TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 376, Number 11, November 2023, Pages 7651–7689
https://doi.org/10.1090/tran/8946
Article electronically published on August 17, 2023

ON THE SMALL NOISE LIMIT IN THE SMOLUCHOWSKI-KRAMERS APPROXIMATION OF NONLINEAR WAVE EQUATIONS WITH VARIABLE FRICTION

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ABSTRACT. We study the validity of a large deviation principle for a class of stochastic nonlinear damped wave equations, including equations of Klein-Gordon type, in the joint small mass and small noise limit. The friction term is assumed to be state dependent. We also provide the proof of the Smolchowski-Kramers approximation for the case of variable friction, non-Lipschitz nonlinear term and unbounded diffusion.

1. INTRODUCTION

In this article we deal with this class of stochastic wave equations with state-dependent damping on a bounded smooth domain $\mathcal{O} \subset \mathbb{R}^d$

$$
\begin{cases} \mu\partial_t^2 u_\mu(t,x) = \Delta u_\mu(t,x) - \gamma(u_\mu(t,x))\partial_t u_\mu(t,x) + f(x, u_\mu(t,x)) \\ \qquad\qquad + \sigma(u_\mu(t,\cdot))\partial_t w^Q(t,x), \\ \\ u_\mu(0,x) = u_0(x), \quad \partial_t u_\mu(0,x) = v_0(x), \qquad u_\mu(t,x) = 0, \quad x \in \partial\mathcal{O}, \end{cases} \tag{1.1}
$$

depending on a parameter $0 < \mu << 1$. Here the friction coefficient γ is strictly positive and bounded and the nonlinearity f is either a Lipschitz-continuous function (in this case we can consider any $d \geq 1$) or a locally Lipschitz-continuous function of the Klein-Gordon type (in this case we can only take $d = 1$). The noise $w^Q(t)$ is a cylindrical Q-Wiener process and σ is a suitable Lipschitz-continuous operator-valued function.

The solution u_μ of equation (1.1) can be seen as the displacement field of some particles in a domain $\mathcal{O}$, subject to interaction forces represented by the Laplacian and to nonlinear reactions represented by f, in the presence of a random external forcing $\sigma(u_\mu(t,\cdot))\partial_t w^Q(t)$ and a state-dependent friction $\gamma(u_\mu(t))\partial_t u_\mu(t)$. A series of papers has investigated the validity of the so-called Smoluchowski-Kramers approximation that describes the limiting behavior of the solution u_μ, as the density μ of the particles vanishes (see [24] and [33]). For the finite dimensional case, the existing literature is quite broad and we refer in particular to [15], [16], [20], [21] and [34] (see also [2], [6], [13] and [25] for systems subject to a magnetic field or

Received by the editors September 10, 2022, and, in revised form, January 19, 2023.
2020 *Mathematics Subject Classification.* Primary 60H15, 60F10, 35R60, 35L15.
Key words and phrases. Smoluchowski-Kramers approximation, large deviations, stochastic nonlinear damped wave equations.
The first author was partially supported by NSF grants DMS-1712934 - *Analysis of Stochastic Partial Differential Equations with Multiple Scales* and DMS-1954299 - *Multiscale Analysis of Infinite-Dimensional Stochastic Systems*.

constrained to stay on a manifold and [7], [23] and [30] for some related multiscaling problems).

In recent years there has been an intense activity dealing with the Smoluchowski-Kramers approximation of infinite dimensional systems. To this purpose, we refer to [4], [5], [8], [31] and [26], [27] and [28] for the case of constant damping term (see also [12] where systems subject to a magnetic field are studied), and to [14] for the case of state-dependent damping. As a matter of fact, these two situations are quite different. When γ is constant, u_μ converges to the solution of the stochastic parabolic problem

$$(1.2)\qquad \begin{cases} \gamma\, \partial_t u(t,x) = \Delta u(t,x) + f(x,u(t,x)) + \sigma(u(t,\cdot))\partial_t w^Q(t,x), \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = v_0(x), \quad u_\mu(t,x) = 0, \quad x \in \partial\mathcal{O}. \end{cases}$$

However, when γ is not constant, because of the interplay between the state-dependent friction and the noise, an extra drift is created and in [14] it has been proven that the limiting equation becomes

$$(1.3)\qquad \begin{cases} \gamma(u(t,x))\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x)) \\ \qquad\qquad - \dfrac{\gamma'(u(t,x))}{2\gamma^2(u(t,x))}\displaystyle\sum_{i=1}^\infty |[\sigma(u(t,\cdot))Qe_i](x)|^2 + \sigma(u(t,\cdot))\partial_t w^Q(t,x), \\ u(0,x) = u_0(x), \qquad u(t)_{|\partial\mathcal{O}} = 0, \end{cases}$$

where $\{Qe_i\}_{i\in\mathbb{N}}$ is a complete orthonormal basis of the reproducing kernel of the noise.

Once proved the validity of the small mass limit, it is important to understand how stable such an approximation is with respect to other important asymptotic features of the two systems, such as for example the long time behavior. To this purpose, in [9] and [4] it is shown that the statistically invariant states of equation (1.1) (in case of constant friction) converge in a suitable sense to the invariant measure of equation (1.2). In the same spirit, the two papers [10] and [11] are devoted to an analysis of the convergence of the quasi-potential that describes, as known, the asymptotics of the exit times and the large deviation principle for the invariant measure.

In the present paper we are interested in studying the validity of a large deviation principle for the following equation

$$(1.4)\qquad \begin{cases} \mu\partial_t^2 u_\mu(t,x) = \Delta u_\mu(t,x) - \gamma(u_\mu(t,x))\partial_t u_\mu(t,x) \\ \qquad\qquad + f(x,u_\mu(t,x)) + \sqrt{\mu}\,\sigma(u_\mu(t,\cdot))\partial_t w^Q(t,x), \\ u_\mu(0,x) = u_0(x), \quad \partial_t u_\mu(0,x) = v_0(x), \quad u_\mu(t,x) = 0, \quad x \in \partial\mathcal{O}, \end{cases}$$

where, together with the mass, we are also assuming that the intensity of the noise vanishes. Our aim is proving that in the joint small mass and small noise limit the family of random variables $\{u_\mu\}_{\mu>0}$ satisfies a large deviation principle in the space $C([0,T];L^p(\mathcal{O}))$ (for some $p > 2$ depending on the dimension d), with respect to the action functional

$$I_T(u) = \frac{1}{2}\left\{\int_0^T \|\varphi(t)\|_H^2\,dt \;:\; u(t) = u^\varphi(t),\ t\in[0,T]\right\},$$

where $u^\varphi(t)$ denotes the solution of the controlled quasi-linear parabolic equation

$$\text{(1.5)}\qquad \begin{cases}\gamma(u(t,x))\partial_t u(t,x) = \Delta u(t,x) + f(x,u(t,x)) + \sigma(u(t,\cdot))\varphi(t,x),\\ u(0,x) = u_0(x), \qquad u(t,x) = 0, \quad x \in \partial\mathcal{O}.\end{cases}$$

This means in particular that, in spite of the fact that in the presence of a non-constant friction coefficient the Smoluchowski-Kramers approximation of equation (1.1) leads to equation (1.3), the large deviation principle is consistent with equation (1.2).

The small parameter we are taking in front of the noise in equation (1.4) is $\sqrt{\mu}$. However, we would like to point out that this is done only for simplicity sake. In fact, $\sqrt{\mu}$ could be replaced by any other positive function $\alpha(\mu)$ such that

$$\lim_{\mu\to 0}\alpha(\mu) = 0, \qquad \lim_{\mu\to 0}\frac{\alpha(\mu)}{\sqrt{\mu}} < \infty,$$

and in this case the speed of the large deviation principle would be $\alpha^2(\mu)$, instead of μ.

Due to the nature of our problem, the weak-convergence approach to large deviation, as developed in [3] for SPDEs, is the ideal tool for our proof. As known, such an approach requires a thorough analysis of the following controlled version of equation (1.4)

(1.6)

$$\begin{cases}\mu\partial_t^2 u_\mu(t,x) = \Delta u_\mu(t,x) - \gamma(u_\mu(t,x))\partial_t u_\mu(t,x) + f(x,u_\mu(t,x)) + \sigma(u_\mu(t,\cdot))Q\varphi(t,x)\\ \qquad\qquad +\sqrt{\mu}\,\sigma(u_\mu(t,\cdot))\partial_t w^Q(t,x), \quad t>0, \qquad x\in\mathcal{O}.\\ u_\mu(0,x) = u_0(x), \quad \partial_t u_\mu(0,x) = v_0(x), \qquad u_\mu(t,x) = 0, \quad x\in\partial\mathcal{O},\end{cases}$$

In [14], only the case of Lipschitz f and bounded σ is considered. However, in relevant models it is important to consider non-Lipschitz nonlinearities. For this reason, in this paper we are considering also nonlinearities f having polynomial growth and satisfying some monotonicity conditions. We would like to stress that this, together with the fact that we allow the diffusion coefficients σ to have linear growth and we have to add a control in equation (1.6), has required the introduction of new arguments, compared with [14], already in the proof of the well-posedness result.

After we have shown that equation (1.6) admits a unique solution u_μ^φ, for every fixed $\mu > 0$ and for every predictable control, we have proven suitable a priori bounds for such solution and its time derivative. Then, we have introduced $\rho_\mu := g(u_\mu^{\varphi_\mu})$, where $g' = \gamma$ and $\{\varphi_\mu\}_{\mu>0}$ is a family of controls all contained $\mathbb{P}$-a.s. in a ball of $L^2([0,T];L^2(\mathcal{O}))$, and we have shown that these estimates imply the tightness of the family $\{\rho_\mu\}_{\mu\in(0,\mu_T)}$ in $C([0,T];H^\delta)$, for some $\mu_T > 0$ and for every $\delta < 1$.

Next, we have shown how, for every sequence $\{\mu_k\}_{k\in\mathbb{N}}$ converging to zero, every limit point ρ of $\{\rho_{\mu_k}\}_{k\in\mathbb{N}}$ is a weak solution of the deterministic controlled problem

$$\begin{cases}\partial_t\rho(t,x) = \operatorname{div}\left[b(\rho(t,x))\nabla\rho(t,x)\right] + f_g(x,\rho(t,x)) + \sigma_g(\rho(t,\cdot))\varphi(t,x),\\ \qquad\qquad\qquad t>0, \qquad x\in\mathcal{O},\\ \rho(0,x) = g(u_0(x)), \qquad \rho(t,x) = 0, \quad x\in\partial\mathcal{O},\end{cases}$$

where $b = 1/\gamma \circ g^{-1}$, $f_g = f \circ g^{-1}$, and $\sigma_g = \sigma \circ g^{-1}$. In order to identify uniquely the limit point and prove that $\{\rho_{\mu_k}\}_{k\in\mathbb{N}}$ converges to ρ, we had first to prove that the equation above has a unique solution. Then, by defining $u := g^{-1}(\rho)$, we have obtained the convergence of $u_\mu^{\varphi_\mu}$ to the solution of the controlled equation (1.6) and this has allowed us to conclude our proof.

Finally, we would like to mention that in Appendix A we have extended the results of [14] and provided a proof of the validity of the Smoluchowski-Kramers approximation for quasi-monotone f having polynomial growth and unbounded diffusion σ (see Hypothesis 4). This has required the proof of quite nontrivial a priori bounds for the solution u_μ and its time derivative $\partial_t u_\mu$, and the introduction of suitable functional spaces where tightness holds and the small-mass limit can be proven.

2. Notations and assumptions

Throughout the present paper $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$, with smooth boundary. We denote by H the Hilbert space $L^2(\mathcal{O})$ and by $\langle\cdot,\cdot\rangle_H$ the corresponding inner product. H^1 is the completion of $C_0^\infty(\mathcal{O})$ with respect to norm

$$\|u\|_{H^1}^2 := \|\nabla u\|_H^2 = \int_{\mathcal{O}} |\nabla u(x)|^2 dx,$$

and H^{-1} is the dual space to H^1. Then H^1, H and H^{-1} are all complete separable metric spaces, and $H^1 \subset H \subset H^{-1}$, with compact embeddings. In what follows, we shall denote

$$\mathcal{H} = H \times H^{-1}, \qquad \mathcal{H}_1 = H^1 \times H.$$

Given the domain $\mathcal{O}$, we denote by $\{e_i\}_{i\in\mathbb{N}} \subset H^1$ the complete orthonormal basis of H which diagonalizes the Laplacian Δ, endowed with Dirichlet boundary conditions on $\partial\mathcal{O}$. Moreover, we denote by $\{-\alpha_i\}_{i\in\mathbb{N}}$ the corresponding sequence of eigenvalues, i.e.

$$\Delta e_i = -\alpha_i e_i, \qquad i \in \mathbb{N}.$$

Next, for every $\delta \in \mathbb{R}$, we denote by H^δ the completion of $C_0^\infty(\mathcal{O})$ with respect to the norm

$$\|u\|_{H^\delta}^2 := \sum_{i=1}^\infty \alpha_i^\delta \langle h, e_i\rangle_H^2.$$

2.1. **The stochastic term.** We assume that $w^Q(t)$ is a cylindrical Q-Wiener process, defined on a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. This means that $w^Q(t)$ can be formally written as

$$w^Q(t) = \sum_{i=1}^\infty Q e_i \beta_i(t),$$

where $\{\beta_i\}_{i\in\mathbb{N}}$ is a sequence of independent standard Brownian motions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, $\{e_i\}_{i\in\mathbb{N}}$ is the complete orthonormal system introduced above that diagonalizes the Laplace operator, endowed with Dirichlet boundary conditions, and $Q : H \to H$ is a bounded linear operator,. When $Q = I$, $w^I(t)$ will be denoted by $w(t)$. In particular, we have $w^Q(t) = Qw(t)$.

In what follows we shall denote by H_Q the set $Q(H)$. H_Q is the reproducing kernel of the noise w^Q and is a Hilbert space, endowed with the inner product

$$\langle Qh, Qk\rangle_{H_Q} = \langle h, k\rangle_H, \qquad h, k \in H.$$

Notice that the sequence $\{Qe_i\}_{i\in\mathbb{N}}$ is a complete orthonormal system in H_Q. Moreover, if U is any Hilbert space containing H_Q such that the embedding of H_Q into U is Hilbert-Schmidt, we have that

$$w^Q \in C([0,T];U). \tag{2.1}$$

Next, we recall that for every two separable Hilbert spaces E and F, $\mathcal{L}_2(E,F)$ denotes the space of Hilbert-Schmidt operators from E into F. $\mathcal{L}_2(E,F)$ is a Hilbert space, endowed with the inner product

$$\langle A,B\rangle_{\mathcal{L}_2(E,F)} = \mathrm{Tr}_E\,[A^\star B] = \mathrm{Tr}_F[BA^\star].$$

As well known, $\mathcal{L}_2(E,F)\subset\mathcal{L}(E,F)$ and

$$\|A\|_{\mathcal{L}(E,F)} \leq \|A\|_{\mathcal{L}_2(E,F)}. \tag{2.2}$$

Hypothesis 1. *The mapping $\sigma : H\to\mathcal{L}_2(H_Q,H)$ is defined by*

$$[\sigma(h)Qe_i](x) = \sigma_i(x,h(x)),\qquad x\in\mathcal{O}\qquad h\in H,\qquad i\in\mathbb{N},$$

for some mapping $\sigma_i : \mathcal{O}\times\mathbb{R}\to\mathbb{R}$. We assume that there exists $L>0$ such that

$$\sup_{x\in\mathcal{O}}\sum_{i=1}^{\infty}|\sigma_i(x,y_1)-\sigma_i(x,y_2)|^2 \leq L\,|y_1-y_2|^2,\qquad y_1,y_2\in\mathbb{R}. \tag{2.3}$$

Moreover,

$$\sup_{x\in\mathcal{O}}\sum_{i=1}^{\infty}|\sigma_i(x,0)|^2 =: \sigma_0^2 < \infty. \tag{2.4}$$

Remark 2.1.

(1) Condition (2.3) implies that σ is Lipschitz continuous. Namely for any $h_1,h_2\in H$

$$\|\sigma(h_1)-\sigma(h_2)\|_{\mathcal{L}_2(H_Q,H)} \leq \sqrt{L}\,\|h_1-h_2\|_H. \tag{2.5}$$

This, together with condition (2.4), implies also that σ has linear growth, that is

$$\|\sigma(h)\|_{\mathcal{L}_2(H_Q,H)} \leq \sqrt{L}\,\|h\|_H + |\mathcal{O}|^{1/2}\sigma_0. \tag{2.6}$$

(2) If σ is constant, then Hypothesis 1 means that σQ is a Hilbert-Schmidt operator in H.

(3) If σ is not constant, Hypothesis 1 is satisfied for example when

$$[\sigma(h)Qk](x) = s(x,h(x))Qk(x),\qquad x\in\mathcal{O},\qquad h,k\in H,$$

for some measurable function $s : \mathcal{O}\times\mathbb{R}\to\mathbb{R}$ such that $s(x,\cdot):\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous, uniformly with respect to $x\in\mathcal{O}$, and for some $Q\in\mathcal{L}(H)$ such that

$$\sum_{i=1}^{\infty}\|Qe_i\|^2_{L^\infty(\mathcal{O})} < \infty. \tag{2.7}$$

In case Q is diagonalizable with respect to the basis $(e_i)_{i\in\mathbb{N}}$, with $Qe_i = \lambda_i e_i$, condition (2.7) reads

$$\sum_{i=1}^{\infty}\lambda_i^2\|e_i\|^2_{L^\infty(\mathcal{O})} < \infty. \tag{2.8}$$

In general (see [18]), we have

$$\|e_i\|_{L^\infty(\mathcal{O})} \leq c\, i^\alpha,$$

for some $\alpha > 0$, and (2.8) becomes

$$\sum_{i=1}^\infty \lambda_i^2\, i^{2\alpha} < \infty.$$

In particular, when $d = 1$ or the domain is a hyperrectangle when $d > 1$ the eigenfunctions $(e_i)_{i\in\mathbb{N}}$ are equi-bounded and (2.8) becomes $Q \in \mathcal{L}_2(H)$.

