpha\gamma} (\partial_\gamma v^\beta) - \Theta^{\gamma\beta} (\partial_\gamma v^\alpha)) \partial_\beta \bar{f} \right) \\ &\equiv \partial_\alpha (U^{\alpha\beta} \partial_\beta \bar{f}).\end{aligned} \quad (\text{B } 23)$$

Next, notice that

$$\begin{aligned}\varrho^{\alpha\beta} &= -\frac{1}{2} J^{\alpha\mu} J^{\beta\nu} \oint \text{d}\mathbf{K} \left(q_\nu \frac{\partial \bar{W}}{\partial z^\mu} - q_\mu \frac{\partial \bar{W}}{\partial z^\nu} \right) \frac{1}{\Omega} \\ &= -\frac{1}{2} J^{\alpha\mu} J^{\beta\nu} \oint \text{d}\mathbf{K} \frac{q_\nu}{\Omega} \frac{\partial \bar{W}}{\partial z^\mu} + \frac{1}{2} J^{\alpha\mu} J^{\nu\beta} \oint \text{d}\mathbf{K} \frac{q_\mu}{\Omega} \frac{\partial \bar{W}}{\partial z^\nu} \\ &= -\frac{1}{2} J^{\alpha\mu} J^{\beta\nu} \left(\frac{\partial}{\partial z^\mu} \oint \text{d}\mathbf{K} \frac{q_\nu \bar{W}}{\Omega} - \oint \text{d}\mathbf{K} q_\nu \bar{W} \frac{\partial}{\partial z^\mu} \frac{1}{\Omega} \right)\end{aligned}$$

$$\begin{aligned}
& -\frac{\partial}{\partial z^\nu} \oint d\mathbf{K} \frac{q_\mu \bar{W}}{\Omega} + \oint d\mathbf{K} q_\mu \bar{W} \frac{\partial}{\partial z^\nu} \frac{1}{\Omega} \Big) \\
& = -\frac{1}{2} J^{\alpha\mu} J^{\beta\nu} \left(\frac{\partial}{\partial z^\mu} \oint d\mathbf{K} \frac{q_\nu \bar{W}}{\Omega} + \frac{\partial v^\lambda}{\partial z^\mu} \frac{\partial}{\partial \Omega} \oint d\mathbf{K} \frac{q_\lambda q_\nu \bar{W}}{\Omega} \right. \\
& \quad \left. - \frac{\partial}{\partial z^\nu} \oint d\mathbf{K} \frac{q_\mu \bar{W}}{\Omega} - \frac{\partial v^\lambda}{\partial z^\nu} \frac{\partial}{\partial \Omega} \oint d\mathbf{K} \frac{q_\lambda q_\mu \bar{W}}{\Omega} \right). \tag{B 24}
\end{aligned}$$

Assuming the notation

$$Q_\mu \doteq \frac{1}{2} \oint d\mathbf{K} \frac{q_\mu \bar{W}}{\Omega}, \quad R_{\mu\nu} \doteq \frac{1}{2} \frac{\partial}{\partial \Omega} \oint d\mathbf{K} \frac{q_\mu q_\nu \bar{W}}{\Omega}, \tag{B 25}$$

one can rewrite (B 24) compactly as follows:

$$\varrho^{\alpha\beta} = J^{\alpha\mu} J^{\beta\nu} (\partial_\nu Q_\mu - \partial_\mu Q_\nu) + J^{\alpha\mu} J^{\beta\nu} ((\partial_\nu v^\lambda) R_{\lambda\mu} - (\partial_\mu v^\lambda) R_{\lambda\nu}). \tag{B 26}$$

Notice also that $\Theta^{\alpha\beta} = 2J^{\alpha\mu} J^{\beta\nu} R_{\mu\nu}$. Hence, for $U^{\alpha\beta}$ introduced in (B 23), one has

$$\begin{aligned}
& U^{\alpha\beta} - J^{\alpha\mu} J^{\beta\nu} (\partial_\nu Q_\mu - \partial_\mu Q_\nu) \\
& = \varrho^{\alpha\beta} - J^{\alpha\mu} J^{\beta\nu} (\partial_\nu Q_\mu - \partial_\mu Q_\nu) - (\Theta^{\gamma\beta} (\partial_\gamma v^\alpha) - \Theta^{\alpha\gamma} (\partial_\gamma v^\beta))/2 \\
& = J^{\alpha\mu} J^{\beta\nu} ((\partial_\nu v^\lambda) R_{\lambda\mu} - (\partial_\mu v^\lambda) R_{\lambda\nu}) + J^{\alpha\mu} J^{\gamma\nu} (\partial_\gamma v^\beta) R_{\mu\nu} - J^{\gamma\mu} J^{\beta\nu} (\partial_\gamma v^\alpha) R_{\mu\nu} \\
& = J^{\alpha\mu} J^{\beta\nu} (\partial_\nu v^\lambda) R_{\lambda\mu} - J^{\alpha\mu} J^{\beta\nu} (\partial_\mu v^\lambda) R_{\lambda\nu} + J^{\alpha\mu} J^{\gamma\nu} (\partial_\gamma v^\beta) R_{\mu\nu} - J^{\gamma\mu} J^{\beta\nu} (\partial_\gamma v^\alpha) R_{\mu\nu} \\
& = J^{\alpha\mu} J^{\beta\lambda} (\partial_\lambda v^\nu) R_{\mu\nu} - J^{\alpha\lambda} J^{\beta\nu} (\partial_\lambda v^\mu) R_{\mu\nu} + J^{\alpha\mu} J^{\gamma\nu} (\partial_\gamma v^\beta) R_{\mu\nu} - J^{\gamma\mu} J^{\beta\nu} (\partial_\gamma v^\alpha) R_{\mu\nu} \\
& = (J^{\alpha\mu} J^{\beta\lambda} J^{\nu\gamma} - J^{\alpha\lambda} J^{\beta\nu} J^{\mu\gamma} + J^{\alpha\mu} J^{\gamma\nu} J^{\beta\lambda} - J^{\gamma\mu} J^{\beta\nu} J^{\alpha\lambda}) (\partial_{\gamma\lambda}^2 \bar{H}) R_{\mu\nu} \\
& = (J^{\alpha\mu} J^{\beta\lambda} J^{\nu\gamma} - J^{\alpha\lambda} J^{\beta\nu} J^{\mu\gamma} - J^{\alpha\mu} J^{\gamma\nu} J^{\lambda\beta} + J^{\mu\gamma} J^{\beta\nu} J^{\alpha\lambda}) (\partial_{\gamma\lambda}^2 \bar{H}) R_{\mu\nu} \\
& = 0, \tag{B 27}
\end{aligned}$$

where we used (3.8) for v^α and the anti-symmetry of $J^{\alpha\beta}$. Therefore,

$$U^{\alpha\beta} = J^{\alpha\mu} J^{\beta\nu} (\partial_\nu Q_\mu - \partial_\mu Q_\nu) = (J^{\alpha\mu} J^{\beta\nu} - J^{\alpha\nu} J^{\beta\mu}) \partial_\nu Q_\mu = -U^{\beta\alpha}, \tag{B 28}$$

and accordingly,

$$\begin{aligned}
& \partial_\alpha U^{\alpha\beta} = (J^{\alpha\mu} J^{\beta\nu} - J^{\alpha\nu} J^{\beta\mu}) \partial_{\nu\alpha}^2 Q_\mu = J^{\alpha\mu} J^{\beta\nu} \partial_{\nu\alpha}^2 Q_\mu \\
& = J^{\nu\mu} J^{\beta\alpha} \partial_{\nu\alpha}^2 Q_\mu = J^{\beta\alpha} \partial_\alpha (J^{\nu\mu} \partial_\nu Q_\mu) = -J^{\alpha\beta} \partial_\alpha (J^{\mu\nu} \partial_\mu Q_\nu) \equiv J^{\alpha\beta} \partial_\alpha \Phi. \tag{B 29}
\end{aligned}$$

Here, $\Phi \doteq -J^{\mu\nu} \partial_\mu Q_\nu$, which is equivalent to (5.24). From (B 29) and the fact that $U^{\alpha\beta} \partial_{\alpha\beta} = 0$ due to the anti-symmetry of $U^{\alpha\beta}$, one has

$$\partial_\alpha (U^{\alpha\beta} \partial_\beta \bar{f}) = J^{\alpha\beta} (\partial_\alpha \Phi) (\partial_\beta \bar{f}) = \{\Phi, \bar{f}\}. \tag{B 30}$$

Hence, (B 23) leads to (5.25).

B.7. Proof of (6.30)

The correlation function

$$\mathfrak{C}_{ss'}(t, \mathbf{x}, \tau, \mathbf{s}; \mathbf{p}, \mathbf{p}') \doteq \overline{g_s(t + \tau/2, \mathbf{x} + \mathbf{s}/2, \mathbf{p}) g_{s'}(t - \tau/2, \mathbf{x} - \mathbf{s}/2, \mathbf{p}')} \tag{B 31}$$

can be readily expressed as

$$\begin{aligned}
\mathfrak{C}_{ss'} &= \left(\delta_{ss'} \sum_{\sigma_s = \sigma'_{s'}} + \sum_{\sigma_s \neq \sigma'_{s'}} \right) \langle \delta(\mathbf{x} + \mathbf{s}/2 - \bar{\mathbf{x}}_{\sigma_s}(t + \tau/2)) \delta(\mathbf{p} - \bar{\mathbf{p}}_{\sigma_s}(t + \tau/2)) \\
&\quad \times \delta(\mathbf{x} - \mathbf{s}/2 - \bar{\mathbf{x}}_{\sigma'_{s'}}(t - \tau/2)) \delta(\mathbf{p}' - \bar{\mathbf{p}}_{\sigma'_{s'}}(t - \tau/2)) \rangle - \mathfrak{C}_{\bar{f}}.
\end{aligned}$$

Here, $\langle \dots \rangle$ is another (in addition to overbar) notation for averaging used in this appendix, the dependence of $\bar{f}_s$ on $(t, \mathbf{x})$ is neglected, and ' $\sigma_s \neq \sigma'_{s'}$ ' denotes that excluded are the terms that have $s' = s$ and $\sigma_s = \sigma'_{s'}$ simultaneously. Aside from this, the summations over σ_s are taken over all $N_s \gg 1$ particles of type s , and the summations over $\sigma_{s'}$ are taken over all $N_{s'} \gg 1$ particles of type s' . Also,

$$\mathfrak{C}_{\bar{f}} \doteq \langle \bar{f}_s(t + \tau/2, \mathbf{x} + \mathbf{s}/2, \mathbf{p}) \bar{f}_{s'}(t - \tau/2, \mathbf{x} - \mathbf{s}/2, \mathbf{p}') \rangle. \quad (\text{B } 32)$$

To the leading order, pair correlations can be neglected. Then,

$$\begin{aligned} \sum_{\sigma_s \neq \sigma'_{s'}} \langle \dots \rangle &= \sum_{\sigma_s \neq \sigma'_{s'}} \underbrace{\langle \delta(\mathbf{x} + \mathbf{s}/2 - \bar{\mathbf{x}}_{\sigma_s}(t + \tau/2)) \delta(\mathbf{p} - \bar{\mathbf{p}}_{\sigma_s}(t + \tau/2)) \rangle}_{\bar{f}_s(t + \tau/2, \mathbf{x} + \mathbf{s}/2, \mathbf{p})/N_s} \\ &\quad \times \underbrace{\langle \delta(\mathbf{x} - \mathbf{s}/2 - \bar{\mathbf{x}}_{\sigma'_{s'}}(t - \tau/2)) \delta(\mathbf{p}' - \bar{\mathbf{p}}_{\sigma'_{s'}}(t - \tau/2)) \rangle}_{\bar{f}_{s'}(t - \tau/2, \mathbf{x} - \mathbf{s}/2, \mathbf{p}')/N_{s'}} \\ &= \frac{\mathfrak{C}_{\bar{f}}}{N_s N_{s'}} \sum_{\sigma_s, \sigma'_{s'}} (1 - \delta_{ss'} \delta_{\sigma_s \sigma'_{s'}}) = (1 - N_s^{-1} \delta_{ss'}) \mathfrak{C}_{\bar{f}} \approx \mathfrak{C}_{\bar{f}}. \end{aligned} \quad (\text{B } 33)$$

Let us also use $\bar{\mathbf{p}}_{\sigma_s}(t + \tau/2) \approx \bar{\mathbf{p}}_{\sigma_s}(t)$. Then,

$$\begin{aligned} \mathfrak{C}_{ss'} &\approx \delta_{ss'} \delta(\mathbf{p} - \mathbf{p}') \sum_{\sigma=1}^{N_s} \langle \delta(\mathbf{x} + \mathbf{s}/2 - \bar{\mathbf{x}}_{\sigma}(t + \tau/2)) \\ &\quad \times \delta(\mathbf{x} - \mathbf{s}/2 - \bar{\mathbf{x}}_{\sigma}(t - \tau/2)) \delta(\mathbf{p} - \bar{\mathbf{p}}_{\sigma}(t)) \rangle. \end{aligned} \quad (\text{B } 34)$$

