

K-THEORY AND THE UNIVERSAL COEFFICIENT THEOREM FOR SIMPLE SEPARABLE EXACT C*-ALGEBRAS NOT ISOMORPHIC TO THEIR OPPOSITES

N. CHRISTOPHER PHILLIPS AND MARIA GRAZIA VIOLA

ABSTRACT. We construct uncountably many mutually nonisomorphic simple separable stably finite unital exact C*-algebras which are not isomorphic to their opposite algebras. In particular, we prove that there are uncountably many possibilities for the K_0 -group, the K_1 -group, and the tracial state space of such an algebra. We show that these C*-algebras satisfy the Universal Coefficient Theorem, which is new even for the already known example of an exact C*-algebra nonisomorphic to its opposite algebra produced in earlier work.

1. INTRODUCTION

In the last two decades there has been much interest in finding examples of simple C*-algebra not isomorphic to their opposite algebras. The motivation for this work is the Elliott classification program for simple C*-algebras, which shows that simple separable nuclear unital C*-algebras that absorb the Jiang-Su algebra Z tensorially and satisfy the Universal Coefficient Theorem are classified up to isomorphism by an invariant consisting of K -theory and tracial information. (See Corollary D in [5], or combine Corollary 4.11 in [15] and Theorem A in [5].) For a unital C*-algebra A , the Elliott invariant is given by

$$\text{Ell}(A) = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), \rho),$$

where $(K_0(A), K_0(A)_+, [1_A], K_1(A))$ is the scaled ordered K -theory of A , $T(A)$ denotes the tracial state simplex, and $\rho: K_0(A) \times T(A) \rightarrow \mathbb{R}$ is the natural pairing map, $\rho([p] - [q], \tau) = \tau(p) - \tau(q)$ for projections $p, q \in M_\infty(A)$ and $\tau \in T(A)$. Since the Elliott invariant of a C*-algebra is the same as that of its opposite algebra, simple C*-algebras which are not isomorphic to their opposite algebra provide examples of simple C*-algebras that are not isomorphic despite having the same Elliott invariant.

In [39] we constructed an example of a simple separable unital exact C*-algebra A not isomorphic to its opposite algebra. The algebra A has a number of nice properties: it is stably finite and approximately divisible, and

Date: 1 September 2022.

2000 Mathematics Subject Classification. 46L35, 46L37, 46L40, 46L54.

N. Christopher Phillips was partially supported by NSF Grants DMS-0701076 and DMS-1101742, and by the Simons Foundation Collaboration Grant for Mathematicians #587103. Maria Grazia Viola was partially supported by a Research Development Fund from Lakehead University and an NSERC Discovery Grant.

it has real rank zero, stable rank one, and a unique tracial state. The order on projections over A is determined by traces, and A tensorially absorbs the Jiang-Su algebra Z . Its K-theory is given by $K_0(A) \cong \mathbb{Z}[\frac{1}{3}]$ and $K_1(A) = 0$. Its Cuntz semigroup is $W(A) \cong \mathbb{Z}[\frac{1}{3}]_+ \amalg (0, \infty)$.

The purpose of this article is to exhibit many examples of simple separable exact C*-algebras not isomorphic to their opposite algebras, and to prove that they satisfy the Universal Coefficient Theorem. (This is new even for the example in [39].) In particular, we prove that there are uncountably many possibilities for the K-theory of such an algebra, while still preserving most of the good properties of the algebra in [39]. For any odd prime q such that -1 is not a square mod q , and for any UHF algebra B stable under tensoring with the q^∞ UHF algebra (the algebra $\bigotimes_{n=1}^\infty M_q$), we produce a simple

separable exact C*-algebra D , not isomorphic to its opposite algebra, with real rank zero and a unique tracial state, such that $K_*(D) \cong K_*(B)$. For any q and B as above, and for any Choquet simplex Δ , we give a simple separable exact C*-algebra D , not isomorphic to its opposite algebra, with real rank one, such that $K_*(D) \cong K_*(B)$, and whose tracial state space is isomorphic to Δ . For any q and B as above, and for any countable abelian group G , we give a simple separable exact C*-algebra D , not isomorphic to its opposite algebra, with real rank zero and a unique tracial state, such that $K_0(D) \cong K_0(B)$ and $K_1(D) \cong G \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{q}]$. We show that all the C*-algebras we construct satisfy the Universal Coefficient Theorem. We give further information on the algebras described above, including showing that the order on projections is determined by traces, computing the Cuntz semigroups, and showing that the algebras have stable rank one and tensorially absorb the q^∞ UHF algebra and the Jiang-Su algebra. The examples described above are not the most general that can be obtained with our method, but are chosen to illustrate the possibilities. There are infinitely many primes q such that -1 is not a square mod q , so there are infinitely many choices for q covered by our examples.

Our results show an essential difference between the class of nuclear C*-algebras absorbing the Jiang-Su algebra and the class of exact C*-algebras absorbing the Jiang-Su algebra, and the importance of nuclearity for the classification of simple C*-algebras.

Question 8.1 in [39] asked whether for any UHF algebra B there exists a simple separable exact C*-algebra D not isomorphic to its opposite algebra that has the same K-theory as B , and the same properties as the algebra of [39]. Our results provide a partial positive answer to this question.

The paper is organized as follows. Section 2 contains preliminaries. In particular, we recall some relevant definitions and constructions involving von Neumann algebras and the Connes invariant. In Section 3 we recall the definition of the continuous Rokhlin property for an action of a finite group G on a separable unital C*-algebra. The model action of G on the UHF algebra of type $\text{card}(G)^\infty$ is an example of an action with this property. Our main result is that if A is a separable, unital C*-algebra satisfying the Universal Coefficient Theorem, and α is an action of a finite abelian group

G on A with the continuous Rokhlin property, then the fixed point algebra A^α and the crossed product $C^*(G, A, \alpha)$ satisfy the Universal Coefficient Theorem. In Theorem 1.10 of [19], Gardella has later extended this result to actions of second countable compact groups. In Section 4 we construct our basic example, one algebra D for each prime q such that -1 is not a square mod q , satisfying the same properties as the algebra in [39], and whose K -theory is given by $K_0(D) \cong \mathbb{Z}[\frac{1}{q}]$ and $K_1(D) = 0$. Moreover, we show that D satisfies the Universal Coefficient Theorem. Section 5 contains the main step towards the proof that these algebras are not isomorphic to their opposites. Each of them has a unique tracial state. We prove that the weak closure of D in the Gelfand-Naimark-Segal representation associated with this tracial state is not isomorphic to its opposite algebra. In Section 6, we tensor these basic examples with other simple separable nuclear unital C^* -algebras. The main result of Section 5 also applies to such tensor products, and we thus obtain the examples described above. Section 7 contains some open problems.

The authors would like to thank E. Gardella for useful discussions about the Universal Coefficient Theorem and for suggesting the argument used to show that the algebra D in Section 4 satisfies the Universal Coefficient Theorem.

Throughout, we denote the circle $\{\zeta \in \mathbb{C} : |\zeta| = 1\}$ by \mathbb{T} . We also denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$ by \mathbb{Z}_n ; the p -adic integers will not be used in this paper.

2. PRELIMINARIES

In this section, we provide some background material about opposite algebras, automorphisms of II_1 factors, the Connes invariant, and the Cuntz semigroup.

First we recall the definition of the opposite algebra and the conjugate algebra of a C^* -algebra A .

Definition 2.1. Let A be a C^* -algebra. The opposite algebra A^{op} is the C^* -algebra which has the same vector space structure, norm, and adjoint as A , while the product of x and y in A^{op} , which we denote by $x \star y$ when necessary, is given by $x \star y = yx$. If $\omega : A \rightarrow \mathbb{C}$ is a linear functional, then we let ω^{op} denote the same map but regarded as a linear functional $\omega^{\text{op}} : A^{\text{op}} \rightarrow \mathbb{C}$. The conjugate algebra A^c is the C^* -algebra whose underlying vector space structure is the conjugate of A , that is, the product of $\lambda \in \mathbb{C}$ and $x \in A^c$ is equal to $\bar{\lambda}x$ (as evaluated in A), and whose ring structure, adjoint, and norm are the same as for A .

Remark 2.2. The map $x \mapsto x^*$ is an isomorphism from A^c to A^{op} .

Notation 2.3. Let A be a C^* -algebra, and let ω be a state on A . We denote the triple consisting of the Gelfand-Naimark-Segal representation, its Hilbert space, and its standard cyclic vector by $(\pi_\omega, H_\omega, \xi_\omega)$.

Also, for any C^* -algebra or von Neumann algebra A and any tracial state τ on A , we denote the usual L^2 -norm by $\|x\|_{2,\tau} = (\tau(x^*x))^{1/2}$ for $x \in A$. When no confusion can arise about the tracial state used, we write $\|x\|_2$.

It seems useful to make explicit the following fact, which has been used implicitly in previous papers.

Lemma 2.4. Let A be a C^* -algebra, and let τ be a tracial state on A . Then τ^{op} is a tracial state on A^{op} and, as von Neumann algebras, we have $\pi_{\tau^{\text{op}}}(A^{\text{op}})'' \cong [\pi_{\tau}(A)'']^{\text{op}}$.

Proof. The functional τ^{op} is a state because A and A^{op} have the same norm and positive elements. It is immediate that τ^{op} is tracial.

Next, we claim that $\|x\|_{2,\tau^{\text{op}}} = \|x\|_{2,\tau}$ for all $x \in A$. Indeed, using the trace property at the third step,

$$(\|x\|_{2,\tau^{\text{op}}})^2 = \tau^{\text{op}}(x^* \star x) = \tau(xx^*) = \tau(x^*x) = (\|x\|_{2,\tau})^2.$$

We can identify $\pi_{\tau}(A)''$ with the set of elements in the Hausdorff completion of A in $\|\cdot\|_{2,\tau}$ which are limits in $\|\cdot\|_{2,\tau}$ of norm bounded sequences in A , and similarly with $\pi_{\tau^{\text{op}}}(A^{\text{op}})''$. It follows from the claim that the identity map of A extends to a linear isomorphism $\pi_{\tau}(A)'' \rightarrow \pi_{\tau^{\text{op}}}(A^{\text{op}})''$, which is easily seen to preserve adjoints and reverse multiplication. \square

To prove that our C^* -algebras are not isomorphic to their opposite algebras, we will need some terminology and results for the automorphisms of a II_1 factor.

Definition 2.5. For any von Neumann algebra M , we denote by $\text{Inn}(M)$ the group of inner automorphisms of M , that is, the automorphisms of the form $\text{Ad}(u)$ for some unitary $u \in M$. Let M be a II_1 factor with separable predual. Denote by τ the unique tracial state on M . An automorphism φ of M is *approximately inner* if there exists a sequence of unitaries $(u_n)_{n \in \mathbb{Z}_{>0}}$ in M such that $\text{Ad}(u_n) \rightarrow \varphi$ pointwise in $\|\cdot\|_2$. Denote by $\overline{\text{Inn}}(M)$ the group of approximately inner automorphisms of M .

Another important class of automorphisms consists of the centrally trivial automorphisms of M .

Definition 2.6. Let M be a II_1 factor with separable predual. Let τ be the unique tracial state on M . Recall that a bounded sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in M is central if $\lim_{n \rightarrow \infty} \|x_n a - a x_n\|_2 = 0$ for all $a \in M$. An automorphism φ of M is said to be *centrally trivial* if $\lim_{n \rightarrow \infty} \|\varphi(x_n) - x_n\|_2 = 0$ for every central sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in M . Let $\text{Ct}(M)$ denote the set of all centrally trivial automorphisms of M .

By the comments following Definition 3.1 in [8], the set $\text{Ct}(M)$ is a normal subgroup of $\text{Aut}(M)$. It is obviously closed.

We recall below from [7] the definition of Connes invariant $\chi(M)$ of a II_1 factor M . In [7], Connes uses centralizing sequences to define the centrally trivial automorphisms. For $\omega \in M_*$ and $x \in M$, we define $[\omega, x] \in M_*$ by $[\omega, x](y) = \omega(xy - yx)$ for $y \in M$. A sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ is then said to be centralizing if $\lim_{n \rightarrow \infty} \|[\omega, x_n]\| = 0$ for all $\omega \in M_*$. In general, centralizing sequences are the right ones to use to define the Connes invariant.

A bounded sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in a II_1 factor M is central if and only if for some (equivalently, for any) strong operator dense subset $S \subseteq M$,

we have $x_n y - y x_n \rightarrow 0$ in the strong operator topology for all $y \in S$. In a II_1 factor, we claim that the central sequences are the same as the centralizing sequences. The implication from (β) to (γ) in Proposition 2.8 of [6] shows that centralizing sequences in M are central. For the reverse, in Proposition 2.8(α) of [6], we take φ there to be the tracial state τ . Since $[\tau, y] = 0$ for all $y \in M$, the implication from (α) to (γ) there shows that central sequences in M are centralizing.

Definition 2.7. Let M be a II_1 factor with separable predual. The *Connes invariant* $\chi(M)$ is the subgroup of the outer automorphism group $\text{Out}(M) = \text{Aut}(M)/\text{Inn}(M)$ obtained as the center of the image under the quotient map of the group of approximately inner automorphisms.

Remark 2.8. If M is isomorphic to its tensor product with the hyperfinite II_1 factor, then $\chi(M)$ is the image in $\text{Aut}(M)/\text{Inn}(M)$ of $\text{Ct}(M) \cap \overline{\text{Inn}(M)}$. See [7].

In general it is not easy to compute the Connes invariant of a II_1 factor. For the hyperfinite II_1 factor R , every centrally trivial automorphism is inner by Theorem 3.2 (1) in [10], so $\chi(R) = \{0\}$. Moreover, any approximately inner automorphism of the free group factor on n generators $\mathcal{L}(\mathbb{F}_n)$ is inner, so $\chi(\mathcal{L}(\mathbb{F}_n)) = \{0\}$. (See [30], or Lemma 3.2 in [52].)

A useful tool to compute the Connes invariant of some II_1 factors is the short exact sequence introduced in [7]. Assume that N is a II_1 factor without nontrivial hypercentral sequences, that is, central sequences that asymptotically commute in the L^2 -norm with every central sequence of N . Let G be a finite subgroup of $\text{Aut}(N)$ such that $G \cap \overline{\text{Inn}(N)} = \{1\}$, and let $\theta: G \rightarrow \text{Aut}(N)$ be the inclusion, regarded as an action of G on N . Define $K = G \cap \text{Ct}(N)$ and let $K^\perp \subseteq \widehat{G}$ be its annihilator, that is,

$$K^\perp = \{f: G \rightarrow \mathbb{T}: f \text{ is a homomorphism and } f|_K = 1\} \subseteq \widehat{G}.$$

For any von Neumann algebra M , let $\xi_M: \text{Aut}(M) \rightarrow \text{Out}(M)$ denote the quotient map. Let $H \subseteq \text{Aut}(N)$ be the subgroup

$$(2.1) \quad H = \{\text{Ad}(u): u \in N \text{ is unitary and } \rho(u) = u \text{ for all } \rho \in G\}.$$

Let $G \vee \text{Ct}(N)$ be the subgroup of $\text{Aut}(N)$ generated by $G \cup \text{Ct}(N)$. (It is closed since G is finite and $\text{Ct}(N)$ is closed and normal.) Taking the closure in the topology of pointwise L^2 -norm convergence, let

$$L = \xi_N((G \vee \text{Ct}(N)) \cap \overline{H}) \subseteq \text{Out}(N).$$

Then the Connes short exact sequence (Theorem 4 of [7]) is

$$(2.2) \quad \{1\} \longrightarrow K^\perp \xrightarrow{\partial} \chi(N \rtimes_\theta G) \xrightarrow{\Pi} L \longrightarrow \{1\}.$$

We briefly describe the maps ∂ and Π in this exact sequence. We follow Section 5 in [31]. For $g \in G$, let $u_g \in N \rtimes_\theta G$ be the standard unitary in the crossed product associated to g , so that $\theta_g = \text{Ad}(u_g)|_N$. Given an element x in $N \rtimes_\theta G$, write it as $\sum_{g \in G} a_g u_g$ with $a_g \in N$ for $g \in G$. For each $\varphi: G \rightarrow \mathbb{T}$

in K^\perp , define $\Delta(\varphi): N \rtimes_\theta G \rightarrow N \rtimes_\theta G$ by

$$\Delta(\varphi) \left(\sum_{g \in G} a_g u_g \right) = \sum_{g \in G} \varphi(g) a_g u_g.$$

Then

$$\Delta(\varphi) \in \text{Ct}(N \rtimes_\theta G) \cap \overline{\text{Inn}(N \rtimes_\theta G)},$$

and $\partial = \xi_{N \rtimes_\theta G} \circ \Delta$.

To define Π , for any $\sigma \in \chi(N \rtimes_\theta G)$ choose an automorphism $\alpha \in \text{Ct}(N \rtimes_\theta G) \cap \overline{\text{Inn}(N \rtimes_\theta G)}$ such that $\xi_{N \rtimes_\theta G}(\alpha) = \sigma$. Since $G \cap \text{Inn}(N) = \{1\}$, by Corollary 6 and Lemma 2 in [30] (or by Lemma 15.42 in [16]) there exist a sequence $(u_n)_{n \in \mathbb{Z}_{>0}}$ of unitaries in N^θ and a unitary $z \in N \rtimes_\theta G$ such that $\alpha = \text{Ad}(z) \circ \left[\lim_{n \rightarrow \infty} \text{Ad}(u_n) \right]$. Set $\varphi_\sigma = \text{Ad}(z^*) \circ \alpha|_N$. One can then show that $\varphi_\sigma \in (G \vee \text{Ct}(N)) \cap \overline{H}$, and that the map $\xi_N(\varphi_\sigma)$ does not depend on the choice of the representative α but only on the class σ . Therefore the map $\Pi: \chi(N \rtimes_\theta G) \rightarrow L$, given by $\Pi(\sigma) = \xi_N(\varphi_\sigma)$, is well defined.

