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Two Vignettes on Students' Symbolizing Activity for Set Relationships

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Mathematicians often use set-builder notation and set diagrams to define and show relationships between sets in proof-related courses. This paper describes various meanings that students might attribute to these representations. Our data consist of students' initial attempts to create and interpret these representations during the first day of a paired teaching experiment. Our analysis revealed that neither student imputed or attributed our desired theoretical meanings to their diagrams or notation. We summarize our findings in two vignettes, one describing students' attributed meanings to instructor-provided set-builder notation and the other describing students' imputed meanings to their personally-created set diagrams to relate pairs of sets.

Keywords: Symbolization, representations, student thinking, sets, set relationships

Introduction and Literature Review

Mathematics is a unique science in which objects of analysis are inaccessible to the five senses and can only be visualized indirectly using various representations (Duval, 2006). Theories of representation abound in the mathematics education literature (e.g., Duval, 1999, 2006; Godino & Font, 2010; Goldin, 2008; Radford, 2013; Vergnaud, 1998). Previous researchers have called for increased visual representations (e.g., Arcavi, 2003) and identified privileged forms of representation for specific mathematical topics (e.g., González-Martín et al., 2011). Diverse representations provide opportunities to construct shared meanings, which invite further investigation into how students invest representations (standard or not) with meaning.

This paper investigates undergraduate students' symbolizing activity about sets and set relationships. We define *symbolizing activity* as a process of mental activities that entails students' creation or interpretation of a perceptible artifact (writing, drawing, gesture, verbalization) to organize, synthesize, or communicate their thinking. We refer to *symbolization* as the status of completing the symbolizing activity and perceptible artifacts as *symbols*. Our definition differs from Tillema's (2010) communication-focused symbolizing activity by including individuals' creation of personal representations to reflect on their thinking.

As part of our investigation into the role of set-based reasoning in students' comprehension of conditional statements (Dawkins, 2017; Dawkins et al., 2021), we created an instructional sequence for students to investigate sets using set-builder notation and diagrams. We present two vignettes detailing the various meanings students imputed to these representations during their initial exposure to these ideas. We provide the following research question to contextualize the vignettes: *What differences in thinking did students exhibit as they (a) made sense of set-builder notation and (b) created set diagrams to describe relationships between sets?*

Theoretical Perspective

One problem with studying representations in isolation from student thinking is that students can impute various ideas to the same symbol. For instance, Gray and Tall (1994) stated that mathematicians utilize algebraic notations (e.g., the numeral 6) fluidly to refer to either a process (e.g., putting together two and four pennies) or a stabilized concept (e.g., a pile of 6 pennies). Thompson (2002) showed differences between individual student meanings for a point on a graph, even after students agreed upon a group definition. Alternately, Eckman and Roh (2022) showed that students might generate one notation to reason about a process they conceive and a different convention to describe a concept they abstract from reflecting on the process.

We propose two constructs to describe the representations we investigate in this paper. First, Eckman and Roh (2022) used the term *personal expression* to describe students' imputation of meaning to a self-generated algebraic expression. In this paper, we expand the definition of *personal expression* to cover all forms of students' mathematical representation. There are two components to personal expressions: a meaning and a perceptible artifact to which the student imputes their meaning. We use the term *meaning* in the constructivist sense (Thompson, 2013; Thompson et al., 2014) that individuals construct and maintain cognitive structures through their experience. A *perceptible artifact* includes any action or product a student produces to convey their meanings (writing, drawing, gesture, verbalization), which another individual might observe with his five senses. Our definition of personal expression is related to de Saussere's (2011) notion of *signifier* and *signified*, which also informed Glasersfeld's (1995) definition of symbol.

