Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education

Editors:
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Brian Katz
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Omaha, Nebraska
February 23 – February 25, 2023

Presented by
The Special Interest Group of the Mathematical Association of America (SIGMAA) for Research in Undergraduate Mathematics Education

CITATION: Cook, S., Katz, B. & Moore-Russo D. (Eds.). (2023). *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*. Omaha, NE.

ISSN: 2474-9346

Explanations and Justifications Regarding Converse Independence

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This paper presents six categories of undergraduate student explanations and justifications regarding the question of whether a converse proof proves a conditional theorem. Two categories of explanation led students to judge that converse proofs cannot so prove, which is the normative interpretation. These judgments depended upon students spontaneously seeking uniform rules of proving across various theorems or assigning a direction to the theorems and proof. The other four categories of explanation led students to affirm that converse proofs prove. We emphasize the rationality of these non-normative explanations to suggest the need for further work to understand how we can help students understand the normative rules of logic.

Keywords: logic, proof, converse independence

Introduction

Proof-based mathematics education research has long attended to how students think about the relationship between mathematical proof and the truth of mathematical claims. Within mathematical practice, deductive proof is the dominant standard for declaring a theorem true. Similarly, we would like students learning proof-based mathematics to adopt normative standards for what kinds of proofs justify a given mathematical claim. In this paper, we specifically consider theorems that are universally quantified conditionals, meaning they can be expressed in the form "For all… , if P, then Q." Since many theorems in undergraduate mathematics are of this form, introduction to proof texts introduce standard proof techniques related to such statements: direct proof, converse proof, contrapositive proof, and proof by contradiction. Of these four types of proof, all prove the given theorem except for the second. A converse proof assumes Q is true and concludes why P must be true as a consequence. We call the principle that the converse proof never proves the given theorem *Converse Independence* (hereafter, "CI"). In common mathematical parlance, the theorem states that P implies Q and the converse proof proves that Q implies P, which are taken to be related, but independent claims.

Previous studies (e.g., Hoyles & Kuchemann, 2002; Yu, et al., 2004) have noted that students often strongly associate a conditional and its converse statement. It has also been suggested that making a distinction between a statement and its converse may be even more difficult when the statement and its converse are both true (Imamoglu & Togrol, 2015), though no explanation was provided for how students should understand CI in this case. CI is meaningfully different for theorems where the statement and converse are both true. Durand-Guerrier (2003) and Dawkins (2019) both discussed how students interpret mathematical statements to refer to the objects that make them true. This referential way of reasoning emphasizes a challenge: how might students distinguish converses that are not distinguished by their truth-values? Our study investigates the question: how do students' reason about the relationship between a converse proof and a conditional theorem, especially in cases where the theorem is true biconditionally?

We answer this research question by characterizing novice undergraduate students' explanations and justifications regarding the relationship between a converse proof and a conditional theorem. These findings help elaborate Hoyles & Kuchemann's (2002) findings of how students might interpret the relationship between converse statements. In other words, our study offers qualitative insights into the previously documented ways students closely associate converse claims. We shall emphasize the coherence and rationality of students' non-normative explanations, which suggest a need for greater attention to this topic in introductory logic.

Literature Review

There are three primary task types that have been used to study how students relate a conditional statement and its converse: inference tasks, statement tasks, and proof tasks. First, some studies consider the inferences that students make or endorse based on a conditional claim. The statement "If P, then Q" is taken to justify inferring Q given knowledge that P is true (often called *modus ponens*). However, according to mathematical logic the conditional statement above does not justify inferring P given knowledge of Q (*affirmation of the consequent*), though students often make this inference or endorse it as appropriate (Evans & Over, 2004; Inglis & Simpson, 2008; Alcock, et al., 2014). Attridge and Inglis (2013) compared inference task performance between UK secondary students studying mathematics and their peers studying English literature (and not mathematics). The year of mathematics study reduced the frequency of *affirmation of the consequent* responses more than the year of English study did, suggesting that general mathematics instruction influences this aspect of CI.

A second type of research task relevant to CI invites students to make judgments about conditional statements and their converse statements. For instance, some scholars (e.g., Hoyles and Kuchemann, 2002; Yu, et al., 2004) have shown that many middle school students believe that conditional statements and their converses say the same thing. Both studies explained this finding in light of the way that students interpreted the truth/falsehood of the statements. Many students affirmed and denied statements using affirming examples (which were provided in the task statement) or counterexamples (which the task did not provide). Accordingly, many students judged the false conditional claim to be true based on affirming examples, so the truth-value did not distinguish the statements.

