UNIQUENESS IN A NAVIER-STOKES-NONLINEAR-SCHRÖDINGER MODEL OF SUPERFLUIDITY

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Abstract. In [JT21], the authors proved the existence of local-in-time weak solutions to the model of superfluidity. The system of governing equations was derived in [Pit59] and couples the nonlinear Schrödinger equation (NLS) and the Navier-Stokes equations (NSE). In this article, we prove two uniqueness theorems for these weak solutions. One of them is the classical weak-strong uniqueness based on a relative entropy method. The other result trades some regularity of the stronger solution for smallness of data and short time/low energy.

Contents

1. Introduction

This article deals with the problem of uniqueness of the local-in-time weak solutions to a model of superfluidity known as "Pitaevskii model" governed by the nonlinear Schrödinger equation (NLS) coupled with the inhomogeneous Navier-Stokes equations (NSE) for incompressible fluids. The Schrödinger equation is used to describe the dynamics of the superfluid phase, while the Navier-Stokes equation is employed to describe the evolution of the normal Helium liquid.

The present article presents two distinct uniqueness results. Motivated by the analytical results in [JT21], we investigate the issue of weak–strong uniqueness in Section 4 presenting a new class of weak solutions with additional regularity properties. There are many results, mostly devoted to the Navier–Stokes system for incompressible fluids, concerning conditional regularity of the weak solutions. Roughly speaking, the weak solutions are regular as soon as they belong to a critical

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regularity class. Results in that direction are presented by Caffarelli et al. [see L. Caffarelli, R.-V. Kohn, L. Nirenberg, Partial regularity of suitable weak solutions of the Navier–Stokes equations, Comm. Pure Appl. Math. 35 (6) (1998) 771–831.] Escauriaza et al. [see L. Escauriaza, G. Seregin, V. Sverak, L3, solutions of Navier Stokes equations and backward uniqueness, Uspekhi Mat. Nauk 58 (350) (2003) 3–44. no. 2, translation in Russian Math. Surveys 58 (2) 211–250.], Prodi [see G. Prodi, Un teorema di unicitá per le equazioni di Navier–Stokes, Ann. Mat. Pura Appl. 48 (1959) 173–182.] , Serrin [see J. Serrin, The Initial Value Problem for the Navier–Stokes Equations, Vol. 9, University of Wisconsin Press, 1963, pp. 69–98.], or more recently, Neustupa et al. [see J.Neustupa, A.Noovotny, P. Penel, An interior regularity of aweak solution to the Navier–Stokes equations in dependence on one component of velocity, in: Topics in Mathematical Fluid Mechanics, in: Quad. Mat., vol. 10, Dept. Math., Seconda Univ., Napoli, Caserta, 2002 pp. 163-183] and [J. Neustupa, M. Pokorny, An interior regularity criterion for an axially symmetric suitable weak solution to the Navier–Stokes equations, J. Math. Fluid Mech. 2 (4) (2000) 381–399.] In the context of compressible fluids related results are presented by Feireisl, Novotný and Sun [see]E. Feireisl, A. Novotny, Y. Sun, Suitable weak solutions to the Navier–Stokes equations of compressible viscous fluids, Nečas Center for Mathematical Modeling, Preprint 2010-019, 2010], Feireisl, Jin, and Novotný [E. Feireisl, B.J. Jin, A. Novotny, Relative entropies, suitable weak solutions, and weak–strong uniqueness for the compressible Navier–Stokes system, J. Math. Fluid Mech. 14 (2012) 717–730.] and by Mellet and Vasseur [see A. Mellet, A. Vasseur, Asymptotic analysis for a Vlasov–Fokker–Planck/compressible Navier–Stokes system of equations, Comm. Math. Phys. 281 (2008) 573–596].

Our analysis is motivated by the pioneering work of Dafermos [see C.M. Dafermos, The second law of thermodynamics and stability, Arch. Ration. Mech. Anal. 70 (1979) 167–179.] and DiPerna [see Ronald DiPerna, Uniqueness of solutions to hyperbolic conservation laws, Indiana Univ. Math. J. 28 (1) (1979) 137–188], the results of Germain [see P. Germain, Weak–strong uniqueness for the isentropic compressible Navier–Stokes system, J. Math. Fluid Mech. 13 (2011) 137–146.]the analysis of Mellet and Vasseur [13] as well as the approach of Feireisl et al. [see E. Feireisl, A. Novotny, Y. Sun, Suitable weak solutions to the Navier–Stokes equations of compressible viscous fluids, Nečas Center for Mathematical Modeling, Preprint 2010-019, 2010]. . By employing Gronwall's argument, a weak–strong uniqueness result is established yielding that a weakly dissipative solution agrees with a classical solution with the same initial data when such a classical solution exists.

We also refer the reader also to Section 2.5 in [Lio96] where a related result is presented in the context of ....fluids. .

The small data assumptions are required to complete a Grönwall's inequality calculation. One of the possibilities arising from the last condition in Theorem 2.4 essentially states that for the system starting at any finite energy (however large), there is always a small enough existence time up to which we can guarantee the uniqueness. The other route is for the energy to be small enough (along with a bound on the existence time that depends on the allowed lower bound of the density field).

1.1. **Notation.** Let $\mathfrak{D}(\Omega)$ be the space of smooth, compactly-supported functions on Ω. Then, $H_0^s(\Omega)$ is the completion of $\mathfrak{D}$ under the Sobolev norm H^s. The more general Sobolev spaces are denoted by $W^{s,p}(\Omega)$, where $s \in \mathbb{R}$ is the derivative index and $1 \leq p \leq \infty$ is the integrability index. A dot on top, like $\dot{H}^s(\Omega)$ or $\dot{W}^{s,p}$, is used when referring to the homogeneous Sobolev spaces.

Consider a 3D vector-valued function $u \equiv (u_1, u_2, u_3)$, where $u_i \in \mathfrak{D}(\Omega), i = 1, 2, 3$. The collection of all divergence-free functions u defines $\mathfrak{D}_d(\Omega)$. Then, $H_d^s(\Omega)$ is the completion of $\mathfrak{D}_d(\Omega)$ under the H^s norm. In addition, to say that a complex-valued wavefunction $\psi \in H^s(\Omega)$ means that its real and imaginary parts are the limits (in the H^s norm) of functions in $\mathfrak{D}(\Omega)$.

The L^2 inner product, denoted by $\langle\cdot,\cdot\rangle$, is sesquilinear (the first argument is complex conjugated, indicated by an overbar) to accommodate the complex nature of the Schrödinger equation. Thus, for example, $\langle\psi, B\psi\rangle = \int_\Omega \bar{\psi} B\psi\, dx$. Needless to say, since the velocity and density are real-valued functions, we will ignore the complex conjugation when they constitute the first argument of the inner product.

We use the subscript x on a Banach space to denote the Banach space is defined over Ω. For instance, L^p_x stands for the Lebesgue space $L^p(\Omega)$, and similarly for the Sobolev spaces: $H^s_{d,x} := H^s_d(\Omega)$. For spaces/norms over time, the subscript t will denote the time interval in consideration, such as $L^p_t := L^p_{[0,T]}$, where T stands for the local existence time unless mentioned otherwise. The Bochner spaces $L^p(0,T;X)$ and $C([0,T];X)$ have their usual meanings, as (L^p and continuous, respectively) maps from $[0,T]$ to a Banach space X.

We also use the notation $X \lesssim Y$ to imply that there exists a positive constant C such that $X \le CY$. The dependence of the constant on various parameters (including the initial data), will be denoted using a subscript as $X \lesssim_{k_1,k_2} Y$ or $X \le C_{k_1,k_2} Y$.

1.2. **Organization of the paper.** In Section 2, we describe the mathematical model and recall the existence results of [JT21], along with statements of the main results. The result involving less regular strong solutions is demonstrated in Section 3, while justifying the additional assumptions used. Finally, Section 4 is devoted to the proof of the more standard weak-strong uniqueness result, using the relative entropy method.

2. Mathematical model and main results

2.1. **Model and recap of existence results.** The nonlinear Schrödinger equation (NLS) is used to describe the dynamics of the superfluid phase, while the incompressible inhomogeneous Navier-Stokes equations (NSE) govern the normal Helium liquid. The "Pitaevskii model" that we consider in this work is as follows. Henceforth, all statements are in the context of a smooth, bounded domain $\Omega \subset \mathbb{R}^3$.

$$\partial_t\psi + \Lambda B\psi = -\frac{1}{2i}\Delta\psi + \frac{\mu}{i}|\psi|^2\psi \tag{NLS}$$

$$B = \frac{1}{2}\left(-i\nabla - u\right)^2 + \mu|\psi|^2 = -\frac{1}{2}\Delta + iu\cdot\nabla + \frac{1}{2}|u|^2 + \mu|\psi|^2 \tag{CPL}$$

$$\partial_t\rho + \nabla\cdot(\rho u) = 2\Lambda\,\mathrm{Re}(\bar{\psi}B\psi) \tag{CON}$$

$$\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla p - \nu\Delta u = -2\Lambda\,\mathrm{Im}(\nabla\bar{\psi}B\psi) + \Lambda\nabla\,\mathrm{Im}(\bar{\psi}B\psi) + \frac{\mu}{2}\nabla|\psi|^4 \tag{NSE}$$

$$\nabla\cdot u = 0 \tag{DIV}$$

We supplement the equations with appropriate initial and boundary conditions[1].

$$\psi(0,x) = \psi_0(x) \qquad u(0,x) = u_0(x) \qquad \rho(0,x) = \rho_0(x) \quad a.e.\ x\in\Omega \tag{INI}$$

[1]For a justification of the exclusion of $t = 0$ in the boundary conditions for the wavefunction, see Remark 2.4 in [JT21]

$$
\begin{aligned}
u = \frac{\partial u}{\partial n} &= 0 \quad a.e.\ (t,x) \in [0,T] \times \partial\Omega \\
\psi = \frac{\partial \psi}{\partial n} = \frac{\partial^2 \psi}{\partial n^2} = \frac{\partial^3 \psi}{\partial n^3} &= 0 \quad a.e.\ (t,x) \in (0,T] \times \partial\Omega
\end{aligned} \tag{BC}
$$

where n is the outward normal direction on the boundary $\partial\Omega$, and T is the local existence time. For more details about the model, refer to the discussion in [JT21] or the derivation in [Pit59]. Weak solutions that we seek are those that satisfy the governing equations in the sense of distributions, for a certain class of test functions.

Definition 2.1 (Weak solutions)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded set with a smooth boundary $\partial\Omega$. For a given time $T > 0$ and $\delta \in (0, \frac{1}{2})$, consider the following test functions:*

(1) a complex-valued scalar field $\varphi \in H^1(0,T; H^1(\Omega))$,
(2) a real-valued, divergence-free (3D) vector field $\Phi \in H^1(0,T; H^1_d(\Omega))$, and
(3) a real-valued scalar field $\sigma \in H^1(0,T; H^1(\Omega))$.