To see that Π is surjective, let $\mu \in L$ and choose $\alpha \in (G \vee \text{Ct}(N)) \cap \overline{H}$ such that $\xi_N(\alpha) = \mu$. Since α is the limit of automorphisms that commute with G , it also commutes with G . It follows that the map

$$(2.3) \quad \eta \left(\sum_{g \in G} a_g u_g \right) = \sum_{g \in G} \alpha(a_g) u_g$$

is an automorphism of $N \rtimes_\theta G$. Moreover, $\eta \in \text{Ct}(N \rtimes_\theta G) \cap \overline{\text{Inn}(N \rtimes_\theta G)}$ and $\Pi(\xi_{N \rtimes_\theta G}(\eta)) = \mu$.

An important concept in the classification of automorphisms of the hyperfinite II_1 factor is the one of obstruction to lifting, defined by Connes in Section 1 of [10]. It will play a key role in showing that our algebras are not isomorphic to their opposites.

Definition 2.9. Let M be a II_1 factor and let α be an automorphism of M . Let n be the smallest nonnegative integer such that there is a unitary $u \in M$ with $\alpha^n = \text{Ad}(u)$. If no power of α is inner, we set $n = 0$. Since M is a factor, it is easy to check that there is $\lambda \in \mathbb{C}$ such that $\lambda^n = 1$ and $\alpha(u) = \lambda u$. (Simply apply $\alpha^{n+1} = \alpha \circ \alpha^n = \alpha^n \circ \alpha$ to any element x in M .) We call λ the *obstruction to lifting* of α , and refer to the pair (n, λ) as the outer invariant of α .

Next, we recall what it means for the order on projections to be determined by traces. Let A be a C^* -algebra and denote by $M_n(A)$ the $n \times n$ matrices with entries in A . Let $M_\infty(A)$ denote the algebraic direct limit of the sequence $(M_n(A), \varphi_n)_{n \in \mathbb{Z}_{>0}}$, in which $\varphi_n: M_n(A) \rightarrow M_{n+1}(A)$ is defined by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Denote by $T(A)$ the set of tracial states of A .

Definition 2.10. We say that *the order on projections over A is determined by traces* if whenever $p_1, p_2 \in M_\infty(A)$ are projections such that $\tau(p_1) < \tau(p_2)$ for every τ in $T(A)$, then p_1 is Murray-von Neumann subequivalent to p_2 .

We conclude this section by recalling the definitions of Cuntz subequivalence and the Cuntz semigroup. See Section 2 of [4] and the references there for the definitions below and the proofs of the assertions made here.

Definition 2.11. Let A be a C^* -algebra and let $a, b \in M_\infty(A)_+$. We say that a is Cuntz subequivalent to b , denoted $a \precsim b$, if there exists a sequence $(v_n)_{n \in \mathbb{Z}_{>0}}$ in $M_\infty(A)$ such that $\lim_{n \rightarrow \infty} \|v_n b v_n^* - a\| = 0$. If $a \precsim b$ and $b \precsim a$ we say that a is Cuntz equivalent to b and write $a \sim b$.

Cuntz equivalence is an equivalence relation, and we write $\langle a \rangle$ for the equivalence class of a .

Definition 2.12. Let A be a C^* -algebra. The *Cuntz semigroup* of A is $W(A) = M_\infty(A)_+ / \sim$. We define a semigroup operation on $W(A)$ by

$$\langle a \rangle + \langle b \rangle = \left\langle \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\rangle$$

and a partial order by $\langle a \rangle \leq \langle b \rangle$ if and only if $a \precsim b$. With this structure $W(A)$ becomes a positively ordered abelian semigroup with identity.

Usually, it is hard to compute the Cuntz semigroup of a C^* -algebra, but the following remark computes $W(A)$ for the C^* -algebras A of interest here.

Remark 2.13. Denote the Jiang-Su algebra by Z . Assume that A is a simple unital exact stably finite C^* -algebra which is Z -stable, that is, $Z \otimes A \cong A$. Let $V(A)$ be the Murray-von Neumann semigroup of A . For any compact convex set Δ , let $\text{LAff}_b(\Delta)_{++}$ denote the set of bounded strictly positive lower semicontinuous affine functions on Δ . By Corollary 5.7 of [4], we have

$$W(A) \cong V(A) \amalg \text{LAff}_b(T(A))_{++}.$$

The addition and order on the disjoint union are defined as follows. On each part of the disjoint union, the addition and order are as usual. For the other cases, for $x \in V(A)$ define $\hat{x}: T(A) \rightarrow [0, \infty)$ by $\hat{x}(\tau) = \tau(x)$ for $\tau \in T(A)$. Now let $x \in V(A)$ and $y \in \text{LAff}_b(T(A))_{++}$. Then $x + y$ is the function $\hat{x} + y \in \text{LAff}_b(T(A))_{++}$. Also, $x \leq y$ if and only if $\hat{x}(\tau) < y(\tau)$ for all $\tau \in T(A)$, and $y \leq x$ if and only if $y(\tau) \leq \hat{x}(\tau)$ for all $\tau \in T(A)$.

3. THE CONTINUOUS ROKHLIN PROPERTY AND THE UNIVERSAL COEFFICIENT THEOREM

We recall the definition of asymptotic homomorphism, and what it means for an action of a finite group on a C^* -algebra to have the continuous Rokhlin property. The main result of this section is that if A is a separable unital C^* -algebra satisfying the Universal Coefficient Theorem, and α is an action of a finite group G on A satisfying the continuous Rokhlin property, then the fixed point algebra A^α and the crossed product $C^*(G, A, \alpha)$ satisfy the Universal Coefficient Theorem. Gardella has later extended this result to actions of second countable compact groups. (See Theorem 1.10 in [19].)

Definition 3.1. Let A be a separable, unital C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ denote an action of a finite group G on A . We say that α has the

continuous Rokhlin property if there exist continuous functions $t \rightarrow e_g^{(t)}$ from $[0, \infty)$ to A , for $g \in G$, such that:

- (1) For each $t \in [0, \infty)$, $(e_g^{(t)})_{g \in G}$ is a family of mutually orthogonal projections with $\sum_{g \in G} e_g^{(t)} = 1$.
- (2) $\lim_{t \rightarrow \infty} \|\alpha_g(e_h^{(t)}) - e_{gh}^{(t)}\| = 0$ for every $g, h \in G$.
- (3) For any $g \in G$ and $a \in A$, we have $\lim_{t \rightarrow \infty} \|e_g^{(t)}a - ae_g^{(t)}\| = 0$.

Lemma 3.2. Let A and B be separable unital C^* -algebras, let G be a finite group, and let $\alpha: G \rightarrow \text{Aut}(A)$ and $\beta: G \rightarrow \text{Aut}(B)$ be actions of G on A and B . Assume that α has the continuous Rokhlin property. Let $A \otimes B$ be any C^* tensor product on which the tensor product action $g \mapsto \alpha_g \otimes \beta_g$ is defined. Then $\alpha \otimes \beta$ has the continuous Rokhlin property.

Proof. Let $(e_g^{(t)})_{g \in G, t \in [0, \infty)}$ be a family of projections in A as in Definition 3.1 for the action α . Then $(e_g^{(t)} \otimes 1)_{g \in G, t \in [0, \infty)}$ is a family of projections in $A \otimes B$ as in Definition 3.1 for the action $\alpha \otimes \beta$. \square

The following example of an action of a finite group with the continuous Rokhlin property will be needed in Proposition 3.9.

Example 3.3. Let G be a topological group, and let $d_1, d_2, \dots \in \mathbb{Z}_{>0}$. Let $\rho = (\rho^{(1)}, \rho^{(2)}, \dots)$ be a sequence of unitary representations $\rho^{(k)}: G \rightarrow L(\mathbb{C}^{d_k})$ of G . Define actions $\nu^{(k)}: G \rightarrow \text{Aut}(L(\mathbb{C}^{d_k}))$ by $\nu_g^{(k)}(a) = \rho^{(k)}(g)a\rho^{(k)}(g)^*$ for $k \in \mathbb{Z}_{>0}$, $g \in G$, and $a \in L(\mathbb{C}^{d_k})$. Let B_ρ be the UHF algebra

$$B_\rho = \bigotimes_{k=1}^{\infty} L(\mathbb{C}^{d_k}),$$

and let $\mu^\rho: G \rightarrow \text{Aut}(B_\rho)$ be the product type action given by

$$\mu_g^\rho = \bigotimes_{k=1}^{\infty} \nu_g^{(k)}.$$

For a fixed unitary representation $\rho: G \rightarrow L(\mathbb{C}^d)$, we abbreviate (ρ, ρ, \dots) to ρ , so that B_ρ is the d^∞ UHF algebra, and the action is given by

$$g \mapsto \mu_g^\rho = \bigotimes_{k=1}^{\infty} \text{Ad}(\rho(g)) \in \text{Aut}(B_\rho).$$

When G is finite with $\text{card}(G) = d$, and ρ is the regular representation $\lambda: G \rightarrow L(l^2(G))$, we write

$$B_G = \bigotimes_{k=1}^{\infty} L(l^2(G)) \quad \text{and} \quad g \mapsto \mu_g^G = \bigotimes_{k=1}^{\infty} \text{Ad}(\lambda(g)) \in \text{Aut}(B_G)$$

and when ρ is the direct sum λ^m of m copies of λ , we write

$$B_{G,m} = \bigotimes_{k=1}^{\infty} L(l^2(G)^m) \quad \text{and} \quad g \mapsto \mu_g^{G,m} = \bigotimes_{k=1}^{\infty} \text{Ad}(\lambda^m(g)) \in \text{Aut}(B_{G,m}).$$

These are product type actions of G on the d^∞ and $(md)^\infty$ UHF algebras.

We fix some notation. For any index set S and $s \in S$, we denote by $\delta_s \in l^2(S)$ the standard basis vector, determined by

$$\delta_s(t) = \begin{cases} 1 & t = s \\ 0 & t \neq s. \end{cases}$$

Lemma 3.4. Let G be a finite group. Then the action $\mu^{G,m}: G \rightarrow \text{Aut}(B_{G,m})$ of Example 3.3 has the continuous Rokhlin property.

Proof. We use the notation above. Recall that λ^m is the direct sum of m copies of the regular representation of G , and define $\nu: G \rightarrow \text{Aut}(L(l^2(G)^m))$ by $\nu_g = \text{Ad}(\lambda^m(g))$.

We begin with a construction involving just two tensor factors. Let

$$v: l^2(G)^m \otimes l^2(G)^m \rightarrow l^2(G)^m \otimes l^2(G)^m$$

be the unitary determined by $v(\xi \otimes \eta) = \eta \otimes \xi$ for $\xi, \eta \in l^2(G)^m$. Equip

$$L(l^2(G)^m \otimes l^2(G)^m) = L(l^2(G)^m) \otimes L(l^2(G)^m)$$

with the action $g \mapsto \nu_g \otimes \nu_g$. Then v is G -invariant. Since $L(l^2(G)^m \otimes l^2(G)^m)^G$ is finite dimensional, there is a continuous path $t \mapsto z_t$ of G -invariant unitaries in $L(l^2(G)^m \otimes l^2(G)^m)$ such that $z_0 = 1$ and $z_1 = v$.

For every $g \in G$, let δ_g be the corresponding standard basis vector in $l^2(G)$, and let $p_g \in L(l^2(G)^m)$ be the projection on the m dimensional subspace spanned by the standard basis vectors $\delta_{g,j} = (0, \dots, 0, \delta_g, 0, \dots, 0)$ for $1 \leq j \leq m$, where δ_g is in the j -th position. Then $\nu_g(p_h) = p_{gh}$ for every $g, h \in G$, and $\sum_{g \in G} p_g = 1$. For $n \in \mathbb{Z}_{\geq 0}$, $t \in [n, n+1]$, and $g \in G$, we define

$$e_g^{(t)} = 1 \otimes 1 \otimes \dots \otimes 1 \otimes z_{t-n}(p_g \otimes 1) z_{t-n}^* \otimes 1 \otimes 1 \otimes 1 \otimes \dots \in B_{G,m},$$

with the expression $z_{t-n}(p_g \otimes 1) z_{t-n}^*$ occupying the two positions $n+1$ and $n+2$ in the tensor product. It is clear that $e_g^{(t)}$ is a projection and that $\sum_{g \in G} e_g^{(t)} = 1$ for all $t \in [0, \infty)$. Since $(\nu_g \otimes \nu_g)(z_{t-n}) = z_{t-n}$ and $\nu_g(p_h) = p_{gh}$,

one easily checks that $\mu_g^{G,m}(e_h^{(t)}) = e_{gh}^{(t)}$ for all $g, h \in G$ and $t \in [0, \infty)$. Finally, if

$$a \in \bigotimes_{k=1}^N L(l^2(G)^m) \subset \bigotimes_{k=1}^\infty L(l^2(G)^m) \quad \text{and} \quad t \geq N,$$

then $e_g^{(t)}$ exactly commutes with a .

Now take $b \in B_{G,m}$. For every $\varepsilon > 0$ there exist $N \geq 1$ and $a \in \bigotimes_{k=1}^N L(l^2(G)^m)$

such that $\|b - a\| < \frac{\varepsilon}{2}$. Suppose $t \geq N$. Then $e_g^{(t)} a = a e_g^{(t)}$ by the previous paragraph, so

$$\|e_g^{(t)} b - b e_g^{(t)}\| \leq 2\|b - a\| + \|e_g^{(t)} a - a e_g^{(t)}\| = 2\|b - a\| < \varepsilon.$$

This completes the proof. \square

The following result will not be used, but it is easy to derive from known results and provides motivation for the idea that the action we construct in Section 4 should have the continuous Rokhlin property. The result we actually need is in Proposition 3.9 below. It is known that there are actions on simple C^* -algebras which have the Rokhlin property but not the continuous Rokhlin property. Giving an example here would take us too far afield.

Proposition 3.5. Let G be a finite group. Let A be a simple separable unital nuclear C^* -algebra satisfying the Universal Coefficient Theorem which, in addition, is either purely infinite or tracially AF in the sense of [37]. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with the Rokhlin property. Assume that for all $g \in G$, the maps $(\alpha_g)_* \in \text{Aut}(K_*(A))$ are the identity maps. Then α has the continuous Rokhlin property.

Proof. Apply Theorem 3.4 of [28] in the purely infinite case, and Theorem 3.5 of [28] in the tracially AF (tracial rank zero) case, to show that α is conjugate to its tensor product with the action μ^G of Example 3.3. Since μ^G has the continuous Rokhlin property (by Lemma 3.4), it follows from Lemma 3.2 that α has the continuous Rokhlin property. \square

We now recall the definition of an asymptotic homomorphism.

Definition 3.6. Let A and B be C^* -algebras. An asymptotic homomorphism from A to B is a family of maps $\psi_t: A \rightarrow B$, indexed by $t \in [0, \infty)$, satisfying the following conditions:

- (1) For all $a \in A$ the map $t \mapsto \psi_t(a)$, from $[0, \infty)$ to B , is continuous.
- (2) For all $a, b \in A$ and $\lambda \in \mathbb{C}$ one has

$$\lim_{t \rightarrow \infty} \|\psi_t(a+b) - \psi_t(a) - \psi_t(b)\| = 0,$$

$$\lim_{t \rightarrow \infty} \|\psi_t(\lambda a) - \lambda \psi_t(a)\| = 0,$$

$$\lim_{t \rightarrow \infty} \|\psi_t(ab) - \psi_t(a)\psi_t(b)\| = 0,$$

and

$$\lim_{t \rightarrow \infty} \|\psi_t(a^*) - \psi_t(a)^*\| = 0.$$

Next we show that, given a separable unital C^* -algebra and an action of a finite group G on A with the continuous Rokhlin property, there exists a unital completely positive asymptotic homomorphism $t \mapsto \psi_t$ from A to A^α which is a left inverse for the inclusion. The following argument was suggested by E. Gardella. It replaces an earlier argument in which ψ_t was not completely positive.

Proposition 3.7. Let A be a separable unital C^* -algebra. Let G be a finite group and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with the continuous Rokhlin property. Denote by A^α the fixed point algebra, and let $\iota: A^\alpha \rightarrow A$ be the canonical inclusion. Then there exists a unital completely positive asymptotic homomorphism $t \mapsto \psi_t: A \rightarrow A^\alpha$ for $t \in [0, \infty)$ such that

$$\lim_{t \rightarrow \infty} \|(\psi_t \circ \iota)(a) - a\| = 0$$

for all $a \in A^\alpha$.

Proof. Given C^* -algebras A and B and a map $\psi: A \rightarrow B$, we denote by $\psi^{(n)}$ the map from $M_n(A)$ to $M_n(B)$ defined by $\psi^{(n)}(a) = (\psi(a_{j,k}))_{j,k=1}^n$ for $a = (a_{j,k})_{j,k=1}^n \in M_n(A)$.

Let $(e_g^{(t)})_{g \in G, t \in [0, \infty)}$ be a family of projections as in Definition 3.1. For $t \in [0, \infty)$ define a map $\rho_t: A \rightarrow A$ by

$$\rho_t(a) = \sum_{g \in G} e_g^{(t)} \alpha_g(a) e_g^{(t)}$$

for $a \in A$. We claim that $t \mapsto \rho_t$ is a unital completely positive asymptotic homomorphism from A to A such that

$$\lim_{t \rightarrow \infty} \|(\rho_t \circ \iota)(a) - a\| = 0$$

for all $a \in A^\alpha$.

