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Large deviations principle for the invariant measures of the 2D stochastic Navier–Stokes equations with vanishing noise correlation

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Abstract
We study the two-dimensional incompressible Navier–Stokes equation on the torus, driven by Gaussian noise that is white in time and colored in space. We consider the case where the magnitude of the random forcing $\sqrt{\epsilon}$ and its correlation scale $\delta(\epsilon)$ are both small. We prove a large deviations principle for the solutions, as well as for the family of invariant measures, as ϵ and $\delta(\epsilon)$ are simultaneously sent to 0, under a suitable scaling.

Keywords Stochastic Navier–Stokes equations · Large deviations · Invariant measures · Quasi-potential

1 Introduction

In the present paper, we consider the two-dimensional incompressible Navier–Stokes equation on the torus $\mathbb{T}^2 = [0, 2\pi]^2$, perturbed by a small additive noise

$$\begin{cases} \partial_t u(t,\xi) + (u(t,\xi)\cdot\nabla)u(t,\xi) = \Delta u(t,\xi) + \nabla p(t,\xi) + \sqrt{\epsilon\, Q_\epsilon}\, \partial_t \eta(t,\xi), \\ \operatorname{div} u(t,\xi) = 0, \qquad u(0,\xi) = u_0(\xi), \qquad u \text{ is periodic in } \mathbb{T}^2. \end{cases} \tag{1.1}$$

The functions $u(t,\xi) \in \mathbb{R}^2$ and $p(t,\xi) \in \mathbb{R}$ denote, respectively, the velocity and the pressure of the fluid at any $(t,\xi) \in \mathbb{R}^+ \times \mathbb{T}^2$. The random forcing $\partial_t \eta(t,\xi)$ is a space-time white noise, while the operator $\sqrt{Q_\epsilon}$ provides spatial correlation to the noise on a scale of size $\delta(\epsilon)$. Here, we are interested in the behavior of Eq. (1.1) as the noise magnitude $\sqrt{\epsilon}$ and the correlation scale $\delta(\epsilon)$ are simultaneously sent to 0.

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In two dimensions, the incompressible Navier–Stokes equation driven by space-time white noise is well-posed only in spaces of negative regularity (see [9]). The driving noise must have more regularity in the spatial variable in order to have function-valued solutions. In our case, we consider a smoothing operator $\sqrt{Q_\epsilon}$ that provides sufficient regularity to interpret Eq. (1.1) in the space $C([0,T];[L^2(\mathbb{T}^2)]^2)$, for any fixed $\epsilon > 0$. In fact, the regularization $\sqrt{Q_\epsilon}$ can be chosen to decay to the identity operator slowly enough for the $\sqrt{\epsilon}$ factor to compensate and produce a function-valued limit.

Under the present assumptions, the $\epsilon \downarrow 0$ limit of Eq. (1.1) in $C([0,T];[L^2(\mathbb{T}^2)]^2)$ is unsurprisingly the corresponding unforced Navier–Stokes equation. A more interesting problem is the quantification of the convergence rate, which can be done using large deviations theory. In [8], it was shown that the solutions to the Leray-projected version of Eq. (1.1) satisfy a large deviations principle in $C([0,T];[L^2(\mathbb{T}^2)]^2)$ with rate function

$$I(u) = \frac{1}{2}\int_0^T \left\|u'(t) + Au(t) + B(u(t))\right\|^2_{[L^2(\mathbb{T}^2)]^2} dt,$$

where A is the Stokes operator and B is the Navier–Stokes nonlinearity. This result was proven using the weak convergence approach developed in [7], which is particularly effective at handling multiple parameter limits. The weak convergence method was also used in [1] and [2] to prove large deviations principles for the stochastic Navier–Stokes with viscosity vanishing at a rate proportional to the strength of the noise, which is believed to be a relevant problem in the study of turbulent fluid dynamics.

If the operator $\sqrt{Q_\epsilon}$ is simultaneously smoothing enough but not too degenerate, then Eq. (1.1) will possess a unique ergodic invariant probability measure (see [13]). In the $\epsilon \downarrow 0$ limit, it can be shown that these measures converge weakly to the Dirac measure at 0. For fixed correlation strength $\delta(\epsilon) = \delta > 0$, it was proven in [3] that the invariant measures also satisfy a large deviations principle in $[L^2(\mathbb{T}^2)]^2$ with rate function given by the quasi-potential

$$U_\delta(x) = \inf\left\{ I_T^\delta(u) \, : \, T > 0, \; u \in C([0,T];[L^2(\mathbb{T}^2)]^2), \; u(0) = 0, \; u(T) = x \right\},$$

where $I_T^\delta : C([0,T];[L^2(\mathbb{T}^2)]^2) \to [0,+\infty]$ is the action functional for the paths, defined by

$$I_T^\delta(u) := \frac{1}{2}\int_0^T \left\| Q_\delta^{-1}\Big(u'(t) + Au(t) + B(u(t))\Big)\right\|^2_{[L^2(\mathbb{T}^2)]^2} dt.$$

This result was generalized in [16] to the case of the Navier–Stokes equations posed on a bounded domain with Dirichlet boundary conditions. In [16] they also considered the case where the equation has a deterministic, time-independent forcing so that the limiting dynamics may have nontrivial point attractors or sets of attractors. Both papers established their results by following the general strategy introduced in [19] for proving large deviations principles for families of invariant measures.

In [4], it was also proven that the quasipotential $U_\delta(x)$, corresponding to the problem on the torus, converges pointwise to

$$U(x) = \|x\|^2_{[H^1(\mathbb{T}^2)]^2},$$

as $\delta \downarrow 0$. This is a consequence of the orthogonality of Au and $B(u)$ in $[L^2(\mathbb{T}^2)]^2$, which in general does not hold for the problem posed on a bounded domain. In some sense, $U(x)$ is what one would expect the quasi-potential for the space-time white noise case to be, if the time-stationary problem were well-posed.

The purpose of this article is to bridge the results of [3] and [4] with the result of [8]. Rather than first taking $\epsilon \downarrow 0$ and then studying what happens as the regularization is removed, we take ϵ and δ to 0 simultaneously. We prove that the invariant measures of Eq. (1.1) satisfy a large deviations principle directly with rate function $U(x)$, under suitable conditions on the regularization $\sqrt{Q_\epsilon}$.

To prove this result, we first prove a large deviations principle for the solutions of Eq. (1.1) in $C([0, T]; [L^2(\mathbb{T}^2)]^2)$, that is uniform with respect to initial conditions in appropriate sets of functions. This is done by proving a large deviations principle for the linearized problem using the weak convergence approach and then transferring this to the nonlinear problem via a suitable generalization of the uniform contraction principle. As well known, when applying the classical contraction principle to obtain the large deviation principle for the solutions of the nonlinear equation (1.1), the mapping that associates the solution of the linear problem to the solution of the nonlinear problem is only required to be continuous. In particular, in our case, this allows for slower decay of the correlation scale $\delta(\epsilon)$ than the one introduced in [8], when proving a large deviation principle for the solutions of (1.1).

However, when proving a large deviation principle for the invariant measures, we need to have a large deviation principle for the solutions of (1.1), which is uniform with respect to initial conditions in a bounded set of $[L^2(\mathbb{T}^2)]^2$. This means that, as we mentioned above, we need a contraction principle that is uniform with respect to initial conditions on a bounded set of $[L^2(\mathbb{T}^2)]^2$. When dealing with uniform contraction principles, the Lipschitz-continuity of the solution mapping is what is usually required. Here, such a mapping is only locally Lipschitz continuous. Thus, in order to make up for the lack of global Lipschitz-continuity we need to have some exponential tightness. Notice that the proof of the exponential tightness of the solutions of the linear problem is not trivial and requires some work, as also the covariance of the noise depends on ϵ and the classical results available in the literature (see e.g. [6]) do not apply.

We would like to mention here that the problem of the validity of a uniform large deviation principle for a general class of SPDEs has been investigated in depth in few papers by Salins and others (see e.g. [17] and [18]). In particular, in [18] a general criterion is given and such criterion applies also to the two-dimensional stochastic Navier–Stokes equation. However, such very general result does not apply to the case of variable covariance and for this reason here we are providing an alternative proof.

Once the uniform large deviation principle for the solutions of equation (1.1) is obtained, the proof of the large deviations principle for the invariant measures follows from several crucial generalizations and modifications of the arguments used in [3], to account for the decaying regularity of the driving noise. In particular, we need another exponential estimate that takes into account of the variable covariance, this time for the H^1-norm of the solution of equation (1.1).

2 Preliminaries

We consider Eq. (1.1) posed on the space of square-integrable, mean zero, space-periodic functions. For an introduction to the 2D Navier–Stokes equations on the torus, see the book [20] by Temam. We follow the notations and conventions used there. Denoting $\mathbb{T}^2 := [0, 2\pi]^2$, we define

$$H := \left\{ f \in [L^2(\mathbb{T}^2)]^2 : \int_{\mathbb{T}^2} f(\xi)d\xi = 0, \quad \mathrm{div} f = 0, \quad f \text{ is periodic in } \mathbb{T}^2 \right\},$$

where the periodic boundary conditions are interpreted in the sense of trace. It can be shown that H is a Hilbert space when endowed with the standard $[L^2(\mathbb{T}^2)]^2$ inner product. We denote the norm and inner product on H by $\|\cdot\|_H$ and $\langle\cdot,\cdot\rangle_H$, respectively.

We denote by $H_{\mathbb{C}}$, the complexification of H, and by $\mathbb{Z}_0^2$ the set $\mathbb{Z}^2 \setminus \{(0,0)\}$. The family $\{e_k\}_{k\in\mathbb{Z}_0^2} \subset H_{\mathbb{C}}$ defined by

$$e_k(\xi) = \frac{1}{2\pi} \frac{(k_2, -k_1)}{\sqrt{k_1^2 + k_2^2}} e^{i\xi\cdot k}, \qquad \xi \in \mathbb{T}^2, \ \ k = (k_1, k_2) \in \mathbb{Z}_0^2,$$

form a complete orthonormal system in $H_{\mathbb{C}}$. Similarly, the family $\{\mathrm{Re}(e_k)\}_{k\in\mathbb{Z}_0^2} \subset H$ form a complete orthonormal system in H. In what follows, we use the basis $\{e_k\}_{k\in\mathbb{Z}_0^2}$ with the implicit assumption that we are only considering the real components.

