Robust Social Welfare Maximization via Information Design in Linear-Quadratic-Gaussian Games

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***Abstract*—Information design involves a designer with the goal of influencing players' actions in an incomplete information game through signals generated from a designed probability distribution so that its objective function is optimized. We consider a setting in which the designer has partial knowledge on players' payoffs, and wants to maximize social welfare. We address the uncertainty about players' preferences by formulating a robust information design problem against the worst-case payoffs. When the players have quadratic payoffs that depend on the actions and an unknown payoff-relevant state, and signals on the state that follow a Gaussian distribution, the information design problem under quadratic design objectives can be stated as a semidefinite program (SDP) (Ui, 2020). Given this fact, we consider ellipsoid perturbations over payoff coefficients in linear-quadratic-Gaussian (LQG) games. We show that we can obtain a similar SDP formulation that approximates the social welfare maximization via robust information design. Numerical experiments identify the relation between the uncertainty level on players' payoffs and the optimal information structures.**

***Index Terms*—Information design, robust optimization, game theory.**

I. Introduction

An INCOMPLETE information game is comprised of multiple players who take actions to maximize their utilities which depend on actions of other players and unknown states. Incomplete information games are used to model federated edge learning [2], electricity spot market [3], cyber-defense in EV charging [4] and traffic flow in communication or transportation networks [5], [6].

Information design problem entails decision over informativeness of signals given to players regarding the payoff state so that induced actions maximize a system level objective. Information designer as an entity commits to an optimal probability distribution of signals conditional on the payoff states before state realization (for an example in pandemic control see Fig. 1). The selected distribution maximizes the designer's objective and adheres to equilibrium constraints. Various entities such as social media companies [7], advertisements platforms [8] and public health agencies [9] could be considered as information designers. In control systems, information design is employed for routing games [10], vehicle-to-vehicle communication [11], and queue management under heterogeneous users [12].

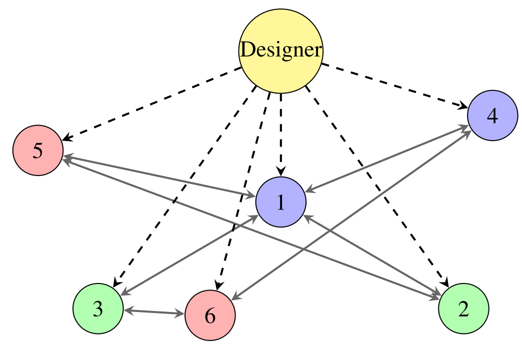

Fig. 1. Information designer sends (dashed arrows) optimally designed signals on the risks of infection from an emerging infectious disease to the players who can be susceptible (blue), infected (red) or recovered (green), so that they follow the recommended health measures, e.g., social distancing or masking that reduce the risk of an outbreak. An individual's infection or disease transmission risk is determined by its contacts (shown by solid edges)–see Example 2. For instance, player 1 (susceptible) has one infected neighbor (player 5) that it can contract the disease from.

In this letter, we propose a robust optimization approach to the information design problem considering the fact that the designer cannot know the players' payoffs exactly. Indeed, while the designer may be knowledgeable about the payoff relevant random state, it may have uncertainty about the payoff coefficients of the players. For instance, in the pandemic control example in Fig. 1, while the public health department may have near-certain information about the potential risks of a disease or intervention, it may not know how the society weights the risks and benefits in their decision-making. Here, we assume the designer has partial knowledge about the players' payoffs, and wants to perform information design over the payoff relevant states.

When the payoffs of the players are unknown, the designer cannot be sure of the rational behavior under a chosen information structure. We formulate this problem as a robust optimization problem where the designer chooses the "best"

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optimal information structure for the worst possible realization of the payoffs. That is, we do not make any assumptions on the distribution of the players' payoff coefficients. Specifically, we assume the players have linear-quadratic payoffs with coefficients unknown by the designer. We further assume that the payoff relevant states and signals generated by the designer come from a Gaussian distribution. In this setting, we show that the robust information design with the goal to maximize social welfare can be approximated with a SDP given ellipsoid perturbations on the payoff coefficients (Theorem 2). The SDP formulation provides a distribution over the actions that the designer can send as signals. The approximation stems from the loss of the obedience condition (incentive compatibility) on the actions suggested by the designer for the realizations of the payoffs other than the worst-case.