*A triplet (ψ, u, ρ) is called a **weak solution** to the Pitaevskii model if:*

(i)

$$
\begin{gathered}
\psi \in L^\infty(0,T; H^1_0(\Omega)) \cap L^2(0,T; H^2(\Omega)) \cap \{B\psi \in L^2(0,T; L^2(\Omega))\} \\
u \in L^\infty(0,T; L^2_d(\Omega)) \cap L^2(0,T; H^1_d(\Omega)) \\
\rho \in L^\infty([0,T] \times \Omega)
\end{gathered} \tag{2.1}
$$

(ii) and they satisfy the governing equations in the sense of distributions for all test functions, i.e.,

$$
\begin{aligned}
-\int_0^T \int_\Omega \left[\psi \partial_t \bar{\varphi} + \frac{1}{2i} \nabla\psi \cdot \nabla\bar{\varphi} - \Lambda \bar{\varphi} B\psi - i\mu \bar{\varphi} |\psi|^2 \psi \right] dx\ dt \\
= \int_\Omega \left[\psi_0 \bar{\varphi}(t=0) - \psi(T)\bar{\varphi}(T) \right] dx
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
-\int_0^T \int_\Omega \left[\rho u \cdot \partial_t \Phi + \rho u \otimes u : \nabla\Phi - \nu \nabla u : \nabla\Phi - 2\Lambda\Phi \cdot \mathrm{Im}(\nabla\bar{\psi} B\psi) \right] dx\ dt \\
= \int_\Omega \left[\rho_0 u_0 \Phi(t=0) - \rho(T) u(T) \Phi(T) \right] dx
\end{aligned} \tag{2.3}
$$

$$
-\int_0^T \int_\Omega \left[\rho \partial_t \sigma + \rho u \cdot \nabla\sigma + 2\Lambda\sigma\, \mathrm{Re}(\bar{\psi} B\psi) \right] dx\ dt = \int_\Omega \left[\rho_0 \sigma(t=0) - \rho(T)\sigma(T) \right] dx \tag{2.4}
$$

where (the initial data) $\psi_0, \rho_0 u_0, \rho_0$ each belong to $L^2(\Omega)$.

In [JT21], the following existence result was proven.

Theorem 2.2 (Local existence)**.** *For any $\delta \in (0, \frac{1}{2})$, let $\psi_0 \in H_0^{\frac{5}{2}+\delta}(\Omega)$ and $u_0 \in H_d^{\frac{3}{2}+\delta}(\Omega)$. Suppose ρ_0 is bounded both above and below a.e. in Ω, i.e., $0 < m \leq \rho_0 \leq M < \infty$. Then, there exists a local existence time T and at least one weak solution (ψ, u, ρ) to the Pitaevskii model. In particular, the weak solutions have the following regularity:*

$$\psi \in C(0,T; H_0^{\frac{5}{2}+\delta}(\Omega)) \cap L^2(0,T; H_0^{\frac{7}{2}+\delta}(\Omega)) \tag{2.5}$$

$$u \in C(0,T; H_d^{\frac{3}{2}+\delta}(\Omega)) \cap L^2(0,T; H_d^2(\Omega)) \tag{2.6}$$

$$\rho \in L^\infty([0,T]\times\Omega) \cap C(0,T; L^2(\Omega)) \tag{2.7}$$

where T depends on $\varepsilon \in (0,m)$, the allowed infimum of the density field.

The solutions also satisfy some bounds/estimates (some derived a priori), summarized here for convenience.

(1) *Superfluid mass bound*:

$$\|\psi\|_{L_x^2}(t) \le \|\psi_0\|_{L_x^2} \quad a.e.\ t \in [0,T] \tag{2.8}$$

(2) *Normal fluid density bound*: Given our choice of T, we know that $0 < \varepsilon < \rho(t,x) < M' := M + m - \varepsilon$ *a.e.* $(t,x) \in [0,T]\times\Omega$.

(3) *Energy equation*:

$$\begin{aligned}\left(\frac{1}{2}\|\sqrt{\rho}u\|_{L_x^2}^2 + \frac{1}{2}\|\nabla\psi\|_{L_x^2}^2 + \frac{\mu}{2}\|\psi\|_{L_x^4}^4\right)(t) + \nu\|\nabla u\|_{L_{[0,t]}^2 L_x^2}^2 + 2\Lambda\|B\psi\|_{L_{[0,t]}^2 L_x^2}^2 \\ = \frac{1}{2}\|\sqrt{\rho_0}u_0\|_{L_x^2}^2 + \frac{1}{2}\|\nabla\psi_0\|_{L_x^2}^2 + \frac{\mu}{2}\|\psi_0\|_{L_x^4}^4 =: E_0 \quad a.e.\ t\in[0,T]\end{aligned} \tag{2.9}$$

(4) *Higher-order energy estimate*:

$$\begin{aligned} X(t) \le 2X_0 \quad a.e.\ t \in [0,T] \qquad &, \qquad \int_0^T Y(\tau)d\tau \le 31X_0 \\ where\ X(t) = 1 + \|\Delta\psi\|_{L_x^2}^2 + \nu\|\nabla u\|_{L_x^2}^2 \quad &, \quad X_0 = X(0) \\ Y = \Lambda\|\nabla(B\psi)\|_{L_x^2}^2 + \|\sqrt{\rho}\partial_t u\|_{L_x^2}^2 &+ \frac{\nu^2}{M'}\|\Delta u\|_{L_x^2}^2 \end{aligned} \tag{2.10}$$

(5) *Highest-order energy estimate*:

$$\begin{aligned} \|\psi\|^2_{H_x^{\frac{5}{2}+\delta}}(t) &\lesssim \|\psi_0\|^2_{H_x^{\frac{5}{2}+\delta}} e^{cQ_T} \quad a.e.\ t\in[0,T] \\ \|B\psi\|_{L^2_{[0,T]}H_x^{\frac{3}{2}+\delta}} &\lesssim \Lambda^{-\frac{1}{2}}(Q_T^{\frac{1}{2}}+1)e^{cQ_T}\|\psi_0\|_{H_x^{\frac{5}{2}+\delta}} \\ \|u\|^2_{H_x^{\frac{3}{2}+\delta}}(t) &\lesssim \|u_0\|^2_{H_x^{\frac{3}{2}+\delta}} + \frac{1}{\nu}\sqrt{\frac{M'}{\varepsilon}}X_0 \\ where\ Q_T = \Lambda\frac{M'}{\nu^2}X_0 + \left(\Lambda\frac{M'}{\nu^2\varepsilon}+\gamma\right)X_0^2T + \Lambda E_1^2 T \quad &, \quad E_1 = \|u_0\|^2_{H_x^{\frac{3}{2}+\delta}} + \|\psi_0\|^2_{H_x^{\frac{5}{2}+\delta}} \end{aligned} \tag{2.11}$$

(6) *Time-derivative bounds*:

$$\begin{gathered}\|\partial_t\psi\|_{L^2_{[0,T]}L^2_x} \lesssim \|B\psi\|_{L^2_{[0,T]}L^2_x} + T^{\frac{1}{2}}\|\psi\|_{L^\infty_{[0,T]}H^2_x} + \mu T^{\frac{1}{2}}\|\psi\|^3_{L^\infty_{[0,T]}H^1_x} \\ \|\partial_t u\|_{L^2_{[0,T]}L^2_x} \lesssim \frac{1}{\sqrt{\varepsilon}}X_0 \\ \|\partial_t\rho\|_{L^2_{[0,T]}H^{-1}_x} \lesssim \left(TE_0M'\right)^{\frac{1}{2}} + \left(\Lambda^{-1}E_0X_0\right)^{\frac{1}{2}}\end{gathered} \tag{2.12}$$

In the derivation of (2.9), we make use of the *non-conservative form* of (NSE). Note that $\tilde{p}$ is simply a modified pressure that includes gradient terms on the RHS of the momentum equation.

$$\rho\partial_t u + \rho u\cdot\nabla u + \nabla\tilde{p} - \nu\Delta u = -2\Lambda\,\mathrm{Im}(\nabla\bar{\psi}B\psi) - 2\Lambda u\,\mathrm{Re}(\bar{\psi}B\psi) \tag{NSE'}$$

2.2. **Uniqueness results.** Given that weak solutions exist, it is instructive to ask whether they are unique and under what additional assumptions (if any). The main results of this article are the two weak-strong uniqueness theorems below, differing from each other in the regularity of the strong solutions. The first of these is a standard weak-strong result, i.e., starting from the same initial conditions, if we have a weak solution and a strong solution (in the precise sense described below), then they are both identical almost everywhere.

Theorem 2.3 (Weak-strong uniqueness)**.** *Let (ψ, u, ρ) be a weak solution as constructed in Theorem 2.2, and suppose there exists a strong solution $(\tilde{\psi}, \tilde{u}, \tilde{\rho})$ to the Pitaevskii model starting from the same initial conditions (such that both these solutions exist up to some time T). Specifically, it is sufficient that the strong solution has the following additional regularity over the weak solution:*

$$\begin{gathered}\tilde{\rho} \in L^2(0,T;W^{1,3}(\Omega)) \quad , \quad \tilde{u} \in L^2(0,T;W^{1,\infty}(\Omega)) \\ \partial_t\tilde{u} \in L^2(0,T;L^\infty(\Omega))\end{gathered} \tag{2.13}$$

Then, the solutions are identical a.e. in $[0,T]\times\Omega$.

Based on the regularity in (2.5), one can easily see that $\partial_t\psi$ is already in $L^2(0,T;L^\infty(\Omega))$, which is why no improvement is needed for the smoothness of $\tilde{\psi}$. In fact, the "weak wavefunction" has enough regularity to be classified as a strong solution; it is the velocity and density that make the solutions in Theorem 2.2 "weak". Furthermore, in the above result, we can observe that the velocity in the strong solution needs to have bounded (in space) spatio-temporal derivatives. As much as this is not unusual to expect from strong solutions, it is possible to trade this rather high regularity for some other conditions in order to obtain uniqueness. This is stated in the next theorem, and will be referred to as "weak-moderate uniqueness" since the notion of strong solution is significantly relaxed.

Theorem 2.4 (Weak-moderate uniqueness)**.** *Let (ψ, u, ρ) be a weak solution as in Theorem 2.2. Suppose there exists a "moderate" solution $(\tilde{\psi}, \tilde{u}, \tilde{\rho})$ starting from the same initial data, such that $\tilde{\rho} \in L^2(0,T;W^{1,3}(\Omega))$, $\partial_t\tilde{u} \in L^2(0,T;L^3(\Omega))$ in addition to the regularity of the weak solutions. Then, the solutions are identical a.e. in $[0,T]\times\Omega$, if the following conditions are satisfied:*

(1) (Small initial velocity and wavefunction)

$$\begin{aligned} \|u_0\|_{H^{\frac{3}{2}+\delta}(\Omega)} &\lesssim \min\{1, \nu^{-\frac{1}{2}}\} \\ \|\psi_0\|_{H^{\frac{5}{2}+\delta}(\Omega)} &\lesssim (\nu\Lambda)^{\frac{1}{2}} \end{aligned} \tag{2.14}$$

(2) (Short existence time and/or low energy)

$$T \lesssim \frac{\varepsilon}{1+\nu\Lambda} \tag{2.15}$$

along with either T or E_0 being small enough that

$$(1+\Lambda)TE_0 + \mu^2 T^2 E_0^6 \lesssim \nu\Lambda \tag{2.16}$$

where $E_0 = \frac{1}{2}\left\|\sqrt{\rho_0}u_0\right\|_{L_x^2}^2 + \frac{1}{2}\|\nabla\psi_0\|_{L_x^2}^2 + \frac{\mu}{2}\|\psi_0\|_{L_x^4}^4$ is the initial energy of the system.