Obviously ρ_t is unital. Moreover, ρ_t is completely positive since if $a = (a_{j,k})_{j,k=1}^n \in M_n(A)_+$ and

$$p_g^{(t)} = \text{diag}(e_g^{(t)}, e_g^{(t)}, \dots, e_g^{(t)})$$

denotes the diagonal matrix with the element $e_g^{(t)}$ everywhere on the diagonal, then

$$\rho_t^{(n)}(a) = \sum_{g \in G} p_g^{(t)} (\alpha_g(a_{j,k}))_{j,k=1}^n p_g^{(t)} \geq 0$$

for every $t \in [0, \infty)$. To show that $\rho = (\rho_t)_{t \in [0, \infty)}$ is an asymptotic homomorphism, observe that $t \mapsto \rho_t(a)$ is clearly continuous for every $a \in A$, and that for every $a, b \in A$ and $\lambda \in \mathbb{C}$ we have $\rho_t(\lambda a + b) = \lambda \rho_t(a) + \rho_t(b)$ and $\rho_t(a^*) = \rho_t(a)^*$. Moreover,

$$\begin{aligned} & \lim_{t \rightarrow \infty} \|\rho_t(ab) - \rho_t(a)\rho_t(b)\| \\ (3.1) \quad &= \lim_{t \rightarrow \infty} \left\| \sum_{g \in G} e_g^{(t)} \alpha_g(ab) e_g^{(t)} - \sum_{g \in G} e_g^{(t)} \alpha_g(a) e_g^{(t)} \alpha_g(b) e_g^{(t)} \right\| \\ &\leq \lim_{t \rightarrow \infty} \sum_{g \in G} \|\alpha_g(a)\| \|\alpha_g(b) e_g^{(t)} - e_g^{(t)} \alpha_g(b)\| = 0. \end{aligned}$$

Lastly, for every $a \in A^\alpha$ we have

$$\begin{aligned} (3.2) \quad & \lim_{t \rightarrow \infty} \|\rho_t(a) - a\| = \lim_{t \rightarrow \infty} \left\| \sum_{g \in G} e_g^{(t)} a e_g^{(t)} - \sum_{g \in G} e_g^{(t)} a \right\| \\ &\leq \lim_{t \rightarrow \infty} \sum_{g \in G} \|a e_g^{(t)} - e_g^{(t)} a\| = 0. \end{aligned}$$

The claim is proved.

It follows from the definition of ρ_t and the relation (2) in Definition 3.1 that

$$(3.3) \quad \lim_{t \rightarrow \infty} \|\alpha_g(\rho_t(a)) - \rho_t(a)\| = 0$$

for all $a \in A$ and $g \in G$.

Now let $E: A \rightarrow A^\alpha$ be the conditional expectation given by

$$E(a) = \frac{1}{\text{card}(G)} \sum_{g \in G} \alpha_g(a)$$

for $a \in A$. For $t \in [0, \infty)$ define a map $\psi_t: A \rightarrow A^\alpha$ by $\psi_t(a) = E(\rho_t(a))$ for $a \in A$. Obviously $\psi_t(a) \in A^\alpha$ for all $a \in A$, and ψ_t is unital, linear, and completely positive. Also, $t \mapsto \psi_t(a)$ is clearly continuous for every $a \in A$. Since G is finite, it follows from (3.3) that $\lim_{t \rightarrow \infty} \|\rho_t(a) - \psi_t(a)\| = 0$ for all $a \in A$. Combining this relation with (3.1) gives $\lim_{t \rightarrow \infty} \|\psi_t(ab) - \psi_t(a)\psi_t(b)\| = 0$ for all $a, b \in A$. Combining it with (3.2) gives $\lim_{t \rightarrow \infty} \|\psi_t(a) - a\| = 0$ for all $a \in A^\alpha$. This completes the proof. \square

Proposition 3.8. Let A be a separable unital C^* -algebra, and let G be a finite group. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with the continuous Rokhlin property. Assume that A satisfies the Universal Coefficient Theorem. Then A^α and $A \rtimes_\alpha G$ satisfy the Universal Coefficient Theorem.

If A is simple and nuclear, then one does not need the continuous Rokhlin property; the Rokhlin property suffices. See Corollary 3.9 of [38] for the crossed product and, for actions of second countable compact groups, see Theorem 3.13 of [18]. Moreover, Gardella, Hirshberg, and Santiago proved (Theorem 4.17 in [20]) that if A is nuclear and satisfies the Universal Coefficient Theorem, G is a second countable compact group with finite covering dimension, and $\alpha: G \rightarrow \text{Aut}(A)$ has finite Rokhlin dimension with commuting towers, then A^α and $A \rtimes_\alpha G$ also satisfy the Universal Coefficient Theorem.

Proof of Proposition 3.8. Denote the suspension of a C^* -algebra A by SA . Let K be the algebra of compact operators. By Theorem 4.2 in [26], for every pair of separable C^* -algebras A and B , the group $KK(A, B)$ is canonically isomorphic to the group of homotopy classes of completely positive asymptotic homomorphisms from $K \otimes SA$ to $K \otimes SB$. Let $\iota: A^\alpha \rightarrow A$ be the inclusion. Proposition 3.7 implies that the group homomorphism $\psi^*: KK(A^\alpha, B) \rightarrow KK(A, B)$ induced by the unital completely positive asymptotic homomorphism $(\psi_t)_{t \in [0, \infty)}$ obtained there satisfies $\iota^* \circ \psi^* = \text{id}_{KK(A^\alpha, B)}$. In particular, ψ^* is naturally split injective with left inverse $\iota^*: KK(A, B) \rightarrow KK(A^\alpha, B)$.

By hypothesis, A satisfies the Universal Coefficient Theorem (Theorem 1.17 of [45]). That is, let B be any separable C^* -algebra. Let

$$\gamma_{A,B}: KK(A, B) \rightarrow \text{Hom}(K_*(A), K_*(B))$$

and

$$\kappa_{A,B}: \text{Ker}(\gamma_{A,B}) \rightarrow \text{Ext}(K_*(A), K_{*+1}(B))$$

be as described before Theorem 1.17 of [45] (and called $\gamma(A, B)$ and $\kappa(A, B)$ in [45] when A and B must be specified). Then $\delta_{A,B} = \kappa_{A,B}^{-1}$ exists, and

there is a (natural) short exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Ext}(K_*(A), K_{*+1}(B)) &\xrightarrow{\delta_{A,B}} KK_*(A, B) \\ &\xrightarrow{\gamma_{A,B}} \text{Hom}(K_*(A), K_*(B)) \longrightarrow 0. \end{aligned}$$

(Naturality is Theorem 4.4 of [45].)

Consider the commutative diagram

$$\begin{array}{ccc} KK(A, B) & \xrightarrow{\gamma_{A,B}} & \text{Hom}(K_*(A), K_*(B)) \\ \downarrow \iota^* & & \downarrow \iota^* \\ KK(A^\alpha, B) & \xrightarrow{\gamma_{A^\alpha, B}} & \text{Hom}(K_*(A^\alpha), K_*(B)), \end{array}$$

in which the vertical maps are induced by $\iota: A^\alpha \rightarrow A$ and the horizontal ones are from the Universal Coefficient Theorem. The map $\gamma_{A,B}$ is surjective because A satisfies the Universal Coefficient Theorem, and the right vertical map is surjective because it is a left inverse of ψ^* , so $\gamma_{A^\alpha, B}$ is surjective.

Now consider the following commutative diagram, in which the horizontal maps are from the Universal Coefficient Theorem:

$$\begin{array}{ccc} \text{Ker}(\gamma_{A,B}) & \xrightarrow{\kappa_{A,B}} & \text{Ext}(K_*(A), K_{*+1}(B)) \\ \psi^* \uparrow & & \psi^* \uparrow \\ \text{Ker}(\gamma_{A^\alpha, B}) & \xrightarrow{\kappa_{A^\alpha, B}} & \text{Ext}(K_*(A^\alpha), K_{*+1}(B)). \end{array}$$

The map $\kappa_{A,B}$ is injective because A satisfies the Universal Coefficient Theorem, and the left vertical map is injective since it has a left inverse ι^* , so $\kappa_{A^\alpha, B}$ is injective.

Lastly, the argument used to prove that $\gamma_{A^\alpha, B}$ is surjective, applied to the commutative diagram

$$\begin{array}{ccc} \text{Ker}(\gamma_{A,B}) & \xrightarrow{\kappa_{A,B}} & \text{Ext}(K_*(A), K_{*+1}(B)) \\ \downarrow \iota^* & & \downarrow \iota^* \\ \text{Ker}(\gamma_{A^\alpha, B}) & \xrightarrow{\kappa_{A^\alpha, B}} & \text{Ext}(K_*(A^\alpha), K_{*+1}(B)), \end{array}$$

shows that $\kappa_{A^\alpha, B}$ is surjective. Therefore A^α satisfies the Universal Coefficient Theorem.

Now we consider the crossed product $A \rtimes_\alpha G$. By the Proposition in [44], A^α is isomorphic to a corner of $A \rtimes_\alpha G$. When α has the Rokhlin property, Corollary 2.15 of [18] implies that this corner is strongly Morita equivalent to $A \rtimes_\alpha G$. (This is saturation of the action, a weaker condition than hereditary saturation as proved in [18].) Since strong Morita equivalence preserves the class of algebras satisfying the Universal Coefficient Theorem, we conclude that $A \rtimes_\alpha G$ satisfies the Universal Coefficient Theorem. \square

The following argument was suggested by E. Gardella.

Proposition 3.9. Let G be a finite group. Let A be a unital separable C^* -algebra which absorbs the UHF algebra of type $\text{card}(G)^\infty$. Suppose the action $\alpha: G \rightarrow \text{Aut}(A)$ has the Rokhlin property. Then:

- (1) Taking μ^G to be the product type action of Example 3.3, we have an equivariant isomorphism

$$(G, A, \alpha) \cong (G, M_{\text{card}(G)^\infty} \otimes A, \mu^G \otimes \alpha).$$

- (2) The action α has the continuous Rokhlin property.

Proof. We prove (1). We will need to cite theorems which use central sequence algebras, so we state notation for them. For a separable unital C^* -algebra A , we define $A^\infty = C_b(\mathbb{Z}_{>0}, A)/C_0(\mathbb{Z}_{>0}, A)$, and we regard A as a subalgebra of A^∞ via its embedding in $C_b(\mathbb{Z}_{>0}, A)$ as the algebra of constant sequences. Then $A' \cap A^\infty$ is the relative commutant of this image of A . (It is written A_∞ in [27]. See Section 2.1 there.) For $\omega \in \beta\mathbb{Z}_{>0} \setminus \mathbb{Z}_{>0}$, if in place of $C_0(\mathbb{Z}_{>0}, A)$ we use

$$\left\{ a = (a_n)_{n \in \mathbb{Z}_{>0}} \in C_b(\mathbb{Z}_{>0}, A) : \lim_{n \rightarrow \omega} a_n = 0 \right\},$$

we call the quotient A^ω . The image of the constant sequences here can also be identified with A , and we again get a relative commutant $A' \cap A^\omega$. There is an obvious surjective map $A^\infty \rightarrow A^\omega$, which gives a unital homomorphism $A' \cap A^\infty \rightarrow A' \cap A^\omega$.

We now fix any $\omega \in \beta\mathbb{Z}_{>0} \setminus \mathbb{Z}_{>0}$.

Since A absorbs $M_{\text{card}(G)^\infty}$, Proposition 2.8 and the comment at the beginning of the proof of Proposition 2.9 in [27] provide an injective unital homomorphism $M_{\text{card}(G)^\infty} \rightarrow A^\infty \cap A'$. Since $M_{\text{card}(G)^\infty}$ is simple, it follows from the previous paragraph that there is an injective unital homomorphism $M_{\text{card}(G)^\infty} \rightarrow A^\omega \cap A'$. Lemma 3.12, the proof of Proposition 3.13 in [35], and the fact that in Lemma 0.5 in [35] the isomorphism is approximately unitarily equivalent to the given homomorphism (see the proof of Proposition A in [43]), now provide a unital isomorphism $\varphi: M_{\text{card}(G)^\infty} \otimes A \rightarrow A$ and unitaries w_n in A for $n \in \mathbb{Z}_{>0}$ such that

$$(3.4) \quad \lim_{n \rightarrow \infty} \|w_n \varphi(1 \otimes a) w_n^* - a\| = 0$$

for every $a \in A$.

Define an action $\beta: G \rightarrow \text{Aut}(A)$ by $\beta_g = \varphi \circ (\mu_g^G \otimes \alpha_g) \circ \varphi^{-1}$ for $g \in G$. We claim that β_g is approximately unitarily equivalent to α_g for every $g \in G$. Set $v_n = w_n \beta_g(w_n^*)$ for $n \in \mathbb{Z}_{>0}$. Let $g \in G$ and $a \in A$. For $n \in \mathbb{Z}_{>0}$ we have

$$\begin{aligned} \|v_n \beta_g(a) v_n^* - \alpha_g(a)\| &= \|w_n \beta_g(w_n^* a w_n) w_n^* - \alpha_g(a)\| \\ &\leq \|w_n \beta_g(w_n^* a w_n) w_n^* - w_n \beta_g(\varphi(1 \otimes a)) w_n^*\| \\ &\quad + \|w_n \beta_g(\varphi(1 \otimes a)) w_n^* - \alpha_g(a)\| \\ &= \|w_n^* a w_n - \varphi(1 \otimes a)\| \\ &\quad + \|w_n \varphi(1 \otimes \alpha_g(a)) w_n^* - \alpha_g(a)\|. \end{aligned}$$

Applying (3.4) to a and $\alpha_g(a)$, we find that $\lim_{n \rightarrow \infty} \|v_n \beta_g(a) v_n^* - \alpha_g(a)\| = 0$.

This proves the claim.

By Theorem 3.5 in [27], there exists an approximately inner automorphism θ such that $\theta \circ \alpha_g \circ \theta^{-1} = \beta_g$ for every $g \in G$. Part (1) follows.

Part (2) is now immediate from Lemma 3.4 and Lemma 3.2. \square

4. THE CONSTRUCTION

In this section we describe a method to construct simple separable C^* -algebras not isomorphic to their opposite algebras. We also show that these C^* -algebras satisfy the Universal Coefficient Theorem. Throughout this section q is a fixed integer with $q \geq 2$. The construction is a generalization of the construction of [39] for $q = 3$. In Section 5, we will restrict q to being an odd prime such that -1 is not a square mod q .

Definition 4.1. Let $q \in \{2, 3, \dots\}$. Define the C^* -algebra A_q to be the reduced free product of q copies of $C([0, 1])$ and the C^* -algebra \mathbb{C}^q , amalgamated over \mathbb{C} , taken with respect to the states given by Lebesgue measure μ on each copy of $C([0, 1])$ and the state given by

$$\omega(c_1, c_2, \dots, c_q) = \frac{1}{q}(c_1 + c_2 + \dots + c_q)$$

on \mathbb{C}^q . That is,

$$A_q = C([0, 1]) \star_r C([0, 1]) \star_r \dots \star_r C([0, 1]) \star_r \mathbb{C}^q.$$

For $k = 1, 2, \dots, q$ we denote by $\varepsilon_k: C([0, 1]) \rightarrow A_q$ the inclusion of the k -th copy of $C([0, 1])$ in A_q . Set

$$(4.1) \quad v = (e^{2\pi i/q}, 1, e^{2(q-1)\pi i/q}, e^{2(q-2)\pi i/q}, \dots, e^{4\pi i/q}) \in \mathbb{C}^q,$$

and regard v as a unitary in A_q via the obvious inclusion.

Lemma 4.2. There exists a unique automorphism $\alpha \in \text{Aut}(A_q)$ such that for all $f \in C([0, 1])$ we have

$$(4.2) \quad \alpha(\varepsilon_1(f)) = \varepsilon_2(f), \quad \alpha(\varepsilon_2(f)) = \varepsilon_3(f), \quad \dots, \quad \alpha(\varepsilon_{q-1}(f)) = \varepsilon_q(f),$$

$$(4.3) \quad \alpha(\varepsilon_q(f)) = \text{Ad}(v)(\varepsilon_1(f)),$$

and

$$(4.4) \quad \alpha(v) = e^{-2\pi i/q} v.$$

Moreover, with v as in (4.1), we have $\alpha^q = \text{Ad}(v)$.

Proof. The proof is the same as that of Lemma 4.6 of [39]. \square

Remark 4.3. The C^* -algebra A_q is unital, separable, simple, exact, and has a unique tracial state. Exactness follows from Theorem 3.2 of [12]. Simplicity and uniqueness of the tracial state follow by applying the corollary on page 431 of [1] several times. Lastly, simplicity and the existence of a faithful tracial state imply that A_q is stably finite.

We also need an action on a UHF algebra. We use a different model than in [39], which has the advantage that the computation of the Connes invariant is more explicit. We start with some notation and a preliminary lemma which we isolate from the main argument for convenience.

Notation 4.4. Let $d \in \mathbb{Z}_{>0}$. Define $\varphi_n: M_{d^n} \rightarrow M_{d^{n+1}}$ by $\varphi_n(x) = \text{diag}(x, x, \dots, x)$ for $x \in M_{d^n}$. Denote by B_d the UHF algebra obtained as the direct limit of the system $(M_{d^n}, \varphi_n)_{n \in \mathbb{Z}_{>0}}$. We identify M_{d^n} with

$$\bigotimes_{k=1}^n M_d \text{ and } B_d \text{ with } \bigotimes_{k=1}^{\infty} M_d.$$

Lemma 4.5. Let $n \in \mathbb{Z}_{>0}$, let $(e_{j,k})_{j,k=1}^n$ be the standard system of matrix units in M_n , and let $s \in M_n$ be the shift unitary $s = e_{1,n} + \sum_{j=2}^n e_{j,j-1}$. Let $v \in M_n$ be a diagonal unitary. Then there are $\gamma \in \mathbb{T}$ and a diagonal unitary $y \in M_n$ such that $yvsy^* = \gamma s$.