Next, we let P be the orthogonal projection from $[L^2(\mathbb{T}^2)]^2$ onto H, known as the Leray projection. We define the Stokes operator by setting

$$Au := -P\Delta u, \quad u \in D(A) := H \cap [W^{2,2}(\mathbb{T}^2)]^2.$$

It is easy to see that A is a diagonal operator on H with respect to the basis $\{e_k\}_{k\in\mathbb{Z}_0^2}$. In particular, for any $k \in \mathbb{Z}_0^2$ we have

$$Ae_k = |k|^2 e_k.$$

Since A is a positive, self-adjoint operator, for any $r \in \mathbb{R}$ we can define the fractional power A^r with domain $D(A^r)$. In fact, it can be shown that $D(A^r)$ is the closure of $\mathrm{span}_{k\in\mathbb{Z}_0^2}\langle e_k\rangle$ with respect to the $[W^{2r,2}(\mathbb{T}^2)]^2$ Sobolev norm. To simplify our notations, we will denote $V^r := D(A^{r/2})$, with the norm given by the $[W^{2r,2}(\mathbb{T}^2)]^2$ Sobolev semi-norm

$$\|u\|_r^2 := \|u\|_{D(A^{r/2})}^2 = \|u\|_{[H^r(\mathbb{T}^2)]^2}^2 = \sum_{k\in\mathbb{Z}_0^2} |k|^{2r} \langle u, e_k\rangle_H^2.$$

In particular, we have that $V^2 = D(A)$ and $V := V^1 = D(A^{1/2})$. For any $r \geq 0$, we denote by V^{-r} the dual space of V^r. In addition, for any $p \geq 1$, we will use the shorthands

$$L^p := [L^p(\mathbb{T}^2)]^2, \quad W^{k,p} := [W^{k,p}(\mathbb{T}^2)]^2.$$

Next, we define the tri-linear form, $b : V \times V \times V \to \mathbb{R}$, by

$$b(u, v, w) := \int_{\mathbb{T}^2} (u(\xi) \cdot \nabla) v(\xi) \cdot w(\xi) d\xi, \quad u, v, w \in V.$$

From standard interpolation inequalities and Sobolev embeddings, it follows that

$$|b(u, v, w)| \leq c \begin{cases} \|u\|_H^{1/2} \|u\|_V^{1/2} \|v\|_V \|w\|_H^{1/2} \|w\|_V^{1/2}, \\ \|u\|_H^{1/2} \|u\|_{V^2}^{1/2} \|v\|_V \|w\|_H, \\ \|u\|_H \|v\|_V \|w\|_H^{1/2} \|w\|_{V^2}^{1/2}, \end{cases} \tag{2.1}$$

for smooth u, v, w. These inequalities can then be extended to the appropriate Sobolev spaces by continuity. We note that the first inequality in (2.1) implies that b is indeed well-defined and continuous on $V \times V \times V$. The tri-linear form b also induces the continuous mappings $B : V \times V \to V'$ and $B : V \to V'$ defined by

$$\langle B(u, v), w \rangle := b(u, v, w),$$
$$B(u) := B(u, u),$$

for $u, v, w \in V$. It can be shown that for any $u, v \in D(A)$

$$B(u, v) = P[(u \cdot \nabla)v],$$

and

$$\langle B(u, v), w \rangle_H = -\langle B(u, w), v \rangle_H, \quad u, v, w \in V, \tag{2.2}$$

which implies that

$$\langle B(u, v), v \rangle_H = 0, \quad u, v \in V.$$

Moreover

$$\langle B(u), Au \rangle_H = 0, \quad u \in D(A). \tag{2.3}$$

Equation (2.2) is still true when considering the problem posed on a bounded domain with Dirichlet boundary conditions. Equation (2.3), on the other hand, only holds for the problem posed on the torus with periodic boundary conditions (for a proof of (2.3), see for example [15]). We note that the proof of our main result relies on

Eq. (2.3) in several places, and hence will not immediately generalize to the case of the Navier–Stokes equation on a bounded domain.

As for the random forcing in Eq. (1.1), we assume that $\eta(t,\xi)$ is a cylindrical Wiener process on the Hilbert space of mean-zero functions in $[L^2(\mathbb{T}^2)]^2$. We then set $w(t) := P\eta(t)$, so that w has the formal expansion

$$w(t,\xi) = \sum_{k\in\mathbb{Z}_0^2} e_k(\xi)\beta_k(t), \quad t \geq 0, \quad \xi \in \mathbb{T}^2,$$

where $\{\beta_k\}_{k\in\mathbb{Z}_0^2}$ are a collection of independent, real-valued Brownian motions on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. We assume that the covariance operator Q_ϵ belongs to $\mathcal{L}(H; H)$ and takes the form

$$Q_\epsilon := (I + \delta(\epsilon)A^\beta)^{-1}, \tag{2.4}$$

for some $\beta > 0$ and $\delta(\epsilon) > 0$. Since we are concerned with the singular noise limit, $\delta(\epsilon)$ will be taken to be a strictly decreasing function of ϵ such that

$$\lim_{\epsilon\to 0} \delta(\epsilon) = 0.$$

Definition (2.4) implies that Q_ϵ is diagonal with respect to the basis $\{e_k\}_{k\in\mathbb{Z}_0^2}$.

Remark 1 In the present paper we only take this particular form of the covariance operator in order to simplify our presentation. The results below can easily be adapted to more general covariance operators with the same smoothing and ergodic properties.

The driving noise, $\sqrt{Q_\epsilon}\, w(t)$, can thus formally be written as the infinite series

$$\sqrt{Q_\epsilon}\, w(t,\xi) = \sum_{k\in\mathbb{Z}_0^2} \sigma_{\epsilon,k}\, e_k(\xi)\, \beta_k(t) := \sum_{k\in\mathbb{Z}_0^2} (1 + \delta(\epsilon)|k|^{2\beta})^{-1/2} e_k(\xi)\, \beta_k(t).$$

Since $\delta(\epsilon)$ converges to zero, as $\epsilon \downarrow 0$, the covariance operator Q_ϵ converges pointwise to the identity operator, as $\epsilon \downarrow 0$. For each fixed $\epsilon > 0$, it is immediate to check that $\sqrt{Q_\epsilon} \in \mathcal{L}(V^r, V^{r+\beta})$. In fact, one can show that

$$\left\| \sqrt{Q_\epsilon} f \right\|_{V^{r+q}} \leq \frac{1}{\sqrt{\delta(\epsilon)}} \, \|f\|_{V^r}, \tag{2.5}$$

for any $r \in \mathbb{R}$, $q \leq \beta$ and $f \in V^r$. Moreover, Q_ϵ is a trace class operator in H if and only if $\beta > 1$. This means that the Wiener process $\sqrt{Q_\epsilon}\, w$ is H-valued only when $\beta > 1$.

By taking the Leray projection on both sides of Eq. (1.1), we obtain the following stochastic evolution problem

$$\begin{cases} du(t) + [Au(t) + B(u(t))]\, dt = \sqrt{\epsilon\, Q_\epsilon}\, dw(t), \\ u(0) = x. \end{cases} \tag{2.6}$$

We assume the initial condition x is an element of H. As is well-known (see [5] or Chapter 15 of [10]), under the assumption that $\beta > 0$, Eq. (2.6) admits a unique generalized solution, $u^x_\epsilon \in C([0, T]; H)$. That is, there exists a progressively measurable process u^x_ϵ taking values in $C([0, T]; H)$, $\mathbb{P}$-a.s. for any $T > 0$, such that

$$\langle u^x_\epsilon(t), h\rangle_H = \langle x, h\rangle_H - \int_0^t \langle u^x_\epsilon(s), Ah\rangle_H \\ - \int_0^t \langle B(u^x_\epsilon(s), h), u_\epsilon(s)\rangle_H + \langle \sqrt{\epsilon\, Q_\epsilon}\, w(t), h\rangle_H, \quad \mathbb{P} - a.s.,$$

for any $h \in D(A)$ and $t \in [0, T]$.

The condition $\beta > 0$ is not enough to ensure the existence and uniqueness of an invariant measure for Eq. (2.6). In the last 25 years there has been an extremely intense activity aimed to the study of the ergodic properties of randomly perturbed PDEs in fluid dynamics and, in particular, of Eq. (1.1). As shown for instance in the monograph [15], a sufficient condition for this is that Q_ϵ be trace-class in H and $\sigma_{\delta(\epsilon),k} \neq 0$ for all k. Notice that if $\beta > 1$, then Q_ϵ is a trace-class operator and in particular,

$$\text{Tr}\, Q_\epsilon \leq c\, \delta_\epsilon^{-1/\beta}, \tag{2.7}$$

for some constant $c = c_\beta > 0$ depending on β. By applying Itô's formula we get

$$\mathbb{E}\, \|u^x_\epsilon(t)\|^2_H + 2\int_0^t \mathbb{E}\|u^x_\epsilon(s)\|^2_V\, ds = \|x\|^2_H + t\,\epsilon\, \text{Tr}\, Q_\epsilon \leq \|x\|^2_H + c_\beta\, t\,\epsilon\, \delta_\epsilon^{-1/\beta}. \tag{2.8}$$

This means that $u^x_\epsilon \in L^2(\Omega; C([0, T]; H) \cap L^2(0, T; V))$ and, in particular, for every $\epsilon > 0$ there exists an invariant measure.

Now, let $\{\nu_\epsilon\}_{\epsilon>0}$ be this family of invariant measures. Each ν_ϵ is ergodic in the sense that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(u^x_\epsilon(t))dt = \int_H f(x)d\nu_\epsilon(x),$$

for all $x \in H$ and Borel-measurable $f : H \to \mathbb{R}$. If

$$\sup_{\epsilon\in(0,1)} \epsilon\, \delta_\epsilon^{-1/\beta} < \infty, \tag{2.9}$$

we have that the family $\{\nu_\epsilon\}_{\epsilon>0}$ is tight in H. Actually, due to (2.8) and the invariance of ν_ϵ, for every $T > 0$ we have

$$\begin{aligned}\int_H \|x\|_V^2 \, d\nu_\epsilon(x) &= \frac{1}{T}\int_0^T \int_H \mathbb{E}\,\|u_\epsilon^x(t)\|_V^2 \, d\nu_\epsilon(x)\, dt\\ &= \frac{1}{T}\int_H \int_0^T \mathbb{E}\,\|u_\epsilon^x(t)\|_V^2 \, dt\, d\nu_\epsilon(x)\\ &\leq \frac{1}{2T}\int_H \|x\|_H^2 \, d\nu_\epsilon(x) + \frac{1}{2} c_\beta \, \epsilon\, \delta_\epsilon^{-1/\beta}\\ &\leq \frac{1}{2T}\int_H \|x\|_V^2 \, d\nu_\epsilon(x) + \frac{1}{2} c_\beta \, \epsilon\, \delta_\epsilon^{-1/\beta}.\end{aligned}$$

Then, thanks to (2.9), if we choose $T > 1$ we get

$$\sup_{\epsilon \in (0,1)} \int_H \|x\|_V^2 \, d\nu_\epsilon(x) < \infty,$$

and this implies the tightness of $\{\nu_\epsilon\}_{\epsilon \in (0,1)}$ in H. In fact, provided that

$$\lim_{\epsilon \to 0} \epsilon\, \delta_\epsilon^{-1/\beta} = 0,$$

we have that

$$\nu_\epsilon \rightharpoonup \delta_0, \qquad \text{as } \epsilon \downarrow 0.$$

The purpose of this paper is to quantify the rate of this convergence through a large deviations principle. To state the main result, we first recall the definition of the large deviations principle. Here we give the Freidlin–Wentzell formulation.

Definition 2.1 Let E be a Banach space. Suppose that $\{\mu_\epsilon\}_{\epsilon>0}$ is a family of probability measures on E and $I : E \to [0, +\infty]$ is a good rate function, meaning that for each $s \geq 0$, the level set $\Phi(s) := \{h \in E : I(h) \leq s\}$ is a compact subset of E. The family $\{\mu_\epsilon\}_{\epsilon>0}$ is said to satisfy a large deviations principle (LDP) in E, with rate function I, if the following hold.

(i) For every $s \geq 0$, $\delta > 0$ and $\gamma > 0$, there exists $\epsilon_0 > 0$ such that

$$\mu_\epsilon(B_E(\varphi, \delta)) \geq \exp\left(-\frac{I(\varphi) + \gamma}{\epsilon}\right),$$

for any $\epsilon \leq \epsilon_0$ and $\varphi \in \Phi(s)$, where $B_E(\varphi, \delta) := \{h \in E : \|h - \varphi\| < \delta\}$.

(ii) For every $s_0 \geq 0$, $\delta > 0$ and $\gamma > 0$, there exists $\epsilon_0 > 0$ such that

$$\mu_\epsilon(B_E^c(\Phi(s), \delta)) \leq \exp\left(-\frac{s - \gamma}{\epsilon}\right),$$

for any $\epsilon \leq \epsilon_0$ and $s \leq s_0$, where $B_E^c(\Phi(s), \delta) := \{h \in E : \text{dist}_E(h, \Phi(s)) \geq \delta\}$.

The main result of this paper is the following.

Theorem 2.1 *Assume that Q_ϵ has the form given in (2.4), for some $\beta > 2$. Moreover, suppose that*

$$\lim_{\epsilon\to 0} \delta(\epsilon) = 0, \qquad \lim_{\epsilon\to 0} \epsilon\, \delta(\epsilon)^{-2/\beta} = 0.$$

Then the family of invariant measures $\{\nu_\epsilon\}_{\epsilon>0}$ of Eq. (2.6) satisfies a large deviations principle in H with rate function given by

$$U(x) = \begin{cases} \|x\|_V^2\,, & x \in V, \\ +\infty, & x \in V \setminus H. \end{cases} \tag{2.10}$$

We remark here that the rate function, $U(x)$, is really the quasipotential corresponding to Eq. (2.6), whose definition is given in Eq. (4.1). The quasi-potential has the explicit representation given in (2.10) in the case the problem is posed on a torus. That formula does not hold in general for the problem posed on a bounded domain with Dirichlet boundary conditions.

