DESCRIPTIVE COMPLEXITY IN CANTOR SERIES

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ABSTRACT. A Cantor series expansion for a real number x with respect to a basic sequence $Q = (q_1, q_2, \dots)$, where $q_i \geq 2$, is a generalization of the base b expansion to an infinite sequence of bases. Ki and Linton in 1994 showed that for ordinary base b expansions the set of normal numbers is a $\mathbf{\Pi}^0_3$-complete set, establishing the exact complexity of this set. In the case of Cantor series there are three natural notions of normality: normality, ratio normality, and distribution normality. These notions are equivalent for base b expansions, but not for more general Cantor series expansions. We show that for any basic sequence the set of distribution normal numbers is $\mathbf{\Pi}^0_3$-complete, and if Q is 1-divergent then the sets of normal and ratio normal numbers are $\mathbf{\Pi}^0_3$-complete. We further show that all five non-trivial differences of these sets are $D_2(\mathbf{\Pi}^0_3)$-complete if $\lim_i q_i = \infty$ and Q is 1-divergent. This shows that except for the trivial containment that every normal number is ratio normal, these three notions are as independent as possible.

1. INTRODUCTION

1.1. **Review of Definability Notions.** In any topological space X, the collection of Borel sets $\mathcal{B}(X)$ is the smallest σ-algebra containing the open sets. They are stratified into levels, the Borel hierarchy, by defining $\mathbf{\Sigma}^0_1 =$ the open sets, $\mathbf{\Pi}^0_1 = \neg\mathbf{\Sigma}^0_1 = \{X - A\colon A \in \mathbf{\Sigma}^0_1\}$ =the closed sets, and for $\alpha < \omega_1$ we let $\mathbf{\Sigma}^0_\alpha$ be the collection of countable unions $A = \bigcup_n A_n$ where each $A_n \in \mathbf{\Pi}^0_{\alpha_n}$ for some $\alpha_n < \alpha$. We also let $\mathbf{\Pi}^0_\alpha = \neg\mathbf{\Sigma}^0_\alpha$. Alternatively, $A \in \mathbf{\Pi}^0_\alpha$ if $A = \bigcap_n A_n$ where $A_n \in \mathbf{\Sigma}^0_{\alpha_n}$ where each $\alpha_n < \alpha$. We also set $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$, in particular $\mathbf{\Delta}^0_1$ is the collection of clopen sets. For any topological space, $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$. All of the collections $\mathbf{\Delta}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ are pointclasses, that is, they are closed under inverse images of continuous functions. A basic fact (see [12]) is that for any uncountable Polish space X, there is no collapse in the levels of the Borel hierarchy, that is, all the various pointclasses $\mathbf{\Delta}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, for $\alpha < \omega_1$, are all distinct. Thus, these levels of the Borel hierarchy can be used to calibrate the descriptive complexity of a set. We say a set $A \subseteq X$ is $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) *hard* if $A \notin \mathbf{\Pi}^0_\alpha$ (resp. $A \notin \mathbf{\Sigma}^0_\alpha$). This says A is "no simpler" than a $\mathbf{\Sigma}^0_\alpha$ set. We say A is $\mathbf{\Sigma}^0_\alpha$-*complete* if $A \in \mathbf{\Sigma}^0_\alpha \setminus \mathbf{\Pi}^0_\alpha$, that is, $A \in \mathbf{\Sigma}^0_\alpha$ and A is $\mathbf{\Sigma}^0_\alpha$ hard. This says A is exactly at the complexity level $\mathbf{\Sigma}^0_\alpha$. Likewise, A is $\mathbf{\Pi}^0_\alpha$-complete if $A \in \mathbf{\Pi}^0_\alpha \setminus \mathbf{\Sigma}^0_\alpha$.

A set $D \subseteq X$ is in the class $D_2(\mathbf{\Pi}^0_3)$ if $D = A \setminus B$ where $A, B \in \mathbf{\Pi}^0_3$. A set D is $D_2(\mathbf{\Pi}^0_3)$-hard if $X \setminus D \notin D_2(\mathbf{\Pi}^0_3)$, and D is $D_2(\mathbf{\Pi}^0_3)$-complete if it is in $D_2(\mathbf{\Pi}^0_3)$ and is $D_2(\mathbf{\Pi}^0_3)$-hard. As with the classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, the class $D_2(\mathbf{\Pi}^0_3)$ has a universal set

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and so is non-selfdual, that is, it is not closed under complements (we will define a particular complete set for $D_2(\mathbf{\Pi}^0_3)$ in §3.2).

H. Ki and T. Linton [13] proved that the set $\mathcal{N}(b)$ of base-b normal numbers (Definition 1 below) is $\mathbf{\Pi}^0_3(\mathbb{R})$-complete. Further work was done by V. Becher, P. A. Heiber, and T. A. Slaman [5] who settled a conjecture of A. S. Kechris by showing that the set of absolutely normal numbers is $\mathbf{\Pi}^0_3(\mathbb{R})$-complete. Furthermore, V. Becher and T. A. Slaman [6] proved that the set of numbers normal in at least one base is $\mathbf{\Sigma}^0_4(\mathbb{R})$-complete. In another direction, D. Airey, S. Jackson, D. Kwietniak, and B. Mance [1], [2] showed that for any dynamical system with a weak form of the specification property, the set of generic points for the system is $\mathbf{\Pi}^0_3$-complete. This result generalizes the Ki-Linton result to many numeration systems other than the standard base b one.

1.2. **Normal Numbers.** We recall the definition of a normal number.

Definition 1. A real number x is *normal of order k in base b* if all blocks of digits of length k in base b occur with relative frequency b^{-k} in the b-ary expansion of x. We denote this set by $\mathcal{N}_k(b)$. Moreover, x is *normal in base b* if it is normal of order k in base b for all natural numbers k. We denote the set of normal numbers in base b by

$$\mathcal{N}(b) := \bigcap_{k\in\omega} \mathcal{N}_k(b).$$

We also wish to mention one of the most fundamental and important results relating to normal numbers in base b. The following is due to D. D. Wall in his Ph.D. dissertation [23].

Theorem 2 (D. D. Wall)**.** *A real number x is normal in base b if and only if the sequence $(b^n x)$ is uniformly distributed mod 1.*

While it is not difficult to prove Theorem 2, its importance in the theory of normal numbers can not be understated. Large portions of the theory of normal numbers in base b make use of Theorem 2. We provide an example of a theorem that provides motivation for the main problem studied in this paper.

Theorem 3 (D. D. Wall)**.** [1] *For all rational numbers q (for the second inclusion we assume also $q \neq 0$) and integers $b \geq 2$, we have*

$$q + \mathcal{N}(b) \subseteq \mathcal{N}(b);$$
$$q\mathcal{N}(b) \subseteq \mathcal{N}(b).$$

That is, normality in base b is preserved by rational addition and multiplication.

Moreover, Theorem 2 suggests a dynamical interpretation of normality which allows the definition of normality to be extended to other expansions such as the regular continued fraction expansion, the Lüroth series expansion, and the β-expansions. See [8] for a basic treatment and introduction to this idea.

In this paper, we are interested in a class of expansions known as the Q-Cantor series expansions that includes the b-ary expansions as a special case, but do not admit an extension of Theorem 2. The study of normal numbers and other statistical properties of real numbers with respect to large classes of Cantor series expansions

[1] A full characterization of $r \in \mathbb{R}$ such that $r + \mathcal{N}(b) \subseteq \mathcal{N}(b)$ is given in [17].

was first done by P. Erdős and A. Rényi in [9] and [10] and by A. Rényi in [18], [19], and [20] and by P. Turán in [21].

The Q-Cantor series expansions, first studied by G. Cantor in [7], are a natural generalization of the b-ary expansions. G. Cantor's motivation to study the Cantor series expansions was to extend the well-known proof of the irrationality of the number $e = \sum 1/n!$ to a larger class of numbers. Results along these lines may be found in the monograph of J. Galambos [11]. If $Q = (q_i)_{i=1}^{\infty}$ is a sequence of integers with each $q_i \geq 2$, then we say that Q is a *basic sequence*. Given a basic sequence $Q = (q_i)_{i=1}^{\infty}$, the *Q-Cantor series expansion* of a real number x is the (unique) [2] expansion of the form

$$x = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{q_1 q_2 \cdots q_i} \tag{1}$$

where $a_0 = \lfloor x \rfloor$ and a_i is in $\{0, 1, \ldots, q_i - 1\}$ for $i \geq 1$ with $a_i \neq q_i - 1$ infinitely often. We abbreviate (1) with the notation $x = a_0.a_1a_2a_3 \ldots$ w.r.t. Q. If $I = [i, j]$ is an interval in $\mathbb{N}$ and the basic sequence Q is understood, we let, with a slight abuse of notation, $x \upharpoonright I$ denote the sequence of digits $a_i, \ldots, a_j$.

For a basic sequence $Q = (q_i)$, a block $B = (e_1, e_2, \cdots, e_k) \in \omega^{<\omega}$, and a natural number j, define

$$\mathcal{I}_{Q,j}(B) = \begin{cases} 1 & \text{if } e_1 < q_j, e_2 < q_{j+1}, \cdots, e_k < q_{j+k-1} \\ 0 & \text{otherwise} \end{cases}$$

and let

$$Q_n(B) = \sum_{j=1}^{n} \frac{\mathcal{I}_{Q,j}(B)}{q_j q_{j+1} \cdots q_{j+k-1}}. \tag{2}$$

Let

$$Q_n^{(k)} := \sum_{j=1}^{n} \frac{1}{q_j q_{j+1} \cdots q_{j+k-1}} \text{ and } T_{Q,n}(x) := \left(\prod_{j=1}^{n} q_j \right) x \bmod 1.$$

$Q_n(B)$ gives the expected number of occurrences of the block B in the Cantor series expansion of x with a starting position in $[1, n]$. We say B has *infinite expectation* if $\lim_{n\to\infty} Q_n(B) = \infty$. $Q_n^{(k)}$ is the expected number of occurrences of 0_k (the length k block of 0s) with a start in $[1, n]$ (which is also the expected number of occurrences of 1_k). We also let $Q_{m,n}(B) = \sum_{j=m}^{n} \frac{\mathcal{I}_{Q,j}(B)}{q_j q_{j+1} \cdots q_{j+k-1}}$, which is expected number of occurrences of B with a start in $[m, n]$.

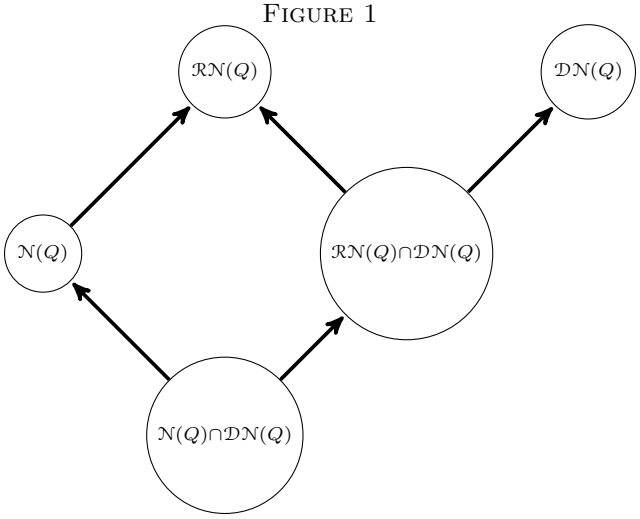
A basic sequence Q is *k-divergent* if $\lim_{n\to\infty} Q_n^{(k)} = \infty$, *fully divergent* if Q is k-divergent for all k, and *k-convergent* if it is not k-divergent. A basic sequence Q is *infinite in limit* if $q_i \to \infty$.

