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Instructional interventions and teacher moves to support student learning of logical principles in mathematical contexts

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This study explores how instructional interventions and teacher moves might support students' learning of logic in mathematical contexts. We conducted an exploratory teaching experiment with a pair of undergraduate students to leverage set-based reasoning for proofs of conditional statements. The students initially displayed a lack of knowledge of contrapositive equivalence and converse independence in validating if a given proof-text proves a given theorem. However, they came to conceive of these logical principles as the teaching experiment progressed. We will discuss how our instructional interventions played a critical role in facilitating students' joint reflection and modification of their reasoning about contrapositive equivalence and converse independence in reading proofs.

Keywords: logic and proof, instructional interventions and teacher moves, contrapositive equivalence, converse independence

The purpose of this study is to explore how students might learn logical principles and how instructional interventions and teacher moves might support students' learning of logic. We focus on two logical principles: contrapositive equivalence and converse independence. By contrapositive equivalence, we refer to a logical principle that a conditional statement has the same truth value as its contrapositive. By converse independence, we mean a logical principle that a conditional statement does not necessarily have the same truth value as its converse. These two logical principles are foundational for mathematical justification: the former provides a logical account that proof of a conditional statement is also a proof of its contrapositive, and the latter provides a logical account that proof of a conditional is not a proof of its converse.

It is critical for students in proof-oriented mathematics courses to know and use contrapositive equivalence and converse independence for their proof activities. However, empirical studies have reported students' challenges with using these logical principles: In Stylianides et al.'s (2004) study, many mathematics undergraduates did not use the contrapositive equivalence as a valid inference. Dawkins et al. (2021) reported a similar phenomenon in which undergraduate students with no proof experience in college conceived that the proof of the contrapositive would not provide a proof of the original conditional statement. Dawkins et al. (2021) also documented calculus students conceiving a proof of a conditional statement as proof of its converse when both the conditional and its converse are true.

While issues with student learning of these logical principles have been studied widely in proof research, these empirical studies have not focused much on how instructional interventions and teacher moves might provide support for students to learn these logical principles (Melhuish et al., 2022; Stylianides & Stylianides, 2017). In this paper, we document a case of two undergraduate students, Carl and Sarah, as a possible account for students coming to understand and might use these two logical principles. We also examined how the instructional interventions and teacher moves we designed and implemented might have played a role in students developing set-based reasoning for these logical principles. This study addresses the following research questions: (1) How might students make progress in learning to use contrapositive equivalence and converse independence by engaging in set-based reasoning? and (2) how, when, and what types of instructional interventions and teacher moves could encourage or facilitate their set-based reasoning for learning these logical principles?

Theoretical Framework

We employ Piaget's genetic epistemology as our theoretical lens for this study. From this perspective, we assume that individual students idiosyncratically organize their experiences within mental schemes (Glasersfeld, 1995; Piaget, 1971; Piaget & Inhelder, 1969). The schemes organized by an individual student's unique experience would provide space of implications for her reasoning (Thompson et al., 2014). On the other hand, individual students' ways of reasoning are not accessible to observers; we, as researchers, propose viable models of their ways of reasoning through their behaviors and utterances. We suggest and use a triad of relationships a student might need to construe when validating a proof-text paired with a conditional statement to be proven (**Error! Reference source not found.**):

- What the student knows: this may include mathematical propositions, logical principles, or any relationship that the student conceives from the given conditional
- What the student construes from the given conditional to be proven: a student might posit a relationship that the given conditional construes (to them).
- What the student believes the given proof-text attempts to prove: A student might posit a relationship that the given proof-text construes (to them).