Introduction to Proof textbooks generally justify CI either using 1) the truth-table or 2) example statements. The former argument (converses have different truth tables) assumes that students are using the truth-table definition to interpret conditional claims, though this has been long shown a poor model of how people interpret such statements (e.g., Evans & Over, 2004; Schroyens, 2010; Inglis, 2006). The second textbook argument is worth considering in more detail. Hammack's (2013) Introduction to Proof textbook provides an instance of this type of argument. It presents one example of a true conditional claim with false converse ("a is a multiple of 6 $\Longrightarrow$ a is divisible by 2") and explains, "Therefore the meanings of $P \Longrightarrow Q$ and $Q \Longrightarrow P$ are in general quite different… a conditional statement and its converse express entirely different things" (p. 44). This justification relies on an implicit meta-theorem about the nature of logic, which we shall call the *Fundamental Warrant of Logic (FWL)*. It states an argument of a given form is only valid if all examples of this argument with true premises have true conclusions. Thus, a single example of some form of argument that yields false results invalidates all other arguments of that same form. In this case, the argument in question would be "If (P⟹Q), then (Q⟹P)." The FWL asserts that the relationship between a statement and its converse must be the same for all conditional statements. The extent to which students agree with this justification or find it compelling is an open question that our study will begin to address.

A third type of task used in research invites students to consider the relationship between a conditional theorem and a converse proof. This is a novel type of task used in the sequence of experiments in which our data originated (see Dawkins & Cook, 2017; Dawkins & Roh, 2022), which we shall describe in the methods section.

Theoretical Framing

Our study draws upon the tradition of Piagetian constructivism both in theoretical orientation (e.g., von Glasersfeld, 1995) and methods (Steffe & Thompson, 2000). Piaget was fond of using logic to model children's reasoning (e.g., Inhelder & Piaget, 1958, 1964). However, the nature of his claims have been widely (mis)understood as claiming that adolescent reasoning comes to conform to formal logic in some strong sense (thought is the mirror of logic). This would violate Piaget's principle of the constructive independence of knowledge. We understand that Piaget used logic as a convenient organizing tool to describe patterns of reasoning (logic was the mirror of thought) that for the student were embedded in more complex systems of meaning (Piaget & Garcia, 2011). Only upon conscious reflection on the form of statements and reasoning can someone abstract conscious understandings of logic (see Beth & Piaget, 1966). A helpful distinction here is between what students construct *in activity* and what they have *reflexively abstracted* to some higher level of re-presentation (von Glasersfeld, 1995). In this study, we do not assume that students have stable re-presentations of logical structure or the abstract relationships between a statement and its converse. Instead, we seek to describe their often tenuous and provisional ways of reasoning about CI that may shift based upon the context of the statements and the task (what Thompson, et al., 2013, called *in the moment meanings*). As such, we present categories of explanations and justifications that students give in particular moments, which are neither mutually exclusive nor necessarily stable throughout their participation in our study. In other words, we seek to understand the subjective rationality of how students begin to reason about such logical relationships.

Methods

The data from this study comes from a larger series of constructivist teaching experiments (Steffe & Thompson, 2000) with undergraduate students from three large public universities in the United States. A total of six teaching experiments were conducted with pairs of students over the course of five years, resulting in a total of 12 participants. None of the students had previously taken a proof course. A key goal of the experiment was to support them in constructing more normative understandings.

The goal of this paper is to characterize students' explanations and justifications regarding whether a converse proof proves a conditional theorem. These explanations and justifications arose in response to What does it prove? tasks. These tasks involve presenting students with a theorem and an associated proof, and students are asked to decide, "Does the proof prove the theorem? Why or why not?" as well as "If the proof does not prove the theorem, what statement does it prove or disprove?" This invites students to attend to the structural relationship between the theorem and the proof and to begin comparing such relationships across theorem/proof pairs. In this paper, we focused on analyzing data from three What does it prove? tasks involving converse proofs (or disproofs). Table 1 presents the three theorems and the structure of the associated proof.

Theorem 1	**Theorem 2**	**Theorem 2'**
For every integer x, if x is a multiple of 6, then x is a multiple of 3.	For any integer x, if x is a multiple of 2 and a multiple of 7, then x is a multiple of 14.	For any integer x, if x is a multiple of 4 and a multiple of 6, then x is a multiple of 24.
Proof 1.2	**Proof 2.2**	**Proof 2.2'**
Let x be an integer that is a multiple of 3. Then x could be 15, which is not a multiple of 6. Thus, it is not necessarily the case that x is a multiple of 6.	Let x be an integer that is a multiple of 14. … Thus, x is a multiple of 2 and a multiple of 7.	Let x be an integer that is a multiple of 24. … Thus, x is a multiple of 4 and a multiple of 6.

Table 1. The three converse What does it prove? tasks studied in this report.