Proof. Write $v = \text{diag}(\beta_1, \beta_2, \dots, \beta_n)$ with $\beta_1, \beta_2, \dots, \beta_n \in \mathbb{T}$. Choose $\gamma \in \mathbb{T}$ such that $\gamma^n = \prod_{j=1}^n \beta_j$. Define $\lambda_j = \bar{\gamma} \beta_j$ for $j = 1, 2, \dots, n$. Then $w = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ satisfies $w = \bar{\gamma} v$. Moreover,

$$(4.5) \quad \prod_{j=1}^n \lambda_j = 1.$$

Define

$$\zeta_1 = \bar{\lambda}_1, \quad \zeta_2 = \overline{\lambda_1 \lambda_2}, \quad \dots \quad \zeta_{n-1} = \overline{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}, \quad \zeta_n = 1,$$

and $y = \text{diag}(\zeta_1, \zeta_2, \dots, \zeta_n)$. Using (4.5), one checks that $ysy^* s^* = w^*$. So $yvsy^* s^* = \gamma v^*$. Since y commutes with v , this gives $yvsy^* = \gamma s$. \square

In the next lemma, B_{q^2} and B_q are of course isomorphic. But we find it convenient to notationally distinguish them: B_{q^2} is the algebra in which we carry out the construction, and B_q is the subalgebra in (9).

Lemma 4.6. Let $q \in \mathbb{Z}_{>0}$ satisfy $q \geq 2$. There exist unitaries $g \in B_{q^2}$ and $u_k \in B_{q^2}$ for $k = -1, 0, 1, 2, \dots$ such that, taking $E_{-1} = C^*(g)$ and $E_n = C^*(g, u_0, u_1, \dots, u_n)$ for $n \in \mathbb{Z}_{\geq 0}$, and using the unique tracial state on B_{q^2} , the following hold:

- (1) For $k = -1, 0, 1, 2, \dots$, $\text{sp}(u_k) = \{\zeta \in \mathbb{T} : \zeta^{q^2} = 1\}$, and for every $\zeta \in \text{sp}(u_k)$, the corresponding spectral projection of u_k has trace $1/q^2$.
- (2) $\text{sp}(g) = \{\zeta \in \mathbb{T} : \zeta^q = 1\}$, and for every $\zeta \in \text{sp}(g)$, the corresponding spectral projection of g has trace $1/q$.
- (3) $u_k g u_k^* = e^{2\pi i/q} g$ for $k \in \{-1, 0\}$.
- (4) $u_k g u_k^* = g$ for $k = 1, 2, \dots$.
- (5) $u_k u_{k+1} u_k^* = e^{-2\pi i/q^2} u_{k+1}$ for $k = -1, 0, 1, 2, \dots$.
- (6) $u_j u_k = u_k u_j$ for $j, k = -1, 0, 1, 2, \dots$ with $|j - k| \geq 2$.
- (7) $E_n \cong (M_{q^{n+1}})^q$ for $n = -1, 0, 1, 2, \dots$.
- (8) If $n \in \{-1, 0, 1, 2, \dots\}$ is odd, then the center $Z(E_n)$ is generated by the order q unitary $w_n = g^*(u_1^*)^q (u_3^*)^q \cdots (u_n^*)^q$ (so $w_{-1} = g^*$), and if n is even then $Z(E_n)$ is generated by the order q unitary $w_n = u_0^q u_2^q u_4^q \cdots u_n^q$.
- (9) $C^*(g, u_0, u_1, u_2, \dots) \cong B_q$.

Proof. For $k = 1, 2, \dots$ let $\sigma_k : M_{q^2} \rightarrow B_{q^2}$ be the map

$$\sigma_k(x) = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes 1 \otimes \cdots,$$

with x in position k . (Thus, $\sigma_1(x) = x \otimes 1 \otimes 1 \otimes \dots$.) Let $\rho: B_{q^2} \rightarrow B_{q^2}$ denote the shift endomorphism of B_{q^2} , determined by $\rho(\sigma_k(x)) = \sigma_{k+1}(x)$ for $k \in \mathbb{Z}_{>0}$ and $x \in M_{q^2}$.

Denote by $(e_{j,k})_{j,k=1}^{q^2}$ the standard system of matrix units in M_{q^2} . Define unitaries $z, s \in M_{q^2}$ by

$$z = \text{diag}(1, e^{2\pi i/q^2}, \dots, e^{2(q^2-1)\pi i/q^2}) \quad \text{and} \quad s = e_{1,q^2} + \sum_{j=2}^{q^2} e_{j,j-1}.$$

Apply Lemma 4.5 with $n = q^2$, with s as given, and with $v = z^*$, getting $\gamma \in \mathbb{T}$. Define unitaries in B_{q^2} by

$$g = (\gamma z^* s)^q \otimes 1 \otimes 1 \otimes \dots,$$

and, for $k = 0, 1, \dots$,

$$u_{2k-1} = \sigma_{k+1}(s) \quad \text{and} \quad u_{2k} = \sigma_{k+1}(z) \sigma_{k+2}(z^*).$$

Thus, for example,

$$u_{-1} = s \otimes 1 \otimes 1 \otimes \dots \quad \text{and} \quad u_0 = z \otimes z^* \otimes 1 \otimes 1 \otimes \dots.$$

Since s is unitarily equivalent to z , the choice of γ using Lemma 4.5 implies that $(\gamma z^* s)^q$ is unitarily equivalent to z^q . So (2) holds if one uses the normalized trace on M_{q^2} . Therefore (2) holds using the tracial state on B_{q^2} . Similar reasoning shows that if k is odd, then (1) holds. One readily checks that $z \otimes z^* \in M_{q^4}$ satisfies (1) if one uses the normalized trace on M_{q^4} , so (1) for even k follows in the same way.

A direct check shows that $szs^* = e^{-2\pi i/q^2} z$. Using this, it is easy to verify the relations (3), (4), (5), and (6) of the statement.

We claim that for $n = -1, 0, 1, \dots$, we have

$$(4.6) \quad E_n = \text{span}(\{g^k u_0^{l_0} u_1^{l_1} \dots u_n^{l_n} : 0 \leq k \leq q-1 \text{ and } 0 \leq l_0, l_1, \dots, l_n \leq q^2-1\}).$$

The claim follows from $g^q = 1$ and $u_n^{q^2} = 1$ (by (2) and (1)), and because, given this, (3), (4), (5), and (6) show that the product of any two of the elements listed in (4.6) is a scalar multiple of another of them.

Let $m \in \{-1, 0, 1, \dots\}$, let $n \in \mathbb{Z}_{\geq 0}$, and let $k, l_0, l_1, \dots, l_m \in \mathbb{Z}$. We compute $u_n(g^k u_0^{l_0} u_1^{l_1} \dots u_m^{l_m}) u_n^*$ using (3), (4), (5), and (6). To this end observe that, by (5), we have $u_k u_{k-1} u_k^* = e^{2\pi i/q^2} u_{k-1}$ for all $k \in \mathbb{Z}_{\geq 0}$. For $m \geq 1$, we then get

$$(4.7) \quad \begin{aligned} & u_n(g^k u_0^{l_0} u_1^{l_1} \dots u_m^{l_m}) u_n^* \\ &= \begin{cases} e^{2\pi i[kq-l_1]/q^2} \cdot g^k u_0^{l_0} u_1^{l_1} \dots u_m^{l_m} & n = 0 \\ e^{2\pi i[l_{n-1}-l_{n+1}]/q^2} \cdot g^k u_0^{l_0} u_1^{l_1} \dots u_m^{l_m} & n = 1, 2, \dots, m-1 \\ e^{2\pi i l_{n-1}/q^2} \cdot g^k u_0^{l_0} u_1^{l_1} \dots u_m^{l_m} & n = m, m+1 \\ g^k u_0^{l_0} u_1^{l_1} \dots u_m^{l_m} & n = m+2, m+3, \dots \end{cases} \end{aligned}$$

The case $m = -1$ is

$$(4.8) \quad u_n g^k u_n^* = \begin{cases} e^{2\pi i k/q} g^k & n = 0 \\ g^k & n = 1, 2, \dots, \end{cases}$$

and the case $m = 0$ is

$$(4.9) \quad u_n (g^k u_0^{l_0}) u_n^* = \begin{cases} e^{2\pi i [kq - l_1]/q^2} \cdot g^k u_0^{l_0} & n = 0 \\ e^{2\pi i l_0/q^2} \cdot g^k u_0^{l_0} & n = 1 \\ g^k u_0^{l_0} & n = 2, 3, \dots \end{cases}$$

Also,

$$(4.10) \quad g(g^k u_0^{l_0} u_1^{l_1} \dots u_m^{l_m}) g^* = e^{-2\pi i l_0/q} \cdot g^k u_0^{l_0} u_1^{l_1} \dots u_m^{l_m}.$$

For $n = -1, 0, 1, \dots$, let w_m be as in (8). Then w_m is a unitary in E_m . By (4) and (6), for both even and odd m , the factors in the definition of w_m commute. Therefore $w_m^q = 1$ by (1) and (2). The relations (4.7), (4.8), (4.9), and (4.10) show that w_m commutes with all the generators of E_m . Therefore w_m is in the center $Z(E_m)$. These relations also show that

$$(4.11) \quad u_{m+1} w_m u_{m+1}^* = \begin{cases} e^{-2\pi i/q} w_m & m \text{ is odd} \\ e^{2\pi i/q} w_m & m \text{ is even.} \end{cases}$$

Combined with $w_m^q = 1$, this shows that $\text{sp}(w_m) = \{\zeta \in \mathbb{T} : \zeta^q = 1\}$. For $m \in \{-1, 0, 1, \dots\}$ and $j \in \{0, 1, \dots, q-1\}$, we now let $p_{m,j}$ be the spectral projection of w_m corresponding to the eigenvalue $e^{2\pi i j/q}$. For fixed m , the projections $p_{m,j}$ are then all unitarily equivalent to each other in B_{q^2} , central in E_m , and sum to 1. In particular, they are all nonzero. It follows that

$$(4.12) \quad E_m = \bigoplus_{j=1}^q p_{m,j} E_m p_{m,j}.$$

For $k \in \mathbb{Z}_{\geq 0}$ define

$$D_k = C^*(u_0 u_2 \dots u_{2k}, u_{2k+1}).$$

We claim that $D_k \cong M_{q^2}$. To see this, use (4.7) (use (4.9) if $k = 0$) to see that

$$u_{2k+1} (u_0 u_2 \dots u_{2k}) u_{2k+1}^* = e^{2\pi i/q^2} u_0 u_2 \dots u_{2k}.$$

Also, $u_{2k+1}^{q^2} = 1$ by (1), and, using (1) and (6), one checks that $(u_0 u_2 \dots u_{2k})^{q^2} = 1$. It is easily seen that the universal C^* -algebra C generated by unitaries v and w satisfying

$$v^{q^2} = w^{q^2} = 1 \quad \text{and} \quad v w v^* = e^{2\pi i/q^2} w$$

is the transformation group C^* -algebra of the action of \mathbb{Z}_{q^2} on \mathbb{Z}_{q^2} by translation, which is isomorphic to M_{q^2} . Since this algebra is simple, the claim follows.

We claim that if $k \neq l$, then D_k commutes with D_l . For the proof, without loss of generality $k < l$. It suffices to prove the following:

- (10) $u_0 u_2 \dots u_{2k}$ commutes with $u_0 u_2 \dots u_{2l}$.
- (11) $u_0 u_2 \dots u_{2k}$ commutes with u_{2l+1} .
- (12) u_{2k+1} commutes with $u_0 u_2 \dots u_{2l}$.

(13) u_{2k+1} commutes with u_{2l+1} .

Of these, (10), (11), and (13) all follow from (6), and (12) follows from (4.7). The claim is proved.

Let $k \in \mathbb{Z}_{\geq 0}$. Since $D_0, D_1, \dots, D_k \subseteq E_{2k+1}$, it follows that there is a unital homomorphism from $D_0 \otimes D_1 \otimes \dots \otimes D_k \cong M_{q^{2k+2}}$ to E_{2k+1} . For $j = 0, 1, \dots, q-1$, recalling that $p_{2k+1,j} \neq 0$ and using (4.12), we see that there is a nonzero unital homomorphism $M_{q^{2k+2}} \rightarrow p_{2k+1,j} E_{2k+1} p_{2k+1,j}$. Therefore $\dim(E_{2k+1}) \geq q^{4k+5}$. It follows from (4.6) that $\dim(E_{2k+1}) \leq q^{4k+5}$. Therefore $\dim(E_{2k+1}) = q^{4k+5}$ and the maps $M_{q^{2k+2}} \rightarrow p_{2k+1,j} E_{2k+1} p_{2k+1,j}$ are isomorphisms, that is, $E_{2k+1} \cong \bigoplus_{j=1}^q M_{q^{2k+2}}$. This is part (7) of the conclusion for odd n , and implies part (8) of the conclusion for odd n .

From (4.6) we also get $\dim(E_{m+1}) \leq q^2 \dim(E_m)$ for $m = -1, 0, 1, \dots$. We have just seen that $\dim(E_m) = q^{2m+3}$ when m is odd, so this relation holds for all $m \in \{-1, 0, 1, \dots\}$. In particular, the elements listed on the right in (4.6) are actually a basis for E_m .

This fact, and the form of the relations (4.7), (4.8), (4.9), and (4.10), implies that $Z(E_m)$ is the linear span of the elements $g^k u_0^{l_0} u_1^{l_1} \dots u_m^{l_m}$ with $0 \leq k \leq q-1$ and $0 \leq l_0, l_1, \dots, l_m \leq q^2 - 1$ which commute with all of g, u_0, u_1, \dots, u_m . Moreover, these relations imply that this happens exactly when $l_0 \in \{0, q, 2q, \dots, q^2 - q\}$, and

$$kq = l_1, \quad l_0 = l_2, \quad l_1 = l_3, \quad \dots, \quad l_{m-2} = l_m, \quad \text{and} \quad l_{m-1} = 0.$$

If m is even, this says $l_j = 0$ for all odd j and there is $r \in \{0, 1, \dots, q-1\}$ such that $l_j = rq$ for all even j . Therefore $\dim(Z(E_m)) = q$. It follows that

$$Z(E_m) = \text{span}(p_{m,0}, p_{m,1}, \dots, p_{m,q-1}) = C^*(w_m),$$

just as we saw above for m odd. This implies part (8) of the conclusion for even n . Conjugation by u_{m+1} permutes the projections $p_{m,j}$ cyclically (by (4.11)) and is an automorphism of E_m (by (4.7), (4.8), and (4.9)), so the summands $p_{m,j} E_m p_{m,j}$ are all isomorphic. Since they are simple, a dimension count shows that they are all isomorphic to $M_{q^{m+1}}$. This is part (7) of the conclusion for even n .

It remains to prove (9). Let $m \in \{-1, 0, 1, \dots\}$, and let $\mu = (\mu_{j,k})_{j,k=1}^q$ be the matrix of partial embedding multiplicities of the inclusions

$$\psi_{j,k}: p_{m,j} E_m p_{m,j} \rightarrow p_{m+1,k} E_{m+1} p_{m+1,k}.$$

We claim that $\mu_{j,k} = 1$ for $j, k = 1, 2, \dots, q$. To prove this, observe that conjugation by u_{m+2} permutes the projections $p_{m+1,k}$ cyclically (by (4.11)) but is the identity on E_m (by (4.7), (4.8), and (4.9)). Therefore, for fixed j , the projections $p_{m,j} p_{m+1,k}$ are all unitarily equivalent in B_{q^2} . Also, $u_{m+1} \in E_{m+1}$ and $w_{m+1} \in Z(E_{m+1})$, so u_{m+1} commutes with the projections $p_{m+1,k}$, while u_{m+1} permutes the projections $p_{m,j}$ cyclically by (4.11). It follows that the projections $p_{m,j} p_{m+1,k}$ are all unitarily equivalent in B_{q^2} for $j, k = 0, 1, \dots, q-1$. Therefore $p_{m,j} p_{m+1,k} \neq 0$ for all j and k . Thus $\mu_{j,k} \geq 1$. Also,

$$\mu \cdot (q^{m+1}, q^{m+1}, \dots, q^{m+1}) = (q^{m+2}, q^{m+2}, \dots, q^{m+2}).$$

For $j = 0, 1, \dots, q-1$ we therefore have $\sum_{k=0}^{q-1} \mu_{j,k} = q$. Since $\mu_{j,k} \geq 1$ for all j and k , the claim follows.

It is now easy to check, for example by computing the K_0 -group, that $\bigcup_{m=-1}^{\infty} E_m \cong B_q$. This is (9). \square

Lemma 4.7. Let $q \in \mathbb{Z}_{>0}$ satisfy $q \geq 2$. Let $g, u_{-1}, u_0, u_1, u_2, \dots \in B_{q^2}$ satisfy the conditions of Lemma 4.6. Set $E = C^*(g, u_0, u_1, u_2, \dots) \subseteq B_{q^2}$. Then $\beta = \text{Ad}(u_{-1})|_E$ is an automorphism of E such that:

- (1) $\beta^q = \text{Ad}(g)$.
- (2) $\beta(g) = e^{2\pi i/q} g$.
- (3) $\beta(u_0) = e^{-2\pi i/q^2} u_0$.
- (4) $\beta(u_n) = u_n$ for $n \in \mathbb{Z}_{>0}$.
- (5) β^k is outer for $k = 1, 2, \dots, q-1$.

Proof. It follows from (3), (5), and (6) of Lemma 4.6 that $u_{-1} E u_{-1}^* = E$, so that β is an automorphism, and that parts (2), (3), and (4) of the conclusion hold. From Lemma 4.6(3) we get $g u_0 g^* = e^{-2\pi i/q}$. This, parts (2), (3), and (4), and the relations (4), (5), and (6) in Lemma 4.6, imply (1).