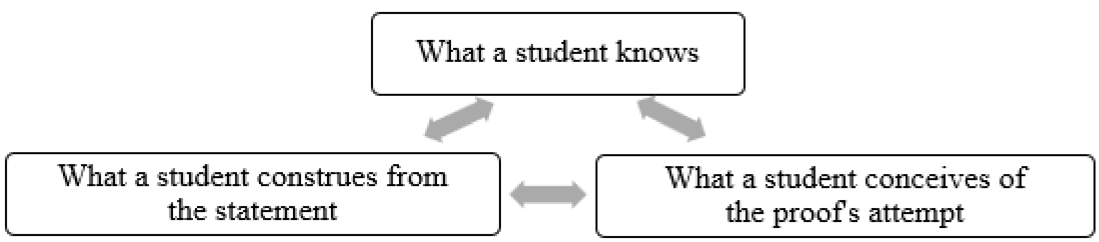

Figure 1. Three relationships a student may construe while reading a proof-text paired with a statement

The three relationships a student construes would likely support her reasoning when examining if the proof-text proves the statement to be proven. However, students may not find compatibility among the relationships they have construed. In such cases, instructional interventions and teacher moves could play a critical role in supporting students in connecting these relationships (e.g., Ellis et al., 2019; Mata-Pereira & da Ponte, 2017). Using this triad as an analytic tool, this paper provides empirical evidence that instructional interventions and teacher moves could leverage students' learning of contrapositive equivalence and converse independence.

Research Methodology

As part of a more extensive study aiming to develop constructivist models of students' abstraction of logic for proof of conditional statements, we have conducted exploratory teaching experiments (Steffe & Thompson, 2000) since 2018. The exploratory aspect of the experiments allowed us to repeat six iterations of the design-implementation-analysis cycle over five years to refine the instructional tasks and our models of students' ways of reasoning about logical principles.

This paper documents our findings from our sixth iteration of the teaching experiments conducted at a large public university in the United States. Two undergraduate students, Carl (engineering major) and Sarah (mathematics major), were recruited from calculus 3 at the beginning of the semester in the Spring of 2022. We selected Carl and Sarah out of 14 students who completed the screening survey (Roh & Lee, 2018) as they met our selection criteria. Their survey responses indicated they would have sufficient mathematical knowledge to comprehend conditional statements and proof-texts in our designed tasks yet need to learn logical principles to validate mathematical proofs. Both students also reported they learned proofs in geometry in high school but had not yet taken any proof-oriented courses in college.

The exploratory teaching experiment was organized once a week for 12 weeks for 75-minute interviews. For the first (intake) and last (exit) interviews, we conducted clinical interviews (Clement, 2000) by meeting each student individually to assess their use of logic to validate proofs of conditional statements. We conducted exploratory teaching interviews for the rest of the ten interviews (Sellers, 2020). We met with both students and implemented the tasks we designed to help students leverage set-based reasoning. The tasks for the teaching interviews consisted of 4 tasks: (1) set theory tasks (defining sets by shared properties); (2) truth conditions tasks (evaluating truth values of conditional statements); (3) what-does-it-prove tasks (reading and comparing proof-texts paired with a conditional statement), and (4) abstraction task (comparing various proofs across different mathematical contents to abstract general proof frames for conditional statements). With these tasks, we tried to create a student-centered learning environment to encourage students' reflection and modification of their reasoning.

Results

At the intake interviews, Carl and Sarah exhibited the opposite of the normative mathematical logic regarding contrapositive equivalence and converse independence. Specifically, they both responded that a proof of the contrapositive of a given conditional does not prove the statement. In contrast, a proof of the converse of a given conditional proves the statement when both the original conditional and its converse are true. Their responses at the intake interviews indicate the absence of these logical principles in the students' reasoning, or at least the intake interview tasks did not evoke the students to use these logical principles. However, their reasoning about logic shifted during the teaching interviews in which we implemented the what-does-it-prove (WDIP) tasks (Days 5-8). Carl and Sarah first made sense of the contrapositive equivalence and later began to make sense of the converse independence. We describe how contrapositive equivalence became these students' knowledge base for proof by contrapositive, yet created resistance to develop converse independence. We also document how our instructional interventions designed to leverage set-based reasoning and teacher moves facilitated students' reflection on mathematical proof and their eventual recognition of converse independence.