When a student creates an expression to organize or synthesize her thinking, she creates a *personal expression*. We consider all non-student generated expressions that require the student to anticipate the expression creator's intended meaning to be *communicative expressions*. There are three components to a communicative expression: (a) the creator's intended meaning, (b) the interpreter's evoked meaning, and (c) the perceptible artifact the creator uses to convey their intended meaning. For instance, Jill might create the personal expression $S = \{x \in \mathbb{Z} | x \text{ is prime}\}$ to denote her image of the set of prime numbers. If Jill presented her personal expression to Jack, Jack would perceive $S = \{x \in \mathbb{Z} | x \text{ is prime}\}$ as a communicative expression to which he would need to assign meaning. However, Jack's evoked meaning from Jill's personal expression may not reflect Jill's intended meaning. In summary, whether a perceptible artifact is a personal or communicative expression is in the eye of the beholder. The expression $S = \{x \in \mathbb{Z} | x \text{ is prime}\}$ is personal to Jill because she created it and communicative to Jack because he must interpret it.

Methodology

The data we present in this paper come from an ongoing project to develop models of students' abstraction of logic for conditional statements (Dawkins, 2017; Dawkins et al., 2021). During the study, we conducted six paired constructivist teaching experiments (Steffe & Thompson, 2000), consisting of 8-12 sessions lasting 60-90 minutes each. We focus on the first day of the Spring 2022 teaching experiment. Our students, who chose the names Sarah and Carl, were enrolled in Calculus 3 at a large public university in the United States. The second author served as the teacher-researcher, with all other authors serving primarily as witnesses.

Students' work was collected via video recording, a shared whiteboard application, and photographs of physical board work. We analyzed the data using the principles of open coding (Strauss & Corbin, 1998). As our initial codes emerged, we realized that some findings aligned with previously proposed constructs (Dawkins et al., in preparation; Sellers et al., 2021). During axial coding, we combined our unique codes with these constructs to describe meanings students exhibited during their symbolizing activity. We synthesized our findings into two vignettes. The first vignette describes meanings that students might attribute to *communicative expressions* of set-builder notation. The second vignette describes how students might construct set diagrams as *personal expressions* to express their image of the relationship between two sets.

Results

For each vignette, we first present a theoretical model of a beneficial meaning for representing sets or set relationships. We then offer two alternative meanings from our data that students exhibited in their symbolizing activity.

Vignette 1: Students' Meanings for Set-builder Notation (Communicative Expressions)

We initially presented Sarah and Carl with pairs of sets (defined using set-builder notation) and asked them to posit relationships between the sets (see Figure 1). Our examples constituted *communicative expressions* because students interpreted our notation. In this vignette, we report on students' imputed meanings to set-builder notation while comparing set pairs α, β and α, γ.

Given the set $\mathbb{T}$ of all triangles, answer these questions about each pair of sets:

1) Is there anything in both sets?
2) Does one set contain all the members of another?
3) Can you say anything more about the relationship between the sets?
4) If you use an oval region to represent one set, how would you portray the other in relation?

$\alpha = \{\Delta ABC \in \mathbb{T} : \Delta ABC \text{ is isosceles}\}$ $\quad$ $\beta = \{\Delta XYZ \in \mathbb{T} : \Delta XYZ \text{ is equilateral}\}$

$\alpha = \{\Delta ABC \in \mathbb{T} : \Delta ABC \text{ is isosceles}\}$ $\quad$ $\gamma = \{\Delta RST \in \mathbb{T} : \angle R \cong \angle S\}$

$\alpha = \{\Delta ABC \in \mathbb{T} : \Delta ABC \text{ is isosceles}\}$ $\quad$ $\eta = \{\Delta JKL \in \mathbb{T} : \Delta JKL \text{ is not isosceles}\}$

Figure 1. An excerpt from Task 1. The prompt is shortened for brevity, and not all pairs of sets are shown.

A beneficial way to interpret communicative expressions for set-builder notation. We first present a theoretical meaning that did not emerge in our data but we considered beneficial for students to compare two sets appropriately (see Figure 2).