It remains to prove (5). Let $k \in \{1, 2, \dots, q-1\}$. For $n \in \mathbb{Z}_{\geq 0}$, let $w_{2n} = u_0^q u_2^q u_4^q \cdots u_{2n}^q$, as in Lemma 4.6(8). Using Lemma 4.6(5) and (6), we get $\beta^k(w_{2n}) = e^{-2\pi i k/q} w_{2n}$, so $\lim_{n \rightarrow \infty} \|\beta^k(w_{2n}) - w_{2n}\| = |1 - e^{-2\pi i k/q}| \neq 0$. However, by Lemma 4.6(8), for any unitary $y \in E$, we have $\lim_{n \rightarrow \infty} \|y w_{2n} y^* - w_{2n}\| = 0$. So $\beta^k \neq \text{Ad}(y)$. \square

Lemma 4.8. Let $q \in \mathbb{Z}_{>0}$ satisfy $q \geq 2$. Let A_q and v be as in Definition 4.1, and let $\alpha \in \text{Aut}(A_q)$ be as in Lemma 4.2. Let g, u_0, u_1, u_2, \dots be as in Lemma 4.6 and, following Lemma 4.6(9), identify B_q with the algebra $E = C^*(g, u_0, u_1, u_2, \dots)$. Let $\beta \in \text{Aut}(B_q)$ correspond to the automorphism $\beta \in \text{Aut}(E)$ of Lemma 4.7. There exists a unitary $w \in C^*(v \otimes g) \subseteq A_q \otimes B_q$ with the following properties:

- (1) $w^q = (v \otimes g)^*$.
- (2) $(\alpha \otimes \beta)(w) = w$.
- (3) w commutes with $1 \otimes (u_0^*)^q g$.
- (4) If we set

$$\gamma = \text{Ad}(w) \circ (\alpha \otimes \beta) \in \text{Aut}(A_q \otimes B_q),$$

then γ generates an action of \mathbb{Z}_q which has the Rokhlin property.

Proof. The construction of w satisfying (1) and (2) is the same as in Lemma 4.8 of [39]. It follows from Lemma 4.6(3) that $v \otimes g$ commutes with $1 \otimes (u_0^*)^q g$, so (3) follows from $w \in C^*(v \otimes g)$. It is straightforward to show that $[\text{Ad}(w) \circ (\alpha \otimes \beta)]^q = \text{id}_{A_q \otimes B_q}$.

Next, we claim that for any $\varepsilon > 0$ and any finite subset $F \subseteq A_q \otimes B_q$, there are projections $e_0, e_1, \dots, e_{q-1} \in A_q \otimes B_q$ such that:

- (5) $\|(\alpha \otimes \beta)(e_k) - e_{k+1}\| < \varepsilon$ for $k = 0, 1, \dots, q-2$ and $\|(\alpha \otimes \beta)(e_{q-1}) - e_0\| < \varepsilon$.
- (6) $\|y e_k - e_k y\| < \varepsilon$ for $k = 0, 1, \dots, q-1$ and all $y \in F$.
- (7) $\sum_{k=0}^{q-1} e_k = 1$.

To prove the claim, for $n \in \mathbb{Z}_{\geq 0}$ let $E_n = C^*(g, u_0, u_1, \dots, u_n)$, as in Lemma 4.6. By a standard approximation argument, it is enough to consider finite subsets of $A_q \otimes B_q$ of the form

$$F = \{a \otimes b : a \in S \text{ and } b \in T\}$$

for $n \in \mathbb{Z}_{\geq 0}$ and finite subsets $S \subseteq A_q$ and $T \subseteq E_{2n}$. Let $\varepsilon > 0$. Let $w_{2n} = u_0^q u_2^q u_4^q \cdots u_{2n}^q$, as in Lemma 4.6(8). By Lemma 4.6(4) and (5) and the formula for β in Lemma 4.7,

$$(4.13) \quad \beta(w_{2n}) = e^{-2\pi i/q} w_{2n}.$$

Let $p_0, p_1, \dots, p_{q-1} \in E_{2n}$ be the spectral projections for w_{2n} , labelled so that $w = \sum_{k=0}^{q-1} e^{2\pi i k/q} p_k$. It follows from (4.13) that $\beta(p_k) = p_{k+1}$ for $k = 0, 1, \dots, q-2$ and $\beta(p_{q-1}) = p_0$. For $k = 0, 1, \dots, q-1$, set $e_k = 1 \otimes p_k$. These projections satisfy conditions (5) and (7). Since $p_k \in Z(E_{2n})$ (by Lemma 4.6(8)), we have $e_k(a \otimes b) = (a \otimes b)e_k$ for all $a \in S$ and $b \in T$. So (6) holds. This proves the claim.

To show that γ satisfies the Rokhlin property, let $F \subseteq A_q \otimes B_q$ be finite and let $\varepsilon > 0$. Construct projections e_0, e_1, \dots, e_{q-1} as in the claim, with $F \cup \{w\}$ in place of F and $\frac{\varepsilon}{2}$ in place of ε . We verify the analogs of (5), (6), and (7) with $\gamma = \text{Ad}(w) \circ (\alpha \otimes \beta)$ in place of $\alpha \otimes \beta$. Only the analog of (5) requires proof. For $k = 0, 1, \dots, q-2$ we have

$$\begin{aligned} \|\gamma(e_k) - e_{k+1}\| &\leq \|\text{Ad}(w)(\alpha \otimes \beta)(e_k) - \text{Ad}(w)(e_{k+1})\| + \|we_{k+1}w^* - e_{k+1}\| \\ &\leq \|(\alpha \otimes \beta)(e_k) - e_{k+1}\| + \|we_{k+1} - e_{k+1}w\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

The proof that $\|(\alpha \otimes \beta)(e_{q-1}) - e_0\| < \varepsilon$ is essentially the same. \square

Definition 4.9. Let A_q be as in Definition 4.1 and let B_q be as in Notation 4.4. Set $C_q = A_q \otimes B_q$, and let γ be the automorphism of Lemma 4.8. We also write γ for the action of \mathbb{Z}_q generated by this automorphism, and define the C*-algebra D_q by $D_q = C_q \rtimes_{\gamma} \mathbb{Z}_q$.

Proposition 4.10. Let $q \in \{2, 3, \dots\}$. The C*-algebra $D_q = C_q \rtimes_{\gamma} \mathbb{Z}_q$ of Definition 4.9 is simple, separable, unital, and exact. It tensorially absorbs the q^∞ UHF algebra B_q and the Jiang-Su algebra Z . Moreover, D_q is approximately divisible, stably finite, has real rank zero and stable rank one, and has a unique tracial state which determines the order on projections over D_q . Also,

$$K_0(D_q) \cong \mathbb{Z}[\frac{1}{q}] \quad \text{and} \quad K_1(D_q) = 0,$$

where the first isomorphism sends $[1]$ to 1, and is an isomorphism of ordered groups. Finally, letting $\mathbb{Z}[\frac{1}{q}]_+$ be the set of nonnegative elements in $\mathbb{Z}[\frac{1}{q}] \subseteq \mathbb{R}$, the Cuntz semigroup of D_q is given by

$$W(D_q) \cong \mathbb{Z}[\frac{1}{q}]_+ \amalg (0, \infty).$$

Proof. We first consider the algebra $C_q = A_q \otimes B_q$ in place of D_q , and we prove that it has most of the properties listed for D_q . The exceptions are that we do not prove stable finiteness or that the order on projections over

C_q is determined by traces, the K-theory is different (and we postpone its calculation), and we do not compute the Cuntz semigroup.

It is obvious that C_q is separable and unital. To prove simplicity of C_q , use simplicity of A_q (Remark 4.3), simplicity and nuclearity of the UHF algebra B_q , and the corollary on page 117 of [47]. (We warn that [47] systematically refers to tensor products as “direct products”.) Exactness of C_q follows from exactness of A_q (Remark 4.3), exactness of B_q , and Proposition 7.1(iii) of [34]. Since A_q and B_q have unique tracial states (the first by Remark 4.3), Corollary 6.13 of [11] (or Lemma 6.1 below) implies that C_q has a unique tracial state. Since A_q is stably finite (Remark 4.3), and B_q is a UHF algebra, Corollary 6.6 of [42] implies that $\text{tsr}(A_q \otimes B_q) = 1$. The algebra B_q is approximately divisible by Proposition 4.1 of [2], so $A_q \otimes B_q$ is approximately divisible. Since C_q is simple, approximately divisible, exact, and has a unique tracial state, it has real rank zero by Theorem 1.4(f) of [2]. The algebra B_q tensorially absorbs B_q , and tensorially absorbs the Jiang-Su algebra Z by Corollary 6.3 of [29]. Therefore C_q tensorially absorbs both algebras.

The algebra D_q is separable and unital because C_q is. Exactness of D_q follows from Proposition 7.1(v) of [34]. Parts (1) and (5) of Lemma 4.7 say that β has period q in $\text{Out}(B_q)$. So, by Theorem 1 in [53], for $k = 1, 2, \dots, q-1$ the automorphism γ^k is outer. Theorem 3.1 of [36] now implies that D_q is simple. Since γ has the Rokhlin property by Lemma 4.8(4), D_q has a unique tracial state by Proposition 4.14 of [38], $\text{tsr}(D_q) = 1$ by Proposition 4.1(1) of [38], D_q is approximately divisible by Proposition 4.5 of [38], and D_q has real rank zero by Proposition 4.1(2) of [38]. Combining Corollary 3.4(1) of [25] with the Rokhlin property, we see that D_q absorbs both B_q and Z . Simplicity of D_q and existence of a tracial state imply stable finiteness.

It now follows from Proposition 2.6 of [39] that the order on projections over D_q is determined by traces.

The computation of $K_0(D_q)$ is done in the same way as in the proof of Proposition 7.2 of [39], and we refer the reader to that article for the many details we omit in the following computation. Here we have

$$K_0(A_q) \cong \mathbb{Z}^q, \quad K_1(A_q) = 0, \quad K_0(B_q) \cong \mathbb{Z}[\tfrac{1}{q}], \quad \text{and} \quad K_1(B_q) = 0,$$

so that the Künneth formula (see [46]) gives

$$K_0(C_q) \cong \mathbb{Z}[\tfrac{1}{q}]^q \quad \text{and} \quad K_1(C_q) = 0.$$

Moreover, by the argument used in the proof of Proposition 7.2 of [39],

$$K_*(D_q) \cong \bigcap_{m=0}^{q-1} \text{Ker}(\text{id} - K_*(\gamma^m)).$$

For $j = 1, 2, \dots, q$, define $r_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{C}^q$ where 1 is in the j -th position. Then the unitary v of (4.1) is

$$v = e^{2\pi i/q} r_1 + r_2 + e^{2(q-1)\pi i/q} r_3 + \dots + e^{4\pi i/q} r_q,$$

and

$$\alpha(v) = r_1 + e^{2(q-1)\pi i/q} r_2 + e^{2(q-2)\pi i/q} r_3 + \dots + e^{2\pi i/q} r_q.$$

This implies that $\alpha(r_j) = r_{j-1}$ for $j = 2, 3, \dots, q$ and that $\alpha(r_1) = r_q$. Since $\text{Ad}(w)$ and β are trivial on K-theory, it follows that $K_0(\gamma): \mathbb{Z}[\frac{1}{q}]^q \rightarrow \mathbb{Z}[\frac{1}{q}]^q$ is given by

$$K_0(\gamma)(\eta_1, \eta_2, \dots, \eta_q) = (\eta_2, \eta_3, \dots, \eta_q, \eta_1).$$

Therefore $\text{id} - K_0(\gamma)$ corresponds to the matrix

$$\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The map $\eta \rightarrow (\eta, \eta, \dots, \eta)$ is an isomorphism from $\mathbb{Z}[\frac{1}{q}]$ to $\text{Ker}(\text{id} - K_0(\gamma))$, and one checks that its image is contained in $\text{Ker}(\text{id} - K_0(\gamma^m))$ for all m such that $0 \leq m \leq q-1$. Therefore this map is an isomorphism from $\mathbb{Z}[\frac{1}{q}]$ to $\bigcap_{m=0}^{q-1} \text{Ker}(\text{id} - K_0(\gamma^m))$.

The computation of the Cuntz semigroup now follows from Remark 2.13 by observing that $V(D_q)$ is the positive part of $K_0(D_q)$ and the uniqueness of the tracial state on D_q implies that $\text{LAff}_b(T(D_q))_{++} = (0, \infty)$. \square

Proposition 4.11. Let C_q and $\gamma: \mathbb{Z}_q \rightarrow \text{Aut}(C_q)$ be as in Definition 4.9. Then $D_q = C_q \rtimes_\gamma \mathbb{Z}_q$ satisfies the Universal Coefficient Theorem.

Proof. By Theorem 1.1 in [24] (see also Theorem 4.1 in [21]), the algebra A_q in Definition 4.1 is KK-equivalent to the full free product

$$E = C([0, 1]) \star_{\mathbb{C}} \cdots \star_{\mathbb{C}} C([0, 1]) \star_{\mathbb{C}} \mathbb{C}^q.$$

It is shown in the proof of Theorem 2.7 in [51] that if A and B are separable unital C^* -algebras, then the suspension $S(A \star_{\mathbb{C}} B)$ of the amalgamated free product is KK-equivalent to the mapping cone of the inclusion $\mathbb{C} \hookrightarrow A \oplus B$. Therefore $A \star_{\mathbb{C}} B$ satisfies the Universal Coefficient Theorem when A and B do. Arguing inductively, we see that E , and therefore also A_q , satisfies the Universal Coefficient Theorem. Since C_q is the tensor product of A_q with a UHF algebra, it too satisfies the Universal Coefficient Theorem.

Observe also that $C_q = A_q \otimes B_q$ absorbs the q^∞ UHF algebra. In addition, by Lemma 4.8(4), the action γ has the Rokhlin property, so Proposition 3.9(2) implies that γ has the continuous Rokhlin property. Using Proposition 3.8, we conclude that $C_q \rtimes_\gamma \mathbb{Z}_q$ satisfies the Universal Coefficient Theorem. \square

5. THE MAIN STEP

Let $D_q = C_q \rtimes \mathbb{Z}_q$ be as in Definition 4.9, and let τ be its unique tracial state. In this section we show that if q is an odd prime such that -1 is not a square mod q , then $\pi_\tau(D_q)''$ is not isomorphic to its opposite algebra. This is the main step in proving that D_q , as well as the tensor product $E \otimes D_q$ for suitable E , is not isomorphic to its opposite algebra.

The following result belongs to the theory of cocycle conjugacy, but we have not found a reference in the literature.

Lemma 5.1. Let M be a factor and let $n \in \mathbb{Z}_{>0}$. Let $\alpha, \beta: \mathbb{Z}_n \rightarrow \text{Aut}(M)$ be actions of \mathbb{Z}_n on M . Write the elements of \mathbb{Z}_n as $0, 1, \dots, n-1$, so that, for example, the automorphisms generating the actions are α_1 and β_1 . Suppose that there is a unitary $y \in M$ such that $\beta_1 = \text{Ad}(y) \circ \alpha_1$. Then there is an isomorphism $\varphi: M \rtimes_{\beta} \mathbb{Z}_n \rightarrow M \rtimes_{\alpha} \mathbb{Z}_n$ which intertwines the dual actions, that is, for all $l \in \widehat{\mathbb{Z}_n}$ we have $\varphi \circ \widehat{\beta}_l = \widehat{\alpha}_l \circ \varphi$.

Proof. For $k \in \mathbb{Z}$ we write $\alpha_k = \alpha_1^k$ and $\beta_k = \beta_1^k$. (This agrees with the notation in the statement when $k \in \{0, 1, \dots, n-1\}$.) For $k \in \mathbb{Z}_{>0}$ define a unitary $y_k \in M$ by

$$y_k = y\alpha_1(y)\alpha_2(y) \cdots \alpha_{k-1}(y).$$

Set $y_0 = 1$, and define $y_k = \alpha_k(y_{-k}^*)$ for $k < 0$. Then one easily checks that $\text{Ad}(y_k) \circ \alpha_k = \beta_k$ for all $k \in \mathbb{Z}$, and moreover that $y_j \alpha_j(y_k) = y_{j+k}$ for all $j, k \in \mathbb{Z}$.

Since $\alpha_n = \beta_n = \text{id}_M$ and M is a factor, we have $y_n \in \mathbb{C} \cdot 1$. So there is a scalar ζ with $|\zeta| = 1$ such that $y_n = \zeta^n \cdot 1$. For $k \in \mathbb{Z}$ define $z_k = \zeta^{-k} y_k$. Then z_k is unitary, and we have $\text{Ad}(z_k) \circ \alpha_k = \beta_k$ for all $k \in \mathbb{Z}$ and $z_j \alpha_j(z_k) = z_{j+k}$ for all $j, k \in \mathbb{Z}$. Moreover, $z_j = z_k$ whenever n divides $j - k$.

Let u_0, u_1, \dots, u_{n-1} be the standard unitaries in the crossed product $M \rtimes_{\alpha} \mathbb{Z}_n$ which implement α , so that for $a \in M \subseteq M \rtimes_{\alpha} \mathbb{Z}_n$ we have $u_k a u_k^* = \alpha_k(a)$ and

$$M \rtimes_{\alpha} \mathbb{Z}_n = \left\{ \sum_{k=0}^{n-1} a_k u_k : a_0, a_1, \dots, a_{n-1} \in M \right\}.$$

Similarly let v_0, v_1, \dots, v_{n-1} be the standard unitaries in $M \rtimes_{\beta} \mathbb{Z}_n$ which implement β . Then there is a unique linear bijection $\varphi: M \rtimes_{\beta} \mathbb{Z}_n \rightarrow M \rtimes_{\alpha} \mathbb{Z}_n$ such that $\varphi(av_k) = az_k u_k$ for $a \in M$ and $k = 0, 1, \dots, n-1$. One checks, using the properties of $(z_k)_{k \in \mathbb{Z}}$, that φ is a homomorphism. Moreover,

$$(\varphi \circ \widehat{\beta}_l)(av_k) = \varphi(e^{2\pi i k l / n} av_k) = e^{2\pi i k l / n} az_k u_k = \widehat{\alpha}_l(az_k u_k) = (\widehat{\alpha}_l \circ \varphi)(av_k)$$

for every $a \in M$ and $k \in \{0, 1, \dots, n-1\}$. Therefore $\varphi \circ \widehat{\beta}_l = \widehat{\alpha}_l \circ \varphi$. \square

Lemma 5.2. Let A and B be C^* -algebras. Let ρ be a state on A and let ω be a state on B . Then $\pi_{\rho \otimes \omega}(A \otimes_{\min} B)'' \cong \pi_{\rho}(A)'' \overline{\otimes} \pi_{\omega}(B)''$.