2.2. **The coefficients γ and f.** Throughout the paper, we shall assume that the friction coefficient satisfies the following condition

Hypothesis 2. *The mapping γ belongs to $C^1_b(\mathbb{R})$ and there exist γ_0 and γ_1 such that*

$$0 < \gamma_0 \leq \gamma(r) \leq \gamma_1, \qquad r \in \mathbb{R}. \tag{2.9}$$

In what follows, we shall define

$$g(r) = \int_0^r \gamma(\sigma)\, d\sigma, \qquad r \in \mathbb{R}.$$

Remark 2.2.

(1) Clearly $g(0) = 0$ and $g'(r) = \gamma(r)$. In particular, due to (2.9), g is uniformly Lipschitz continuous on $\mathbb{R}$.

(2) The function g is strictly increasing and

$$(g(r_1) - g(r_2))\,(r_1 - r_2) \geq \gamma_0\, |r_1 - r_2|^2, \qquad r_1, r_2 \in \mathbb{R}.$$

As far as the nonlinearity f is concerned, in this paper we shall consider two situations: f is Lipschitz continuous and $\mathcal{O}$ is a bounded smooth domain in $\mathbb{R}^d$, for any arbitrary $d \geq 1$, or f is only locally Lipschitz continuous with polynomial growth and $\mathcal{O}$ is a bounded interval in $\mathbb{R}$.

Hypothesis 3. *The mapping $f : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is measurable and there exists $c > 0$ such that*

$$\sup_{x\in\mathcal{O}} |f(x,r) - f(x,s)| \leq c\,|r - s|, \qquad r, s \in \mathbb{R}.$$

Moreover

$$\sup_{x\in\mathcal{O}} |f(x,0)| < \infty.$$

In what follows, for every function $u : \mathcal{O} \to \mathbb{R}$, we shall denote

$$F(u)(x) = f(x, u(x)), \qquad x \in \mathcal{O}.$$

Hypothesis 4. *We have $\mathcal{O} = [0, L]$ and the mapping $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ is measurable and satisfies the following conditions.*

(1) *There exist $\theta > 1$ and $c_1 > 0$ such that for every $r \in \mathbb{R}$*

$$\sup_{x\in[0,L]} |f(x,r)| \leq c_1 \left(1 + |r|^\theta\right), \qquad \sup_{x\in[0,L]} |\partial_r f(x,r)| \leq c_1 \left(1 + |r|^{\theta-1}\right). \tag{2.10}$$

Moreover, there exists $c_2 > 0$ such that for every $r \in \mathbb{R}$ and $x \in [0, L]$

$$\mathfrak{f}(x,r) := \int_0^r f(x,s)\, ds \leq c_2 \left(1 - |r|^{\theta+1}\right). \tag{2.11}$$

(2) *For every* $x \in [0, L]$*, the function* $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ *is differentiable and*

$$\sup_{(x,r)\in[0,L]\times\mathbb{R}} \partial_r f(x, r) \leq 0. \tag{2.12}$$

(3) *For every* $r \in \mathbb{R}$*, the function* $f(\cdot, r) : [0, L] \to \mathbb{R}$ *is differentiable and*

$$\sup_{x\in[0,L]} |\partial_x f(x, r)| \leq c\,(1 + |r|)\,, \qquad r \in \mathbb{R}.$$

Remark 2.3.

(1) A typical example of a function f satisfying Hypothesis 4 is

$$f(r) = -a\,|r|^{\theta-1}r.$$

(2) When $d = 1$, we have that $H^1 \hookrightarrow L^\infty(\mathcal{O})$, and then we get the fundamental fact that $F(u) \in H$, for every $u \in H^1$.

(3) In the existing literature the well-posedness of stochastic semi-linear wave equations having polynomial nonlinearities is not restricted to space dimension $d = 1$. However, here the presence of a state-dependent friction coefficient, and the fact that we are not just interested in the well-posedness of equation (1.1), but also in the small mass and small noise limits, makes our analysis more complicated and we can only handle the case of space dimension $d = 1$.

(4) We are assuming (2.12) just for the sake of simplicity. In fact, our results remain true under the condition

$$\sup_{(x,r)\in[0,L]\times\mathbb{R}} \partial_r f(x, r) < \infty.$$

(5) From (2.11) and (2.12), it is not hard to show that for every $r \in \mathbb{R}$

$$\sup_{x\in[0,L]} r f(x, r) \leq c_2\Big(1 - |r|^{\theta+1}\Big). \tag{2.13}$$

Indeed, if we consider the function

$$G(x, r) := \mathfrak{f}(x, r) - r f(x, r), \qquad r \in \mathbb{R},\ x \in [0, L],$$

then for every $x \in [0, L]$, $\partial_r G(x, r) = -r\partial_r f(x, r) \geq 0$ if $r > 0$, and $\partial_r G(x, r) \leq 0$ if $r < 0$. Note that $G(x, 0) = 0$, we have $G(x, r) \geq 0$, and thus (2.13) follows from (2.11).

(6) Thanks to (2.12) we have

$$\langle F(u) - F(v), u - v\rangle_H \leq 0, \qquad u, v \in H^1.$$

In particular, there exists some $c > 0$ such that

$$\langle F(u), u\rangle_H \leq c\,\|u\|_H, \qquad u \in H^1. \tag{2.14}$$

(7) Due to (2.10), for every $u, v \in H^1$, we have

$$\|F(u) - F(v)\|_H^2 \leq c \int_{\mathcal{O}} \Big(1 + |u(x)|^{2(\theta-1)} + |v(x)|^{2(\theta-1)}\Big)\, |u(x) - v(x)|^2\, dx,$$

so that

$$\|F(u) - F(v)\|_H \leq c\,\big(1 + \|u\|_{H^1}^{\theta-1} + \|v\|_{H^1}^{\theta-1}\big)\, \|u - v\|_H. \tag{2.15}$$

In particular, we have

$$\|F(u)\|_H \leq c\,\big(1 + \|u\|_{H^1}^{\theta}\big)\,, \qquad u \in H^1.$$

(8) In the same way

$$\|F(u)-F(v)\|_{L^1} \leq c\int_{\mathcal{O}} \left(1+|u(x)|^{\theta-1}+|v(x)|^{\theta-1}\right)|u(x)-v(x)|\,dx \\ \leq c\left(1+\|u\|^{\theta-1}_{L^{2(\theta-1)}}+\|v\|^{\theta-1}_{L^{2(\theta-1)}}\right)\|u-v\|_H. \tag{2.16}$$

Now, by proceeding as in [29] (see also [5]), for every $n\in\mathbb{N}$ and $x\in\mathcal{O}$ we define

$$f_n(x,r) := \begin{cases} f(x,n)+(r-n)\partial_r f(x,n), & \text{if } r\geq n,\\ f(x,r), & \text{if } r\in[-n,n],\\ f(x,-n)+(r+n)\partial_r f(x,-n), & \text{if } r\leq -n.\end{cases} \tag{2.17}$$

Clearly, for every $n\in\mathbb{N}$, the mapping $f_n(x,\cdot):\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous, uniformly with respect to $x\in[0,L]$, and

$$f_n(x,r)=f(x,r),\qquad x\in[0,L],\quad r\in[-n,n].$$

Moreover,

$$\sup_{x\in[0,L]}|f_n(x,r)|\leq c\left(1+|r|^\theta\right),\qquad \sup_{x\in[0,L]}|\partial_r f_n(x,r)|\leq c\left(1+|r|^{\theta-1}\right), \tag{2.18}$$

for some constant c independent of n, and there exists $n_0\in\mathbb{N}$ such that for every $n\geq n_0$

$$\mathfrak{f}_n(x,r):=\int_0^r f_n(x,s)\,ds\leq c\left(1-n^{\theta+1}\right),\qquad r\in\mathbb{R},\qquad x\in[0,L]. \tag{2.19}$$

3. The problem and the method

As we mentioned in Section 1, we are interested in the study of the validity of a large deviation principle, as $\mu\downarrow 0$ for the family $\{\mathcal{L}(u_\mu)\}_{\mu>0}$, where u_μ is the solution of equation (1.1). Our final goal is proving the following result.

Theorem 3.1. *Assume Hypotheses* 1 *and* 2 *and either Hypothesis* 3 *or Hypothesis* 4 *and fix $p<\infty$, if $d=1,2$, and $p<2d/(d-2)$, if $d>2$. Then, for every $(u_0,v_0)\in\mathcal{H}_1$ and $T>0$, the family $\{\mathcal{L}(u_\mu)\}_{\mu>0}$ satisfies a large deviation principle in $C([0,T];L^p(\mathcal{O}))$, as $\mu\downarrow 0$, with action functional*

$$I_T(u)=\frac{1}{2}\left\{\int_0^T\|\varphi(t)\|_H^2\,dt\ :\ u(t)=u^\varphi(t),\ t\in[0,T]\right\}, \tag{3.1}$$

where $u^\varphi(t)$ denotes the unique weak solution to the quasi-linear parabolic equation

$$\begin{cases}\partial_t u(t,x)=\gamma^{-1}(u(t,x))\left[\Delta u(t,x)+f(x,u(t,x))+\sigma(u(t,\cdot))\varphi(t,x)\right],\\ u(0,x)=u_0(x),\qquad u(t,x)=0,\qquad x\in\partial\mathcal{O}.\end{cases} \tag{3.2}$$

Theorem 3.1 is proved by following the classical weak convergence approach to large deviations, as developed for SPDEs in [3]. To this purpose, we need first to introduce some notations. For every $T>0$, we denote by $\mathcal{P}_T$ the set of predictable processes in $L^2(\Omega\times[0,T];H)$, and for every $M>0$ we introduce the sets

$$\mathcal{S}_{T,M}:=\left\{\varphi\in L^2_w(0,T;H)\ :\ \|\varphi\|_{L^2([0,T];H)}\leq M\right\},$$

and

$$\Lambda_{T,M}:=\{\varphi\in\mathcal{P}_T\ :\ \varphi\in\mathcal{S}_{T,M},\ \mathbb{P}-\text{a.s.}\}.$$

Next, for every $\varphi \in \mathcal{P}_T$ we consider the controlled version of equation (1.1)

(3.3)
$$\begin{cases} \mu\partial_t^2 u_\mu(t,x) = \Delta u_\mu(t,x) - \gamma(u_\mu(t,x))\partial_t u_\mu(t,x) + f(x,u_\mu(t,x)) \\ \qquad\qquad + \sigma(u_\mu(t,\cdot))Q\varphi(t,x) + \sqrt{\mu}\,\sigma(u_\mu(t,\cdot))\partial_t w^Q(t,x), \quad t>0, \quad x\in\mathcal{O}, \\ u_\mu(0,x) = u_0(x), \quad \partial_t u_\mu(0,x) = v_0(x), \qquad u_\mu(t,x)=0, \quad x\in\partial\mathcal{O}. \end{cases}$$

The well-posedness of the equation above has been proven in [14] in the case the nonlinearity f is Lipschitz-continuous, the diffusion coefficient σ is bounded and the control $\varphi = 0$. In what follows, we will prove that also under the more general conditions we are assuming for f and σ, the following result holds.

Theorem 3.2. *Under Hypotheses* 1 *and* 2 *and either Hypothesis* 3 *or Hypothesis* 4, *for every* $T, M > 0$ *and* $\varphi \in \Lambda_{T,M}$ *and for every initial condition* $(u_0, v_0) \in \mathcal{H}_1$, *there exists a unique adapted process* $(u_\mu, v_\mu) \in L^2(\Omega, C([0,T];\mathcal{H}_1))$ *which solves the systems of equations*

(3.4)
$$\begin{cases} u_\mu(t,x) = u_0(x) + \int_0^t v_\mu(s,x)\,ds, \\ \mu\, v_\mu(t,x) = \mu\, v_0(x) + \displaystyle\int_0^t \left[\Delta u_\mu(s,x) - \gamma(u_\mu(s,x))v_\mu(s,x) + f(x,u_\mu(s,x))\right. \\ \qquad\qquad \left. +\sigma(u_\mu(s))Q\varphi(s,x)\right] ds + \sqrt{\mu}\displaystyle\int_0^t \sigma(u_\mu(s))\,dw^Q(s). \end{cases}$$

Once proved Theorem 3.2, we introduce the following two conditions.

(C1) Let $\{\varphi_\mu\}_{\mu>0}$ be an arbitrary family of processes in $\Lambda_{T,M}$ such that
$$\lim_{\mu\to 0}\varphi_\mu = \varphi, \qquad \text{in distribution in} \quad L^2_w(0,T;H),$$
where $L^2_w(0,T;H)$ is the space $L^2([0,T];H)$ endowed with the weak topology and $\varphi \in \Lambda_{T,M}$. Then, for every $p < \infty$ we have
$$\lim_{\mu\to 0} u_\mu^{\varphi_\mu} = u^\varphi, \qquad \text{weakly in} \quad C([0,T], L^p(\mathcal{O})),$$
where $u_\mu^{\varphi_\mu}$ is the solution to (3.3), corresponding to the control φ_μ, and u^φ is the solution to (3.2), corresponding to the control φ.

(C2) For every $T, R > 0$ and $p < \infty$, the level sets $\Phi_{T,R} = \{I_T \le R\}$ are compact in the space $C([0,T];L^p(\mathcal{O}))$.

As shown in [3], Conditions (C1) and (C2) imply the validity of a Laplace principle with action functional I_T in the space $C([0,T];L^p(\mathcal{O}))$ for the family $\{u_\mu\}_{\mu>0}$. Due to the compactness of the level sets $\Phi_{T,R}$ stated in (C1) this is equivalent to the validity of Theorem 3.1.

Thus, in what follows our strategy will be first proving Theorem 3.2, for every fixed $\mu > 0$, and then proving conditions (C1) and (C2).

4. Well-posedness of equation (3.4)

In Theorem 3.2 the parameter $\mu > 0$ is fixed. This means that in this section, without any loss of generality, we can assume $\mu = 1$. If we denote
$$\eta := v + g(u), \qquad z = (u, \eta),$$

then system (3.4) can be rewritten as the following abstract stochastic evolution equation

$$dz(t) = [A(z(t)) + B_\varphi(t, z(t))]\, dt + \Sigma(z(t))dw^Q(t), \quad z(0) = (u_0, v_0 + g(u_0)), \tag{4.1}$$

where

$$A(u, \eta) = (-g(u) + \eta, \Delta u + F(u)), \qquad (u, \eta) \in D(A) = \mathcal{H}_1,$$
$$B_\varphi(t, (u, \eta)) = (0, \sigma(u)Q\varphi(t)), \qquad (u, \eta) \in \mathcal{H}, \quad t \in [0, T],$$

and

$$\Sigma(u, \eta) = (0, \sigma(u)), \qquad (u, \eta) \in \mathcal{H}.$$

This means that the adapted $\mathcal{H}_1$-valued process $z(t) = (u(t), \eta(t))$ is the unique solution to the equation

$$z(t) = (u_0, v_0 + g(u_0)) + \int_0^t [A(z(s)) + B_\varphi(s, z(s))]\, ds + \int_0^t \Sigma(z(s))\, dw^Q(s), \tag{4.2}$$

if and only if the adapted $\mathcal{H}_1$-valued process $(u(t), v(t)) = (u(t), -g(u(t)) + \eta(t))$ is the unique solution of

$$\begin{cases} u(t, x) = u_0(x) + \displaystyle\int_0^t v(s, x)\, ds, \\ v(t, x) = \mu\, v_0(x) + \displaystyle\int_0^t [\Delta u(s, x) - \gamma(u(s, x))v(s, x) + f(x, u(s, x)) \\ \qquad\qquad +\sigma(u(s, \cdot))Q\varphi(s, x)]\, ds + \displaystyle\int_0^t \sigma(u(s, \cdot))\, dw^Q(s). \end{cases} \tag{4.3}$$

In our proof of Theorem 3.2 we first assume that $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous and then we consider the case Hypothesis 4 holds.

4.1. **The case when f satisfies Hypothesis 3.** In [14, Section 3] an analogous result has been proved, in the case $\varphi = 0$ (and hence $B_\varphi = 0$) and σ. Here, we extend the arguments used in [14] to consider the case of an arbitrary $\varphi \in \Lambda_{T,M}$, σ having linear growth. As in [14, Section 3], the arguments we are using here are based on classical tools from the theory of monotone non-linear operators and we refer to the monograph [1] for all details.