In Bayesian persuasion, in which there is a single player [13], robustness is explored in the worst-case, online and various other settings [14], [15], [16], [17], [18]. For instance, [19] considers information design where the designer learns unknown payoffs via auctions. Instead, here we consider the multi-player setting, i.e., information design, and assume the game is unknown. In our setting, the designer maximizes the worst-case objective given rational behavior.

Notation: We use $A_{i,j}$ to denote the element in the ith row and jth column of matrix A. We use $\bullet$ to represent the Frobenius product, e.g., $A \bullet B = \sum_{i=1}^{m}\sum_{j=1}^{m} A_{i,j}B_{i,j}$ for $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{m\times m}$. We represent the set of $m \times m$ symmetric and symmetric positive semi-definite matrices using P^m and P^m_+, respectively. $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. I and O indicate the identity and zero matrices, respectively. $\mathbf{1}$ is a column vector of all ones.

II. Generic Robust Information Design Problem for Welfare Maximization

An incomplete information game involves a set of $n \in \mathbb{N}^+$ players belonging to the set $\mathcal{N} := \{1, \ldots, n\}$, each of which selects actions $a_i \in \mathcal{A}_i$ to maximize the expectation of its payoff function $u_i^\theta(a, \gamma)$ where $a \equiv (a_i)_{i\in\mathcal{N}} \in \mathcal{A}$ is the action profile, $\gamma \equiv (\gamma_i)_{i\in\mathcal{N}} \in \Gamma$ is the payoff state vector, and $\theta \in \Theta$ is a payoff parameter. Players know the payoff parameter θ, but they do not know the payoff state γ. Player i forms expectation about the payoff state γ based on the prior on the state ψ and its signal/type $\omega_i \in \Omega_i$.

The information designer does not know the payoff parameter θ, but is more informed about the payoff state γ than the players. Specifically, an information designer aims to maximize a system level objective function $f^\theta : \mathcal{A} \times \Gamma \to \mathbb{R}$, e.g., social welfare, that depends on the actions of the players (a), and the state realization (γ) by deciding on an information structure ζ belonging to the set of probability distributions over the signals $\mathcal{Z}$. That is, ζ is a conditional probability on the signals $\{\omega_i\}_{i\in\mathcal{N}}$ given the payoff state vector γ, i.e., $(P(\omega \,|\, \gamma))$ belonging to the space of all such conditional probability distributions $\mathcal{Z}$. The information structure determines the fidelity of signals $\{\omega_i\}_{i\in\mathcal{N}}$ that will be revealed to the players given a realization of the payoff state vector γ.

We introduce social welfare as a design objective.

Definition 1 (Social Welfare): Social welfare design objective is the sum of individual utility functions,

$$f^\theta(a, \gamma) = \sum_{i=1}^{n} u_i^\theta(a, \gamma). \tag{1}$$

Social welfare is a common design objective used in congestion [6], global [20] or public goods games [9].

We represent the incomplete information game given $\theta \in \Theta$ and a prior ψ on the state γ by the tuple $G_\theta := \{\mathcal{N}, \mathcal{A}, \Gamma, \{u_i^\theta\}_{i\in\mathcal{N}}, \{\omega_i\}_{i\in\mathcal{N}}, \zeta, \psi\}$. We use $\mathcal{G}_\Theta := \{G_\theta : \theta \in \Theta\}$ to refer to the set of possible games.

A strategy of player i maps each possible value of the private signal $\omega_i \in \Omega_i$ to an action $s_i(\omega_i) \in \mathcal{A}_i$, i.e., $s_i : \Omega_i \to \mathcal{A}_i$. A strategy profile $s = (s_i)_{i\in\mathcal{N}}$ is a Bayesian Nash equilibrium (BNE) with information structure ζ of the game G_θ, if it satisfies the following inequality

$$E_\zeta[u_i^\theta(s_i(\omega_i), s_{-i}, \gamma)|\omega_i] \geq E_\zeta[u_i^\theta(a_i', s_{-i}, \gamma)|\omega_i], \tag{2}$$

for all $a_i' \in \mathcal{A}_i, \omega_i \in \Omega_i, i \in \mathcal{N}$, and $s_{-i} = (s_j(\omega_j))_{j\neq i}$ is the equilibrium strategy of all the players except player i, and E_ζ is the expectation operator with respect to the distribution ζ and the prior ψ. We denote the set of BNE strategies in a game G_θ with $BNE(G_\theta)$.