Proof. The proof is straightforward (one starts by identifying $H_{\rho \otimes \omega}$ with $H_{\rho} \otimes H_{\omega}$), and is omitted. \square

For $q \in \{2, 3, \dots\}$ let $D_q = C_q \rtimes_{\gamma} \mathbb{Z}_q$ be the C^* -algebra of Definition 4.9. Our next step is to show that D_q is not isomorphic to its opposite algebra whenever q is an odd prime such that -1 is not a square mod q , by associating to D_q a II_1 factor T_q , and computing the Connes invariant of T_q .

For $q \in \{2, 3, \dots\}$, set

$$N_q = [\star_1^q L^{\infty}([0, 1])] \star \mathcal{L}(\mathbb{Z}_q),$$

and for $k = 1, 2, \dots, q$ denote by $\bar{\varepsilon}_k: L^\infty([0, 1]) \rightarrow N_q$ the inclusion of the k -th free factor $L^\infty([0, 1])$ in N_q . Let v be the element of $\mathcal{L}(\mathbb{Z}_q) \cong \mathbb{C}^q$ defined in (4.1) and let $\bar{\alpha}$ be the automorphism of N_q given by

$$(5.1) \quad \bar{\alpha}(\bar{\varepsilon}_1(f)) = \bar{\varepsilon}_2(f), \quad \bar{\alpha}(\bar{\varepsilon}_2(f)) = \bar{\varepsilon}_3(f), \quad \dots, \quad \bar{\alpha}(\bar{\varepsilon}_{q-1}(f)) = \bar{\varepsilon}_q(f),$$

$$(5.2) \quad \bar{\alpha}(\bar{\varepsilon}_q(f)) = \text{Ad}(v)(\bar{\varepsilon}_1(f))$$

for all $f \in L^\infty([0, 1])$, and

$$(5.3) \quad \bar{\alpha}(v) = e^{-2\pi i/q} v.$$

Thus $\bar{\alpha}^q = \text{Ad}(v)$. Then N_q is the weak operator closure of the image of A_q under the Gelfand-Naimark-Segal representation coming from the unique tracial state on A_q (see Remark 4.3), and $\bar{\alpha} \in \text{Aut}(N_q)$ is an extension of the automorphism α defined in (4.2), (4.3), and (4.4). Identify B_q with $C^*(g, u_0, u_1, u_2, \dots)$ as in Lemma 4.6(9). Let ω be the unique tracial state on B_q , and let R_0 be the weak operator closure $\pi_\omega(B_q)''$. Then R_0 is isomorphic to the hyperfinite II_1 factor R . Denote by $\bar{\beta}$ the extension to R_0 of the automorphism β of Lemma 4.7. Then, with $g \in B_q \subseteq \pi_\omega(B_q)''$ being as in Lemma 4.6, from Lemma 4.7(1) and (2) we get $\bar{\beta}^q = \text{Ad}(g)$ and $\bar{\beta}(g) = e^{2\pi i/q} g$. It is well known, and easy to see, that these relations imply that $\bar{\beta}$ has period q in $\text{Out}(R_0)$.

Let w be as in Lemma 4.8. The automorphism $\bar{\gamma} = \text{Ad}(w) \circ (\bar{\alpha} \otimes \bar{\beta})$ generates an action, which we also call $\bar{\gamma}$, of \mathbb{Z}_q on $N_q \bar{\otimes} R_0$. Since $\bar{\beta}$ has period q in $\text{Out}(R_0)$, Corollary 1.14 in [33] (or Theorem 13.1.16 in [32]) implies that $\bar{\gamma}^k$ is outer for $k = 1, 2, \dots, q-1$. By Proposition 13.1.5(ii) of [32], the crossed product

$$(5.4) \quad T_q = (N_q \bar{\otimes} R_0) \rtimes_{\bar{\gamma}} \mathbb{Z}_q$$

is a factor of type II_1 .

Remark 5.3. Let G be a discrete group containing a nonabelian free group and such that its von Neumann algebra $\mathcal{L}(G)$ is a factor. Let R be the hyperfinite II_1 factor. By the proof of Proposition 3.5 in [52], any central sequence in $\mathcal{L}(G) \bar{\otimes} R$ has the form $(1 \otimes x_n)_{n \in \mathbb{Z}_{>0}} + (y_n)_{n \in \mathbb{Z}_{>0}}$ for a central sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in R and a sequence $(y_n)_{n \in \mathbb{Z}_{>0}}$ in $\mathcal{L}(G) \bar{\otimes} R$ such that $\lim_{n \rightarrow \infty} \|y_n\|_2 = 0$. Moreover, $\mathcal{L}(G) \bar{\otimes} R$ has no nontrivial hypercentral sequences, that is, any hypercentral sequence in $\mathcal{L}(G) \bar{\otimes} R$ has the form $(\lambda_n \cdot 1)_{n \in \mathbb{Z}_{>0}} + (y_n)_{n \in \mathbb{Z}_{>0}}$ for a sequence $(\lambda_n)_{n \in \mathbb{Z}_{>0}}$ in \mathbb{C} and a sequence $(y_n)_{n \in \mathbb{Z}_{>0}}$ in $\mathcal{L}(G) \bar{\otimes} R$ such that $\lim_{n \rightarrow \infty} \|y_n\|_2 = 0$.

Using the Connes exact sequence in (2.2), we now compute the Connes invariant of T_q . The argument follows Section 5 in [52], with suitable changes. We reproduce it here for the convenience of the reader.

Proposition 5.4. Let q be an odd prime and let $T_q = (N_q \bar{\otimes} R_0) \rtimes_{\bar{\gamma}} \mathbb{Z}_q$ be the II_1 factor defined in (5.4). Then $\chi(T_q) \cong \mathbb{Z}_{q^2}$. Moreover, the unique subgroup of order q in $\chi(T_q)$ is the image of the action $\sigma: \widehat{\mathbb{Z}}_q \rightarrow \text{Aut}(T_q)$ obtained as the dual action on $T_q = (N_q \bar{\otimes} R_0) \rtimes_{\bar{\gamma}} \mathbb{Z}_q$.

Proof. Denote by \mathbb{F}_q the free group on q generators. For the first part of the proof, we use the facts and notation in the discussion after Lemma 5.2. By abuse of notation, we write \mathbb{Z}_q for the image of \mathbb{Z}_q in $\text{Aut}(N_q \overline{\otimes} R_0)$ under $\bar{\gamma}$. Since $N_q \cong \mathcal{L}(\mathbb{F}_q \star \mathbb{Z}_q)$ is full by Lemma 3.2 in [52] and $\bar{\alpha} \notin \text{Inn}(N_q)$, Corollary 3.3 in [9] implies that $\mathbb{Z}_q \cap \text{Inn}(N_q \overline{\otimes} R_0) = \{1\}$. Moreover, by Remark 5.3, the II_1 factor $N_q \overline{\otimes} R_0$ has no nontrivial hypercentral sequences.

To compute the Connes invariant of T_q , we first compute the subgroups K^\perp and L introduced after Remark 2.8.

By Remark 5.3, any central sequence in $N_q \overline{\otimes} R_0$ has the form $(1 \otimes x_n)_{n \in \mathbb{Z}_{>0}} + (y_n)_{n \in \mathbb{Z}_{>0}}$ for a central sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ in R_0 and a sequence $(y_n)_{n \in \mathbb{Z}_{>0}}$ in $N_q \overline{\otimes} R_0$ such that $\lim_{n \rightarrow \infty} \|y_n\|_2 = 0$. Since the trace on $N_q \overline{\otimes} R_0$ is unique, it is γ -invariant, so also $\lim_{n \rightarrow \infty} \|\bar{\gamma}(y_n)\|_2 = 0$. Therefore $\bar{\gamma} \in \text{Ct}(N_q \overline{\otimes} R_0)$ if and only if $\bar{\beta} \in \text{Ct}(R_0)$. Since $\text{Ct}(R_0) = \text{Inn}(R_0)$ by Theorem 3.2(1) in [10], and $\bar{\beta}$ is outer, it follows that $K = \mathbb{Z}_q \cap \text{Ct}(N_q \overline{\otimes} R_0)$ is trivial. Thus $K^\perp \cong \mathbb{Z}_q$.

We next compute L . Using the notation of Lemma 4.6, we have $R_0 = \{g, u_0, u_1, u_2, \dots\}''$ and $\bar{\beta} = \text{Ad}(u_{-1})$. Let $\xi: \text{Aut}(N_q \overline{\otimes} R_0) \rightarrow \text{Out}(N_q \overline{\otimes} R_0)$ denote the quotient map. We claim that $L \cong \mathbb{Z}_q$ and that a generator of L is given by

$$\mu = \xi(\text{id} \otimes [\text{Ad}(u_0^*) \circ \bar{\beta}]).$$

We prove the claim. Using $(\alpha \otimes \beta)(w) = w$ (by Lemma 4.8(2)) at the first step,

$$\begin{aligned} \bar{\gamma}^{-1} \circ (\text{id} \otimes [\text{Ad}(u_0^*) \circ \bar{\beta}]) &= \text{Ad}(w^*) \circ (\bar{\alpha}^{-1} \otimes \bar{\beta}^{-1}) \circ (\text{id} \otimes [\text{Ad}(u_0^*) \circ \bar{\beta}]) \\ &= \text{Ad}(w^*(1 \otimes u_0^*)) \circ (\bar{\alpha}^{-1} \otimes \text{id}). \end{aligned}$$

By Remark 5.3, this automorphism is in $\text{Ct}(N_q \overline{\otimes} R_0)$, so that

$$\text{id} \otimes [\text{Ad}(u_0^*) \circ \bar{\beta}] \in \mathbb{Z}_q \vee \text{Ct}(N_q \overline{\otimes} R_0).$$

For $n \in \mathbb{Z}_{>0}$ define $y_n = u_0 u_1^* u_2 u_3^* \cdots u_{2n}$, which is a unitary in R_0 . Using Lemma 4.6(3) and (4), we get $y_n g y_n^* = e^{2\pi i/q} g$. Using Lemma 4.6(5) and (6), we get

$$y_n u_k y_n^* = \begin{cases} e^{-2\pi i/q^2} u_k & k = 0 \\ u_k & k = 1, 2, \dots, 2n-1 \\ e^{2\pi i/q^2} u_k & k = 2n \\ e^{-2\pi i/q^2} u_k & k = 2n+1 \\ u_k & k = 2n+2, 2n+3, \dots \end{cases}$$

(The calculation for $k = 1, 2, \dots, 2n-1$ depends on the parity of k .) Comparing these formulas with Lemma 4.7, we see that $\lim_{n \rightarrow \infty} y_n x y_n^* = \beta(x)$ for all $x \in \{g, u_0, u_1, u_2, \dots\}$, and hence for all $x \in B_q$. Therefore $\lim_{n \rightarrow \infty} \|\text{Ad}(y_n)(x) - \bar{\beta}(x)\|_2 = 0$ for all $x \in R_0$. Also, Lemma 4.7(3) and (4) imply $\bar{\beta}(y_n) = e^{-2\pi i/q^2} y_n$ for $n \in \mathbb{Z}_{>0}$. Set $z_n = u_0^* y_n$. Then

$$\text{id} \otimes [\text{Ad}(u_0^*) \circ \bar{\beta}] = \lim_{n \rightarrow \infty} \text{Ad}(1 \otimes z_n) \quad \text{and} \quad \bar{\beta}(z_n) = z_n.$$

It follows (recalling H from (2.1)) that

$$\text{id} \otimes [\text{Ad}(u_0^*) \circ \bar{\beta}] \in (\mathbb{Z}_q \vee \text{Ct}(N_q \overline{\otimes} R_0)) \cap \bar{H}.$$

So $\mu = \xi(\text{id} \otimes [\text{Ad}(u_0^*) \circ \bar{\beta}]) \in L$.

To finish the proof of the claim, since μ has order q , we show that μ generates L . Given an automorphism $\varphi \in (\mathbb{Z}_q \vee \text{Ct}(N_q \bar{\otimes} R_0)) \cap \bar{H}$, there exists $k \in \{0, 1, \dots, q-1\}$ such that $\varphi \circ \bar{\gamma}^k$ is centrally trivial. By the same argument as used to prove Proposition 3.6 in [52], there are $\nu \in \text{Aut}(N_q)$ and a unitary $z \in N_q \bar{\otimes} R_0$ such that $\varphi \circ \bar{\gamma}^k = \text{Ad}(z) \circ (\nu \otimes \text{id})$. Thus, with $x = z(\nu \otimes \text{id})((w^*)^k)$, we have

$$\varphi = \text{Ad}(z) \circ (\nu \otimes \text{id}) \circ \text{Ad}(w^*)^k \circ (\bar{\alpha}^{-k} \otimes \bar{\beta}^{-k}) = \text{Ad}(x) \circ ([\nu \circ \bar{\alpha}^{-k}] \otimes \bar{\beta}^{-k}).$$

Since $\varphi \in \bar{H} \subseteq \overline{\text{Inn}(N_q \bar{\otimes} R_0)}$ and N_q is full, Corollary 3.3 in [9] implies that $\nu \circ \bar{\alpha}^{-k} \in \text{Inn}(N_q)$, so that $\nu \circ \bar{\alpha}^{-k} = \text{Ad}(c)$ for some unitary $c \in N_q$. Thus $\varphi = \text{Ad}(x(c \otimes 1)) \circ (\text{id} \otimes \bar{\beta}^{-k})$ differs from a power of $\text{id} \otimes [\text{Ad}(u_0^*) \circ \bar{\beta}]$ only by an inner automorphism. This concludes the proof of the claim.

Next consider the Connes short exact sequence (2.2):

$$\{1\} \longrightarrow \mathbb{Z}_q \xrightarrow{\partial} \chi(T_q) \xrightarrow{\Pi} \mathbb{Z}_q \longrightarrow \{1\}.$$

Taking $\mu = \xi(\text{id} \otimes [\text{Ad}(u_0^*) \circ \bar{\beta}])$, let $\eta \in \text{Ct}(T_q) \cap \overline{\text{Inn}(T_q)}$ be the automorphism defined in (2.3). Since the only possibilities for $\chi(T_q)$ are \mathbb{Z}_{q^2} and $\mathbb{Z}_q \oplus \mathbb{Z}_q$, to complete the proof it is enough to show that $\xi(\eta)$ does not have order q , that is, $\eta^q \notin \text{Inn}(T_q)$.

By Lemma 4.7(2) and (3), we have

$$(\bar{\alpha} \otimes \bar{\beta})(1 \otimes (u_0^*)^q g) = e^{4\pi i/q} (1 \otimes (u_0^*)^q g).$$

Lemma 4.8(3) gives $\text{Ad}(w)(1 \otimes (u_0^*)^q g) = 1 \otimes (u_0^*)^q g$. Hence, $\bar{\gamma}(1 \otimes (u_0^*)^q g) = e^{4\pi i/q} (1 \otimes (u_0^*)^q g)$. Let y be the standard unitary (of order q) implementing the generating automorphism $\text{Ad}(w) \circ (\bar{\alpha} \otimes \bar{\beta})$ of the action $\bar{\gamma}$ in the crossed product $T_q = (N_q \bar{\otimes} R_0) \rtimes_{\bar{\gamma}} \mathbb{Z}_q$, so that $\text{Ad}(w) \circ (\bar{\alpha} \otimes \bar{\beta}) = \text{Ad}(y)$. Then

$$\text{Ad}(1 \otimes (u_0^*)^q g)(y) = e^{-4\pi i/q} y.$$

Recall from Lemma 4.7(3) and (1) that $\bar{\beta}(u_0) = e^{-2\pi i/q^2} u_0$ (which implies that $\bar{\beta}$ commutes with $\text{Ad}(u_0^*)$) and $\bar{\beta}^q = \text{Ad}(g)$. For $a_0, a_1, \dots, a_{q-1} \in N_q \bar{\otimes} R_0$, it follows that

$$\begin{aligned} \eta^q \left(\sum_{k=0}^{q-1} a_k y^k \right) &= \sum_{k=0}^{q-1} (1 \otimes [\text{Ad}(u_0^*)^q \circ \bar{\beta}^q])(a_k) y^k = \sum_{k=0}^{q-1} (1 \otimes \text{Ad}((u_0^*)^q g))(a_k) y^k \\ &= \text{Ad}(1 \otimes (u_0^*)^q g) \left(\sum_{k=0}^{q-1} e^{4\pi i k/q} a_k y^k \right). \end{aligned}$$

Since q is odd, we conclude that, up to an inner automorphism, η^q is the dual action at a nontrivial element of the dual group. Therefore η^q is outer. \square

Using the Connes invariant, we can now show that the C*-algebra D_q is not isomorphic to its opposite algebra whenever q is an odd prime such that -1 is not a square mod q .

Proposition 5.5. Let q be any odd prime such that -1 is not a square mod q . Let $D_q = C_q \rtimes_{\gamma} \mathbb{Z}_q$ be the C*-algebra of Definition 4.9. Let τ be the unique tracial state on D_q (Proposition 4.10), and let π_{τ} be the

Gelfand-Naimark-Segal representation associated to τ . Then the von Neumann algebra $\pi_\tau(D_q)''$ is not isomorphic to its opposite algebra.