3 Large deviation principle for the paths

The proof of Theorem 2.1 requires a large deviations principle for the solutions to Eq. (2.6). One such large deviations principle is proven in [8], but here we have to proceed differently in order to obtain a result that is uniform with respect to initial conditions in bounded subsets of H. Unlike in [8], we first prove a large deviation principle for the linearized Ornstein–Uhlenbeck process in the space $C([0, T]; L^4)$, and then transfer it back to the appropriate Navier–Stokes process by means of the contraction principle.

3.1 LDP for the Ornstein–Uhlenbeck process

Assume that Q_ϵ has the form given in (2.4), for some $\beta > 0$. For every $\epsilon > 0$, let z_ϵ denote the mild solution to the equation

$$\begin{cases} dz_\epsilon + Az_\epsilon dt = \sqrt{\epsilon\, Q_\epsilon}\, dw(t), \\ z_\epsilon(0) = 0. \end{cases} \tag{3.1}$$

It is well-known that z_ϵ is given by the stochastic convolution

$$z_\epsilon(t) = \int_0^t S(t-s)\sqrt{\epsilon\, Q_\epsilon}\, dw(s), \qquad t \geq 0,$$

where $\{S(t)\}_{t\geq 0}$ is the analytic semigroup generated by the operator $-A$ on H. It can be shown that $z_\epsilon \in L^p(\Omega; C([0, T]; V^r))$, for any $r < \beta$ and $p \geq 1$ (e.g. see [11]). In this subsection, we prove that the family $\{z_\epsilon\}_{\epsilon>0}$ satisfies a large deviations principle in $C([0,T];L^4\cap H)$. To do so, we first prove that the stochastic convolution z_ϵ converges to 0 in $L^p(\Omega;C([0,T];L^p))$, as $\epsilon\downarrow 0$, for every $p\geq 1$.

Lemma 3.1 *For any $\epsilon>0$, the process z_ϵ has trajectories in $C([0,T];L^p)$, $\mathbb{P}$-a.s. for any $p\in[1,\infty)$. Moreover,*

$$\lim_{\epsilon\to 0}\epsilon\log\frac{1}{\delta(\epsilon)}=0\implies\lim_{\epsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}\|z_\epsilon(t)\|^p_{L^p\cap H}=0. \tag{3.2}$$

Proof Fix any $p<\infty$. Thanks to the Burkholder–Davis–Gundy inequality and the uniform boundedness of the basis $\{e_k\}_{k\in\mathbb{Z}^2_0}$, we have

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}\|z_\epsilon(t)\|^p_{L^p}&=\epsilon^{p/2}\,\mathbb{E}\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)\sqrt{Q_\epsilon}dw(s)\right\|^p_{L^p}\\&\leq\epsilon^{p/2}\int_{\mathbb{T}^2}\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t\sum_{k\in\mathbb{Z}^2_0}e^{-|k|^2(t-s)}\sigma_{\delta(\epsilon),k}e_k(\xi)d\beta_k(s)\right|^p d\xi\\&\leq c_p\,\epsilon^{p/2}\int_{\mathbb{T}^2}\Big(\sum_{k\in\mathbb{Z}^2_0}\sigma^2_{\delta(\epsilon),k}|e_k(\xi)|^2\int_0^T e^{-2|k|^2 s}ds\Big)^{p/2}d\xi\\&\leq c_p\,\epsilon^{p/2}\Big(\sum_{k\in\mathbb{Z}^2_0}\frac{1}{|k|^2(1+\delta(\epsilon)|k|^{2\beta})}\Big)^{p/2}<\infty.\end{aligned}$$

Therefore, (3.2) follows by noting that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^2_0}\frac{1}{|k|^2(1+\delta(\epsilon)|k|^{2\beta})}&\leq c\int_1^\infty\frac{1}{r(1+\delta(\epsilon)r^\beta)}dr=c\int_{\delta(\epsilon)^{1/\beta}}^\infty\frac{1}{r(1+r^\beta)}dr\\&\leq c\int_{\delta(\epsilon)^{1/\beta}}^1\frac{dr}{r}+c\int_1^\infty\frac{dr}{r^{\beta+1}}\leq\frac{c}{\beta}\log\frac{1}{\delta(\epsilon)}+\frac{c}{\beta},\end{aligned}$$

for some constant $c=c_\beta$ depending on β. □

To prove that the family $\{z_\epsilon\}_{\epsilon>0}$ satisfies a large deviations principle in $C([0,T];L^4\cap H)$ we use the weak convergence approach, as developed for SPDEs in [7]. This approach involves proving convergence of the solutions to a sequence of controlled versions of the equations. For $\varphi\in L^2(\Omega;L^2(0,T;H))$, we denote by $z_{\epsilon,\varphi}$ the solution to the equation

$$dz_{\epsilon,\varphi}(t)+Az_{\epsilon,\varphi}(t)\,dt=\sqrt{\epsilon\,Q_\epsilon}\,dw(t)+\sqrt{Q_\epsilon}\,\varphi(t)\,dt,\quad z_{\epsilon,\varphi}(0)=0,$$

and we denote by z_φ the solution to the so-called skeleton equation

$$\frac{dz_\varphi}{dt}(t)+Az_\varphi(t)=\varphi(t),\qquad z_\varphi(0)=0. \tag{3.3}$$

Notice that $z_{\epsilon,\varphi}$ can be written as

$$\mathcal{G}_\epsilon\left(\sqrt{\epsilon}\left(w+\frac{1}{\sqrt{\epsilon}}\int_0^\cdot \varphi(s)\,ds\right)\right),$$

where $\mathcal{G}_\epsilon$ denotes the measurable mapping that associates to w the mild solution of the equation

$$dz(t)+Az(t)\,dt=\sqrt{Q_\epsilon}\,dw(t),\quad z(0)=0$$

This means that we are in the setting covered in [7]. Actually, Theorem 6 of [7] implies the following result.

Theorem 3.1 *The family $\{\mathcal{L}(z_\epsilon)\}_{\epsilon>0}$ satisfies a large deviations principle in $C([0,T];L^4\cap H)$, with rate function*

$$J_T(z)=\frac{1}{2}\inf\left\{\int_0^T\|\varphi(t)\|_H^2\,dt\ :\ \varphi\in L^2(0,T;H),\, z=z_\varphi\right\},\tag{3.4}$$

if the following two conditions hold for any $M\in[0,\infty)$.

(i) The set

$$\Phi(M):=\Big\{z\in C([0,T];L^4\cap H)\ :\ z=z_\varphi,\ \varphi\in L^2(0,T;H),\ \frac{1}{2}\int_0^T\|\varphi(t)\|_H^2\,dt\leq M\Big\}$$

is a compact subset of $C([0,T];L^4\cap H)$.

(ii) For every $\{\varphi_\epsilon\}_{\epsilon\geq 0}\subset L^2(\Omega;L^2(0,T;H))$, such that

$$\sup_{\epsilon\in(0,1)}\frac{1}{2}\int_0^T\|\varphi_\epsilon(t)\|_H^2\,dt\leq M,\ \mathbb{P}-a.s.,\tag{3.5}$$

if φ_ϵ converges to φ_0 in distribution with respect to the weak topology of $L^2(0,T;H)$, as $\epsilon\downarrow 0$, then $z_{\epsilon,\varphi_\epsilon}$ converges to z_{φ_0} in distribution in $C([0,T];L^4\cap H)$, as $\epsilon\downarrow 0$.

Thus, to prove the large deviations principle it remains to prove conditions (i) and (ii) in the above theorem. Note that condition (i) is precisely the statement that J_T is a good rate function.

Theorem 3.2 *Assume that*

$$\lim_{\epsilon\to 0}\delta(\epsilon)=0,\quad \lim_{\epsilon\to 0}\epsilon\log\frac{1}{\delta(\epsilon)}=0.$$

Then the family $\{\mathcal{L}(z_\epsilon)\}_{\epsilon>0}$ *of solutions to Eq.* (3.1) *satisfies a large deviations principle in* $C([0,T];L^4\cap H)$ *with rate function*

$$J_T(z)=\begin{cases}\frac{1}{2}\int_0^T \left\|z'(t)+Az(t)\right\|_H^2 dt & \textit{if } z\in W^{1,2}(0,T;H)\cap L^2(0,T;D(A)),\\ +\infty & \textit{otherwise}.\end{cases} \tag{3.6}$$

Proof In view of Theorem 3.1, it suffices to show that conditions (i) and (ii) in Theorem 3.1 hold true. Equality of the rate functions defined in Eqs. (3.4) and (3.6) follows immediately from the fact that $z_\psi=z_\varphi$ implies that $\psi=\varphi$.
Step 1. We first verify condition (i). Suppose that $z\in\Phi(M)$, so that $z=z_\varphi$ for some $\varphi\in L^2(0,T;H)$ satisfying

$$\frac{1}{2}\int_0^T \|\varphi(t)\|_H^2\,dt\le M. \tag{3.7}$$

For any $\zeta\in(0,1)$, the function $z_\varphi(t)=\int_0^t S(t-s)\varphi(s)ds$ can be rewritten as $z_\varphi=\Gamma_\zeta(Y_\zeta(\varphi))$ where

$$\Gamma_\zeta(Y)(t):=c_\zeta\int_0^t (t-s)^{\zeta-1}S(t-s)Y(s)ds,$$

for $c_\zeta=\sin(\zeta\pi)/\pi$ and

$$Y_\zeta(\varphi)(s):=\int_0^s (s-r)^{-\zeta}S(s-r)\varphi(r)dr.$$

It is possible to show that for any $\zeta\in(0,\frac{1}{2})$, $p\ge 2$, $\rho\in(0,1)$ and $\delta\in(0,\frac{1}{2})$ such that $\delta+\dfrac{\rho}{2}<\zeta-\dfrac{1}{p}$,

$$\Gamma_\zeta: L^p(0,T;H)\to C^\delta([0,T];V^\rho), \tag{3.8}$$

is a continuous linear mapping (see Appendix A of [10]). Moreover, by Young's inequality we have that

$$\begin{aligned}\left\|Y_\zeta(\varphi)\right\|^p_{L^p(0,T;H)}&=\int_0^T\left\|\int_0^s (s-r)^{-\zeta}S(s-r)\varphi(r)dr\right\|_H^p ds\\ &\le\int_0^T\Big(\int_0^s (s-r)^{-\zeta}\,\|\varphi(r)\|_H\,dr\Big)^p ds\\ &\le\Big(\int_0^T t^{-\frac{2\zeta p}{p+2}}dt\Big)^{\frac{p+2}{2}}\,\|\varphi\|^p_{L^2(0,T;H)},\end{aligned} \tag{3.9}$$

which is finite provided that $\zeta < \frac{1}{2} + \frac{1}{p}$. Hence $z \in C^\delta(0, T; V^\rho)$ for any δ, ρ satisfying $\delta + \frac{\rho}{2} < \frac{1}{2}$. Moreover, thanks to (3.7),

$$\|z\|_{C^\delta(0,T;V^\rho)} \leq c_{p,\delta,\rho}\sqrt{M},$$

so that $\Phi(M)$ is a bounded set in $C^\delta(0, T; V^\rho)$ and thus a compact set in $C([0, T]; L^4 \cap H)$.

Step 2. Next, we verify condition (ii) in Theorem 3.1. Let $M > 0$ and let $\{\varphi_\epsilon\}_{\epsilon \geq 0}$ be a sequence in $L^2(\Omega; L^2(0, T; H))$ satisfying (3.5). Thanks to the Skorokhod theorem, there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t\geq 0}, \bar{\mathbb{P}})$, a cylindrical Wiener process $\bar{w}(t)$, and collection $\{\bar{\varphi}_\epsilon\}_{\epsilon\geq 0}$ in $L^2(\bar{\Omega}; L^2(0, T; H))$ such that φ_ϵ and $\bar{\varphi}_\epsilon$ have the same distributions and

$$\lim_{\epsilon\to 0} \bar{\varphi}_\epsilon = \bar{\varphi}_0, \ \bar{\mathbb{P}} - \text{a.s.},$$

with respect to the weak topology of $L^2(0, T; H)$. If we show that $z_{\epsilon,\bar{\varphi}_\epsilon}$ converges to $z_{\bar{\varphi}_0}$ in $L^4(\Omega; C([0, T]; L^4 \cap H))$, then condition (ii) will follow.

