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Differences in Students' Beliefs and Knowledge Regarding Mathematical Proof: Comparing Novice and Experienced Provers

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Learning to interpret proofs is an important milepost in the maturity and development of students of higher mathematics. A key learning objective in proof-based courses is to discern whether a given proof is a valid justification of its underlying claim. In this study, we presented students with conditional statements and associated proofs and asked them to determine whether the proofs proved the statements and to explain their reasoning. Prior studies have found that inexperienced provers often accept the proof of a statement's converse and reject proofs by contraposition, which are both erroneous determinations. Our study contributes to the literature by corroborating these findings and suggesting a connection between students' reading comprehension and proof validation behaviors and their beliefs about mathematical proof and mathematical knowledge base.

Keywords: logic and proof, belief about mathematical proof, mathematical knowledge base

Learning to interpret proofs is an important milepost in students' mathematical development and maturity. Such development is especially crucial since proof is a structure unique to the field of mathematics (Balacheff, 2008; Fawcett, 1938). Though much variability exists in how transition-to-proof courses are delivered in the U.S., over 80% of them attend to principles of formal logic (David & Zazkis, 2020). Presumably among the many facets of students' development with regard to proof is their ability to correctly discern whether a proof justifies a given theorem. While undergraduate students' comprehension and behaviors have been a focus of research in the reading and validation of proofs (e.g., Dawkins & Zazkis, 2021; Selden & Selden, 2003), this study aims to provide insights of how students' beliefs and knowledge about proofs might be associated with their reading and validation thereof. Building on the existing research (e.g., Dawkins & Roh, 2022), this study explores the accuracy with which students validate theorem-proof pairs, the reasons they offer for their decisions, and the similarities and differences exhibited by students with different degrees of proof experience. By studying similarities and differences between these groups, we address the following research question: What differences exist between novice and experienced provers in how they read proofs and characterize the relationship between proofs and theorems?

Theoretical Perspective

In this study, we employ the lens of radical constructivism (Glasersfeld, 1988). Under this view, knowledge does not objectively reflect reality, rather it is stored in the mind of an individual learner who has organized their activity and experience idiosyncratically into schemes. As such, we designed our investigations to understand our participants' schemes regarding proofs of theorems in order to build models for their thinking.

As a way of organizing and interpreting our findings about students' schemes regarding proof, we introduce the constructs of *beliefs about mathematical proof* and *mathematical knowledge base*, both of which may differ from student to student. The former refers to general notions that students hold regarding the practice of proving or the properties that a proof should have. For example, a student might believe that a proof needs to make explicit the structure of their proof e.g., direct proof, contrapositive, while another would accept a proof which only implies the structure. The latter refers to content-specific knowledge that students accept without justification. For example, a student might conceive that the sum of two continuous functions is continuous and accept a proof that used this argument as valid. Another student might reject a proof which doesn't justify this claim. In either case, such knowledge is only relevant in proofs that pertain to functions and their analysis.

Though ideas in one's mathematical knowledge base do not require justification, students were asked on multiple occasions to explain why certain ideas were true. To more fully describe their understanding, we rely on *warrants* (Toulmin, 1958), the reason a prover gives for why their evidence is germane to their argument. In particular, we use the *warrant-types* described by Inglis et al. (2007) to make sense of our participants' knowledge bases.

Research Methodology

As part of a larger study, we conducted clinical interviews (Clement, 2000) with undergraduate students with various levels of proof experience at a large public university in the United States from spring 2020 to spring 2022. We recruited eight students who had already taken at least two proof-oriented courses by spring 2020. We labeled these participants experienced provers. To compare and contrast these provers' conceptions about proof, in the springs of 2021 and 2022, we recruited four students who had not yet taken any proof-oriented mathematics courses at the university level, labeling them novice provers. The second author of this paper served as the interviewer of all participants while the remaining authors served as witnesses. In the discussion of results, we label participants with E or N (indicating their experienced or novice prover classification), a number from 1-8, and a pseudonym.