Teacher Moves Leveraging Student Progress in the Contrapositive Equivalence

On Day 5, the first day for the WDIP tasks, the interviewer presented Theorem 1 ("For every integer x, if x is a multiple of 6, then x is a multiple of 3") with three associated proof-texts, Proof 1.1 (direct proof), Proof 1.2 (disproof of converse), and Proof 1.3 (proof of the contrapositive). Carl and Sarah immediately attended to the first and last lines of Proof 1.1 and said that Proof 1.1 would prove Theorem 1. Afterward, these students frequently examined if the first line and last line of other proof-texts matched the if-part (the premise) and the then-part (the conclusion) of the theorem to be proven, respectively. These students' attention to the first and last lines of the proof-texts enabled them to find the compatibility among what relationship they know about the premise and conclusion of the given theorem, what relationship the given theorem describes, and direct proof attempts to prove the given theorem.

On the other hand, these students' tendency to check the matches between the first line of a proof-text and the premise of the theorem statement may have hindered them from discerning why proof of contrapositive indeed proves the given theorem, even though proof of contrapositive does not start with the if-part of the given theorem statement.

Sarah: I said it [Proof 1.3] proves something different. I guess it [Proof 1.3] proves the complements of the original one that if x is not a multiple of 3, then x is not a multiple of 6, which [is] base[d] off of how we were seeing the complement, remember? That could possibly be true [inaudible].

Carl: Yeah. I said the same thing […] It's, yes, if we're allowed to say that if A is a subset of B, and B's complement is a subset of A's complement.

In the dialogue above, Sarah and Carl used set languages, such as subsets and complement sets. Carl used letters A and B to name the truth sets for the premise and conclusion of Theorem 1 and used these letters to interpret what Theorem 1 says and what Proof 1.3 proves in terms of subset relationships. Sarah then claimed that what Proof 1.3 attempts to prove is the contrapositive of Theorem 1 and could also be true. Carl agreed to interpret Theorem 1 and Proof 1.3 in terms of subset relationships if they were allowed to say the set relationships $A \subseteq B$ and $B^c \subseteq A^c$. However, the students' use of set language itself did not indicate their use of contrapositive equivalence. They responded that both Theorem 1 and its contrapositive are true in this case but drew two different diagrams to represent what Theorem 1 construed to them and what they believed Proof 1.3 attempts to prove. Sarah drew a diagram to represent what Theorem 1 meant to her: the truth set P of the premise of Theorem 1 is a subset of the truth set Q of the conclusion of Theorem 1. She also drew another diagram for Proof 1.3, in which Q^c is contained in P^c (**Error! Reference source not found.** left).

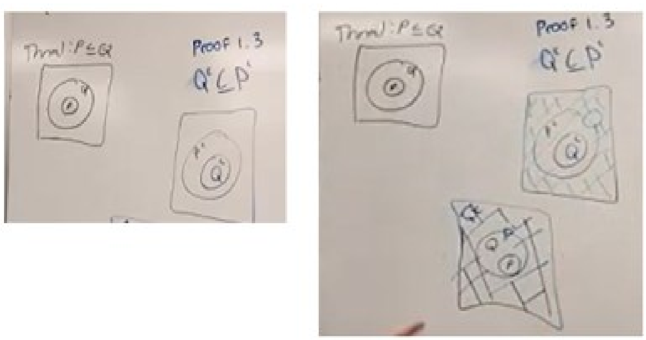

Figure 2. Sarah's diagrams for Theorem 1 & Proof 1.3: (left) initial; (right) after the teacher intervention