The small data assumptions are required to complete a Grönwall's inequality calculation. One of the possibilities arising from the last condition in Theorem 2.4 essentially states that for the system starting at any finite energy (however large), there is always a small enough existence time up to which we can guarantee the uniqueness. The other route is for the energy to be small enough (along with a bound on the existence time that depends on the allowed lower bound of the density field).

Remark 2.5. *For fixed values of ν and Λ, it was argued in Section 2.1 of [JT21] that the local existence time T grows with decreasing ε. However, from (2.15), it is clear that for a unique solution, we require T to decrease when ε is reduced. This demonstrates a tug-of-war between the existence and weak-moderate uniqueness in the Pitaevskii model.*

It is to be noted that in the proof of Theorem 2.2, we already established that for a weak solution, $\partial_t u \in L^2(0,T;L^2)$. So, as far as the (time derivative of the) "moderate" velocity goes, we have only assumed a slight increase in the spatial integrability, from L^2 to L^3 (as opposed to L^∞ in the first result). Also noteworthy is the absence of any additional (over the weak solution) bound on the spatial derivatives of $\tilde{u}$. The regularity of the "moderate" density remains unchanged from Theorem 2.3.

Summarized below are some well-known Sobolev embeddings (see Chapter 5 of [Eva10]) that will be repeatedly utilized in the calculations that follow. We will begin with a proof of the second theorem in the next section.

Lemma 2.6 (Sobolev embeddings)**.** *For Ω a smooth, bounded subset of $\mathbb{R}^3$,*

(1) $H^1(\Omega) \subset L^6(\Omega)$
(2) $H^s(\Omega) \subset L^\infty(\Omega)\ \forall\ s > \frac{3}{2}\ \ ;\ \ H^s(\mathbb{R}) \subset L^\infty(\mathbb{R})\ \forall\ s > \frac{1}{2}$
(3) $H^s(\Omega) \Subset H^{s'}(\Omega)\ \forall\ s, s' \in \mathbb{R}, s > s'$

3. Weak-moderate uniqueness (Proof of Theorem 2.4)

In the subsequent proof, we will use the difference between the weak and moderate solutions as the test functions, so as obtain a Grönwall-type estimate for the $L^\infty_{[0,T]}L^2_x$ norm of the difference of two solutions. This is possible in the case of the velocity and wavefunction simply because of their regularities. In the case of the density, it is possible to achieve it by working with smooth approximations and passing to the limit. Finally, observe that the increased spatial regularity of $\tilde{\rho}$ results in an increased temporal regularity too.

$$\begin{aligned} \|\partial_t\tilde{\rho}\|_{L^2_{[0,T]}L^2_x} &\lesssim \|\tilde{u}\cdot\nabla\tilde{\rho}\|_{L^2_{[0,T]}L^2_x} + \left\|\bar{\tilde{\psi}}\tilde{B}\tilde{\psi}\right\|_{L^2_{[0,T]}L^2_x} \\ &\lesssim \|\tilde{u}\|_{L^\infty_{[0,T]}L^6_x}\|\tilde{\rho}\|_{L^2_{[0,T]}W^{1,3}_x} + \left\|\tilde{\psi}\right\|_{L^\infty_{[0,T]}L^\infty_x}\left\|\tilde{B}\tilde{\psi}\right\|_{L^2_{[0,T]}L^2_x} < \infty \end{aligned} \tag{3.1}$$

The notion of the weak solution holds for all times $0 \le t \le T$, where T is the chosen local existence time. Considering (2.2), (2.3) and (2.4) up to a time $t \le T$, and differentiating with respect to t, we arrive at:

$$-\int_\Omega \left[\psi\partial_t\bar{\varphi} + \frac{1}{2i}\nabla\psi\cdot\nabla\bar{\varphi} - \Lambda\bar{\varphi}B\psi - i\mu\bar{\varphi}|\psi|^2\psi\right] = -\frac{d}{dt}\int_\Omega \psi(t)\bar{\varphi}(t) \tag{3.2}$$

$$-\int_\Omega \left[\rho u\cdot\partial_t\Phi + \rho u\otimes u : \nabla\Phi - \nu\nabla u : \nabla\Phi - 2\Lambda\Phi\cdot \mathrm{Im}(\nabla\bar{\psi}B\psi)\right] = -\frac{d}{dt}\int_\Omega \rho(t)u(t)\cdot\Phi(t) \tag{3.3}$$

$$-\int_\Omega \left[\rho\partial_t\sigma + \rho u\cdot\nabla\sigma + 2\Lambda\sigma\,\mathrm{Re}(\bar{\psi}B\psi)\right] = -\frac{d}{dt}\int_\Omega \rho(t)\sigma(t) \tag{3.4}$$

We write the above time-differentiated weak formulation for both the solutions (ψ, u, ρ) and $(\tilde{\psi}, \tilde{u}, \tilde{\rho})$, setting the test functions to be $\varphi = \psi - \tilde{\psi}$, $\Phi = u - \tilde{u}$, and $\sigma = \rho - \tilde{\rho}$. Now, we subtract each of the equations (3.2), (3.3) and (3.4) from their counterparts formed by replacing the triplet $(\tilde{\psi}, \tilde{u}, \tilde{\rho})$. We will analyze the resulting equations below. In what follows, κ is a small positive real number, used to extract out the dissipative terms $\|\nabla\Phi\|^2_{L^2_x}$ and $\|\nabla\psi\|^2_{L^2_x}$, and small enough that they may be absorbed into the dissipative terms on the LHS of the momentum and Schrödinger equations, respectively.

3.1. **The wavefunction equation.** While dealing with the Schrödinger equation, we will also consider the real part after subtracting the equations for the two different weak solutions. Subsequently, a little rearrangement yields:

$$\frac{d}{dt}\frac{1}{2}\|\varphi\|^2_{L^2_x} + \Lambda\,\mathrm{Re}\int_\Omega \overline{\varphi}B_L\varphi + \Lambda\mu\int_\Omega |\varphi|^2\left|\tilde{\psi}\right|^2 = -\Lambda\,\mathrm{Re}\int_\Omega \overline{\varphi}(B-\tilde{B})\tilde{\psi} - \mu\,\mathrm{Im}\int_\Omega \overline{\varphi}\left(|\varphi|^2 + \overline{\varphi}\tilde{\psi}\right)\tilde{\psi} \tag{3.5}$$

Here, $B_L := B - \mu|\psi|^2$ is the linear (in ψ) part of the coupling operator. It is known from Lemma 2.7 in [JT21] that the second term on the LHS is non-negative. This term will now be expanded (using the definition of B_L) to obtain our "dissipation", which will absorb similar quantities on the RHS. A combination of Hölder's and Young's inequalities, and dropping some non-negative terms from the LHS, leads to:

$$\frac{d}{dt}\frac{1}{2}\|\varphi\|_{L_x^2}^2 + \frac{\Lambda}{2}\|\nabla\varphi\|_{L_x^2}^2 + \Lambda\mu\|\varphi\|_{L_x^4}^4 \leq \Lambda \underbrace{\|\overline{\varphi}u\cdot\nabla\varphi\|_{L_x^1}}_{\boxed{\text{A-1}}} + \Lambda \underbrace{\left\|\overline{\varphi}(u+\tilde{u})\cdot\Phi\tilde{\psi}\right\|_{L_x^1}}_{\boxed{\text{A-2}}} + \Lambda \underbrace{\left\|\overline{\varphi}\Phi\cdot\nabla\tilde{\psi}\right\|_{L_x^1}}_{\boxed{\text{A-3}}}$$
$$+ (1+\Lambda)\mu \underbrace{\left\||\overline{\varphi}|^2\overline{\varphi}\tilde{\psi}\right\|_{L_x^1}}_{\boxed{\text{A-4}}} + (1+\Lambda)\mu \underbrace{\left\|(\overline{\varphi})^2(\tilde{\psi})^2\right\|_{L_x^1}}_{\boxed{\text{A-5}}} \tag{3.6}$$

To arrive at (3.6), we expressed $B - \tilde{B}$ as shown below, and we will have occasion to reuse this.

$$\begin{aligned} B - \tilde{B} &= \frac{1}{2}\left(|u|^2 - |\tilde{u}|^2\right) + i(u-\tilde{u})\cdot\nabla + \mu\left(|\psi|^2 - \left|\tilde{\psi}\right|^2\right) \\ &= \frac{1}{2}(u+\tilde{u})\cdot\Phi + i\Phi\cdot\nabla + \mu\left(|\varphi|^2 + \overline{\varphi}\tilde{\psi} + \varphi\overline{\tilde{\psi}}\right) \end{aligned} \tag{3.7}$$

Now, we will look at each of the terms in (3.6).

(i)

$$\boxed{\text{A-1}} \lesssim \|\varphi\|_{L_x^2}\|u\|_{L_x^\infty}\|\nabla\varphi\|_{L_x^2} \lesssim (\kappa\Lambda)^{-1}\|u\|_{L_x^\infty}^2\|\varphi\|_{L_x^2}^2 + \kappa\Lambda\|\nabla\varphi\|_{L_x^2}^2$$

(ii)

$$\begin{aligned} \boxed{\text{A-2}} &\lesssim \|\varphi\|_{L_x^2}\|u+\tilde{u}\|_{L_x^6}\|\Phi\|_{L_x^2}\left\|\tilde{\psi}\right\|_{L_x^6} \\ &\lesssim (\kappa\Lambda)^{-1}\|\nabla(u+\tilde{u})\|_{L_x^2}^2\left\|\nabla\tilde{\psi}\right\|_{L_x^2}^2\|\Phi\|_{L_x^2}^2 + \kappa\Lambda\|\nabla\varphi\|_{L_x^2}^2 \end{aligned}$$

(iii)

$$\begin{aligned} \boxed{\text{A-3}} &\lesssim \|\varphi\|_{L_x^4}\|\Phi\|_{L_x^4}\left\|\nabla\tilde{\psi}\right\|_{L_x^2} \\ &\lesssim \left\|\nabla\tilde{\psi}\right\|_{L_x^2}^2\|\varphi\|_{L_x^4}^2 + \|\Phi\|_{L_x^4}^2 \\ &\lesssim (\kappa\Lambda)^{-3}\left\|\nabla\tilde{\psi}\right\|_{L_x^2}^8\|\varphi\|_{L_x^2}^2 + (\kappa\nu)^{-3}\|\Phi\|_{L_x^2}^2 + \kappa\nu\|\nabla\Phi\|_{L_x^2}^2 + \kappa\Lambda\|\nabla\varphi\|_{L_x^2}^2 \end{aligned}$$

In going to the last step, we have used the 3D version of the Ladyzhenskaya inequality: $\|f\|_{L_x^4}^2 \lesssim \|f\|_{L_x^2}^{\frac{1}{2}}\|\nabla f\|_{L_x^2}^{\frac{3}{2}}$, followed by Young's, in order to extract the required dissipative terms.