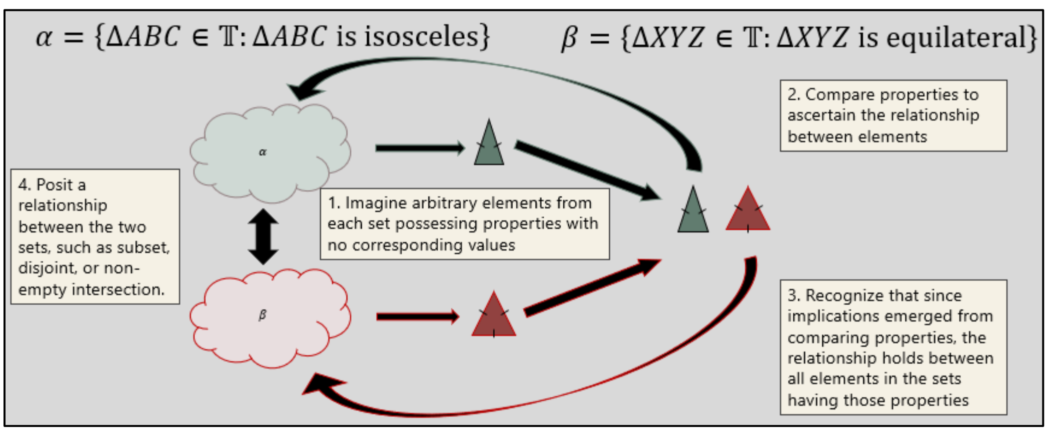

Figure 2. A productive meaning and resulting method for comparing sets α and β.

A student determining the relationship between sets α and β might first imagine arbitrary elements, ΔABC and ΔXYZ, from each set (Figure 2, step 1). These arbitrary elements contain no specific measurements for characteristics such as angle measure. The student would then compare the properties of the two elements to determine their relationship (Figure 2, step 2). For instance, the student might imagine that since the isosceles triangle has at least two equal sides and the equilateral triangle has exactly three equal sides, the equilateral triangle can be considered isosceles. The student would then infer that since every element in set β has exactly three equal sides, all equilateral triangles can be considered isosceles (Figure 2, step 3). Finally, the student would conclude that if all equilateral triangles are isosceles, then set β must be a subset of set α (Figure 2, step 4). Conventionally, we call this meaning for ΔABC or ΔXYZ an *arbitrary particular*. However, our theoretically propitious meaning was distinct from the meanings exhibited by Carl and Sarah. In the following data-driven examples, we show two meanings these students attributed to set-builder notation while comparing pairs of sets.

Meaning 1a: Particular ΔABC. When Sarah read the teacher-researcher's communicative expressions $\alpha = \{\Delta ABC \in \mathbb{T}: \Delta ABC \text{ is isosceles}\}$ and $\beta = \{\Delta XYZ \in \mathbb{T}: \Delta XYZ \text{ is equilateral}\}$, she imagined specific triangles with corresponding values and labels unique to each triangle.

Sarah: I have a question. (Interviewer 1: Ok.) The left-hand side set (α) is congruent to the right-hand side set (β) but they're two different triangles…or are they the same triangle? Like, they are different sets of triangles, right?

(omitted dialogue)

Interviewer 2: Sarah, just to clarify, was your question in part about set α has the letters ABC and set β has the letters XYZ?

Sarah: Yes.

Interviewer 2: And so, because there are different letters, you weren't sure if the triangles were the same triangles?

Sarah: Yeah, I just got a little bit confused on that.

In this example, Sarah questioned interviewer 1 (second author) whether two triangles from sets α and β (which she considered congruent) could be regarded as the same triangle when comparing sets. Eventually, interviewer 2 (first author) asked whether Sarah's confusion emanated from denoting elements of set α with ΔABC and elements of set β with ΔXYZ, which she confirmed. In other words, Sarah comprehended that a triangle ΔABC from set α could be congruent to a triangle ΔXYZ in set β but was unsure whether ΔABC could exist within set β because the vertices of ΔABC in the communicative expression for set α were not labeled with the letters for set β. We call Sarah's evoked meaning for the expression ΔABC a *particular* triangle. We compare Sarah's meaning with our theoretical meaning in the vignette 1 summary.