Since f is assumed to be Lipschitz continuous, we have

$$\|A(z)\|_{\mathcal{H}} \le c\,(1 + \|z\|_{\mathcal{H}_1}), \qquad z \in D(A), \tag{4.4}$$

and, as shown in [14, Lemma 3.1], there exists $\kappa \in \mathbb{R}$ such that

$$\langle A(z_1) - A(z_2), z_1 - z_2 \rangle_{\mathcal{H}} \le \kappa\, \|z_1 - z_2\|^2_{\mathcal{H}}.$$

Moreover, for every $\lambda > 0$ small enough

$$\text{Range}\,(I - \lambda A) = \mathcal{H}.$$

This means that the operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is *quasi-m-dissipative*. In particular, this implies that there exists $\lambda_0 > 0$ such that

$$J_\lambda := (I - \lambda A)^{-1}, \qquad \lambda \in (0, \lambda_0),$$

is a well-defined Lipschitz-continuous mapping in $\mathcal{H}$ and we can introduce the *Yosida approximation* of A, defined as

$$A_\lambda(z) := A(J_\lambda(z)) = \frac{1}{\lambda}\,(J_\lambda(z) - z), \qquad z \in \mathcal{H}.$$

Notice that

$$\langle A_\lambda(z_1) - A_\lambda(z_2), z_1 - z_2\rangle_{\mathcal{H}} \leq \frac{\kappa}{1-\lambda\kappa}\,\|z_1 - z_2\|^2_{\mathcal{H}}, \tag{4.5}$$

and

$$\|A_\lambda(z_1) - A_\lambda(z_2)\|_{\mathcal{H}} \leq \frac{2}{\lambda(1-\lambda\kappa)}\|z_1 - z_2\|_{\mathcal{H}}. \tag{4.6}$$

Moreover, for every $z \in D(A)$

$$\|A^\lambda(z)\|_{\mathcal{H}} \leq \frac{1}{1-\lambda\kappa}\,\|A(z)\|_{\mathcal{H}},$$

so that for every $z \in D(A)$

$$\|J_\lambda(z) - z\|_{\mathcal{H}} = \lambda\|A^\lambda(z)\| \leq \frac{\lambda}{1-\lambda\kappa}\|A(z)\|_{\mathcal{H}},$$

and

$$\lim_{\lambda\to 0}\|A_\lambda(z) - A(z)\|_{\mathcal{H}} = 0.$$

In [14, Proof of Theorem 3.2], it has been shown that there exists some $\lambda_1 \in (0,\lambda_0)$ such that for every $\lambda \in (0,\lambda_1)$

$$\langle A_\lambda(z), z\rangle_{\mathcal{H}_1} \leq -\frac{\gamma_0}{2}\,\|J_\lambda(z)_1\|^2_{H^1} + c\,\|J_\lambda(z)_2\|^2_{H^{-1}}, \qquad z \in \mathcal{H}_1. \tag{4.7}$$

Furthermore, for every $\lambda, \nu \in (0,\lambda_0)$ and $z_1, z_2 \in \mathcal{H}_1$ it holds

$$\langle A_\lambda(z_1) - A_\nu(z_2), z_1 - z_2\rangle_{\mathcal{H}} \leq \|z_1 - z_2\|^2_{\mathcal{H}} + c\,(\lambda+\nu)\left(\|z_1\|^2_{\mathcal{H}_1} + \|z_2\|^2_{\mathcal{H}_1} + 1\right). \tag{4.8}$$

Concerning the random operator B_φ, according to Hypothesis 1 for every $t \in [0,T]$ we have that $B_\varphi(t,\cdot) : \mathcal{H} \to \mathcal{H}_1$ is well-defined and, in view of (2.5), for any $z_1, z_2 \in \mathcal{H}$

$$\|B_\varphi(t,z_1) - B_\varphi(t,z_2)\|_{\mathcal{H}_1} = \|(\sigma(u_1) - \sigma(u_2))\,Q\varphi(t)\|_H \leq \sqrt{L}\,\|u_1 - u_2\|_H\,\|\varphi(t)\|_H. \tag{4.9}$$

Finally, Hypothesis 1 implies that the mapping $\Sigma : \mathcal{H} \to \mathcal{L}_2(H_Q, \mathcal{H}_1)$ is well-defined and due to (2.5) for any $z_1, z_2 \in \mathcal{H}$

$$\|\Sigma(z_1) - \Sigma(z_2)\|_{\mathcal{L}_2(H_Q,\mathcal{H}_1)} = \|\sigma(u_1) - \sigma(u_2)\|_{\mathcal{L}_2(H_Q,H)} \leq \sqrt{L}\,\|u_1 - u_2\|_H. \tag{4.10}$$

Step 1. For every $\lambda \in (0,\lambda_0)$ the approximating problem

$$dz(t) = [A_\lambda(z(t)) + B_\varphi(t,z(t))]\,dt + \Sigma(z(t))dw^Q(t), \qquad z(0) = (u_0, v_0 + g(u_0)) \tag{4.11}$$

admits a unique solution $z_\lambda \in L^2(\Omega; C([0,T];\mathcal{H}))$.

Proof of Step 1. According to (4.6), we have

$$\begin{aligned}\int_0^T \|A_\lambda(z_1(s)) - A_\lambda(z_2(s))\|^2_{\mathcal{H}}\,ds &\leq \frac{c}{\lambda^2}\int_0^T \|z_1(s) - z_2(s)\|^2_{\mathcal{H}}\,ds \\ &\leq \frac{c_T}{\lambda^2}\sup_{t\in[0,T]}\|z_1(t) - z_2(t)\|^2_{\mathcal{H}}.\end{aligned} \tag{4.12}$$

According to (4.9), if $\varphi \in \Lambda_{T,M}$, we have

(4.13)
$$\int_0^T \|B_\varphi(t,z_1)-B_\varphi(t,z_2)\|^2_{\mathcal{H}_1}\,dt \leq L\int_0^T \|u_1(t)-u_2(t)\|^2_H\,\|\varphi(t)\|^2_H\,dt \leq L\,M^2 \sup_{t\in[0,T]}\|z_1(t)-z_2(t)\|^2_{\mathcal{H}}, \qquad \mathbb{P}-\text{a.s.}$$

Finally, according to (4.10), we have

(4.14)
$$\mathbb{E}\sup_{t\in[0,T]}\left\|\int_0^t(\Sigma(z_1(s))-\Sigma(z_2(s))dw^Q(s)\right\|^2_{\mathcal{H}_1} \leq c\int_0^T \mathbb{E}\,\|\Sigma(z_1(s))-\Sigma(z_2(s))\|^2_{\mathcal{L}_2(H_Q,\mathcal{H}_1)}\,ds \leq T L\,\mathbb{E}\sup_{t\in[0,T]}\|z_1(t)-z_2(t)\|^2_{\mathcal{H}}.$$

Therefore, in view of (4.12), (4.13) and (4.14), for every $\lambda\in(0,\lambda_0)$ the mapping

$$\Phi_\lambda(z)(t)=(u_0,v_0+g(u_0))+\int_0^t[A_\lambda(z(s))+B_\varphi(s,z(s))]\,ds+\int_0^t\Sigma(z(s))\,dw^Q(s)$$

is Lipschitz continuous from $L^2(\Omega;C([0,T];\mathcal{H}))$ into itself, and then for every $\lambda\in(0,\lambda_0)$ equation (4.11) admits a unique solution $z_\lambda\in L^2(\Omega;C([0,T];\mathcal{H}))$.

Step 2. There exists $c_T>0$ such that

$$\mathbb{E}\sup_{t\in[0,T]}\|z_\lambda(t)\|^2_{\mathcal{H}_1}\leq c_T\left(1+\|z_0\|^2_{\mathcal{H}_1}\right),\qquad \lambda\in(0,\lambda_1). \tag{4.15}$$

Proof of Step 2. As a consequence of Itô's formula, we have

$$\|z_\lambda(t)\|^2_{\mathcal{H}_1}=\|z_0\|^2_{\mathcal{H}_1}+2\int_0^t\langle A_\lambda(z_\lambda(s)),z_\lambda(s)\rangle_{\mathcal{H}_1}\,ds+2\int_0^t\langle B_\varphi(s,z_\lambda(s)),z_\lambda(s)\rangle_{\mathcal{H}_1}\,ds + \int_0^t\|\Sigma(z_\lambda(s))\|^2_{\mathcal{L}_2(H_Q,\mathcal{H}_1)}ds+2\int_0^t\langle z_\lambda(s),\Sigma(z_\lambda(s))dw^Q(s)\rangle_{\mathcal{H}_1}.$$

Due to (4.7), we have

(4.16)
$$\int_0^t\langle A_\lambda(z_\lambda(s)),z_\lambda(s)\rangle_{H^1}\,ds\leq-\frac{\gamma_0}{2}\int_0^t\|J_\lambda(z_\lambda(s))_1\|^2_{H^1}\,ds+c\int_0^t\|J_\lambda(z_\lambda(s))\|^2_{\mathcal{H}}\,ds \leq -\frac{\gamma_0}{2}\int_0^t\|J_\lambda(z_\lambda(s))_1\|^2_{H^1}\,ds+c\int_0^t\|z_\lambda(s)\|^2_{\mathcal{H}}\,ds.$$

Next, recalling that $\varphi\in\Lambda_{T,M}$, due to (4.9) for every $\delta>0$ we have

(4.17)
$$\left|\int_0^t\langle B_\varphi(s,z_\lambda(s)),z_\lambda(s)\rangle_{\mathcal{H}_1}ds\right|\leq\delta\int_0^t\|B_\varphi(s,z_\lambda(s))\|^2_{\mathcal{H}_1}\,ds+\frac{c}{\delta}\int_0^t\|z_\lambda(s)\|^2_{\mathcal{H}_1}\,ds \leq\delta\,c_M\left(1+\sup_{r\in[0,t]}\|z_\lambda(r)\|^2_{\mathcal{H}}\right)+\frac{c}{\delta}\int_0^t\|z_\lambda(s)\|^2_{\mathcal{H}_1}\,ds,\qquad\mathbb{P}-\text{a.s.}$$

In the same way, thanks to (4.10) for every $\delta > 0$

(4.18)

$$\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r \langle z_\lambda(s),\Sigma(z_\lambda(s))\ dw^Q(s)\rangle_{\mathcal{H}_1}\right|$$
$$\leq c\,\mathbb{E}\left(\sup_{r\in[0,t]}\|z_\lambda(r)\|^2_{\mathcal{H}_1}\int_0^t\|\Sigma(z_\lambda(s))\|^2_{\mathcal{L}_2(H_Q,\mathcal{H}_1)}ds\right)^{\frac{1}{2}}$$
$$\leq \delta\,\mathbb{E}\sup_{r\in[0,t]}\|z_\lambda(r)\|^2_{\mathcal{H}_1}+\frac{c}{\delta}\int_0^t\mathbb{E}\|z_\lambda(s)\|^2_{\mathcal{H}_1}\,ds+\frac{c_T}{\delta}.$$

Finally, due to (4.10) we get

$$\int_0^t\|\Sigma(z_\lambda(s))\|^2_{\mathcal{L}_2(H_Q,\mathcal{H}_1)}\,ds\leq c_T\left(1+\int_0^t\mathbb{E}\|z_\lambda(s)\|^2_{\mathcal{H}}\,ds\right). \tag{4.19}$$

Therefore, if we choose $\delta > 0$ small enough, from (4.16), (4.17), (4.18) and (4.19) we get

$$\mathbb{E}\sup_{r\in[0,t]}\|z_\lambda(r)\|^2_{\mathcal{H}_1}+\frac{\gamma_0}{2}\int_0^t\mathbb{E}\|J_\lambda(z_\lambda(s))_1\|^2_{H^1}\,ds\leq c_T\int_0^t\mathbb{E}\sup_{r\in[0,s]}\|z_\lambda(r)\|^2_{\mathcal{H}_1}\,ds+c_T.$$

Now, Gronwall's lemma allows to obtain (4.15).

Step 3. There exists $z\in L^2(\Omega;C([0,T];\mathcal{H}))$ such that

$$\lim_{\lambda\to 0}\mathbb{E}\sup_{t\in[0,T]}\|z_\lambda(t)-z(t)\|^2_{\mathcal{H}}=0. \tag{4.20}$$

Proof of Step 3. For every $\lambda,\nu\in(0,\lambda_1)$, we denote $\rho_{\lambda,\nu}(t):=z_\lambda(t)-z_\nu(t)$. We have

$$d\rho_{\lambda,\nu}(t)=\big[A_\lambda(z_\lambda(t))-A_\nu(z_\nu(t))\big]dt$$
$$+\big[B_\varphi(t,z_\lambda(t))-B_\varphi(t,z_\nu(t))\big]dt+\big[\Sigma(z_\lambda(t))-\Sigma(z_\nu(t))\big]dw^Q(t).$$

We have

$$\|\rho_{\lambda,\nu}(t)\|^2_{\mathcal{H}}=2\int_0^t\langle A_\lambda(z_\lambda(s))-A_\nu(z_\nu(s)),\rho_{\lambda,\nu}(s)\rangle_{\mathcal{H}}ds$$
$$+2\int_0^t\langle B_\varphi(s,z_\lambda(s))-B_\varphi(s,z_\nu(s)),\rho_{\lambda,\nu}(s)\rangle_{\mathcal{H}}ds$$
$$+\int_0^t\|\Sigma(z_\lambda(s))-\Sigma(z_\nu(s))\|^2_{\mathcal{L}_2(H_Q,\mathcal{H})}\,ds$$
$$+2\int_0^t\langle\rho_{\lambda,\nu}(s),(\Sigma(z_\lambda(s))-\Sigma(z_\nu(s)))\,dw^Q(s)\rangle_{\mathcal{H}}=:\sum_{k=1}^4 I_k(t).$$

Due to (4.8), we have

$$|I_1(t)|\leq c\int_0^t\|\rho_{\lambda,\nu}(s)\|^2_{\mathcal{H}}ds+c(\lambda+\nu)\int_0^t\big(\|z_\lambda(s)\|^2_{\mathcal{H}_1}+\|z_\nu(s)\|^2_{\mathcal{H}_1}+1\big)\,ds, \tag{4.21}$$

and due to (4.10) we have

$$|I_3(t)|\leq c\int_0^t\|\rho_{\lambda,\nu}(s)\|^2_{\mathcal{H}}\,ds. \tag{4.22}$$

Moreover, by proceeding as in the proof of (4.17) and (4.18), for every $\delta > 0$ we have

(4.23)

$$\mathbb{E}\sup_{r\in[0,t]}|I_2(r)| + \mathbb{E}\sup_{r\in[0,t]}|I_4(r)| \leq \delta\,\mathbb{E}\sup_{r\in[0,t]}\|\rho_{\lambda,\nu}(r)\|^2_{\mathcal{H}} + \frac{c}{\delta}\int_0^t \mathbb{E}\,\|\rho_{\lambda,\nu}(r)\|^2_{\mathcal{H}}\,dr.$$

Therefore, if we choose $\delta > 0$ sufficiently small, from (4.21), (4.22) and (4.23), we obtain

$$\begin{aligned}\mathbb{E}\sup_{r\in[0,t]}\|\rho_{\lambda,\nu}(r)\|^2_{\mathcal{H}} \leq & c\int_0^t \mathbb{E}\sup_{r\in[0,s]}\|\rho_{\lambda,\nu}(r)\|^2_{\mathcal{H}}\,ds \\ & + c\,(\lambda+\nu)\int_0^t \left(\|z_\lambda(s)\|^2_{\mathcal{H}_1} + \|z_\nu(s)\|^2_{\mathcal{H}_1} + 1\right) ds,\end{aligned}$$

and the Gronwall lemma, together with (4.15), gives

$$\begin{aligned}\mathbb{E}\sup_{r\in[0,T]}\|\rho_{\lambda,\nu}(r)\|^2_{\mathcal{H}} \leq & c_T(\lambda+\nu)\int_0^T \left(\|z_\lambda(s)\|^2_{\mathcal{H}_1} + \|z_\nu(s)\|^2_{\mathcal{H}_1} + 1\right) ds \\ & \leq c_T(\lambda+\nu)\left(1 + \|z_0\|^2_{\mathcal{H}}\right).\end{aligned}$$

This implies that

$$\lim_{\lambda,\nu\to 0}\mathbb{E}\sup_{r\in[0,T]}\|\rho_{\lambda,\nu}(r)\|^2_{\mathcal{H}} = 0,$$

so that the family $\{z_\lambda\}_{\lambda\in(0,\lambda_1)}$ is Cauchy and (4.20) follows.

Step 4. There exists a unique solution $z \in L^2(\Omega; C([0,T];\mathcal{H}_1))$ for equation (4.1).

Proof of Step 4. For every $\lambda \in (0,\lambda_0)$ we have that z_λ satisfies equation (4.11). Then, by proceeding as in [14, Proof of Theorem 3.2], we take the limit, as λ goes to zero, of both sides of (4.11) in $L^2(\Omega; C([0,T];\mathcal{H}_{-1}))$ and thanks to (4.20) we obtain that z satisfies the equation

$$z(t) = (u_0, v_0 + g(u_0)) + \int_0^t [A(z(s)) + B_\varphi(s, z(s))]\,ds + \int_0^t \Sigma(z(s))\,dw^Q(s),$$

and $z \in L^2(\Omega; L^\infty(0,T;\mathcal{H}_1))$.

Next, by using again arguments analogous to those used in [14, Proof of Theorem 3.2], we can show that z has continuous trajectories and is the unique solution of equation (4.2).

4.2. **The case when f satisfies Hypothesis 4.** In view of what we have seen in Subsection 4.1, for every $n \in \mathbb{N}$ and for every $\varphi \in \Lambda_{T,M}$ and $(u_0, v_0) \in \mathcal{H}_1$ there exists a unique solution $(u_n, v_n) \in L^2(\Omega; C([0,T];\mathcal{H}_1))$ for the equation

(4.24)

$$\begin{cases} u_n(t,x) = u_0(x) + \displaystyle\int_0^t v_n(s,x)\,ds, \\ v_n(t,x) = \mu\,v_0(x) + \displaystyle\int_0^t [\Delta u_n(s,x) - \gamma(u_n(s,x))v_n(s,x) + f_n(x, u(s,x)) \\ \qquad\qquad + \sigma(u_n(s,\cdot))Q\varphi(s,x)]\,ds + \displaystyle\int_0^t \sigma(u(s,\cdot))\,dw^Q(s,x), \end{cases}$$

where f_n is the function defined in (2.17).

For every $n \in \mathbb{N}$, we define

$$\tau_n := \inf \{t \geq 0 \ : \ \|u_n(t)\|_{H^1} \geq n/C\},$$

with $\inf \emptyset = +\infty$, where $C > 0$ is a constant such that $\|\cdot\|_{L^\infty(0,L)} \leq C\|\cdot\|_{H^1}$. Clearly $\{\tau_n\}_{n\in\mathbb{N}}$ is an increasing sequence of stopping times.

We denote $\tau := \sup_{n\in\mathbb{N}} \tau_n$, and for every $\omega \in \Omega$ and $t < \tau(\omega) \wedge T$, define

$$z(t)(\omega) := z_n(t)(\omega), \quad \text{if } t < \tau_n(\omega) \leq T.$$

Notice that this is a good definition, as $f_n(r) = f_m(r)$, for every $n \leq m$ and $|r| \leq n$. Moreover, since for $\omega \in \Omega$ and $t \leq \tau_n(\omega) \wedge T$

$$\|u_n(t)(\omega)\|_{L^\infty([0,L])} \leq C\|u_n(t)(\omega)\|_{H^1} \leq n,$$

we have that $f_n(u(s)(\omega)) = f(u(s)(\omega))$. This means $z(t) = z_n(t)$ solves equation (4.3) for $t \leq \tau_n \wedge T$.