(iv)

$$\begin{aligned} \boxed{\text{A-4}} &\lesssim \|\varphi\|_{L_x^4}^3\left\|\tilde{\psi}\right\|_{L_x^4} \\ &\lesssim \frac{\Lambda\mu}{2}\|\varphi\|_{L_x^4}^4 + (\Lambda\mu)^{-1}\left\|\tilde{\psi}\right\|_{L_x^4}^2\|\varphi\|_{L_x^4}^2 \\ &\lesssim \frac{\Lambda\mu}{2}\|\varphi\|_{L_x^4}^4 + (\Lambda\mu)^{-4}(\kappa\Lambda)^{-3}\left\|\tilde{\psi}\right\|_{L_x^4}^8\|\varphi\|_{L_x^2}^2 + \kappa\Lambda\|\nabla\varphi\|_{L_x^2}^2 \end{aligned}$$

where we have once again used the Ladyzhenskaya inequality. Note that the first term in the last line can be absorbed into the LHS of (3.6).

(v)

$$\boxed{\text{A-5}} \lesssim \|\varphi\|_{L^4_x}^2 \left\|\tilde{\psi}\right\|_{L^4_x}^2 \lesssim (\kappa\Lambda)^{-3} \left\|\tilde{\psi}\right\|_{L^4_x}^8 \|\varphi\|_{L^2_x}^2 + \kappa\Lambda \|\nabla\varphi\|_{L^2_x}^2$$

Combining these estimates into (3.6), and dropping the $\|\varphi\|_{L^4_x}^4$ term, gives us:

$$\frac{d}{dt}\|\varphi\|_{L^2_x}^2 + \frac{\Lambda}{2}\|\nabla\varphi\|_{L^2_x}^2 \leq h_1(t)\left[\|\varphi\|_{L^2_x}^2 + \|\Phi\|_{L^2_x}^2\right] + \kappa\nu\|\nabla\Phi\|_{L^2_x}^2 + \kappa\Lambda\|\nabla\varphi\|_{L^2_x}^2 \tag{3.8}$$

where $h_1 \in L^1_{[0,T]}$.

3.2. **The momentum equation.** Subtracting the two momentum equations, we get:

$$\begin{aligned} -\int_\Omega \rho\Phi\cdot\partial_t\Phi - \int_\Omega \sigma\tilde{u}\cdot\partial_t\Phi - \int_\Omega [\rho\Phi\otimes u + \rho\tilde{u}\otimes\Phi + \sigma\tilde{u}\otimes\tilde{u}] : \nabla\Phi + \nu\|\nabla\Phi\|_{L^2_x}^2 \\ + 2\Lambda\int_\Omega \Phi\cdot\operatorname{Im}\left(\nabla\overline{\psi}B\psi - \nabla\overline{\tilde{\psi}}\tilde{B}\tilde{\psi}\right) = -\frac{d}{dt}\int_\Omega \rho|\Phi|^2 - \frac{d}{dt}\int_\Omega \sigma\tilde{u}\cdot\Phi \end{aligned} \tag{3.9}$$

The first term on the LHS can be rewritten since we see that $\frac{|\Phi|^2}{2}$ satisfies the requirements to be a test function for the density field in (3.4).

$$-\int_\Omega \rho\Phi\cdot\partial_t\Phi = -\int_\Omega \rho\partial_t\frac{|\Phi|^2}{2} = \int_\Omega \rho u\cdot\nabla\frac{|\Phi|^2}{2} + \Lambda\operatorname{Re}\int_\Omega |\Phi|^2\overline{\psi}B\psi - \frac{d}{dt}\int_\Omega \rho\frac{|\Phi|^2}{2}$$

Further simplification using Hölder's and Young's inequalities (to appropriately absorb the dissipation $\|\nabla\Phi\|_{L^2_x}^2$ from the RHS into the LHS) yields,

$$\left.\begin{aligned} &\tfrac{d}{dt}\tfrac{1}{2}\left\|\sqrt{\rho}\Phi\right\|_{L^2_x}^2 \\ &+\tfrac{\nu}{2}\|\nabla\Phi\|_{L^2_x}^2 \end{aligned}\right\} \begin{aligned} \leq &\frac{1}{\nu}\underbrace{\left[\|\rho\Phi\otimes u\|_{L^2_x}^2 + \|\rho\tilde{u}\otimes\Phi\|_{L^2_x}^2 + \|\sigma\tilde{u}\otimes\tilde{u}\|_{L^2_x}^2\right]}_{\boxed{\text{B-1}}+\boxed{\text{B-2}}+\boxed{\text{B-3}}} \\ &- 2\Lambda\underbrace{\int_\Omega \Phi\cdot\operatorname{Im}\left(\nabla\overline{\varphi}B\psi + \nabla\overline{\tilde{\psi}}B\varphi + \nabla\overline{\tilde{\psi}}(B-\tilde{B})\tilde{\psi}\right)}_{\boxed{\text{B-4}}+\boxed{\text{B-5}}+\boxed{\text{B-6}}} \\ &- \underbrace{\int_\Omega \partial_t(\sigma\tilde{u})\cdot\Phi}_{\boxed{\text{B-7}}} - \underbrace{\int_\Omega \rho u\cdot\nabla\frac{|\Phi|^2}{2}}_{\boxed{\text{B-8}}} - \Lambda\underbrace{\int_\Omega |\Phi|^2\operatorname{Re}\overline{\psi}B\psi}_{\boxed{\text{B-9}}} \end{aligned} \tag{3.10}$$

Each of the terms on the RHS can be bounded from above as follows (using the Hölder and Young inequalities, and the Some of these terms are more straightforward than others.

(i)

$$\boxed{\text{B-1}} \lesssim \|\rho\|_{L^\infty_x}^2 \|u\|_{L^\infty_x}^2 \|\Phi\|_{L^2_x}^2$$

(ii)

$$\boxed{\text{B-2}} \lesssim \|\rho\|_{L^\infty_x}^2 \|\tilde{u}\|_{L^\infty_x}^2 \|\Phi\|_{L^2_x}^2$$

(iii)

$$\boxed{\text{B-3}} \lesssim \|\tilde{u}\|_{L_x^\infty}^4 \|\sigma\|_{L_x^2}^2$$

(iv)

$$\boxed{\text{B-4}} \lesssim \|\nabla\varphi\|_{L_x^2} \|B\psi\|_{L_x^\infty} \|\Phi\|_{L_x^2} \lesssim (\kappa\Lambda)^{-1} \|B\psi\|_{L_x^\infty}^2 \|\Phi\|_{L_x^2}^2 + \kappa\Lambda \|\nabla\varphi\|_{L_x^2}^2$$

(v) Expanding $B - \tilde{B}$ as in (3.7),

$$\begin{aligned}
\boxed{\text{B-6}} &= \int_\Omega \Phi \cdot \operatorname{Im}\left[\nabla\overline{\tilde{\psi}} \left\{ \frac{1}{2}(u+\tilde{u})\cdot\Phi + i\Phi\cdot\nabla + \mu(|\varphi|^2 + \overline{\varphi}\tilde{\psi} + \varphi\overline{\tilde{\psi}})\tilde{\psi} \right\} \tilde{\psi}\right] \\
&\lesssim \left\|\nabla\tilde{\psi}\right\|_{L_x^\infty} \|u+\tilde{u}\|_{L_x^\infty} \|\Phi\|_{L_x^2}^2 + \left\|\nabla\tilde{\psi}\right\|_{L_x^\infty}^2 \|\Phi\|_{L_x^2}^2 \\
&\quad + \left\|\nabla\tilde{\psi}\right\|_{L_x^\infty} \left\|\tilde{\psi}\right\|_{L_x^\infty} \|\Phi\|_{L_x^2} \|\varphi\|_{L_x^2} \left(\|\varphi\|_{L_x^\infty} + \left\|\tilde{\psi}\right\|_{L_x^\infty} \right) \\
&\lesssim \left(\left\|\nabla\tilde{\psi}\right\|_{L_x^\infty} \|u+\tilde{u}\|_{L_x^\infty} + \left\|\nabla\tilde{\psi}\right\|_{L_x^\infty}^2 \right) \|\Phi\|_{L_x^2}^2 \\
&\qquad + \left\|\tilde{\psi}\right\|_{L_x^\infty}^2 \left(\|\psi\|_{L_x^\infty}^2 + \left\|\tilde{\psi}\right\|_{L_x^\infty}^2 \right) \|\varphi\|_{L_x^2}^2
\end{aligned}$$

Estimating the remaining terms requires some of the additional regularity/small-data assumptions listed in Theorem 2.4.

(vi)

$$\begin{aligned}
\boxed{\text{B-5}} &= \operatorname{Im}\int_\Omega \Phi\cdot\nabla\overline{\tilde{\psi}} \left(-\frac{1}{2}\Delta\varphi + \frac{1}{2}|u|^2\varphi + iu\cdot\nabla\varphi + \mu|\psi|^2\varphi \right) \\
&= \boxed{\text{B-5.1}} + \boxed{\text{B-5.2}} + \boxed{\text{B-5.3}} + \boxed{\text{B-5.4}}
\end{aligned}$$

- Integrating by parts on the first term,

$$\boxed{\text{B-5.1}} = \frac{1}{2} \underbrace{\int_\Omega \operatorname{Im} \nabla(\Phi\cdot\nabla)\tilde{\psi}\cdot\nabla\varphi}_{\boxed{\text{B-5.1.1}}} + \frac{1}{2} \underbrace{\int_\Omega \operatorname{Im} \Phi\cdot\Delta\tilde{\psi}\nabla\varphi}_{\boxed{\text{B-5.1.2}}}$$

$$\boxed{\text{B-5.1.1}} \lesssim \left\|\nabla\tilde{\psi}\right\|_{L_x^\infty} \|\nabla\Phi\|_{L_x^2} \|\nabla\varphi\|_{L_x^2} \lesssim (\kappa\nu)^{-1} \left\|\nabla\tilde{\psi}\right\|_{L_x^\infty}^2 \|\nabla\varphi\|_{L_x^2}^2 + \kappa\nu \|\nabla\Phi\|_{L_x^2}^2$$

In order to get the first term on the RHS to also be absorbed into the dissipation from the Schrödinger equation, we will require $(\kappa\nu)^{-1}\left\|\nabla\tilde{\psi}\right\|_{L_x^\infty}^2(t) \leq \kappa\Lambda$. For now, we will assume this, and justify it in Section 3.4.

$$\boxed{\text{B-5.1.2}} \lesssim \left\|\Delta\tilde{\psi}\right\|_{L_x^\infty} \|\Phi\|_{L_x^2} \|\nabla\varphi\|_{L_x^2} \lesssim (\kappa\Lambda)^{-1} \left\|\Delta\tilde{\psi}\right\|_{L_x^\infty}^2 \|\Phi\|_{L_x^2}^2 + \kappa\Lambda \|\nabla\varphi\|_{L_x^2}^2$$