At this point, the interviewer intervened with Carl and Sarah by inviting them to use only one diagram for both Theorem 1 and Proof 1.3. This teacher intervention enabled Sarah to use her diagram for Theorem 1 to shade the region corresponding to the complements of P and Q. (i.e., If $P \subseteq Q$, then $Q^c \subseteq P^c$). By Instructor's request, Sarah was also able to use her diagram for Proof 1.3 to describe Theorem 1 (i.e., If $Q^c \subseteq P^c$, then $P \subseteq Q$). Sarah's revised diagrams (**Error! Reference source not found.**, right) are indicative of her progress in the conceptualization of contrapositive equivalence (P $\subseteq Q$ iff $Q^c \subseteq P^c$). After looking at Sarah' revised diagrams, Carl agreed with Sarah and illustrated his diagram, which was similar to

Sarah's diagrams. But he also added another case in which if two sets A and B are equal, then A's complement and B's complement are also equal (If P $= Q$, then $Q^c = P^c$). Figure 3 illustrates our model of Sarah's reasoning in which she began to use contrapositive equivalence as her new knowledge to comprehend a proof of contrapositive as a proof of the original conditional. From that point, the contrapositive equivalence became robust knowledge for Carl and Sarah for the rest of the teaching experiment. Here, we see the teacher intervention supported these students to use what they came to know (contrapositive equivalence) to connect

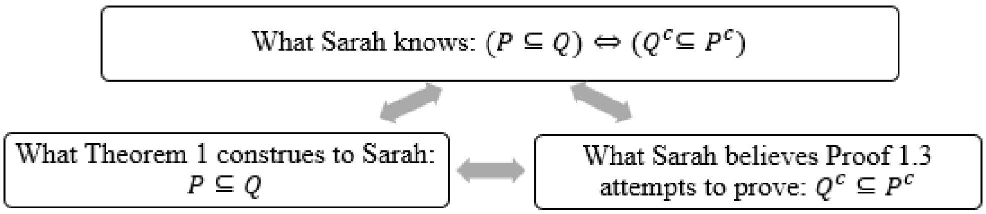

Figure 3. A model of Sarah's way of reasoning with contrapositive equivalence

what relationship Theorem 1 construed to them ($P \subseteq Q$) with what they believed Proof 1.3 attempts to prove ($Q^c \subseteq P^c$).

Teacher Moves Leveraging Student Progress in Converse Independence

On Day 6, the interviewer presented Theorem 2 (for any integer x, if x is a multiple of 2 and 7, then x is a multiple of 14) with two associated proof-texts: Proof 2.1 (proof of converse) and Proof 2.2 (direct proof). Carl and Sarah responded that Proof 2.1 proves Theorem 2 despite the reversed order because they believed Theorem 2 and its converse are both true. To be more specific, Carl explained that he knew the set of all multiples of 2 and 7 (P) is the same set as the set of all multiples of 14 (Q), i.e., $P = Q$, and to him, Theorem 2 interprets the subset relationship $P \subseteq Q$. Carl also believed Proof 2.1 proves the reversed subset relationship $Q \subseteq P$. While Theorem 2 and Proof 2.1 form different subset relationships, he could infer Theorem 2 from Proof 2.1 by substituting P to Q and Q to P in the subset relationship $Q \subseteq P$. He said that since he already knows $P = Q$, by using his knowledge, he could infer Theorem 2 from what Proof 2.1 proves.