In this letter, the designer does not make any distributional assumptions on the payoff parameter θ, and aims to select the best signal distribution for the worst case scenario, i.e.,

$$\min_{\theta\in\Theta}\max_{\zeta\in\mathcal{Z}} E_\zeta[f^\theta(s, \gamma)] \quad \text{s.t.} \quad s \in BNE(G_\theta). \tag{3}$$

The outer optimization problem in (3) evaluates to the designer's objective under the worst possible payoff parameter realization, and BNE actions given a signal distribution ζ. The designer wants to do the best it can to maximize the system objective assuming the realization of the worst-case scenario. We note that the information design problem is not a Stackelberg (leader-follower) game, since the players are not strategic against the designer's strategy and objective [13].

We denote the optimal solution to (3) by ζ^*. Given the robust optimal information structure ζ^*, the information design timeline is given in the following:

1) Designer notifies players about ζ^*
2) Realization of the payoff parameter θ and payoff state γ, and subsequent draw of signals w_i, $\forall i \in \mathcal{N}$ from $\zeta^*(\omega, \gamma)$
3) Players take action according to BNE strategies under information structure ζ^* in game G_θ.

The generic robust information design problem in (3) is not tractable in general. In the following we make assumptions on the payoff structure and the signal distribution to attain a tractable formulation.

A. Linear-Quadratic-Gaussian (LQG) Games

An LQG game corresponds to an incomplete information game with quadratic payoff functions and Gaussian information structures. Specifically, each player $i \in \mathcal{N}$ decides on its action $a_i \in \mathcal{A}_i \equiv \mathbb{R}$ according to a payoff function

$$u_i^\theta(a, \gamma) = -H_{i,i}a_i^2 - 2\sum_{j\neq i} H_{i,j}a_i a_j + 2\gamma_i a_i \tag{4}$$

where $\mathcal{A} \equiv \mathbb{R}^n$ and $\Gamma \equiv \mathbb{R}^n$ that is a quadratic function of player i's action, and is bilinear with respect to a_i and a_j, and a_i and γ. We collect the coefficients of the quadratic payoff function in a matrix $H = [H_{i,j}]_{n\times n}$. The payoff parameter θ, unknown to the designer in (4), is the coefficients matrix H, i.e., $\theta \equiv H$. We note that the utility in (4) can have other terms that depend on a_{-i} or γ, but not on player i's action a_i.

Payoff state γ follows a Gaussian distribution, i.e., $\gamma \sim \psi(\mu, \Sigma)$ where ψ is a multivariate normal probability distribution with mean $\mu \in \mathbb{R}^n$ and covariance matrix Σ. Each player $i \in \mathcal{N}$ receives a private signal $\omega_i \in \Omega_i \equiv \mathbb{R}$. We define the information structure of the game $\zeta(\omega|\gamma)$ as the conditional distribution of $\omega \equiv (\omega_i)_{i\in N}$ given γ. We assume the joint distribution over the random variables (ω, γ) is Gaussian; thus, ζ is a Gaussian distribution.

Next, we provide two examples of LQG games.

Example 1 (Beauty Contest Game): Payoff function of player i is given by

$$u_i^\theta(a, \gamma) = -(1-\theta)(a_i - \gamma)^2 - \theta(a_i - \bar{a}_{-i})^2, \tag{5}$$

where $\theta \in [0, 1]$ and $\bar{a}_{-i} = \sum_{j\neq i} a_j/(n-1)$ represents the average action of other players. The first term in (5) represents the players' urge to take actions close to the payoff state γ. The second term accounts for players' tendency towards taking actions in compliance with others. The constant θ gauges the importance between the two terms. The payoff captures settings where the valuation of a good depends on both the performance of the company and what others think about its value [20].

Example 2 (Social Distancing Game): Player i's action $a_i \in \mathbb{R}^+ \cup \{0\}$ is its social distancing effort to avoid the infectious disease contraction/transmission (see also Fig. 1). The risk of infection depends on unknown disease specific parameters, e.g., severity, infection rate, and the social distancing actions individuals in contact with player i. We define the payoff function of player i as follows,

$$u_i^\theta(a, \gamma) = -H_{i,i}a_i^2 - (1 - \delta_i a_i) r_i(a, \gamma) \tag{6}$$

where the risk of infection is $r_i := \gamma - 2\sum_{i\neq j} H_{i,j}a_j$, $0 < \delta_i < 1$ is the risk reduction coefficient. In the definition of risk r_i, γ denotes the risk rate of the disease such as infection rate or severity, and $H_{i,j}$ determines the contacts of player i and the intensity of the contacts. The first term in (6) represents the cost of social distancing. The second term in (6) denotes the overall risk of infection that scales with the player's social distancing efforts.