Next, notice that

$$\begin{aligned} &\langle \delta(\mathbf{x} + \mathbf{s}/2 - \bar{\mathbf{x}}_{\sigma}(t + \tau/2)) \delta(\mathbf{x} - \mathbf{s}/2 - \bar{\mathbf{x}}_{\sigma}(t - \tau/2)) \delta(\mathbf{p} - \bar{\mathbf{p}}_{\sigma}(t)) \rangle \\ &= \langle \delta(\mathbf{s} + \bar{\mathbf{x}}_{\sigma}(t - \tau/2) - \bar{\mathbf{x}}_{\sigma}(t + \tau/2)) \delta(\mathbf{x} - \mathbf{s}/2 - \bar{\mathbf{x}}_{\sigma}(t - \tau/2)) \delta(\mathbf{p} - \bar{\mathbf{p}}_{\sigma}(t)) \rangle \\ &= \langle \delta(\mathbf{s} + \bar{\mathbf{x}}_{\sigma}(t - \tau/2) - \bar{\mathbf{x}}_{\sigma}(t + \tau/2)) \delta(\mathbf{x} - (\bar{\mathbf{x}}_{\sigma}(t + \tau/2) + \bar{\mathbf{x}}_{\sigma}(t - \tau/2))/2) \delta(\mathbf{p} - \bar{\mathbf{p}}_{\sigma}(t)) \rangle \\ &\approx \langle \delta(\mathbf{s} - \mathbf{v}_s(t, \bar{\mathbf{x}}_{\sigma}, \bar{\mathbf{p}}_{\sigma}) \tau) \delta(\mathbf{x} - \bar{\mathbf{x}}_{\sigma}(t)) \delta(\mathbf{p} - \bar{\mathbf{p}}_{\sigma}(t)) \rangle \\ &\approx \delta(\mathbf{s} - \mathbf{v}_s(t, \mathbf{x}, \mathbf{p}) \tau) \langle \delta(\mathbf{x} - \bar{\mathbf{x}}_{\sigma}(t)) \delta(\mathbf{p} - \bar{\mathbf{p}}_{\sigma}(t)) \rangle \\ &= \delta(\mathbf{s} - \mathbf{v}_s(t, \mathbf{x}, \mathbf{p}) \tau) \bar{f}_s(t, \mathbf{x}, \mathbf{p})/N_s. \end{aligned} \quad (\text{B } 35)$$

Hence,

$$\mathfrak{C}_{ss'} = \delta_{ss'} \delta(\mathbf{p} - \mathbf{p}') \delta(\mathbf{s} - \mathbf{v}_s(t, \mathbf{x}, \mathbf{p}) \tau) F_s(t, \mathbf{x}, \mathbf{p}), \quad (\text{B } 36)$$

where we used $\bar{f}_s \approx F_s$. Therefore,

$$\begin{aligned} \mathfrak{G}_{ss'}(t, \mathbf{x}, \omega, \mathbf{k}; \mathbf{p}, \mathbf{p}') &= \int \frac{d\tau}{2\pi} \frac{d\mathbf{s}}{(2\pi)^n} e^{i\omega\tau - i\mathbf{k} \cdot \mathbf{s}} \mathfrak{C}_{ss'}(t, \mathbf{x}, \tau, \mathbf{s}; \mathbf{p}, \mathbf{p}') \\ &\approx \frac{1}{(2\pi)^n} \delta_{ss'} \delta(\mathbf{p} - \mathbf{p}') F_s(t, \mathbf{x}, \mathbf{p}). \end{aligned} \quad (\text{B } 37)$$

B.8. Proof of (9.80)

Using the symmetry $\mathcal{U}^{\alpha\beta\gamma\delta} = \mathcal{U}^{\beta\alpha\gamma\delta} = \mathcal{U}^{\beta\alpha\delta\gamma}$, one readily obtains from (6.47) that

$$\Delta = \frac{1}{2P^0} \int d\mathbf{k} \bar{g}_{\beta\gamma} p_{\alpha} p_{\delta} \mathcal{U}^{\alpha\beta\gamma\delta} + \frac{1}{8} \oint d\mathbf{k} \mathcal{J}, \quad (\text{B } 38)$$

$$\mathcal{J} = -\frac{\partial'}{\partial \mathbf{p}} \cdot \left(\frac{\mathbf{k} \mathfrak{E}}{P^0 \varpi} \right) - \frac{1}{(P^0)^2} \frac{\partial \mathfrak{E}}{\partial p_0} + \frac{\bar{g}^{00} \mathfrak{E}}{(P^0)^3}, \quad (\text{B } 39)$$

where $\varpi \doteq k_\rho p^\rho = P^0(\mathbf{k} \cdot \mathbf{v} - \omega)$ and the prime in ∂' denotes that p_0 is considered as a function of $\mathbf{p}$ at differentiation. One can also write this as follows:

$$\mathcal{J} = \frac{\mathbf{k}}{\varpi} \cdot \left(\frac{\partial' P_0}{\partial \mathbf{p}} \right) \frac{\mathfrak{E}}{(P^0)^2} - \frac{1}{P^0} \frac{\partial'}{\partial \mathbf{p}} \cdot \left(\frac{\mathbf{k} \mathfrak{E}}{\varpi} \right) - \frac{1}{(P^0)^2} \frac{\partial \mathfrak{E}}{\partial p_0} + \frac{\bar{g}^{00} \mathfrak{E}}{(P^0)^3}. \quad (\text{B } 40)$$

As shown in (Garg & Dodin 2020, appendix B), the following equality is satisfied:

$$\frac{\mathbf{k}}{\varpi} \cdot \left(\frac{\partial' P_0}{\partial \mathbf{p}} \right) = \frac{1}{\varpi} \frac{\partial \varpi}{\partial p_0} - \frac{\bar{g}^{00}}{P^0}. \quad (\text{B } 41)$$

Also notice that

$$\begin{aligned} \frac{\partial'}{\partial \mathbf{p}} \cdot \left(\frac{\mathbf{k} \mathfrak{E}}{\varpi} \right) &= \frac{\partial}{\partial \mathbf{p}} \cdot \left(\frac{\mathbf{k} \mathfrak{E}}{\varpi} \right) + \frac{\partial P_0}{\partial \mathbf{p}} \cdot \frac{\partial}{\partial p_0} \left(\frac{\mathbf{k} \mathfrak{E}}{\varpi} \right) \\ &= \frac{\partial}{\partial \mathbf{p}} \cdot \left(\frac{\mathbf{k} \mathfrak{E}}{\varpi} \right) - \mathbf{k} \cdot \mathbf{v} \frac{\partial}{\partial p_0} \left(\frac{\mathfrak{E}}{\varpi} \right), \end{aligned} \quad (\text{B } 42)$$

where we used Hamilton's equation $\partial_{\mathbf{p}} P_0 = -\partial_{\mathbf{p}} H = -\mathbf{v}$. Therefore,

$$\mathcal{J} = \frac{1}{\varpi} \frac{\partial \varpi}{\partial p_0} \frac{\mathfrak{E}}{(P^0)^2} - \frac{1}{P^0} \frac{\partial}{\partial \mathbf{p}} \cdot \left(\frac{\mathbf{k} \mathfrak{E}}{\varpi} \right) + \frac{\mathbf{k} \cdot \mathbf{v}}{P^0} \frac{\partial}{\partial p_0} \left(\frac{\mathfrak{E}}{\varpi} \right) - \frac{1}{(P^0)^2} \frac{\partial \mathfrak{E}}{\partial p_0}. \quad (\text{B } 43)$$

The first and the last terms can be merged; then, one obtains

$$\begin{aligned} \mathcal{J} &= -\frac{1}{P^0} \frac{\partial}{\partial \mathbf{p}} \cdot \left(\frac{\mathbf{k} \mathfrak{E}}{\varpi} \right) + \frac{\mathbf{k} \cdot \mathbf{v}}{P^0} \frac{\partial}{\partial p_0} \left(\frac{\mathfrak{E}}{\varpi} \right) - \frac{\varpi}{(P^0)^2} \frac{\partial}{\partial p_0} \left(\frac{\mathfrak{E}}{\varpi} \right) \\ &= -\frac{1}{P^0} \frac{\partial}{\partial \mathbf{p}} \cdot \left(\frac{\mathbf{k} \mathfrak{E}}{\varpi} \right) + \frac{\omega}{P^0} \frac{\partial}{\partial p_0} \left(\frac{\mathfrak{E}}{\varpi} \right) \\ &= -\frac{1}{P^0} \frac{\partial}{\partial p_\lambda} \left(\frac{k_\lambda \mathfrak{E}}{\varpi} \right). \end{aligned} \quad (\text{B } 44)$$

In combination with (B 38), this leads to (9.80).

Appendix C. Properties of the collision operator

Here, we prove the properties of the collision operator discussed in section 6.8. To shorten the calculations, we introduce two auxiliary functions,

$$\begin{aligned} \mathcal{Z}_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') &\doteq \pi \delta(\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}_{s'}) \mathcal{Q}_{ss'}(\mathbf{k} \cdot \mathbf{v}_s, \mathbf{k}; \mathbf{p}, \mathbf{p}'), \\ \mathcal{F}_{ss'}(\mathbf{p}, \mathbf{p}') &\doteq \frac{\partial F_s(\mathbf{p})}{\partial p_j} F_{s'}(\mathbf{p}') - F_s(\mathbf{p}) \frac{\partial F_{s'}(\mathbf{p}')}{\partial p'_j}, \end{aligned} \quad (\text{C } 1)$$

which have the following properties:

$$\mathcal{Z}_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') = \mathcal{Z}_{s's}(\mathbf{k}; \mathbf{p}', \mathbf{p}), \quad \mathcal{F}_{ss'}(\mathbf{p}, \mathbf{p}') = -\mathcal{F}_{s's}(\mathbf{p}', \mathbf{p}). \quad (\text{C } 2)$$

C.1. Momentum conservation

Momentum conservation is proven as follows. Using integration by parts, one obtains

$$\begin{aligned} &\sum_s \int d\mathbf{p} p_l \mathcal{C}_s \\ &= \sum_{s, s'} \int d\mathbf{p} p_l \frac{\partial}{\partial p_i} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' k_i k_j \mathcal{Z}_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') \mathcal{F}_{ss'}(\mathbf{p}, \mathbf{p}') \end{aligned}$$

$$= - \sum_{s,s'} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p} d\mathbf{p}' k_l k_j Z_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') \mathcal{F}_{ss'}(\mathbf{p}, \mathbf{p}'). \quad (\text{C } 3)$$

Now we swap the dummy variables $s \leftrightarrow s'$ and $\mathbf{p} \leftrightarrow \mathbf{p}'$ and then apply (C2):

$$\begin{aligned} & \sum_s \int d\mathbf{p} p_l \mathcal{C}_s \\ &= - \sum_{s',s} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' d\mathbf{p} k_l k_j Z_{s's}(\mathbf{k}; \mathbf{p}', \mathbf{p}) \mathcal{F}_{s's}(\mathbf{p}', \mathbf{p}) \\ &= \sum_{s',s} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' d\mathbf{p} k_l k_j Z_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') \mathcal{F}_{ss'}(\mathbf{p}, \mathbf{p}'). \end{aligned} \quad (\text{C } 4)$$

The expression on the right-hand side of (C4) is minus that in (C3). Hence, both are zero, which proves that $\sum_s \int d\mathbf{p} p_l \mathcal{C}_s = 0$.

C.2. Energy conservation

Energy conservation is proven similarly, using that $v_s^i = \partial \mathcal{H}_s / \partial p_i$ and the fact that $\mathbf{k} \cdot \mathbf{v}_s$ and $\mathbf{k} \cdot \mathbf{v}'_{s'}$ are interchangeable due to the presence of $\delta(\mathbf{k} \cdot \mathbf{v}_s - \mathbf{k} \cdot \mathbf{v}'_{s'})$ in $Z_{ss'}$:

$$\begin{aligned} & \sum_s \int d\mathbf{p} \mathcal{H}_s \mathcal{C}_s \\ &= \sum_{s,s'} \int d\mathbf{p} \mathcal{H}_s \frac{\partial}{\partial p_i} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' k_i k_j Z_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') \mathcal{F}_{ss'}(\mathbf{p}, \mathbf{p}') \\ &= - \sum_{s,s'} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p} d\mathbf{p}' (\mathbf{k} \cdot \mathbf{v}_s) k_j Z_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') \mathcal{F}_{ss'}(\mathbf{p}, \mathbf{p}') \\ &= - \sum_{s',s} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' d\mathbf{p} (\mathbf{k} \cdot \mathbf{v}'_{s'}) k_j Z_{s's}(\mathbf{k}; \mathbf{p}', \mathbf{p}) \mathcal{F}_{s's}(\mathbf{p}', \mathbf{p}) \\ &= \sum_{s',s} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' d\mathbf{p} (\mathbf{k} \cdot \mathbf{v}_s) k_j Z_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') \mathcal{F}_{ss'}(\mathbf{p}, \mathbf{p}'). \end{aligned} \quad (\text{C } 5)$$

Like in the previous case, the third and the fifth lines are minus each other, whence $\sum_s \int d\mathbf{p} \mathcal{H}_s \mathcal{C}_s = 0$.