- $$\boxed{\text{B-5.2}} \lesssim \|u\|_{L^\infty_x}^2 \left\|\nabla\tilde{\psi}\right\|_{L^\infty_x} \|\Phi\|_{L^2_x} \|\varphi\|_{L^2_x} \lesssim \|u\|_{L^\infty_x}^4 \|\Phi\|_{L^2_x}^2 + \left\|\tilde{\psi}\right\|_{L^\infty_x}^2 \|\varphi\|_{L^2_x}^2$$

- $$\begin{aligned}\boxed{\text{B-5.3}} &\lesssim \|u\|_{L^\infty_x}^2 \left\|\nabla\tilde{\psi}\right\|_{L^\infty_x} \|\Phi\|_{L^2_x} \|\nabla\varphi\|_{L^2_x} \\ &\lesssim (\kappa\Lambda)^{-1} \|u\|_{L^\infty_x}^2 \left\|\nabla\tilde{\psi}\right\|_{L^\infty_x}^2 \|\Phi\|_{L^2_x}^2 + \kappa\Lambda \|\nabla\varphi\|_{L^2_x}^2\end{aligned}$$

- $$\boxed{\text{B-5.4}} \lesssim \left\|\Delta\tilde{\psi}\right\|_{L^\infty_x} \|\Phi\|_{L^2_x} \|\nabla\varphi\|_{L^2_x} \lesssim (\kappa\Lambda)^{-1} \left\|\tilde{\Delta\psi}\right\|_{L^\infty_x}^2 \|\Phi\|_{L^2_x}^2 + \kappa\Lambda \|\nabla\varphi\|_{L^2_x}^2$$

(vii)

$$\boxed{\text{B-7}} = \underbrace{-\int_\Omega \partial_t\tilde{u}\cdot\sigma\Phi}_{\boxed{\text{B-7.1}}} \underbrace{- \int_\Omega \partial_t\sigma(\tilde{u}\cdot\Phi)}_{\boxed{\text{B-7.2}}}$$

- $$\begin{aligned}\boxed{\text{B-7.1}} &\lesssim \|\partial_t\tilde{u}\|_{L^3_x} \|\sigma\|_{L^2_x} \|\Phi\|_{L^6_x} \\ &\lesssim \|\partial_t\tilde{u}\|_{L^3_x} \|\sigma\|_{L^2_x} \|\nabla\Phi\|_{L^2_x} \\ &\lesssim (\kappa\nu)^{-1} \|\partial_t\tilde{u}\|_{L^3_x}^2 \|\sigma\|_{L^2_x}^2 + \kappa\nu \|\nabla\Phi\|_{L^2_x}^2\end{aligned}$$

- By subtracting the continuity equations for ρ and $\tilde{\rho}$, we obtain

  $$\partial_t\sigma + \Phi\cdot\nabla\tilde{\rho} + u\cdot\nabla\sigma = 2\Lambda\,\mathrm{Re}\left(\overline{\psi}B\psi - \overline{\tilde{\psi}}\tilde{B}\tilde{\psi}\right)$$

  which allows us to rewrite $\boxed{\text{B-7.2}}$ as

  $$\boxed{\text{B-7.2}} = \underbrace{\int_\Omega (\tilde{u}\cdot\Phi)\left[-\Phi\cdot\nabla\tilde{\rho} - u\cdot\nabla\sigma + 2\Lambda\,\mathrm{Re}\left(\overline{\psi}B\psi - \overline{\tilde{\psi}}\tilde{B}\tilde{\psi}\right)\right]}_{\boxed{\text{B-7.2.1}}+\boxed{\text{B-7.2.2}}+\boxed{\text{B-7.2.3}}}$$

  Each of these terms will now be estimated.

  $$\begin{aligned}\boxed{\text{B-7.2.1}} &\lesssim \|\tilde{u}\|_{L^\infty_x} \|\nabla\tilde{\rho}\|_{L^3_x} \|\Phi\|_{L^2_x} \|\Phi\|_{L^6_x} \\ &\lesssim (\kappa\nu)^{-1} \|\tilde{u}\|_{L^\infty_x}^2 \|\nabla\tilde{\rho}\|_{L^3_x}^2 \|\Phi\|_{L^2_x}^2 + \kappa\nu \|\nabla\Phi\|_{L^2_x}^2\end{aligned}$$

  This is a suitable estimate due to the regularity of the stronger solution: $\tilde{\rho} \in L^2_{[0,T]} W^{1,3}_x$. For the second term, we will additionally interpolate the L^3_x norm between the L^2_x and L^6_x norms: $\|\nabla\tilde{u}\|_{L^3_x}^2 \lesssim \|\nabla\tilde{u}\|_{L^2_x} \left\|D^2\tilde{u}\right\|_{L^2_x}$.

$$\begin{aligned}
\boxed{\text{B-7.2.2}} &= \int_\Omega \sigma u \cdot \nabla(\Phi \cdot \tilde{u}) = \int_\Omega \sigma u \cdot (\nabla \Phi) \cdot \tilde{u} + \int_\Omega \sigma u \cdot (\nabla \tilde{u}) \cdot \Phi \\
&\lesssim \|u\|_{L_x^\infty} \|\tilde{u}\|_{L_x^\infty} \|\sigma\|_{L_x^2} \|\nabla \Phi\|_{L_x^2} + \|u\|_{L_x^\infty} \|\sigma\|_{L_x^2} \|\nabla \tilde{u}\|_{L_x^3} \|\Phi\|_{L_x^6} \\
&\lesssim (\kappa\nu)^{-1} \left(\|\tilde{u}\|_{L_x^\infty}^2 + \|\nabla \tilde{u}\|_{L_x^2} \left\|D^2 \tilde{u}\right\|_{L_x^2} \right) \|u\|_{L_x^\infty}^2 \|\sigma\|_{L_x^2}^2 \\
&\qquad + \kappa\nu \|\nabla \Phi\|_{L_x^2}^2
\end{aligned}$$

Finally,

$$\boxed{\text{B-7.2.3}} = \underbrace{2\Lambda \int_\Omega (\tilde{u} \cdot \Phi) \operatorname{Re} \left(\overline{\varphi} B \psi + \overline{\tilde{\psi}} B \varphi + \overline{\tilde{\psi}} (B - \tilde{B}) \tilde{\psi} \right)}_{\boxed{\text{B-7.2.3.1}} + \boxed{\text{B-7.2.3.2}} + \boxed{\text{B-7.2.3.3}}}$$

These are, in turn, bounded as follows. The third term is dealt with in a manner mirroring that of $\boxed{\text{B-6}}$.

$$\boxed{\text{B-7.2.3.1}} \lesssim \|\tilde{u}\|_{L_x^\infty} \|\Phi\|_{L_x^2} \|\varphi\|_{L_x^2} \|B\psi\|_{L_x^\infty} \lesssim \|\tilde{u}\|_{L_x^\infty}^2 \|\Phi\|_{L_x^2}^2 + \|B\psi\|_{L_x^\infty}^2 \|\varphi\|_{L_x^2}^2$$

$$\begin{aligned}
\boxed{\text{B-7.2.3.3}} \lesssim \|\tilde{u}\|_{L_x^\infty} \left\|\nabla \tilde{\psi}\right\|_{L_x^\infty} &\left(\|u + \tilde{u}\|_{L_x^\infty} + \left\|\nabla \tilde{\psi}\right\|_{L_x^\infty} + \|\tilde{u}\|_{L_x^\infty} \left\|\nabla \tilde{\psi}\right\|_{L_x^\infty} \right) \|\Phi\|_{L_x^2}^2 \\
&+ \left\|\tilde{\psi}\right\|_{L_x^\infty}^2 \left(\|\psi\|_{L_x^\infty}^2 + \left\|\tilde{\psi}\right\|_{L_x^\infty}^2 \right) \|\varphi\|_{L_x^2}^2
\end{aligned}$$

The middle term requires to be broken down further, much like $\boxed{\text{B-5}}$.

$$\boxed{\text{B-7.2.3.2}} = \underbrace{\operatorname{Re} \int_\Omega (\tilde{u} \cdot \Phi) \overline{\tilde{\psi}} \left[-\frac{1}{2} \Delta \varphi + \frac{1}{2} |u|^2 \varphi + i u \cdot \nabla \varphi + \mu |\psi|^2 \varphi \right]}_{\boxed{\text{B-7.2.3.2.1}} + \boxed{\text{B-7.2.3.2.2}} + \boxed{\text{B-7.2.3.2.3}} + \boxed{\text{B-7.2.3.2.4}}}$$

We integrate by parts on the first term, and analyze the remaining terms just as before.

$$\begin{aligned}
\boxed{\text{B-7.2.3.2.1}} &= \frac{1}{2} \operatorname{Re} \left[\int_\Omega (\nabla \Phi \cdot \tilde{u}) \cdot \tilde{\psi} \nabla \varphi + \int_\Omega (\nabla \tilde{u} \cdot \Phi) \cdot \tilde{\psi} \nabla \varphi + \int_\Omega (\Phi \cdot \tilde{u}) \nabla \tilde{\psi} \cdot \nabla \varphi \right] \\
&\lesssim (\kappa\Lambda)^{-1} \left[\|\nabla \tilde{u}\|_{L_x^\infty}^2 \left\|\tilde{\psi}\right\|_{L_x^\infty}^2 + \|\tilde{u}\|_{L_x^\infty}^2 \left\|\nabla \tilde{\psi}\right\|_{L_x^\infty}^2 \right] \|\Phi\|_{L_x^2}^2 \\
&\qquad + (\kappa\Lambda)^{-1} \|\tilde{u}\|_{L_x^\infty}^2 \left\|\tilde{\psi}\right\|_{L_x^\infty}^2 \|\nabla \Phi\|_{L_x^2}^2 + \kappa\Lambda \|\nabla \varphi\|_{L_x^2}^2
\end{aligned}$$

To be able to absorb the $\|\nabla \Phi\|_{L_x^2}^2$ term into the dissipation (at the very end), we require $(\kappa\Lambda)^{-1} \|\tilde{u}\|_{L_x^\infty}^2(t) \left\|\tilde{\psi}\right\|_{L_x^\infty}^2(t) \le \kappa\nu$. As was done in $\boxed{\text{B-5.1.1}}$, we will assume this for now and justify it in Section 3.4.

$$\boxed{\text{B-7.2.3.2.2}} \lesssim \|\tilde{u}\|^2_{L^\infty_x}\|u\|^4_{L^\infty_x}\|\Phi\|^2_{L^2_x} + \left\|\tilde{\psi}\right\|^2_{L^\infty_x}\|\varphi\|^2_{L^2_x}$$

$$\boxed{\text{B-7.2.3.2.3}} \lesssim (\kappa\Lambda)^{-1}\|\tilde{u}\|^2_{L^\infty_x}\|u\|^2_{L^\infty_x}\left\|\tilde{\psi}\right\|^2_{L^\infty_x}\|\Phi\|^2_{L^2_x} + \kappa\Lambda\|\nabla\varphi\|^2_{L^2_x}$$

$$\boxed{\text{B-7.2.3.2.4}} \lesssim \|\tilde{u}\|^2_{L^\infty_x}\|\Phi\|^2_{L^2_x} + \|\psi\|^2_{L^\infty_x}\left\|\tilde{\psi}\right\|^2_{L^\infty_x}\|\varphi\|^2_{L^2_x}$$

(viii) Using (3.1), and the vector identity $\nabla\frac{|\Phi|^2}{2} = \Phi\cdot\nabla\Phi - (\nabla\times\Phi)\times\Phi$,

$$\boxed{\text{B-8}} \lesssim \|\rho\|_{L^\infty_x}\|u\|_{L^\infty_x}\|\Phi\|_{L^2_x}\|\nabla\Phi\|_{L^2_x} \lesssim (\kappa\nu)^{-1}\|\rho\|^2_{L^\infty_x}\|u\|^2_{L^\infty_x}\|\Phi\|^2_{L^2_x} + \kappa\nu\|\nabla\Phi\|^2_{L^2_x}$$

(viv)

$$\boxed{\text{B-9}} \lesssim \|\psi\|_{L^\infty_x}\|B\psi\|_{L^\infty_x}\|\Phi\|^2_{L^2_x}$$

Returning to (3.10), and using the ensuing estimates, we arrive at:

$$\varepsilon\frac{d}{dt}\|\Phi\|^2_{L^2_x} + \frac{\nu}{2}\|\nabla\Phi\|^2_{L^2_x} \le h_2(t)\left[\|\varphi\|^2_{L^2_x} + \|\Phi\|^2_{L^2_x} + \|\sigma\|^2_{L^2_x}\right] + \kappa\nu\|\nabla\Phi\|^2_{L^2_x} + \kappa\Lambda\|\nabla\varphi\|^2_{L^2_x} \tag{3.11}$$

where $h_2 \in L^1_{[0,T]}$. Also, note that we have replaced the density in the first term of the LHS by its minimum value.