In the examples above, there is a common payoff state, i.e., $\gamma_i = \gamma$ for $i \in \mathcal{N}$.

Next we state the main structural assumption on the unknown payoff parameter H of the LQG game.

Assumption 1: We assume the following affine perturbation structure on the payoff matrix H,

$$H_{i,j} = [H_0]_{i,j} + v_{i,j}\epsilon_{i,j}, \quad \forall i, j \in \mathcal{N} \tag{7}$$

where H_0 is the nominal payoff matrix, $v_{i,j} \in \mathbb{R}$, is an element of the unknown perturbation matrix $v \in \mathbb{R}^{n\times n}$ which covers a given closed and convex perturbation set $\mathcal{V}$ such that $0 \in \mathcal{V}$ and $\epsilon_{i,j}$ is the constant shift.

We note that while the actual payoff parameters H are unknown to the designer, they are known by the players. The designer only knows the nominal payoff matrix H_0, potentially obtained from past data.

B. From Signal to Action Distributions

We define the distribution of actions induced by the information structure under a given strategy profile as follows.

Definition 2 (Action Distribution): An action distribution is the probability of observing an action profile $a \in \mathcal{A}$ when players follow a strategy profile s under ζ, which can be computed as

$$\phi(a|\gamma) = \sum_{\omega:s(\omega)=a} \zeta(\omega|\gamma). \tag{8}$$

According to the definition, the probability of observing the action profile a is the sum of the conditional probabilities of all signal profiles ω under ζ that induce action profile a given the strategy profile s.

Definition 3 (Equilibrium Action Distribution Set): The set of *equilibrium* action distributions induced by BNE strategies under an information structure $\zeta \in \mathcal{Z}$ for game G_θ is

$$B_\theta(\zeta) = \{\phi : \phi \text{ satisfies (8) for } s \in BNE(G_\theta) \text{ given } \zeta \in \mathcal{Z}\}. \tag{9}$$

We begin by stating the BNE condition in (2) by a set of linear constraints for LQG games given the payoff matrix H.

Lemma 1: Define the covariance matrix $X \in P_+^{2n}$ as

$$X := \begin{bmatrix} var(a) & cov(a, \gamma) \\ cov(\gamma, a) & var(\gamma) \end{bmatrix}. \tag{10}$$

For a given payoff matrix H where $H+H^T$ is positive definite, the BNE condition in (2) can be written as the following set of equality constraints,

$$\sum_{j\in\mathcal{N}} H_{i,j}X_{i,j} - X_{i,n+i} = 0, \quad i \in \mathcal{N} \tag{11}$$

where $X_{i,j} = cov(a_i, a_j)$ for $i \le n$, and $j \le n$, and $X_{i,n+i} = cov(a_i, \gamma_i)$.

Proof: See the Appendix. ■

The condition in (11) ensures that X is a Bayesian correlated equilibrium (BCE)—see [21] for a definition. When θ is known, we can state the designer's maximization problem in (3) as the determination of an action distribution subject to the constraint that actions belong to $B_\theta(\zeta)$, i.e., $\max_{\phi\in B_\theta(\zeta)} E_\phi[f(a, \gamma)]$. Indeed, we can state the design problem as a SDP using X in (10) as the decision variable, subject to the BCE constraints in (11)—see [1]. In such a case, the players would not benefit from deviating from the recommended actions because they would satisfy the obedience condition as per the revelation principle, see [21, Proposition 1]. However, this principle does not apply in the setting where θ is chosen adversarially. Next, we address this issue in the finite scenario and ellipsoid perturbation settings.

III. Robust Information Design Under Finite Scenarios

In the following, we express the robust information design problem under a finite set of scenarios as a mixed integer SDP using action distributions.

Theorem 1 (Finite-Case): Suppose Assumption 1 holds. Let the design objective $f^\theta(a, \gamma)$ be quadratic in its arguments with the coefficients stored in matrix $F \in \mathbb{R}^{2n\times 2n}$, i.e., $f^\theta(a, \gamma) = [a \quad \gamma]^T F [a \quad \gamma]$. Assume the design objective coefficients do not depend on H. Consider a finite perturbation vector with C scenarios, and let $v_c \in \mathbb{R}^{n\times n}$ refer to perturbation vectors corresponding to one of the scenarios $c \in \mathcal{C} = \{1, \ldots, C\}$. The following mixed-integer SDP formulation relaxes the BCE conditions in the robust information design problem (3):