Proof. We claim that $T_q \cong \pi_\tau(D_q)''$. Let σ be the unique tracial state on A_q (Remark 4.3), and let ω be the unique tracial state on the UHF algebra B_q . We have an obvious map $C_q = A_q \otimes B_q \rightarrow N_q \bar{\otimes} R_0$ which intertwines γ and $\bar{\gamma}$. Lemma 5.2 shows that this map induces an isomorphism $\pi_{\sigma \otimes \omega}(C_q)'' \cong N_q \bar{\otimes} R_0$. Since the group is finite, taking crossed products by \mathbb{Z}_q gives an isomorphism $T_q \cong \pi_\tau(D_q)''$, as claimed.

To show that T_q is not isomorphic to its opposite algebra, we give a recipe which starts with a factor P , just given as a factor of type II_1 with certain properties (see (1), (2), (3), and (4) below), and produces a subset $S_q(P)$ of \mathbb{Z}_q , which we identify with $\{0, 1, \dots, q-1\}$. The important point is that this recipe does not depend on knowing any particular element, automorphism, etc. of P . That is, if we start with some other factor of type II_1 which is isomorphic to P , then we get the same subset of $\{0, 1, \dots, q-1\}$, regardless of the choice of isomorphism. When -1 is not a square mod q , we will show that the recipe also applies to P^{op} and gives a different subset, from which it will follow that $T_q^{\text{op}} \not\cong T_q$.

We describe the construction first, postponing the proofs that the steps can be carried out and the result is independent of the choices made. Let P be a factor of type II_1 with separable predual. Let $\chi(P)$ denote the Connes invariant of P as in Definition 2.7, and assume that P satisfies the following properties:

- (1) $\chi(P) \cong \mathbb{Z}_{q^2}$.
- (2) The unique subgroup of $\chi(P)$ of order q is the image of a subgroup (not necessarily unique) of $\text{Aut}(P)$ isomorphic to \mathbb{Z}_q .
- (3) Let $\rho: \mathbb{Z}_q \rightarrow \text{Aut}(P)$ come from a choice of the subgroup and isomorphism in (2). Form the crossed product $P \rtimes_\rho \mathbb{Z}_q$, and let $\widehat{\rho}: \widehat{\mathbb{Z}_q} \rightarrow \text{Aut}(P \rtimes_\rho \mathbb{Z}_q)$ be the dual action. Then for every nontrivial element $l \in \widehat{\mathbb{Z}_q}$, the automorphism $\widehat{\rho}_l \in \text{Aut}(P \rtimes_\rho \mathbb{Z}_q)$ has a factorization $\varphi \circ \psi$, in which φ is an approximately inner automorphism and ψ is a centrally trivial automorphism.
- (4) For any nontrivial element $l \in \widehat{\mathbb{Z}_q}$ and any factorization $\widehat{\rho}_l = \varphi \circ \psi$ as in (3), there is a unitary $z \in P \rtimes_\rho \mathbb{Z}_q$ such that $\psi^q = \text{Ad}(z)$, and there is $k \in \{0, 1, \dots, q-1\}$ such that $\psi(z) = e^{2\pi i k/q} z$. (See the obstruction to lifting of Definition 2.9.)

For a type II_1 factor P which satisfies (1), (2), (3), and (4), we take $S_q(P)$ to be the set of all values of $k \in \{0, 1, \dots, q-1\}$ which appear in (4) for any choice of the action $\rho: \mathbb{Z}_q \rightarrow \text{Aut}(P)$, any nontrivial element $l \in \widehat{\mathbb{Z}_q}$, and any choice of the factorization $\widehat{\rho}_l = \varphi \circ \psi$ as in (3). We think of $S_q(P)$ as a subset of \mathbb{Z}_q in the obvious way.

We claim that the crossed product $P \rtimes_\rho \mathbb{Z}_q$ is uniquely determined up to isomorphism and the dual action $\widehat{\rho}: \widehat{\mathbb{Z}_q} \rightarrow \text{Aut}(P \rtimes_\rho \mathbb{Z}_q)$ is uniquely determined up to conjugacy and automorphisms of \mathbb{Z}_q . There are two ambiguities in the choice of ρ . If we change the isomorphism of \mathbb{Z}_q with the subgroup of $\chi(P)$ of order q , we are modifying ρ by an automorphism of \mathbb{Z}_q . The crossed

product $M \rtimes_{\rho} \mathbb{Z}_q$ is the same, and the dual action is modified by the corresponding automorphism of $\widehat{\mathbb{Z}_q}$. Now suppose that we fix an isomorphism of \mathbb{Z}_q with the subgroup of $\chi(P)$ of order q , but choose a different lift $\bar{\rho}$ to a homomorphism $\mathbb{Z}_q \rightarrow \text{Aut}(P)$. Then Lemma 5.1 implies that the crossed products are isomorphic and the dual actions are conjugate. This proves the claim.

Since if the dual action changes by conjugation by an automorphism, the automorphisms in the decomposition of (3) also change by conjugation by an automorphism, and the obstruction to lifting is unchanged by conjugation, it follows that changing the dual action by conjugation leaves $S_q(P)$ invariant. This shows that $S_q(P)$ can be computed by fixing a particular choice of $\rho: \mathbb{Z}_q \rightarrow \text{Aut}(P)$.

Next we check that if P satisfies (1), (2), (3), and (4), then so does P^{op} . For this purpose, we use the von Neumann algebra P^c described in Definition 2.1, which is isomorphic to P^{op} by Remark 2.2. Scalar multiplication enters in the definition of $S_q(P)$ in only two places. The first is the definition of the dual action $\widehat{\rho}: \widehat{\mathbb{Z}_q} \rightarrow \text{Aut}(P \rtimes_{\rho} \mathbb{Z}_q)$. However, the change is easily undone by applying the automorphism $l \mapsto -l$ of $\widehat{\mathbb{Z}_q}$. The other place is in the definition of the obstruction to lifting. So P^c satisfies the conditions (1), (2), (3), and (4), and we get

$$(5.5) \quad S_q(P^c) = \{-l: l \in S_q(P)\},$$

where we are treating $S_q(\cdot)$ as a subset of \mathbb{Z}_q .

In the rest of the proof, we show that the II_1 factor $T_q = (N_q \overline{\otimes} R_0) \rtimes_{\bar{\gamma}} \mathbb{Z}_q$ satisfies (1), (2), (3), and (4), and that moreover $S_q(T_q)$ can be computed using, for each nontrivial element $l \in \widehat{\mathbb{Z}_q}$, just one choice of the factorization $\widehat{\rho}_l = \varphi \circ \psi$ in (3) and one choice of the unitary z in (4). We then finish by computing $S_q(T_q)$.

By Proposition 5.4 we have $\chi(T_q) \cong \mathbb{Z}_{q^2}$, and the unique subgroup of order q in $\chi(T_q)$ is the image of the action $\sigma: \widehat{\mathbb{Z}_q} \rightarrow \text{Aut}(T_q)$ obtained as the dual action on $T_q = (N_q \overline{\otimes} R_0) \rtimes_{\bar{\gamma}} \mathbb{Z}_q$. Thus, there exists at least one choice for ρ , namely σ composed with some isomorphism $\mathbb{Z}_q \rightarrow \widehat{\mathbb{Z}_q}$. Therefore T_q satisfies property (2).

By Takesaki's duality theory (see Theorem 4.5 of [48]), with $\lambda(g)$ denoting the left regular representation of \mathbb{Z}_q on $l^2(\mathbb{Z}_q)$, there is an isomorphism

$$T_q \rtimes_{\rho} \mathbb{Z}_q \cong (N_q \overline{\otimes} R_0) \otimes B(l^2(\mathbb{Z}_q))$$

which identifies $g \mapsto \widehat{\rho}_g$ with the tensor product $g \mapsto \bar{\gamma}_g \otimes \text{Ad}(\lambda(g)^*)$.

Now let $l \in \widehat{\mathbb{Z}_q}$. We claim that $\widehat{\rho}_l$ can be written as $\varphi \circ \psi$ for an approximately inner automorphism φ and a centrally trivial automorphism ψ , and that this factorization is unique up to inner automorphisms. This will imply property (3). We first consider uniqueness, which is equivalent to showing that every automorphism which is both approximately inner and centrally trivial is in fact inner. Since N_q is full by Lemma 3.2 in [52], the decomposition described in the proof of Lemma 3.6 of [52] can be used to

show that every automorphism of $N_q \overline{\otimes} R_0$ which is both approximately inner and centrally trivial is in fact inner. Uniqueness now follows because $(N_q \overline{\otimes} R_0) \otimes B(l^2(\mathbb{Z}_q)) \cong N_q \overline{\otimes} R_0$.

For existence, since the approximately inner automorphisms are a normal subgroup of $\text{Aut}(T_q \rtimes_\rho \mathbb{Z}_q)$, it suffices to take l to be the standard generator of $\widehat{\mathbb{Z}_q}$. Equivalently, consider $\bar{\gamma} \otimes \text{Ad}(\lambda(1)^*)$. We will take

$$\varphi = (\text{Ad}(w) \circ (\text{id}_{N_q} \otimes \bar{\beta})) \otimes \text{Ad}(\lambda(1)^*) \quad \text{and} \quad \psi = (\bar{\alpha} \otimes \text{id}_{R_0}) \otimes \text{id}_{B(l^2(\mathbb{Z}_q))}.$$

It is clear that $\bar{\gamma} \otimes \text{Ad}(\lambda(1)^*) = \varphi \circ \psi$. The automorphism φ is approximately inner because, by construction, $\bar{\beta}$ is approximately inner. (In fact, by Theorem XIV.2.16 of [49], every automorphism of R is approximately inner.) To see that ψ is centrally trivial, we observe that by Remark 5.3 every central sequence in $N_q \overline{\otimes} R_0$ has the form $(1 \otimes x_n)_{n=1}^\infty + (y_n)_{n=1}^\infty$ for a central sequence $(x_n)_{n=1}^\infty$ in R_0 and a sequence $(y_n)_{n=1}^\infty$ in $N_q \overline{\otimes} R_0$ such that $\lim_{n \rightarrow \infty} \|y_n\|_2 = 0$. Since the trace on $N_q \overline{\otimes} R_0$ is unique, it is ψ -invariant, so also $\lim_{n \rightarrow \infty} \|\psi(y_n)\|_2 = 0$, which implies that ψ is centrally trivial. This proves the claim.

The obstruction to lifting for ψ (as in property (4)) is independent of the choice of the unitary z implementing ψ^q because $T_q \rtimes_\sigma \widehat{\mathbb{Z}_q}$ is a factor. By the proof of Proposition 1.4 of [10] it is unchanged if ψ is replaced by $\text{Ad}(y) \circ \psi$ for any unitary $y \in T_q \rtimes_\sigma \widehat{\mathbb{Z}_q}$. The centrally trivial factor in the decomposition of any automorphism of $T_q \rtimes_\sigma \widehat{\mathbb{Z}_q}$, in particular, of $(\bar{\gamma} \otimes \text{Ad}(\lambda(1)^*))^l$, is determined up to inner automorphisms. Since, moreover, φ and ψ commute up to an inner automorphism, it follows that we can compute $S_q(T_q)$ by simply computing the obstructions to lifting for all powers ψ^l for a fixed choice of ψ and for $l = 1, 2, \dots, q-1$. We can take

$$\psi = (\bar{\alpha} \otimes \text{id}_{R_0}) \otimes \text{id}_{B(l^2(\mathbb{Z}_q))},$$

for which $z = v \otimes 1 \otimes 1$ has already been shown to be a unitary with $\psi^q = \text{Ad}(z)$ and $\psi(z) = e^{-2\pi i/q} z$. Now one uses Equations (5.1), (5.2), and (5.3) to check that

$$(\psi^l)^q = \text{Ad}(z^l) \quad \text{and} \quad \psi^l(z^l) = e^{-2\pi i l^2/q} z^l.$$

Therefore (identifying $\{0, 1, \dots, q-1\}$ with \mathbb{Z}_q in the usual way)

$$S_q(T_q) = \{-l^2 : l \in \mathbb{Z}_q \setminus \{0\}\}.$$

As observed in Equation (5.5) above, $S_q(T_q^{\text{op}})$ is then given by

$$S_q(T_q^{\text{op}}) = \{l^2 : l \in \mathbb{Z}_q \setminus \{0\}\}.$$

If q is an odd prime such that -1 is not a square mod q , we have $S_q(T_q^{\text{op}}) \neq S_q(T_q)$, whence $T_q^{\text{op}} \not\cong T_q$. \square

6. C*-ALGEBRAS NOT ISOMORPHIC TO THEIR OPPOSITE ALGEBRAS

We use the results of Section 5 to produce a number of examples of simple exact C*-algebras not isomorphic to their opposite algebras and which satisfy the Universal Coefficient Theorem.

Lemma 6.1. Let A and B be unital C^* -algebras and assume that B has a unique tracial state τ . Then the map $\sigma \mapsto \sigma \otimes \tau$ is an affine weak* homeomorphism from the tracial state space $T(A)$ of A to the tracial state space $T(A \otimes B)$ of $A \otimes B$.

We can't use Proposition 6.12 in [11], because (see Proposition 2.7 in [11]) it assumes that $T(A)$ is finite dimensional.

Proof. It is easy to check that the map $\sigma \mapsto \sigma \otimes \tau$ is injective, by considering $(\sigma \otimes \tau)(a \otimes 1)$ for $a \in A$.

We prove surjectivity. Let ρ be a tracial state on $A \otimes B$. Define a tracial state σ on A by $\sigma(a) = \rho(a \otimes 1)$ for $a \in A$. We claim that $\sigma \otimes \tau = \rho$. It suffices to verify equality on $a \otimes b$ for $a \in A_+$ and $b \in B$. So let $a \in A_+$. Define a tracial positive linear functional $\nu_a: B \rightarrow \mathbb{C}$ by $\nu_a(b) = \rho(a \otimes b)$ for $b \in B$. By uniqueness of τ , there is $\lambda(a) \geq 0$ such that $\nu_a = \lambda(a)\tau$. Then

$$\lambda(a) = \nu_a(1) = \rho(a \otimes 1) = \sigma(a).$$

Thus for all $b \in B$, we have

$$(\sigma \otimes \tau)(a \otimes b) = \lambda(a)\tau(b) = \nu_a(b) = \rho(a \otimes b).$$

This completes the proof of surjectivity, and shows that the inverse map is given by $\rho \mapsto \rho|_{A \otimes 1_B}$.

It is obvious that $\rho \mapsto \rho|_{A \otimes 1_B}$ is affine and is continuous for the weak* topologies. Since both tracial state spaces are compact and Hausdorff, it follows that $\sigma \mapsto \sigma \otimes \tau$ is an affine homeomorphism. \square

Proposition 6.2. Let q be any odd prime such that -1 is not a square mod q . Let D_q be the C^* -algebra of Definition 4.9. Let E be a simple separable unital nuclear stably finite C^* -algebra. Then $E \otimes D_q$ is exact and $E \otimes D_q \not\cong (E \otimes D_q)^{\text{op}}$.

Proof. Exactness follows from Proposition 7.1(iii) of [34].

Let τ be the unique tracial state on D_q (Proposition 4.10). Let R be the hyperfinite II_1 factor. We claim that

$$(6.1) \quad R \overline{\otimes} \pi_\tau(D_q)'' \cong \pi_\tau(D_q)''. \quad \square$$

To prove the claim, let ω be the unique tracial state on B_q . By Proposition 4.10, there is an isomorphism $\varphi: D_q \rightarrow B_q \otimes D_q$. Since $(\omega \otimes \tau) \circ \varphi$ is a tracial state on D_q , we have $(\omega \otimes \tau) \circ \varphi = \tau$. Therefore $\pi_{\omega \otimes \tau}(B_q \otimes D_q)'' \cong \pi_\tau(D_q)''$. Since $\pi_{\omega \otimes \tau}(B_q \otimes D_q)'' \cong \pi_\omega(B_q)'' \overline{\otimes} \pi_\tau(D_q)''$ by Lemma 5.2, and $\pi_\omega(B_q)'' \cong R$, the claim follows.

We now claim that we may assume that $E \otimes B_q \cong E$. Indeed, $B_q \otimes D_q \cong D_q$ by Proposition 4.10, so that $(E \otimes B_q) \otimes D_q \cong E \otimes D_q$. Accordingly, we may assume that E is infinite dimensional. By Corollary 9.14 of [23], there is a tracial state on E . So the Krein-Milman Theorem provides an extreme tracial state on E .

For any extreme tracial state σ on E , we have

$$(6.2) \quad \pi_\sigma(E)'' \cong R.$$

Using Lemma 5.2 at the first step, (6.2) at the second step, and (6.1) at the third step, we then get

$$(6.3) \quad \pi_{\sigma \otimes \tau}(E \otimes D_q)'' \cong \pi_{\sigma}(E)'' \overline{\otimes} \pi_{\tau}(D_q)'' \cong R \overline{\otimes} \pi_{\tau}(D_q)'' \cong \pi_{\tau}(D_q)''. \quad \square$$

Now suppose that there is an isomorphism $\psi: (E \otimes D_q)^{\text{op}} \rightarrow E \otimes D_q$. Let σ be any extreme tracial state on E . It follows from Lemma 6.1 that $\sigma \otimes \tau$ is an extreme tracial state on $E \otimes D_q$. Therefore $(\sigma \otimes \tau) \circ \psi$ is an extreme tracial state on $(E \otimes D_q)^{\text{op}} \cong E^{\text{op}} \otimes D_q^{\text{op}}$. Lemma 6.1 now provides an extreme tracial state ρ on E^{op} such that $(\sigma \otimes \tau) \circ \psi = \rho \otimes \tau^{\text{op}}$. The state ρ^{op} is clearly extreme. Using (6.3) at the first step, Lemma 2.4 at the fourth step, Lemma 5.2 at the fifth step, and (6.3) with ρ^{op} in place of σ at the sixth step, we therefore get

$$\begin{aligned} \pi_{\tau}(D_q)'' &\cong \pi_{\sigma \otimes \tau}(E \otimes D_q)'' \cong \pi_{(\sigma \otimes \tau) \circ \psi}((E \otimes D_q)^{\text{op}})'' \\ &= \pi_{\rho \otimes \tau^{\text{op}}}((E \otimes D_q)^{\text{op}})'' \cong [\pi_{\rho^{\text{op}} \otimes \tau}(E \otimes D_q)'']^{\text{op}} \\ &\cong [\pi_{\rho^{\text{op}}}(E)'' \overline{\otimes} \pi_{\tau}(D_q)'']^{\text{op}} \cong [\pi_{\tau}(D_q)'']^{\text{op}}. \end{aligned}$$

This contradicts Proposition 5.5. \square

We use Proposition 6.2 to give many examples of simple separable exact C^* -algebras not isomorphic to their opposite algebras. Many other variations are possible. The ones we give are chosen to demonstrate the possibilities of nontrivial K_1 , of K_0 being the same as that of many different UHF algebras, of real rank one rather than zero, and of having many tracial states.