For a block $B = (e_1, \ldots, e_k)$ as above we let $|B| = k$ denote the length of B, and let $\|B\| = (e_1 + 1) + \cdots + (e_k + 1)$. For $1 \leq t \leq k$ we let $B(t) = e_t$ denote the tth element of the block B.

For x a real with Q-Cantor series expansion $a_0.a_1a_2 \cdots$, we let $N_n^Q(B, x)$ be the number of i with $1 \leq i \leq n$ such that $x \upharpoonright [i, i + |B| - 1] = B$. We let $N_{m,n}^Q(B, x)$ be the number of $i \in [m, n]$ with $x \upharpoonright [i, i + |B| - 1] = B$. This counts the number of occurrences of the block B with a start in the interval $[m, n]$.

[2] Uniqueness can be proved in the same way as for the b-ary expansions.

FIGURE 1



Motivated by Theorem 2, we make the following definitions of normality for Cantor series expansions.

Definition 4. A real number x is *Q-normal of order k* if for all blocks B of length k such that $\lim_{n\to\infty} Q_n(B) = \infty$,

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n(B)} = 1. \tag{3}$$

We let $\mathcal{N}_k(Q)$ be the set of numbers that are Q-normal of order k. The real number x is *Q-normal* if $x \in \mathcal{N}(Q) := \bigcap_{k=1}^{\infty} \mathcal{N}_k(Q)$.

Definition 5. A real number x is *Q-ratio normal of order k* (here we write $x \in \mathcal{RN}_k(Q)$) if for all blocks B_1 and B_2 of length k such that $\lim_{n\to\infty} \min(Q_n(B_1), Q_n(B_2)) = \infty$, we have

$$\lim_{n\to\infty} \frac{N_n^Q(B_1,x)/Q_n(B_1)}{N_n^Q(B_2,x)/Q_n(B_2)} = 1. \tag{4}$$

We say that x is *Q-ratio normal* if $x \in \mathcal{RN}(Q) := \bigcap_{k=1}^{\infty} \mathcal{RN}_k(Q)$.

Definition 6. A real number x is *Q-distribution normal* if the sequence $(T_{Q,n}(x))_{n=0}^{\infty}$ is uniformly distributed mod 1. Let $\mathcal{DN}(Q)$ be the set of Q-distribution normal numbers.

We note that by Theorem 2, the analogous versions of the above definitions are equivalent for the b-ary expansions. The situation is far more interesting in the case that Q is infinite in limit and fully divergent.

It was proved in [16] that the directed graph in Figure 1 gives the complete containment relationships between these notions when Q is infinite in limit and fully divergent. The vertices are labeled with all possible intersections of one, two, or three choices of the sets $\mathcal{N}(Q)$, $\mathcal{RN}(Q)$, and $\mathcal{DN}(Q)$, where we know that $\mathcal{N}(Q) = \mathcal{N}(Q) \cap \mathcal{RN}(Q)$ and $\mathcal{N}(Q) \cap \mathcal{DN}(Q) = \mathcal{N}(Q) \cap \mathcal{DN}(Q) \cap \mathcal{RN}(Q)$. The set labeled on vertex A is a subset of the set labeled on vertex B if and only if there is a

directed path from A to B. For example, $\mathcal{N}(Q) \cap \mathcal{DN}(Q) \subseteq \mathcal{RN}(Q)$, so all numbers that are Q-normal and Q-distribution normal are also Q-ratio normal.

We remark that all inclusions suggested from Figure 1 are either easily proved ($\mathcal{N}(Q) \subseteq \mathcal{RN}(Q)$) or are trivial. The difficulty comes in showing a lack of inclusion. The most challenging of these is to prove that there is a basic sequence Q where $\mathcal{RN}(Q) \cap \mathcal{DN}(Q) \backslash \mathcal{N}(Q) \neq \emptyset$.

As the equivalence of these definitions is so key to the study of normality in base b, it is natural to ask how "independent" these sets are. There have been two approaches to measure this. First, it is natural to ask if, for example, there is a simple condition $P(x)$ where if x is Q-normal and $P(x)$ also holds, we will have the x is Q-distribution normal (or any other permutation of definitions of normality). One example of such an attempt to find a condition $P(x)$ is motivated by Theorem 3. This theorem strongly fails when Q is infinite in limit and fully divergent: Q-distribution normality is preserved only by non-zero integer multiplication while Q-normality and Q-ratio normality aren't even preserved by integer multiplication. In fact, the easiest ways to construct members of $\mathcal{N}(Q) \backslash \mathcal{DN}(Q)$ is to use the techniques presented in [16] to construct members of the (surprisingly) non-empty set

$$\{x \in \mathcal{N}(Q) : \forall n \geq 2 \; nx \notin \mathcal{N}(Q)\}.$$

This motivated Samuel Roth to ask the third author if it is true that $nx \in \mathcal{N}(Q)$ for all natural numbers n implies that $x \in \mathcal{DN}(Q)$ at the 2012 RTG conference: Logic, Dynamics and Their Interactions, with a Celebration of the Work of Dan Mauldin in Denton, Texas. This question was later strongly shown to be false in [4] as it was shown that there exist basic sequences Q such that

$$\dim_{\mathrm{H}} \left(\{x \in \mathbb{R} : rx + s \in \mathcal{N}(Q) \backslash \mathcal{DN}(Q) \forall r \in \mathbb{Q} \backslash \{0\}, s \in \mathbb{Q}\} \right) = 1.$$

Any other attempt to find such an additional condition that would allow one to get from one form of normality to another has thus far failed.

The second method has been to attempt to find the "size" of the difference sets suggested by Figure 1. All of these difference sets are meager and have zero measure. The Hausdorff dimension of most of these difference sets has been calculated in [3]. In particular, when Q is infinite in limit and fully divergent, all non-empty difference sets except for $\mathcal{N}(Q) \backslash \mathcal{DN}(Q)$ are known to have full Hausdorff dimension. The Hausdorff dimension of $\mathcal{N}(Q) \backslash \mathcal{DN}(Q)$ is known to have full Hausdorff dimension only for a small class of infinite in limit and fully divergent basic sequences Q.

Another approach to measuring the difference sets is to determine the exact descriptive complexity of these sets. If we show that for two of these classes (for a given Q) the difference set is $D_2(\mathbf{\Pi}^0_3)$-complete, then the difference has maximum logical complexity. One of the main results of this paper, Theorem 15, is to establish this fact for the five non-trivial difference sets. As we mention in some examples below, this can be used to rule out potential theorems connecting the different classes.

Remark 7. When Q is infinite in limit and k-divergent, conditions (3) and (4) can be replaced by

$$\lim_{n \to \infty} \frac{N_n^Q(B, x)}{Q_n^{(k)}} = 1 \text{ and } \lim_{n \to \infty} \frac{N_n^Q(B_1, x)}{N_n^Q(B_2, x)} = 1,$$

respectively. This class of expansions will be important for us throughout this paper. Moreover, $\mathcal{N}_k(Q) = \mathcal{RN}_k(Q) = \mathbb{R}$ if and only if Q is k-convergent.

Remark 8. Assuming $Q = (q_i)$ is infinite in limit, we have the following easy facts about distribution normality. A real x with Q-Cantor series expansion $a_0.a_1a_2\cdots$ is in $\mathcal{DN}(Q)$ iff the sequence $(\frac{a_i}{q_i})$ is uniformly distributed mod 1. If $y = b_0.b_1b_2\cdots$ and $\lim_{i\to\infty} \frac{a_i - b_i}{q_i} = 0$, then $x \in \mathcal{DN}(Q)$ iff $y \in \mathcal{DN}(Q)$. Also, if $\{i\colon a_i \neq b_i\}$ has density 0, then $x \in \mathcal{DN}(Q)$ iff $y \in \mathcal{DN}(Q)$.

We will need the following theorem of [15].

Theorem 9. *The sets $\mathcal{N}(Q), \mathcal{RN}(Q)$, and $\mathcal{DN}(Q)$ are sets of full measure for all basic sequences Q.*[3]

1.3. **Statement of Results.** We will prove the following theorems. First we address the complexity of the various normality classes themselves. Theorems 10 and 11 can be seen as generalizations of the Ki-Linton result as $\mathcal{N}(Q)$, $\mathcal{RN}(Q)$, and $\mathcal{DN}(Q)$ all coincide when Q is the constant b sequence. The proofs, particularly that of Theorem 11, have however significant extra complications.

Theorem 10. *For all basic sequences Q, the set $\mathcal{DN}(Q)$ is $\mathbf{\Pi}^0_3$-complete.*

Theorem 11. *The sets $\mathcal{N}(Q)$ and $\mathcal{RN}(Q)$ are $\mathbf{\Pi}^0_3$-complete if Q is 1-divergent, and clopen if Q is 1-convergent. Moreover, $\mathcal{N}_k(Q)$ and $\mathcal{RN}_k(Q)$ are $\mathbf{\Pi}^0_3$-complete if Q is k-divergent and clopen if Q is k-convergent.*

We can extend Theorem 11 to show the following.

Theorem 12. *Let $\mathcal{C}$ be a collection of blocks. Then the set*

$$\mathcal{N}_{\mathcal{C}}(Q) = \left\{ x \in \mathbb{R}\colon \lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n(B)} = 1 \ \forall B \in \mathcal{C} \text{ such that } \lim_{n\to\infty} Q_n(B) = \infty \right\}$$

is $\mathbf{\Pi}^0_3$-complete if there exists $B \in \mathcal{C}$ such that $\lim_{n\to\infty} Q_n(B) = \infty$ and clopen otherwise. Similarly, the set

$$\mathcal{RN}_{\mathcal{C}}(Q) = \left\{ x \in \mathbb{R}\colon \lim_{n\to\infty} \frac{N_n^Q(B_1,x)/Q_n(B_1)}{N_n^Q(B_2,x)/Q_n(B_2)} = 1 \ \forall B_1, B_2 \in \mathcal{C} \text{ such that } |B_1| = |B_2| \text{ and } \lim_{n\to\infty} \min(Q_n(B_1), Q_n(B_2)) = \infty \right\}$$

is $\mathbf{\Pi}^0_3$-complete if there exist $B_1, B_2 \in \mathcal{C}$ such that $\lim_{n\to\infty} \min(Q_n(B_1), Q_n(B_2)) = \infty$ and $\mathcal{C}$ satisfies the following hypothesis:

($\star$) For every $B \in \mathcal{C}$ with infinite expectation there is a block $B' \in \mathcal{C}$ of infinite expectation with $|B'| = |B|$ and an integer $1 \leq t \leq |B|$ such that $|B(t) - B'(t')| > 1$ for all $1 \leq t' \leq |B'|$.

If there do not exist $B_1, B_2 \in \mathcal{C}$ such that $\lim_{n\to\infty} \min(Q_n(B_1), Q_n(B_2)) = \infty$, then $\mathcal{RN}_{\mathcal{C}}(Q)$ is clopen.