To simplify our notation, we dispense with the bars. Now, for any $t \geq 0$, we have

$$\begin{aligned} z_{\epsilon,\varphi_\epsilon}(t) - z_\varphi(t) &= \sqrt{\epsilon}\int_0^t S(t-s)\sqrt{Q_\epsilon}\, dw(s) + \int_0^t S(t-s)\Big[\sqrt{Q_\epsilon}\varphi_\epsilon(s) - \varphi(s)\Big]ds \\ &=: J_1^\epsilon(t) + J_2^\epsilon(t). \end{aligned}$$

Thanks to Lemma 3.1, we have

$$\lim_{\epsilon\to 0} \mathbb{E}\|J_1^\epsilon\|^4_{C([0,T];L^4\cap H)} = 0.$$

To handle the control terms, we observe that $\sqrt{Q_\epsilon}\,\varphi_\epsilon$ converges to φ weakly in $L^2(0, T; H)$. Indeed, for any $h \in L^2(0, T; H)$, it follows that

$$\begin{aligned} \Big|\langle\sqrt{Q_\epsilon}\varphi_\epsilon - \varphi, h\rangle_{L^2(0,T;H)}\Big| &= \Big|\langle\varphi_\epsilon, (\sqrt{Q_\epsilon} - I)h\rangle_{L^2(0,T;H)} + \langle\varphi_\epsilon - \varphi, h\rangle_{L^2(0,T;H)}\Big| \\ &\leq \sqrt{2M}\,\Big\|(\sqrt{Q_\epsilon} - I)h\Big\|_{L^2(0,T;H)} + \Big|\langle\varphi_\epsilon - \varphi, h\rangle_{L^2(0,T;H)}\Big|, \end{aligned}$$

which converges to 0, $\mathbb{P}-$a.s., as $\epsilon \to 0$, since Q_ϵ converges to $\mathbb{1}$ pointwise in H and φ_ϵ converges to φ weakly. Moreover, we already showed in Step 1 that the solution map $\Gamma : L^2(0, T; H) \to C([0, T]; L^4 \cap H)$ given by

$$\Gamma(\varphi)(t) = \int_0^t S(t-s)\varphi(s)ds,$$

is a compact operator. Since compact operators map weakly convergent sequences to strongly convergent sequences, it follows

$$\lim_{\epsilon\to 0} \|J_2^\epsilon\|_{C([0,T];L^4\cap H)} = 0, \quad \mathbb{P}-\text{a.s.}$$

Moreover, we have

$$\sup_{\epsilon>0} \|J_2^\epsilon\|_{C([0,T];L^4\cap H)} < \infty, \quad \mathbb{P}-a.s.$$

so that

$$\lim_{\epsilon\to 0} \mathbb{E}\|J_2^\epsilon\|^4_{C([0,T];L^4\cap H)} = 0.$$

□

3.2 Uniform LDP for the Navier–Stokes process

To obtain a uniform large deviations principle for the solutions to Eq. (2.6), we will apply a suitable uniform contraction principle to the large deviations principle for the solutions to Eq. (3.1).

For every $x \in H$, let $\mathcal{F}_x : L^4(0,T;L^4\cap H) \to C([0,T];H)$ be the mapping that associates to any $z \in L^4(0,T;L^4\cap H)$ the solution to the equation

$$\begin{cases} du(t) + Au(t)dt + B\Big(u(t)+z(t)\Big)dt = 0, \\ u(0) = x \in H. \end{cases}$$

In particular, we see that $I + \mathcal{F}_x$ maps a trajectory of z_ϵ to a trajectory of u_ϵ^x.

The proof of the following result is from [5]. Here we give a brief sketch of it to emphasize the right dependence on the initial conditions.

Lemma 3.2 *[5] For every fixed $T > 0$, the mappings $\mathcal{F}_x : L^4(0,T;L^4\cap H) \to C([0,T];H)$ are Lipschitz continuous on balls, uniformly over x in bounded sets of H. That is, for any $r > 0$ and $R > 0$, there exists a constant $L_{r,R} > 0$ such that*

$$\sup_{x\in B_H(r)} \|\mathcal{F}_x(f) - \mathcal{F}_x(g)\|_{C([0,T];H)} \le L_{r,R}\,\|f-g\|_{L^4(0,T;L^4\cap H)}, \quad f,g \in B_{L^4(0,T;L^4\cap H)}(R). \tag{3.10}$$

Proof For every $f, g \in B_{L^4(0,T;L^4\cap H)}(R)$, we define $u := \mathcal{F}_x(f) - \mathcal{F}_x(g)$. By proceeding as in [5], we have

$$\begin{aligned}
&\|u(t)\|_H^2 + \int_0^t \|u(s)\|_V^2\, ds \\
&\leq c\, \|g - f\|^2_{L^4(0,t;L^4)} \Big[\|\mathcal{F}_x(f)\|_{L^\infty(0,T;H)} \|\mathcal{F}_x(f)\|_{L^2(0,T;V)} \\
&\quad + \|\mathcal{F}_x(g)\|_{L^\infty(0,T;H)} \|\mathcal{F}_x(g)\|_{L^2(0,T;V)} + \|f\|^2_{L^4(0,T;L^4)} + \|g\|^2_{L^4(0,T;L^4)} \Big] \\
&\quad + c \int_0^t \Big[\|\mathcal{F}_x(g)(s)\|_V^2 + \|g(s)\|_{L^4}^4 \Big] \|u(s)\|_H^2\, ds.
\end{aligned}$$

Now, for an arbitrary $f \in L^4(0,T;L^4)$

$$\begin{aligned}
&\frac{1}{2}\, \|\mathcal{F}_x(f)(t)\|_H^2 + \int_0^t \|\mathcal{F}_x(f)(t)\|_V^2\, ds \\
&\leq \frac{1}{2}\, \|x\|_H^2 + \int_0^t \Big[|b(\mathcal{F}_x(f)(s), f(s), \mathcal{F}_x(f)(s))| + |b(f(s), f(s), \mathcal{F}_x(f)(s))| \Big] ds \\
&\leq \frac{1}{2}\, \|x\|_H^2 + \int_0^t \Big[\|\mathcal{F}_x(f)(s)\|_H^{1/2} \|\mathcal{F}_x(f)(s)\|_V^{3/2} \|f(s)\|_{L^4} + \|f(s)\|_{L^4}^2 \|\mathcal{F}_x(f)(s)\|_V \Big] ds \\
&\leq \frac{1}{2}\, \|x\|_H^2 + \frac{1}{2} \int_0^t \|\mathcal{F}_x(f)(s)\|_V^2\, ds + c \int_0^t \|f(s)\|_{L^4}^4 \|\mathcal{F}_x(f)(s)\|_H^2\, ds + c\, \|f\|^4_{L^4(0,t;L^4)},
\end{aligned} \tag{3.11}$$

which implies that

$$\|\mathcal{F}_x(f)(t)\|_H^2 + \int_0^t \|\mathcal{F}_x(f)(s)\|_V^2\, ds \leq (\|x\|_H^2 + \|f\|^4_{L^4(0,t;L^4)}) \exp\Big(\|f\|^4_{L^4(0,t;L^4)} \Big).$$

This implies that if $f, g \in B_{L^4(0,T;L^4)}(R)$, and $x \in B_H(r)$, there exists $L_{r,R} > 0$ such that

$$\begin{aligned}
&\|u(t)\|_H^2 + \textstyle\int_0^t \|u(s)\|_V^2\, ds \\
&\quad \leq L_{r,R}\, \|g - f\|^2_{L^4(0,t;L^4)} \exp\Big(c \textstyle\int_0^t \Big[\|\mathcal{F}_x(g)(s)\|_V^2 + \|g(s)\|_{L^4}^4 \Big] ds \Big).
\end{aligned} \tag{3.12}$$

By using again (3.11) to estimate $\mathcal{F}_x(g)$, we obtain (3.10) □

In the proof of the main result, Theorem 2.1, two different non-equivalent formulations of the uniform large deviations principle will be required. A thorough comparative analysis of the different formulations of uniform LDPs is given in the paper [17]. We state the definitions for the two forms needed in this paper, using the same notations and conventions as in [17]. This first definition can also be found in [14].

Definition 3.1 Let E be a Banach space and let D be some non-empty set. Suppose that for each $x \in D$, $\{\mu_\epsilon^x\}_{\epsilon>0}$ is a family of probability measures on E and $I^x : E \to [0, +\infty]$ is a good rate function. The family $\{\mu_\epsilon^x\}_{\epsilon>0}$ is said to satisfy a Freidlin–Wentzell uniform large deviations principle in E with rate functions I^x, uniformly with respect to $x \in D$, if the following statement holds.

(i) For any $s \geq 0$, $\delta > 0$ and $\gamma > 0$, there exists $\epsilon_0 > 0$ such that

$$\inf_{x\in D}\left(\mu_\epsilon^x(B_E(\varphi,\delta)) - \exp\Big(-\frac{I^x(\varphi)+\gamma}{\epsilon}\Big)\right) \geq 0, \qquad \epsilon \leq \epsilon_0,$$

for any $\varphi \in \Phi^x(s)$, where $\Phi^x(s) := \{h \in E : I^x(h) \leq s\}$.

(ii) For any $s_0 \geq 0$, $\delta > 0$ and $\gamma > 0$, there exists $\epsilon_0 > 0$ such that

$$\sup_{x\in D}\mu_\epsilon^x(B_E^c(\Phi^x(s),\delta)) \leq \exp\Big(-\frac{s-\gamma}{\epsilon}\Big), \quad \epsilon \leq \epsilon_0,$$

for any $s \leq s_0$, where

$$B_E^c(\Phi^x(s),\delta) = \{h \in E : \mathrm{dist}_E(h,\Phi^x(s)) \geq \delta\}.$$

We will use the following modification of the contraction principle that allows to get a large deviation principle which is uniform with respect to some parameters.