Each clinical interview lasted between 60 and 120 minutes. Some interviews in the spring of 2020 were conducted in person in a space other than their regular classroom while the rest of the interviews were conducted remotely. To facilitate retrospective analysis, we video- and audio-recorded all interviews. Participants completed all annotations on tablet computers, allowing us to collect digital copies of their work.

Interview Tasks

The tasks for the clinical interviews consisted of a series of theorem-proof pairs. After showing a theorem and at least one proof associated with it, we asked a student participant to think aloud while reading and interpreting them. When the student had indicated that they had sufficiently reviewed the theorem and proof, we asked whether the proof proves the theorem. If they determined that the proof did not prove the theorem, we asked if there were other statements that it proved. As the student responded to our questions, the interviewer asked follow-up questions in tandem to understand the student's reasoning for their decision.

While we asked all student participants the same questions for each theorem-proof pair, these pairs were not the same across the three all data collection periods. In spring 2020, we offered our experienced provers five different theorems (Theorems 1, 2, 4, 6, and 9 in Figure 1), each of which was accompanied by two or three different proofs. In spring 2021, we provided one theorem (labeled Theorem in Figure 1) to two of our novice provers and four associated proofs. In spring 2022, we presented the remaining novice provers four different theorems (Theorems $\alpha, \beta, \gamma,$ and δ in Figure 1), each with a single proof of its converse or contrapositive.

For all of the theorems in Figure 1 below, we provided students additional information such as relevant definitions or supporting theorems which may be needed for their reading of the proofs. We informed students that the proofs we provided were mathematically valid, but proofs associated with a theorem may not necessarily prove the theorem.

Theorems presented to experienced provers in Spring 2020:

Theorem 1: If x is a multiple of 6, then x is a multiple of 3.
(Associated proofs are direct, disproof of converse, and contraposition)
Theorem 2: If x is a multiple of 2 and a multiple of 7, then x is a multiple of 14.
(Associated proofs are direct and proof of converse)
Theorem 4: If $ABCD$ is a rhombus, then the diagonal AC forms two congruent isosceles triangles.
(Associated proofs are direct and disproof of converse)
Theorem 6: For any line segment AB, if a point X is on the perpendicular bisector of AB, then $AX = BX$.
(Associated proofs prove the converse and prove directly)
Theorem 9: If f and g are continuous on $[a, b]$, $f(a) = g(b)$, and $f(b) = g(a)$, then there is a c in $[a, b]$ such that $f(c) = g(c)$.
(Associated proofs are direct, disproof of converse, and contraposition)

Theorem presented to novice provers in Spring 2021:

Theorem: For any integer x, if x is not a multiple of 3, then $x^2 - 1$ is a multiple of 3.
(Associated proofs are direct, inverse, converse, and contrapositive)

Theorems presented to novice provers in Spring 2022:

Theorem α: Given a line segment AB, for all points X, if X is on the perpendicular bisector of AB, then $AX = BX$.
(Associated proof proves the converse)
Theorem β: For any triangle XYZ, if no two angles are congruent, then the triangle is scalene.
(Associated proof proves the contrapositive)
Theorem γ: For any integer x, if x is a multiple of 4 and a multiple of 21, then x is a multiple of 84.
(Associated proof proves the converse)
Theorem δ: For any integer x, if x is not a multiple of 3, then it cannot be written as the sum of three consecutive integers.
(Associated proof proves the contrapositive)

Figure 1. Theorems and types of proofs associated with them

Data Collection and Analysis

To facilitate our analysis, we transcribed each interview and created detailed field notes to describe how students processed each proof. We analyzed data in hopes of building a theory grounded in the available data (Strauss & Corbin, 1998). We first coded each line of each transcript by describing student behavior e.g., reviewing given definitions, drawing diagrams, deciding on the validity of a proof. We further coded the transcripts to attend to students' reasoning underlying their responses to questions, which revealed five different phenomena which we present in more detail shortly. These phenomena gave rise to two ways of categorizing students' conceptions – belief about mathematical proof and mathematical knowledge base.