Though we engaged in the experiments overall to promote set-based reasoning (Dawkins, 2017), these tasks were implemented without instructor guidance toward the normative way of reasoning regarding CI. Since our goal was to categorize students' initial responses regarding CI, the analysis in this paper approached the data as task-based interviews (focusing only on the initial implementation of the three tasks above). Consistent with teaching experiment methodology, the researchers continuously generated and tested hypotheses about students' ways of reasoning. We thus had developed models of how each student reasoned about CI, and were struck by the coherence of students' explanations and justifications, especially those that were non-normative (i.e., argued that the converse proof did prove the theorem). Using comparative coding methods (Glaser, et al., 1968), we identified six categories for students' explanations and justifications for whether a converse proof proved the original theorem (henceforth we shall say "does prove" or "does not prove" always referring to the theorem that is converse to the proof). As noted above, these ways of reasoning were not always stable, nor were they mutually exclusive. However, some of the non-normative categories of explanation persisted with certain participants, suggesting they may be important for future work.

Results

In this section we describe the six categories of explanations and justifications regarding CI. The first two categories led students to the normative interpretation that converse proofs do not prove while the latter four led them to affirm that the converse does prove. Overall, four of the 12 students initially decided Proof 2.2 did not prove Theorem 2.

Category 1: Proof Rules Should be Universal

Students who were observed using this category of reasoning believed that proof rules should be universal, that is, a proof rule cannot be accepted unless it works in all cases (similar to Hammack's, 2013, explanation that implicitly invokes the FWL). Students using this category acknowledged that a statement and its converse can have different truth values (Theorem 1) and used this to conclude that there are cases where one can prove the converse of an implication while the implication itself is false (like Theorem 2'). Therefore, they claimed that a converse proof cannot be accepted as proof of an implication because it would only lead to the correct conclusion when the implication and its converse are both true (Theorem 2). One student, Theo (all names are pseudonyms), provided a justification for why Proof 2.2 did not prove Theorem 2 that fit into this category. He explained (imagining the truth-sets of the hypothesis and conclusion as circular regions):

> In this case, the circle of the "if" exists inside the "then," but it encompasses the whole "then." They're the same set. So, in 2.2, when we switch them around, the "if" and the "then" are still the same sets, but, if we had an example where the "if" is a subspace of the space, and then you switch it around, it's not necessarily true. In this case, it is, but, in general, if you switch them, it might not work.

Theo recognized that the truth sets were the same in this case, and thus each was contained in the other. He used this to explain why the implication and its converse are both true for Theorem 2. It is important that Theo treated all conditional theorems and proofs as instances of the same relationship, and judged that the relationship should remain invariant across the different theorems. Therefore, he claims that the proof of a converse cannot be accepted as proof of an implication because it would not always lead to the correct conclusion.

Category 2: Proofs Should Match the Theorem Direction

Students who used this category of reasoning understood that conditional statements and proofs both had an inherent direction. Accordingly, they judged that a proof should match the theorem's direction. This is distinct from the first category because it was not based on comparing proof relationships across different theorems. One student, Moria, gave an explanation that serves as an example of this category:

> I don't feel like the converse is inherently proving the theorem and if you do that across it's like the converse is proving the theorem… I think it has to do with the trickle down what you're starting with necessarily, if I start with 14, it's going to be a multiple of 14, that's just, it's not going to prove anything… Using those two [Proof 2.2 proving Theorem 2] is saying, "if you have a multiple of 14, it will be a multiple of 14" is what that kind of 2.2 to the theorem 2 is saying.

Moria claimed that combining Proof 2.2 and Theorem 2 would be tautological because the hypothesis of the proof is the same as the conclusion of the theorem. Since she viewed theorems and implications as paths from one claim to another, she argued that the proof began at the ending point of the theorem, which is "not going to prove anything." This led her to claim that Proof 2.2 did not prove Theorem 2 (affirming CI).

Category 3: Equivalent Properties Allows for Substitution

Students who were observed using this category of justification claimed that the properties in the theorem's hypothesis and the properties in the theorem's conclusion were equivalent. They then used this to argue that the direction of the proof did not matter because the properties being related were the same. One student, April, justified that proof 2.2 proved theorem 2 by writing the equation "2*7*k=14*k" and claiming that this shows that the properties "multiple of 2 and multiple of 7" and "multiple of 14" are equivalent. Another student, Jean-Luc, also provided this type of explanation. His partner had argued for CI, but he responded:

> I guess we're having trouble because multiple 14 breaks down into seven and two and then seven and two break down right into 14. So, I feel like it's saying the same thing, but I totally understand what you're saying. It's reversed.

While Jean-Luc acknowledged that the direction of the proof was the reverse of the direction of the theorem, his belief that "multiple of 14" and "multiple of 2 and multiple of 7" are "saying the same thing" led him to see no conflict between the proof and the theorem. Since he viewed "multiple of 14" and "multiple of 2 and multiple of 7" as being the same property, he treated them as though they were synonymous. This sense of identity led him to deny CI since the order did not distinguish the proof from the theorem.