3.3. The continuity equation.

In this case, we get

$$-\int_\Omega \sigma\partial_t\sigma - \int_\Omega(\rho u - \tilde{\rho}\tilde{u})\cdot\nabla\sigma - 2\Lambda\int_\Omega \sigma\,\mathrm{Re}\left(\overline{\psi}B\psi - \overline{\tilde{\psi}}\tilde{B}\tilde{\psi}\right) = -\frac{d}{dt}\int_\Omega|\sigma|^2$$

which can be rewritten (after simplifying the first term on the LHS) as:

$$\frac{d}{dt}\frac{1}{2}\|\sigma\|^2_{L^2_x} = \underbrace{\int_\Omega \tilde{\rho}\Phi\cdot\nabla\sigma}_{\boxed{\text{C-1}}} - 2\Lambda\,\mathrm{Re}\underbrace{\int_\Omega \sigma\overline{\varphi}B\psi + \sigma\overline{\tilde{\psi}}(B-\tilde{B})\tilde{\psi}}_{\boxed{\text{C-2}}+\boxed{\text{C-3}}} \tag{3.12}$$

Remark 3.1. *The above calculations can be rigorously justified by considering the difference between the equations for ρ^N (the approximate densities in Section 4.3 of [JT21]) and $\tilde{\rho}$. Passing to the limit $N\to\infty$ leaves us with the desired equation for $\sigma = \rho - \tilde{\rho}$, since we know that $\rho \in C(0,T;L^2_x)$.*

(i) Once again, recalling that we have assumed in Theorem 2.4 that $\tilde{\rho} \in L^2_{[0,T]} W^{1,3}_x$. Thus,

$$\boxed{\text{C-1}} = -\int_\Omega \nabla\tilde{\rho}\cdot\Phi\sigma \lesssim \|\nabla\tilde{\rho}\|_{L^3_x}\|\Phi\|_{L^6_x}\|\sigma\|_{L^2_x} \lesssim (\kappa\nu)^{-1}\|\nabla\tilde{\rho}\|^2_{L^3_x}\|\sigma\|^2_{L^2_x} + \kappa\nu\|\nabla\Phi\|^2_{L^2_x}$$

Observe that σ should always be considered in the L^2_x norm while using Hölder's, since there is no dissipation term for the density.

(ii) For the second term, apart from the above, we also interpolate the L^3_x norm between the L^2_x and L^6_x norms.

$$\begin{aligned}\boxed{\text{C-2}} \lesssim \|\sigma\|_{L^2_x}\|\varphi\|_{L^6_x}\|B\psi\|_{L^3_x} &\lesssim \|\sigma\|_{L^2_x}\|\nabla\varphi\|_{L^2_x}\|B\psi\|^{\frac{1}{2}}_{L^2_x}\|\nabla B\psi\|^{\frac{1}{2}}_{L^2_x} \\ &\lesssim (\kappa\Lambda)^{-1}\|\sigma\|^2_{L^2_x}\|B\psi\|_{L^2_x}\|\nabla B\psi\|_{L^2_x} + \kappa\Lambda\|\nabla\varphi\|^2_{L^2_x}\end{aligned}$$

where once again, we choose κ small enough for the absorption of the dissipation term, this time into the corresponding term from the Schrödinger equation.

(iii) Similar analysis on the third term yields

$$\begin{aligned}\boxed{\text{C-3}} &\lesssim \|\sigma\|_{L^2_x}\left\|\overline{\tilde{\psi}}(B-\tilde{B})\tilde{\psi}\right\|_{L^2_x} \\ &\lesssim \|\sigma\|_{L^2_x}\Bigg[\left\|\tilde{\psi}\right\|_{L^\infty_x}\|u+\tilde{u}\|_{L^3_x}\|\Phi\|_{L^6_x} + \|\Phi\|_{L^6_x}\left\|\tilde{\psi}\right\|_{L^6_x}\left\|\nabla\tilde{\psi}\right\|_{L^6_x} \\ &\qquad + \left\|\tilde{\psi}\right\|^2_{L^\infty_x}\|\varphi\|^2_{L^4_x} + \|\varphi\|_{L^6_x}\left\|\tilde{\psi}\right\|_{L^\infty_x}\left\|\tilde{\psi}\right\|^2_{L^6_x}\Bigg] \\ &\lesssim \kappa^{-1}\|\sigma\|^2_{L^2_x}\Bigg[\nu^{-1}\left\|\tilde{\psi}\right\|^2_{L^\infty_x}\|u+\tilde{u}\|_{L^2_x}\|\nabla u+\nabla\tilde{u}\|_{L^2_x} + \nu^{-1}\left\|\nabla\tilde{\psi}\right\|^2_{L^2_x}\left\|D^2\tilde{\psi}\right\|^2_{L^2_x} \\ &\qquad + \Lambda^{-1}\left\|\tilde{\psi}\right\|^2_{L^\infty_x}\left\|\nabla\tilde{\psi}\right\|^4_{L^2_x}\Bigg] + (\kappa\Lambda)^{-3}\|\sigma\|^4_{L^2_x}\left\|\tilde{\psi}\right\|^8_{L^\infty_x}\|\varphi\|^2_{L^2_x} \\ &\qquad + \kappa\nu\|\nabla\Phi\|^2_{L^2_x} + \kappa\Lambda\|\nabla\varphi\|^2_{L^2_x}\end{aligned}$$

In going to the last step, we have used the Ladyzhenskaya inequality, followed by Young's. A consequence of this is that $\|\sigma\|_{L^2_x}$ is now raised to the power 4, as opposed to 2 which is needed for a Grönwall's inequality argument (coming up next). So, we split it as follows: $\|\sigma\|^4_{L^2_x} = \|\rho-\tilde{\rho}\|^2_{L^2_x}\|\sigma\|^2_{L^2_x} \lesssim \left(\|\rho\|^2_{L^2_x} + \|\tilde{\rho}\|^2_{L^2_x}\right)\|\sigma\|^2_{L^2_x}$. We also replace $\|\varphi\|^2_{L^2_x}$ by the upper bound $\|\psi\|^2_{L^2_x} + \left\|\tilde{\psi}\right\|^2_{L^2_x}$.

Using the above estimates, we can simplify (3.12) to read:

$$\frac{d}{dt}\|\sigma\|^2_{L^2_x} \leq h_3(t)\|\sigma\|^2_{L^2_x} + \kappa\nu\|\nabla\Phi\|^2_{L^2_x} + \kappa\Lambda\|\nabla\varphi\|^2_{L^2_x} \tag{3.13}$$

where $h_3 \in L^1_{[0,T]}$.

3.4. **Small data and short time/low energy.** In Section 3.2, in the analysis of $\boxed{\text{B-5.1.1}}$ and $\boxed{\text{B-7.2.3.2.1}}$, we assumed the following to hold a.e. $t \in [0,T]$:

$$(\kappa\nu)^{-1}\left\|\nabla\tilde{\psi}\right\|^2_{L^\infty_x}(t) \le \kappa\Lambda \tag{3.14}$$

$$(\kappa\Lambda)^{-1}\|\tilde{u}\|^2_{L^\infty_x}(t)\left\|\tilde{\psi}\right\|^2_{L^\infty_x}(t) \le \kappa\nu \tag{3.15}$$

Let us begin with a discussion of (3.14). From here on, we will drop the tilde on the functions in the above conditions (since they must hold for both the weak solutions). Also, for the remainder of this section, we will denote $s = \frac{3}{2} + \delta$.

Due to the embedding $H_x^{\frac{3}{2}+\delta+k} \subset C_x^k$, a sufficient condition for (3.14) to hold is:

$$\|\psi\|^2_{L^\infty_t H^{s+1}_x} \lesssim \nu\Lambda \tag{3.16}$$

We can derive such a bound using the Lions-Magenes lemma (see Lemma 1.2 in Chapter 3 of [Tem77]).

$$\begin{aligned}
\|\psi\|^2_{H^{s+1}_x}(t) &= \|\psi_0\|^2_{H^{s+1}_x} + 2\int_0^t \langle \partial_t\psi, \psi\rangle_{H^{-s-1}\times H^{s+1}} \\
&\le \|\psi_0\|^2_{H^{s+1}_x} + 2T^{\frac{1}{2}}\|\partial_t\psi\|_{L^2_t H^{-s-1}_x}\|\psi\|_{L^\infty_t H^{s+1}_x} \\
&\lesssim \|\psi_0\|^2_{H^{s+1}_x} + T\|\partial_t\psi\|^2_{L^2_t H^{-s-1}_x} + \frac{1}{2}\|\psi\|^2_{L^\infty_t H^{s+1}_x}
\end{aligned}$$

implying

$$\|\psi\|^2_{L^\infty_t H^{s+1}_x} \lesssim \|\psi_0\|^2_{H^{s+1}_x} + T\|\partial_t\psi\|^2_{L^2_t L^2_x} \tag{3.17}$$

Now, we will derive a slightly better bound for $\|\partial_t\psi\|_{L^2_t L^2_x}$ than that in (2.12), so that everything may be expressed in terms of the initial energy E_0. Multiplying (NLS) by $\partial_t\overline{\psi}$, and integrating over Ω,

$$\frac{i}{2}\frac{d}{dt}\|\nabla\psi\|^2_{L^2_x} + \|\partial_t\psi\|^2_{L^2_x} = -\Lambda\int_\Omega \partial_t\overline{\psi}B\psi + \frac{\mu}{i}\int_\Omega \partial_t\overline{\psi}|\psi|^2\psi$$

We now take the real and imaginary parts of this, and then add the two equations. In the process, we will use the fact that $\operatorname{Re}\left(\partial_t\overline{\psi}|\psi|^2\psi\right) = \frac{1}{4}\partial_t|\psi|^4$.