C.3. H-theorem

From (6.72) and (6.73), one has

$$\left(\frac{d\sigma}{dt} \right)_{\text{coll}} = - \sum_s \int d\mathbf{p} (1 + \ln F_s(\mathbf{p})) \mathcal{C}_s = - \sum_s \int d\mathbf{p} \ln F_s(\mathbf{p}) \mathcal{C}_s, \quad (\text{C } 6)$$

where we used particle conservation, $\int d\mathbf{p} \mathcal{C}_s = 0$. Then,

$$\begin{aligned} \left(\frac{d\sigma}{dt} \right)_{\text{coll}} &= - \sum_{ss'} \int d\mathbf{p} \ln F_s(\mathbf{p}) \frac{\partial}{\partial p_i} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p}' k_i k_j Z_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') \mathcal{F}_{ss'}(\mathbf{p}, \mathbf{p}') \\ &= \sum_{ss'} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p} d\mathbf{p}' k_i k_j \frac{\partial \ln F_s(\mathbf{p})}{\partial p_i} Z_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') \mathcal{F}_{ss'}(\mathbf{p}, \mathbf{p}'). \end{aligned} \quad (\text{C } 7)$$

Let us swap the dummy variables $s \leftrightarrow s'$ and $\mathbf{p} \leftrightarrow \mathbf{p}'$ and then apply (C2) to obtain

$$\left(\frac{d\sigma}{dt} \right)_{\text{coll}} = - \sum_{ss'} \int \frac{d\mathbf{k}}{(2\pi)^n} \int d\mathbf{p} d\mathbf{p}' k_i k_j \frac{\partial \ln F_{s'}(\mathbf{p}')}{\partial p'_i} Z_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') \mathcal{F}_{ss'}(\mathbf{p}, \mathbf{p}'). \quad (\text{C } 8)$$

Upon comparing (C8) with (C7), one can put the result in a symmetrized form:

$$\left(\frac{d\sigma}{dt}\right)_{\text{coll}} = \frac{1}{2} \sum_{ss'} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p} d\mathbf{p}' k_i k_j Z_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') \mathcal{F}_{ss'}(\mathbf{p}, \mathbf{p}') \times \left(\frac{\partial \ln F_s(\mathbf{p})}{\partial p_i} - \frac{\partial \ln F_{s'}(\mathbf{p}')}{\partial p'_i} \right). \quad (\text{C } 9)$$

But notice that

$$\mathcal{F}_{ss'}(\mathbf{p}, \mathbf{p}') = \left(\frac{\partial \ln F_s(\mathbf{p})}{\partial p_j} - \frac{\partial \ln F_{s'}(\mathbf{p}')}{\partial p'_j} \right) F_s(\mathbf{p}) F_{s'}(\mathbf{p}'). \quad (\text{C } 10)$$

Thus,

$$\left(\frac{d\sigma}{dt}\right)_{\text{coll}} = \frac{1}{2} \sum_{ss'} \int \frac{d\mathbf{k}}{(2\pi)^n} d\mathbf{p} d\mathbf{p}' Z_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}') F_s(\mathbf{p}) F_{s'}(\mathbf{p}') \times \left(\mathbf{k} \cdot \frac{\partial \ln F_s(\mathbf{p})}{\partial \mathbf{p}} - \mathbf{k} \cdot \frac{\partial \ln F_{s'}(\mathbf{p}')}{\partial \mathbf{p}'} \right)^2 \geq 0. \quad (\text{C } 11)$$

Appendix D. Conservation laws for on-shell waves

Here, we prove the momentum-conservation theorem (7.87) and the energy-conservation theorem (7.89) for QL interactions of plasmas with on-shell waves.

D.1. Momentum conservation

Let us multiply (7.83) by k_l and integrate over $\mathbf{k}$. Then, one obtains

$$\begin{aligned} 0 &= \int d\mathbf{k} k_l \frac{\partial J}{\partial t} + \int d\mathbf{k} k_l \frac{\partial (v_g^i J)}{\partial x^i} - \int d\mathbf{k} k_l \frac{\partial}{\partial k_i} \left(\frac{\partial w}{\partial x^i} J \right) - 2 \int d\mathbf{k} k_l \gamma J \\ &= \frac{\partial}{\partial t} \int d\mathbf{k} k_l J + \frac{\partial}{\partial x^i} \int d\mathbf{k} k_l v_g^i J + \int d\mathbf{k} \frac{\partial w}{\partial x^l} J - 2 \int d\mathbf{k} w \gamma J. \end{aligned} \quad (\text{D } 1)$$

Similarly, multiplying (7.84) by $\mathcal{H}_s$ and integrating over $\mathbf{p}$ yields

$$\begin{aligned} 0 &= \int d\mathbf{p} p_l \frac{\partial F_s}{\partial t} + \int d\mathbf{p} p_l \frac{\partial (v_s^i F_s)}{\partial x^i} - \int d\mathbf{p} p_l \frac{\partial}{\partial p_i} \left(\frac{\partial \mathcal{H}_s}{\partial x^i} F_s \right) \\ &\quad - \int d\mathbf{p} p_l \frac{\partial}{\partial p_i} \left(\mathcal{D}_{s,ij} \frac{\partial F_s}{\partial p_j} \right) - \int d\mathbf{p} p_l \mathcal{C}_s \\ &= \frac{\partial}{\partial t} \int d\mathbf{p} p_l F_s + \frac{\partial}{\partial x^i} \int d\mathbf{p} p_l v_s^i F_s + \frac{\partial}{\partial x^l} \int d\mathbf{p} \Delta_s F_s + \int d\mathbf{p} \frac{\partial H_{0s}}{\partial x^l} F_s \\ &\quad - \int d\mathbf{p} \Delta_s \frac{\partial F_s}{\partial x^l} + \int d\mathbf{p} \mathcal{D}_{s,lj} \frac{\partial F_s}{\partial p_j} - \int d\mathbf{p} p_l \mathcal{C}_s. \end{aligned} \quad (\text{D } 2)$$

Let us sum up (D2) over species and also add it with (D1). The contribution of the collision integral disappears due to (6.71), so one obtains

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \left(\sum_s \int d\mathbf{p} p_l F_s + \int d\mathbf{k} k_l J \right) + \frac{\partial}{\partial x^i} \left(\sum_s \int d\mathbf{p} p_l v_s^i F_s + \int d\mathbf{k} k_l v_g^i J \right) \\ &\quad + \sum_s \frac{\partial}{\partial x^l} \int d\mathbf{p} \Delta_s F_s + \sum_s \int d\mathbf{p} \frac{\partial H_{0s}}{\partial x^l} F_s \end{aligned}$$

$$\begin{aligned}
& + \sum_s \int d\mathbf{p} \mathbb{D}_{s,lj} \frac{\partial F_s}{\partial p_j} - 2 \int d\mathbf{k} k_l \gamma J \\
& - \sum_s \int d\mathbf{p} \Delta_s \frac{\partial F_s}{\partial x^l} + \int d\mathbf{k} J \frac{\partial w}{\partial x^l}.
\end{aligned} \tag{D3}$$

Next, notice that

$$\begin{aligned}
2 \int d\mathbf{k} k_l \gamma J &= 2\pi \sum_s \int d\mathbf{p} d\mathbf{k} k_l k_j \frac{|\boldsymbol{\alpha}_s^\dagger \boldsymbol{\eta}|^2}{\partial_\omega \Lambda} J \delta(w - \mathbf{k} \cdot \mathbf{v}_s) \frac{\partial F_s}{\partial p_j} \\
&= \sum_s \int d\mathbf{p} \mathbb{D}_{s,lj} \frac{\partial F_s}{\partial p_j}.
\end{aligned} \tag{D4}$$

Also, assuming that $\boldsymbol{\Xi}_0$, $|\boldsymbol{\alpha}_s^\dagger \boldsymbol{\eta}|^2$, and $\boldsymbol{\eta}^\dagger \boldsymbol{\wp}_s \boldsymbol{\eta}$ are independent of $\mathbf{x}$ and using (7.80), one gets

$$\begin{aligned}
& \sum_s \int d\mathbf{p} \Delta_s \frac{\partial F_s}{\partial x^l} \\
&= \sum_s \int d\mathbf{p} \frac{\partial F_s}{\partial x^l} \left(\frac{\partial}{\partial p_i} \oint d\omega d\mathbf{k} \frac{k_i}{2(\omega - \mathbf{k} \cdot \mathbf{v}_s)} |\boldsymbol{\alpha}_s^\dagger \boldsymbol{\eta}|^2 (h(\mathbf{k}) + h(-\mathbf{k})) \delta(\omega - w(\mathbf{k})) \right. \\
&\quad \left. + \frac{1}{2} \int d\omega d\mathbf{k} (\boldsymbol{\eta}^\dagger \boldsymbol{\wp}_s \boldsymbol{\eta}) (h(\mathbf{k}) + h(-\mathbf{k})) \delta(\omega - w(\mathbf{k})) \right) \\
&= -\frac{1}{2} \sum_s \oint d\omega d\mathbf{k} d\mathbf{p} (h(\mathbf{k}) + h(-\mathbf{k})) \delta(\omega - w(\mathbf{k})) \frac{k_i |\boldsymbol{\alpha}_s^\dagger \boldsymbol{\eta}|^2}{\omega - \mathbf{k} \cdot \mathbf{v}_s} \frac{\partial^2 F_s}{\partial x^l \partial p_i} \\
&\quad + \frac{1}{2} \sum_s \oint d\omega d\mathbf{k} d\mathbf{p} (h(\mathbf{k}) + h(-\mathbf{k})) \delta(\omega - w(\mathbf{k})) \frac{\partial}{\partial x^l} (\boldsymbol{\eta}^\dagger \boldsymbol{\wp}_s \boldsymbol{\eta} F_s) \\
&= -\sum_s \int d\omega d\mathbf{k} h(\mathbf{k}) \delta(\omega - w(\mathbf{k})) \frac{\partial}{\partial x^l} \oint d\mathbf{p} \left(\frac{k_i |\boldsymbol{\alpha}_s^\dagger \boldsymbol{\eta}|^2}{\omega - \mathbf{k} \cdot \mathbf{v}_s} \frac{\partial F_s}{\partial p_i} - \boldsymbol{\eta}^\dagger \boldsymbol{\wp}_s \boldsymbol{\eta} F_s \right) \\
&= -\sum_s \int d\omega d\mathbf{k} h(\mathbf{k}) \delta(\omega - w(\mathbf{k})) \frac{\partial (\boldsymbol{\eta}^\dagger \boldsymbol{\Xi} \boldsymbol{\eta})}{\partial x^l} \\
&= -\sum_s \int d\mathbf{k} h(\mathbf{k}) \frac{\partial \Lambda(w(\mathbf{k}), \mathbf{k})}{\partial x^l} \\
&= \sum_s \int d\mathbf{k} J \frac{\partial w}{\partial x^l},
\end{aligned} \tag{D5}$$

where we also used (7.75b). Substituting (D4) and (D5) into (D3) leads to (7.87).

D.2. Energy conservation

Let us multiply (7.83) by w and integrate over $\mathbf{k}$. Then, one obtains

$$\begin{aligned}
0 &= \int d\mathbf{k} w \frac{\partial J}{\partial t} + \int d\mathbf{k} w \frac{\partial (v_g^i J)}{\partial x^i} - \int d\mathbf{k} w \frac{\partial}{\partial k_i} \left(\frac{\partial w}{\partial x^i} J \right) - 2 \int d\mathbf{k} w \gamma J \\
&= \frac{\partial}{\partial t} \int d\mathbf{k} w J - \int d\mathbf{k} \frac{\partial w}{\partial t} J + \frac{\partial}{\partial x^i} \int d\mathbf{k} w v_g^i J - \int d\mathbf{k} \frac{\partial w}{\partial x^i} v_g^i J \\
&\quad + \int d\mathbf{k} v_g^i \frac{\partial w}{\partial x^i} J - 2 \int d\mathbf{k} w \gamma J \\
&= \frac{\partial}{\partial t} \int d\mathbf{k} w J + \frac{\partial}{\partial x^i} \int d\mathbf{k} w v_g^i J - \int d\mathbf{k} \frac{\partial w}{\partial t} J - 2 \int d\mathbf{k} w \gamma J.
\end{aligned} \tag{D6}$$

Similarly, multiplying (7.84) by $\mathcal{H}_s$ and integrating over $\mathbf{p}$ yields

$$\begin{aligned}
0 &= \int d\mathbf{p} \mathcal{H}_s \frac{\partial F_s}{\partial t} + \int d\mathbf{p} \mathcal{H}_s \frac{\partial(v_s^i F_s)}{\partial x^i} - \int d\mathbf{p} \mathcal{H}_s \frac{\partial}{\partial p_i} \left(\frac{\partial \mathcal{H}_s}{\partial x^i} F_s \right) \\
&\quad - \int d\mathbf{p} \mathcal{H}_s \frac{\partial}{\partial p_i} \left(D_{s,ij} \frac{\partial F_s}{\partial p_j} \right) - \int d\mathbf{p} \mathcal{H}_s C_s \\
&= \frac{\partial}{\partial t} \int d\mathbf{p} \mathcal{H}_s F_s - \int d\mathbf{p} \frac{\partial \mathcal{H}_s}{\partial t} F_s + \frac{\partial}{\partial x^i} \int d\mathbf{p} \mathcal{H}_s v_s^i F_s - \int d\mathbf{p} \frac{\partial \mathcal{H}_s}{\partial x^i} v_s^i F_s \\
&\quad + \int d\mathbf{p} v_s^i \frac{\partial \mathcal{H}_s}{\partial x^i} F_s + \int d\mathbf{p} v_s^i D_{s,ij} \frac{\partial F_s}{\partial p_j} - \int d\mathbf{p} \mathcal{H}_s C_s \\
&= \frac{\partial}{\partial t} \int d\mathbf{p} \mathcal{H}_s F_s + \frac{\partial}{\partial x^i} \int d\mathbf{p} \mathcal{H}_s v_s^i F_s - \int d\mathbf{p} \frac{\partial H_{0s}}{\partial t} F_s - \int d\mathbf{p} \frac{\partial \Delta_s}{\partial t} F_s \\
&\quad + \int d\mathbf{p} v_s^i D_{s,ij} \frac{\partial F_s}{\partial p_j} - \int d\mathbf{p} \mathcal{H}_s C_s \\
&= \frac{\partial}{\partial t} \int d\mathbf{p} H_{0s} F_s + \frac{\partial}{\partial x^i} \int d\mathbf{p} H_{0s} v_s^i F_s + \frac{\partial}{\partial x^i} \int d\mathbf{p} \Delta_s v_s^i F_s - \int d\mathbf{p} \frac{\partial H_{0s}}{\partial t} F_s \\
&\quad + \int d\mathbf{p} \Delta_s \frac{\partial F_s}{\partial t} + \int d\mathbf{p} v_s^i D_{s,ij} \frac{\partial F_s}{\partial p_j} - \int d\mathbf{p} \mathcal{H}_s C_s. \tag{D 7}
\end{aligned}$$