Theorem 6.3. Let q be an odd prime such that -1 is not a square mod q . Then there exists a simple separable unital exact C^* -algebra A not isomorphic to its opposite algebra which is approximately divisible and stably finite, has stable rank one, tensorially absorbs the q^{∞} UHF algebra and the Jiang-Su algebra, and has the property that traces determine the order on projections over A . In addition, A has the following properties:

- (1) $K_0(A) \cong \mathbb{Z}[\frac{1}{q}]$ with $[1_A] \mapsto 1$ and $K_0(A)_+ \rightarrow \mathbb{Z}[\frac{1}{q}] \cap [0, \infty)$.
- (2) $K_1(A) = 0$.
- (3) $W(A) \cong \mathbb{Z}[\frac{1}{q}]_+ \amalg (0, \infty)$.
- (4) A has real rank zero.
- (5) A has a unique tracial state.
- (6) A satisfies the Universal Coefficient Theorem.

Proof. Take A to be the C^* -algebra D_q of Definition 4.9. Then $A \not\cong A^{\text{op}}$ by Proposition 5.5 (or equivalently by Proposition 6.2 with $E = \mathbb{C}$). All the other properties follow from Proposition 4.10 and Proposition 4.11. \square

Theorem 6.4. Let q be an odd prime such that -1 is not a square mod q . Let B be any UHF algebra whose “supernatural number” is divisible by arbitrarily large powers of q . Then there exists a C^* -algebra A as in Theorem 6.3, except that (1) and (3) are replaced by:

- (1) $K_0(A) \cong K_0(B)$ as a scaled ordered group.
- (3) $W(A) \cong K_0(B)_+ \amalg (0, \infty)$.

Proof. Let D_q be as in Definition 4.9. Take $A = B \otimes D_q$. Then exactness of A and $A \not\cong A^{\text{op}}$ follows from Proposition 6.2. Since D_q satisfies the Universal Coefficient Theorem, and B belongs to the nuclear bootstrap category, A satisfies the Universal Coefficient Theorem.

The condition on B implies that $K_0(B) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{q}] \cong K_0(B)$. Moreover, $\text{Tor}_1^{\mathbb{Z}}(K_*(B), \mathbb{Z}[\frac{1}{q}])$ is clearly zero. Since B is in the bootstrap class, the Künneth formula of [46] gives $K_0(A) \cong K_0(B)$ and $K_1(A) = 0$.

It is obvious that A is separable and unital. Simplicity of A follows from simplicity of B and D_q and nuclearity of B , by the corollary on page 117 in [47]. (We warn that this reference systematically refers to tensor products as “direct products”.) Since D_q has a unique tracial state (by Proposition 4.10), Lemma 6.1 implies that A has a unique tracial state. Combined with simplicity, this gives stable finiteness. The algebra A absorbs both the UHF algebra B_q and Z because D_q does (by Proposition 4.10). Since A is stably finite, B_q is a UHF algebra, and $B_q \otimes A \cong A$, Corollary 6.6 of [42] implies that $\text{tsr}(A) = 1$. The algebra D_q is approximately divisible by Proposition 4.10. So $A = B \otimes D_q$ is approximately divisible. Since A is simple, approximately divisible, exact, and has a unique tracial state, it has real rank zero by Theorem 1.4(f) of [2]. It follows from Proposition 2.6 of [39] that the order on projections over A is determined by traces. The computation of $W(A)$ follows from the computation of $K_0(A)$ above, $Z \otimes A \cong A$, and Remark 2.13. \square

Theorem 6.5. Let q be an odd prime such that -1 is not a square mod q . Let G be any countable abelian group. Then there exists a C^* -algebra A as in Theorem 6.3, except that (2) is replaced by:

$$(2) \quad K_1(A) \cong G \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{q}].$$

Proof. Choose, using Theorem 4.20 of [14], a simple unital AH algebra E with a unique tracial state, such that $K_0(E) \cong \mathbb{Z}[\frac{1}{q}]$, with $[1_E] \mapsto 1$, and such that $K_1(E) \cong G$. Let D_q be as in Definition 4.9. Take $A = E \otimes D_q$. Using E in place of B , proceed as in the proof of Theorem 6.4. The only difference is in the computation of $K_*(A)$. We have

$$K_0(E) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{q}] \cong \mathbb{Z}[\frac{1}{q}] \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{q}] \cong \mathbb{Z}[\frac{1}{q}],$$

and

$$\text{Tor}_1^{\mathbb{Z}}(K_0(E), \mathbb{Z}[\frac{1}{q}]) = 0 \quad \text{and} \quad \text{Tor}_1^{\mathbb{Z}}(K_1(E), \mathbb{Z}[\frac{1}{q}]) = 0.$$

So the Künneth formula of [46] implies that $K_*(A)$ is as claimed. \square

Theorem 6.6. Let q be an odd prime such that -1 is not a square mod q . Let Δ be any Choquet simplex with more than one point. Then there exists a C^* -algebra A as in Theorem 6.3, except that (3), (4), and (5) are replaced by:

- (3) $W(A) \cong \mathbb{Z}[\frac{1}{q}]_+ \amalg \text{LAff}_b(\Delta)_{++}$.
- (4) A has real rank one.
- (5) $T(A) \cong \Delta$.

Proof. Using Theorem 3.9 of [50], choose a simple unital AI algebra E such that $K_0(E) \cong \mathbb{Z}[\frac{1}{q}]$, with $[1_E] \mapsto 1$, and $T(E) \cong \Delta$. Let D_q be as in Definition 4.9. Take $A = E \otimes D_q$. Using E in place of B , proceed as in the proof of Theorem 6.4. The differences are as follows. Here, since D_q has a unique tracial state (by Proposition 4.10), Lemma 6.1 gives an affine homeomorphism from $T(E) \cong \Delta$ to $T(A)$. The computation of $W(A)$ is as before, but the answer is different because $T(A) \cong \Delta$ instead of being a point. Since there is only one state on the scaled ordered group $K_0(A)$, all tracial states must agree on all projections in A . Since Δ has more than one point, the projections in A do not distinguish the tracial states. So A does not have real rank zero by Theorem 1.4(e) in [2] and Theorem 5.11 in [22]. However, we still get $\text{tsr}(A) = 1$, so A has real rank at most 1 by Proposition 1.2 of [3]. \square

Remark 6.7. Each of Theorem 6.4, Theorem 6.5, and Theorem 6.6 (separately) gives uncountably many mutually nonisomorphic C^* -algebras satisfying the Universal Coefficient Theorem.

7. OPEN QUESTIONS

Question 7.1. Let q be an odd prime such that -1 is a square mod q . Is it still true that D_q , as in Definition 4.9, is not isomorphic to its opposite algebra?

The invariant we use, the obstruction to lifting, no longer distinguishes D_q and $(D_q)^{\text{op}}$, but this does not mean that they are isomorphic.

Even if $D_q \cong (D_q)^{\text{op}}$, different methods might give a positive answer to the following question.

Question 7.2. Let q be an odd prime such that -1 is a square mod q . Does there exist a simple separable unital exact stably finite C^* -algebra A not isomorphic to its opposite algebra such that $K_0(A) \cong \mathbb{Z}[\frac{1}{q}]$ and $K_1(A) = 0$?

Of course, we would really like to get all the other properties in Theorem 6.3 as well, in particular, unique tracial state, real rank zero, and $B_q \otimes A \cong A$.

Question 7.3. Does there exist a simple separable unital exact stably finite C^* -algebra A not isomorphic to its opposite algebra such that $K_0(A) \cong \mathbb{Z}$, with $[1_A] \mapsto 1$, and $K_1(A) = 0$?

Such an algebra would have no nontrivial projections.

Question 7.4. Does there exist a simple separable unital purely infinite C^* -algebra A not isomorphic to its opposite algebra?

Question 7.5. Does there exist a simple separable unital nuclear C^* -algebra A not isomorphic to its opposite algebra?

Under an additional axiom of set theory, nonseparable examples are known [17]. By classification, a separable example can't both absorb the Jiang-Su algebra tensorially and satisfy the Universal Coefficient Theorem.

ACKNOWLEDGMENTS

Some of this work was carried out during a summer school held at the CRM in Bellaterra, during a visit to the Fields Institute in Toronto, and during a visit by the second author to the University of Oregon. Both authors are grateful to the CRM and the Fields Institute for their hospitality. The second author would also like to thank the University of Oregon for its hospitality.

REFERENCES

- [1] D. Avitzour, *Free products of C^* -algebras*, Trans. Amer. Math. Soc. **271** (1982), 423–435.
- [2] B. Blackadar, A. Kumjian, and M. Rørdam, *Approximately central matrix units and the structure of non-commutative tori*, K-Theory **6** (1992), 267–284.
- [3] L. G. Brown and G. K. Pedersen, *C^* -algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131–149.
- [4] N. P. Brown, F. Perera, and A. S. Toms, *The Cuntz semigroup, the Elliott conjecture, and dimension functions on C^* -algebras*, J. Reine Angew. Math. **621** (2008), 191–211.
- [5] J. Castillejos, S. Evington, A. Tikuisis, S. White, and W. Winter, *Nuclear dimension of simple C^* -algebras*, Invent. Math. **224** (2021), 245–290.
- [6] A. Connes, *Almost periodic states and factors of type III_1* , J. Funct. Anal. **16** (1974), 415–445.
- [7] A. Connes, *Sur la classification des facteurs de type II*, C. R. Acad. Sci. Paris Sér. A **281** (1975), 13–15.
- [8] A. Connes, *A factor not anti-isomorphic to itself*, Ann. Math. (2) **101** (1975), 536–554.
- [9] A. Connes, *Classification of injective factors: Cases II_1 , II_∞ , III_λ , $\lambda \neq 1$* , Ann. Math. (2) **104** (1976), 73–115.
- [10] A. Connes, *Periodic automorphisms of the hyperfinite factor of type II_1* , Acta Sci. Math. (Szeged) **39** (1977), 39–66.
- [11] J. Cuntz and G. K. Pedersen, *Equivalence and traces on C^* -algebras*, J. Funct. Anal. **33** (1979), 135–164.
- [12] K. Dykema, *Exactness of reduced amalgamated free product C^* -algebras*, Forum Math. **16** (2004), 161–180.
- [13] K. J. Dykema and M. G. Viola, *Non-outer conjugate \mathbb{Z}_{p^2} -actions on free product factors*, J. Operator Theory **57** (2007), 267–301.
- [14] G. A. Elliott and G. Gong, *On the classification of C^* -algebras of real rank zero, II*, Ann. Math. (2) **144** (1996), 497–610.
- [15] G. A. Elliott, G. Gong, H. Lin, and Z. Niu, *On the classification of simple amenable C^* -algebras with finite decomposition rank, II*, preprint (arXiv: 1507.03437v3 [math.OA]).
- [16] D. E. Evans and Y. Kawahigashi, *Quantum Symmetries on Operator Algebras*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.
- [17] I. Farah and I. Hirshberg, *Simple nuclear C^* -algebras not isomorphic to their opposites*, Proc. Natl. Acad. Sci. USA **114** (2017), 6244–6249.
- [18] E. Gardella, *Crossed products by compact group actions with the Rokhlin property*, J. Noncommutative Geom. **11** (2017), 1593–1626.
- [19] E. Gardella, *Equivariant KK -theory and the continuous Rokhlin property*, preprint (arXiv:1406.1208v3 [math.OA]).
- [20] E. Gardella, I. Hirshberg, and L. Santiago, *Rokhlin dimension: duality, tracial properties, and crossed products*, Ergodic Th. Dynam. Syst. **41** (2021), 408–460.
- [21] E. Germain, *KK -theory of reduced free product C^* -algebras*, Duke Math. J. **82** (1996), 707–723.

- [22] U. Haagerup, *Quasitraces on exact C^* -algebras are traces*, C. R. Math. Rep. Acad. Sci. Canada **36** (2014), 67–92.
- [23] U. Haagerup and S. Thorbjørnsen, *Random matrices and K -theory for exact C^* -algebras*, Documenta Math. **4** (1999), 341–450 (electronic).
- [24] K. Hasegawa, *KK -equivalences for amalgamated free product C^* -algebras*, Int. Math. Res. Not. **2016** (2016), 7619–7636.
- [25] I. Hirshberg and W. Winter, *Rokhlin actions and self-absorbing C^* -algebras*, Pacific J. Math. **233** (2007), 125–143.
- [26] T. Houghton-Larsen and K. Thomsen, *Universal (co) homology theories*, K-theory **16** (1999), 1–27.
- [27] M. Izumi, *Finite group actions on C^* -algebras with the Rokhlin Property, I* Duke Math. J. **122** (2004), 233–280.
- [28] M. Izumi, *Finite group actions on C^* -algebras with the Rohlin property. II*, Adv. Math. **184** (2004), 119–160.
- [29] X. Jiang and H. Su, *On a simple unital projectionless C^* -algebra*, Amer. J. Math. **121** (1999), 359–413.
- [30] V. F. R. Jones, *Notes on the Connes invariant $\chi(M)$* , unpublished notes.
- [31] V. F. R. Jones, *A II_1 factor anti-isomorphic to itself but without involutory antiautomorphism*, Math. Scand. **46** (1980), 103–117.
- [32] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Vol. II. Advanced theory, Graduate Studies in Mathematics, 16, American Mathematical Society, Providence, RI, 1997.
- [33] R. R. Kallman, *A generalization of free action*, Duke Math. J. **36** (1969), 781–789.
- [34] E. Kirchberg, *Commutants of unitaries in UHF algebras and functorial properties of exactness*, J. reine angew. Math. **452** (1994), 39–77.
- [35] E. Kirchberg and N. C. Phillips, *Embedding of exact C^* -algebras in the Cuntz algebra \mathcal{O}_2* , J. reine angew. Math. **525** (2000), 17–53.
- [36] A. Kishimoto, *Outer automorphisms and reduced crossed products of simple C^* -algebras*, Commun. Math. Phys. **81** (1981), 429–435.
- [37] H. Lin, *Tracially AF C^* -algebras*, Trans. Amer. Math. Soc. **353** (2001), 693–722.
- [38] H. Osaka and N. C. Phillips, *Crossed products by finite group actions with the Rokhlin property*, Math Z. **270** (2012), 19–42.
- [39] N. C. Phillips and M. G. Viola, *A simple separable exact C^* -algebra not anti-isomorphic to itself*, Math. Ann. **355** (2013), 783–799.
- [40] F. Rădulescu, *An invariant for subfactors in the von Neumann algebra of a free group*, pages 213–239 in: *Free probability theory (Waterloo, ON, 1995)*, Fields Inst. Commun. vol. 12, Amer. Math. Soc., Providence, RI, 1997.
- [41] M. Rørdam, *The stable and the real rank of \mathcal{Z} -absorbing C^* -algebras*, Internat. J. Math. **15** (2004), 1065–1084.
- [42] M. Rørdam, *On the structure of simple C^* -algebras tensored with a UHF-algebra*, J. Funct. Anal. **100** (1991), 1–17.
- [43] M. Rørdam, *A short proof of Elliott’s Theorem: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$* , C. R. Math. Rep. Acad. Sci. Canada **16** (1994), 31–36.
- [44] J. Rosenberg, *Appendix to O. Bratteli’s paper on “Crossed products of UHF algebras”*, Duke Math. J. **46** (1979), 25–26.
- [45] J. Rosenberg and C. Schochet, *The Künneth theorem and the Universal Coefficient Theorem for Kasparov’s generalized K -functor*, Duke Math. J. **55** (1987), 431–474.
- [46] C. Schochet, *Topological methods for C^* -algebras II: geometric resolutions and the Künneth formula*, Pacific J. Math. **98** (1982), 443–458.
- [47] M. Takesaki, *On the cross-norm of the direct product of C^* -algebras*, Tôhoku Math. J. (2) **16** (1964), 111–122.
- [48] M. Takesaki, *Duality for crossed products and the structure of von Neumann algebras of type III*, Acta Math. **13** (1973), 249–310.
- [49] M. Takesaki, *Theory of Operator Algebras III*, Springer-Verlag, Berlin, etc., 2003.
- [50] K. Thomsen, *Inductive limits of interval algebras: The tracial state space*, Amer. J. Math. **116** (1998), 605–620.

- [51] K. Thomsen, *On the KK -theory and the E -theory of amalgamated free products of C^* -algebras*, J. Funct. Anal. **201** (2003), 30–56.
- [52] M. G. Viola, *On a subfactor construction of a factor non-antiisomorphic to itself*, Internat. J. Math. **15** (2004), 833–854.
- [53] S. Wassermann, *Tensor products of $*$ -automorphisms of C^* -algebras*, Bull. London Math. Soc. **7** (1975), 65–70.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE OR 97403-1222,
USA

DEPARTMENT OF MATHEMATICAL SCIENCES, LAKEHEAD UNIVERSITY, 500 UNIVERSITY
AVENUE, ORILLIA ON L3V 0B9, CANADA
E-mail address: mviola@lakeheadu.ca