$$\min_{y_c\in\{0,1\}, c\in\mathcal{C}} \max_{X\in P_+^{2n}} F \bullet X \tag{12}$$

$$\text{s.t.} \quad \sum_{c=1}^{C} y_c = 1, \tag{13}$$

$$y_c(R_{0,l} \bullet X + \sum_{(i,j)\in\mathcal{Y}_l} [v_c]_{i,j}\epsilon_{i,j}X_{i,j}) = 0, \forall l \in \mathcal{N}, c \in \mathcal{C} \tag{14}$$

$$M_{k,l} \bullet X = cov(\gamma_k, \gamma_l), \quad \forall k, l \in \mathcal{N} \text{ with } k \le l, \tag{15}$$

where X is defined in (10), $R_{0,l} \in P^{2n}, l \in \mathcal{N}$ is given as:

$$[R_{0,l}]_{i,j} = \begin{cases} [H_0]_{l,l} & \textit{if} \quad i = j = l, \\ [H_0]_{l,j}/2 & \textit{if} \quad i = l, 1 \le j \le n, j \ne l, \\ -1/2 & \textit{if} \quad i = l, j = n + l, \\ [H_0]_{i,l}/2 & \textit{if} \quad j = l, 1 \le i \le n, i \ne l \\ -1/2 & \textit{if} \quad j = l, i = n + l, \\ 0 & \text{otherwise}, \end{cases} \tag{16}$$

and $M_{k,l} \in P^{2n}$ is given as:

$$[M_{k,l}]_{i,j} = \begin{cases} 1/2 & \text{if } k < l, i = n + k, j = n + l, \\ 1/2 & \text{if } k < l, i = n + l, j = n + k, \\ 1 & \text{if } k = l, i = n + k, j = n + l, \\ 0 & \text{otherwise}, \end{cases} \tag{17}$$

and $\mathcal{Y}_l$ refer to the elements of the perturbation vector with

$$\mathcal{Y}_l := \{\{i, j\} : i = j = l \vee i = l, 1 \le j \le n, j \ne l \\ \vee j = l, 1 \le i \le n, i \ne l\}. \tag{18}$$

Proof: We can express the expected quadratic objective using the Frobenius product as follows,

$$E_\phi[f(a, \gamma)] = E_\phi\left[\begin{bmatrix} a^T, & \gamma^T \end{bmatrix} F \begin{bmatrix} a \\ \gamma \end{bmatrix}\right] \tag{19}$$

$$= F \bullet X \tag{20}$$

where $F = \begin{bmatrix} [F]_{1,1} & [F]_{1,2} \\ [F]_{1,2} & [F]_{2,2} \end{bmatrix} \in P^{2n}$, and note that $[F]_{i,j}$ denotes the i, jth $n \times n$ submatrix.

Let c^* be the worst-case scenario from the perspective of the designer. The designer chooses X^* that maximizes its objective $F \bullet X$ subject to rational behavior of players in the worst case scenario. As per Lemma 1, we have

$$\sum_{j\in\mathcal{N}} H_{i,j}X^*_{i,j} - X^*_{i,n+i} = 0, \quad \forall i \in \mathcal{N} \tag{21}$$

$$\sum_{j\in\mathcal{N}} ([H_0]_{i,j} + [v_{c^*}]_{i,j}\epsilon_{i,j})X^*_{i,j} - X^*_{i,n+i} = 0, \forall i \in \mathcal{N}. \tag{22}$$

We rewrite (22) in terms of matrices $R_{0,l}, \forall l \in \mathcal{N}$ as in (16) and X as in (10) to obtain (14). Minimization over $y_c, \{1, 2, .., C\}$ enforces the constraint c^* among the set of constraints in (14) to be selected. Constraint (15) corresponds to the assignment of $var(\gamma)$ to $[X]_{2,2}$. Constraint (15) is not affected by perturbations to H. While X^* satisfies the BCE condition in (11) for c^*, it does not satisfy it for $c \in \mathcal{C} \setminus c^*$. ■

According to the formulation in (12)-(15), the solution entails finding the covariance matrix X that maximizes $F \bullet X$ for the worst possible scenario. We note that an alternative equivalent formulation can entail C covariance matrices, i.e., $X_1, \ldots, X_C$, and leave out the integer variables $\{y_c\}_{c=1,\ldots,C}$.