Let us sum up (D 7) over species and also add it with (D 6). The contribution of the collision integral disappears due to (6.71), so one obtains

$$\begin{aligned}
0 &= \frac{\partial}{\partial t} \left(\sum_s \int d\mathbf{p} H_{0s} F_s + \int d\mathbf{k} w J \right) + \frac{\partial}{\partial x^i} \left(\sum_s \int d\mathbf{p} H_{0s} v_s^i F_s + \int d\mathbf{k} w v_g^i J \right) \\
&\quad + \frac{\partial}{\partial x^i} \sum_s \int d\mathbf{p} \Delta_s v_s^i F_s - \sum_s \int d\mathbf{p} \frac{\partial H_{0s}}{\partial t} F_s \\
&\quad + \sum_s \int d\mathbf{p} v_s^i D_{s,ij} \frac{\partial F_s}{\partial p_j} - 2 \int d\mathbf{k} w \gamma J \\
&\quad + \sum_s \int d\mathbf{p} \Delta_s \frac{\partial F_s}{\partial t} - \int d\mathbf{k} J \frac{\partial w}{\partial t}. \tag{D 8}
\end{aligned}$$

Next, notice that

$$\begin{aligned}
2 \int d\mathbf{k} w \gamma J &= 2\pi \sum_s \int d\mathbf{p} d\mathbf{k} w k_j \frac{|\boldsymbol{\alpha}_s^\dagger \boldsymbol{\eta}|^2}{\partial_\omega \Lambda} J \delta(w - \mathbf{k} \cdot \mathbf{v}_s) \frac{\partial F_s}{\partial p_j} \\
&= \sum_s \int d\mathbf{p} v_s^i D_{s,ij} \frac{\partial F_s}{\partial p_j}. \tag{D 9}
\end{aligned}$$

Also, assuming that Ξ_0 , $|\boldsymbol{\alpha}_s^\dagger \boldsymbol{\eta}|^2$, and $\boldsymbol{\eta}^\dagger \boldsymbol{\wp}_s \boldsymbol{\eta}$ are independent of t and using (7.80), one gets

$$\begin{aligned}
&\sum_s \int d\mathbf{p} \Delta_s \frac{\partial F_s}{\partial t} \\
&= \sum_s \int d\mathbf{p} \frac{\partial F_s}{\partial t} \left(\frac{\partial}{\partial p_i} \oint d\omega d\mathbf{k} \frac{k_i}{2(\omega - \mathbf{k} \cdot \mathbf{v}_s)} |\boldsymbol{\alpha}_s^\dagger \boldsymbol{\eta}|^2 (h(\mathbf{k}) + h(-\mathbf{k})) \delta(\omega - w(\mathbf{k})) \right. \\
&\quad \left. + \frac{1}{2} \int d\omega d\mathbf{k} (\boldsymbol{\eta}^\dagger \boldsymbol{\wp}_s \boldsymbol{\eta}) (h(\mathbf{k}) + h(-\mathbf{k})) \delta(\omega - w(\mathbf{k})) \right)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \sum_s \oint d\omega d\mathbf{k} d\mathbf{p} (h(\mathbf{k}) + h(-\mathbf{k})) \delta(\omega - w(\mathbf{k})) \frac{k_i |\boldsymbol{\alpha}_s^\dagger \boldsymbol{\eta}|^2}{\omega - \mathbf{k} \cdot \mathbf{v}_s} \frac{\partial^2 F_s}{\partial t \partial p_i} \\
&\quad + \frac{1}{2} \sum_s \oint d\omega d\mathbf{k} d\mathbf{p} (h(\mathbf{k}) + h(-\mathbf{k})) \delta(\omega - w(\mathbf{k})) \frac{\partial}{\partial t} (\boldsymbol{\eta}^\dagger \boldsymbol{\wp}_s \boldsymbol{\eta} F_s) \\
&= - \sum_s \int d\omega d\mathbf{k} h(\mathbf{k}) \delta(\omega - w(\mathbf{k})) \frac{\partial}{\partial t} \oint d\mathbf{p} \left(\frac{k_i |\boldsymbol{\alpha}_s^\dagger \boldsymbol{\eta}|^2}{\omega - \mathbf{k} \cdot \mathbf{v}_s} \frac{\partial F_s}{\partial p_i} - \boldsymbol{\eta}^\dagger \boldsymbol{\wp}_s \boldsymbol{\eta} F_s \right) \\
&= - \sum_s \int d\omega d\mathbf{k} h(\mathbf{k}) \delta(\omega - w(\mathbf{k})) \frac{\partial (\boldsymbol{\eta}^\dagger \boldsymbol{\Xi} \boldsymbol{\eta})}{\partial t} \\
&= - \sum_s \int d\mathbf{k} h(\mathbf{k}) \frac{\partial \Lambda(w(\mathbf{k}), \mathbf{k})}{\partial t} \\
&= \sum_s \int d\mathbf{k} J \frac{\partial w}{\partial t}, \tag{D 10}
\end{aligned}$$

where we also used (7.75a). Substituting (D 9) and (D 10) into (D 8) leads to (7.89).

Appendix E. Uniqueness of the entropy-preserving distribution

Here, we prove that the Boltzmann–Gibbs distribution is the only distribution for which the entropy density σ is conserved. According to (C 11), σ is conserved when

$$\delta(\mathbf{k} \cdot (\mathbf{v}_s - \mathbf{v}'_{s'})) (\mathbf{k} \cdot \mathbf{G}_{ss'}(\mathbf{p}, \mathbf{p}'))^2 = 0 \tag{E 1}$$

(for all $\mathbf{p}$, $\mathbf{p}'$, and $\mathbf{k}$, as well as all s and s'), where

$$\mathbf{G}_{ss'}(\mathbf{p}, \mathbf{p}') \doteq \frac{\partial \ln F_s(\mathbf{p})}{\partial \mathbf{p}} - \frac{\partial \ln F_{s'}(\mathbf{p}')}{\partial \mathbf{p}'}. \tag{E 2}$$

Let us decompose the vector $\mathbf{G}_{ss'}(\mathbf{p}, \mathbf{p}')$ into components parallel and perpendicular to the vector $\mathbf{v}_s - \mathbf{v}'_{s'}$:

$$\mathbf{G}_{ss'}(\mathbf{p}, \mathbf{p}') = \lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'}) (\mathbf{v}_s - \mathbf{v}'_{s'}) + \mathbf{G}_{ss'}^\perp(\mathbf{p}, \mathbf{p}'), \tag{E 3}$$

where $\lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'})$ is a scalar function. (Because the velocities are functions of the momenta, one can as well consider $\lambda_{ss'}$ as a function of $\mathbf{p}$ and $\mathbf{p}'$.) Due to the presence of the delta function in (E 1), the contribution of the first term to (E 1) is zero, so (E 1) can be written as

$$\delta(\mathbf{k} \cdot (\mathbf{v}_s - \mathbf{v}'_{s'})) (\mathbf{k} \cdot \mathbf{G}_{ss'}^\perp(\mathbf{p}, \mathbf{p}'))^2 = 0. \tag{E 4}$$

By considering this formula for $\mathbf{k}$ parallel to $\mathbf{G}_{ss'}^\perp(\mathbf{p}, \mathbf{p}')$ (and thus perpendicular to $\mathbf{v}_s - \mathbf{v}'_{s'}$), one finds that $\mathbf{G}_{ss'}^\perp(\mathbf{p}, \mathbf{p}') = 0$. Combined with (E 2) and (E 3), this yields

$$\frac{\partial \ln F_s(\mathbf{p})}{\partial \mathbf{p}} - \frac{\partial \ln F_{s'}(\mathbf{p}')}{\partial \mathbf{p}'} = \lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'}) (\mathbf{v}_s - \mathbf{v}'_{s'}). \tag{E 5}$$

Also, by swapping $\mathbf{p} \leftrightarrow \mathbf{p}'$ and $s \leftrightarrow s'$, one finds that

$$\lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'}) = \lambda_{s's}(\mathbf{v}'_{s'}, \mathbf{v}_s). \tag{E 6}$$

Equation (E 5) yields, in particular, that[†]

$$\frac{\partial \ln F_s(\mathbf{p})}{\partial p_2} - \frac{\partial \ln F_{s'}(\mathbf{p}')}{\partial p'_2} = \lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'}) (v_{s,2} - v'_{s',2}), \tag{E 7a}$$

[†] The idea of this argument was brought to author's attention by G. W. Hammett and is taken from (Landreman 2017), where it is applied to single-species plasmas with a specific $\mathcal{H}_s$.

$$\frac{\partial \ln F_s(\mathbf{p})}{\partial p_3} - \frac{\partial \ln F_{s'}(\mathbf{p}')}{\partial p'_3} = \lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'}) (v_{s,3} - v'_{s',3}), \quad (\text{E } 7b)$$

where we have assumed some coordinate axes in the momentum and velocity space labeled $(1, 2, 3, \dots)$. Then,

$$\frac{\partial^2 \ln F_s(\mathbf{p})}{\partial p_2 \partial v_{s,1}} = \frac{\partial \lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'})}{\partial v_{s,1}} (v_{s,2} - v'_{s',2}), \quad (\text{E } 8a)$$

$$\frac{\partial^2 \ln F_s(\mathbf{p})}{\partial p_3 \partial v_{s,1}} = \frac{\partial \lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'})}{\partial v_{s,1}} (v_{s,3} - v'_{s',3}), \quad (\text{E } 8b)$$

where the derivative with respect to $v_{s,1}$ is taken at fixed $v_{s,i \neq 1}$ and at fixed $\mathbf{v}'_{s'}$. Due to (E 5), $\lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'})$ is continuous for all F_s and $F_{s'}$. (Here we consider only physical distributions, which are always differentiable.) Then, (E 8) leads to

$$\frac{1}{v_{s,2} - v'_{s',2}} \frac{\partial^2 \ln F_s(\mathbf{p})}{\partial p_2 \partial v_{s,1}} = \frac{1}{v_{s,3} - v'_{s',3}} \frac{\partial^2 \ln F_s(\mathbf{p})}{\partial p_3 \partial v_{s,1}}. \quad (\text{E } 9)$$

By differentiating this with respect to $v'_{s',2}$, one obtains

$$\frac{\partial^2 \ln F_s(\mathbf{p})}{\partial p_2 \partial v_{s,1}} = 0, \quad (\text{E } 10)$$

whence (E 8a) yields

$$\frac{\partial \lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'})}{\partial v_{s,1}} = 0. \quad (\text{E } 11)$$

By repeating this argument for other axes and for $\mathbf{v}'$ instead of $\mathbf{v}$, one can also extend (E 11) to

$$\frac{\partial \lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'})}{\partial \mathbf{v}} = 0, \quad \frac{\partial \lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'})}{\partial \mathbf{v}'} = 0. \quad (\text{E } 12)$$

Hence, $\lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'})$ is actually independent of the velocities; i.e. $\lambda_{ss'}(\mathbf{v}_s, \mathbf{v}'_{s'}) = \lambda_{ss'}$. Using this along with (E 6), one also finds that

$$\lambda_{ss'} = \lambda_{s's}. \quad (\text{E } 13)$$

Let us rewrite (E 5) as follows:

$$\frac{\partial \ln F_s(\mathbf{p})}{\partial \mathbf{p}} - \lambda_{ss'} \mathbf{v}_s = \frac{\partial \ln F_{s'}(\mathbf{p}')}{\partial \mathbf{p}'} - \lambda_{s's} \mathbf{v}'_{s'}. \quad (\text{E } 14)$$

Here, the left-hand side is independent of $\mathbf{p}'$ and the right-hand side is independent of $\mathbf{p}$, so both must be equal to some vector

$$\boldsymbol{\mu}_{ss'} = \boldsymbol{\mu}_{s's} \quad (\text{E } 15)$$

that is independent of both $\mathbf{p}$ and $\mathbf{p}'$. Because $\mathbf{v}_s = \partial_{\mathbf{p}} \mathcal{H}_s$, this is equivalent to

$$\ln F_s(\mathbf{p}) - \lambda_{ss'} \mathcal{H}_s(\mathbf{p}) = \boldsymbol{\mu}_{ss'} \cdot \mathbf{p} + \eta_{ss'} \quad (\text{E } 16a)$$

(and similarly for $\mathbf{p}'$), where the integration constant $\eta_{ss'}$ is independent of both $\mathbf{p}$ and $\mathbf{p}'$. This is supposed to hold for any s' , so one can also write

$$\ln F_s(\mathbf{p}) - \lambda_{ss''} \mathcal{H}_s(\mathbf{p}) = \boldsymbol{\mu}_{ss''} \cdot \mathbf{p} + \eta_{ss''}, \quad (\text{E } 16b)$$

where s'' is any other species index. Subtracting equations (E 16) from each other gives

$$(\lambda_{ss'} - \lambda_{ss''}) \mathcal{H}_s(\mathbf{p}) = (\boldsymbol{\mu}_{ss'} - \boldsymbol{\mu}_{ss''}) \cdot \mathbf{p} + \eta_{ss'} - \eta_{ss''}. \quad (\text{E } 17)$$

By differentiating this with respect to $\mathbf{p}$, one finds

$$(\lambda_{ss'} - \lambda_{ss''})\mathbf{v}_s = \boldsymbol{\mu}_{ss'} - \boldsymbol{\mu}_{ss''}. \quad (\text{E } 18)$$

By differentiating this further with respect to $\mathbf{v}_s$, one obtains $\lambda_{ss'} = \lambda_{ss''}$. Then, (E 18) yields $\boldsymbol{\mu}_{ss'} = \boldsymbol{\mu}_{ss''}$, and (E 17) yields $\eta_{ss'} = \eta_{ss''}$. In other words, the functions $\lambda_{ss'}$, $\boldsymbol{\mu}_{ss'}$, and $\eta_{ss'}$ are independent of their second index and thus can as well be written as

$$\lambda_{ss'} = \lambda_s, \quad \boldsymbol{\mu}_{ss'} = \boldsymbol{\mu}_s, \quad \eta_{ss'} = \eta_s. \quad (\text{E } 19)$$

But then, (E 13) and (E 15) also yield $\lambda_s = \lambda_{s'} \equiv \lambda$ and $\boldsymbol{\mu}_s = \boldsymbol{\mu}_{s'} \equiv \boldsymbol{\mu}$. Therefore, (E 16) can be written as

$$F_s(\mathbf{p}) = \text{const}_s \times \exp(\lambda \mathcal{H}_s(\mathbf{p}) + \boldsymbol{\mu} \cdot \mathbf{p}), \quad (\text{E } 20)$$

which is the Boltzmann–Gibbs distribution (section 8.1). This proves that a plasma that conserves its entropy density necessarily has the Boltzmann–Gibbs distribution.