Remark 13. The $\mathbf{\Pi}^0_3$-completeness of $\mathcal{N}_{\mathcal{C}}$ holds for general $\mathcal{C}$ (and all Q), but the $\mathbf{\Pi}^0_3$-completeness of $\mathcal{RN}_{\mathcal{C}}$ requires the extra hypothesis ($\star$) on $\mathcal{C}$. We do not know if this extra hypothesis is necessary. For base b expansions, normality and ratio normality coincide so the extra assumption is not needed, but we do not know for general Q.

[3]The definitions of Q-normality and Q-ratio normality that were used were less general, but the general result holds with only small modification for our more general definition.

Remark 14. The proof of Theorem 12 will also show the $\mathbf{\Pi}^0_3$-completeness for a variation of ratio normality which we call *strong ratio normality*. Here we remove the restriction that $|B_1| = |B_2|$ in the above definition of $\mathcal{RN}_{\mathcal{C}}(Q)$. We can accordingly relax the $(\star)$ assumption by removing the requirement that $|B'| = |B|$.

The next theorem addresses the complexity of the difference sets. We note that the hypotheses on Q are necessary as the various normality classes coincide for base b expansions (where Q is not infinite in limit) and when Q is not 1-divergent then $\mathcal{N}(Q)$ and $\mathcal{RN}(Q)$ are clopen.

Theorem 15. *Assume that Q is infinite in limit and 1-divergent. Then the sets $\mathcal{DN}(Q)\setminus\mathcal{N}(Q)$, $\mathcal{DN}(Q)\setminus\mathcal{RN}(Q)$, $\mathcal{N}(Q)\setminus\mathcal{DN}(Q)$, $\mathcal{RN}(Q)\setminus\mathcal{DN}(Q)$, and $\mathcal{RN}(Q)\setminus\mathcal{N}(Q)$ are all $D_2(\mathbf{\Pi}^0_3)$-complete.*

Theorem 15 imposes limitations on the relationships between the classes $\mathcal{N}(Q)$, $\mathcal{RN}(Q)$, and $\mathcal{DN}(Q)$. For example, consider the sets $\mathcal{N}(Q)$ and $\mathcal{RN}(Q)$. Since $\mathcal{RN}(Q) \setminus \mathcal{N}(Q)$ is $D_2(\mathbf{\Pi}^0_3)$-complete, there cannot be a $\mathbf{\Sigma}^0_3$ set A such that $A \cap \mathcal{RN}(Q) = \mathcal{N}(Q)$ (as otherwise we would have $\mathcal{RN}(Q) \setminus \mathcal{N}(Q) = \mathcal{RN}(Q) \setminus A \in \mathbf{\Pi}^0_3$, a contradiction). Thus, no $\mathbf{\Sigma}^0_3$ condition can be added to the assumption of ratio normality to give the set of normal numbers. Equivalently, anytime a $\mathbf{\Sigma}^0_3$ set contains $\mathcal{N}(Q)$ (or $\mathcal{DN}(Q)$), then it must contain elements of $\mathcal{RN}(Q) \setminus \mathcal{N}(Q)$ and $\mathcal{DN}(Q)\setminus\mathcal{N}(Q)$ (resp. $\mathcal{N}(Q)\setminus\mathcal{DN}(Q)$, and so $\mathcal{RN}(Q)\setminus\mathcal{DN}(Q)$). For example, though $\mathcal{N}(Q)$ has Lebesgue measure one, any $\mathbf{\Pi}^0_2$ set of measure one which contains $\mathcal{N}(Q)$ must contain an element of $\mathcal{RN}(Q)\setminus\mathcal{N}(Q)$, as well as $\mathcal{DN}(Q)\setminus\mathcal{N}(Q)$. Many naturally occurring sets of reals A are defined by conditions which result in them being $\mathbf{\Sigma}^0_3$ sets. Examples include countable sets, co-countable sets, the class BA of *badly approximable* numbers (which is a $\mathbf{\Sigma}^0_2$ set), the Liouville numbers (which is a $\mathbf{\Pi}^0_2$ set), and the set of $x \in [0,1]$ where a particular continuous function $f\colon [0,1] \to \mathbb{R}$ is not differentiable. In all these cases the theorem implies that either the set omits some normal number, or else contains a number which is ratio normal but not normal (and likewise for $\mathcal{DN}(Q)$). Of course, many of these statements are easy to see directly, but the point is that they all follow immediately from the general complexity result, Theorem 15.

Previous work of Mance [14] had shown that all of the non-trivial difference sets (all except $\mathcal{N}(Q) \setminus \mathcal{RN}(Q)$, which is trivially empty) are non-empty assuming Q is infinite in limit k-divergent for all k. Thus, Theorem 15 strengthens this in two ways: we relax the hypothesis to Q being 1-divergent, and we show the difference sets are actually $D_2(\mathbf{\Pi}^0_3)$-complete.

To mention another application of Theorem 15, consider (relative to a fixed basic sequence Q which is infinite in limit and 1-divergent) the following weakening of distribution normality. Say a real x is *ϵ-weakly distribution normal* (for $\epsilon > 0$) if there is an ℓ such that for all $N \geq \ell$ and all intervals $(a,b) \subseteq (0,1)$

$$\left| \frac{1}{N} \#\{n < N\colon q_0 \cdots q_{n-1} x \mod 1 \in [a,b]\} - (b-a) \right| \leq \epsilon.$$

The set $\mathcal{WDN}(Q)_\epsilon$ of ϵ-weakly distributional normal numbers is easily a $\mathbf{\Sigma}^0_2$ set. It therefore follows from Theorem 15 that $\mathcal{N}(Q) \cap \mathcal{WDN}(Q)_\epsilon \setminus \mathcal{DN}(Q)$ is non-empty.

For one more example, let $\mathcal{R}(Q)$ denote the set of *rich* numbers. These are the x such that every block $B \in \omega^{<\omega}$ occurs in the Q-ary expansion of x. The set $\mathcal{R}(Q)$ is easily a $\mathbf{\Pi}^0_2$ set, and contains the ratio normal numbers (and so also the normal

numbers). From Theorem 15 it therefore follows that $\mathcal{DN}(Q) \cap \mathcal{R}(Q) \setminus \mathcal{N}(Q)$ is non-empty, in other words, distribution normal and rich does not imply normal (if this failed, then $\mathcal{DN}(Q) \setminus \mathcal{N}(Q)$ would be equal to $\mathcal{DN}(Q) \setminus \mathcal{R}(Q)$ which a $\mathbf{\Pi}^0_3$ set, contradicting Theorem 15).

2. $\mathbf{\Pi}^0_3$-completeness of the normality classes

Throughout, $Q = (q_i)_{i=1}^\infty$ will denote a *basic sequence*, that is, a sequence of integers q_i with $q_i \geq 2$ for all i.

Note that the $\mathbf{\Pi}^0_3$-completeness of the set $\mathcal{N}(Q) \cap [0,1]$ (and likewise for $\mathcal{RN}(Q) \cap [0,1]$ and $\mathcal{DN}(Q) \cap [0,1]$) immediately implies the $\mathbf{\Pi}^0_3$-completeness of $\mathcal{N}(Q)$, since if $\mathcal{N}(Q)$ were in $\mathbf{\Sigma}^0_3$, then so would be $\mathcal{N}(Q) \cap [0,1]$. Similarly, the $\mathbf{\Pi}^0_3$-completeness of the difference sets restricted to $[0,1]$ (for example $\mathcal{RN}(Q) \setminus \mathcal{N}(Q) \cap [0,1]$) implies the $\mathbf{\Pi}^0_3$-completeness of the difference set (e.g. $\mathcal{RN}(Q) \setminus \mathcal{N}(Q)$). So, for the rest of the paper we will restrict our attention to reals in $[0,1]$, that is, when we write $\mathcal{N}(Q)$ etc., we will henceforth mean $\mathcal{N}(Q) \cap [0,1]$.

The basic sequence Q forms the set of bases for the expansion of an real $x \in [0,1]$ into a Cantor series

$$x = \sum_{i=1}^{\infty} \frac{a_i}{q_1 \cdots q_i},$$

where $0 \leq a_i < q_i$. Recall we abbreviate the above equation by writing $x = .a_1 a_2 \cdots$ when Q is understood. Let X_Q be the set of all sequences $(a_i)_{i=1}^\infty$ with $0 \leq a_i < q_i$. X_Q is a compact Polish space with the product of the discrete topologies on the sets $\{0, 1, \ldots, q_i - 1\}$. We let $\varphi_2 \colon X_Q \to [0,1]$ be the map $\varphi_2((a_i)) = .a_1 a_2 \ldots$. Our reduction maps will always be of the form $\varphi(x) = \varphi_2 \circ \varphi_1$, where $\varphi_1 \colon \omega^\omega \to X_Q$ will vary from proof to proof.

We first prove the completeness result for distribution normality, Theorem 10.

Proof of Theorem 10. Let $P = \{x \in \omega^\omega \colon \lim_n x(n) = \infty\}$. It is well-known that P is $\mathbf{\Pi}^0_3$-complete. We define a continuous $\varphi \colon \omega^\omega \to [0,1]$ which will be a reduction of P to $\mathcal{DN}(Q)$, that is, such that $P = \varphi^{-1}(\mathcal{DN}(Q))$. This suffices to show that $\mathcal{DN}(Q)$ is $\mathbf{\Pi}^0_3$-complete. Again, φ will be of the form $\varphi = \varphi_2 \circ \varphi_1$ where φ_2 is as above.

Let $0 = b_0 < b_1 < b_2 < \cdots$ be a sufficiently fast-growing sequence from $\mathbb{N}$, so that in particular $\lim_n \frac{b_0 + \cdots + b_{n-1}}{b_n} = 0$. Let $I_n = [b_{n-1}, b_n)$, which we call the nth block of $\mathbb{N}$.

Fix a $z = (z_i)_{i=1}^\infty \in X_Q$ such that $\varphi_2(z) = .z_1 z_2 \cdots \in \mathcal{DN}(Q)$. We will use z as a "reference point" from which we make certain modifications depending on the point $x \in \omega^\omega$.

Fix $x \in \omega^\omega$ and we describe the construction for $\varphi_1(x)$. Let $x'(n) = \min\{x(n), n\}$. Clearly $x \in P$ iff $x' \in P$. Consider the nth block I_n. We may assume the b_n grow fast enough so that for all n and all $k \leq n+2$, for all $b \geq b_{n-1}$ we have that

$$\left| \frac{1}{b} \# \left\{ i < b \colon T_{Q,i}(z) \in \left[0, \frac{1}{k}\right) \right\} - \frac{1}{k} \right| < \frac{1}{2n}. \tag{5}$$

For $i \in I_n$ we define $a_i = (\varphi_1(x))(i)$ as follows. For $i \in I_n$, if $T_{Q,i}(z) \notin [0, \frac{1}{x'(n)+2})$, then we set $a_i = z_i$. If $T_{Q,i}(z) \in [0, \frac{1}{x'(n)+2})$, then we set a_i to be the least integer in $[z_i, q_i)$ such that $\frac{a_i}{q_i} > \frac{1}{x'(n)+2}$.

This defines the map φ_1, and it is clear that φ_1, and thus φ, is continuous. We show φ is a reduction of P to $\mathcal{DN}(Q)$.