Theorem 3.3 *(Uniform Contraction Principle) Let D be some nonempty set. Assume the family of measures $\{\gamma_\epsilon\}_{\epsilon>0}$ satisfies a large deviations principle on a Banach space F with good rate function $J : F \to [0,+\infty]$. Moreover, assume that there exists another Banach space G such that $F \subset G$ with continuous embedding, such that the family $\{\gamma_\epsilon\}_{\epsilon>0}$ is exponentially tight in G, that is for every $s > 0$ there exists $R_s > 0$ and $\epsilon_s > 0$ such that*

$$\gamma_\epsilon(B_G(R)\cap F) \geq 1 - \exp(-\frac{s}{\epsilon}), \quad \epsilon \leq \epsilon_s. \tag{3.13}$$

Next, suppose that $\{\Lambda^x\}_{x\in D}$ is a family of continuous mappings from F to a Banach space E. We assume that Λ^x are Lipschitz continuous on all balls of G, uniformly over $x \in D$, i.e. for every $R > 0$ there exists some $L_R > 0$ such that

$$\sup_{x\in D}\ \sup_{\varphi_1,\varphi_2\in B_G(R)\cap F}\frac{\|\Lambda^x(\varphi_1)-\Lambda^x(\varphi_2)\|_E}{\|\varphi_1-\varphi_2\|_F} = L_R. \tag{3.14}$$

Then the family of push-forward measures $\{\mu_\epsilon^x\}_{\epsilon>0}$ defined by $\mu_\epsilon^x := \gamma_\epsilon \circ (\Lambda^x)^{-1}$ satisfies a Freidlin–Wentzell uniform large deviations principle in E with rate functions I^x uniformly with respect to $x \in D$, where I^x is given by

$$I^x(\varphi) := \inf\{J(\psi) : \psi \in F,\ \varphi = \Lambda^x(\psi)\}.$$

Proof *Lower Bound.* Fix $s \geq 0$, $\delta > 0$ and $\gamma > 0$. For each $x \in D$, let $\varphi^x \in E$ be such that $I^x(\varphi^x) \leq s$. Therefore, for each $x \in D$ there exists $\psi^x \in F$ such that $\varphi^x = \Lambda^x(\psi^x)$ and $J(\psi^x) \leq I^x(\varphi^x) + \gamma/2$. Since J is a good rate function and $J(\psi^x) \leq s + \gamma/2$, we have

$$\sup_{x\in D}|\psi^x|_F =: \kappa < \infty. \tag{3.15}$$

Now, for every $R > 0$ we have

$$\mu_\epsilon^x(B_E(\varphi^x, \delta)) = \gamma_\epsilon\left(\left\{f \in F : \left\|\Lambda^x(f) - \varphi^x\right\|_E < \delta\right\}\right)$$
$$\geq \gamma_\epsilon\left(\left\{f \in B_G(R) \cap F : \left\|\Lambda^x(f) - \varphi^x\right\|_E < \delta\right\}\right).$$

Then, due to (3.14) and (3.15), there exists some $L_R = L_R(\delta, \kappa) > 0$ such that

$$\mu_\epsilon^x(B_E(\varphi^x, \delta)) \geq \gamma_\epsilon\left(\left\{f \in B_G(R) \cap F : \left\|f - \psi^x\right\|_F < L_R\right\}\right)$$
$$\geq \gamma_\epsilon\left(\left\{f \in F : \left\|f - \psi^x\right\|_F < L_R\right\}\right) - \gamma_\epsilon\left([B_F(R) \cap F]^c\right).$$

Thanks to the exponential tightness (3.13), we can fix $\bar{R} > 0$ such that for some $\epsilon_1 > 0$

$$\gamma_\epsilon\left([B_G(\bar{R}) \cap F]^c\right) \leq \frac{1}{2}\exp\left(-\frac{J(\psi^x) + \gamma/4}{\epsilon}\right), \quad \epsilon \leq \epsilon_1.$$

Next, we fix $\epsilon_2 > 0$ such that

$$\gamma_\epsilon\left(\left\{f \in F : \left\|f - \psi^x\right\|_F < L_{\bar{R}}\right\}\right) \geq \exp\left(-\frac{J(\psi^x) + \gamma/4}{\epsilon}\right), \quad \epsilon \leq \epsilon_2,$$

so that, for some $\epsilon_0 \leq \epsilon_1 \wedge \epsilon_2$, which is independent of $x \in D$, we get

$$\mu_\epsilon^x(B_E(\varphi^x, \delta)) \geq \frac{1}{2}\exp\left(-\frac{J(\psi^x) + \gamma/4}{\epsilon}\right)$$
$$\geq \exp\left(-\frac{J(\psi^x) + \gamma/2}{\epsilon}\right) \geq \exp\left(-\frac{I(\varphi^x) + \gamma}{\epsilon}\right), \quad \epsilon \leq \epsilon_0.$$

Upper Bound. Fix $s_0 \geq 0$, $\delta > 0$ and $\gamma > 0$ and observe that

$$\mu_\epsilon^x(B_E^c(\Phi^x(s), \delta)) = \gamma_\epsilon\left(\left\{f \in F : \inf_{\varphi \in E\,:\,I^x(\varphi) \leq s}\left\|\Lambda^x(f) - \varphi\right\|_E \geq \delta\right\}\right).$$

For every $R > 0$ and $s > 0$ there exists $L_R = L_R(\delta, s) > 0$ such that for every $f \in B_G(R) \cap F$ if there exists $\psi \in F$ such that

$$J(\psi) \leq s, \quad \|f - \psi\|_F \leq L_R,$$

then, for every $x \in D$

$$I^x(\Lambda^x(\psi)) \leq J(\psi) \leq s, \quad \|\Lambda^x(f) - \Lambda^x(\psi)\|_E < \delta.$$

Therefore

$$\mu_\epsilon^x(B_E^c(\Phi^x(s), \delta)) \leq \gamma_\epsilon\left(\left\{f \in B_G(R) \cap F : \inf_{\psi \in F\,:\,J(\psi) \leq s}\|f - \psi\|_F \geq L_R\right\}\right)$$
$$+ \gamma_\epsilon\left([B_G(R) \cap F]^c\right).$$

Now, by using again the exponential tightness (3.13), we can find $\bar{R} > 0$ and $\epsilon_1 > 0$ such that

$$\gamma_\epsilon\left([B_G(\bar{R}) \cap F]^c\right) \leq \exp\left(-\frac{s-2\gamma}{\epsilon}\right), \quad \epsilon \leq \epsilon_1,$$

so that

$$\mu_\epsilon^x\left(B_E^c(\Phi^x(s),\delta)\right) \leq \gamma_\epsilon\left(\left\{f \in F : \inf_{\psi \in F\,:\,J(\psi)\leq s} \|f-\psi\|_E \geq L_{\bar{R}}\right\}\right) + \exp\left(-\frac{s-\gamma/2}{\epsilon}\right), \quad \epsilon \leq \epsilon_1.$$

Therefore, we can find $\epsilon_0 \leq \epsilon_1$, independent of $x \in D$, such that

$$\mu_\epsilon^x\left(B_E^c(\Phi^x(s),\delta)\right) \leq 2\exp\left(-\frac{s-\gamma/2}{\epsilon}\right) \leq \exp\left(-\frac{s-\gamma}{\epsilon}\right), \quad \epsilon \leq \epsilon_0.$$

□

To define the Navier–Stokes rate function, we first define the Hamiltonian

$$\mathcal{H}(u) := u' + Au + B(u), \qquad u \in D(\mathcal{H}) := W^{1,2}(0,T;V^{-1}) \cap L^2(0,T;V)).$$

For $u \in D(\mathcal{H})$ the nonlinearity $B(u)$ is a well-defined element of $L^2(0,T;V^{-1})$. Now, for any $x \in H$ and $u \in C([0,T];H)$, we define

$$I^x(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \|\mathcal{H}(u)(t)\|_H^2\,dt & \text{if } \mathcal{H}(u) \in L^2(0,T;H), \text{ and } u(0)=x, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 3.4 *Assume that $\beta > 1$ and*

$$\lim_{\epsilon\to 0} \delta(\epsilon) = 0, \qquad \sup_{\epsilon>0} \epsilon\,\delta(\epsilon)^{-1/\beta} < \infty. \tag{3.16}$$

If u_ϵ^x is the solution to Eq. (2.6), then for any $R > 0$, the family $\{\mathcal{L}(u_\epsilon^x)\}_{\epsilon>0}$ satisfies a Freidlin–Wentzell uniform large deviations principle in $C([0,T];H)$ with rate functions I^x, uniformly with respect to $x \in B_H(R)$.

Before proving the theorem above, we notice that if $D \subset H$ is a bounded set and $\{z^x\}_{x\in D} \subset C([0,T];L^4 \cap H)$, then

$$\sup_{x\in D} J_T(z^x) < \infty \Longrightarrow \sup_{x\in D} \|z^x\|_{C([0,T];L^4\cap H)} < \infty. \tag{3.17}$$

Moreover, as proven in the following lemma, the family of measures defined by $\gamma_\epsilon = \mathcal{L}(z_\epsilon)$ satisfies the exponential estimate (3.13), with $F = C([0,T]; L^4 \cap H)$ and $G = L^4(0,T; L^4)$.

Lemma 3.3 (Exponential Estimate I) *Under the same assumptions of Theorem 3.4, for any $s > 0$ there exist $\epsilon_s > 0$ and $R_s > 0$ such that*

$$\gamma_\epsilon \left(B_{L^4(0,T;L^4)}(R_s)\right) \geq 1 - \exp\left(-\frac{s}{\epsilon}\right), \qquad \epsilon \leq \epsilon_s.$$

Proof For every $x \in L^4$, we define

$$f(x) = \left(1 + \|x\|_{L^4}^4\right)^{1/4}.$$

The function $f : L^4 \to \mathbb{R}$ is twice differentiable and

$$Df(x) = f(x)^{-3} x^3, \quad D^2 f(x) = 3\, f(x)^{-3} \left(x^2 - f(x)^{-4}\, x^3 \otimes x^3\right)$$

(here and in what follows, for every $x \in L^p$ and $q \leq p$ we denote $x^q = (x_1^q, x_2^q)$). Next, for every $\epsilon > 0$, we define

$$F_\epsilon(x) = \exp\left(\frac{f(x)}{\epsilon}\right), \quad x \in L^4.$$

We have

$$DF_\epsilon(x) = \frac{1}{\epsilon}\, F_\epsilon(x) Df(x) = \frac{1}{\epsilon}\, F_\epsilon(x) f(x)^{-3}\, x^3, \tag{3.18}$$

and

$$\begin{aligned} D^2 F_\epsilon(x) &= \frac{1}{\epsilon^2} F_\epsilon(x) Df(x) \otimes Df(x) + \frac{1}{\epsilon} F_\epsilon(x) D^2 f(x) \\ &= F_\epsilon(x) \left[\frac{1}{\epsilon^2}\, f^{-6}(x)\, x^3 \otimes x^3 + \frac{3}{\epsilon}\, f(x)^{-3} \left(x^2 - f(x)^{-4} x^3 \otimes x^3\right)\right]. \end{aligned} \tag{3.19}$$

As a consequence of Itô's formula, we have

$$\begin{aligned} \mathbb{E}\, F_\epsilon(z_\epsilon(t)) &= \exp\left(\epsilon^{-1}\right) \\ &\quad + \mathbb{E} \int_0^t \left[DF_\epsilon(z_\epsilon(s)) A z_\epsilon(s) + \frac{\epsilon}{2} \sum_{k=1} D^2 F_\epsilon(z_\epsilon(s)) (\sqrt{Q_\epsilon} e_k, \sqrt{Q_\epsilon} e_k)\right] ds \\ &= \exp\left(\epsilon^{-1}\right) + \mathbb{E} \int_0^t \left[DF_\epsilon(z_\epsilon(s)) A z_\epsilon(s) + \frac{\epsilon}{2} \sum_{k=1} \sigma_{\epsilon,k}^2\, D^2 F_\epsilon(z_\epsilon(s))(e_k, e_k)\right] ds. \end{aligned} \tag{3.20}$$

Thanks to (3.19) and (2.7), we have

$$\begin{aligned}\sum_{k=1}^{\infty}\sigma_{\epsilon,k}^2\, D^2F_\epsilon(z_\epsilon(s))(e_k,e_k) &\leq \frac{1}{\epsilon^2}\,F_\epsilon(z_\epsilon(s))f(z_\epsilon(s))^{-6}\sum_{k=1}^{\infty}\sigma_{\epsilon,k}^2\left|\langle z_\epsilon(s)^3,e_k\rangle\right|^2\\&\quad+\frac{3}{\epsilon}\,F_\epsilon(z_\epsilon(s))\,f(z_\epsilon(s))^{-3}\sum_{k=1}^{\infty}\sigma_{\epsilon,k}^2\langle z_\epsilon(s)^2,e_k^2\rangle\\&\leq \frac{1}{\epsilon^2}\,F_\epsilon(z_\epsilon(s))\,f(z_\epsilon(s))^{-6}\|z_\epsilon(s)^3\|_{L^2}^2\\&\quad+\frac{3}{4\pi^2\epsilon}\,F_\epsilon(z_\epsilon(s))\,f(z_\epsilon(s))^{-3}\,\mathrm{Tr}\,Q_\epsilon^2\,\|z_\epsilon(s)\|_{L^2}^2\\&\leq \frac{1}{\epsilon^2}\,F_\epsilon(z_\epsilon(s))\left[f(z_\epsilon(s))^{-6}\|z_\epsilon(s)^2\|_{L^3}^3+\frac{3}{4\pi^2}\,\epsilon\,\delta(\epsilon)^{-1/\beta}\right].\end{aligned}\tag{3.21}$$