Appendix F. Total momentum and energy

Here, we show that the total momentum and energy in the OC–wave representation equals the total momentum and energy in the particle–field representation.

F.1. Nonrelativistic electrostatic interactions

F.1.1. Momentum

Assuming the notation $\mathcal{P}_l \doteq \sum_s \int d\mathbf{p} p_l \bar{f}_s$ and using (9.17) for $\boldsymbol{\Theta}_s$, one can represent the OC momentum density as follows:

$$\begin{aligned} \sum_s \int d\mathbf{p} p_l F_s &= \mathcal{P}_l + \frac{1}{2} \sum_s \int d\mathbf{p} p_l \frac{\partial}{\partial p_i} \left(\Theta_{s,ij} \frac{\partial \bar{f}_s}{\partial p_j} \right) \\ &\approx \mathcal{P}_l - \frac{1}{2} \sum_s \int d\mathbf{p} \Theta_{s,lj} \frac{\partial F_s}{\partial p_j} \\ &= \mathcal{P}_l - \int d\mathbf{k} h(\mathbf{k}) \frac{\partial}{\partial \vartheta} \sum_s e_s^2 \oint d\mathbf{p} \frac{k_l}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{p}} \Big|_{\vartheta=0} \\ &= \mathcal{P}_l - \int d\mathbf{k} k_l h(\mathbf{k}) \frac{\partial}{\partial \vartheta} \left(\frac{k^2 (\epsilon_{\parallel \text{H}}(w(\mathbf{k}) + \vartheta, \mathbf{k}) - 1)}{4\pi} \right) \Big|_{\vartheta=0} \\ &= \mathcal{P}_l - \int d\mathbf{k} k_l J, \end{aligned} \quad (\text{F } 1)$$

where we substituted (9.16). This leads to (9.18).

F.1.2. Energy

Assuming the notation $\mathcal{K} \doteq \sum_s \int d\mathbf{p} H_{0s} \bar{f}_s$ and using (9.17) for $\boldsymbol{\Theta}_s$, one can represent the OC energy density as follows:

$$\begin{aligned} \sum_s \int d\mathbf{p} H_{0s} F_s &= \mathcal{K} + \frac{1}{2} \sum_s \int d\mathbf{p} \frac{p^2}{2m_s} \frac{\partial}{\partial p_i} \left(\Theta_{s,ij} \frac{\partial \bar{f}_s}{\partial p_j} \right) \\ &\approx \mathcal{K} - \frac{1}{2} \sum_s \int d\mathbf{p} v_s^i \Theta_{s,ij} \frac{\partial F_s}{\partial p_j} \\ &= \mathcal{K} - \int d\mathbf{k} h(\mathbf{k}) \frac{\partial}{\partial \vartheta} \sum_s e_s^2 \oint d\mathbf{p} \frac{\mathbf{k} \cdot \mathbf{v}}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{p}} \Big|_{\vartheta=0}. \end{aligned} \quad (\text{F } 2)$$

Notice that

$$\begin{aligned} \frac{\partial}{\partial \vartheta} \frac{\mathbf{k} \cdot \mathbf{v}}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} &= \frac{\partial}{\partial \vartheta} \left(-1 + \frac{w(\mathbf{k}) + \vartheta}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \right) \\ &= \frac{1}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} + w(\mathbf{k}) \frac{\partial}{\partial \vartheta} \frac{1}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta}. \end{aligned} \quad (\text{F } 3)$$

Then,

$$\begin{aligned} \sum_s \int d\mathbf{p} H_{0s} F_s &= \mathcal{K} - \int d\mathbf{k} w(\mathbf{k}) h(\mathbf{k}) \frac{\partial}{\partial \vartheta} \sum_s e_s^2 \oint d\mathbf{p} \frac{\mathbf{k}}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \cdot \frac{\partial F_s}{\partial \mathbf{p}} \Big|_{\vartheta=0} \\ &\quad - \int d\mathbf{k} h(\mathbf{k}) \sum_s e_s^2 \oint d\mathbf{p} \frac{\mathbf{k}}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s} \cdot \frac{\partial F_s}{\partial \mathbf{p}} \\ &= \mathcal{K} - \int d\mathbf{k} w(\mathbf{k}) h(\mathbf{k}) \frac{\partial}{\partial \vartheta} \left(\frac{k^2 (\epsilon_{\parallel \mathbf{H}}(w(\mathbf{k}) + \vartheta, \mathbf{k}) - 1)}{4\pi} \right) \Big|_{\vartheta=0} \\ &\quad - \int d\mathbf{k} h(\mathbf{k}) \frac{k^2 (\epsilon_{\parallel \mathbf{H}}(w(\mathbf{k}), \mathbf{k}) - 1)}{4\pi}. \end{aligned} \quad (\text{F } 4)$$

Using (9.14) and (9.16), one obtains that the sum of the OC and wave energy is given by

$$\begin{aligned} \sum_s \int d\mathbf{p} H_{0s} F_s + \int d\mathbf{k} w J &= \mathcal{K} + \int d\mathbf{k} h(\mathbf{k}) \frac{k^2}{4\pi} \\ &= \mathcal{K} + \sum_{\sigma} \frac{\bar{k}_{\sigma}^2 |\check{\varphi}_{\sigma}|^2}{16\pi} \\ &= \sum_s \int d\mathbf{p} \left(\frac{p^2}{2m_s} + e_s \bar{\varphi}_s \right) \bar{f}_s + \frac{1}{8\pi} \tilde{\mathbf{E}}^{\dagger} \tilde{\mathbf{E}}, \end{aligned} \quad (\text{F } 5)$$

where we also substituted (7.64).

F.2. Relativistic electromagnetic interactions

F.2.1. Momentum

Let us assume the notation $\mathcal{P} \doteq \sum_s \int d\mathbf{p} \mathbf{p} \bar{f}_s$ and

$$\chi(\omega, \mathbf{k}) \doteq \sum_s \frac{4\pi e_s^2}{\omega^2} \oint d\mathbf{p} \frac{\mathbf{v}_s \mathbf{v}_s^{\dagger}}{\omega - \mathbf{k} \cdot \mathbf{v}_s} \mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{p}} = \epsilon(\omega, \mathbf{k}) - \mathbf{1} + \frac{\mathbf{w}_p}{\omega^2}. \quad (\text{F } 6)$$

Then, using (9.52) for Θ_s , one can represent the OC momentum density as follows:

$$\begin{aligned} \sum_s \int d\mathbf{p} p_l F_s &\approx \mathcal{P}_l - \frac{1}{2} \sum_s \int d\mathbf{p} \Theta_{s,lj} \frac{\partial F_s}{\partial p_j} \\ &= \mathcal{P}_l - \int d\mathbf{k} k_l h(\mathbf{k}) \frac{\partial}{\partial \vartheta} \sum_s e_s^2 \oint d\mathbf{p} \frac{1}{w^2(\mathbf{k})} \frac{(\boldsymbol{\eta}^{\dagger} \mathbf{v}_s \mathbf{v}_s^{\dagger} \boldsymbol{\eta})}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{p}_j} \Big|_{\vartheta=0} \\ &= \mathcal{P}_l - \int d\mathbf{k} \frac{k_l h(\mathbf{k})}{4\pi \omega^2(\mathbf{k})} \boldsymbol{\eta}^{\dagger} \frac{\partial}{\partial \omega} (\omega^2 \chi(\omega, \mathbf{k})) \boldsymbol{\eta} \Big|_{\omega=w(\mathbf{k})} \\ &= \mathcal{P}_l + \int d\mathbf{k} \frac{k_l h(\mathbf{k})}{2\pi \omega(\mathbf{k})} - \int d\mathbf{k} \frac{k_l h(\mathbf{k})}{4\pi \omega^2(\mathbf{k})} \boldsymbol{\eta}^{\dagger} \frac{\partial}{\partial \omega} (\omega^2 \epsilon(\omega, \mathbf{k})) \boldsymbol{\eta} \Big|_{\omega=w(\mathbf{k})} \\ &= \mathcal{P}_l + \int d\mathbf{k} \frac{k_l}{4\pi \omega(\mathbf{k})} (h(\mathbf{k}) + h(-\mathbf{k})) - \int d\mathbf{k} k_l J, \end{aligned} \quad (\text{F } 7)$$

where we substituted (9.51) and used (7.23). Next, let us rewrite (9.54) as

$$\mathcal{P} = \mathcal{P}^{(\text{kin})} + \overline{(\tilde{\mathbf{A}}/c)} \sum_s e_s \int d\mathbf{p} f_s = \mathcal{P}^{(\text{kin})} + \frac{1}{4\pi c} \overline{\tilde{\mathbf{A}}(\nabla \cdot \tilde{\mathbf{E}})}, \quad (\text{F } 8)$$

where the last equality is due to Gauss's law. This gives

$$\mathcal{P}_l - \mathcal{P}_l^{(\text{kin})} = \frac{1}{4\pi} \overline{(-i\hat{\omega}^{-1}\tilde{E}_l)(\partial_j \tilde{E}^j)} = -\frac{i}{4\pi} \overline{(\hat{\omega}^{-1}\tilde{E}_l)(\partial_j \tilde{E}^j)^*}. \quad (\text{F } 9)$$

Then, using (2.53) and also (7.63) for $\mathbf{U}$, one obtains

$$\begin{aligned} \mathcal{P}_l - \mathcal{P}_l^{(\text{kin})} &\approx -\frac{i}{4\pi} \int d\omega d\mathbf{k} \omega^{-1} \mathbf{U}_l^j(\omega, \mathbf{k}) (ik_j)^* \\ &\approx -\int d\mathbf{k} \frac{\mathbf{k} \cdot \boldsymbol{\eta}^*}{4\pi w(\mathbf{k})} \eta_l(h(\mathbf{k}) + h(-\mathbf{k})), \end{aligned} \quad (\text{F } 10)$$

and thus (F 7) can be written as follows:

$$\begin{aligned} \sum_s \int d\mathbf{p} \mathbf{p} F_s + \int d\mathbf{k} \mathbf{k} J &\approx \mathcal{P} + \int d\mathbf{k} \frac{\mathbf{k}}{4\pi w(\mathbf{k})} (h(\mathbf{k}) + h(-\mathbf{k})) \\ &\approx \mathcal{P}^{(\text{kin})} + \int d\mathbf{k} \frac{1}{4\pi w(\mathbf{k})} (\mathbf{k} - \boldsymbol{\eta}(\mathbf{k} \cdot \boldsymbol{\eta}^*)) (h(\mathbf{k}) + h(-\mathbf{k})) \\ &= \mathcal{P}^{(\text{kin})} + \text{re} \int d\mathbf{k} \frac{h(\mathbf{k})}{2\pi w(\mathbf{k})} \boldsymbol{\eta}^* \times (\mathbf{k} \times \boldsymbol{\eta}), \end{aligned} \quad (\text{F } 11)$$

where we used (7.7) and (7.23) again. For an eikonal wave (9.57), which has $h(\mathbf{k}) = \delta(\mathbf{k} - \bar{\mathbf{k}}) |\bar{\mathbf{E}}|^2/4$ (section 7.4.1), this gives

$$\text{re} \int d\mathbf{k} \frac{h(\mathbf{k})}{2\pi w(\mathbf{k})} \boldsymbol{\eta}^* \times (\mathbf{k} \times \boldsymbol{\eta}) = \frac{1}{8\pi c} \text{re} \left(\check{\mathbf{E}}^* \times \left(\frac{c\bar{\mathbf{k}}}{\bar{\omega}} \times \check{\mathbf{E}} \right) \right) = \frac{\check{\mathbf{E}} \times \check{\mathbf{B}}}{4\pi c}. \quad (\text{F } 12)$$

In case of a broadband spectrum, the same equality applies as well, because contributions of the individual eikonal waves to both left-hand side and the right-hand side are additive. (Alternatively, one can invoke (2.53) again.) This leads to (9.53).