$$\frac{1}{2}\frac{d}{dt}\|\nabla\psi\|^2_{L^2_x} + \|\partial_t\psi\|^2_{L^2_x} + \frac{\mu}{4}\frac{d}{dt}\|\psi\|^4_{L^4_x} = -\Lambda\operatorname{Re}\int_\Omega \partial_t\overline{\psi}B\psi - \Lambda\operatorname{Im}\int_\Omega \partial_t\overline{\psi}B\psi + \mu\operatorname{Im}\int_\Omega \partial_t\overline{\psi}|\psi|^2\psi$$

Using Hölder's inequality and Sobolev embedding, we get:

$$\frac{1}{2}\frac{d}{dt}\|\nabla\psi\|^2_{L^2_x} + \frac{1}{2}\|\partial_t\psi\|^2_{L^2_x} \lesssim \Lambda^2\|B\psi\|^2_{L^2_x} + \mu^2\|\nabla\psi\|^6_{L^2_x}$$

Integrating over time, and using (2.9), we arrive at:

$$\|\partial_t\psi\|^2_{L^2_t L^2_x} \lesssim (1+\Lambda)E_0 + \mu^2 T E_0^6 \tag{3.18}$$

Using (3.17) and (3.18), we conclude that (3.16) may now be replaced by another sufficient condition:

$$\|\psi_0\|^2_{H^{s+1}_x} + (1+\Lambda)TE_0 + \mu^2 T^2 E_0^6 \lesssim \nu\Lambda \tag{3.19}$$

It is obvious that this can be satisfied for when E_0 (or T) and $\|\psi_0\|_{H^{s+1}_x}$ are small enough. Here, we will establish one such choice. Let the constant of proportionality in (3.19) be c. First, select E_0 such that $TE_0(1+\Lambda) \leq \frac{c\nu\Lambda}{3}$, i.e., $E_0 \leq \frac{c\nu}{3T}$. Then, update[2] E_0 so that $\mu^2 T^2 E_0^6 \leq \frac{c\nu\Lambda}{3}$ as well. Using the previous bound for E_0, this reduces to $\mu^2 T E_0^5 \leq \Lambda$, i.e., $E_0 \leq \Lambda^{\frac{1}{5}}\mu^{-\frac{2}{5}}T^{-\frac{1}{5}}$. Finally, choose the initial wavefunction to satisfy $\|\psi_0\|^2_{H^{s+1}_x} \leq \frac{c\nu\Lambda}{3}$.

Now, we will move on to (3.15). In this case, the analysis for the wavefunction is very similar and we will not discuss it. For the velocity, we follow a similar approach as well, and make use of (2.10). After some simplification, we arrive at:

$$\left[\|u_0\|^2_{H^s_x} + \frac{TX_0}{\varepsilon}\right]\left[\|\psi_0\|^2_{H^s_x} + (1+\Lambda)TE_0 + \mu^2T^2E_0^6\right] \lesssim \nu\Lambda \tag{3.20}$$

where $X_0 = 1 + \nu\|\nabla u_0\|^2_{L^2_x} + \|\Delta\psi_0\|^2_{L^2_x} \leq 1 + \nu\|u_0\|^2_{H^s_x} + \|\psi_0\|^2_{H^{s+1}_x}$.

The second term on the LHS is already controlled by (3.19). Since $X_0 \geq 1$, the only way we can control the first term on the LHS is by choosing a T small enough that

$$\|u_0\|^2_{H^s_x} + \frac{T}{\varepsilon}(1 + \nu\|u_0\|^2_{H^s_x} + \|\psi_0\|^2_{H^{s+1}_x}) \lesssim 1$$

Since $\|\psi_0\|^2_{H^{s+1}_x} \lesssim \nu\Lambda$, we choose $\|u_0\|_{H^s_x} \lesssim \min\{1, \nu^{-1}\}$, and then, we pick $T \lesssim \frac{\varepsilon}{1+\nu\Lambda}$. The other possibility, of course, is that for a given E_0, we simply select T small enough (along with the appropriate smallness of $\|u_0\|_{H^s_x}$ and $\|\psi_0\|_{H^{s+1}_x}$) to satisfy (3.14) and (3.15).

Having discussed the smallness assumption, we now proceed to add (3.8), (3.11) and (3.13). By choosing κ sufficiently small, we can ensure all the dissipation terms are absorbed by the LHS. What remains is:

$$\frac{d}{dt}\left[\|\varphi\|^2_{L^2_x} + \|\Phi\|^2_{L^2_x} + \|\sigma\|^2_{L^2_x}\right] \lesssim (h_1+h_2+h_3)(t)\left[\|\varphi\|^2_{L^2_x} + \|\Phi\|^2_{L^2_x} + \|\sigma\|^2_{L^2_x}\right] \tag{3.21}$$

Using Grönwall's inequality, and the fact that $\|\varphi\|_{L^2_x} = \|\Phi\|_{L^2_x} = \|\sigma\|_{L^2_x} = 0$ at $t = 0$ completes the proof of Theorem 2.4.

□

[2]Recall that E_0 depends on μ, so it is not correct to infer a dependence of E_0 bound on μ.

4. Weak-strong uniqueness (Proof of Theorem 2.3)

We shall now work with a pair of solutions, one weak and the other strong (in the sense indicated in (2.13)). The approach makes use of the energy structure of the system and is motivated by the classical results of Section 2.5 in [Lio96]. We will begin with the wavefunction, by acting with the gradient operator on (NLS) for each of the two solutions ψ and $\tilde{\psi}$. Then, we will subtract the equations and integrate against $\nabla(\psi-\tilde{\psi})$. The real part of the resulting equation is given below. The analysis is a combination of the calculations in Section 3.2 of [JT21] and Section 3.1 above, in that we will be looking at the energy (H^1_x norm) difference between the two wavefunctions.

$$\frac{d}{dt}\frac{1}{2}\left\|\nabla(\psi-\tilde{\psi})\right\|^2_{L^2_x} = -\Lambda\operatorname{Re}\int_x \nabla(\psi-\tilde{\psi})\cdot\nabla(B\psi-\tilde{B}\tilde{\psi}) + \mu\operatorname{Im}\int_x \nabla(\psi-\tilde{\psi})\cdot\nabla\left(|\psi|^2\psi-\left|\tilde{\psi}\right|^2\tilde{\psi}\right)$$

Integrating by parts on the RHS, we get a Laplacian term in each of the integrals. In the first integral, we observe that $B\psi-\tilde{B}\tilde{\psi}=-\frac{1}{2}\Delta(\psi-\tilde{\psi})+\ldots$, so that this gives us a "dissipation" term along with other terms. In the second term of the RHS, we extract the $\Delta(\psi-\tilde{\psi})$ in L^2_x, and absorb it into the aforementioned dissipation term. After some manipulation, we arrive at:

$$\left.\begin{array}{r}\frac{d}{dt}\left\|\nabla(\psi-\tilde{\psi})\right\|^2_{L^2_x}\\ +\Lambda\left\|D^2(\psi-\tilde{\psi})\right\|^2_{L^2_x}\end{array}\right\}\lesssim \frac{\Lambda}{\varepsilon}\left[\|u+\tilde{u}\|^2_{L^\infty_x}\left\|\tilde{\psi}\right\|^2_{L^\infty_x}+\|\nabla\psi\|^2_{L^\infty_x}\right]\|\sqrt{\rho}(u-\tilde{u})\|^2_{L^2_x}$$
$$+\left[\Lambda\left(\|u\|^4_{L^\infty_x}+\|\tilde{u}\|^2_{L^\infty_x}\right)+ \mu(1+\Lambda)\left(\|\psi\|^4_{L^\infty_x}+\left\|\tilde{\psi}\right\|^4_{L^\infty_x}\right)\right]\left\|\nabla(\psi-\tilde{\psi})\right\|^2_{L^2_x} \tag{4.1}$$

Now, we move on to the momentum equation. The modus operandi here follows Section 2.5 of [Lio96]. In what follows, $\Psi=-2\Lambda u\operatorname{Re}(\overline{\psi}B\psi)$ and $\Psi'=-2\Lambda\operatorname{Im}(\nabla\overline{\psi}B\psi)$. We begin from (2.3), setting $\Phi=\tilde{u}$. This way, we can arrive at an expression for $\int_x \rho u\cdot\tilde{u}$.

$$\int_x \rho u\cdot\tilde{u}+\nu\int_{t,x}\nabla u:\nabla\tilde{u}=\int_x\rho_0|u_0|^2+\int_{t,x}\rho u\cdot(\partial_t\tilde{u}+\tilde{u}\cdot\nabla\tilde{u})+\int_{t,x}\Psi'\cdot\tilde{u} \tag{4.2}$$

We write (NSE') for the strong solution, and rearrange it to get:

$$\rho\partial_t\tilde{u}+\rho u\cdot\nabla\tilde{u}+\nabla\tilde{p}-\nu\Delta\tilde{u}=\tilde{\Psi}+\tilde{\Psi}'+(\rho-\tilde{\rho})\partial_t\tilde{u}+(\rho-\tilde{\rho})\tilde{u}\cdot\nabla\tilde{u}+\rho(u-\tilde{u})\cdot\nabla\tilde{u} \tag{4.3}$$

Multiplying (4.3) by u and integrating,

$$\int_{t,x}\rho u\partial_t\tilde{u}+\int_{t,x}\rho u\otimes u:\nabla\tilde{u}+\nu\int_{t,x}\nabla u:\nabla\tilde{u}=\int_{t,x}\tilde{\Psi}\cdot u+\int_{t,x}\tilde{\Psi}'\cdot u$$
$$+\int_{t,x}(\rho-\tilde{\rho})u\cdot(\partial_t\tilde{u}+\tilde{u}\cdot\nabla\tilde{u})+\int_{t,x}\rho(u-\tilde{u})\otimes u:\nabla\tilde{u} \tag{4.4}$$

Adding (4.2) and (4.4),

$$
\begin{aligned}
\int_x \rho u \cdot \tilde{u} + 2\nu \int_{t,x} \nabla u : \nabla \tilde{u} = \int_x \rho_0 |u_0|^2 + \int_{t,x} (\rho - \tilde{\rho}) u \cdot (\partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u}) \\
+ \int_{t,x} \rho (u - \tilde{u}) \otimes u : \nabla \tilde{u} + \int_{t,x} \left[\Psi' \cdot \tilde{u} + \tilde{\Psi} \cdot u + \tilde{\Psi}' \cdot u \right]
\end{aligned}
\tag{4.5}
$$

Next, we take the inner product of both (NSE) and (NSE') with u, use incompressibility, and add them, to arrive at the energy equation for the normal fluid alone.

$$
\frac{d}{dt} \frac{1}{2} \|\sqrt{\rho} u\|_{L^2_x}^2 + \nu \|\nabla u\|_{L^2_x}^2 = -2\Lambda \int_\Omega u \cdot \operatorname{Im}(\nabla \bar{\psi} B \psi) - \Lambda \int_\Omega |u|^2 \operatorname{Re}(\bar{\psi} B \psi) \tag{4.6}
$$