Step 1. There exists $c_T > 0$ independent of $n \in \mathbb{N}$ such that

(4.25)
$$\mathbb{E} \sup_{t\in[0,T]} \|u(t\wedge\tau_n)\|_H^2 + \int_0^T \mathbb{E}\,\|u(t\wedge\tau_n)\|_{H^1}^2\,dt + \int_0^T \mathbb{E}\,\|u(t\wedge\tau_n)\|_{L^{\theta+1}}^{\theta+1}\,dt$$
$$\leq c_T\Big(1 + \int_0^T \mathbb{E}\,\|u(t\wedge\tau_n)\|_H^2\,dt + \int_0^T \mathbb{E}\,\|v(t\wedge\tau_n)\|_H^2\,dt + \mathbb{E}\sup_{t\in[0,T]}\|v(t\wedge\tau_n)\|_H^2\Big).$$

Proof of Step 1. Recall that for $t < \tau_n \leq T$, $z(t) = z_n(t)$ is a solution of equation (4.3), by proceeding as in [14, Proof of Lemma 4.1], we have

(4.26)
$$\frac{\gamma_0}{4}\,\|u(t\wedge\tau_n)\|_H^2 + \int_0^{t\wedge\tau_n} \|u(s)\|_{H^1}^2 \leq c + c\,\|v(t\wedge\tau_n)\|_H^2 + \int_0^{t\wedge\tau_n} \|v(s)\|_H^2\,ds$$
$$+ \int_0^{t\wedge\tau_n} \langle F(u(s)), u(s)\rangle_H ds + \int_0^{t\wedge\tau_n} \langle u(s), \sigma(u(s))Q\varphi(s)\rangle_H ds$$
$$+ \int_0^{t\wedge\tau_n} \langle u(s), \sigma(u(s))dw^Q(s)\rangle_H.$$

Thanks to (2.13), we have

$$\int_0^{t\wedge\tau_n} \langle F(u(s)), u(s)\rangle_H ds \leq -c_2 \int_0^{t\wedge\tau_n} \|u(s)\|_{L^{\theta+1}}^{\theta+1}\,ds + c_2 t. \tag{4.27}$$

Moreover, by proceeding as in the proof of (4.17) and (4.18), for every $\delta > 0$ we have

$$\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^{r\wedge\tau_n}\langle u(s), \sigma(u(s))Q\varphi(s)\rangle_H ds\right| + \mathbb{E}\sup_{r\in[0,t]}\left|\int_0^{r\wedge\tau_n}\langle u(s), \sigma(u(s))dw^Q(s)\rangle_H\right|$$
$$\leq \delta\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|_H^2 + \frac{c}{\delta}\int_0^t \mathbb{E}\|u(s\wedge\tau_n)\|_H^2\,ds.$$

Therefore, if we choose $\delta > 0$ sufficiently small above, this, together with (4.26) and (4.27) allows to conclude that (4.25) holds true.

Step 2. There exists $c_T>0$ independent of $n\in\mathbb{N}$ such that

(4.28)

$$\mathbb{E}\sup_{t\in[0,T]}\|(u(t\wedge\tau_n),v(t\wedge\tau_n))\|^2_{\mathcal{H}_1}+\mathbb{E}\sup_{t\in[0,T]}\|u(t\wedge\tau_n)\|^{\theta+1}_{L^{\theta+1}}+\gamma_0\int_0^T\mathbb{E}\|v(s)\|^2_H\,ds\leq c_T\left(1+\|u_0\|^{\theta+1}_{H^1}+\|v_0\|^2_H\right).$$

Proof of Step 2. From the Itô formula we have

$$\begin{aligned}\frac{1}{2}\Big[\|u(t\wedge\tau_n)\|^2_{H^1}+\|v(t\wedge\tau_n)\|^2_H\Big]&=\frac{1}{2}\Big[\|u_0\|^2_{H^1}+\|v_0\|^2_H\Big]-\int_0^{t\wedge\tau_n}\langle\gamma(u(s))v(s),v(s)\rangle_H\,ds\\&+\int_{\mathcal{O}}\mathfrak{f}(x,u_n(t\wedge\tau_n,x))\,dx-\int_{\mathcal{O}}\mathfrak{f}(x,u_0(x))\,dx+\int_0^{t\wedge\tau_n}\langle\sigma(u(s))Q\varphi(s),v(s)\rangle_H\,ds\\&+\int_0^{t\wedge\tau_n}\langle\sigma(u(s))dw^Q(s),v(s)\rangle_H+\frac{1}{2}\int_0^{t\wedge\tau_n}\|\sigma(u(s))\|^2_{\mathcal{L}_2(H_Q,H)}\,ds.\end{aligned}$$

Thanks to (2.11) we have

$$\int_{\mathcal{O}}\mathfrak{f}(x,u(t\wedge\tau_n,x))\,dx\leq c-c_2\,\|u(t\wedge\tau_n)\|^{\theta+1}_{L^{\theta+1}},$$

and

$$\left|\int_{\mathcal{O}}\mathfrak{f}(x,u_0(x))\,dx\right|\leq c\left(1+\int_{\mathcal{O}}|u_0(x)|^{\theta+1}\,dx\right)=c\left(1+\|u_0\|^{\theta+1}_{L^{\theta+1}}\right)\leq c\left(1+\|u_0\|^{\theta+1}_{H^1}\right).$$

Therefore, due to (2.6),

$$\begin{aligned}&\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|^2_{H^1}+\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|^{\theta+1}_{L^{\theta+1}}+\sup_{r\in[0,t]}\|v(r\wedge\tau_n)\|^2_H\\&\quad+\gamma_0\int_0^{t\wedge\tau_n}\|v(s)\|^2_H\,ds\\&\leq c\left(1+\|u_0\|^{\theta+1}_{H^1}+\|v_0\|^2_H\right)+\sup_{r\in[0,t]}\left|\int_0^{r\wedge\tau_n}\langle\sigma(u(s))Q\varphi(s),v(s)\rangle_H\,ds\right|\\&\quad+\sup_{r\in[0,t]}\left|\int_0^{r\wedge\tau_n}\langle\sigma(u(s))dw^Q(s),v(s)\rangle_H\right|+c\int_0^{t\wedge\tau_n}\|u(s)\|^2_H\,ds.\end{aligned}$$

Since $\varphi\in\Lambda_{T,M}$, by proceeding as in (4.17) and (4.18), this implies that for every $\delta>0$

$$\begin{aligned}&\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|^2_{H^1}+\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|^{\theta+1}_{L^{\theta+1}}\\&\quad+\mathbb{E}\sup_{r\in[0,t]}\|v(r\wedge\tau_n)\|^2_H+\gamma_0\mathbb{E}\int_0^{t\wedge\tau_n}\|v(s)\|^2_H\,ds\\&\leq c\left(1+\|u_0\|^{\theta+1}_{H^1}+\|v_0\|^2_H\right)+\delta\,\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|^2_H\\&\quad+\frac{c}{\delta}\int_0^t\mathbb{E}\|v(s\wedge\tau_n)\|^2_H\,ds+c\int_0^t\mathbb{E}\,\|u(s\wedge\tau_n)\|^2_H\,ds.\end{aligned}$$

In particular, if we choose δ small enough we have

(4.29)

$$\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|^2_{H^1}+\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|^{\theta+1}_{L^{\theta+1}}+\mathbb{E}\sup_{r\in[0,t]}\|v(r\wedge\tau_n)\|^2_H$$
$$\leq c\left(1+\|u_0\|^{\theta+1}_{H^1}+\|v_0\|^2_H\right)+c\int_0^t\mathbb{E}\|v(s\wedge\tau_n)\|^2_H\,ds+c\int_0^t\mathbb{E}\,\|u(s\wedge\tau_n)\|^2_H\,ds,$$

and if we choose δ large enough we have

$$\begin{aligned}&\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|^{\theta+1}_{L^{\theta+1}}+\mathbb{E}\sup_{r\in[0,t]}\|v(r\wedge\tau_n)\|^2_H+\gamma_0\mathbb{E}\int_0^{t\wedge\tau_n}\|v(s)\|^2_H\,ds\\&\qquad\leq c\left(1+\|u_0\|^{\theta+1}_{H^1}+\|v_0\|^2_H\right)+c\int_0^t\mathbb{E}\,\|u(s\wedge\tau_n)\|^2_H\,ds.\end{aligned}\tag{4.30}$$

Combining (4.30) with (4.25) yields that

$$\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|^2_H\leq c\int_0^t\mathbb{E}\,\|u(s\wedge\tau_n)\|^2_H\,ds.\tag{4.31}$$

Thanks to Gronwall's lemma, (4.28) follows from (4.29).

Step 3. There exists $z=(u,v)\in L^2(\Omega;C([0,T];\mathcal{H}_1))$ solution to problem (4.3) such that

(4.32)

$$\mathbb{E}\sup_{t\in\,[0,T]}\|(u(t),v(t))\|^2_{\mathcal{H}_1}+\mathbb{E}\sup_{t\in[0,T]}\|u(t)\|^{\theta+1}_{L^{\theta+1}}$$
$$+\gamma_0\int_0^T\mathbb{E}\|v(s)\|^2_H\,ds\leq c_T\left(1+\|u_0\|^{\theta+1}_{H^1}+\|v_0\|^2_H\right).$$

Proof of Step 3. According to (4.28), for every $T>0$ we have

$$\mathbb{P}(\tau_n\leq T)\leq\frac{C^2}{n^2}\mathbb{E}\big(\,\|u(\tau_n)\|^2_{H^1}\,;\tau_n\leq T\big)\leq\frac{C^2}{n^2}\mathbb{E}\sup_{t\in[0,T]}\|u(t\wedge\tau_n)\|^2_{H^1}\leq\frac{c}{n^2},$$

so that

$$\lim_{n\to\infty}\mathbb{P}(\tau_n\leq T)=0,$$

and hence $\mathbb{P}(\tau=\infty)=1$. This implies for every $t\in[0,T]$, $z(t\wedge\tau_n)\to z(t)$, $\mathbb{P}$-a.s. as $n\to\infty$, so that z belongs to $L^2(\Omega;C([0,T];\mathcal{H}_1))$ and solves equation (4.3). By taking the limit as $n\to\infty$ in (4.28), we get (4.32).

Step 4. The solution z is unique in $L^2(\Omega;C([0,T];\mathcal{H}_1))$.

Proof of Step 4. Let z_1 and z_2 be two solutions of equation (4.1) in $L^2(\Omega;C([0,T];\mathcal{H}_1))$. For every $R>0$, we define

$$\tau_R:=\tau_{1,R}\wedge\tau_{2,R},$$

where

$$\tau_{i,R}:=\inf\left\{t\geq 0\;:\;\|u_i(t)\|_{H^1}\geq R\right\},\qquad i=1,2.$$

Since z_1 and z_2 belong to $L^2(\Omega;C([0,T];\mathcal{H}_1))$, we have

$$\lim_{R\to\infty}\mathbb{P}\left(\tau_R<T\right)=0.\tag{4.33}$$

Now, if we define $\rho = z_1 - z_2$, from the Itô formula we have

$$\begin{aligned}\|\rho(t\wedge\tau_R)\|^2_{\mathcal{H}} = 2\int_0^{t\wedge\tau_R} &\langle A(z_1(s)) - A(z_2(s)), \rho(s)\rangle_{\mathcal{H}} ds\\ &+ 2\int_0^{t\wedge\tau_R} \langle B_\varphi(s, z_1(s)) - B_\varphi(s, z_2(s)), \rho(s)\rangle_{\mathcal{H}} ds\\ &+ \int_0^{t\wedge\tau_R} \|\Sigma(z_1(s)) - \Sigma(z_2(s))\|^2_{\mathcal{L}_2(H_Q,\mathcal{H})}\, ds\\ &+2\int_0^{t\wedge\tau_R} \langle \rho(s), (\Sigma(z_1(s)) - \Sigma(z_2(s)))\, dw^Q(s)\rangle_{\mathcal{H}} =: \sum_{k=1}^4 I_k(t\wedge\tau_R).\end{aligned}$$

We have

$$\begin{aligned}\langle A(z_1(s)) - A(z_2(s)), \rho(s)\rangle_{\mathcal{H}} =&-\langle (g(u_1(s)) - g(u_2(s)), u_1(s) - u_2(s)\rangle_H\\ &+\langle F(u_1(s)) - F(u_2(s)), \eta_1(s) - \eta_2(s)\rangle_{H^{-1}},\end{aligned}$$

so that, according to (2.15)

$$\begin{aligned}|\langle A(z_1(s)) -\ A(z_2(s)), \rho(s)\rangle_{\mathcal{H}}| &\leq c\,\|\rho(s)\|^2_{\mathcal{H}} + c\,\|F(u_1(s)) - F(u_2(s))\|^2_H\\ &\leq c\,\|\rho(s)\|^2_{\mathcal{H}} + c\left(1 + \|u_1(s)\|^{2(\theta-1)}_{H^1} + \|u_2(s)\|^{2(\theta-1)}_{H^1}\right)\|u_1(s) - u_2(s)\|^2_H.\end{aligned}$$

This implies that

$$\sup_{r\in[0,t]} |I_1(r\wedge\tau_R)| \leq c(R)\int_0^{t\wedge\tau_R} \|\rho(s\wedge\tau_R)\|^2_{\mathcal{H}}\, ds. \tag{4.34}$$

Moreover, by proceeding as in (4.17) and (4.18), for every $\delta > 0$ we have

$$\mathbb{E}\sup_{r\in[0,t]} (|I_2(r\wedge\tau_R)| + |I_4(r\wedge\tau_R)|) \leq \delta\mathbb{E}\sup_{r\in[0,t]} \|\rho(r\wedge\tau_R)\|^2_{\mathcal{H}} + \frac{c}{\delta}\int_0^t \mathbb{E}\|\rho(s\wedge\tau_R)\|^2_{\mathcal{H}}\, ds. \tag{4.35}$$

Therefore, since

$$\sup_{r\in[0,t]} |I_3(r\wedge\tau_R)| \leq c\int_0^{t\wedge\tau_R} \|\rho(s\wedge\tau_R)\|^2_{\mathcal{H}}\, ds,$$

thanks to (4.34) and (4.35), if we fix some $\delta > 0$ small enough, we get

$$\mathbb{E}\sup_{r\in[0,t]} \|\rho(r\wedge\tau_R)\|^2_{\mathcal{H}} \leq c(R)\int_0^t \mathbb{E}\|\rho(r\wedge\tau_R)\|^2_{\mathcal{H}}\, dr.$$

This implies that for every $R > 0$

$$\mathbb{E}\sup_{r\in[0,T]} \|\rho(r\wedge\tau_R)\|^2_{\mathcal{H}} = 0.$$

In view of (4.33), by taking the limit as $R\uparrow\infty$ this gives $\mathbb{E}\sup_{r\in[0,T]}\|\rho(r)\|^2_{\mathcal{H}} = 0$ and uniqueness follows.

5. A priori bounds and tightness

In the previous section we have proved that for any $\mu > 0$ and any $T > 0$ there exists a unique solution $(u_\mu, \partial_t u_\mu) \in L^2(\Omega; C([0,T];\mathcal{H}_1))$ to system (4.3). Our purpose here is proving a bound for $(u_\mu, \partial_t u_\mu)$, which is uniform with respect to μ.

Lemma 5.1. *Under Hypotheses* 1 *and* 2 *and either Hypothesis* 3 *or Hypothesis* 4, *for every* $T, M > 0$ *and for every initial condition* $(u_0, v_0) \in \mathcal{H}_1$, *there exist* $c_T > 0$ *and* $\mu_T > 0$ *such that for every* $\varphi \in \Lambda_{T,M}$ *and* $\mu \in (0, \mu_T)$

$$\mathbb{E} \sup_{t\in[0,T]} \|u_\mu(t)\|^2_{H^1} + \mathbb{E} \sup_{t\in[0,T]} \|u_\mu(t)\|^{\theta+1}_{L^{\theta+1}} + \mu\, \mathbb{E} \sup_{t\in[0,T]} \|\partial_t u_\mu(t)\|^2_H + \int_0^T \mathbb{E}\|\partial_t u_\mu(t)\|^2_H\, dt \leq c_T. \tag{5.1}$$

Proof. We give our proof in case Hypothesis 4 holds and we leave to the reader the proof in case Hypothesis 3 holds.

Step 1. There exists $c_T > 0$ such that for all $\mu \in (0,1)$

$$\mathbb{E} \sup_{r\in[0,t]} \|u_\mu(r)\|^2_H + \int_0^t \mathbb{E}\|u_\mu(s)\|^2_{H^1}\, ds + \int_0^t \mathbb{E}\, \|u_\mu(s)\|^{\theta+1}_{L^{\theta+1}}\, ds \leq c_T \left(1 + \int_0^t \mathbb{E}\|u_\mu(s)\|^2_H ds + \mu \int_0^t \mathbb{E}\|\partial_t u_\mu(s)\|^2_H ds + \mu^2 \mathbb{E} \sup_{r\in[0,t]} \|\partial_t u_\mu(r)\|^2_H\right). \tag{5.2}$$

Proof of Step 1. As shown in [14, Proof of Lemma 4.1], for every $\mu \in (0,1)$ we have

$$\frac{\gamma_0}{4}\|u_\mu(t)\|^2_H + \int_0^t \|u_\mu(s)\|^2_{H^1} ds \leq c + c\mu^2\|\partial_t u_\mu(t)\|^2_H + \mu \int_0^t \|\partial_t u_\mu(s)\|^2_H ds + \int_0^t \langle F(u_\mu(s)), u_\mu(s)\rangle_H ds + \int_0^t \langle u_\mu(s), \sigma(u_\mu(s))Q\varphi_\mu(s)\rangle_H ds + \sqrt{\mu}\int_0^t \langle u_\mu(s), \sigma(u_\mu(s))dw^Q(s)\rangle_H. \tag{5.3}$$

Due to (2.13), we have

$$\int_0^t \langle F(u_\mu(s)), u_\mu(s)\rangle_H ds \leq -c_2 \int_0^t \|u_\mu(s)\|^{\theta+1}_{L^{\theta+1}}\, ds + c_2 t. \tag{5.4}$$

Moreover, by proceeding as in the proof of (4.17) and (4.18), for every $\delta > 0$ we have

$$\mathbb{E} \sup_{r\in[0,t]} \left|\int_0^r \langle u_\mu(s), \sigma(u_\mu(s))Q\varphi_\mu(s)\rangle_H ds\right| + \sqrt{\mu}\,\mathbb{E} \sup_{r\in[0,t]} \left|\int_0^t \langle u_\mu(s), \sigma(u_\mu(s))dw^Q(s)\rangle_H\right| \leq \delta\mathbb{E} \sup_{r\in[0,t]} \|u_\mu(r)\|^2_H + \frac{c}{\delta}\int_0^t \mathbb{E}\|u_\mu(s)\|^2_H\, ds.$$

Therefore, if we choose $\delta > 0$ sufficiently small above, this, together with (5.4) and (5.3) allows to conclude that (5.2) holds true.