Moreover, thanks to (3.18) we have

$$\begin{aligned}DF_\epsilon(z_\epsilon(s))Az_\epsilon(s) &= \frac{1}{\epsilon}\,F_\epsilon(z_\epsilon(s))f(z_\epsilon(s))^{-3}\langle z_\epsilon(s)^3,Az_\epsilon(s)\rangle\\&=-\frac{3}{\epsilon}\,F_\epsilon(z_\epsilon(s))f(z_\epsilon(s))^{-3}\langle\nabla z_\epsilon(s)z_\epsilon(s)^2,\nabla z_\epsilon(s)\rangle\\&=-\frac{3}{4\epsilon}\,F_\epsilon(z_\epsilon(s))f(z_\epsilon(s))^{-3}\|z_\epsilon(s)^2\|_{H^1}^2.\end{aligned}$$

Therefore, putting this together with (3.21), we conclude

$$\begin{aligned}&DF_\epsilon(z_\epsilon(s))Az_\epsilon(s)+\frac{\epsilon}{2}\sum_{k=1}\sigma_{\epsilon,k}^2\,D^2F_\epsilon(z_\epsilon(s))(e_k,e_k)\\&\leq\frac{1}{\epsilon}\,F_\epsilon(z_\epsilon(s))\left[\frac{1}{2}f(z_\epsilon(s))^{-6}\|z_\epsilon(s)^2\|_{L^3}^3-\frac{3}{4}f(z_\epsilon(s))^{-3}\|z_\epsilon(s)^2\|_{H^1}^2+\frac{3}{8\pi^2}\,\epsilon\,\delta(\epsilon)^{-1/\beta}\right].\end{aligned}\tag{3.22}$$

By interpolation, we have

$$\|z_\epsilon(s)^2\|_{L^3}^3\leq\|z_\epsilon(s)^2\|_H^2\|z_\epsilon(s)^2\|_{H^1}\leq\|z_\epsilon(s)^2\|_H\|z_\epsilon(s)^2\|_{H^1}^2=\|z_\epsilon(s)\|_{L^4}^2\|z_\epsilon(s)^2\|_{H^1}^2,$$

and this implies

$$\begin{aligned}f(z_\epsilon(s))^{-6}\|z_\epsilon(s)^2\|_{L^3}^3&\leq f(z_\epsilon(s))^{-3}\|z_\epsilon(s)\|_{L^4}^2\,f(z_\epsilon(s))^{-3}\|z_\epsilon(s)^2\|_{H^1}^2\\&\leq f(z_\epsilon(s))^{-3}\|z_\epsilon(s)^2\|_{H^1}^2.\end{aligned}$$

In particular

$$\begin{aligned}&\tfrac{1}{2}f(z_\epsilon(s))^{-6}\|z_\epsilon(s)^2\|_{L^3}^3-\frac{3}{4}f(z_\epsilon(s))^{-3}\|z_\epsilon(s)^2\|_{H^1}^2\leq-\frac{1}{4}\,f(z_\epsilon(s))^{-3}\|z_\epsilon(s)^2\|_{H^1}^2\\&\leq-\frac{1}{4}\,f(z_\epsilon(s))^{-3}\|z_\epsilon(s)\|_{L^4}^4=-\frac{1}{4}\,f(z_\epsilon(s))+\frac{1}{4}\,f(z_\epsilon(s))^{-3}.\end{aligned}$$

Due to (3.20) and (3.22), this allows to conclude that

$$\mathbb{E}\, F_\epsilon(z_\epsilon(t)) \le \exp\left(\epsilon^{-1}\right) + \frac{1}{4}\int_0^t \mathbb{E} F_\epsilon(z_\epsilon(s))\left[-\frac{f(z_\epsilon(s))}{\epsilon} + \frac{1}{\epsilon} + \frac{3}{2\pi^2}\delta(\epsilon)^{-1/\beta}\right] ds.$$

Since $e^x(a-x) \le e^{a-1}$, for every $x \ge 0$ and $a \ge 1$, it follows

$$\mathbb{E}\, F_\epsilon(z_\epsilon(t)) \le \exp\left(\epsilon^{-1}\right) + t\exp\left(\frac{1+\epsilon\,\delta(\epsilon)^{-1/\beta}}{\epsilon}\right). \tag{3.23}$$

Now, since the function

$$r \in [0,+\infty) \mapsto h(r) := \exp\left(\frac{(1+r)^{1/4}}{\epsilon}\right) \in [1,\infty),$$

is convex when $\epsilon \le 1/3$, and increasing, for any $R > 0$ we have

$$\begin{aligned}
\mathbb{P}\left(\|z_\epsilon\|_{L^4(0,T,L^4)} \ge R\right) &= \mathbb{P}\left(\frac{1}{T}\int_0^T \|z_\epsilon(s))\|_{L^4}^4\, ds \ge \frac{R^4}{T}\right)\\
&= \mathbb{P}\left(h\left(\frac{1}{T}\int_0^T \|z_\epsilon(s))\|_{L^4}^4\, ds\right) \ge h\left(\frac{R^4}{T}\right)\right)\\
&\le \mathbb{P}\left(\frac{1}{T}\int_0^T F_\epsilon(z(s))\, ds \ge \exp\left(\frac{(1+R^4/T)^{1/4}}{\epsilon}\right)\right)\\
&\le \exp\left(-\frac{(1+R^4/T)^{1/4}}{\epsilon}\right)\frac{1}{T}\int_0^T \mathbb{E} F_\epsilon(z(s))\, ds.
\end{aligned}$$

Then, thanks to (3.23) and (3.16), we can find $R_s > 0$ sufficiently large and $\epsilon_s > 0$ sufficiently small such that

$$\begin{aligned}
\mathbb{P}\left(\|z_\epsilon\|_{L^4(0,T,L^4)} \ge R_s\right) &\le \exp\left(-\frac{(1+R_s^4/T)^{1/4}}{\epsilon}\right)\left(\exp\left(\epsilon^{-1}\right) + \frac{T}{2}\exp\left(\frac{1+\epsilon\,\delta(\epsilon)^{-1/\beta}}{\epsilon}\right)\right)\\
&\le \exp\left(-\frac{(1+R_s^4/T)^{1/4}}{\epsilon}\right)\exp\left(\frac{1+2\epsilon\,\delta(\epsilon)^{-1/\beta}}{\epsilon}\right)\\
&\le \exp\left(-\frac{s}{\epsilon}\right), \quad \epsilon \le \epsilon_s.
\end{aligned}$$

□

Proof of Theorem 3.4 First of all, notice that $u_\epsilon^x = (I + \mathcal{F}_x)(z_\epsilon)$. Lemma 3.2 implies that the mapping

$$I + \mathcal{F}_x : C([0,T]; L^4 \cap H) \to C([0,T]; H),$$

is Lipschitz on balls of $L^4(0,T;L^4)$, uniformly on bounded sets of H. Therefore, thanks to Theorem 3.3 applied to the Banach spaces $F = C([0,T]; L^4 \cap H)$, $G = L^4(0,T;L^4)$ and $E = C([0,T]; H)$, and thanks to Theorem 3.2, the family $\{(I +$

$\mathcal{F}_x)(z_\epsilon)\} = \{u_\epsilon^x\}$ satisfies a Freidlin–Wentzell uniform large deviations principle in $C([0, T]; H)$, with rate function

$$I_T^x(u) = \inf \left\{ J_T(z) : u = z + \mathcal{F}_x(z),\ z \in W^{1,2}(0, T; H) \cap L^2(0, T; D(A)) \right\}.$$

If $u \in D(\mathcal{H})$ and $u(0) = x$, then $\mathcal{H}(u) \in L^2(0, T; V^{-1})$ and u is a weak solution to

$$\begin{cases} du(t) + [Au(t) + B(u(t))]dt = \mathcal{H}(u)(t)dt, \\ u(0) = x. \end{cases} \tag{3.24}$$

Note that $u \in D(\mathcal{H})$ implies that Eq. (3.3) with forcing $\varphi = \mathcal{H}(u)$ has a unique weak solution $z_\varphi \in X$. In particular this also implies that $\mathcal{F}_x(z_\varphi) \in D(\mathcal{H})$ and $u = z_\varphi + \mathcal{F}_x(z_\varphi)$. This decomposition is unique. Indeed, if $u = z + \mathcal{F}_x(z)$ for some other $z \in D(\mathcal{H})$, then $u - \mathcal{F}_x(z)$ would again be a weak solution to Eq. (3.3) with forcing $\varphi = \mathcal{H}(u)$ so that $z = z_\varphi$. This implies that

$$J_T(z_\varphi) = \frac{1}{2} \int_0^T \|\mathcal{H}(u)(t)\|_H^2\, dt,$$

whenever $\mathcal{H}(u) \in L^2(0, T; H)$. □

Remark 2 Notice that both the proof of Theorem 3.2 and the proof of Theorem 3.4 do not require periodic boundary conditions.

The second definition for the uniform large deviations principle is given by the following. It can be found in [12].

Definition 3.2 Let E be a Banach space and let D be some non-empty set. Suppose that for each $x \in D$, $\{\mu_\epsilon^x\}_{\epsilon>0}$ is a family of probability measures on E and $I^x : E \to [0, +\infty]$ is a good rate function. The family $\{\mu_\epsilon^x\}_{\epsilon>0}$ is said to satisfy a Dembo–Zeitouni uniform large deviations principle in E with rate functions I^x uniformly with respect to $x \in D$ if the following hold.

(i) For any $\gamma > 0$ and open set $G \subset E$, there exists $\epsilon_0 > 0$ such that

$$\inf_{x\in D} \mu_\epsilon^x(G) \geq \exp\left(-\frac{1}{\epsilon} \left[\sup_{y\in D} \inf_{u\in G} I^y(u) + \gamma \right] \right), \quad \epsilon \leq \epsilon_0.$$

(ii) For any $\gamma > 0$ and closet set $F \subset E$, there exists $\epsilon_0 > 0$ such that

$$\sup_{x\in D} \mu_\epsilon^x(F) \leq \exp\left(-\frac{1}{\epsilon} \left[\inf_{y\in D} \inf_{u\in G} I^y(u) - \gamma \right] \right), \quad \epsilon \leq \epsilon_0.$$

Corollary 3.1 *Assume that*

$$\lim_{\epsilon\to 0} \delta(\epsilon) = 0, \quad \lim_{\epsilon\to 0} \epsilon \log \frac{1}{\delta(\epsilon)} = 0.$$

Let $K \subset H$ be a compact set. Then the family $\{\mathcal{L}(u_\epsilon^x)\}_{\epsilon>0}$ of solutions to Eq. (2.6) satisfies a Dembo–Zeitouni uniform large deviations principle in $C([0,T];H)$ with rate functions $\{I^x\}_{x\in K}$ uniformly with respect to $x \in K$.