First suppose that $x \notin P$. Then there is an i_0 such that for infinitely many n we have $x'(n) = i_0$. For such n we have that for all $i \in I_n$ that $T_{Q,i}(\varphi(x)) \notin [0, \frac{1}{x'(n)+2})$, and thus $T_{Q,i}(\varphi(x)) \notin [0, \frac{1}{i_0+2})$. This follows from the facts that

$$T_{Q,i}(\varphi(x)) \geq T_{Q,i}(z),$$

and if $T_{Q,i}(z) \in [0, \frac{1}{x'(n)+2})$ then by definition of a_i we have that $T_{Q,i}(\varphi(x)) \geq \frac{a_i}{q_i} > \frac{1}{x'(n)+2}$. Since $\frac{b_0+\cdots+b_{n-1}}{b_n}$ tends to 0, it follows that $\varphi(x) \notin \mathcal{DN}(Q)$.

Assume now $x \in P$, and we show that $\varphi(x) \in \mathcal{DN}(Q)$. From Remark 8 it suffices to show that $\{i \colon a_i \neq z_i\}$ has density 0. Fix $\epsilon > 0$. Since $x \in P$, $\lim_{n\to\infty} x'(n) = \infty$. Fix n_0 large enough so that $\frac{1}{x'(n)+2} < \frac{\epsilon}{2}$ for all $n \geq n_0$. If $n \geq n_0$, then for all $k \in I_n$ we have that

(6)
$$\left|\frac{1}{k}\#\left\{i < k \colon T_{Q,i}(z) \in \left[0, \frac{1}{x'(n_0)+2}\right)\right\} - \frac{1}{x'(n_0)+2}\right| < \frac{1}{2n} \leq \frac{1}{x'(n_0)+2} < \frac{\epsilon}{2}.$$

For $n \geq n_0$, the $i \in I_n$ for which $a_i \neq z_i$ are the i for which $T_{Q,i}(z) \in [0, \frac{1}{x'(n)+2})$, which is a subset of the $i \in I_n$ for which $T_{Q,i}(z) \in [0, \frac{1}{x'(n_0)+2})$, for n large enough. From Equation (6) it follows that for all large enough n and $k > b_n$ that

$$\frac{1}{k}\|\{i < k \colon a_i \neq z_i\}\| < \epsilon.$$

Thus, $\{i \colon a_i \neq z_i\}$ has density 0. □

We next prove the completeness result, Theorem 11, for the classes $\mathcal{N}(Q)$ and $\mathcal{RN}(Q)$.

Proof of Theorem 11. Suppose Q is 1-divergent and we first show that $\mathcal{N}(Q)$ and $\mathcal{RN}(Q)$ are $\mathbf{\Pi}^0_3$-complete. Let $P \subseteq \omega^\omega$ be the $\mathbf{\Pi}^0_3$-complete set as before. Fix $z = (z_i)_{i=1}^\infty \in X_Q$ such that $\varphi_2(z) = .z_1 z_2, \cdots \in \mathcal{N}(Q)$.

We say a block $B \in \omega^k$ is *good* if $\lim_n Q_n(B) = \infty$, that is, the block B has an infinite expectation. Recall that if $B = (e_1, \ldots, e_k)$, then $\|B\| = \sum_{1\leq i\leq k}(e_i + 1)$.

We let $0 = b_0 < b_1 < \cdots$ be a sufficiently fast-growing sequence so that the following hold:

(1) $b_n - b_{n-1} > 2^n b_{n-1}$.
(2) $\frac{1}{Q_m(B)}|N^Q_m(B, z) - Q_m(B)| < \frac{1}{n}$ for any good B with $\|B\| \leq n$, and any $m \geq b_{n-1}$.
(3) $\frac{b_{n-1}}{Q_{b_n}(B)} < \frac{1}{4^n}$ for any good B with $\|B\| \leq n$.

We define the map $\varphi_1 \colon \omega^\omega \to X_Q$, and our final reduction map will be $\varphi = \varphi_2 \circ \varphi_1$ where φ_2 is as in the proof of Theorem 10. Let $I_n = [b_{n-1}, b_n)$. For $x \in \omega^\omega$, let $x'(n) = \max\{27, \min\{x(n), n\}\}$. We define $\varphi_1(x) \restriction I_n$ as follows.

Let $B_0, B_1, \ldots B_p$ enumerate the good blocks which occur among the first $\lfloor \sqrt[6]{x'(n)} \rfloor$ many blocks, where we order the blocks first by $\|B\|$ and then lexicographically. Note that this ordering of the blocks has order-type ω and the ith block in this ordering has length $\leq i$. So, for $j \leq p$ we have $|B_j| \leq \sqrt[6]{x'(n)} \leq \sqrt[6]{n}$.

For each $0 \leq i \leq p$ let $m(i) \in [b_{n-1}, b_n)$ be the least m so that $N_{m,b_n}(B_i, z) \leq \frac{2}{\sqrt[3]{x'(n)}} Q_{b_n}(B_i)$. Note that $N_{m(i),b_n}(B_i, z) \geq \frac{3}{2\sqrt[3]{x'(n)}} Q_{b_n}(B_i)$. Let $m = \max\{m(i) \colon 0 \leq$

$i \le p\}$. Let i_0 be such that $m(i_0) = m$. We note that $i_0 = i_0(n)$ depends on n, but as n is fixed for the rest of the definition of $\varphi_1(x) \restriction I_n$, we will just write i_0.

Consider the block B_{i_0}. We say a block B_j is *sparse* relative to B_{i_0} if $Q_{b_n}(B_j) < \frac{1}{4|B_j|4^{|B_j|}\sqrt{x'(n)}} Q_{b_n}(B_{i_0})$. Let $A \subseteq I_n$ be the set of $i \in [m, b_n]$ such that $z \restriction [i, i + |B_{i_0}|-1] = B_{i_0}$. Let r be the digit altering function defined as follows. If $B_{i_0}(1) = 0$, then r maps 0 to 1 and leaves all other values fixed. If $B_{i_0}(1) \neq 0$ then r maps $B_{i_0}(1)$ to $B_{i_0}(1) - 1$ and leaves all other values fixed. Since the block B_{i_0} is fixed for the rest of the definition, the function r is also. Note that r applied to a valid Q expansion results in a valid Q expansion. Also, r is at most 2-to-1, so each block B has most $2^{|B|}$ many preimages under r (we apply r to a block by applying it to each digit).

Let $A' \subseteq A$ be those $i \in A$ such that for all $j < p$ such that B_j is sparse relative to B_{i_0}, and all $q < |B_j|$, $z \restriction [i-q, \dots, i-q+|B_j|-1] \neq B_j$ and $z \restriction [i-q, \dots, i-q+|B_j|-1] \notin B_j^S$, where $S \subseteq |B_j|$ and B_j^S is the set of blocks B such that $r(B) = B_j$ (here $r(B)$ means apply r to all of the digits of B).

Note that if z' is obtained by applying r to the digits z_i for i in a subset of A', then $z \restriction I_n$ and $z' \restriction I_n$ have the same number of occurrences of the block B_j for $j = 1, \dots, p$.

Note that

$$|A'| \ge \frac{3}{2\sqrt[3]{x'(n)}} Q_{b_n}(B_{i_0}) - \sum_{j<p} \sum_{S \subseteq |B_j|} \sum_{r_S(B) = B_j} |B_j| N^Q_{m,b_n}(B, z)$$

where the first sum ranges over the j such that B_j is sparse relative to B_{i_0}, and the third sum ranges over blocks B with $r_S(B) = B_j$, where r_S applies r to the digits in the set S. Since r either lowers a digit or changes a 0 to a 1, it follows that if $r_S(B) = B_j$ then $Q_k(B) \le Q_k(B_j)$ for any k.

Thus,

$$\begin{aligned}
|A'_n| &\ge \frac{3}{2\sqrt[3]{x'(n)}} Q_{b_n}(B_{i_0}) - \sum_{j<p} |B_j| 4^{|B_j|} N^Q_{b_n}(B_j) \\
&\ge \frac{3}{2\sqrt[3]{x'(n)}} Q_{b_n}(B_{i_0}) - (1 + \frac{1}{n}) \sum_{j<p} |B_j| 4^{|B_j|} Q_{b_n}(B_j) \\
&\ge \frac{3}{2\sqrt[3]{x'(n)}} Q_{b_n}(B_{i_0}) - (1 + \frac{1}{n}) \sum_{j<p} |B_j| 4^{|B_j|} \left(\frac{1}{4|B_j|4^{|B_j|}\sqrt{x'(n)}} Q_{b_n}(B_{i_0}) \right) \\
&\ge \frac{3}{2\sqrt[3]{x'(n)}} Q_{b_n}(B_{i_0}) - \frac{1}{2} \frac{\sqrt[6]{x'(n)}}{\sqrt{x'(n)}} Q_{b_n}(B_{i_0}) \\
&= \frac{3}{2\sqrt[3]{x'(n)}} Q_{b_n}(B_{i_0}) - \frac{1}{2\sqrt[3]{x'(n)}} Q_{b_n}(B_{i_0}) \\
&\ge \frac{1}{\sqrt[3]{x'(n)}} Q_{b_n}(B_{i_0})
\end{aligned}$$

for all large enough n. Let A'' be the last $\frac{1}{x'(n)} Q_{b_n}(B_{i_0})$ elements of A'.

For $i \in I_n$, let

$$\varphi_1(x)(i) = \begin{cases} r(z_i) & \text{if } i \in A'' \\ z_i & \text{otherwise} \end{cases}$$

Note that $\varphi_1(x) \upharpoonright I_n$ is obtained from $z \upharpoonright I_n$ by changing certain digits, the number of such changes being $|A''| = \frac{1}{x'(n)} Q_{b_n}(B_{i_0})$. Note that the least element i of A'' is at least the number m from above (since $A'' \subseteq A \subseteq [m, b_n]$) and

$$\begin{aligned} N_m(B_{i_0}, z) &\geq N_{b_n}(B_{i_0}, z) - N_{m,b_n}(B_{i_0}, z \\ &\geq N_{b_n}(B_{i_0}, z) - \frac{2}{\sqrt[3]{x'(n)}} Q_{b_n}(B_{i_0}) \\ &\geq \left((1 - \frac{1}{n}) - \frac{2}{\sqrt[3]{x'(n)}} \right) Q_{b_n}(B_{i_0}), \end{aligned}$$

since $m = m(i_0)$ and using the definition of $m(i_0)$. Since $x'(n) \geq 27$, $N_m(B_{i_0}, z) \geq \frac{1}{3} Q_{b_n}(B_{i_0})$ and in particular $m \geq \frac{1}{3} Q_{b_n}(B_{i_0})$. From property (3) of the b_n it follows that the first element of A'' is at least $\frac{4^n}{3} b_{n-1} > 2^{n-1} b_{n-1}$, for all large enough n.