F.2.2. Energy

Assuming the notation $\mathcal{K} \doteq \sum_s \int d\mathbf{p} H_{0s} \bar{f}_s$ and using (9.52) for $\boldsymbol{\Theta}_s$, one can represent the OC energy density as follows:

$$\begin{aligned} \sum_s \int d\mathbf{p} H_{0s} F_s &\approx \mathcal{K} - \frac{1}{2} \sum_s \int d\mathbf{p} v_s^i \Theta_{s,ij} \frac{\partial F_s}{\partial p_j} \\ &\approx \mathcal{K} - \int d\mathbf{k} h(\mathbf{k}) \frac{\partial}{\partial \vartheta} \sum_s e_s^2 \oint d\mathbf{p} \frac{(\mathbf{k} \cdot \mathbf{v}_s)}{w^2(\mathbf{k})} \frac{(\boldsymbol{\eta}^\dagger \mathbf{v}_s \mathbf{v}_s^\dagger \boldsymbol{\eta})}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{p}} \Big|_{\vartheta=0}. \end{aligned}$$

Using (F 3) and (F 6) for $\boldsymbol{\chi}$, one further obtains

$$\begin{aligned} \sum_s \int d\mathbf{p} H_{0s} F_s &= \mathcal{K} - \int d\mathbf{k} \frac{h(\mathbf{k})}{4\pi w(\mathbf{k})} \boldsymbol{\eta}^\dagger \frac{\partial}{\partial \vartheta} \left(\sum_s 4\pi e_s^2 \oint d\mathbf{p} \frac{\mathbf{v}_s \mathbf{v}_s^\dagger}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s + \vartheta} \mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{p}} \right) \boldsymbol{\eta} \Big|_{\vartheta=0} \\ &\quad - \int d\mathbf{k} \frac{h(\mathbf{k})}{4\pi} \boldsymbol{\eta}^\dagger \left(\sum_s \frac{4\pi e_s^2}{w^2(\mathbf{k})} \oint d\mathbf{p} \frac{\mathbf{v}_s \mathbf{v}_s^\dagger}{w(\mathbf{k}) - \mathbf{k} \cdot \mathbf{v}_s} \mathbf{k} \cdot \frac{\partial F_s}{\partial \mathbf{p}} \right) \boldsymbol{\eta} \end{aligned}$$

$$\begin{aligned}
&= \mathcal{K} - \int d\mathbf{k} \frac{h(\mathbf{k})}{4\pi w(\mathbf{k})} \eta^\dagger \frac{\partial(\omega^2 \chi(\omega, \mathbf{k}))}{\partial \omega} \eta \Big|_{\omega=w(\mathbf{k})} - \int d\mathbf{k} \frac{h(\mathbf{k})}{4\pi} \eta^\dagger \chi \eta \\
&= \mathcal{K} - \int d\mathbf{k} \frac{h(\mathbf{k})}{4\pi w(\mathbf{k})} \eta^\dagger \frac{\partial(\omega^2 \epsilon(\omega, \mathbf{k}))}{\partial \omega} \eta \Big|_{\omega=w(\mathbf{k})} + \int d\mathbf{k} \frac{h(\mathbf{k})}{2\pi} \\
&\quad - \int d\mathbf{k} \frac{h(\mathbf{k})}{4\pi} \eta^\dagger \left(\epsilon(w(\mathbf{k}), \mathbf{k}) - 1 + \frac{\mathbf{w}_p}{w^2(\mathbf{k})} \right) \eta \\
&= \mathcal{K} - \int d\mathbf{k} w J + \int d\mathbf{k} \frac{3h(\mathbf{k})}{4\pi} - \int d\mathbf{k} \frac{h(\mathbf{k})}{4\pi} \eta^\dagger \epsilon(w(\mathbf{k}), \mathbf{k}) \eta - \int d\mathbf{k} \frac{h(\mathbf{k})}{4\pi w^2(\mathbf{k})} \eta^\dagger \mathbf{w}_p \eta.
\end{aligned}$$

Using (9.50) and proceeding as in section F.2.1, one can also cast this as follows:

$$\sum_s \int d\mathbf{p} H_{0s} F_s + \int d\mathbf{k} w J = \mathcal{K} + \frac{1}{8\pi} (3\widetilde{\mathbf{E}}^\dagger \widetilde{\mathbf{E}} - \widetilde{\mathbf{B}}^\dagger \widetilde{\mathbf{B}}) - \frac{1}{8\pi c^2} \widetilde{\mathbf{A}}^\dagger \mathbf{w}_p \widetilde{\mathbf{A}}. \quad (\text{F } 13)$$

Now, notice that

$$\begin{aligned}
\mathcal{K} &= \sum_s \int d\mathbf{p} \overline{H_{0s}(\mathbf{p}) f_s^{(\text{kin})}(\mathbf{p} - e_s \widetilde{\mathbf{A}}/c)} \\
&= \sum_s \int d\mathbf{p} \overline{H_{0s}(\mathbf{p} + e_s \widetilde{\mathbf{A}}/c) f_s^{(\text{kin})}(\mathbf{p})} \\
&\approx \mathcal{K}^{(\text{kin})} + \overline{(\widetilde{\mathbf{A}}/c) \cdot \sum_s e_s \int d\mathbf{p} \mathbf{v}_s f_s^{(\text{kin})}} + \sum_s \frac{e_s^2}{2c^2} \int d\mathbf{p} \overline{(\widetilde{\mathbf{A}} \mu_s^{-1} \widetilde{\mathbf{A}}) f_s^{(\text{kin})}} \\
&\approx \mathcal{K}^{(\text{kin})} + \frac{1}{c} \overline{\widetilde{\mathbf{A}} \cdot \widetilde{\mathbf{j}}} + \frac{1}{8\pi c^2} \overline{\widetilde{\mathbf{A}}^\dagger \mathbf{w}_p \widetilde{\mathbf{A}}}, \quad (\text{F } 14)
\end{aligned}$$

where $\widetilde{\mathbf{j}}$ is the oscillating-current density. From Ampere's law,

$$\begin{aligned}
\frac{1}{c} \overline{\widetilde{\mathbf{A}} \cdot \widetilde{\mathbf{j}}} &= \overline{\frac{(-i\hat{\omega}^{-1} \widetilde{\mathbf{E}})^\dagger}{4\pi} (i c \hat{\mathbf{k}} \times \widetilde{\mathbf{B}} + i \hat{\omega} \widetilde{\mathbf{E}})} \\
&\approx -\frac{\widetilde{\mathbf{E}}^\dagger \widetilde{\mathbf{E}}}{4\pi} - \frac{\widetilde{\mathbf{E}}}{4\pi} \cdot \left(\frac{c \hat{\mathbf{k}}}{\hat{\omega}} \times \widetilde{\mathbf{B}} \right) \\
&\approx -\frac{\widetilde{\mathbf{E}}^\dagger \widetilde{\mathbf{E}}}{4\pi} - \left(\widetilde{\mathbf{E}} \times \frac{c \hat{\mathbf{k}}}{\hat{\omega}} \right) \cdot \frac{\widetilde{\mathbf{B}}}{4\pi} \\
&\approx \frac{1}{4\pi} (\widetilde{\mathbf{B}}^\dagger \widetilde{\mathbf{B}} - \widetilde{\mathbf{E}}^\dagger \widetilde{\mathbf{E}}). \quad (\text{F } 15)
\end{aligned}$$

Substituting (F 14) and (F 15) into (F 13) leads to (9.55).

Appendix G. Selected notation

This paper uses the following notation (also see section 2 for the index convention):

Symbol	Definition	Explanation
■		placeholder
■*		complex conjugate
■ ⁻¹		inverse
■ [†]		Hermitian adjoint
■ ^{-†}	(■ [†]) ⁻¹	inverse Hermitian adjoint

$\blacksquare^T$		transpose
$\blacksquare a$	section 4.2	auxiliary notation
$\blacksquare^{(\mu)}$	section 6.5	contribution from the microscopic part
$\blacksquare^{(m)}$	section 6.5	contribution from the macroscopic part
$\bar{\blacksquare}$		average part or, for eikonal waves, a quantity evaluated on the local wavevector
$\tilde{\blacksquare}$		oscillatory part
$\bar{\blacksquare}$		macroscopic part
$\tilde{\blacksquare}$		microscopic part
$\hat{\blacksquare}$		operator
$\dot{\blacksquare}$		time derivative
$\circ\blacksquare$	(2.23)	Fourier image
$\circ\tilde{\blacksquare}$		envelope of an eikonal (or monochromatic) wave
$\blacksquare_A$	section 2.1.2	anti-Hermitian part
$\blacksquare_H$	section 2.1.2	Hermitian part
$\partial_{\blacksquare}$	$\partial/\partial\blacksquare$	partial derivative (but $\partial_i \doteq \partial/\partial x^i$, $\partial_\alpha \doteq \partial/\partial z^\alpha$, $\partial_a \doteq \partial/\partial X^a$)
∂^a, ∂^i	$\partial/\partial K_a, \partial/\partial p_i$	partial derivative with respect to a lower-index quantity
$\tilde{\partial}_{\blacksquare}$	(4.27)	auxiliary notation
d_t	(3.32), (7.36)	convective time derivative
$\{\blacksquare, \blacksquare\}_x$	(2.32)	Poisson bracket on $(\mathbf{x}, \mathbf{k})$
$\{\blacksquare, \blacksquare\}$	(2.56)	Poisson bracket on $(\mathbf{x}, \mathbf{k})$
$[\blacksquare, \blacksquare]$		commutator
$\langle \blacksquare \blacksquare \rangle$	(2.1), (2.59)	inner product on $\mathcal{H}_x$ or on $\mathcal{H}_X$
$\doteq$		definition
$\cdot$	section 2.1.3	scalar product
$\star$	(2.30)	Moyal product on $(\mathbf{x}, \mathbf{k})$
$\star$	(2.72)	Moyal product on $(\mathbf{X}, \mathbf{K})$
$i0$	section 4.2	i times an infinitesimally small positive number
$\oint$		principal-value integral
eigv		eigenvalue
im		imaginary part
pv	(4.24)	auxiliary notation
re		real part
oper_X		operator corresponding to a Weyl symbol on $(\mathbf{X}, \mathbf{K})$
oper_x		operator corresponding to a Weyl symbol on $(\mathbf{x}, \mathbf{k})$
symb		same as symb_X or symb_x when the two are equal
symb_X		Weyl symbol of an operator on $\mathcal{H}_X$
symb_x		Weyl symbol of an operator on $\mathcal{H}_x$
sgn		sign
tr		trace
Γ, Γ_s	section 6.7	part of a collision operator
Δ_s	section 6.6	particle's total ponderomotive energy in on-shell waves
$\Theta^{\alpha\beta c}$	(5.9)	auxiliary notation
$\Theta^{\alpha\beta}$	(5.22)	dressing function (since section 5.3)
Θ_{ij}, Θ	(5.42)	dressing function (a part of $\Theta^{\alpha\beta}$)
Λ	section 7.1	dispersion function (one of Λ_b)
Λ_b	section 7.1	b th eigenvalue of Ξ
Π_s	table 1	OC momentum flux density of species s
Π_w	table 1	wave momentum flux density

Ξ	section 6.3	dispersion matrix
$\hat{\Xi}$	(6.13), (6.17)	dispersion operator
Ξ_0	section 6.3	vacuum dispersion matrix
$\hat{\Xi}_0$	(6.1)	vacuum dispersion operator
Ξ_p	(6.21)	Weyl symbol of $\hat{\Xi}_p$
$\hat{\Xi}_p$	section 6.1	auxiliary operator
Φ, Φ_s	(5.24), (5.43)	ponderomotive energy
Ψ^i, Ψ	section 6.1	generic interaction field
$\tilde{\Psi}_c^i, \tilde{\Psi}_c$	(7.17)	complexified interaction field
Ω	(4.17)	auxiliary notation
$\alpha_{s,i}, \alpha_s$	section 6.1	Weyl symbols of $\hat{\alpha}_{s,i}$ and $\hat{\alpha}_s$
$\hat{\alpha}_{s,i}, \hat{\alpha}_s$	section 6.1	coupling operators
γ	(7.47), (7.86)	linear dissipation rate as a function of $(t, \mathbf{x}, \mathbf{k})$
$\overline{\gamma}$	section 7.3	local linear dissipation rate of an eikonal wave
δ		Kronecker symbol or delta function
ϵ	section 3.1.1	geometrical-optics parameter
ϵ	(9.42)	dielectric tensor
$\epsilon_{\parallel}, \epsilon_{\perp}$	(9.9)	parallel and transverse permittivity
ε	section 3.1.1	small parameter proportional to the oscillation amplitude
$\boldsymbol{\eta}$	section 7.1	polarization vector (one of $\boldsymbol{\eta}_b$)
$\boldsymbol{\eta}_b$	section 7.1	b th eigenvector of $\boldsymbol{\Xi}$
θ		eikonal phase
κ	section 7.2.3	auxiliary notation
κ_x, κ_p	(3.2)	characteristic inverse scales in $\mathbf{x}$ and $\mathbf{p}$, respectively
ρ_s		charge density of species s
$\varrho^{\alpha\beta}$	(5.18)	auxiliary notation
$\wp_s$	(6.22)	Weyl symbol of $\hat{\wp}_s$
$\hat{\wp}_s$	(6.22)	coupling operator
σ	(6.73)	entropy density
$\varsigma_{\mathbf{k}}$	(7.82)	sign of the action density
φ		electrostatic potential
ψ, Ψ		any field
ω		coordinate in the frequency space dual to t
$\overline{\omega}$	$-\partial_t \theta$	local frequency of an eikonal wave
$\hat{\omega}$	$i\partial_t$	frequency operator
C	(3.5)	Fourier image of W
$C_{\bullet}, \mathbf{C}_{\bullet}$	(2.77), (2.82)	Fourier images of $W_{\bullet}$ and $\mathbf{W}_{\bullet}$
$\mathcal{C}_s$	section 6.8	collision operator of species s
$\mathbf{C}_{\bullet}, \mathbf{C}_{\bullet}$	(2.44), (2.52)	Fourier images of $W_{\bullet}$ and $\mathbf{W}_{\bullet}$
$D^{\alpha\beta}$	section 5.1	Weyl symbol of $\hat{D}^{\alpha\beta}$
$\mathcal{D}^{\alpha\beta}$	(5.17)	phase-space-diffusion coefficient
$\mathcal{D}_{ij}, \mathbf{D}_s$	(5.41), (6.77a)	momentum-diffusion coefficient (part of $\mathcal{D}^{\alpha\beta}$)
$\mathcal{D}_0^{\alpha\beta}$	(5.8)	auxiliary notation
$\hat{D}^{\alpha\beta}$	(3.34)	diffusion operator on $\mathcal{H}_X$
$\mathcal{E}_s$	table 1	OC energy density of species s
$\mathcal{E}_w$	table 1	wave energy density (also see (7.42) for eikonal waves)
F, F_s	(5.27), (5.38)	OC distribution functions
$\mathfrak{F}_s$	(6.60)	polarization drag for species s