Step 2. For every $T > 0$ there exist $c_T > 0$ and $\mu_T > 0$ such that

$$\sup_{\mu\in(0,\mu_T)} \mathbb{E} \sup_{t\in[0,T]} \|u_\mu(t)\|^2_H \leq c_T, \tag{5.5}$$

and (5.1) holds.

Proof of Step 2. As in the previous section, from Itô's formula, we have

$$\|u_\mu(t)\|^2_{H^1} + \mu\|\partial_t u_\mu(t)\|^2_H = \|u_0\|^2_{H^1} + \mu\|v_0\|^2_H - \int_0^t \langle \gamma(u_\mu(s))\partial_t u_\mu(s), \partial_t u_\mu(s)\rangle_H\, ds$$
$$+\int_{\mathcal{O}} \mathfrak{f}(x, u_\mu(t,x))\, dx - \int_{\mathcal{O}} \mathfrak{f}(x, u_0(x))\, dx + \int_0^t \langle \sigma(u_\mu(s))Q\varphi(s), \partial_t u_\mu(s)\rangle_H\, ds$$
$$+\sqrt{\mu}\int_0^t \langle \sigma(u_\mu(s))dw^Q(s), \partial_t u_\mu(s)\rangle_H + \int_0^t \|\sigma(u_\mu(s))\|^2_{\mathcal{L}_2(H_Q,H)}\, ds.$$

Due to (2.6), (2.9) and (2.11), this gives

$$\|u_\mu(t)\|^2_{H^1} + c\|u_\mu(s)\|^{\theta+1}_{L^{\theta+1}} + \mu\|\partial_t u_\mu(t)\|^2_H + \gamma_0 \int_0^t \|\partial_t u_\mu(s)\|^2_H\, ds$$
$$\leq c\left(1 + \|u_0\|^{\theta+1}_{H^1} + \mu\|v_0\|^2_H\right) + \int_0^t \|\sigma(u_\mu(s))\|^2_{\mathcal{L}_2(H_Q,H)}\, ds$$
$$+ \int_0^t \langle \sigma(u_\mu(s))Q\varphi(s), \partial_t u_\mu(s)\rangle_H\, ds + \sqrt{\mu}\int_0^t \langle \sigma(u_\mu(s))dw^Q(s), \partial_t u_\mu(s)\rangle_H.$$

Therefore, by proceeding as in Step 1, for every $\delta > 0$ we have

(5.6)

$$\mathbb{E}\sup_{r\in[0,t]} \|u_\mu(r)\|^2_{H^1} + \mathbb{E}\sup_{r\in[0,T]} \|u_\mu(r)\|^{\theta+1}_{L^{\theta+1}} + \mu\,\mathbb{E}\sup_{r\in[0,t]} \|\partial_t u_\mu(r)\|^2_H + \int_0^t \mathbb{E}\|\partial_t u_\mu(s)\|^2_H\, ds$$
$$\leq c\left(1 + \|u_0\|^{\theta+1}_{H^1} + \mu\|v_0\|^2_H\right) + \delta\mathbb{E}\sup_{r\in[0,t]} \|u_\mu(r)\|^2_H + \frac{c}{\delta}\int_0^t \mathbb{E}\|\partial_t u_\mu(s)\|^2_H\, ds$$
$$+ c\int_0^t \mathbb{E}\|u_\mu(s)\|^2_H\, ds.$$

If we take $\delta > 0$ small enough in (5.6), we get

$$\begin{aligned} &\mathbb{E}\sup_{r\in[0,t]} \|u_\mu(r)\|^2_{H^1} + \mathbb{E}\sup_{r\in[0,T]} \|u_\mu(r)\|^{\theta+1}_{L^{\theta+1}} + \mu\,\mathbb{E}\sup_{r\in[0,t]} \|\partial_t u_\mu(r)\|^2_H \\ &\quad\leq c\left(1 + \int_0^t \mathbb{E}\|\partial_t u_\mu(s)\|^2_H\, ds\right) + c\int_0^t \mathbb{E}\|u_\mu(s)\|^2_H\, ds, \end{aligned} \tag{5.7}$$

and if we take $\delta > 0$ large enough we get

$$\mathbb{E}\sup_{r\in[0,T]} \|u_\mu(r)\|^{\theta+1}_{L^{\theta+1}} + \mu\,\mathbb{E}\sup_{r\in[0,t]} \|\partial_t u_\mu(r)\|^2_H + \int_0^t \mathbb{E}\|\partial_t u_\mu(s)\|^2_H\, ds \leq c\left(1 + \mathbb{E}\sup_{r\in[0,t]} \|u_\mu(r)\|^2_H\right). \tag{5.8}$$

By combining together (5.2) and (5.8), we can fix $\mu_T > 0$ such that for every $\mu \in (0, \mu_T)$

$$\mathbb{E}\sup_{r\in[0,t]} \|u_\mu(r)\|^2_H \leq c_T\left(1 + \int_0^t \mathbb{E}\|u_\mu(s)\|^2_H ds\right),$$

which implies (5.5). Thus, from (5.5), (5.7) and (5.8), we obtain (5.1).

□

Now, for every $T > 0$ and $\mu > 0$ we define

$$\rho_\mu(t,x) = g(u_\mu(t,x)), \quad (t,x) \in [0,T] \times \mathcal{O}.$$

According to Hypothesis 2, we know that

$$|g(r)| \leq \gamma_1 |r|, \qquad |g'(r)| \leq \gamma_1, \qquad r \in \mathbb{R}$$

so that for every $\mu > 0$ and $t \in [0,T]$,

$$\|\rho_\mu(t)\|_H \leq \gamma_1 \|u_\mu(t)\|_H, \qquad \|\rho_\mu(t)\|_{H^1} \leq \gamma_1 \|u_\mu(t)\|_{H^1}, \\ \|\partial_t \rho_\mu(t)\|_H \leq \gamma_1 \|\partial_t u_\mu(t)\|_H. \tag{5.9}$$

Since the function g is strictly increasing, it is invertible and we have

$$u_\mu(t,x) = g^{-1}(\rho_\mu(t,x)), \quad (t,x) \in [0,T] \times \mathcal{O},$$

which implies that

$$\Delta u_\mu(t,x) = \operatorname{div}\left[\nabla g^{-1}(\rho_\mu(t))\right] = \operatorname{div}\left[\frac{1}{\gamma(g^{-1}(\rho_\mu(t)))} \nabla \rho_\mu(t)\right].$$

Moreover, by the definition of ρ_μ,

$$\nabla \rho_\mu(t) = \gamma(u_\mu(t)) \nabla u_\mu(t), \qquad \partial_t \rho_\mu(t) = \gamma(u_\mu(t)) \partial_t u_\mu(t).$$

This means that if we integrate equation (3.3) (with φ_μ) with respect to t we have

$$\rho_\mu(t) + \mu \partial_t u_\mu(t) = g(u_0) + \mu v_0 + \int_0^t \operatorname{div}\big[b(\rho_\mu(s)) \nabla \rho_\mu(s)\big] ds + \int_0^t F_g(\rho_\mu(s)) ds \\ + \int_0^t \sigma_g(\rho_\mu(s)) Q \, \varphi_\mu(s) ds + \sqrt{\mu} \int_0^t \sigma_g(\rho_\mu(s)) dw^Q(s), \tag{5.10}$$

where for every $r \in \mathbb{R}$ and $x \in \mathcal{O}$

$$b(r) := \frac{1}{\gamma(g^{-1}(r))}, \qquad f_g(x,r) := f(x, g^{-1}(r)), \tag{5.11}$$

and for every $u \in H^1$

$$F_g(u) = f_g \circ u, \qquad \sigma_g(u) := \sigma(g^{-1} \circ u). \tag{5.12}$$

Theorem 5.2. *Assume Hypotheses 1 and 2 and either Hypothesis 3 or Hypothesis 4 and fix an arbitrary $T > 0$ and $(u_0, v_0) \in \mathcal{H}_1$. Then, for any family of predictable controls $\{\varphi_\mu\}_{\mu \in (0,\mu_T)} \subset \Lambda_{T,M}$, the family of probabilities $\{\mathcal{L}(\rho_\mu)\}_{\mu \in (0,\mu_T)}$ is tight in $C([0,T]; H^\delta)$, for every $\delta < 1$.*

Proof. According to (5.1) and (5.9), we have that

$$\mathbb{E} \sup_{r \in [0,t]} \|\rho_\mu(r)\|_{H^1}^2 + \int_0^t \mathbb{E} \|\partial_t \rho_\mu(s)\|_H^2 \, ds \leq c_T, \qquad \mu \in (0, \mu_T).$$

This means that for every $\epsilon > 0$ there exists $L_\epsilon > 0$ such that if we denote by K_ϵ the ball of radius L_ϵ in $C([0,T]; H^1) \cap W^{1,2}([0,T]; H)$, then

$$\inf_{\mu \in (0,\mu_T)} \mathbb{P}(\rho_\mu \in K_\epsilon) \geq 1 - \epsilon.$$

This allows to conclude as, due to the Aubin-Lions lemma, the set K_ϵ is compact in $C([0,T]; H^\delta)$, for every $\delta < 1$. □

6. The limit controlled problem

In order to prove conditions (C1) and (C2), we need first to understand better the controlled quasi-linear parabolic problem
(6.1)
$$\begin{cases}\partial_t\rho(t,x)=\operatorname{div}\left[b(\rho(t,x))\nabla\rho(t,x)\right]+f_g(x,\rho(t,x))+\sigma_g(\rho(t,\cdot))\varphi(t,x), & t>0, \quad x\in\mathcal{O},\\ \rho(0,x)=g(u_0(x)), \qquad \rho(t,x)=0, \quad x\in\partial\mathcal{O}.\end{cases}$$

In view of Hypothesis 2, we have

$$\frac{1}{\gamma_1}\,|r|\leq|g^{-1}(r)|\leq\frac{1}{\gamma_0}|r|,\qquad r\in\mathbb{R}.$$

When Hypothesis 4 holds, thanks to (2.10), this means that for every $u\in L^{\theta+1}([0,L])$

$$\|F_g(u)\|_{H^{-1}}\leq c\int_0^L(1+|g^{-1}(u(x))|^\theta)\,dx\leq c\left(1+\|u\|_H^{\frac{2}{\theta-1}}\|u\|_{L^{\theta+1}}^{\frac{(\theta+1)(\theta-2)}{\theta-1}}\right). \tag{6.2}$$

Next, again due to Hypothesis 2 we have

$$\frac{1}{\gamma_1}\leq\frac{g^{-1}(r)}{r}\leq\frac{1}{\gamma_0},\qquad\qquad r\in\mathbb{R},$$

so that, thanks to Hypothesis 4, we get,

$$f_g(r)r=f(g^{-1}(r))g^{-1}(r)\cdot\frac{r}{g^{-1}(r)}\leq c\left(1-|r|^{\theta+1}\right),\qquad r\in\mathbb{R}$$

(here we define $g^{-1}(0)/0=1/\gamma(0)$). In particular, we have

$$\langle F_g(u),u\rangle_H\leq c\left(1-\|u\|_{L^{\theta+1}}^{\theta+1}\right),\qquad u\in H^1. \tag{6.3}$$

Moreover, since g^{-1} is increasing and f is decreasing, we have that f_g is decreasing, so that

$$\langle F_g(u_1)-F_g(u_2),u_1-u_2\rangle_H\leq 0. \tag{6.4}$$

Finally, thanks to Hypotheses 2 and 1,

$$|b(r)|\leq\frac{1}{\gamma_0},\qquad b(r)\geq\frac{1}{\gamma_1},\qquad |b(r)-b(s)|\leq c\,|r-s|,\qquad r,s\in\mathbb{R}, \tag{6.5}$$

and

$$\|\sigma_g(u_1)-\sigma_g(u_2)\|_{\mathcal{L}(H_Q,H)}\leq c\,\|u_1-u_2\|_H,\qquad u_1,u_2\in H. \tag{6.6}$$

Definition 6.1. A function $\rho\in L^2([0,T];H^1)$ is a weak solution to equation (6.1) if for every test function $\psi\in C_0^\infty(\mathcal{O})$ and $t\in[0,T]$

$$\langle\rho(t),\psi\rangle_H=\langle g(u_0),\psi\rangle_H-\int_0^t\langle b(\rho(s))\nabla\rho(s),\nabla\psi\rangle_H ds \tag{6.7}$$

$$+\int_0^t\langle F_g(\rho(s))+\sigma_g(\rho(s))Q\varphi(s),\psi\rangle_H\,ds. \tag{6.8}$$

Proposition 6.2. *Assume Hypotheses 1 and 2 and either Hypothesis 3 or Hypothesis 4 and fix any $T>0$ and $\varphi\in L^2([0,T];H)$. Then, for every $u_0\in H^1$ there is at most one weak solution $\rho\in L^2([0,T];H^1)$ to equation (6.1).*

Proof. We adapt here the method introduced in the proof of [14, Theorem 6.2]. To this purpose, let $(\phi_n)_{n\in\mathbb{N}}$ be the sequence of twice differentiable functions constructed in [22, Theorem 3.1] such that

$$\phi_n'(0)=0,\qquad |\phi_n'(r)|\leq 1,\qquad 0\leq \phi_n''(r)\leq \frac{2}{n|r|},\qquad r\in\mathbb{R}, \tag{6.9}$$

and

$$\lim_{n\to\infty}\sup_{r\in\mathbb{R}}\big|\phi_n(r)-|r|\big|=0. \tag{6.10}$$

Now, suppose $\rho_1,\rho_2\in L^2([0,T];H^1)$ are both solutions to (6.1). Using integration by parts, for any fixed positive superharmonic function $\psi\in C_0^\infty(\mathcal{O})$, we have

$$\begin{aligned}
&\big\langle \phi_n(\rho_1(t)-\rho_2(t)),\psi\big\rangle_H\\
&=\int_0^t \big\langle \phi_n'(\rho_1(s)-\rho_2(s))(F_g(\rho_1(s))-F_g(\rho_2(s))),\psi\big\rangle_H ds\\
&\quad+\int_0^t \big\langle \phi_n'(\rho_1(s)-\rho_2(s))\Big(\sigma_g(\rho_1(s))-\sigma_g(\rho_2(s))\Big)Q\varphi(s),\psi\big\rangle_H ds\\
&\quad-\int_0^t \big\langle \phi_n''(\rho_1(s)-\rho_2(s))\nabla(\rho_1(s)-\rho_2(s))\Big(b(\rho_1(s))\nabla\rho_1(s)-b(\rho_2(s))\nabla\rho_2(s)\Big),\psi\big\rangle_H ds\\
&\quad-\int_0^t \big\langle \phi_n'(\rho_1(s)-\rho_2(s))\Big(b(\rho_1(s))\nabla\rho_1(s)-b(\rho_2(s))\nabla\rho_2(s)\Big),\nabla\psi\big\rangle_H ds\\
&=:\sum_{k=1}^4 I_{k,n}(t).
\end{aligned} \tag{6.11}$$

In the case when f satisfies Hypothesis 3, by the Lipschitz continuity of F_g, we have

$$I_{1,n}(t)\leq c\int_0^t \big\langle|\rho_1(s)-\rho_2(s)|,\psi\big\rangle_H ds.$$

In the case when f satisfies Hypothesis 4, thanks to the fact that $F_g'\leq 0$, $\phi_n''\geq 0$ and $\psi\geq 0$, we have $I_{1,n}(t)\leq 0$.

Next, by proceeding as in the proof of [14, Theorem 6.2], we know that

$$I_{3,n}(t)+I_{4,n}(t)\leq \frac{c_T}{n}\big(\,\|\psi\|_{H^1}+\|\psi\|_{L^\infty}\,\big)\int_0^t\Big(\,\|\rho_1(s)\|^2_{H^1}+\|\rho_2(s)\|^2_{H^1}\Big)ds.$$

And moreover, it follows from (2.3) that

$$\begin{aligned}
I_{2,n}(t)&=\int_0^t\int_{\mathcal{O}}\phi_n'(\rho_1(s)-\rho_2(s))\Big[\Big(\sigma_g(\rho_1(s))-\sigma_g(\rho_2(s))\Big)Q\varphi(s)\Big]\psi(x)dxds\\
&\leq \int_0^t\int_{\mathcal{O}}\sum_{i=1}^\infty\Big|\varphi_i(s)\Big(\sigma_i(x,u_1(s))-\sigma_i(x,u_2(s))\Big)\Big|\psi(x)dxds\\
&\leq c\int_0^t\|\varphi(s)\|_H\int_{\mathcal{O}}\Big(\sum_{i=1}^\infty\Big|\sigma_i(x,u_1(s))-\sigma_i(x,u_2(s))\Big|^2\Big)^{\frac12}\psi(x)dxds\\
&\leq C\int_0^t\|\varphi(s)\|_H\big\langle|\rho_1(s)-\rho_2(s)|,\psi\big\rangle_H ds,
\end{aligned}$$

where $\varphi_i(t):=\langle\varphi(t),e_i\rangle_H$.