Suppose first that $x \notin P$. There is a least ℓ, which we call ℓ_0, such that for infinitely many n we have that $x'(n) = \ell_0$. At such a stage n in the construction, we consider the first $\sqrt[6]{\ell_0}$ many blocks. So, for infinitely many such n we may assume that the block B_{i_0} is fixed, that is, the value of i_0 does not depend on n along this subsequence. Then for large enough such n we have:

$$\begin{aligned} N^Q_{b_n}(B_{i_0}, \varphi_1(x)) &\leq b_{n-1} + N^Q_{b_n}(B_{i_0}, z) - \frac{1}{\ell_0} Q_{b_n}(B_{i_0}) \\ &\leq b_{n-1} + Q_{b_n}(B_{i_0}) \left(1 + \frac{1}{n}\right) - \frac{1}{\ell_0} Q_{b_n}(B_{i_0}) \\ &\leq \left(\frac{1}{4^n} + 1 + \frac{1}{n} - \frac{1}{\ell_0} \right) Q_{b_n}(B_{i_0}) \\ &\leq \left(1 - \frac{1}{2\ell_0} \right) Q_{b_n}(B_{i_0}) \end{aligned} \tag{7}$$

This shows $\varphi(x) \notin \mathcal{N}(Q)$ when $x \notin P$. Consider the block B_{i_0} which is fixed along the subsequence. If $B_{i_0}(1) \geq 1$, then we obtain $\varphi_1(x) \upharpoonright I_n$ from $z \upharpoonright I_n$ by lowering certain occurrences of the digit $B_{i_0}(1)$ to $B_{i_0}(1) - 1$. This will not decrease the number of occurrences of the block 0_k, where $k = |B_{i_0}|$. So, $N^Q_{b_n}(0_k, \varphi_1(x)) \geq N^Q_{b_n}(0_k, z) \geq Q_{b_n}(0_k)(1 - \epsilon)$, for small ϵ (say $\epsilon < \frac{1}{3\ell_0}$) and all large enough n. This, along with Equation 7, shows $\varphi_1(x) \notin \mathcal{RN}(Q)$. If $B_{i_0}(1) = 0$, then r maps 0 to 1 and leaves all other digits unchanged. This cannot decrease the number of occurrences of of the block 1_k. So, $N^Q_{b_n}(1_k, \varphi_1(x)) \geq N^Q_{b_n}(1_k, z) \geq Q_{b_n}(1_k)(1 - \epsilon)$ which again shows $\varphi_1(x) \notin \mathcal{RN}(Q)$.

Suppose now that $x \in P$ so that $\lim_n x'(n) = \infty$. Let B be a good block, and fix $\epsilon > 0$. Let $n_0 > |B|$ be such that for all $n \geq n_0$, $x'(n)$ is large enough that B is one of the first $\sqrt[6]{x'(n)}$ many blocks. Consider now $n \geq n_0$ and corresponding interval I_n. Let $\delta > 0$ and assume n is sufficiently large and inductively that we have shown $|N^Q_{b_{n-1}}(B, z) - N^Q_{b_{n-1}}(B, \varphi_1(x))| \leq \delta Q_{b_{n-1}}(B)$.

Consider first the case B is sparse at stage n relative to B_{i_0}. Let p be as before, so $p \leq \sqrt[6]{x'(n)}$. Then $B = B_j$ for some $j \leq p$. So, for any $i \in I_n$ we have that if $z \upharpoonright [i, i + |B| - 1] = B$ or $\varphi_1(x) \upharpoonright [i, i + |B| - 1] = B$ then $[i, i + |B| - 1] \cap A'_n = \emptyset$. Since $x \upharpoonright I_n$ is obtained from $z \upharpoonright I_n$ by changing the value only at points of $A''_n \subseteq A'_n$, it follows that for any $k \in [b_{n-1}, b_n]$ that $|N^Q_k(B, z) - N^Q_k(B, \varphi_1(x))| = |N^Q_{b_{n-1}}(B, z) - N^Q_{b_{n-1}}(B, \varphi_1(x))|$. So we have for $k \in [b_{n-1}, b_n]$ and large enough n:

(8)
$$\begin{aligned}|N_k^Q(B,\varphi_1(x)) - Q_k(B)| &\le |N_k^Q(B,z) - Q_k(B)| + |N_{b_{n-1}}^Q(B,\varphi_1(x)) - N_{b_{n-1}}^Q(B,z)| \\ &\le |N_k^Q(B,z) - Q_k(B)| + \delta Q_{b_{n-1}}(B) \\ &\le \frac{1}{2^n}Q_k(B) + \delta Q_k(B) \\ &\le Q_k(B)\left(\frac{1}{2^n} + \delta\right),\end{aligned}$$

which verifies normality for the block B. Since $|N_{b_n}^Q(B,z) - N_{b_n}^Q(B,\varphi_1(x))| = |N_{b_{n-1}}^Q(B,z) - N_{b_{n-1}}^Q(B,\varphi_1(x))|$, the inductive hypothesis at b_n follows immediately.

Consider next the case that $B = B_j$ is not sparse at stage n relative to B_{i_0}. From the definition of i_0 and m we have that

$$Q_m(B) \ge Q_{m_j}(B_j) \ge \left(1 - \frac{2}{\sqrt[3]{x'(n)}}\right) Q_{b_n}(B_j). \tag{9}$$

From the definition of being sparse relative to B_{i_0} we have that

$$Q_{b_n}(B) \ge \frac{1}{4|B|4^{|B|}\sqrt{x'(n)}} Q_{b_n}(B_{i_0}). \tag{10}$$

Recall that $\varphi_1(x) \upharpoonright I_n$ and $z \upharpoonright I_n$ only differ on A'', and $\min(A'') \ge m$.

Now let $k \in [b_{n-1}, b_n]$, and we estimate $|N_k^Q(B,\varphi_1(x)) - Q_k(B)|$. If $k < m$ then $|N_k^Q(B,\varphi_1(x)) - Q_k(B)| \le |N_k^Q(B,z) - Q_k(B)| + |N_{b_{n-1}}^Q(B,z) - N_{b_{n-1}}^Q(B,\varphi_1(x))| \le \left(\frac{1}{2^n} + \delta\right) Q_k(B)$, which verifies normality for B.

So, assume $k \ge m$. We have

$$\begin{aligned}|N_k^Q(B,\varphi_1(x)) - Q_k(B)| &\le b_{n-1} + \frac{1}{x'(n)}Q_{b_n}(B_{i_0}) + |N_k^Q(B,z) - Q_k(B)| \\ &\le \frac{1}{2^n}Q_{b_n}(B) + \frac{1}{x'(n)}Q_{b_n}(B_{i_0}) + \frac{1}{2^n}Q_k(B) \\ &\le \frac{1}{2^n}\frac{1}{1 - \frac{2}{\sqrt[3]{x'(n)}}}Q_m(B) + \frac{1}{x'(n)}Q_{b_n}(B_{i_0}) + \frac{1}{2^n}Q_k(B) \\ &\le \frac{1}{2^{n-2}}Q_m(B) + \frac{1}{x'(n)}Q_{b_n}(B_{i_0}) + \frac{1}{2^n}Q_k(B) \\ &\le \frac{1}{2^{n-3}}Q_k(B) + \frac{1}{x'(n)}4|B|4^{|B|}\sqrt{x'(n)}Q_{b_n}(B) \\ &\le \frac{1}{2^{n-3}}Q_k(B) + \frac{1}{x'(n)}4|B|4^{|B|}\sqrt{x'(n)}\frac{1}{1 - \frac{2}{\sqrt[3]{x'(n)}}}Q_m(B) \\ &\le \frac{1}{2^{n-3}}Q_k(B) + \frac{12|B|4^{|B|}}{\sqrt{x'(n)}}Q_k(B)\end{aligned}$$

Since $x'(n) \to \infty$, this shows normality for the block B. Similarly, letting $k = b_n$ we have $|N_{b_n}^Q(B,\varphi_1(x)) - N_{b_n}^Q(B,z)| \le b_{n-1} + \frac{1}{x'(n)}Q_{b_n}(B_{i_0}) \le \frac{1}{4^n}Q_{b_n}(B) + \frac{8|B|4^{|B|}}{\sqrt{x'(n)}}Q_{b_n}(B) \le \delta Q_{b_n}(B)$, which verifies the inductive hypothesis at b_n.

This completes the proof that $\mathcal{N}(Q)$ and $\mathcal{RN}(Q)$ are $\mathbf{\Pi}^0_3$-complete assuming Q is 1-divergent. If Q is 1-convergent, then every x is in $\mathcal{N}(Q)$ and $\mathcal{RN}(Q)$, so the conclusion of Theorem 11 holds trivially. □

The second part of Theorem 11 concerning $\mathcal{N}_k(Q)$ and $\mathcal{RN}_k(Q)$ and the proof of Theorem 12 are slight generalizations of the proof of Theorem 11 given above. Since the proofs are similar we just sketch the differences.

Proof of Theorem 12. We use the notation and terminology of the proof of Theorem 11. We may assume that all of the blocks $B \in \mathcal{C}$ have infinite expectation, that is, $\lim_{n\to\infty} Q_n(B) = \infty$. At stage n of the construction we again let $p = \lfloor \sqrt[6]{x'(n)} \rfloor$, and let $B_1, \ldots, B_p$ enumerate the first p many blocks of $\mathcal{C}$. We define the block B_{i_0} as before, maximizing the value of $m(i)$ for $1 \leq i \leq p$. For the first part of Theorem 12 we may use the same digit changing function r as in the proof of Theorem 11. If $x \in P$, then the proof of Theorem 11 shows that $\varphi(x) \in \mathcal{N}_{\mathcal{C}}(Q)$. If $x \notin P$, then for infinitely many n the value of i_0 will be constant, and B_{i_0} is a block in $\mathcal{C}$. As in Equation 7, this gives an $\epsilon > 0$ such that for infinitely many n we have $|N^Q_{b_n}(B_{i_0}, \varphi_1(x)) - Q_{b_n}(B_{i_0})| > \epsilon Q_{b_n}(B_{i_0})$. Thus, $\varphi(x) \notin \mathcal{N}_{\mathcal{C}}(Q)$. For the second statement of Theorem 12 we modify the argument as the blocks 0_k and 1_k used in the proof of Theorem 11 may not be in $\mathcal{C}$. The additional hypothesis of Theorem 12, however, guarantees the existence of a block $B_j \in \mathcal{C}$ and an integer t such that $|B_{i_0}(t) - B_j(t')| > 1$ for all t'. As in the argument after Equation 7, we modify the definition of $\varphi_1(x) \upharpoonright I_n$ to change by 1 all occurrences of $B_{i_0}(t)$ in $z \upharpoonright I_n$ which correspond to a possible occurrence of B_{i_0} (that is, the integers $i \in I_n$ where $z \upharpoonright [i-t+1, i+|B_{i_0}|] = B_{i_0}$). This will not affect the number of occurrences of the block B_j in I_n. This gives that $\varphi(x) \notin \mathcal{RN}_{\mathcal{C}}(Q)$.

□

3. Proof of Theorem 15

We will show the $D_2(\mathbf{\Pi}^0_3)$-completeness of the non-trivial combinations of the form $A \setminus B$ where A, B are one of $\mathcal{N}(Q)$, $\mathcal{DN}(Q)$, $\mathcal{RN}(Q)$. There are 5 non-trivial combinations, as $\mathcal{N}(Q) \subseteq \mathcal{RN}(Q)$. Section 3.1 handles four of these cases, which are essentially done by the same proof. The fifth case, $\mathcal{RN}(Q) \setminus \mathcal{N}(Q)$, is more complicated and will be handled in Section 3.2. We note that the $D_2(\mathbf{\Pi}^0_3)$-completeness of the sets $A \setminus B$ implies that the sets of the form $A \cap B$ or of the form $A \cup B$ (for $A, B \in \mathcal{N}(Q), \mathcal{DN}(Q), \mathcal{RN}(Q)$) are $\mathbf{\Pi}^0_3$-complete by the following simple lemma.

Lemma 16. If A, B are $\mathbf{\Pi}^0_3$ and $A \setminus B$ is $D_2(\mathbf{\Pi}^0_3)$-complete, then $A \cup B$ and $A \cap B$ are $\mathbf{\Pi}^0_3$-complete.