G, G_s	section 4.2	Weyl symbols of $\hat{G}$ and $\hat{G}_s$
G_0	(4.23)	approximation of G to the zeroth order in ϵ
$\hat{G}, \hat{G}_s$	(3.25)	effective Green's operators on $\mathcal{H}_X$
$\mathfrak{G}_{ss'}$	(6.30)	spectrum of the correlations between g_s and $g_{s'}$
$\hat{\mathcal{G}}$	section 3.2	Green's operator on $\mathcal{H}_X$
$\mathcal{I}$	(7.27)	action density of an eikonal wave
J	section 7.4.3	phase-space action density
$J^{\alpha\beta}, \mathbf{J}$	(2.57)	canonical Poisson structure
$\mathcal{J}^i, \mathcal{J}$	(7.28)	action flux density of an eikonal wave
H, H_s		particle Hamiltonian
$\mathcal{H}_s$	(5.40)	OC Hamiltonian of species s
$\mathcal{H}_{\mathbf{x}}$		Hilbert space formed by functions on $\mathbf{x}$
$\mathcal{H}_X$		Hilbert space formed by functions on $\mathbf{X}$
$\mathbf{K}$	$(-\omega, \mathbf{q})$	coordinate in the wavevector space dual to $\mathbf{X}$
$\hat{\mathbf{K}}$	$(-\hat{\omega}, \hat{\mathbf{q}})$	wavevector operator on $\mathcal{H}_X$
$\hat{\mathcal{L}}$	(3.14)	extended Liouvillian (up to a factor i)
$\hat{\mathcal{L}}_s$	(6.2)	coupling operator
$\hat{\mathcal{L}}_X$	(2.73)	same as the Poisson bracket on $(\mathbf{X}, \mathbf{K})$
$\hat{\mathcal{L}}_{\mathbf{x}}$	(2.32)	same as the Poisson bracket on $(\mathbf{x}, \mathbf{k})$
$\mathcal{L}$	(6.6)	Ψ -dependent part of the plasma Lagrangian density
$\mathcal{L}_0$	(6.1)	Lagrangian density of Ψ in vacuum
M		number of components of Ψ or of another vector field
N	$2n + 1$	dimension of the extended phase space $\mathbf{X}$
$\mathcal{N}, \mathcal{N}_s$	(5.39)	OC density
$\mathcal{O}$		big O ('at most of the order of')
$P_s^i, \mathbf{P}_s$	table 1	OC momentum density of species s
$P_w^i, \mathbf{P}_w$	table 1	wave momentum density (also see (7.42) for eikonal waves)
$Q_s^i, \mathbf{Q}_s$	table 1	OC energy flux density of species s
$Q_w^i, \mathbf{Q}_w$	table 1	wave energy flux density
$\mathcal{Q}_{ss'}$	section 6.8	symmetrized coefficient in the collision operator
$\hat{\mathbf{R}}_s$	(6.2)	coupling operator
$\mathbb{R}$		real axis
S	section 7.1	action integral
S_{ad}	(6.16)	adiabatic action integral
$\mathbf{S}$	(8.9)	spectrum of the macroscopic oscillations
$\hat{T}_\tau$	(3.21)	shift operator (see also section 4.1)
$\mathcal{U}^{ij}, \mathbf{U}$	(6.78)	average Wigner function of the macroscopic field $\tilde{\Psi}(\mathbf{x})$
$\mathbf{U}_{c\pm}, \mathbf{U}_c$	section 7.4.2	average Wigner matrix of $\tilde{\Psi}_c$ and $\tilde{\Psi}_c^*$ ($\mathbf{U}_c \equiv \mathbf{U}_{c+}$)
$V^a, \mathbf{V}$	section 3.2	unperturbed velocity in the $\mathbf{X}$ space
$\mathcal{V}_n$		volume of n -dimensional homogeneous plasma
W	(3.4)	Wigner function of $\tilde{H}(\mathbf{X})$
$W_\bullet, \mathbf{W}_\bullet$	(2.76), (2.81)	Weyl symbol of $\hat{W}_\bullet$ (Wigner function or matrix)
$\hat{W}$	(3.3)	density operator on $\mathcal{H}_X$ of $\tilde{H}$
$\hat{W}_\bullet, \hat{\mathbf{W}}_\bullet$	(2.75), (2.80)	density operator on $\mathcal{H}_X$ of a given field
W, W_s	section 5.6	Wigner functions of $\tilde{H}$ and $\tilde{H}_s$ with $\mathbf{p}$ as a parameter
$W_\bullet, \mathbf{W}_\bullet$	(2.43), (2.51)	Weyl symbol of $\hat{W}_\bullet$ (Wigner function or matrix)
$\hat{W}_\bullet, \hat{\mathbf{W}}_\bullet$	(2.42), (2.50)	density operator on $\mathcal{H}_X$ of a given field
$\mathfrak{W}$	section 6.4	average Wigner matrix of the microscopic field $\tilde{\Psi}(\mathbf{x})$
$X^a, \mathbf{X}$	$(t, \mathbf{z})$	coordinate in the extended phase space
$\hat{X}^a, \hat{\mathbf{X}}$	$(\hat{t}, \hat{\mathbf{z}})$	operator of the position in the extended phase space

$\mathcal{X}_{ss'}$	(6.36)	Weyl symbol of $\hat{\mathcal{X}}_{ss'}$
$\hat{\mathcal{X}}_{ss'}$	(6.35)	coupling operators on $\mathcal{H}_x$ that enter $\tilde{U}$
d		differential
e_s		charge of species s
f, f_s		distribution function
g, g_s	section 4.3	initial conditions for $\tilde{f}$ and $\tilde{f}_s$
h	(7.64), (7.74)	rescaled phase-space action density
$h_{c\pm}, h_c$	section 7.4.2	auxiliary notation ($h_{c+} \equiv h_c$)
$k_i, \mathbf{k}$		coordinate in the wavevector space dual to $\mathbf{x}$
$\bar{k}_i, \bar{\mathbf{k}}$	$\partial_i \theta, \partial_{\mathbf{x}} \theta$	local wavevector of an eikonal wave
$\hat{k}_i, \hat{\mathbf{k}}$	$-i\partial_i, -i\partial_{\mathbf{x}}$	wavevector operator
$\mathbf{k}$	$(-\omega, \mathbf{k})$	coordinate in the wavevector space dual to $\mathbf{x}$
$\bar{\mathbf{k}}$	$(-\bar{\omega}, \bar{\mathbf{k}})$	local spacetime-wavevector of an eikonal wave
$\hat{\mathbf{k}}$	$-i\partial_{\mathbf{x}}$	spacetime-wavevector operator
$ \mathbf{k}\rangle$	(2.16), (2.17)	eigenvector of $\hat{\mathbf{k}}$ corresponding to the eigenvalue $\mathbf{k}$
ℓ_τ	section 4.1.1	displacement in $\mathbf{X}$ along unperturbed characteristics
m_s		mass of species s
n	$\dim \mathbf{x}$	number of spatial dimensions
n	$n + 1$	number of spacetime dimensions
$p_i, \mathbf{p}$		coordinate in the momentum space
$\hat{p}_i, \hat{\mathbf{p}}$	$\mathbf{p}$	position operator corresponding to the coordinate $\mathbf{p}$
$q^i, \mathbf{q}$	$(\mathbf{k}, \mathbf{r})$	coordinate in the wavevector space dual to $\mathbf{z}$
$\hat{q}^i, \hat{\mathbf{q}}$	$(\hat{\mathbf{k}}, \hat{\mathbf{r}})$	wavevector operator corresponding to the coordinate $\mathbf{z}$
$r_i, \mathbf{r}$		coordinate in the wavevector space dual to $\mathbf{p}$
$\hat{r}^i, \hat{\mathbf{r}}$	$-i\partial^i, -i\partial_{\mathbf{p}}$	wavevector operator corresponding to the coordinate $\mathbf{p}$
s		species index
t		time
$\hat{t}$	t	time operator
u^α	(3.8)	oscillating part of the phase-space velocity
$\hat{u}^\alpha$	(3.30)	u^α as an operator on $\mathcal{H}_X$
$v^\alpha, v^i, \mathbf{v}$	(3.8) (5.32)	average velocity in phase space or in physical space, or, since section 5.4, OC velocity
$v_g^i, \mathbf{v}_g$	(7.25)	group velocity as a function of $(t, \mathbf{x}, \mathbf{k})$
$\bar{v}_g^i, \bar{\mathbf{v}}_g$	(7.29)	local group velocity of an eikonal wave
w	(7.22)	eikonal-wave frequency as a function of $(t, \mathbf{x}, \mathbf{k})$
$x^i, \mathbf{x}$		coordinate in space
$\bar{x}^i, \bar{\mathbf{x}}$		ray coordinate in space
$\hat{x}^i, \hat{\mathbf{x}}$	$\mathbf{x}$	space-position operator
$\mathbf{x}^i, \mathbf{x}$	$(t, \mathbf{x})$	coordinate in spacetime
$\hat{\mathbf{x}}^i, \hat{\mathbf{x}}$	$\mathbf{x}$	spacetime-position operator
$ \mathbf{x}\rangle$	(2.16), (2.17)	eigenvector of $\hat{\mathbf{x}}$ corresponding to the eigenvalue $\mathbf{x}$
$z^\alpha, \mathbf{z}$	$(\mathbf{x}, \mathbf{p})$	coordinate in phase space
$\hat{z}^\alpha, \hat{\mathbf{z}}$	$(\hat{\mathbf{x}}, \hat{\mathbf{p}})$	phase-space position operator

REFERENCES

- ANDERSSON, L., JOUDIOUX, J., OANCEA, M. A. & RAJ, A. 2021 Propagation of polarized gravitational waves. *Phys. Rev. D* **103**, 044053.
- BALAKIN, A. A., DODIN, I. Y., FRAIMAN, G. M. & FISCH, N. J. 2016 Backward Raman amplification of broad-band pulses. *Phys. Plasmas* **23**, 083115.
- BESSE, N., ELSKENS, Y., ESCANDE, D. F. & BERTRAND, P. 2011 Validity of quasilinear theory: refutations and new numerical confirmation. *Plasma Phys. Control. Fusion* **53**, 025012.
- BINNEY, J. & TREMAINE, S. 2008 *Galactic Dynamics: Second Edition*. Princeton: Princeton Univ. Press.
- BLOKH, K. Y., RODRÍGUEZ-FORTUÑO, F. J., NORI, F. & ZAYATS, A. V. 2015 Spin-orbit interactions of light. *Nat. Photonics* **9**, 796.
- BRIZARD, A. J., COOK, D. R. & KAUFMAN, A. N. 1993 Wave-action conservation for pseudo-Hermitian fields. *Phys. Rev. Lett.* **70**, 521.
- CARTWRIGHT, N. D. 1976 A non-negative Wigner-type distribution. *Physica A: Stat. Mech. Appl.* **83**, 210.
- CARY, J. R. & BRIZARD, A. J. 2009 Hamiltonian theory of guiding-center motion. *Rev. Mod. Phys.* **81**, 693.
- CARY, J. R. & KAUFMAN, A. N. 1977 Ponderomotive force and linear susceptibility in Vlasov plasma. *Phys. Rev. Lett.* **39**, 402.
- CARY, J. R. & KAUFMAN, A. N. 1981 Ponderomotive effects in collisionless plasma: a Lie transform approach. *Phys. Fluids* **24**, 1238.
- CATTO, P. J., LEE, J. & RAM, A. K. 2017 A quasilinear operator retaining magnetic drift effects in tokamak geometry. *J. Plasma Phys.* **83**, 905830611.
- CHAVANIS, P.-H. 2012 Kinetic theory of long-range interacting systems with angle-action variables and collective effects. *Physica A* **391**, 3680.
- CREWS, D. W. & SHUMLAK, U. 2022 On the validity of quasilinear theory applied to the electron bump-on-tail instability. *Phys. Plasmas* **29**, 043902.
- DEWAR, R. L. 1972 A Lagrangian theory for nonlinear wave packets in a collisionless plasma. *J. Plasma Phys.* **7**, 267.
- DEWAR, R. L. 1973 Oscillation center quasilinear theory. *Phys. Fluids* **16**, 1102.
- DEWAR, R. L. 1977 Energy-momentum tensors for dispersive electromagnetic waves. *Austral. J. Phys.* **30**, 533.
- DODIN, I. Y. 2014 On variational methods in the physics of plasma waves. *Fusion Sci. Tech.* **65**, 54.
- DODIN, I. Y. & FISCH, N. J. 2010a On generalizing the K - χ theorem. *Phys. Lett. A* **374**, 3472.
- DODIN, I. Y. & FISCH, N. J. 2010b On the evolution of linear waves in cosmological plasmas. *Phys. Rev. D* **82**, 044044.
- DODIN, I. Y. & FISCH, N. J. 2012 Axiomatic geometrical optics, Abraham–Minkowski controversy, and photon properties derived classically. *Phys. Rev. A* **86**, 053834.
- DODIN, I. Y. & FISCH, N. J. 2014 Ponderomotive forces on waves in modulated media. *Phys. Rev. Lett.* **112**, 205002.
- DODIN, I. Y., GEYKO, V. I. & FISCH, N. J. 2009 Langmuir wave linear evolution in inhomogeneous nonstationary anisotropic plasma. *Phys. Plasmas* **16**, 112101.
- DODIN, I. Y., RUIZ, D. E., YANAGIHARA, K., ZHOU, Y. & KUBO, S. 2019 Quasioptical modeling of wave beams with and without mode conversion. I. Basic theory. *Phys. Plasmas* **26**, 072110.
- DODIN, I. Y., ZHMOGINOV, A. I. & RUIZ, D. E. 2017 Variational principles for dissipative (sub)systems, with applications to the theory of linear dispersion and geometrical optics. *Phys. Lett. A* **381**, 1411.
- DRUMMOND, W. E. & PINES, D. 1962 Non-linear stability of plasma oscillations. *Nucl. Fusion Suppl.* **3**, 1049.
- ERIKSSON, L.-G. & HELANDER, P. 1994 Monte Carlo operators for orbit-averaged Fokker–Planck equations. *Phys. Plasmas* **1**, 308.
- ESCANDE, D. F., BÉNISTI, D., ELSKENS, Y., ZARZOSO, D. & DOVEIL, F. 2018 Basic microscopic plasma physics from N -body mechanics. *Rev. Mod. Plasma Phys.* **2**, 1, arXiv: 1805.11408.
- FETTERMAN, A. J. & FISCH, N. J. 2008 α channeling in a rotating plasma. *Phys. Rev. Lett.* **101**, 205003.
- FISCH, N. J. 1987 Theory of current drive in plasmas. *Rev. Mod. Phys.* **59**, 175.