Meaning 1b: Spontaneous particular ΔABC. When Carl compared the the communicative expressions ΔABC and ΔXYZ, he imagined various possible pairings between elements in set α and set β and the relationships that might occur for each comparison.

Interviewer 1: What do you think, Carl (about the relationship between sets α and β)?

Carl: Yeah. Um, I thought that it was. I don't think of like, the sets. I thought it more like, it could, like have a good chance of being 100% the same triangle. But also, there's also a good chance that it's close, similar, but not quite. Like 70 or so percent chance.

Interviewer 1: You're talking about one specific triangle?

Carl: Yeah, like comparing ABC to XYZ.

Carl's explanation shows that he was considering two distinct situations: (1) comparing a triangle with exactly two equal sides from set α with a triangle from set β and (2) comparing a triangle with exactly three equal sides from set α with a triangle from set β. Carl's probabilistic language also indicates that he imagined how often the triangles he selected spontaneously were likely to be in both sets. We thus say that Carl's evoked meaning for the expression ΔABC was of a *spontaneous particular* and not an *arbitrary particular* triangle. We further discuss how Carl's *spontaneous particular* meaning emerged in his set diagram personal expressions in vignette 2.

Carl's *spontaneous particular* meaning ΔABC and ΔXYZ is analogous to what Sellers et al. (2021) called an *MQ4* meaning for a quantified variable. Students exhibit an *MQ4* meaning when they spontaneously select elements within a domain of universal discourse, make inferences without exhausting all elements, and may or may not repeat this process to make (potentially different) inferences about other elements.

Summary of vignette 1. In this vignette, we have shown three meanings that a student might have for ΔABC in the communicative expression $\alpha = \{\Delta ABC \in \mathbb{T}: \Delta ABC\ is$ isosceles$\}$. A student exhibiting the *arbitrary particular* meaning (see Figure 2) imagines triangles defined by the properties described in the set-builder notation and would be capable of making general comparisons between sets. Sarah's meaning of *particular* triangles allowed her to imagine elements of sets α, β and γ (Figure 2, step 1) and make rudimentary comparisons between these elements (Figure 2, step 2). However, her image of triangles with fixed values for various characteristics precluded her from discerning the appropriate relationship between the entire sets of objects (Figure 2, steps 3, 4). Carl's meaning of *spontaneous particular* triangles allowed him to imagine random pairings of elements from sets α and β (Figure 2, step 1) and compare them (Figure 2, step 2). In effect, Carl constructed relationships of likelihood, not relationships of necessity as are privileged in mathematical logic, which are essential for proving.

Vignette 2: Students' Meanings for Set diagrams (Personal Expressions)

We also invited Sarah and Carl to construct set diagrams to represent the relationships they envisioned between pairs of sets (see Figure 1, question 4). We consider the diagrams that Carl and Sarah generated (even if they exhibited the conventions of Euler diagrams) to constitute their *personal expressions* for organizing their thinking about various pairs of sets.

Regions that partition: A beneficial way to construct a set diagram. We first present a beneficial meaning that did not emerge in our data which a student might leverage to construct a set diagram to compare sets α and β (see Figure 3). First, the student would imagine the universe of discourse, $\mathbb{T}$, the set of all triangles. The student might then draw a box to metaphorically gather all triangles into an enclosed entity (Figure 3, part 1). Second, the student would consider set α, the set of isosceles triangles. The student might represent α by drawing an oval region inside the box to simultaneously gather all isosceles triangles and partition them from other triangles (Figure 3, part 2). The student would recognize that the region outside the oval denoting α constitutes the complement set to α (α^c). Third, the student would utilize the *arbitrary particular* meaning to determine that all elements of β exist within set α (Figure 3, part 3).