Proof In view of Theorem 2.7 of [17], to prove equivalence of the two uniform large deviation principles over a compact subset of H, it suffices to show that for every fixed $s \geq 0$ the mapping

$$x \in H \mapsto \Phi^x(s) := \{u \in C([0,T];H) : I^x(u) \leq s\},$$

is continuous with respect to the Hausdorff metric. That is, we must show that for any $\{x_n\}_{n=1}^\infty \subset H$ such that $x_n \to x \in H$,

$$\lim_{n\to\infty} \max\left(\sup_{u\in\Phi^{x_n}(s)} \mathrm{dist}_{C([0,T];H)}(u,\Phi^x(s)),\ \sup_{u\in\Phi^{x}(s)} \mathrm{dist}_{C([0,T];H)}(u,\Phi^{x_n}(s))\right) = 0.$$

This is immediately implied by the continuity of the Navier–Stokes equations with respect to initial conditions. Indeed, suppose that u_φ^x is a solution to the equation,

$$\begin{cases} du(t) + (Au(t) + B(u(t)))dt = \varphi(t)dt, \\ u(0) = x, \end{cases} \tag{3.25}$$

with driving force $\varphi \in L^2(0,T;H)$. Then by standard energy estimates (see for instance, [15]), we have

$$\begin{aligned} &\sup_{0\leq t\leq T} \left\|u_\varphi^x(t) - u_\varphi^y(t)\right\|_H^2 + \int_0^T \left\|u_\varphi^x(t) - u_\varphi^y(t)\right\|_V^2 dt \\ &\quad\leq \|x-y\|_H^2 \exp\left(c\int_0^T \left\|u_\varphi^y(t)\right\|_V^2 dt\right) \\ &\quad\leq \|x-y\|_H^2 \exp\left(c\big[\,\|y\|_H^2 + \|\varphi\|_{L^2(0,T;H)}^2\,\big]\right). \end{aligned}$$

Now, if $u \in \Phi^x(s)$, then $\varphi_u := \mathcal{H}(u) \in L^2(0,T;H)$, $\frac{1}{2}\|\varphi_u\|_{L^2(0,T;H)}^2 \leq s$ and u solves Eq. (3.24). But then, the weak solution $v \in W^{1,2}(0,T;V^{-1}) \cap L^2(0,T;V)$ to

$$\begin{cases} dv(t) + [Av(t) + B(v(t))]dt = \varphi_u(t)dt, \\ v(0) = y, \end{cases}$$

belongs to $\Phi^y(s)$. Therefore,

$$\mathrm{dist}_{C([0,T];H)}(u,\Phi^y(s)) \leq \|u-v\|_{C([0,T];H)} \leq c_s(\|y\|_H)\ \|x-y\|_H\,,$$

for some continuous increasing function $c_s : [0, +\infty) \to [0, +\infty)$. Since this is true for arbitrary $u \in \Phi^x(s)$, it follows that

$$\sup_{u\in\Phi^x(s)} \text{dist}_{C([0,T];H)}(u, \Phi^y(s)) \leq c_s(\|y\|_H)\ \|x-y\|_H\,,$$

which implies the result, since $\sup_{n\in\mathbb{N}} \|x_n\|_H < \infty$. □

4 Proof of Theorem 2.1

We start this section with the description of the quasi-potential associated with Eq. (2.6). To simplify notation, for any $T > 0$ we will denote

$$I_T(u) := \frac{1}{2}\int_0^T \|\mathcal{H}(u)(t)\|_H^2\, dt,$$

whenever $\mathcal{H}(u) \in L^2(0, T; H)$. In addition, we set

$$I_T^y(u) := \begin{cases} I_T(u), & \text{if } u(0) = y,\\ +\infty, & \text{otherwise.}\end{cases}$$

The quasi-potential $U : H \to [0, +\infty]$ is defined as

$$U(x) := \inf\{I_T(u) : T > 0, u \in C([0, T]; H), u(0) = 0, u(T) = x\}. \tag{4.1}$$

For any $x \in H$, $U(x)$ gives the minimum action of all paths that start at 0 and end at x. Since 0 is an asymptotically attracting equilibria for the Navier–Stokes equations, $U(x)$ will govern the long-time dynamics and asymptotic behavior of the invariant measures.

In the particular case of the Navier–Stokes equations on the torus, the orthogonality of $B(u)$ and Au can be taken advantage of to provide an explicit formula for the quasipotential. In fact, as proven in [4, Theorem 7.1] we have that for any $x \in H$

$$U(x) = \begin{cases} \|x\|_V^2\,, & x \in V,\\ +\infty, & x \in H \setminus V.\end{cases} \tag{4.2}$$

Now, we proceed with the proof of Theorem 2.1. Some of the steps of the proof are analogous to those used in [3, Theorem 4.5], where a large deviation principle for the invariant measures of the $2D$ stochastic Navier–stokes equation is studied, under the assumption that the covariance of the noise does not depend on ϵ. In those steps our arguments will be less detailed and we refer the reader to [3]. On the other hand, our arguments will be fully detailed in those steps of the proof that deviate from [3], and require new arguments and techniques.

4.1 Lower bound

Proposition 4.1 *Under the assumptions of Theorem 2.1, the family of invariant measures $\{\nu_\epsilon\}_{\epsilon>0}$ of Eq. (2.6) satisfies the large deviations principle lower bound in H with rate function $U(x)$. That is, for any $x \in H$, $\delta > 0$ and $\gamma > 0$, there exists $\epsilon_0 > 0$ such that*

$$\nu_\epsilon(B_H(x,\delta)) \geq \exp\Big(-\frac{U(x)+\gamma}{\epsilon}\Big), \qquad \epsilon \leq \epsilon_0.$$

Proof Fix $x \in H$, and any $\delta > 0$, $\gamma > 0$ and $T > 0$. We assume that $U(x) < \infty$ or else there is nothing to prove. Suppose that $\{v^y\}_{y\in H} \subset C([0,T];H)$ is a family of paths satisfying

$$\sup_{y\in H} \left\| v^y(T) - x \right\|_H < \delta/2.$$

Thanks to the invariance of ν_ϵ, we have

$$\begin{aligned}
\nu_\epsilon(B_H(x,\delta)) &= \int_H \mathbb{P}(\left\| u_\epsilon^y(T) - x \right\|_H < \delta)\, d\nu_\epsilon(y) \\
&\geq \int_H \mathbb{P}(\left\| u_\epsilon^y - v^y \right\|_{C([0,T];H)} < \delta/2)\, d\nu_\epsilon(y) \\
&\geq \int_{B_H(0,R)} \mathbb{P}(\left\| u_\epsilon^y - v^y \right\|_{C([0,T];H)} < \delta/2)\, d\nu_\epsilon(y) \\
&\geq \nu_\epsilon(B_H(0,R)) \inf_{y\in B_H(0,R)} \mathbb{P}(\left\| u_\epsilon^y - v^y \right\|_{C([0,T];H)} < \delta/2).
\end{aligned}$$

Since the invariant measures are becoming concentrated around 0, as $\epsilon \downarrow 0$, we have

$$\lim_{\epsilon\to 0} \nu_\epsilon(B_H(0,R)) = 1,$$

for any $R > 0$. Thus, we can pick $\epsilon_1(R) > 0$ small enough that

$$\nu_\epsilon(B_H(0,R)) \geq \frac{1}{2}, \quad \epsilon \leq \epsilon_1(R).$$

Thanks to Theorem 3.4, a Freidlin–Wentzell uniform large deviations principle holds. Then, for every $s_0 > 0$ there exists $\epsilon_2(R) > 0$ such that for any $v^y \in C([0,T];H)$ with $I_T^y(v^y) \leq s_0$,

$$\inf_{y\in B_H(0,R)} \mathbb{P}\left(\left\| u_\epsilon^y - v^y \right\|_{C([0,T];H)} < \delta/2\right) \geq \inf_{y\in B_H(0,R)} \exp\left(-\frac{1}{\epsilon}\left[I_T^y(v^y) + \gamma/2\right]\right),$$

for every $\epsilon \leq \epsilon_2(R)$. Therefore, to complete the proof, it remains to find a T large enough that for each $y \in B_H(0,R)$, there exists a path $v^y \in C([0,T];H)$ with $v^y(0) = y$ that satisfies

(a) $I_T(v^y) \le U(x) + \gamma/2$,
(b) $\|v^y(T) - x\|_H < \delta/2$.

The paths we choose are the solutions u_φ^y to the controlled Navier Stokes equations, Eq. (3.25), with initial condition $y \in H$ and control $\varphi \in L^2(0, T; H)$, defined by

$$\varphi(t) = \begin{cases} 0 & \text{if } 0 \le t \le T_1, \\ \bar{\varphi}(t - T_1) & \text{if } T_1 \le t \le T_1 + T_2, \end{cases}$$

with T_1 and T_2 to be chosen. Here, $\bar{\varphi} \in C([0, T_2]; H)$ is a path such that $u_{\bar{\varphi}}^0(0) = 0$ and $u_{\bar{\varphi}}^0(T_2) = x$ with $I_{T_2}(u_{\bar{\varphi}}^0) \le U(x) + \gamma/2$. Such a T_2 and $\bar{\varphi}$ exist by the definition of the quasipotential U. Meanwhile, $T_1 = T_1(\lambda)$ is taken large enough that the solutions $\{u_0^y\}_{y \in B_H(0,R)}$ to the unforced Navier–Stokes equations satisfy

$$\sup_{y \in B_H(0,R)} \left\| u_0^y(T_1) \right\|_H < \lambda,$$

for some small λ. Clearly point (a) is satisfied since the path contributes nothing to the action integral on the interval $[0, T_1]$. Point (b) follows by noting that the controlled Navier Stokes equations are continuous with respect to initial conditions. Indeed, since $u_0^y(T_1) \in B_H(0, \lambda)$, we have by a standard estimate (for example see Proposition 2.1.25 of [15]) that

$$\begin{aligned} \left\| u_\varphi^y(T_1 + T_2) - x \right\|_H &\le \sup_{z \in B_H(0,\lambda)} \left\| u_{\bar{\varphi}}^z(T_2) - u_{\bar{\varphi}}^0(T_2) \right\|_H \\ &\le \sup_{z \in B_H(0,\lambda)} \|z\|_H \exp\left(c \|z\|_H^2 + c \|\bar{\varphi}\|_{L^2(0,T_2;H)}^2 \right). \end{aligned}$$

This implies point (b) if λ is taken small enough. We conclude the proof upon taking $\epsilon_0 := \min(\epsilon_1, \epsilon_2)$. □

4.2 Upper bound

Proposition 4.2 *Under the assumptions of Theorem 2.1, the family of invariant measures $\{\nu_\epsilon\}_{\epsilon>0}$ of Eq. (2.6) satisfies the large deviations principle upper bound in H with rate function $U(x)$. That is, for any $s \ge 0$, $\delta > 0$ and $\gamma > 0$, there exists $\epsilon_0 > 0$ such that*

$$\nu_\epsilon\left(\{h \in H : \mathrm{dist}_H(h, \Phi(s)) > \delta\}\right) \le \exp\left(-\frac{s-\gamma}{\epsilon}\right), \qquad \epsilon \le \epsilon_0.$$

where

$$\Phi(s) := \{y \in H : U(y) \le s\}.$$

The proof requires the following three lemmas.

Lemma 4.1 *(*Exponential Estimate II*) Assume that Q has the form given in* (2.4) *for some $\beta > 2$. Moreover, suppose that*

$$\lim_{\epsilon\to 0}\delta(\epsilon)=0, \qquad \sup_{\epsilon>0}\epsilon\,\delta(\epsilon)^{-2/\beta}<\infty.$$

Then for any $s > 0$ there exist $\epsilon_s > 0$ and $R_s > 0$ such that

$$\nu_\epsilon(B_V(0,R_s))\geq 1-\exp\left(-\frac{s}{\epsilon}\right), \qquad \epsilon\leq\epsilon_s.$$

Proof Fix $R > 0$, $\epsilon > 0$ and $\gamma > 0$ and let u^0_ϵ be the solution of Eq. (2.6). Thanks to the ergodicity of ν_ϵ, we have

$$\begin{aligned}\nu_\epsilon(B^c_V(0,R)) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{P}(u^0_\epsilon(s)\in B^c_V(0,R))\,ds\\ &\leq \exp\Big(-\frac{R^2}{2\epsilon}\Big)\frac{1}{T}\limsup_{T\to\infty}\int_0^T \mathbb{E}\exp\left(\frac{\left\|u^0_\epsilon(s)\right\|^2_V}{2\epsilon}\right)ds. \end{aligned}\tag{4.3}$$

To estimate the expectation of the exponential, we apply the Ito formula to the functional $F_\epsilon : \mathbb{R}\times V\to\mathbb{R}$ defined by

$$F_\epsilon(t,v)=\exp\left(t+\frac{\|v\|^2_V}{2\epsilon}\right),$$

whose derivatives are given by

$$D_tF_\epsilon(t,u)=F_\epsilon(t,u),$$

and

$$D_uF_\epsilon(t,u)=\frac{1}{\epsilon}F_\epsilon(t,u)u, \quad D^2_uF_\epsilon(t,u)=\frac{1}{\epsilon^2}F_\epsilon(t,u)u\otimes u+\frac{1}{\epsilon}F_\epsilon(t,u)I.$$