Proof. Suppose that $A \cup B$ were $\mathbf{\Sigma}^0_3$. Then $A \setminus B = (A \cup B) \setminus B$ would be $\mathbf{\Sigma}^0_3$, a contradiction. Likewise, if $A \cap B$ were $\mathbf{\Sigma}^0_3$ then $A \setminus B = A \setminus (A \cap B)$ would be $\mathbf{\Pi}^0_3$, a contradiction. □

Lastly, we note that since $\mathcal{N}(Q), \mathcal{RN}(Q)$, and $\mathcal{DN}(Q)$ are sets of full measure by Theorem 9, their intersections are nonempty. We will freely use this fact without mentioning it.

3.1. Completeness of $\mathcal{DN}(Q) \setminus \mathcal{N}(Q)$, $\mathcal{DN}(Q) \setminus \mathcal{RN}(Q)$, $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$, and $\mathcal{RN}(Q) \setminus \mathcal{DN}(Q)$.

Theorem 17. *Let Q be infinite in limit and 1-divergent. Then the sets $\mathcal{DN}(Q) \setminus \mathcal{N}(Q)$, $\mathcal{DN}(Q) \setminus \mathcal{RN}(Q)$, $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$, and $\mathcal{RN}(Q) \setminus \mathcal{DN}(Q)$ are all $D_2(\mathbf{\Pi}^0_3)$-complete.*

Proof. Let $C = \{x \in \omega^\omega \colon x(2n) \to \infty\}$, $D = \{x \in \omega^\omega \colon x(2n+1) \to \infty\}$. It is easy to see that $C \setminus D$ is $D_2(\mathbf{\Pi}^0_3)$-complete.

Fix a fast growing sequence $\{b_n\}$, so in particular $(2^n \sum_{i<n} b_i)/b_n \to 0$. Recall 0_k denotes the sequence of length k consisting of all 0s. We introduce two basic operations which can be performed on an interval $I \in \mathbb{N}^{[a,b]}$ of digits:

$\Theta_{k,\ell}$: Let $A \subseteq [a,b]$ be the set of j which start an occurrence of 0_k, that is, $(I(j), \dots, I(j+k-1)) = 0_k$. Let $A' \subseteq A$ be the last $\lfloor \frac{|A|}{\ell} \rfloor$ many elements of A. For each $j \in A'$, change the digit $I(j)$ from a 0 to a 1.

Ξ_k: For every $j \in [a,b]$ with $I(j) \in [\frac{k}{k+1} q_j, q_j]$, change the digit from $I(j)$ to $q_j - 1$.

For the difference hierarchy results we will use both operations, exploiting the fact that, roughly speaking, they allow us to modify normality/ratio normality and distribution normality independently.

Let $z \in \mathcal{DN}(Q) \cap \mathcal{RN}(Q)$, and let $(z_i)_{i=1}^\infty \in X_Q$ be the digits of the Q-Cantor series expansion of z.

We suppose the b_n are chosen so that for all $k \leq 2n$ such that Q is k-divergent we have $Q^{(k)}_{b_n} > 2^n b_{n-1}$ and $\forall m \geq b_n$ $|N^Q_m(0_k, z) - Q^{(k)}_m| < \frac{1}{2^n} Q^{(k)}_m$.

Suppose first that $\lim_n Q^{(k)}_n = \infty$ for all k, that is, Q is fully divergent. Given $x \in \omega^\omega$, we define $\varphi_1(x) \in X_Q$ as follows. Let $x'(n) = \max\{2, \min\{x(n), n\}\}$. Consider the interval of digits $z \restriction I_n$, where $I_n = [b_{n-1}, b_n)$. We let $\varphi_1(x) \restriction [b_{n-1}, b_n)$ be given by starting with $z \restriction I_n$ and applying the operation $\Theta_{x'(2n), x'(2n)}$ and then the operation $\Xi_{x'(2n+1)}$ to it.

Recall $\varphi_2 \colon X_Q \to [0,1]$ is the continuous map

$$\varphi_2(d_1, d_2 \dots) = \sum_{i=1}^\infty \frac{d_i}{q_1 \cdots q_i}.$$

We show that $\varphi = \varphi_2 \circ \varphi_1$ is a reduction from $C \setminus D$ to $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$. In fact, we show that $x \in C$ iff $\varphi(x) \in \mathcal{N}(Q)$ and $x \in D$ iff $\varphi(x) \in \mathcal{DN}(Q)$.

Since $q_i \to \infty$, the Θ operation does not affect distribution normality as it involves changing each digit in the Q Cantor series expansion by at most 1 (see Remark 8). Also, as $q_i \to \infty$ we have that the Ξ operation does not effect normality, since for any block of digits B we have that $|N^Q_n(B, z) - N^Q_n(B, y)|$ is bounded with n (regardless of x), where y is the result of applying the Ξ operation in all of the I_n.

First suppose $x \in C$, so $x(2n) \to \infty$. Let w be the result of applying just the first operation $\Theta_{x'(2n), x'(2n)}$ to z in each of the intervals I_n. We claim that $w \in \mathcal{N}(Q)$. Consider a block B of digits, and let $k = |B|$ denote its length. First note that for

any $\epsilon > 0$ all large enough n we have

$$\begin{aligned}|N^Q_{b_n}(B,w) - N^Q_{b_n}(B,z)| &\leq b_{n-1} + \frac{2|B|}{x'(2n)} N^Q_{b_n}(0_{x'(2n)}, z) \\ &\leq b_{n-1} + \frac{2|B|}{x'(2n)} \left(1 + \frac{1}{2^n}\right) Q_{b_n}(0_{x'(2n)}) \\ &\leq b_{n-1} + \frac{2|B|}{x'(2n)} 2Q_{b_n}(B) \\ &\leq \epsilon Q_{b_n}(B).\end{aligned}$$

Since $z \in \mathcal{N}(Q)$, for large enough n we have $|N^Q_{b_n}(B,w) - Q_{b_n}(B)| < 2\epsilon Q_{b_n}(B)$. Fix $\epsilon > 0$, and consider now n large enough and $k \in [b_{n-1}, b_n)$. Let m be the first element of A' in I_n. Note that $|A| \geq 2^n b_{n-1}$, and so $m \geq 2^{n-1} b_{n-1}$ as $x'(2n) \geq 2$. If $k < m$ then we have

$$\begin{aligned}|N^Q_k(B,w) - Q_k(B)| &\leq |N^Q_{b_{n-1}}(B,w) - Q_{b_{n-1}}(B)| + |N^Q_{b_{n-1},k}(B,w) - Q_{b_{n-1},k}(B)| \\ &\leq 2\epsilon Q_{b_{n-1}}(B) + |N^Q_{b_{n-1},k}(B,z) - Q_{b_{n-1},k}(B)| \\ &\leq 2\epsilon Q_{b_{n-1}}(B) + |N^Q_k(B,z) - Q_k(B)| + |N^Q_{b_{n-1}}(B,z) - Q_{b_{n-1}}(B)| \\ &\leq \left(2\epsilon + \frac{2}{2^n}\right) Q_k(B).\end{aligned}$$

If $k \geq m$, then for large enough n first note that we have

$$\begin{aligned}Q_k(B) \geq Q_k(0_{x'(2n)}) &\geq (1 - \frac{1}{2^n}) N^Q_k(0_{x'(2n)}, z) \\ &\geq (1 - \frac{1}{2^n})(1 - \frac{1}{x'(2n)}) N^Q_{b_n}(0_{x'(2n)}, z) \\ &\geq (1 - \frac{1}{2^n})^2 (1 - \frac{1}{x'(2n)}) Q_{b_n}(0_{x'(2n)}) \\ &\geq \frac{1}{2} Q_{b_n}(0_{x'(2n)}) \\ &\geq 2^{n-1} b_{n-1}.\end{aligned}$$

So we have

$$\begin{aligned}|N^Q_k(B,w) - Q_k(B)| &\leq b_{n-1} + \frac{2|B|}{x'(2n)} N^Q_k(0_{x'(2n)}, z) + |N^Q_k(B,z) - Q_k(B)| \\ &\leq \frac{2}{2^n} Q_k(B) + \frac{2|B|}{x'(2n)} N^Q_k(0_{|B|}, z) + \frac{1}{2^n} Q_k(B) \\ &\leq \frac{3}{2^n} Q_k(B) + \frac{3|B|}{x'(2n)} Q_k(B) \\ &\leq \epsilon Q_k(B).\end{aligned} \tag{11}$$

This shows that $w \in \mathcal{N}(Q)$.

If $x \notin C$, say $x(2n) = c$ for infinitely many n, then for infinitely many n we have

$$\begin{aligned} N^Q_{b_n}(0_c, w) &\leq N^Q_{b_n}(0_c, z) + b_{n-1} - \frac{1}{c} N^Q_{b_{n-1},b_n}(0_c, z) \\ &\leq \left(1 + \frac{1}{2^n}\right) Q_{b_n}(0_c) + 2b_{n-1} - \frac{1}{c} N^Q_{b_n}(0_c, z) \\ &\leq \left(1 + \frac{3}{2^n}\right) Q_{b_n}(0_c) - \frac{1}{c}\left(1 - \frac{1}{2^n}\right) Q_{b_n}(0_c) \\ &\leq \left(1 - \frac{1}{2c}\right) Q_{b_n}(0_c). \end{aligned} \tag{12}$$

On the other hand, the block 1_c occurs in $w \restriction [0, b_n)$ at least as many times as it does in $z \restriction [0, b_n)$. So, $N^Q_{b_n}(1_c, w) \geq N^Q_{b_n}(1_c, z) \geq (1 - \frac{1}{2^n}) Q_{b_n}(1_c)$. Since $Q_{b_n}(0_c) = Q_{b_n}(1_c)$, it follows that $w \notin \mathcal{RN}(Q)$ (and also $w \notin \mathcal{N}(Q)$).

If $x \in D$, so $x(2n+1) \to \infty$, then $\varphi(x) \in \mathcal{DN}(Q)$. We use the fact that if $u = .u_1 u_2 \cdots \in \mathcal{DN}(Q)$ and $v = .v_1 v_2 \ldots$ is such that $|(u_i - v_i)/q_i| \to 0$, then $v \in \mathcal{DN}(Q)$ (see Remark 8). If $x \notin D$, then $\varphi(x) \notin \mathcal{DN}(Q)$ since for infinitely many intervals $[b_{n-1}, b_n)$ we have that $(\varphi_1(x))(i)/q_i \notin [1 - \epsilon, 1 - \frac{\epsilon}{2}]$, where $\epsilon = \frac{1}{c+1}$ and $x(2n+1) = c$ for infinitely many n.

So we have that if $x \in C$ then $\varphi(x) \in \mathcal{N}(Q)$, and if $x \notin C$ then $\varphi(x) \notin \mathcal{RN}(Q)$. Also, $x \in D$ iff $\varphi(x) \in \mathcal{DN}(Q)$. Thus, in the last two cases of the theorem, φ is a reduction of $C \setminus D$ to the desired difference set. For the first two cases of the theorem, φ is a reduction of $D \setminus C$ to the desired difference set. This completes the proof of Theorem 17 in the case where Q is fully divergent.

Assume now that that there is a largest integer k_0 such that Q is k_0-divergent. We again obtain $\varphi_1(x) \restriction [b_{n-1}, b_n)$ by applying two operations. One of these is $\Xi_{x'(2n+1)}$, where the operation Ξ is as before. For the other, we use operation $\Theta_{k_0, x'(2n)}$.