Category 4: Set Equality has No Direction

Students who provided this category of explanation acknowledged that the set of objects that satisfied the hypothesis was equivalent to the set of objects that satisfied the conclusion. They then used this to argue that since the hypothesis and conclusion were referring to the same set of objects, they were interchangeable. This category is very similar to the previous; they differ with regard to whether the student is attending to sets of objects or properties. Mathematically speaking, the sets of objects are truly equal while the properties differ in definition. For example, the set of integers that are multiples of both 2 and 7 is the same as the set of integers that are multiples of 14. Students giving such justification thus did not find grounds to distinguish a converse proof from the conditional theorem. One student, Phil, reasoned in this manner:

> I was saying I feel like it's true because all multiples of 2 and 7 are all multiples of 14. So, if it's a multiple of 2 and a multiple of 7, then it's going to be a multiple of 14 because multiples of 2 and 7 are 14, 28, and 42, and then to whatever degree you want to go to.

Phil claimed that Proof 2.2 still proves Theorem 2 because both the hypothesis and the conclusion are referring to the same set of objects. Therefore, switching them around in a statement does not change the meaning of the statement.

Category 5: The Theorem, the Proof, and my Knowledge All Agree

Students who used this category of reasoning focused on what they knew to be true when deciding whether a converse proof proved. If there were no contradictions between the theorem, the proof, and what they knew to be true, then they affirmed that the proof proved. In biconditional situations, this led them to affirm the converse proof. One student, Carl, used this category of reasoning to justify why Proof 2.2 proved Theorem 2:

> I said that it proves that if x is a multiple of 14, then x is a multiple of 2 and 7. Which is kind of like the opposite, but not really, of Theorem 2. But because the statement is true it doesn't really matter.

Carl argued that because Theorem 2 and its converse are both true, the order of the proof "doesn't really matter." In other words, Carl did not distinguish the truth of the claim from the efficacy of the proof in justifying the claim, since both the theorem and the proof described accurate facts about the same objects and properties. This reflects the common phenomenon that people judge arguments very differently depending upon whether they believe the conclusion of the argument (Inglis & Simpson, 2007; Evans & Feeny, 2004).

Category 6: Error-free Proofs Prove

Students who used this category of justification did not attend to the direction of the proof at all. Instead, students only focused on dissecting the line-by-line content of the proof for errors. Two participants, Joaquin and Violet, had a long discussion about whether the inferences made within the proof were valid. They sought to determine whether they believed each line in the proof followed from the previous line. Since they concluded that this was the case, they also concluded that proof 2.2 did in fact prove theorem 2. Interestingly, they frequently stated that the theorem was "true" and the proof was "true," seemingly assigning the same epistemic roles to the two kinds of text. Whereas the previous type of explanation notes the reversed order and dismisses its importance, this kind of explanation does not entail attention to an explicit link between the proof and the theorem beyond discussing the same topics.

Discussion

We present six categories of student explanation and justification regarding the relationship between a converse proof and a conditional theorem. There is some overlap with Hoyles and Kuchemann's (2002) description of four types of reasoning about CI: converses are the same in general, converses are distinct in general, converses are the same with reference to data, and converses are distinct with reference to data. Our first category matches the view that converses are distinct in general. It depended upon students perceiving all conditional theorems and proofs as instances of the same relationship (in accordance with the FWL). Our second category is similar, though it depended upon students according some significance to the direction of a theorem and proof. Our latter four categories all correspond to the view that converses are the same with reference to data. We did not observe the view that converses are the same in general (possibly since we presented Theorem 1) or that converses are distinct with reference to data for Theorem 2 (since this theorem is true biconditionally).

Students rejecting CI (for Theorem 2) variously attended to properties, sets of objects, or to their judgments about what is true. In each case, they did not see a reason to distinguish converse proofs from the given theorem, even when their partners provided explanations from Categories 1 and 2. Indeed, we are struck by the coherence of these non-normative explanations. Once students perceive some kind of identity (of property or set), it is hard to understand why they would perceive it as having some direction. Furthermore, explanations in Categories 5 and 6 raise more fundamental issues of how students learn to distinguish what a proof accomplishes from what they believe to be the case. It is worth noting that many students providing explanations in Categories 3-5 recognized the reversed order and also recognized that the shared structure among the proofs that follow converse order. Nevertheless, they affirmed that converse proofs can prove the theorem and that this relationship differed by context. We interpret this as a rejection of the FWL as applied to this type of proof, which poses a major challenge for the teaching of logic and proof that we hope will be addressed in future research in this area.

Acknowledgments

This research was funded by NSF DUE #1954613 and #1954768.

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