This obviously holds as an a priori estimate, valid for the approximate fields (ψ^N, u^N, ρ^N), but given the regularity of the solution, we can easily pass to the limit to see that it is accurate for the weak solutions too. From this equation, integrating on $[0,t]$ for $0 \le t \le T$, we can obtain an equation for $\int_x \rho u \cdot u$.

$$
\frac{1}{2} \int_x \rho |u|^2 + \nu \int_{t,x} |\nabla u|^2 = \frac{1}{2} \int_x \rho_0 |u_0|^2 + \int_{t,x} \left[\frac{1}{2} \Psi \cdot u + \Psi' \cdot u \right] \tag{4.7}
$$

Finally, multiply (4.3) by $\tilde{u}$ and integrate over $[0,t] \times \Omega$. Since $\tilde{u}$ is a strong solution, $\frac{1}{2}|\tilde{u}|^2$ can work as a test function. Therefore, we use (2.4) to simplify, and the resulting equation is:

$$
\begin{aligned}
\frac{1}{2} \int_x \rho |\tilde{u}|^2 + \nu \int_{t,x} |\nabla \tilde{u}|^2 = \frac{1}{2} \int_x \rho_0 |u_0|^2 + \int_{t,x} (\rho - \tilde{\rho}) \tilde{u} \cdot (\partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u}) \\
+ \int_{t,x} \rho (u - \tilde{u}) \otimes \tilde{u} : \nabla \tilde{u} + \int_{t,x} \left[\tilde{\Psi} + \tilde{\Psi}' \right] \cdot \tilde{u} + \Lambda \operatorname{Re} \int_{t,x} \overline{\psi} B \psi |\tilde{u}|^2
\end{aligned}
\tag{4.8}
$$

Add (4.7) and (4.8), and subtract (4.5). Then, differentiate the result with respect to t.

$$
\begin{aligned}
\frac{d}{dt} \frac{1}{2} \|\sqrt{\rho}(u - \tilde{u})\|_{L^2_x}^2 + \nu \|\nabla (u - \tilde{u})\|_{L^2_x}^2 = -\int_x (\rho - \tilde{\rho})(u - \tilde{u}) \cdot (\partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u}) \\
- \int_x \rho (u - \tilde{u}) \otimes (u - \tilde{u}) : \nabla \tilde{u} - \Lambda \operatorname{Re} \int_x (\overline{\psi} B \psi - \overline{\tilde{\psi}} \tilde{B} \tilde{\psi})(|u|^2 - |\tilde{u}|^2) \\
- \Lambda \operatorname{Re} \int_x \overline{\tilde{\psi}} \tilde{B} \tilde{\psi} |u - \tilde{u}|^2 - 2\Lambda \operatorname{Im} \int_x (\nabla \overline{\psi} B \psi - \nabla \overline{\tilde{\psi}} \tilde{B} \tilde{\psi}) \cdot (u - \tilde{u})
\end{aligned}
\tag{4.9}
$$

Reduction to a form that is amenable to the Grönwall's inequality can be achieved using Hölder's and Young's inequalities to extract quadratic terms of interest. The only detail worth mentioning here is for the term

$$
\int_x (\overline{\psi} B \psi - \overline{\tilde{\psi}} \tilde{B} \tilde{\psi})(|u|^2 - |\tilde{u}|^2) \le \left\| \overline{\psi} B \psi - \overline{\tilde{\psi}} \tilde{B} \tilde{\psi} \right\|_{L^2_x} \|u + \tilde{u}\|_{L^\infty_x} \frac{1}{\sqrt{\varepsilon}} \|\sqrt{\rho}(u - \tilde{u})\|_{L^2_x}
$$

The wavefunction part[3] on the RHS is split as:

$$
\left\| \overline{\psi} B \psi - \overline{\tilde{\psi}} \tilde{B} \tilde{\psi} \right\|_{L^2_x} \le \left\| (\overline{\psi} - \overline{\tilde{\psi}}) B \psi \right\|_{L^2_x} + \left\| \overline{\tilde{\psi}} (B - \tilde{B}) \psi \right\|_{L^2_x} + \left\| \overline{\tilde{\psi}} \tilde{B} (\psi - \tilde{\psi}) \right\|_{L^2_x}
$$

[3]The same argument also works for the final term on the RHS of (4.9).

The first term is easy to handle since $B\psi \in L^2_t L^\infty_x$. The second term is treated using (3.7), except that the difference in velocities is now replaced by $\frac{\sqrt{\rho}}{\sqrt{\varepsilon}}(u-\tilde{u})$. For the final term, we can factor out a dissipative contribution, since $\tilde{B}(\psi-\tilde{\psi}) = -\frac{1}{2}\Delta(\psi-\tilde{\psi}) + \dots$

Therefore, the final contribution from the momentum equation can be summarized thusly.

$$\left.\begin{aligned} &\tfrac{d}{dt}\left\|\sqrt{\rho}(u-\tilde{u})\right\|^2_{L^2_x} \\ &+\nu\|\nabla(u-\tilde{u})\|^2_{L^2_x} \end{aligned}\right\} \lesssim \left[\|\partial_t \tilde{u}\|^2_{L^\infty_x} + \|\tilde{u}\|^2_{L^\infty_x}\|\nabla\tilde{u}\|^2_{L^\infty_x}\right]\|\rho-\tilde{\rho}\|^2_{L^2_x}$$
$$\begin{aligned} &+\Bigg[\max\{\kappa^{-1},1\}\frac{\Lambda}{\varepsilon}\|u+\tilde{u}\|^2_{L^\infty_x}\left(1+\|\psi\|^2_{L^\infty_x}+\left\|\tilde{\psi}\right\|^2_{L^\infty_x}\right) \\ &\qquad +\max\{\kappa^{-1},1\}\frac{\Lambda}{\varepsilon}\left(\|\nabla\psi\|^2_{L^\infty_x}+\left\|\nabla\tilde{\psi}\right\|^2_{L^\infty_x}\right) \\ &\quad +\|\nabla\tilde{u}\|_{L^\infty_x}+\frac{1+\Lambda}{\varepsilon}+\frac{\Lambda}{\varepsilon}\left\|\tilde{\psi}\right\|_{L^\infty_x}\left\|\tilde{B}\tilde{\psi}\right\|_{L^\infty_x}\Bigg]\left\|\sqrt{\rho}(u-\tilde{u})\right\|^2_{L^2_x} \\ &+\Lambda\Bigg[\mu^2\|\psi\|^2_{L^\infty_x}\left(1+\|\psi\|^2_{L^\infty_x}\right)+\mu^2\left\|\tilde{\psi}\right\|^2_{L^\infty_x}\left(1+\left\|\tilde{\psi}\right\|^2_{L^\infty_x}\right) \\ &\qquad +\|\tilde{u}\|^2_{L^\infty_x}\left(1+\|\tilde{u}\|^2_{L^\infty_x}\right)+\|B\psi\|^2_{L^\infty_x}\Bigg]\left\|\nabla(\psi-\tilde{\psi})\right\|^2_{L^2_x} \\ &+\kappa\nu\|\nabla(u-\tilde{u})\|^2_{L^2_x}+\kappa\Lambda\left\|D^2(\psi-\tilde{\psi})\right\|^2_{L^2_x} \end{aligned} \tag{4.10}$$

Finally, in the case of the continuity equation, we take the difference of (CON) written for each of the solutions, multiply by $\rho-\tilde{\rho}$ and integrate over Ω. Recall that we have established the weak solution $\rho \in C([0,T];L^2_x)$, so all the forthcoming manipulations are allowed.

$$\frac{d}{dt}\frac{1}{2}\|\rho-\tilde{\rho}\|^2_{L^2_x} = -\int_x \rho(u-\tilde{u})\cdot\nabla\tilde{\rho} + 2\Lambda\,\mathrm{Re}\int_x(\rho-\tilde{\rho})\left(\overline{\psi}B\psi-\overline{\tilde{\psi}}\tilde{B}\tilde{\psi}\right)$$

In the first term on the RHS, we can replace ρ by $(\rho-\tilde{\rho})$, because the velocities u and $\tilde{u}$ are divergence-free. Then,

$$\frac{d}{dt}\|\rho-\tilde{\rho}\|^2_{L^2_x} \lesssim \|\rho-\tilde{\rho}\|_{L^2_x}\|u-\tilde{u}\|_{L^6_x}\|\nabla\tilde{\rho}\|_{L^3_x} + \Lambda\|\rho-\tilde{\rho}\|_{L^2_x}\left\|\overline{\psi}B\psi-\overline{\tilde{\psi}}\tilde{B}\tilde{\psi}\right\|_{L^2_x}$$

The handling of the last term on the RHS was already discussed above in the context of the momentum equation. So, we end up with:

$$
\begin{aligned}
&\frac{d}{dt}\|\rho-\tilde{\rho}\|_{L_x^2}^2 \lesssim \left[(\kappa\nu)^{-1}\|\tilde{\rho}\|_{W_x^{1,3}}^2+\kappa^{-1}\Lambda+\Lambda\right]\|\rho-\tilde{\rho}\|_{L_x^2}^2 \\
&+\Lambda\left[\|B\psi\|_{L_x^\infty}^2+\left\|\tilde{\psi}\right\|_{L_x^\infty}^2\|\tilde{u}\|_{L_x^\infty}^2\left(1+\|\tilde{u}\|_{L_x^\infty}^2\right)+\mu^2\left(\|\psi\|_{L_x^\infty}^6+\left\|\tilde{\psi}\right\|_{L_x^\infty}^6\right)\right]\left\|\nabla(\psi-\tilde{\psi})\right\|_{L_x^2}^2 \\
&\qquad+\frac{\Lambda}{\varepsilon}\left\|\tilde{\psi}\right\|_{L_x^\infty}^2\left[\|u+\tilde{u}\|_{L_x^\infty}^2\|\psi\|_{L_x^\infty}^2+\|\nabla\psi\|_{L_x^\infty}^2\right]\|\sqrt{\rho}(u-\tilde{u})\|_{L_x^2}^2 \\
&\qquad\qquad+\kappa\nu\|\nabla(u-\tilde{u})\|_{L_x^2}^2+\kappa\Lambda\left\|D^2(\psi-\tilde{\psi})\right\|_{L_x^2}^2
\end{aligned} \tag{4.11}
$$

The last step is to add (4.1), (4.10) and (4.11), and choose κ small enough that the dissipation terms on the RHS are absorbed into the LHS. This leaves us with:

$$
\begin{aligned}
&\frac{d}{dt}\left[\left\|\nabla(\psi-\tilde{\psi})\right\|_{L_x^2}^2+\|\sqrt{\rho}(u-\tilde{u})\|_{L_x^2}^2+\|\rho-\tilde{\rho}\|_{L_x^2}^2\right] \\
&\qquad\leq H(t)\left[\left\|\nabla(\psi-\tilde{\psi})\right\|_{L_x^2}^2+\|\sqrt{\rho}(u-\tilde{u})\|_{L_x^2}^2+\|\rho-\tilde{\rho}\|_{L_x^2}^2\right]
\end{aligned} \tag{4.12}
$$

where $H(t)$ is integrable on $[0,T]$. The weak-strong uniqueness thus follows from Grönwall's inequality, because both the solutions originate from the same initial conditions.

□

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