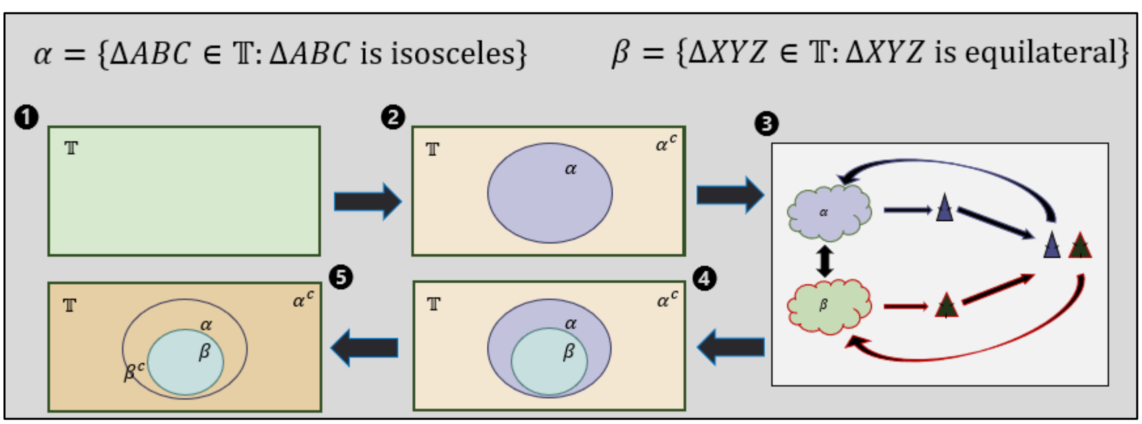

Figure 3. A productive method to produce a set diagram comparing sets α and β.

The student might then draw an oval region to represent set β within his previously created region for α (Figure 3, part 4). The student would realize that his actions (a) gather equilateral triangles from the universal set $\mathbb{T}$ into the oval region β and (b) denote the regions outside the oval for set β as the set of all non-equilateral triangles (β^c; Figure 3, part 5). We have previously used *regions that partition* to describe students' set diagrams constructed through this propitious meaning (Dawkins et al., in preparation). In the following data-driven examples, we report two other meanings that Sarah and Carl attributed to their diagrams comparing set pairs α,η and α, γ.

Meaning 2a: Regions that gather. When Sarah drew her set diagram *personal expression* to relate sets α and η (which are disjoint sets), she drew one oval region to represent α and a second, non-overlapping oval region to represent η (see Figure 4). Sarah then drew isosceles and equilateral triangles to represent the elements she imagined in set α and a scalene triangle to represent the elements she imagined in set η (see Figure 4). When the interviewer asked Sarah what she imagined the region outside α and η to represent, Sarah responded that this area was irrelevant but she could imagine the region as empty if she chose. Sarah's comment indicates that she created her personal expression to represent solely her images of α, η, and their relationship.

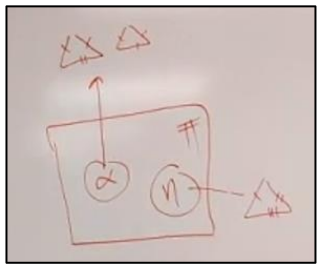

Figure 4. Sarah's set diagram comparing sets α and η (regions that gather).

We have previously defined students' diagrams to which they gave meaning solely to the inside of drawn regions as *regions that gather* (Dawkins et al., in preparation). A student who creates a diagram to express *regions that gather* typically ignores areas outside their drawn regions (i.e., no partitioning). These students use set diagrams to highlight sets of interest, not construe relationships between all the elements within the universal domain.

Meaning 2b: Regions that distinguish. While comparing sets α and γ, Carl initially concluded the two sets were equal and drew a single ovular region (see Figure 5). Carl then claimed that he could further clarify his diagram by drawing a second oval region inside the first. Carl explained that the new oval region denoted instances where he was comparing triangles from α and γ with exactly three congruent sides. He then stated that the region outside of the interior oval (but inside the exterior oval) represented instances where he was comparing triangles with exactly two congruent sides. In other words, Carl introduced a local partition to distinguish two classes of elements he perceived within his gathered set containing elements of sets α and γ. We use the term *regions that distinguish* to describe Carl's separation-of-elements-into-cases meanings he imputed within a locally gathered region of his set diagram.