Therefore, we have for every $t \geq 0$

$$\begin{aligned}\left\langle \phi_n(\rho_1(t) - \rho_2(t)), \psi \right\rangle_H &\leq \frac{c_T\left(\|\psi\|_{H^1} + \|\psi\|_{L^\infty} \right)}{n} \int_0^t \left(\|\rho_1(s)\|^2_{H^1} + \|\rho_2(s)\|^2_{H^1} \right) ds \\ &\quad + C \int_0^t \left(1 + \|\varphi(s)\|_H \right) \left\langle |\rho_1(s) - \rho_2(s)|, \psi \right\rangle_H ds.\end{aligned}$$

Taking $n \to \infty$, we obtain

$$\left\langle |\rho_1(t) - \rho_2(t)|, \psi \right\rangle_H \leq C \int_0^t \left(1 + \|\varphi(s)\|_H \right) \left\langle |\rho_1(s) - \rho_2(s)|, \psi \right\rangle_H ds,$$

and then by the Gronwall lemma, we conclude

$$\left\langle |\rho_1(t) - \rho_2(t)|, \psi \right\rangle_H = 0, \quad t \geq 0.$$

Finally, due to the arbitrariness of positive superharmonic test function $\psi \in C_0^\infty(\mathcal{O})$, we have $\rho_1 = \rho_2$. This completes the proof. □

Proposition 6.3. *Assume Hypotheses* 1 *and* 2 *and either Hypothesis* 3 *or Hypothesis* 4 *and fix any* $T > 0$, $u_0 \in H^1$ *and* $\varphi \in L^2([0, T]; H)$. *Suppose that* ρ *is the weak solution of equation* (6.1). *Then, if we define* $u := g^{-1}(\rho)$, *we have that* $u \in L^2([0, T]; H^1)$ *is a weak solution to equation* (3.2).

Proof. Due to the fact that $\rho \in L^2([0, T]; H^1)$ and g^{-1} is differentiable with bounded derivative, we have that $u \in L^2([0, T]; H^1)$ and

$$\nabla(g(u(t)) = g'(u(t))\nabla u(t) = \gamma(u(t))\nabla u(t).$$

Recalling how b, F_g and σ_g were defined, this implies

$$\begin{aligned}S_\varphi(t, \rho(t)) &:= \operatorname{div}\left[b(\rho(t))\nabla\rho(t)\right] + F_g(\rho(t)) + \sigma_g(\rho(t))Q\varphi(t) \\ &= \operatorname{div}\left[\frac{1}{\gamma(u(t))}\nabla(g(u(t))\right] + f(u(t)) + \sigma(u(t))Q\varphi(t) \\ &= \Delta u(t) + f(u(t)) + \sigma(u(t))Q\varphi(t)\end{aligned}$$

in H^{-1} sense. Moreover, by mollifying ρ with respect to t and x and then by taking the limit, we have that

$$\partial_t u \in L^2([0, T]; H^{-1})$$

and

$$\partial_t \rho(t) = g'(u(t))\partial_t u(t) = \gamma(u(t))\partial_t u(t),$$

in H^{-1} sense. We can now conclude, as we know that $\partial_t \rho(t) = \mathcal{S}_\varphi(t, \rho(t))$, in H^{-1} sense. □

7. Proof of Theorem 3.1

In order to prove Theorem 3.1, we will show that the conditions (C1) and (C2) that we introduced in Section 3 are both satisfied. We will consider here the case the nonlinearity f satisfies Hypothesis 4 and we leave to the reader to adapt our proof to the case f satisfies Hypothesis 3.

Theorem 7.1. *Under Hypotheses* 1 *and* 2 *and either Hypothesis* 3 *or Hypothesis* 4, *condition* (C1) *holds.*

Proof. Let us fix $T, M > 0$ and let $\{\varphi_\mu\}$ be a family of processes in $\Lambda_{T,M}$ such that

$$\lim_{\mu\to 0} \varphi_\mu = \varphi, \qquad \text{in distribution in } L^2_w(0,T;H),$$

where $L^2_w(0,T;H)$ is the space $L^2([0,T];H)$ endowed with the weak topology and $\varphi \in \Lambda_{T,M}$.

For every sequence $\{\mu_k\}_{k\in\mathbb{N}}$ converging to 0, as $k \uparrow \infty$, we denote $\rho_k = g(u_k)$, where $u_k := u_{\mu_k}^{\varphi_{\mu_k}}$ is the solution of equation (3.3), corresponding to the control φ_{μ_k}. Thanks to Theorem 5.2 and Lemma 5.1, for an arbitrary $\delta < 1$ we have that the family

$$\{\mathcal{L}(\rho_k,\ \mu_k \partial_t u_k, \varphi_{\mu_k})\}_{k\in\mathbb{N}} \subset \mathcal{P}\left(C([0,T];H^\delta) \times C([0,T];H) \times \mathcal{S}_{T,M}\right)$$

is tight. We denote by ρ a weak limit point for the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ and we denote

$$\Lambda := C([0,T];H^\delta) \times C([0,T];H) \times \mathcal{S}_{T,M} \times C([0,T],U)),$$

where U is the Hilbert space such that (2.1) holds. By the Skorokhod Theorem there exist random variables

$$\mathcal{Y} = (\hat\rho, 0, \hat\varphi, \hat w^Q), \qquad \mathcal{Y}_k = \left(\hat\rho_k, \hat\theta_k, \hat\varphi_k, \hat w^Q_k\right), \qquad k\in\mathbb{N},$$

defined on a probability space $(\hat\Omega, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t\in[0,T]}, \hat{\mathbb{P}})$, such that

$$\mathcal{L}(\mathcal{Y}) = \mathcal{L}(\rho, 0, \varphi, w^Q), \qquad \mathcal{L}(\mathcal{Y}_k) = \mathcal{L}(\rho_k,\ \mu_k\partial_t u_k, \varphi_{\mu_k}, w^Q), \qquad k\in\mathbb{N},$$

and such that

$$\lim_{k\to\infty} \mathcal{Y}_k = \mathcal{Y} \quad \text{in } \ \Lambda, \qquad \hat{\mathbb{P}} - \text{a.s.} \tag{7.1}$$

For every $k\in\mathbb{N}$ and $\psi \in H^2$, we have

$$\begin{aligned}\langle \hat\rho_k(t) + \hat\theta_k(t), \psi\rangle_H = \langle g(u_0) + \mu_k v_0, \psi\rangle_H + \int_0^t \langle \operatorname{div}\left[b(\hat\rho_k(s))\nabla\hat\rho_k(s)\right], \psi\rangle_H\, ds\\ + \int_0^t \langle F_g(\hat\rho_k(s)), \psi\rangle_H\, ds + \int_0^t \langle \sigma_g(\hat\rho_k(s))Q\,\hat\varphi_k(s), \psi\rangle_H\, ds\\ + \sqrt{\mu_k}\int_0^t \langle \sigma_g(\hat\rho_{\mu_k}(s))d\hat w^Q_k(s), \psi\rangle_H.\end{aligned}$$

Thanks to (7.1), for every $t\in[0,T]$ we have

$$\lim_{k\to\infty} \langle \hat\rho_k(t) + \hat\theta_k(t), \psi\rangle_H = \langle \hat\rho(t), \psi\rangle_H, \qquad \hat{\mathbb{P}} - \text{a.s.} \tag{7.2}$$

Next, if we define $\hat u_k := g^{-1}(\hat\rho_k)$ and $\hat u := g^{-1}(\hat\rho)$, we have

$$\begin{aligned}&\int_0^t \langle b(\hat\rho_k(s))\nabla\hat\rho_k(s), \nabla\psi\rangle_H ds - \int_0^t \langle b(\hat\rho(s))\nabla\hat\rho(s), \nabla\psi\rangle_H\, ds\\ &\quad = \int_0^t \langle \nabla\hat u_k(s), \nabla\psi\rangle_H ds - \int_0^t \langle \nabla\hat u(s), \nabla\psi\rangle_H ds = -\int_0^t \langle(\hat u_k(s) - \hat u(s)), \Delta\psi\rangle_H ds.\end{aligned}$$

In particular, since (7.1) implies the $\mathbb{P}$-a.s. convergence of $\hat u_k$ to $\hat u$ in $C([0,T];H)$, we get that

$$\lim_{k\to\infty} \int_0^t \langle b(\hat\rho_k(s))\nabla\hat\rho_k(s), \nabla\psi\rangle_H\, ds = \int_0^t \langle b(\hat\rho(s))\nabla\hat\rho(s), \nabla\psi\rangle_H\, ds, \qquad \hat{\mathbb{P}} - \text{a.s.} \tag{7.3}$$

It is immediate to check that estimate (2.16) extends to F_g. Thus we have

$$\left|\int_0^t \langle F_g(\hat\rho_k(s)),\psi\rangle_H\,ds - \int_0^t \langle F_g(\hat\rho(s)),\psi\rangle_H\,ds\right| \leq \int_0^t \|F_g(\hat\rho_k(s)) - F_g(\hat\rho(s))\|_{L^1}\,ds\,\|\psi\|_{H^1}$$
$$\leq c\int_0^t \left(1+\|\hat\rho_k(s)\|^{\theta-1}_{L^{2(\theta-1)}} + \|\hat\rho(s)\|^{\theta-1}_{L^{2(\theta-1)}}\right)\|\hat\rho_k(s)-\hat\rho(s)\|_H\,ds\,\|\psi\|_{H^1},$$

so that, thanks to (7.1),

$$\lim_{k\to\infty}\int_0^t \langle F_g(\hat\rho_k(s)),\psi\rangle_H\,ds = \int_0^t \langle F_g(\hat\rho(s)),\psi\rangle_H\,ds, \qquad \hat{\mathbb{P}}-\text{a.s.} \tag{7.4}$$

Now, for any $h\in L^2(0,T;H)$, due to (6.6)

$$\left|\int_0^t \langle \sigma_g(\hat\rho(s))Qh(s),\psi\rangle_H ds\right| \leq c\|\psi\|_H\left(\int_0^T (1+\|\hat\rho(s)\|_H^2)ds\right)^{\frac12}\left(\int_0^T \|h(s)\|_H^2 ds\right)^{\frac12}$$

and this implies that the mapping

$$h\in L^2([0,T];H)\mapsto \int_0^t \langle \sigma_g(\hat\rho(s)))Qh(s),\psi\rangle_H\,ds\in\mathbb{R}$$

is a linear functional so that, thanks to (7.1), for every fixed $t\geq 0$

$$\lim_{k\to\infty}\int_0^t \langle \sigma_g(\hat\rho(s))Q\hat\varphi_k(s),\psi\rangle_H ds = \int_0^t \langle \sigma_g(\hat\rho(s))Q\hat\varphi(s),\psi\rangle_H\,ds, \qquad \hat{\mathbb{P}}-\text{a.s.} \tag{7.5}$$

Moreover, we have

$$\left|\int_0^t \langle(\sigma_g(\hat\rho_k(s))-\sigma_g(\hat\rho(s)))Q\hat\varphi_k(s),\psi\rangle_H ds\right|$$
$$\leq \|\psi\|_H\int_0^t \|\sigma_g(\hat\rho_k(s))-\sigma_g(\hat\rho(s))\|_{\mathcal{L}(H_Q,H)}\|\hat\varphi_k(s)\|_H\,ds$$
$$\leq c\,\|\psi\|_H\|\hat\rho_k-\hat\rho\|_{L^2([0,T];H)}\|\hat\varphi_k\|_{L^2([0,T];H)},$$

and by using again (7.1) we get

$$\lim_{k\to\infty}\int_0^t \langle(\sigma_g(\hat\rho_k(s))-\sigma_g(\hat\rho(s)))Q\hat\varphi_k(s),\psi\rangle_H ds = 0, \qquad \hat{\mathbb{P}}-\text{a.s.}$$

This, together with (7.5), implies

$$\lim_{k\to\infty}\int_0^t \langle \sigma_g(\hat\rho_k(s))Q\hat\varphi_k(s),\psi\rangle_H ds = \int_0^t \langle \sigma_g(\hat\rho(s))Q\hat\varphi(s),\psi\rangle_H ds, \qquad \hat{\mathbb{P}}-\text{a.s.} \tag{7.6}$$

Finally, since

$$\sup_{k\in\mathbb{N}}\hat{\mathbb{E}}\sup_{t\in[0,T]}\left|\int_0^t \langle \sigma_g(\hat\rho_{\mu_k}(s))d\hat w_k^Q(s),\psi\rangle_H\right|^2$$
$$\leq c\sup_{k\in\mathbb{N}}\|\psi\|_H^2\int_0^T\left(1+\hat{\mathbb{E}}\|\hat\rho_{\mu_k}(s)\|_H^2\right)ds<\infty,$$

we conclude that

$$\lim_{k\to\infty}\sqrt{\mu_k}\sup_{t\in[0,T]}\hat{\mathbb{E}}\left|\int_0^t \langle \sigma_g(\hat\rho_{\mu_k}(s))d\hat w_k^Q(s),\psi\rangle_H\right|^2 = 0.$$

This, together with (7.2), (7.3), (7.4) and (7.6), implies that

$$\langle \hat\rho(t), \psi\rangle_H = \langle g(u_0), \psi\rangle_H - \int_0^t \langle b(\hat\rho(s))\nabla\hat\rho(s), \nabla\psi\rangle_H ds$$
$$\int_0^t \left[\langle F_g(\hat\rho(s)), \psi\rangle_H + \langle \sigma_g(\hat\rho(s))Q\hat\varphi(s), \psi\rangle_H\right] ds.$$

As proven in Proposition 6.2, the equation above has at most one solution. Thus, for every sequence $\{\mu_k\}_{k\in\mathbb{N}} \downarrow 0$ the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ converges in distribution to the solution ρ of equation (6.1) with respect to the strong topology of $C([0,T];H^\delta)$, for any arbitrary $\delta < 1$, and hence, due to the embedding of H^1 into $L^p(\mathcal{O})$, with respect to the strong topology of $C([0,T];L^p(\mathcal{O}))$, for every $p < \infty$, if $d = 1, 2$ and $p < 2d/(d-2)$, if $d > 2$. In particular, this implies that the sequence $\{u_k\}_{k\in\mathbb{N}}$ converges in distribution to the solution u^φ of equation (3.2) with respect to the strong topology of $C([0,T];L^p(\mathcal{O}))$ and condition (C1) holds.

As a consequence of the arguments used above to prove condition (C1), we have that the mapping

$$\varphi \in L^2_w(0,T;H) \mapsto u^\varphi \in C([0,T];L^p(\mathcal{O}))$$

is continuous. Therefore, since $\Lambda_{T,M}$ is compact in $L^2_w(0,T;H)$, for every $M > 0$, we have that

$$\Phi_{T,R} = \{I_T \leq R\} = \{u^\varphi \ : \ \varphi \in \Lambda_{T,2R^2}\}$$

is compact, and condition (C2) follows. □

Appendix A. The small mass limit for system (1.1)

The Smoluchowski-Kramers approximation for system (1.1) has been studied in [14], in the case f is Lipschitz-continuous and σ is bounded. Here, we prove an analogous result when σ is unbounded and f has polynomial growth (see Hypothesis 4).

In what follows, we shall assume that Hypotheses 1, 2 and Hypothesis 4 hold. By applying Theorem 3.2 with the control $\varphi = 0$, we have that for every $T > 0$ and every $(u_0, v_0) \in \mathcal{H}_1$ there exists a unique adapted solution u_μ to equation (1.1), such that $(u_\mu, \partial_t u_\mu) \in L^2(\Omega; C([0,T];\mathcal{H}_1))$.

For every $a \in [0,1)$ we denote

$$X(a) := \bigcap_{q<q(a)} L^q(0,T;H^a), \qquad Y(a) := \bigcap_{p<2/a} L^p(0,T;H^a),$$

where

$$q(a) := \frac{2(\theta+1)}{2+(\theta-1)a}.$$

Theorem A.1. *Assume Hypotheses 1, 2 and 4 hold, and for every $\mu > 0$, let u_μ denote the unique solution to equation (1.1), with the initial conditions $(u_0, v_0) \in \mathcal{H}_1$.*

(1) *If $\theta \in (1,3)$, then for every $a \in [0,1)$ and $\eta > 0$ we have*

$$\lim_{\mu\to 0} \mathbb{P}\Big(\|u_\mu - u\|_{X(a)} > \eta \Big) = 0,$$

where $u \in L^2(\Omega; L^2([0,T];H^1))$ is the unique solution to equation (1.3), with initial datum u_0.

(2) *If we assume*

(A.1)
$$\begin{cases} \exists\, \rho \in [0,(\theta+1)/4) \ \ \text{such that} \ \ \|\sigma(h)\|_{\mathcal{L}_2(H_Q,H)} \leq c\big(1+\|h\|_H^\rho\big), \qquad h\in H, \\ \theta \in (1,3], \quad \text{if } \rho>0, \qquad \theta\in(1,5), \quad \text{if } \rho=0, \end{cases}$$

then for every $a\in[0,1)$ *and* $\eta>0$ *we have*

$$\lim_{\mu\to 0}\mathbb{P}\Big(\,\|u_\mu-u\|_{Y(a)}>\eta\Big)=0.$$

A.1. **Energy estimates.** One of the key ingredients in our proof of Theorem A.1 is the tightness of $\{u_\mu\}_{\mu\in(0,1)}$ in suitable functional spaces. This will require the following a priori bounds.

Lemma A.2. *Assume Hypotheses* 1, 2 *and* 4 *hold, and fix* $T>0$ *and* $(u_0,v_0)\in\mathcal{H}_1$.

(1) *There exist* $\mu_T\in(0,1)$ *and* $c_T>0$ *such that for every* $\mu\in(0,\mu_T)$,

$$\mathbb{E}\sup_{t\in[0,T]}\|u_\mu(t)\|_H^2+\mathbb{E}\int_0^T\|u_\mu(t)\|_{H^1}^2\,dt+\mathbb{E}\int_0^T\|u_\mu(t)\|_{L^{\theta+1}}^{\theta+1}\,dt\leq c_T, \tag{A.2}$$

and

$$\mathbb{E}\sup_{t\in[0,T]}\|u_\mu(t)\|_{H^1}^2+\mathbb{E}\sup_{t\in[0,T]}\|u_\mu(t)\|_{L^{\theta+1}}^{\theta+1}+\mu\mathbb{E}\sup_{t\in[0,T]}\|\partial_t u_\mu(t)\|_H^2+\mathbb{E}\int_0^T\|\partial_t u_\mu(t)\|_H^2\,dt \leq \frac{c_T}{\mu}. \tag{A.3}$$

(2) *If, in addition, condition* (A.1) *holds, then for every* $\mu\in(0,\mu_T)$

$$\mathbb{E}\sup_{t\in[0,T]}\|u_\mu(t)\|_{H^1}^2+\mathbb{E}\sup_{t\in[0,T]}\|u_\mu(t)\|_{L^{\theta+1}}^{\theta+1}+\mu\mathbb{E}\sup_{t\in[0,T]}\|\partial_t u_\mu(t)\|_H^2\leq\frac{c_T}{\mu^\beta}, \tag{A.4}$$

where

$$\beta=\beta(\rho):=\frac{\theta+1}{2(\theta+1-2\rho)}.$$

Remark A.1.