Results

The goal for our research was to characterize the differences between how novice and experienced provers understood proofs, theorems, and the relationship between them. We begin by discussing their commonalities in order to provide a reference for their differences. We found that students' comprehension and validation of proofs are associated with their beliefs about mathematics proof and mathematical knowledge base. Various sub-categories of each construct emerged from our analysis. In particular, we found two different sub-categories of students' beliefs and three sub-categories of their knowledge. Once we identified all students with these categories, we compared and contrasted novice and experienced provers. Our findings are summarized below (See Table 1).

Table 1. Summary of conceptions about mathematical proof

Conception	Phenomenon	Novice	Experienced
Beliefs about Mathematical Proof	Valid proofs require logically sequenced arguments.	✓	✓
	Valid proofs require correct overall structure.	×	✓
Mathematical Knowledge Base	Arguments rely on empirical evidence.	✓	×
	Arguments rely on definitions.	×	✓
	Arguments rely on logically sound principles.	×	✓

Summary of Conceptions of Proof of Novice and Experienced Provers

Both groups of provers exhibited a belief that valid proofs require correctly sequenced arguments, yet only experienced provers believed that proofs must also follow the correct structure i.e., assumptions and conclusions are correctly identified. Regarding mathematical knowledge base, novice provers primarily argued using empirical evidence while experienced provers preferred arguments based on definitions. Lastly, experienced provers alone showed consistent sensitivity to logically sound principles.

Beliefs about Mathematical Proof

This category pertains to what students generally believe a prover should do when formulating a proof or what properties a proof ought to include. Our findings refer specifically to the characteristics that students believe contribute to the validity of a proof, or lack thereof.

Valid proofs require logically sequenced arguments. A logically sequenced argument is such that each line in a proof is both justified by the ones that precede it and justifies the ones that follow. Put alternately, students attended to the body of the proof without necessarily attending to the assumptions and conclusions. In each of the following excerpts, one each from a novice and experienced prover, participants discussed why this coherent flow is necessary in a valid proof.

Interviewer: "Can you explain why [this proof doesn't prove the theorem]?"

Priya (E4): "Because they're not justifying their steps. When they don't justify their steps, the steps they've omitted don't indicate that they understand what's going on. It just seems like they're fudging because they know where they need to go."

Interviewer: "Can you explain why the proof proves the theorem?"

Carl (N2): "If x wasn't on the perpendicular bisector…Therefore, the lines AX and BX would not be equal. Because it is on the perpendicular bisector, triangles AMX and BMX would be equal. By the SAS theorem, the triangles would be equal."

Priya referred to the desired conclusion as "where they need to go," thereby acknowledging a proof framework, but rejected the arguments used to arrive there. Carl accepted his proof based solely on the links between arguments and did not attend to the hypothesis and conclusion.

Valid proofs require correct overall structure. If the category above pertains to the body of a proof, this category pertains to its head and tail, the assumptions and conclusions, respectively. The belief that a proof's validity depends on its hypotheses and conclusions, and the flow from the former to the latter, was prevalent only among experienced provers.

Interviewer: "Can you explain why [the proof doesn't prove the theorem]?"

Nate (E6): "In proof 1.2, your given assumption is actually what we're trying to prove…This statement is what we're needing at the end of our proof…So they're starting at the opposite end of the proof…This is saying that if x is a multiple of 3, then it's not a multiple of 6, which is not what we're actually trying to prove."

Interviewer: "What is your top criterion for this to be a valid proof?"

Heather (E2): "It starts by assuming that the if condition is true."

Nate and Heather each attended to how the proof began and ended. Nate rejected the proof because it assumed the wrong premise. Heather cited the correct assumption as her top validation condition. In each case, the student attended to the ends of the proof rather than the body thereof.

Mathematical Knowledge Base

A student's knowledge base refers to the information that they have at their disposal which helps them read, interpret, and formulate proofs. Most relevant to this study is the set of content-specific tools that students have which allows them to analyze and compare arguments.