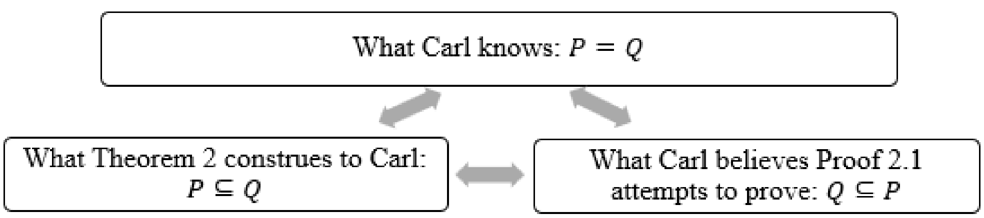

Figure 4. A model of Carl's reasoning about why Proof 2.1 proves Theorem 2

On Day 7, the interviewer presented Theorem 4 (Given any quadrilateral ABCD, if ABCD is a kite and parallelogram, then ABCD is a rhombus) with Theorem 4.1 (proof of converse) and Theorem 4.2 (direct proof). Carl continued claiming, and Sarah agreed, that although Proof 4.1 proves the converse of Theorem 4, it also proves Theorem 4 because both Theorem 4 and its converse are true. We infer that these students meant the two subset relationships $P \subseteq Q$ and $Q \subseteq P$ are indistinguishable to them when $P = Q$. In our model, they conceived that Proof 4.1 proves the converse of Theorem 4. But since they already knew the premise and the conclusion of Theorem 4 represents the same truth sets ($P = Q$), they went further to infer that Proof 4.1 proves Theorem 4 as well.

On Day 7, Carl claimed, and Sarah agreed, that they should be allowed to use what they know without justification. He compared proof of converse with proof of contrapositive. Although Proof 1.3 does not justify contrapositive equivalence, Carl accepted that Proof 1.3 (proof of contrapositive) proves Theorem 1 because he knew contrapositive equivalence. Carl then claimed with an analogy that we should also accept Proofs 2.1 and 4.1 (proofs of converse) even though these proofs do not provide justification for $P = Q$, since he already knew these sets were equal. Furthermore, Carl claimed, and Sarah agreed, that proof does not necessarily justify explicitly what they already know. This is similar to how one may apply a known theorem without reproving it. The distinction between the ways mathematicians cite prior knowledge and how Carl wanted to cite prior knowledge is quite subtle, and we see Carl's reasoning as subjectively rational. The more central question was: how would Carl and Sarah justify $P = Q$ instead of saying they already know it?

On Day 8, the interviewer presented Proof 4.4 () as an instructional intervention, which resembled our model of Carl's reasoning on Day 7. As a version of proof of converse, Proof 4.4 explicitly stated the equal set relationship that "we already know" without justification. Both students responded that Proof 4.4 would prove Theorem 4 directly because it explicitly stated $P = Q$ in the proof-text and thus provided warrants ($P = Q$) to support Proof 4.1 ($Q \subseteq P$) infers Theorem 4 ($P \subseteq Q$).

Proof 4.4: Let P and Q be subsets of $\mathbb{Q}u$ defined as follows:
$P = \{ABCD \in \mathbb{Q}u: ABCD\ is\ a\ kite\ and\ paralleogram\}$;
$Q = \{ABCD \in \mathbb{Q}u: ABCD\ is\ a\ rhombus\}$.
From Proof 4.1, we proved $Q \subseteq P$.
And we already know $P = Q$.
Then $P \subseteq Q$.
Thus, given any quadrilateral $\blacksquare ABCD$, if $\blacksquare ABCD$ is a kite and is a parallelogram, then $\blacksquare ABCD$ is a rhombus.

Figure 5. Proof 4.4: A model of Carl's reasoning in accepting Proof 4.1 (proof of converse) as a proof of Theorem 4

While the interviewer acknowledged that the students already knew $P = Q$, she invited them to state what it would mean for two sets P and Q to be equal. Students' responses to this instructional intervention uncovered the absence of meaning for equal sets in these students' reasoning: They were not sure how to say two sets are equal. Carl merely suggested that two sets are equal ($P = Q$) when their complements are equal ($P^c = Q^c$) (see **Error! Reference source not found.**, left). Sarah then suggested combining Theorem 4 and its converse (see **Error! Reference source not found.**, right) as a meaning for equal sets.