(1) If the mapping σ is bounded, then condition (A.1) is satisfied for $\rho=0$, in which case $\beta=1/2$, so that for every $\mu\in(0,\mu_T)$

$$\sqrt{\mu}\,\mathbb{E}\sup_{t\in[0,T]}\left(\|u_\mu(t)\|_{H^1}^2+\|u_\mu(t)\|_{L^{\theta+1}}^{\theta+1}+\mu\,\|\partial_t u_\mu(t)\|_H^2\right)\leq c_T.$$

(2) When condition (A.1) holds with $\rho=0$, in order to prove (A.4) we can take any $\theta>1$.

(3) If $\rho\in[0,(\theta+1)/4)$, then $\beta<1$. Therefore, if condition (A.1) holds, due to (A.4) we have

$$\lim_{\mu\to0}\mu^2\mathbb{E}\sup_{t\in[0,T]}\|\partial_t u_\mu(t)\|_H^2=0, \tag{A.5}$$

which is the same bound proven in [14].

Proof. Estimates (A.2) and (A.3) can be proved by proceeding as in the proof of Lemma 5.1. Thus, we will only prove (A.4) under condition (A.1).

For every $\mu \in (0,1)$ and $t \in [0,T]$, define

$$L_\mu(t) := \|u_\mu(t)\|_{H^1}^2 + \int_0^L \Big(c_2 - \mathfrak{f}(x, u_\mu(t,x))\Big)dx + \mu\,\|\partial_t u_\mu(t)\|_H^2\,,$$

where the function $\mathfrak{f}$ and the constant c_2 have been introduced in Hypothesis 4. Due to (2.11), we have that

$$L_\mu(t) \geq \|u_\mu(t)\|_{H^1}^2 + \|u_\mu(t)\|_{L^{\theta+1}}^{\theta+1} + \mu\,\|\partial_t u_\mu(t)\|_H^2, \qquad \mathbb{P}-\text{a.s.}$$

Thus, we obtain (A.4) once we have proved that

$$\mu^\beta \mathbb{E} \sup_{t\in[0,T]} L_\mu(t) \leq c_T. \tag{A.6}$$

Assume (A.6) is not true. Then there is a sequence $(\mu_k)_{k\in\mathbb{N}} \subset (0,1)$ converging to 0, as $k \to \infty$, such that

$$\lim_{k\to\infty} \mu_k^\beta\, \mathbb{E} \sup_{t\in[0,T]} L_{\mu_k}(t) = +\infty. \tag{A.7}$$

For every $k \in \mathbb{N}$, the mapping $t \mapsto L_{\mu_k}(t)$ is continuous $\mathbb{P}$-a.s., so that there exists a random time $t_k \in [0,T]$ such that

$$L_{\mu_k}(t_k) = \sup_{t\in[0,T]} L_{\mu_k}(t).$$

As a consequence of the Itô formula, if s is any random time such that $\mathbb{P}(s \leq t_k) = 1$, we have

$$L_{\mu_k}(t_k) - L_{\mu_k}(s) \leq \frac{1}{\mu_k}\int_s^{t_k} \|\sigma(u_{\mu_k}(r))\|_{\mathcal{L}_2(H_Q,H)}^2\,dr + 2\big(M_k(t_k) - M_k(s)\big),$$

where

$$M_k(t) := \int_0^t \big\langle \partial_t u_{\mu_k}(r), \sigma(u_{\mu_k}(r))dw^Q(r)\big\rangle_H.$$

Thanks to Young's inequality, since $2\beta \geq 1$ and $\mu_k < 1$, we have

$$\begin{aligned}
\frac{1}{\mu_k}\int_s^{t_k} \|\sigma(u_{\mu_k}(r))\|_{\mathcal{L}_2(H_Q,H)}^2\,dr &\leq \frac{c}{\mu_k}\int_s^{t_k}\Big(1 + \|u_{\mu_k}(r)\|_H^{2\rho}\Big)dr \\
&\leq c\left(\frac{t_k - s}{\mu_k} + \frac{t_k - s}{\mu_k^{\frac{\theta+1}{\theta+1-2\rho}}} + \int_0^T \|u_{\mu_k}(t)\|_H^{\theta+1}\,dt\right) \\
&\leq c\left(\frac{t_k - s}{\mu_k^{2\beta}} + \int_0^T \|u_{\mu_k}(t)\|_H^{\theta+1}\,dt\right).
\end{aligned}$$

Therefore, if we define

$$U_k := \int_0^T \|u_{\mu_k}(t)\|_{L^{\theta+1}}^{\theta+1}\,dt, \quad \text{and} \quad M_k := \sup_{t\in[0,T]} |M_k(t)|,$$

we can fix a constant κ_T independent of k, with $L_{\mu_k}(0) \leq \kappa_T$, such that

$$L_{\mu_k}(t_k) - L_{\mu_k}(s) \leq \kappa_T\Big(\frac{t_k - s}{\mu_k^{2\beta}} + U_k\Big) + 4M_k.$$

In particular, if we take $s=0$, we have

$$t_k \geq \frac{\mu_k^{2\beta}}{\kappa_T}\Big(L_{\mu_k}(t_k) - 4M_k - \kappa_T(U_k+1)\Big) =: \frac{\mu_k^{2\beta}}{\kappa_T}\delta_k. \tag{A.8}$$

On the set $E_k := \big\{\delta_k > 0\big\}$, for any $s \in \big[t_k - \frac{\mu_k^{2\beta}}{2\kappa_T}\delta_k, t_k\big]$ we have

$$L_{\mu_k}(s) \geq L_{\mu_k}(t_k) - \frac{1}{2}\delta_k - \kappa_T U_k - 4M_k = \frac{1}{2}\Big[L_{\mu_k}(t_k) - 4M_k - \kappa_T(U_k-1)\Big].$$

Hence, if we define

$$I_k := \int_0^T L_{\mu_k}(s)ds,$$

recalling how δ_k was defined in (A.8), we have

$$I_k \geq \int_{t_k - \frac{\mu_k^{2\beta}}{2\kappa_T}\delta_k}^{t_k} L_{\mu_k}(s)ds \geq \frac{\mu_k^{2\beta}}{4\kappa_T}\Big[\Big(L_{\mu_k}(t_k) - 4M_k - \kappa_T U_k\Big)^2 - \kappa_T^2\Big],$$

so that

$$\mathbb{E}(I_k) \geq \mathbb{E}(I_k; E_k) \geq \mathbb{E}\left(\frac{\mu_k^{2\beta}}{4C_T}\Big(L_{\mu_k}(t_k) - 4M_k - C_T U_k\Big)^2; E_k\right) - \frac{\mu_k^{2\beta}}{4}C_T. \tag{A.9}$$

Now, thanks to condition (A.1), and estimates (A.2) and (A.3), we have

$$\begin{aligned}
\mathbb{E}(M_k) &\leq \mathbb{E}\left(\int_0^T \|\partial_t u_{\mu_k}(t)\|_H^2 \, \|\sigma(u_{\mu_k}(t))\|^2_{\mathcal{L}_2(H_Q,H)}\, dt\right)^{\frac{1}{2}}\\
&\leq \mathbb{E}\left[\sup_{t\in[0,T]} \|\sigma(u_{\mu_k}(t))\|_{\mathcal{L}_2(H_Q,H)} \left(\int_0^T \|\partial_t u_{\mu_k}(t)\|_H^2\, dt\right)^{\frac{1}{2}}\right]\\
&\leq c\,\mathbb{E}\sup_{t\in[0,T]} \|\sigma(u_{\mu_k}(t))\|^{\frac{2}{\rho}}_{\mathcal{L}_2(H_Q,H)} + c\,\mathbb{E}\Big(\int_0^T \|\partial_t u_{\mu_k}(t)\|_H^2\, dt\Big)^{\frac{1}{2-\rho}}\\
&\leq c\left(1 + \mathbb{E}\sup_{t\in[0,T]} \|u_{\mu_k}(t)\|_H^2\right) + c\Big(\int_0^T \mathbb{E}\,\|\partial_t u_{\mu_k}(t)\|_H^2\, dt\Big)^{\frac{1}{2-\rho}} \leq \frac{C_T}{\mu_k^{\frac{1}{2-\rho}}}.
\end{aligned}$$

In particular, since (A.1) implies

$$\beta = \frac{\theta+1}{2(\theta+1-2\rho)} \geq \frac{1}{2-\rho},$$

we have that

$$\limsup_{k\to\infty} \mu_k^{\beta}\mathbb{E}(M_k) < +\infty. \tag{A.10}$$

Moreover, by (A.2) and (A.3), we have

$$\sup_k \mathbb{E}(U_k) < \infty. \tag{A.11}$$

Therefore, due to (A.8), (A.10) and (A.11), as a consequence of (A.7) we have

$$\lim_{k\to\infty} \mu_k^{\beta}\mathbb{E}(\delta_k) = +\infty. \tag{A.12}$$

Now, we have

$$\mu_k^{\beta}\mathbb{E}(\delta_k) = \mu_k^{\beta}\mathbb{E}(\delta_k; E_k) \leq \mathbb{E}\big(\mu_k^{\beta}(\delta_k+\kappa_T); E_k\big) \leq \Big[\mathbb{E}\big(\mu_k^{2\beta}(\delta_k+\kappa_T)^2; E_k\big)\Big]^{\frac{1}{2}},$$

so that, thanks to (A.9) we have

$$\mathbb{E}(I_k) \geq \frac{1}{4\kappa_T}\mathbb{E}\big(\mu_k^{2\beta}(\delta_k+\kappa_T)^2; E_k\big) - \frac{\mu_k^{2\beta}\kappa_T}{4} \geq \frac{1}{4\kappa_T}\left[\mu_k^{\beta}\mathbb{E}(\delta_k)\right]^2 - \frac{\mu_k^{2\beta}\kappa_T}{4}.$$

Due to (A.12), this implies

$$\lim_{k\to\infty}\mathbb{E}(I_k) = +\infty.$$

However, the limit above is not possible. Actually, since we

$$L_{\mu_k}(t) \leq \|u_{\mu_k}(t)\|_{H^1}^2 + \mu_k\,\|\partial_t u_{\mu_k}(t)\|_H^2 + L\,c\Big(1+\|u_{\mu_k}(t)\|_{L^{\theta+1}}^{\theta+1}\Big), \quad \mathbb{P}\text{-a.s.}$$

as a consequence of (A.2) and (A.3) we have

$$\sup_{k\in\mathbb{N}}\mathbb{E}(I_k) < +\infty,$$

and this gives a contradiction. In particular, this means that claim (A.6) is true, and (A.4) holds. □

A.2. **Tightness.** As in Section 5, for every $T>0$ and $\mu>0$ we have defined

$$\rho_\mu(t,x) = g(u_\mu(t,x)), \quad (t,x)\in[0,T]\times[0,L],$$

and, by integrating equation (1.1) with respect to t, we got

(A.13)

$$\rho_\mu(t) + \mu\partial_t u_\mu(t) = g(u_0) + \mu v_0 + \int_0^t \operatorname{div}\big[b(\rho_\mu(s))\nabla\rho_\mu(s)\big]ds + \int_0^t F_g(\rho_\mu(s))ds + \int_0^t \sigma_g(\rho_\mu(s))dw^Q(s),$$

where we recall that b, f_g, F_g and σ_g were defined in (5.11) and (5.12).

Definition A.2. Let E be a Banach space with norm $\|\cdot\|_E$. Given $r>1$ and $\lambda\in(0,1)$, we denote by $W^{\lambda,r}(0,T;E)$ be the Banach space of all $u\in L^p(0,T;E)$ such that

$$[u]_{W^{\lambda,r}(0,T;E)} := \int_0^T\int_0^T \frac{\|u(t)-u(s)\|_E^r}{|t-s|^{1+\lambda r}}dtds < \infty,$$

endowed with the norm

$$\|u\|_{W^{\lambda,r}(0,T;E)}^r = \int_0^T \|u(t)\|_E^r\,dt + [u]_{W^{\lambda,r}(0,T;E)}.$$

It is possible to prove that if $\lambda r<1$, $p\leq r/(1-\lambda r)$ and $1\leq r\leq p$, then $W^{\lambda,r}(0,T;E)\subset L^p(0,T;E)$ and there exists some $c>0$ such that for all $u\in W^{\lambda,r}(0,T;E)$

(A.14) $$\|\tau_h(u)-u\|_{L^p(0,T-h;E)} \leq c\,h^\lambda T^{1/p-1/r}\,[u]_{W^{\lambda,r}(0,T;E)}, \qquad h>0,$$

where

$$\tau_h(u)(t) = u(t+h), \qquad t\in[-h,T-h]$$

(see [32, Lemma5]).

Proposition A.3. *Assume Hypotheses 1, 2 and 4 hold, and fix any $T>0$ and $(u_0,v_0)\in\mathcal{H}_1$, and an arbitrary sequence $(\mu_k)_{k\in\mathbb{N}}\subset(0,1)$ that converges to 0.*

(1) *The family of probability measures $\big(\mathcal{L}(\rho_{\mu_k})\big)_{k\in\mathbb{N}}$ is tight in $X(a)$, for every $a\in[0,1)$.*

(2) *If condition* (A.1) *holds, then the family of probability measures* $\big(\mathcal{L}(\rho_{\mu_k})\big)_{k\in\mathbb{N}}$ *is tight in* $Y(a)$, *for every* $a \in [0,1)$.

Remark A.3.

(1) By taking $a = 0$, we have

$$(\mathcal{L}(\rho_{\mu_k}))_{k\in\mathbb{N}} \quad \text{is tight in} \quad \bigcap_{q<\theta+1} L^q(0,T;H)$$

and thanks to the embedding $H^a \hookrightarrow C([0,L])$, for $a \in (1/2,1)$, we have

$$\big(\mathcal{L}(\rho_{\mu_k})\big)_{k\in\mathbb{N}} \quad \text{is tight in} \quad \bigcap_{q<4(\theta+1)/(\theta+3)} L^q(0,T;C([0,L])).$$

(2) When condition (A.1) holds, we have that

$$\big(\mathcal{L}(\rho_{\mu_k})\big)_{k\in\mathbb{N}} \quad \text{is tight in} \quad \bigcap_{p<\infty} L^p(0,T;H).$$

Proof. For every $0 \le t_1 \le t_2 \le T$, we have

$$\begin{aligned}
\mathbb{E}\left\|\int_{t_1}^{t_2} \operatorname{div}\big[b(\rho_\mu(s))\nabla\rho_\mu(s)\big]ds\right\|_{H^{-1}}^{(\theta+1)/\theta}
&\le \left(\mathbb{E}\left\|\int_{t_1}^{t_2} \operatorname{div}\big[b(\rho_\mu(s))\nabla\rho_\mu(s)\big]ds\right\|_{H^{-1}}^2\right)^{(\theta+1)/2\theta}\\
&\le C(t_2-t_1)^{(\theta+1)/2\theta}\left(\mathbb{E}\int_0^T \|\rho_\mu(s)\|_{H^1}^2\,ds\right)^{(\theta+1)/2\theta},
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left\|\int_{t_1}^{t_2} F_g(\rho_{\mu_k}(s))ds\right\|_{H^{-1}}^{(\theta+1)/\theta}
&\le C\mathbb{E}\left(\int_{t_1}^{t_2}\left(1+\|u_\mu(s)\|_{\theta+1}^{\theta}\right)ds\right)^{(\theta+1)/\theta}\\
&\le C(t_2-t_1)^{1/\theta}\mathbb{E}\int_0^T\left(1+\|u_\mu(s)\|_{\theta+1}^{\theta+1}\right)ds,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left\|\int_{t_1}^{t_2}\sigma_g(\rho_\mu(s))dw^Q(s)\right\|_{H^{-1}}^{(\theta+1)/\theta} &\le \left(\mathbb{E}\left\|\int_{t_1}^{t_2}\sigma_g(\rho_\mu(s))dw^Q(s)\right\|_{H^{-1}}^2\right)^{(\theta+1)/2\theta}\\
&\le C(t_2-t_1)^{(\theta+1)/2\theta}\left(1+\mathbb{E}\sup_{t\in[0,T]}\|u_\mu(t)\|_H^2\right)^{(\theta+1)/2\theta}.
\end{aligned}$$

In view of (A.2) and (A.13), it is not difficult to show that for every $\lambda \in (0,1/(\theta+1))$,

$$\sup_{\mu\in(0,\mu_T)} \mathbb{E}[\rho_\mu + \mu\partial_t u_\mu]_{W^{\lambda,\theta_0}(0,T;H^{-1})}^{\theta_0} < \infty, \tag{A.15}$$

where

$$\theta_0 := \frac{\theta+1}{\theta} \in (1,2).$$

Moreover, by (A.2) and (A.3) we have

$$\sup_{\mu\in(0,\mu_T)} \mathbb{E}\,\|\rho_\mu+\mu\partial_t u_\mu\|^2_{L^\infty([0,T];H)}<\infty. \tag{A.16}$$

Therefore, from (A.15) and (A.16) we conclude that for every $\epsilon>0$, there exists $L_1(\epsilon)>0$ such that, if we define

$$K_1^\epsilon=\left\{f:[0,T]\times\mathbb{R}\to\mathbb{R}:[f]_{W^{\lambda,\theta_0}(0,T;H^{-1})}+\|f\|_{L^\infty(0,T;H)}\leq L_1(\epsilon)\right\},$$

then

$$\inf_{\mu\in(0,\mu_T)}\mathbb{P}\big(\rho_\mu+\mu\partial_t u_\mu\in K_1^\epsilon\big)>1-\epsilon/4.$$

According to (A.14), we have that for every $p<(\theta+1)/\theta$

$$\lim_{h\to 0}\|\tau_h f-f\|_{L^p(0,T-h;H^{-1})}=0,\qquad f\in K_1^\epsilon.$$

Hence, in view of [32, Theorem 6], we have that K_1^ϵ is relatively compact in $L^q(0,T,H^{-\delta})$, for every $q<\infty$ and $\delta>0$.