If $x \notin C$, then $\varphi(x) \notin \mathcal{RN}(Q)$ as before. If $x \in C$, that is, $x(2n) \to \infty$, then the argument of Equation 11 shows that $\lim_k \frac{|N^Q_k(B,w) - Q_k(B)|}{Q_k(B)} = 0$ for any block B of length $\leq k_0$. Since this accounts for all of the blocks of infinite expectation, we have that $w \in \mathcal{N}(Q)$. Since the second operation does not affect normality, it follows that $\varphi(x) \in \mathcal{N}(Q)$.

As before, we have that $x \in D$ iff $\varphi(x) \in \mathcal{DN}(Q)$. So we again have that $x \in C$ implies $\varphi(x) \in \mathcal{N}(Q)$, $x \notin C$ implies $\varphi(x) \notin \mathcal{RN}(Q)$, and $x \in D$ iff $\varphi(x) \in \mathcal{DN}(Q)$. Thus, as in the previous case φ gives the desired reductions.

□

3.2. **Completeness of $\mathcal{RN}(Q) \setminus \mathcal{N}(Q)$.** We will need to define a class of functions $\psi_{P,Q}$ in order to prove the last case of Theorem 15. Let $P = (p_i)$ and $Q = (q_i)$ be basic sequences. If $x = a_0.a_1a_2\cdots$ w.r.t. P, then put

$$\psi_{P,Q}(x) = \sum_{i=1}^{\infty} \frac{\min(a_i, q_i - 1)}{q_1 \cdots q_i}.$$

We will need the following theorem of [16].

Theorem 18. *Suppose that P and Q are basic sequences which are infinite in limit. If $x = a_0.a_1a_2\cdots$ w.r.t. P satisfies $a_i < q_i - 1$ for infinitely many i, then for every*

block B

$$N_i^Q(B, \psi_{P,Q}(x)) = N_i^P(B, x) + O(1).$$

While Theorem 18 is not difficult to prove, it has been an essential tool in proving some of the more difficult theorems about Q-normal numbers.

Recall that for a basic sequence Q and block B that $Q_n(B)$ (see Equation 2) denotes the expected number of occurrences of B with a starting position in $[1, n]$ with respect to the basic sequence Q. Since we will be dealing with several basic sequences in this section, we extend this notation in a natural manner. Namely, if P (or R) are basic sequences, then we let $P_n(B)$ (or $R_n(B)$) denote the expected number of occurrences of B with a starting position in $[1, n]$ with respect to P (or R). We similarly use the notation $N_n^P(B, z)$ to denote the number of occurrences of B in the P-Cantor series for z with starting position in $[1, n]$.

We will use the following lemma about concatenating intervals of normal sequences for different basic sequences.

Lemma 19. Let P, Q be basic sequences which are infinite in limit and assume that $\limsup_i \frac{p_i}{q_i}$ is finite. Let $u \in \mathcal{N}(P)$, and $v \in \mathcal{N}(Q)$. Let $B \in \omega^{<\omega}$ have infinite expectation with respect to P and Q, and let $\epsilon > 0$. Then there is an i_0 such that if $i' > i \geq i_0$ then $|N_{i'}^R(B, w) - R_{i'}(B)| < \epsilon R_{i'}(B)$ where $w \upharpoonright [0, i) = u$, $w \upharpoonright [i, \infty) = v$, $R \upharpoonright [0, i) = P$, and $R \upharpoonright [i, \infty) = Q$.

Proof. Fix C such that $\frac{p_i}{q_i} \leq C$ for all i. Since $\frac{p_i}{q_i} \leq C$, and B has infinite expectation with respect to P and Q, for large enough i we have that $Q_i(B) \leq 2C^{|B|}P_i(B)$. Let i_0 be such that for all $i \geq i_0$ we have that $|N_i^Q(B, v) - Q_i(B)| < \epsilon' Q_i(B)$ and $|N_i^P(B, u) - P_i(B)| < \epsilon' P_i(B)$ where $\epsilon' = \frac{\epsilon}{3+4C^{|B|}}$. Let $i' > i \geq i_0$. We then have

$$\begin{aligned}
|N_{i'}^R(B, w) - R_{i'}(B)| &\leq |N_i^P(B, u) - P_i(B)| + |N_{i,i'}^Q(B, v) - Q_{i,i'}(B)| \\
&\leq \epsilon' P_i(B) + |N_i^Q(B, v) - Q_i(B)| + |N_{i'}^Q(B, v) - Q_{i'}(B)| \\
&\leq \epsilon' P_i(B) + \epsilon' Q_i(B) + \epsilon' Q_{i'}(B) \\
&\leq \epsilon' R_i(B) + 2\epsilon' Q_{i'}(B) \\
&\leq \epsilon' R_i(B) + 2\epsilon'(Q_i(B) + Q_{i,i'}(B)) \\
&\leq \epsilon' R_i(B) + 2\epsilon'(2C^{|B|}P_i(B) + Q_{i,i'}(B)) \\
&\leq \epsilon' R_i(B) + 2\epsilon'(1 + 2C^{|B|})R_{i'}(B) \\
&\leq \epsilon'(3 + 4C^{|B|})R_{i'}(B) \leq \epsilon R_{i'}(B).
\end{aligned}$$

□

We now prove the following theorem which gives the last case of Theorem 15.

Theorem 20. *Let Q be a basic sequence which is infinite in limit and 1-divergent. Then the set $\mathcal{RN}(Q) \setminus \mathcal{N}(Q)$ is $D_2(\mathbf{\Pi}_3^0)$-complete.*

Proof. Let $C, D \subseteq \omega^\omega$ be as in Section 3.1. For $k, n \in \mathbb{N}$, recall that $Q_n^{(k)}$ denotes the sum $Q_n^{(k)} = \sum_{i=1}^n \frac{1}{q_i q_{i+1} \cdots q_{i+k-1}}$.

For each m, let $P_m = (\max(2, \lfloor \frac{m+1}{m+2} q_i \rfloor))_{i=1}^\infty$. Recall $(P_m)_i(B)$ denotes the expectation of B in the first i digits of P_m, as in Equation 2, and $(P_m)_i^{(k)}$ denote

the expectation of 0_k in the first i digits of P_m. We fix for the rest of the proof $w_m \in \mathcal{N}(P_m)$, which we identify with a P_m-Cantor series expansion.

Given a strictly increasing sequence $\{b_n\}$ (which we will choose below) and an $x \in \omega^\omega$ we define a new basic sequence $P_x = P(x, \{b_n\}, Q) = (p_i^x)$ as follows. Let $x'(n) = \min\{x(n), n\}$. Let $p_i^x = P_{x'(2n+1)}(i) = \max(2, \lfloor \frac{x'(2n+1)+1}{x'(2n+1)+2} q_i \rfloor)$ for $i \in [b_{n-1}, b_n)$. Note that $p_i \le q_i$ for all i, and if $x(2n+1) \to \infty$ then $\lim_i \frac{p_i^x}{q_i} = 1$. Also $Q_n^{(k)} \le (P_x)_n^{(k)}$ for all k, n.

Claim 21. Suppose Q is k-divergent. If $x(2n+1) \to \infty$ then $\lim_i \frac{Q_i^{(k)}}{(P_x)_i^{(k)}} \to 1$. If $x(2n+1)$ does not tend to ∞ and $\lim_n \frac{Q_{b_{n-1}}^{(k)}}{Q_{b_{n-1},b_n}^{(k)}} = 0$, there is a subsequence of $\frac{Q_i^{(k)}}{(P_x)_i^{(k)}}$ which is bounded away from 1.

Proof. If $x(2n+1) \to \infty$, then $\ell_n = \frac{x(2n+1)+1}{x(2n+1)+2} \to 1$ and so p_i^x/q_i tends to 1. Note that $\left(\frac{1}{q_j q_{j+1} \cdots q_{j+k-1}}\right) / \left(\frac{1}{p_j p_{j+1} \cdots p_{j+k-1}}\right) = (\ell_n)^k$ for all $j \in [b_n, b_{n+1})$. We then have that for all k that $Q_i^{(k)}/(P_x)_i^{(k)} \to 1$ using the simple fact that if $c_j, d_j \ge 0$, $\sum_j c_j = \infty$, $\sum_j d_j = \infty$, and $c_j/d_j \to 1$, then $\left(\sum_{j=1}^i c_j\right) / \left(\sum_{j=1}^i d_j\right) \to 1$.

If $x(2n+1)$ does not tend to ∞, then $\exists u < 1$ such that $\ell_n \le u$ for infinitely many n. So, for infinitely many n we have that $Q_{b_{n-1},b_n}^{(k)} \le u^k (P_x)_{b_{n-1},b_n}^{(k)}$. Thus, for infinitely many n

$$\begin{aligned}
\frac{Q_{b_n}^{(k)}}{(P_x)_{b_n}^{(k)}} &= \frac{Q_{b_{n-1}}^{(k)} + Q_{b_{n-1},b_n}^{(k)}}{(P_x)_{b_{n-1}}^{(k)} + (P_x)_{b_{n-1},b_n}^{(k)}} \le \frac{Q_{b_{n-1}}^{(k)} + Q_{b_{n-1},b_n}^{(k)}}{Q_{b_{n-1}}^{(k)} + (P_x)_{b_{n-1},b_n}^{(k)}} \le \frac{Q_{b_{n-1}}^{(k)} + Q_{b_{n-1},b_n}^{(k)}}{Q_{b_{n-1}}^{(k)} + (\frac{1}{u^k}) Q_{b_{n-1},b_n}^{(k)}} \\
&\le \frac{Q_{b_{n-1}}^{(k)} + (\frac{1}{u^k}) Q_{b_{n-1},b_n}^{(k)} + (1 - \frac{1}{u^k}) Q_{b_{n-1},b_n}^{(k)}}{Q_{b_{n-1}}^{(k)} + (\frac{1}{u^k}) Q_{b_{n-1},b_n}^{(k)}} \\
&= 1 - \left(\frac{1}{u^k} - 1\right) \frac{Q_{b_{n-1},b_n}^{(k)}}{Q_{b_{n-1}}^{(k)} + (\frac{1}{u^k}) Q_{b_{n-1},b_n}^{(k)}} \\
&\le 1 - \left(\frac{1}{u^k} - 1\right) \frac{Q_{b_{n-1},b_n}^{(k)}}{(\frac{2}{u^k}) Q_{b_{n-1},b_n}^{(k)}} = 1 - \left(\frac{1}{u^k} - 1\right) \frac{u^k}{2} \\
&= 1 - \frac{1}{2}(1 - u^k)
\end{aligned}$$

□

First assume that Q is k-divergent for all k. Using Lemma 19 we then inductively pick the b_n satisfying the following:

(1) For all $m_1, m_2 \le n+1$ and all $i > b_{n-1}$, let $P = P(m_1, m_2, i)$ be defined by: $P \upharpoonright [0, b_{n-1}) = P_{m_1} \upharpoonright [0, b_{n-1})$ and $P \upharpoonright [b_{n-1}, i) = P_{m_2} \upharpoonright [b_{n-1}, i)$. Let $w \upharpoonright [0, b_{n-1}) = w_{m_1}$ and $w \upharpoonright [b_{n-1}, i) = w_{m_2} \upharpoonright [b_{n-1}, i)$. Then for any B with $\|B\| \le n+1$ we have $|N_i^P(B, w) - P_i(B)| < \frac{1}{2^n} P_i(B)$.
(2) $(P_m)_{b_{n-1},b_n}(0_{2n}) > 2^n b_{n-1}$ for all $m \le n$.