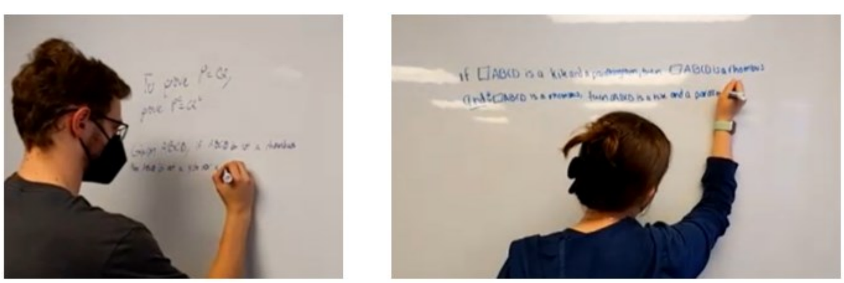

Figure 6. Students wrote their meaning of equal sets: Left: Carl; Right: Sarah

After the interviewer invited Carl and Sarah to state their meaning of equal sets, Sarah shifted her interpretation, revealing a sense of circularity. In particular, her interpretation of equal sets by this conjunction of a conditional and its converse helped her to realize the absence of justification for $P = Q$ in Proof 4.4. When Carl asked if "we are given $P = Q$ from Proof 4.4 or not," Sarah responded to Carl that "No, Proof 4.4 isn't even really proving $P = Q$. It's just saying it is." When Carl asked again if that ($P = Q$) is "given or inferred," Sarah again responded to Carl that "it [Proof 4.4] never says [justifies] $P = Q$." From this student dialogue, we could see Sarah's reasoning had evolved regarding the converse independence. While engaging in the sequence of activities and responding to the interviewer's prompts, she concluded that we should use only what we had already proved without justification again. Since they already justified contrapositive equivalence by diagrams (Figure 2), Sarah believed that they could say Proof 1.3 (proof of contrapositive) proves Theorem 1 even though Proof 1.3 does not justify the contrapositive equivalence. However, she contended that "Proofs 2.1 and 4.1 (proofs of converse) do not necessarily prove their original theorems because the theorems ask "if P, then Q" ($P \subseteq Q$) but the proofs instead prove "if Q, then P" ($Q \subseteq P$). To prove the theorem to be proven, the proofs would have to prove $P = Q$ so we can assume $P \subseteq Q$ *and* $Q \subseteq P$." While Carl did not accept Sarah's claim on Day 8, he exhibited his acceptance of converse independence at the exit interview. Specifically, Carl determined, "Proof γ (proof of converse) proves $B \subseteq A$ but doesn't really prove Theorem γ ($A \subseteq B$) [because] Proof γ lies on the reader inferring $A = B$. Basically, if proof γ added an extra line, proving $A = B$, then it would be equal to it. Then it would prove the theorem. [but] I don't think they do it."

Discussion

In this paper, we documented how Carl and Sarah generalized contrapositive equivalence and converse independence across proofs of conditional statements as they were engaging with the WDIP tasks. The interviewer's prompting to use only one diagram to interpret two subset relationships helped the students make the line of inference between Proof 1.3 (proof of contrapositive) and Theorem 1 explicit, such that they affirmed it by their "prior knowledge" regarding the contrapositive equivalence. Later, Carl used this idea to justify why proof of converse proves the original conditional. For him, proof of contrapositive and proof of converse both relied on his prior knowledge, which he called "prove indirectly." Indeed, in one case, his prior knowledge was logical knowledge about the generalizable contrapositive relationship; in the other, it was local mathematical knowledge that the situation described by Theorem 4 related two equal sets of quadrilaterals. By introducing Proof 4.4, the interviewer's move of reflecting the form of Carl's justification back to the students allowed them to move forward in critiquing the justification. However, to see the conflict, which is what mathematicians usually call "circularity," Sarah needed to interpret set equality as the conjunction of two subset claims. This made it clearer how asserting $P = Q$ without proof was tantamount to asserting $P \subseteq Q$ without proof. Understanding the logic of the relationships between proofs and theorems is quite challenging. Still, we see how these instructional moves supported Carl and Sarah in apprehending the structure of their own arguments to evaluate them more precisely. We are pleased with how our interventions allowed these students to wrestle deeply with these matters of justification, though more work is needed to explore this arena of learning.

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