Arguments rely on empirical evidence. Under this approach, students cited particular examples to substantiate their claims. On several occasions, novice provers used one or more particular examples directly before declaring that a theorem was indeed valid.

Interviewer: "Can you explain why [proof 1 proves for any integer x, if x is not a multiple of 3, then $x^2 - 1$ is a multiple of 3]?"

Joaquin (N4): "I didn't realize that the proof would approach the problem like this…It says let x be an integer that is not a multiple of 3…We could pick 8, 7 even…For me personally, I experimented with some numbers. For example, I let k equal 1."

Joaquin's acceptance of the theorem stemmed from his ability to satisfy it with several spontaneously chosen examples. Though he convinced himself of the validity of the proof inductively, we do not necessarily claim that he would have accepted a proof by example. Nonetheless, his empirical reasoning was fairly common among novice provers.

Arguments rely on definitions. Rather than using particular examples, provers in this category reasoned arbitrarily i.e., using examples which represent all examples. Put alternately, experienced provers reasoned from definitions and properties rather than from examples.

Interviewer: "Can you explain in your own words what this theorem states?"

Priya (E4): "Given any integer x, if x satisfies the property of being a multiple of 6, meaning there is some number that multiplied by 6 gives you x,…There is another number that when multiplied by 3 gives you x."

Whereas Joaquin reasoned via empirical evidence, Priya reasoned arbitrarily and directly from the definitions. Joaquin's and Priya's preferred modes of reasoning were common among other novice and experienced provers, respectively.

Arguments rely on logically sound principles. Provers in this category were adept at employing logic, most notably for this study contrapositive equivalence and converse independence (CE/CI). Novice provers did not consistently exhibit understanding of CE/CI. Nevertheless, the manners in which experienced provers justified these principles varied greatly.

Interviewer: "Can you explain why [this proves if x is not a multiple of 3, then $x^2 - 1$ is a multiple of 3]?" (proof proves the converse)

Violet (N3): "They're showing that x is not a multiple of 3 by saying that x equals k times 3…But it can't be since in the theorem it says that x is not a multiple of 3…I think it [proves the theorem] because they're showing in their work that x is a multiple of 3, because they're assuming that it's a multiple of 3."

Note that Violet attended to the arguments in the body of the proof but exhibited no sensitivity to the overall structure of the proof.

Experienced provers, on the whole, reliably recognized CE/CI. Significant differences however existed in the way they justify these ideas. For example, some participants took CE/CI as given, but did not provide a justification.

Interviewer: "Why do you think that since this disproves the converse that it does not prove the theorem?"

Priya (E4): "Because the converse is not logically equivalent to the original."

I: "What do you mean that they are not equivalent?"

Priya (E4): "That's a good question. Like how do I know that two things aren't logically equivalent? I guess at this point, that's just an inherent fact to me."

Provers who reasoned about CE/CI in this fashion perhaps viewed CE/CI as a belief rather than knowledge since it is neither requires nor is accompanied by warrant.

Other experienced provers were able to warrant CE/CI with concrete examples which were specific to a particular context. These contexts were not always overtly mathematical in nature, as shown by the excerpt below.

Interviewer: "You said this proves if Q, then P, right? My question is why a proof of if Q then P is not a proof of if P then Q."

Mark (E7): "Let's say I say that if an animal is a blue jay, then it is a bird…This is the example I always think of when I have to think of if-then statements."

Though not explicitly stated by the student, it can be reasonably presumed that since the statement he gave had a false converse, its purpose is to illustrate general converse independence through a particular example. Note that while empirical evidence was primarily used by experienced provers, this was not exclusively the case.

Finally, our participants also justified their knowledge of CE/CI through abstract warrants which were not beholden to any particular contexts. Such justifications most often took the form of truth tables, subset relationships, and logical manipulations (see Figure 2).