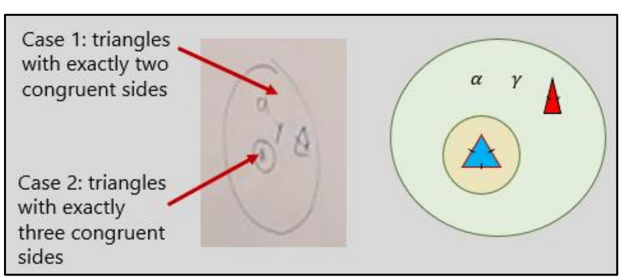

Figure 5. Carl's set diagram for comparing sets α and γ and a digital reproduction (regions that distinguish).

Summary of Vignette 2. In this vignette, we have shown three meanings that students might attribute to the regions they draw in a set diagram *personal expression*. Sarah's meaning allowed her to imagine gathering elements of a similar type, and she drew *regions that gather* to denote her grouping action. Sarah's meanings allowed her to create oval regions to represent the sets α and η and successfully determined that the sets were disjoint (Figure 3, steps 2-4). However, Sarah indicated that the regions outside the ovals were irrelevant to the task. Later in the interview, Sarah claimed that exterior regions contained non-examples of the sets but gave no indication that she considered these elements as the complement of the sets portrayed by her ovular regions (Figure 3, steps 2, 5). We also note that some students creating *regions that gather* may not include a box representing the universe of discourse (see Dawkins et al., 2021; Figure 3, step 1). Carl's meaning allowed him to imagine sorting comparisons between classes of elements, and he drew *regions that distinguish* to denote his sorting of these cases. Carl provided meaning to all areas within his outermost region (Figure 2, steps 2-4). However, Carl created his set diagrams not to partition the universe into sets possessing or not possessing properties (Figure 3, steps 2, 5) but to sort comparisons of set elements. Finally, a student imagining gathering elements into one set while simultaneously creating a complement set draws *regions that partition* to denote this partitioning action.

Discussion and Conclusion

Our research question for this paper was related to students' differences in meaning when creating *personal expressions* and interpreting *communicative expressions* for sets and set relationships. In vignette 1 we described three meanings, one theoretical and two emerging from our data, that students might possess for communicative expressions of set-builder notation presented to them by an instructor. In vignette 2 we described three meanings, one theoretical and two emerging from our data, that students might attribute to personal expressions they create to diagrammatically represent set relationships. Our results show that students can (a) invest only one portion of a conventional meaning to an expression (e.g., regions that gather, particular) or (b) aatribute meanings that allow local comparisons of element classes but fail to support claims about set relationships (e.g., spontaneous particular, regions that distinguish).

Our findings further work done by previous mathematics educators. For instance, we provided an expanded definition of Eckman and Roh's (2022) *personal expression* and proposed *communicative expressions* to describe the role of symbols in mathematical communication. We also added *regions that distinguish* to Dawkins et al.'s (in preparation) description of the meanings students attribute to set diagrams. Finally, we utilized Sellers et al.'s (2021) *MQ4 meaning* to inform our descriptions of *spontaneous particular* and *regions that distinguish*, extending their MQ framework beyond the context of interpreting quantified variables.

Our vignettes also have relevance for instructors. Vignette 1 highlights that students may attribute very different meanings to a communicative expression of set-builder notation than their instructor intended when creating the symbol as a personal expression. Vignette 2 reveals that students' imputed meanings to local regions of their personal expression set diagrams may vary across students and differ from convention. Therefore, we recommend that instructors regularly facilitate classroom discussions about personal and communicative expressions and the potential meanings the expression creators and interpreters attribute to these expressions.

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