Next, due to (A.3), we have

$$\lim_{\mu\to 0}\mathbb{E}\,\|\mu\partial_t u_\mu\|^2_{L^2([0,T];H)}=0,$$

hence for every sequence $(\mu_k)_{k\in\mathbb{N}}\subset(0,\mu_T)$ converging to zero, there exists a compact K_2^ϵ in $L^2([0,T];H)$ such that

$$\mathbb{P}\big(\mu_k\partial_t u_{\mu_k}\in K_2^\epsilon\big)>1-\epsilon/4,\qquad k\in\mathbb{N}.$$

Since $L^2([0,T];H)\subset L^{\theta_0}([0,T];H^{-\delta})$, for $\delta>0$, we have that K_2^ϵ is also compact in $L^{\theta_0}([0,T];H^{-\delta})$, which implies that $K_1^\epsilon+K_2^\epsilon$ is relatively compact in $L^{\theta_0}([0,T];H^{-\delta})$, and for every $k\in\mathbb{N}$,

$$\mathbb{P}\big(\rho_{\mu_k}\in K_1^\epsilon+K_2^\epsilon\big)\geq 1-\epsilon/2.$$

Moreover, thanks to estimate (A.2) there exists $L_2(\epsilon)>0$ such that, if we define

$$K_3^\epsilon=\Big\{f:[0,T]\times\mathbb{R}\to\mathbb{R}:\|f\|_{L^{\theta+1}([0,T];H)}\leq L_2(\epsilon)\Big\},$$

then

$$\inf_{\mu\in(0,\mu_T)}\mathbb{P}\big(\rho_\mu\in K_3^\epsilon\big)\geq 1-\epsilon/4,$$

and thus

$$\inf_{k\in\mathbb{N}}\mathbb{P}\Big(\rho_{\mu_k}\in\big(K_1^\epsilon+K_2^\epsilon\big)\cap K_3^\epsilon\Big)\geq 1-3\epsilon/4.$$

By using again [32, Theorem 6], $\big(K_1^\epsilon+K_2^\epsilon\big)\cap K_3^\epsilon$ is relatively compact in $L^p([0,T];H^{-\delta})$ for every $\delta>0$ and $p<\theta+1$. This implies that the family of probability measures $\big(\mathcal{L}(\rho_{\mu_k})\big)_{k\in\mathbb{N}}$ is tight in $L^p([0,T];H^{-\delta})$, for every $\delta>0$ and $p<\theta+1$.

Now, according to [32, Theorem 1], we have

$$\lim_{h\to 0}\sup_{f\in\left(K_1^\epsilon+K_2^\epsilon\right)\cap K_3^\epsilon}\|\tau_h f-f\|_{L^p([0,T];H^{-\delta})}=0.$$

Furthermore, since

$$\sup_{\mu\in[0,\mu_T]}\mathbb{E}\,\|\rho_\mu\|^2_{L^2([0,T];H^1)}<\infty,$$

there exists $L_3(\epsilon)>0$ such that, if we define

$$K_4^\epsilon=\Big\{f:[0,T]\times\mathbb{R}\to\mathbb{R}\ :\ \|f\|_{L^2([0,T];H^1)}\leq L_3(\epsilon)\Big\},$$

then

$$\inf_{\mu\in(0,\mu_T)} \mathbb{P}\big(\rho_\mu \in K_4^\epsilon\big) \geq 1-\epsilon/4.$$

Thus, if we define

$$K^\epsilon = \big(K_1^\epsilon + K_2^\epsilon\big) \cap K_3^\epsilon \cap K_4^\epsilon,$$

we have

$$\inf_{k\in\mathbb{N}} \mathbb{P}(\rho_{\mu_k} \in K^\epsilon) \geq 1-\epsilon.$$

For every $a \in [0,1)$ and for any $\delta > 0$, by interpolation

$$\|u\|_{H^a} \leq C(a,\delta)\, \|u\|_{H^{-\delta}}^{\frac{1-a}{1+\delta}}\, \|u\|_{H^1}^{\frac{a+\delta}{1+\delta}}.$$

Thus, according to [32, Theorem 7], we have K^ϵ is relatively compact in $L^q(0,T;H^a)$, where $q = q(a,\delta,p)$ satisfies

$$\frac{1}{q} = \frac{1-a}{p(1+\delta)} + \frac{a+\delta}{2(1+\delta)}, \qquad \delta > 0, \quad p < \theta+1.$$

This means that K^ϵ is relatively compact in $L^q(0,T;H^a)$, for every $q < q(a)$, where

$$q(a) = \frac{2(\theta+1)}{2+(\theta-1)a}, \qquad a \in [0,1),$$

so that $\big(\mathcal{L}(\rho_{\mu_k})\big)_{k\in\mathbb{N}}$ is tight in $X(a)$.

Now let us assume that the condition (A.1) holds. Thanks to (A.5) we know that

$$\lim_{\mu\to 0} \mathbb{E}\, \|\mu \partial_t u_\mu\|_{L^\infty([0,T];H)} = 0.$$

Since $L^\infty(0,T;H) \subset L^q(0,T;H^{-\delta})$, for every $q < \infty$ and $\delta > 0$, we can proceed as in the proof of part (1), and we have that $\big(\mathcal{L}(\rho_{\mu_k})\big)_{k\in\mathbb{N}}$ is tight in $L^p([0,T];H^{-\delta})$ for every $p < \infty$ and $\delta > 0$. Finally, since

$$\text{(A.17)} \qquad \sup_{\mu\in[0,\mu_T]} \mathbb{E}\, \|\rho_\mu\|^2_{L^2([0,T];H^1)} < \infty,$$

by using the same argument as in the proof of part (1), we have that for every $a \in [0,1)$, $\big(\mathcal{L}(\rho_{\mu_k})\big)_{k\in\mathbb{N}}$ is tight in $L^q([0,T];H^a)$, where $q = q(a,\delta,p)$ satisfies

$$\frac{1}{q} = \frac{1-a}{p(1+\delta)} + \frac{a+\delta}{2(1+\delta)}, \qquad \delta > 0, \quad p < \infty.$$

This implies that $\big(\mathcal{L}(\rho_{\mu_k})\big)_{k\in\mathbb{N}}$ is tight in $L^q([0,T];H^a)$, for every $q < 2/a$, and the proof of part (2) follows. $\square$

A.3. **The limiting problem.** Here we will prove the uniqueness of solutions for the following equation

$$\begin{cases} \gamma(u(t,x))\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x)) - \dfrac{\gamma'(u(t,x))}{2\gamma^2(u(t,x))} \displaystyle\sum_{i=1}^\infty |[\sigma(u(t,\cdot)Qe_i](x)|^2 \\ \qquad\qquad + \sigma(u(t,\cdot))\partial_t w^Q(t,x), \\ u(0,x) = u_0(x), \qquad u(t,0) = u(t,L) = 0. \end{cases}$$

To this purpose, we shall first study the following quasi-linear parabolic equation

(A.18)
$$\begin{cases}\partial_t\rho(t,x) = \operatorname{div}\big[b(\rho(t,x))\nabla\rho(t,x)\big] + f_g(x,\rho(t,x)) + \sigma_g(\rho(t,\cdot))dw^Q(t,x),\\[2ex] \rho(0,x) = g(u_0(x)), \qquad \rho(t,0)=\rho(t,L)=0.\end{cases}$$

Definition A.4. An $(\mathcal{F}_t)_{t\geq 0}$ adapted process $\rho\in L^2(\Omega;L^2(0,T;H^1))$ is a solution of equation (A.18) if for every test function $\psi\in C_0^\infty([0,L])$

(A.19)
$$\begin{aligned}\langle\rho(t),\psi\rangle_H = \langle g(u_0),\psi\rangle_H &+ \int_0^t \langle b(\rho(s))\nabla\rho(s),\nabla\psi\rangle_H ds\\ &+\int_0^t\langle F_g(\rho(s)),\psi\rangle_H ds + \int_0^t\langle\sigma_g(\rho(s))dw^Q(s),\psi\rangle_H.\end{aligned}$$

By proceeding as in the proof of [14, Theorem 6.2], combined with similar arguments as in the proof of Proposition 6.2, we have the following result.

Proposition A.5. *Under Hypotheses* 1, 2 *and* 4, *there is at most one solution* $\rho\in L^2(\Omega;L^2(0,T;H^1))$ *to equation* (A.18).

Moreover, by proceeding as in the proof of [14, Theorem 7.1], we have the following result.

Proposition A.4. *Assume Hypotheses* 1 *and* 2 *and either Hypothesis* 3 *or Hypothesis* 4 *and fix any* $T>0$, $u_0\in H^1$. *Suppose that* $\rho\in L^2(\Omega;L^2(0,T;H^1))$ *is the solution of equation* (A.18). *Then, if we define* $u:=g^{-1}(\rho)$, *we have that* u *belongs to* $L^2(\Omega;L^2(0,T;H^1))$ *and is a solution to equation* (1.3). *Furthermore, equation* (1.3) *admits at most one solution* $u\in L^2(\Omega;L^2(0,T;H^1))$.

A.4. **Proof of Theorem A.1.** For every sequence $\{\mu_k\}_{k\in\mathbb{N}}\subset(0,\mu_T)$ converging to 0 as $k\to\infty$, we denote

$$u_k := u_{\mu_k} \quad\text{and}\quad \rho_k := g(u_k),\quad k\in\mathbb{N}.$$

In view of the first part of Proposition A.3, if we define

$$X := \bigcap_{0\leq a<1} X(a),$$

we have that the family

$$\big\{\mathcal{L}(\rho_k,\ \mu_k\partial_t u_k)\big\}_{k\in\mathbb{N}}\subset\mathcal{P}\Big(X\times L^2(0,T;H)\Big)$$

is tight.

We denote by ρ a weak limit point for the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ and we denote

$$\mathcal{K} := X\times L^2(0,T;H)\times C([0,T],U),$$

where U is the Hilbert space such that the embedding $H_Q\subset U$ is Hilbert-Schmidt. According to the Skorokhod Theorem there exist random variables

$$\mathcal{Y} = (\hat\rho,0,\hat w^Q),\qquad \mathcal{Y}_k = \left(\hat\rho_k,\hat\theta_k,\hat w_k^Q\right),\quad k\in\mathbb{N},$$

defined on a probability space $(\hat\Omega,\hat{\mathcal{F}},\{\hat{\mathcal{F}}_t\}_{t\in[0,T]},\hat{\mathbb{P}})$, such that

$$\mathcal{L}(\mathcal{Y}) = \mathcal{L}(\rho,0,w^Q),\qquad \mathcal{L}(\mathcal{Y}_k) = \mathcal{L}(\rho_k,\ \mu_k\partial_t u_k, w^Q),\quad k\in\mathbb{N},$$

and such that

$$\lim_{k\to\infty} \mathcal{Y}_k = \mathcal{Y} \text{ in } \mathcal{K}, \quad \hat{\mathbb{P}}\text{-a.s.} \tag{A.20}$$

In particular,

$$\lim_{k\to\infty} \Big(\|\hat{\rho}_k - \hat{\rho}\|_{L^p([0,T];H)} + \|\hat{\rho}_k - \hat{\rho}\|_{L^q([0,T];C([0,L]))} \Big) = 0, \quad \hat{\mathbb{P}}\text{-a.s.} \tag{A.21}$$

for every $p < \theta + 1$ and every $q < 4(\theta+1)/(\theta+3)$. Moreover, thanks to (A.17), we have that $\hat{\rho} \in L^2(\Omega; L^2([0,T]; H^1))$, and, taking possibly a subsequence,

$$\lim_{k\to\infty} \hat{\rho}_k = \hat{\rho}, \quad \text{in } L^2_w([0,T]; H^1), \quad \hat{\mathbb{P}}\text{-a.s.},$$

where $L^2_w([0,T]; H^1)$ is the space $L^2([0,T]; H^1)$ endowed with the weak topology.

By proceeding as in the proof of [14, Theorem 7.1], thanks to Proposition A.4, in order to prove Theorem A.1, it is sufficient to show that $\hat{\rho}$ solves the parabolic equation (A.18).

For every $k \in \mathbb{N}$ and $\psi \in C_0^\infty([0,L])$, we have

$$\begin{aligned} \langle \hat{\rho}_k(t) + \hat{\theta}_k(t), \psi\rangle_H &= \langle g(u_0) + \mu_k v_0, \psi\rangle_H - \int_0^t \langle b(\hat{\rho}_k(s))\nabla\hat{\rho}_k(s), \nabla\psi\rangle_H ds \\ &\quad + \int_0^t \langle F_g(\hat{\rho}_k(s)), \psi\rangle_H ds + \int_0^t \langle \sigma_g(\hat{\rho}_k(s)) d\hat{w}_k^Q(s), \psi\rangle_H, \quad \hat{\mathbb{P}}\text{-a.s.} \end{aligned} \tag{A.22}$$

Since $\hat{\rho}_k + \hat{\theta}_k$ converges to $\hat{\rho}$ in $L^2(0,T;H)$, $\hat{\mathbb{P}}$-a.s., we have

$$\lim_{k\to\infty} \int_0^t \langle \hat{\rho}_k(s) + \hat{\theta}_k(s), \psi\rangle_H \, ds = \int_0^t \langle \hat{\rho}(s), \psi\rangle_H \, ds, \quad t \in [0,T], \quad \hat{\mathbb{P}}\text{-a.s.} \tag{A.23}$$

As in the proof of [14, Theorem 7.1], we have that

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \left| \int_0^t \langle b(\hat{\rho}_k(s))\nabla\hat{\rho}_k(s), \nabla\psi\rangle_H ds - \int_0^t \langle b(\hat{\rho}(s))\nabla\hat{\rho}(s), \nabla\psi\rangle_H ds \right| = 0, \ \hat{\mathbb{P}}\text{-a.s.} \tag{A.24}$$

Next, as (2.16) extends to F_g, for each $k \in \mathbb{N}$,

$$\begin{aligned} &\left| \int_0^t \langle F_g(\hat{\rho}_k(s)) - F_g(\hat{\rho}(s)), \psi\rangle_H \right| \\ &\le c\,\|\psi\|_{H^1} \int_0^t \Big(1 + \|\hat{\rho}_k(s)\|^{\theta-1}_{L^{2(\theta-1)}} + \|\hat{\rho}(s)\|^{\theta-1}_{L^{2(\theta-1)}}\Big) \|\hat{\rho}_k(s) - \hat{\rho}(s)\|_H \, ds \\ &\le c\,\|\psi\|_{H^1} \Big(1 + \|\hat{\rho}_k\|^{\theta-1}_{L^{q(\theta-1)}([0,T];C([0,L]))} + \|\hat{\rho}\|^{\theta-1}_{L^{q(\theta-1)}([0,T];C([0,L]))}\Big) \|\hat{\rho}_k - \hat{\rho}\|_{L^p([0,T];H)}, \end{aligned}$$

for any p, q satisfying $p^{-1} + q^{-1} = 1$. Now, if $\theta \in (1,3)$, we can fix

$$\frac{4(\theta+1)}{7 + 2\theta - \theta^2} < p < \theta + 1,$$

so that $q(\theta-1) < 4(\theta+1)/(\theta+3)$. Then thanks to (A.21), we have

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \left| \int_0^t \langle F_g(\hat{\rho}_k(s)), \psi\rangle_H ds - \int_0^t \langle F_g(\hat{\rho}(s)), \psi\rangle_H ds \right| = 0, \quad \hat{\mathbb{P}}\text{-a.s.} \tag{A.25}$$

Finally, since

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \left\| \hat{w}_k^Q(t) - \hat{w}^Q(t) \right\|_U = 0, \quad \mathbb{P}\text{-a.s.}$$

and

$$\lim_{k\to\infty} \|\hat\rho_k - \hat\rho\|_{L^2([0,T];H)} = 0, \quad \mathbb{P}\text{-a.s.}$$

with the uniform estimate

$$\sup_{k\in\mathbb{N}} \mathbb{E} \sup_{t\in[0,T]} \|\hat\rho_k(t)\|_H^2 < \infty,$$

by [19, Corollary 4.5] we have that

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \left| \int_0^t \langle \sigma_g(\hat\rho_k(s)) d\hat w_k^Q(s), \psi\rangle_H - \int_0^t \langle \sigma_g(\hat\rho(s)) d\hat w^Q(s), \psi\rangle_H \right| = 0, \tag{A.26}$$

in probability.

Therefore, combining (A.23)–(A.26), if we integrate with respect to time both sides of equation (A.22) and take the limit as $k \to \infty$, it follows that for every $\psi \in C_0^\infty([0,L])$ and $t \in [0,T]$,

$$\int_0^t \langle \hat\rho(s), \psi\rangle_H \, ds = \int_0^t \Bigg[\langle g(u_0), \psi\rangle_H - \int_0^s \langle b(\hat\rho(r))\nabla\hat\rho(r), \nabla\psi\rangle_H dr + \int_0^s \langle F_g(\hat\rho(r)), \psi\rangle_H dr + \int_0^s \langle \sigma_g(\hat\rho(r)) d\hat w^Q(r), \psi\rangle_H \Bigg] ds, \quad \hat{\mathbb{P}}\text{-a.s.}$$

Due to the arbitrariness of $t \in [0,T]$, this means that $\hat\rho \in L^2(\Omega; X \cap L^2(0,T;H^1))$ solves equation (A.18) with initial data u_0, and the first part of the theorem is proved.

We omit the proof of the second part as it is analogous to the one we have just seen. We only notice that in order to prove (A.25) we need that $q(\theta-1) < 4$. In particular, we need $4/(\theta-1) > 1$, and this is satisfied if $\theta \in (1,5)$.

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