Mark (E7): "It works from logic that for an implication to be true, either the hypothesis is false or the conclusion is true. Since they have the same truth table, we know that the statements are going to be equivalent." (see figure 2, left image)

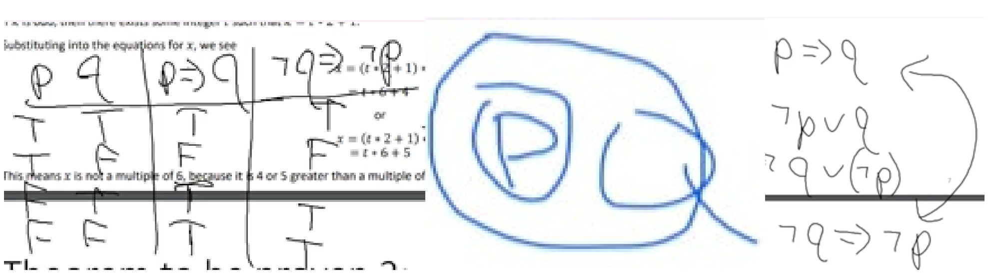

Figure 2: Mark's truth table and Euler diagram as well as Nate's syntactic argument

Interviewer: Can you explain why the proof demonstrates [if Q, then not P]?"

Mark (E7): "So it could be the case that x is a multiple of 3 and not a multiple of 6. It could be the case the Q and we don't know anything about P. So, what the theorem says is that if I am anything in P, then I will also be in Q. But what this shows is that if I'm in Q, then I might not be in P." (see Figure 2, middle image)

Interviewer: "How can you tell that the proof of the contrapositive also proves the theorem?"
Nate (E6): "I think the easiest way would be through logic. The statement is P implies Q. That's the same is not P or Q. Then, if we do double negation, we get not Q implies not P. So, these two are exactly the same." (see Figure 2, right image)

Discussion and Conclusion

The goals of this study were to characterize the ways in which undergraduate students interpret proof-texts, their relationships to underlying theorems, and to describe the differences between novice and experienced provers. Though our tasks were designed to gauge reading comprehension through student behaviors, we learned much about their conceptions of proof, suggesting that the phenomena are related.

With regard to beliefs about mathematical proof, both groups of provers asserted that a logical linking of ideas should be present in a proof. Our research is thereby consistent with prior literature (e.g., Ko & Knuth, 2013; Selden & Selden, 2003; Dawkins & Zazkis, 2021). Similarly, we found that students with more mathematical development were more likely to attend to the assumptions that are made at the outset of the proof and the overall structure, which is also consistent with prior studies (e.g., Heinze & Reiss, 2003; Weber, 2008). Indeed, experienced provers validated proofs correctly more often than novice ones, though experienced provers make occasional errors. Our findings in this regard support the work of Inglis and Alcock (2012).

The results of our study also highlight the different ways in which our participants justify the ideas of contrapositive equivalence and converse independence. Our findings are consistent with prior literature. In their discussion of modeling arguments, Inglis et al. (2007) discuss the warrants used by graduate students of number theory. Though their participants were more mathematically developed than ours, parallels exist between our findings and theirs. Inglis et al. (2007) do not discuss participants who offer no warrant, but our provers who readily asserted the ideas of CE/CI but could give no reason for their validity exhibit what Krupnik et al. (2018) call *psychological knowledge*, a belief that an idea is true which the knower cannot justify. Our participants who warranted CE/CI with a single example e.g., the blue jay, parallel what Inglis et al. (2007) call the inductive warrant-type, wherein a prover evaluates a conjecture using one or more specific examples. Inglis et al. (2007) also describe a structural-intuitive warrant-type, wherein a prover uses a mental or visual structure to support a conjecture. This is consistent with our experienced provers who used set-theoretic notions and Euler diagrams to justify CE/CI. Inglis et al. (2007) describe the deductive warrant-type as reasoning solely from axioms. Since our provers who used truth tables and logical manipulations were relying on the base relationships between propositions in an implication, their reasoning was consistent with this warrant-type.

Our findings suggest that students with formal training in proof validate proofs with greater reliability but greater attention may be paid to the justification for CE/CI. Ongoing studies are testing instructional interventions using set-theoretic activities to effect deeper conceptual understanding of proof structures (Dawkins et al., in preparation).

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