Given $x \in \omega^\omega$, we define $\varphi_1(x)$ as follows. Suppose $\varphi_1(x) \upharpoonright b_{n-1}$ has been defined. Let $y \upharpoonright [b_{n-1}, b_n) = w_{x(2n+1)} \upharpoonright [b_{n-1}, b_n)$. Then we perform the operation $\Theta_{x'(2n),x'(2n)}$ of Section 3.1 on $y \upharpoonright [b_{n-1}, b_n)$ to produce $\varphi_1(x) \upharpoonright [b_{n-1}, b_n)$. This defines $\varphi_1(x) \upharpoonright [b_{n-1}, b_n)$. Doing this for all blocks $[b_{n-1}, b_n)$ produces $\varphi_1(x)$.

If $x \notin C$, that is $x(2n)$ does not tend to ∞, then $\varphi(x) \notin \mathcal{RN}(Q)$. This is because if $k = \liminf x(2n)$, then there will be infinitely many n for which 0_k occurs in $[0, b_n)$ at most

(13)
$$\begin{aligned} N_{b_n}^{P_{x'(2n+1)}}(0_k, w_{x'(2n+1)})(1 - \frac{1}{k}) + b_{n-1} &\leq (P_{x'(2n+1)})_{b_n}^{(k)}(1 - \frac{1}{k})(1 + \frac{1}{2^n}) + b_{n-1} \\ &\leq (P_{x'(2n+1)})_{b_n}^{(k)}(1 - \frac{1}{k} + \frac{1}{2^{n-1}}) \\ &\leq (P_{x'(2n+1)})_{b_n}^{(k)}(1 - \frac{1}{2k}) \end{aligned}$$

many times while 1_k occurs at least

(14)
$$\begin{aligned} N_{b_n}^{P_{x'(2n+1)}}(1_k, w_{x'(2n+1)}) - b_{n-1} &\geq (P_{x'(2n+1)})_{b_n}^{(k)}(1 - \frac{1}{2^n}) - b_{n-1} \\ &\geq (P_{x'(2n+1)})_{b_n}^{(k)}(1 - \frac{1}{2^{n-1}}) \end{aligned}$$

many times.

If $x(2n) \to \infty$ but $x(2n+1)$ does not tend to infinity, then $\varphi(x) \in \mathcal{RN}(Q) \setminus \mathcal{N}(Q)$. To see this, first note that the point y as above is in $\mathcal{RN}(Q)$. Recall P_x is defined by $P_x \upharpoonright [b_{n-1}, b_n) = P_{x(2n+1)} \upharpoonright [b_{n-1}, b_n)$. We show that $y \in \mathcal{N}(P_x)$, which implies $y \in \mathcal{RN}(Q)$. For any B and for large enough n and for any $b_{n-1} \leq i < b_n$ we have from property (1) of the b_n:

$$|N_i^{P'}(B, y') - P_i'(B)| < \frac{1}{2^n} P_i'(B), \tag{15}$$

where y' and P' are defined by:

$$\begin{aligned} y' \upharpoonright [0, b_{n-1}) &= w_{x(2n-1)} \upharpoonright [0, b_{n-1}) \\ y' \upharpoonright [b_{n-1}, i) &= w_{x(2n+1)} \upharpoonright i \\ P' \upharpoonright [0, b_{n-1}) &= P_{x(2n-1)} \\ P' \upharpoonright [b_{n-1}, i) &= P_{x(2n+1)} \upharpoonright [b_{n-1}, i). \end{aligned}$$

Also, from property (2) of the b_n we have:

$$|N_i^{P'}(B, y') - N_i^{P_x}(B, y)| \leq b_{n-2} < \frac{1}{2^n}(P_x)_{b_{n-1}}(B) \leq \frac{1}{2^n}(P_x)_i(B). \tag{16}$$

Finally,

$$|P_i'(B) - (P_x)_i(B)| \leq b_{n-2} < \frac{1}{2^n}(P_x)_i(B). \tag{17}$$

From Equations 15, 16 and 17 we have $|N_i^{P_x}(B, y) - (P_x)_i(B)| \leq \frac{4}{2^n}(P_x)_i(B)$. This shows that $y \in \mathcal{RN}(Q)$. Since $x(2n+1)$ does not tend to infinity, then from Claim 21 there is a subsequence on which $\frac{Q_i^{(k)}}{(P_x)_i^{(k)}}$ is bounded away from 1. Since y is P_x-normal, we have that y is not Q-normal.

The operation applied to y to produce $\varphi_1(x)$ does not affect normality or ratio normality if $x(2n) \to \infty$ (this is just as in Section 3.1). So, $\varphi(x) \in \mathcal{RN}(Q) \setminus \mathcal{N}(Q)$.

Finally, if $x(n) \to \infty$, then as above $y \in \mathcal{N}(P_x)$. As $x(2n+1) \to \infty$, we have from Claim 21 that $\lim_i \frac{Q_i^{(k)}}{(P_x)_i^{(k)}} \to 1$ and it follows that $y \in \mathcal{N}(Q)$. Since $x(2n) \to \infty$ as well, from the argument in Section 3.1 we also have that $y = \varphi(x) \in \mathcal{N}(Q)$.

So, in all cases we have that $x \in C \setminus D$ iff $\varphi(x) \in \mathcal{RN}(Q) \setminus \mathcal{N}(Q)$.

Suppose now that there is a largest integer k_0 such that Q is k_0-divergent. The proof is essentially identical to that above. We let P_m be as before, and now let w_m be $\leq k_0$ normal with respect to P_m, that is, for all B of length $\leq k_0$ we have $\lim_i \frac{N_i(B,w_m)}{(P_m)_i(B)} = 1$. We define $\varphi_1(x)$ by first defining y exactly as before (using the values $x(2n+1)$). We then modify y to $\varphi_1(x)$, (using $x(2n)$) but in a slightly different manner. Namely, we get $\varphi_1(x) \upharpoonright [b_{n-1}, b_n)$ from $y \upharpoonright [b_{n-1}, b_n)$ as follows. Let $A \subseteq [b_{n-1}, b_n)$ be the integers i in this interval such that $y \upharpoonright [i, i+k_0-1) = 0_{k_0}$. Let $A' \subseteq A$ be the last $\lfloor \frac{|A|}{x(2n)} \rfloor$ many elements of A. For each $i \in A'$ we change $y(i)$ from a 0 to a 1, and for all other i in this interval we set $\varphi_1(x)(i) = y(i)$.

If $x(2n)$ does not tend to infinity, then easily $\varphi(x) \notin \mathcal{RN}(Q)$ as $\frac{N_i^{P_x}(1_{k_0}, \varphi_1(x))}{N_i^{P_x}(0_{k_0}, \varphi_1(x))}$ does not tend to 1. If $x(2n)$ tends to infinity, then we easily have that $\varphi(x)$ is in $\mathcal{N}(Q)$ (or $\mathcal{RN}(Q)$) iff y is in $\mathcal{N}(Q)$ (resp. $\mathcal{RN}(Q)$). In this case, as above, we have that if $x(2n+1) \to \infty$ then $y \in \mathcal{N}(Q)$, and if $x(2n+1)$ does not tend to infinity then $y \in \mathcal{RN}(Q) \setminus \mathcal{N}(Q)$. So, in all cases we have $x \in C \setminus D$ iff $\varphi(x) \in \mathcal{RN}(Q) \setminus \mathcal{N}(Q)$. □

3.3. **Further Discussion.** Theorem 17 can be extended further. First, the hypothesis that $Q = (q_i)$ in infinite in limit can be weakened to the following condition studied by T. Šalát [22]: $\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{q_i} = 0$. This condition is equivalent to saying that there is a set $D \subseteq \mathbb{N}$ of density 0 such that $(q_i)_{i \notin D}$ tends to infinity (see Theorem 1.20 of [24]). Since changing a sequence on a set of density 0 may affect normality and ratio normality, we must now use the argument of Theorem 11. At stage n of the construction of $\varphi_1(x) \upharpoonright I_n$, we again use two operations $\Theta'_{x'(2n)}$ and $\Xi'_{x'(2n+1)}$. The first operation $\Theta'_{x'(2n)}$ is the operation implicitly described in the proof of Theorem 11. That is, we define the block B_{i_0} exactly as in that proof, and define the sets $A, A', A'' \subseteq I_n$ as in that proof. We then eliminate the occurrences of the block B_{i_0} at the points of A'' by applying the digit changing function r as in Theorem 11. Let w be the result of applying this first operation to z (so w is the $\varphi_1(x)$ of Theorem 11). The proof of Theorem 11 did not require that Q be infinite in limit, and so we have that $x(2n) \to \infty$ implies $w \in \mathcal{N}(Q)$ and $x(2n) \nrightarrow \infty$ implies $w \notin \mathcal{RN}(Q)$. The function r changes digits by at most 1, and does not affect distribution normality using Remark 8 and the fact that D has density 0 (changing a sequence on a set of density 0 does not affect distribution normality). So, $w \in \mathcal{DN}(Q)$. The second operation $\Xi'_{x'(2n+1)}$ is the operation $\Xi_{x'(2n+1)}$ of Theorem 17 except we only apply the operation to digits not in D. We let $\varphi_1(x)$ be the result of applying these operations to w. The operations $\Xi'_{x'(2n+1)}$ do not affect normality or distribution normality as $q_i \to \infty$ off of D, and so for every block B, $|N_m^Q(B, \varphi_1(x)) - N_m^Q(B, w)|$ is bounded with m. As in Theorem 17 we have that $\varphi_1(x) \in \mathcal{DN}(Q)$ iff $x(2n+1) \to \infty$. So, $\varphi = \varphi_2 \circ \varphi_1$ is a reduction of $C \setminus D$ (or $D \setminus C$ depending on the case) to the desired set.

Second, we can prove the version of Theorem 17 with $\mathcal{N}(Q)$ and $\mathcal{RN}(Q)$ replaced with $\mathcal{N}_k(Q)$ and $\mathcal{RN}_k(Q)$, provided we assume that Q is k-divergent (and infinite

in limit, or more generally $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \frac{1}{q_i} = 0$). We proceed as above except in defining the block B_{i_0} used in the first operation, we only consider the first $\sqrt[6]{x'(2n)}$ many good blocks $B_1, \ldots, B_p$ of length k. This makes sense since there is some block of length k, namely 0_k, which has infinite expectation. If w again denotes the result of applying the first operation in all of the I_n, then the proof of Theorem 11 shows that if $x(2n) \to \infty$ then $|N_m^Q(B,w) - N_m^Q(B,z)|/Q_m(B) \to 0$ for all blocks B of length k. It follows that if $x(2n) \to \infty$ then $w \in \mathcal{N}_k(Q)$ and if $x(2n) \nrightarrow \infty$ then $w \notin \mathcal{RN}_k(Q)$. Also, $w \in \mathcal{DN}(Q)$ as above. The second operation works exactly as in the above argument, so $\varphi_1(x) \in \mathcal{DN}(Q)$ iff $x(2n+1) \to \infty$. So, $\varphi = \varphi_2 \circ \varphi_1$ again gives the desired reduction.

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