- FISCH, N. J. & RAX, J. M. 1992 Interaction of energetic alpha-particles with intense lower hybrid waves. *Phys. Rev. Lett.* **69**, 612.
- FRAIMAN, G. M. & KOSTYUKOV, I. YU. 1995 Influence of external inhomogeneous static fields on interaction between beam of charged-particles and packet of electromagnetic waves. *Phys. Plasmas* **2**, 923.
- GALEEV, A. A. & SAGDEEV, R. Z. 1985 *Theory of Weakly Turbulent Plasma, Part 4 in 'Basic Plasma Physics I'*. New York: North-Holland, edited by A. A. Galeev and R. N. Sudan.
- GAPONOV, A. V. & MILLER, M. A. 1958 Potential wells for charged particles in a high-frequency electromagnetic field. *Zh. Eksp. Teor. Fiz.* **34**, 242.
- GARG, D. & DODIN, I. Y. 2020 Average nonlinear dynamics of particles in gravitational pulses: effective Hamiltonian, secular acceleration, and gravitational susceptibility. *Phys. Rev. D* **102**, 064012.
- GARG, G. & DODIN, I. Y. 2021a Gauge-invariant gravitational waves in matter beyond linearized gravity, arXiv: 2106.05062.
- GARG, G. & DODIN, I. Y. 2021b Gauge invariants of linearized gravity with a general background metric, arXiv: 2105.04680.
- GARG, G. & DODIN, I. Y. 2022 Gravitational wave modes in matter, arXiv: 2204.09095.
- HAMILTON, C. 2020 A simple, heuristic derivation of the Balescu-Lenard kinetic equation for stellar systems. *Mon. Notices Royal Astron. Soc.* **501**, 3371.
- HAYES, W. D. 1973 Group velocity and nonlinear dispersive wave propagation. *Proc. R. Soc. Lond. A* **332**, 199.
- HIZANIDIS, K., MOLVIG, K. & SWARTZ, K. 1983 A retarded time superposition principle and the relativistic collision operator. *J. Plasma Phys.* **30**, 223.
- KAUFMAN, A. N. 1972 Quasilinear diffusion of an axisymmetric toroidal plasma. *Phys. Fluids* **15**, 1063.
- KAUFMAN, A. N. 1987 Phase-space-Lagrangian action principle and the generalized K - χ theorem. *Phys. Rev. A* **36**, 982.
- KAUFMAN, A. N. & HOLM, D. D. 1984 The Lie-transformed Vlasov action principle: relativistically covariant wave propagation and self-consistent ponderomotive effects. *Phys. Lett. A* **105**, 277.
- KENNEL, C. F. & ENGELMANN, F. 1966 Velocity space diffusion from weak plasma turbulence in a magnetic field. *Phys. Fluids* **9**, 2377.
- KENTWELL, G. W. 1987 Oscillation-center theory at resonance. *Phys. Rev. A* **35**, 4703.
- KENTWELL, G. W. & JONES, D. A. 1987 The time-dependent ponderomotive force. *Phys. Rep.* **145**, 319.
- KRALL, N. A. & TRIVELPIECE, A. W. 1973 *Principles of Plasma Physics*. New York: McGraw-Hill.
- LANDAU, L. D. & LIFSHITZ, E. M. 1976 *Mechanics*. Oxford: Butterworth-Heinemann.
- LANDREMAN, M. 2017 The H theorem for the Landau-Fokker-Planck collision operator. Unpublished.
- LEE, J., SMITHE, D., WRIGHT, J. & BONOLI, P. 2018 A positive-definite form of bounce-averaged quasilinear velocity diffusion for the parallel inhomogeneity in a tokamak. *Plasma Phys. Control. Fusion* **60**, 025007.
- LICHTENBERG, A. J. & LIEBERMAN, M. A. 1992 *Regular and Chaotic Dynamics*. New York: Springer-Verlag, second edition.
- LIFSHITZ, E. M. & PITAEVSKII, L. P. 1981 *Physical Kinetics*. New York: Pergamon Press.
- LITTLEJOHN, R. G. 1979 A guiding center Hamiltonian: a new approach. *J. Math. Phys.* **20**, 2445.
- LITTLEJOHN, R. G. 1981 Hamiltonian formulation of guiding center motion. *Phys. Fluids* **24**, 1730.
- LITTLEJOHN, R. G. 1983 Variational principles of guiding centre motion. *J. Plasma Phys.* **29**, 111.
- LITTLEJOHN, R. G. 1986 The semiclassical evolution of wave packets. *Phys. Rep.* **138**, 193.
- LIU, C. & DODIN, I. Y. 2015 Nonlinear frequency shift of electrostatic waves in general collisionless plasma: unifying theory of fluid and kinetic nonlinearities. *Phys. Plasmas* **22**, 082117.

- MAGORRIAN, J. 2021 Stellar dynamics in the periodic cube. *Mon. Notices Royal Astron. Soc.* **507**, 4840.
- MCDONALD, S. W. 1988 Phase-space representations of wave equations with applications to the eikonal approximation for short-wavelength waves. *Phys. Rep.* **158**, 337.
- MCDONALD, S. W. 1991 Wave kinetic equation in a fluctuating medium. *Phys. Rev. A* **43**, 4484.
- MCDONALD, S. W., GREBOGI, C. & KAUFMAN, A. N. 1985 Locally coupled evolution of wave and particle distribution in general magnetoplasma geometry. *Phys. Lett. A* **111**, 19.
- MCDONALD, S. W. & KAUFMAN, A. N. 1985 Weyl representation for electromagnetic waves: The wave kinetic equation. *Phys. Rev. A* **32**, 1708.
- MOTZ, H. & WATSON, C. J. H. 1967 The radio-frequency confinement and acceleration of plasmas. *Adv. Electron. Electron Phys.* **23**, 153.
- MOYAL, J. E. 1949 Quantum mechanics as a statistical theory. *Proc. Cambridge Philosoph. Soc.* **45**, 99.
- MYNICK, H. E. 1988 The generalized Balescu-Lenard collision operator. *J. Plasma Phys.* **39**, 303.
- OANCEA, M. A., JOUDIOUX, J., DODIN, I. Y., RUIZ, D. E., PAGANINI, C. F. & ANDERSSON, L. 2020 Gravitational spin Hall effect of light. *Phys. Rev. D* **102**, 024075.
- OCHS, I. E. 2021 Controlling and exploiting perpendicular rotation in magnetized plasmas. PhD thesis, Princeton University.
- OCHS, I. E. & FISCH, N. J. 2021a Nonresonant diffusion in alpha channeling. *Phys. Rev. Lett.* **127**, 025003.
- OCHS, I. E. & FISCH, N. J. 2021b Wave-driven torques to drive current and rotation. *Phys. Plasmas* **28**, 102506.
- OCHS, I. E. & FISCH, N. J. 2022 Momentum conservation in current drive and alpha-channeling-mediated rotation drive, arXiv: 2201.07853.
- PINSKER, R. I. 2001 Introduction to wave heating and current drive in magnetized plasmas. *Phys. Plasmas* **8**, 1219.
- RIGAS, I., SÁNCHEZ-SOTO, L. L., KLIMOV, A., ŘEHÁČEK, J. & HRADIL, Z. 2011 Orbital angular momentum in phase space. *Ann. Phys.* **326**, 426.
- REGISTER, A. & OBERMAN, C. 1968 On the kinetic theory of stable and weakly unstable plasma. Part 1. *J. Plasma Phys.* **2**, 33.
- REGISTER, A. & OBERMAN, C. 1969 On the kinetic theory of stable and weakly unstable plasma. Part 2. *J. Plasma Phys.* **3**, 119.
- ROSTOKER, N. 1964 Superposition of dressed test particles. *Phys. Fluids* **7**, 479.
- RUIZ, D. E. 2017 Geometric theory of waves and its applications to plasma physics. PhD thesis, Princeton University, arXiv:1708.05423.
- RUIZ, D. E. & DODIN, I. Y. 2015a First-principles variational formulation of polarization effects in geometrical optics. *Phys. Rev. A* **92**, 043805.
- RUIZ, D. E. & DODIN, I. Y. 2015b On the correspondence between quantum and classical variational principles. *Phys. Lett. A* **379**, 2623.
- RUIZ, D. E. & DODIN, I. Y. 2017a Extending geometrical optics: a Lagrangian theory for vector waves. *Phys. Plasmas* **24**, 055704.
- RUIZ, D. E. & DODIN, I. Y. 2017b Ponderomotive dynamics of waves in quasiperiodically modulated media. *Phys. Rev. A* **95**, 032114.
- RUIZ, D. E., GLINSKY, M. E. & DODIN, I. Y. 2019 Wave kinetic equation for inhomogeneous drift-wave turbulence beyond the quasilinear approximation. *J. Plasma Phys.* **85**, 905850101.
- SCHLICKESER, R. & YOON, P. H. 2014 Quasilinear theory of general electromagnetic fluctuations in unmagnetized plasmas. *Phys. Plasmas* **21**, 092102.
- SCHMIT, P. F., DODIN, I. Y. & FISCH, N. J. 2010 Controlling hot electrons by wave amplification and decay in compressing plasma. *Phys. Rev. Lett.* **105**, 175003.
- SILIN, V. P. 1961 Collision integral for charged particles. *Zh. Eksp. Teor. Fiz.* **40**, 1768.
- STIX, T. H. 1992 *Waves in Plasmas*. New York: AIP, second edition.
- TRACY, E. R., BRIZARD, A. J., RICHARDSON, A. S. & KAUFMAN, A. N. 2014 *Ray Tracing and Beyond: Phase Space Methods in Plasma Wave Theory*. New York: Cambridge University Press.

- TRIGGER, S. A., ERSHKOVICH, A. I., VAN HEIJST, G. J. F. & SCHRAM, P. P. J. M. 2004 Kinetic theory of Jeans instability. *Phys. Rev. E* **69**, 066403.
- VEDENOV, A. A., VELIKHOV, E. P. & SAGDEEV, R. Z. 1961 Nonlinear oscillations of rarified plasma. *Nucl. Fusion* **1**, 82.
- WEIBEL, E. S. 1981 Quasi-linear theory without the random phase approximation. *Phys. Fluids* **24**, 413.
- WHITHAM, G. B. 1974 *Linear and Nonlinear Waves*. New York: Wiley.
- WONG, H. V. 2000 Particle canonical variables and guiding center Hamiltonian up to second order in the Larmor radius. *Phys. Plasmas* **7**, 73.
- YASSEEN, F. 1983 Quasilinear theory of inhomogeneous magnetized plasmas. *Phys. Fluids* **26**, 468.
- YASSEEN, F. & VACLAVIK, J. 1986 Quasilinear theory of uniformly magnetized inhomogeneous plasmas: electromagnetic fluctuations. *Phys. Fluids* **29**, 450.
- YE, H. & KAUFMAN, A. N. 1992 Self-consistent theory for ion gyroresonance. *Phys. Fluids B* **4**, 1735.
- YOON, P. H., ZIEBELL, L. F., KONTAR, E. P. & SCHLICKEISER, R. 2016 Weak turbulence theory for collisional plasmas. *Phys. Rev. E* **93**, 033203.
- ZAKHAROV, V. E., L'VOV, V. S. & FALKOVICH, G. 1992 *Kolmogorov Spectra of Turbulence I: Wave Turbulence*. New York: Springer-Verlag.
- ZHU, H. & DODIN, I. Y. 2021 Wave-kinetic approach to zonal-flow dynamics: recent advances. *Phys. Plasmas* **28**, 032303.