Formal application of the Ito formula to the solution u^x_ϵ to Eq. (2.6) implies that

$$\begin{aligned}\mathbb{E}F_\epsilon(t,u^x_\epsilon(t)) &= F_\epsilon(0,x)+\mathbb{E}\int_0^t\Big[D_tF_\epsilon(s,u^x_\epsilon(s))+\langle D_uF_\epsilon(s,u^x_\epsilon(s)),-Au^x_\epsilon(s)-B(u^x_\epsilon(s))\rangle_V\\ &\quad+\frac{\epsilon}{2}\sum_{k\in\mathbb{Z}^2_0}^{\infty}\langle D^2_uF_\epsilon(s,u^x_\epsilon(s))Q_\epsilon e_k,e_k\rangle_V\Big]ds\\ &= F_\epsilon(0,x)+\mathbb{E}\int_0^t F_\epsilon(s,u^x_\epsilon(s))\Big[1-\frac{1}{\epsilon}\left\|u^x_\epsilon(s)\right\|^2_{V^2}\\ &\quad+\frac{\epsilon}{2}\sum_{k\in\mathbb{Z}^2_0}\frac{1}{\epsilon^2}\left(\left|\langle u^x_\epsilon(s),\sqrt{Q_\epsilon}e_k\rangle_V\right|^2+\frac{1}{\epsilon}\langle Q_\epsilon e_k,e_k\rangle_V\right)\Big]ds\end{aligned}$$

$$
\begin{aligned}
&= F_\epsilon(0,x) + \mathbb{E}\int_0^t F_\epsilon(s,u_\epsilon^x(s))\Big[1-\frac{1}{\epsilon}\left\|u_\epsilon^x(s)\right\|_{V^2}^2 \\
&\quad + \frac{1}{2}\sum_{k\in\mathbb{Z}_0^2}\Big(\frac{1}{\epsilon}\sigma_{\epsilon,k}^2|\langle u_\epsilon^x(s), Ae_k\rangle_H|^2 + |k|^2\sigma_{\epsilon,k}^2\Big)\Big]ds \\
&\le F_\epsilon(0,x) + \mathbb{E}\int_0^T F_\epsilon(s,u_\epsilon^x(s))\Big[1-\frac{1}{2\epsilon}\left\|u_\epsilon^x(s)\right\|_{V^2}^2 + \frac{1}{2}\sum_{k\in\mathbb{Z}_0^2}|k|^2\sigma_{\epsilon,k}^2\Big]ds,
\end{aligned}
$$

where in the second line we used identity (2.3) to dispose of the nonlinearity and in the fourth line we used that $|\sigma_{\epsilon,k}|\le 1$, for any $k\in\mathbb{Z}_0^2$ and $\epsilon>0$.

We remark that the use of the Ito formula on the V-valued process u_ϵ^x is justified since we are assuming additional regularity ($\beta>2$) of the noise. For a rigorous justification, see for instance the proof of Proposition 2.4.12 of [15].

Now, since $\beta>2$, we have

$$
P_\epsilon := \sum_{k\in\mathbb{Z}_0^2}|k|^2\sigma_{\epsilon,k}^2 = \sum_{k\in\mathbb{Z}_0^2}\frac{|k|^2}{1+\delta(\epsilon)|k|^{2\beta}} \le c\int_1^\infty \frac{r}{1+\delta(\epsilon)r^\beta}\,dr \le c\,\delta(\epsilon)^{-2/\beta}.
$$

Therefore, thanks to the Poincaré inequality and the fact that $e^x(a-x)\le\exp(a-1)$, for every $a>1$ and $x\ge 0$, it follows that

$$
\begin{aligned}
\mathbb{E}F_\epsilon(t,u_\epsilon^x(t)) &\le \exp\Big(\frac{\|x\|_V^2}{2\epsilon}\Big) + \mathbb{E}\int_0^t \exp(s)\exp\Big(\frac{\left\|u_\epsilon^x(s)\right\|_V^2}{2\epsilon}\Big)\Big(1+\frac{1}{2}P_\epsilon - \frac{\left\|u_\epsilon^x(s)\right\|_V^2}{2\epsilon}\Big)ds \\
&\le \exp\left(\frac{\|x\|_V^2}{2\epsilon}\right) + \int_0^t \exp(s)\exp\left(\frac{1}{2}P_\epsilon\right)ds.
\end{aligned}
$$

Hence,

$$
\mathbb{E}\exp\Big(\frac{\left\|u_\epsilon^x(t)\right\|_V^2}{2\epsilon}\Big) \le \exp\Big(-t+\frac{\|x\|_V^2}{2\epsilon}\Big) + \exp\Big(\frac{1}{2}P_\epsilon\Big).
$$

Finally, using Eq. (4.3), we see that

$$
\begin{aligned}
\nu_\epsilon(B_V^c(0,R)) &\le \exp\Big(-\frac{R^2}{\epsilon}\Big)\limsup_{T\to\infty}\frac{1}{T}\int_0^T\Big[e^{-t}+\exp\Big(\frac{1}{2}P_\epsilon\Big)\Big]dt \\
&= \exp\Big(-\frac{R^2}{\epsilon}+\frac{P_\epsilon}{2}\Big) \le \exp\Big(-\frac{R^2 - C\epsilon\,\delta_\epsilon^{-2/\beta}}{\epsilon}\Big),
\end{aligned}
$$

which completes the proof of the lemma, since $\epsilon\,\delta(\epsilon)^{-2/\beta}\to 0$, as $\epsilon\downarrow 0$. □

Lemma 4.2 *For any $\delta>0$ and $s>0$, there exist $\lambda>0$ and $T>0$ such that for any $t\ge T$ and $z\in C([0,t];H)$,*

$$
|z(0)|_H<\lambda,\quad I_t(z)\le s \implies \mathrm{dist}_H(z(t),\Phi(s))<\delta,
$$

where $\Phi(s) := \{x \in H : U(x) \leq s\}$.

Lemma 4.3 *For any* $s > 0$, $\delta > 0$ *and* $r > 0$, *let* λ *be as in Lemma 4.2. Then there exists* $N \in \mathbb{N}$ *large enough that*

$$u \in H_{r,s,\delta}(N) \implies I_T(u) \geq s,$$

where the set $H_{r,s,\delta}(n)$ *is defined for* $N \in \mathbb{N}$ *by*

$$H_{r,s,\delta}(N) := \left\{u \in C([0,N];H),\ \|u(0)\|_H \leq r,\ \|u(j)\|_H \geq \lambda,\ j = 1,\dots,N\right\}.$$

The proofs of Lemmas 4.2 and 4.3 depend only on the properties of the deterministic Navier–Stokes equation and can be found in [3] (see Lemmas 7.2 and 7.3).

Proof of Proposition 4.2 Fix any $s > 0, \delta > 0$ and $\gamma > 0$ and let R_s be as in Lemma 4.1. Due to the invariance of ν_ϵ, for any $t \geq 0$ we have

$$\begin{aligned}
\nu_\epsilon\left(\{h \in H : \mathrm{dist}_H(h, \Phi(s)) \geq \delta\}\right) &= \int_H \mathbb{P}\left(\mathrm{dist}_H(u_\epsilon^y(t), \Phi(s)) \geq \delta\right) d\nu_\epsilon(y)\\
&= \int_{B_V^c(0,R_s)} \mathbb{P}\left(\mathrm{dist}_H(u_\epsilon^y(t), \Phi(s)) \geq \delta\right) d\nu_\epsilon(y)\\
&\quad + \int_{B_V(0,R_s)} \mathbb{P}\left(\mathrm{dist}_H(u_\epsilon^y(t), \Phi(s)) \geq \delta,\ u_\epsilon^y \in H_{R_s,s,\delta}(N)\right) d\nu_\epsilon(y)\\
&\quad + \int_{B_V(0,R_s)} \mathbb{P}\left(\mathrm{dist}_H(u_\epsilon^y(t), \Phi(s)) \geq \delta,\ u_\epsilon^y \notin H_{R_s,s,\delta}(N)\right) d\nu_\epsilon(y)\\
&=: K_1 + K_2 + K_3.
\end{aligned}$$

Now, thanks to Lemma 4.1 we know that

$$K_1 \leq \nu_\epsilon(B_V^c(0,R_s)) \leq \exp\left(-\frac{s}{\epsilon}\right).$$

Next, let N be as in Lemma 4.3. Since $H_{R_s,s,\delta}(N)$ is a closed set in $C([0,N];H)$ and $B_V(0,R_s)$ is a compact subset of H, the Dembo–Zeitouni uniform large deviation principle over compact sets, Corollary 3.1, implies that there exists $\epsilon_0 > 0$ such that

$$\begin{aligned}
K_2 &\leq \sup_{y \in B_V(0,R_s)} \mathbb{P}(u_\epsilon^y \in H_{R_s,s,\delta}(N))\\
&\leq \exp\left(-\frac{1}{\epsilon}\left[\inf_{z \in B_V(0,R_s)} \inf_{h \in H_{R_s,s,\delta}(N)} I_t^z(h) - \gamma\right]\right),
\end{aligned}$$

for any $\epsilon \leq \epsilon_0$. Hence, by Lemma 4.3,

$$K_2 \leq \exp(-\frac{1}{\epsilon}[s-\gamma]).$$

To address K_3, we use the Markov property of u_ϵ to stop the process at integer times. We then have

$$\begin{aligned} K_3 &= \int_{B_V(0,R_s)} \mathbb{P}\left(\bigcup_{j=1}^{N} \left\{ |u_\epsilon^y(j)|_H < \lambda \right\} \bigcap \left\{ \mathrm{dist}_H(u_\epsilon^y(t), \Phi(s)) \geq \delta \right\} \right) d\nu_\epsilon(y) \\ &\leq \sum_{j=1}^{N} \int_{B_V(0,R_s)} \mathbb{P}\left(\left\{ |u_\epsilon^y(j)|_H < \lambda \right\} \bigcap \left\{ \mathrm{dist}_H(u_\epsilon^y(t), \Phi(s)) \geq \delta \right\} \right) d\nu_\epsilon(y) \\ &\leq \sum_{j=1}^{N} \sup_{y \in B_H(0,\lambda)} \mathbb{P}(\mathrm{dist}_H(u_\epsilon^y(t-j), \Phi(s)) \geq \delta). \end{aligned}$$

In order to use the uniform LDP of Theorem 3.4, we must convert this event at time $t-j$ to an event in $C([0,t-j];H)$. To do so, we pick t large enough that Lemma 4.2 applies for $\delta/2$. Then, if $y \in B_H(\lambda)$

$$\begin{aligned} &\mathrm{dist}_H(u_\epsilon^y(t-j), \Phi(s)) \geq \delta \\ &\qquad \Longrightarrow \inf\left\{ \left\| u_\epsilon^y - v \right\|_{C([0,t-j];H)} : \|v(0)\|_H < \lambda, I_{t-j}(v) \leq s \right\} \geq \frac{\delta}{2} \\ &\qquad \Longrightarrow \mathrm{dist}_{C([0,t-j];H)}\left(u_\epsilon^y, \Psi^y(s) \right) \geq \delta/2, \end{aligned}$$

where

$$\Psi^y(s) := \{ v \in C([0,t-j];H) : v(0) = y, I_{t-j}(v) \leq s \}.$$

Then, by Theorem 3.4, there exists $\epsilon_{0,j}$ such that for any $\epsilon \leq \epsilon_{0,j}$,

$$\begin{aligned} &\sup_{y \in B_H(0,\lambda)} \mathbb{P}(\mathrm{dist}_H(u_\epsilon^y(t-j), \Phi(s)) \geq \delta) \\ &\qquad \leq \sup_{y \in B_H(0,\lambda)} \mathbb{P}(\mathrm{dist}_{C([0,t-j];H)}\left(u_\epsilon^y, \Psi^y(s) \right) \geq \delta/2) \\ &\qquad \leq \exp\left(-\frac{s-\gamma}{\epsilon} \right). \end{aligned}$$

Hence, for any $\epsilon < \min(\epsilon_0, \epsilon_{0,1}, \ldots, \epsilon_{0,N})$ it follows that

$$K_3 \leq N \exp\left(-\frac{s-\gamma}{\epsilon} \right),$$

which implies the result. □

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