The formulation in (12)-(15) relaxes the BCE condition for scenarios that are not the worst-case. For illustration purposes, consider $C = 2$ scenarios. Assume $c = 1$ is the worst case scenario, i.e., $y_1 = 1$ and $y_2 = 0$. In such a case, X^* will be a BCE for $c = 1$ exactly, while the BCE condition in (11) will be approximately satisfied for $c = 2$. Specifically, we have

$$\sum_{j\in\mathcal{N}} ([H_0]_{i,j} + [v_2]_{i,j}\epsilon_{i,j})X^*_{i,j} - X^*_{i,n+i}$$

$$= \sum_{j\in\mathcal{N}} ([H_0]_{i,j} + [v_2]_{i,j}\epsilon_{i,j} + [v_1]_{i,j}\epsilon_{i,j} - [v_1]_{i,j}\epsilon_{i,j})X^*_{i,j} - X^*_{i,n+i} \tag{23}$$

$$= \sum_{j\in\mathcal{N}} ([v_2]_{i,j}\epsilon_{i,j} - [v_1]_{i,j}\epsilon_{i,j})X^*_{i,j} > 0, \quad \forall i \in \mathcal{N}. \tag{24}$$

We can interpret this relation as the optimal solution X^* being an approximate BNE for the good scenario $c = 2$.

Remark 1: The standard robust optimization problem in (3) requires that X^* is feasible for every $\theta \in \Theta$. The formulation for this problem would entail getting rid of the integer variables from the formulation in (12)-(15), i.e., $y_c = 1$ for each (14) and removing (13). This formulation may restrict the feasibility region drastically, as is often the issue with robust optimization problems with equality constraints [22].

IV. Robust Welfare Maximizing Information Design Under Ellipsoid Uncertainty

We assume the following ellipsoidal structural form for the perturbation vectors in (7) for $l \in \mathcal{N}$,

$$\mathcal{V}_l = \text{Ball}_\rho = \{v : ||v_l||_2 \le \rho, v_l = \{[v_l]_{i,j}\}_{\{i,j\}\in\mathcal{Y}_l}\}, \tag{25}$$

where $\rho \ge 0$ and $\mathcal{Y}_l$ is given in (18). Under convex continuous uncertainty sets as the one above, the number of scenarios C is infinite. Thus, the formulation in Theorem 1 where we enforce BCE constraints in (11) exactly for the worst-case scenario, and annul the other cases using integer variables is not viable. Moreover, enforcing the BCE constraints in (11) for all perturbations $v \in \mathcal{V}_l$ may limit the solution space drastically as per Remark 1. Instead, here we relax the BCE constraint in (11) as follows

$$|\sum_{j\in\mathcal{N}} H_{i,j}X_{i,j} - X_{i,n+i}| \le \alpha, \quad i \in \mathcal{N} \tag{26}$$

where $\alpha \ge 0$ is a finite constant. This relaxation guarantees an approximate tractable solution to the information design problem in which the designer aims to maximize social welfare under ellipsoidal perturbations.

Theorem 2: Consider the social welfare in (1) as the designer's objective $f^\theta(a, \gamma)$. Assume H is given by (7) and perturbation vectors $v_l, \forall l \in \mathcal{N}$ exhibit ellipsoid uncertainty (25). Consider the following SDP for $\alpha \ge 0$:

$$\max_{X\in P^{2n}_+, t} \quad t \tag{27}$$

$$\text{s.t.} \quad F_0 \bullet X - \frac{n^2\rho}{2n-1}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(\epsilon_{i,j}X_{i,j})^2} \ge t, \tag{28}$$

$$R_{0,l} \bullet X + \rho\sqrt{\sum_{(i,j)\in\mathcal{Y}_l}(\epsilon_{i,j}X_{i,j})^2} \le \alpha, \quad \forall l \in \mathcal{N} \tag{29}$$

$$-R_{0,l} \bullet X + \rho\sqrt{\sum_{(i,j)\in\mathcal{Y}_l}(\epsilon_{i,j}X_{i,j})^2} \le \alpha, \quad \forall l \in \mathcal{N} \tag{30}$$

$$M_{k,l} \bullet X = cov(\gamma_k, \gamma_l), \quad \forall k, l \in \mathcal{N} \text{ with } k \leq l, \tag{31}$$

where the matrices $R_{0,l}$ and $M_{k,l}$ are as defined in (16) and (17), respectively, and $F_0 = \begin{bmatrix} -H_0 & I \\ I & O \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$. If the worst-case scenario is realized and $\alpha = 0$, then the designer's objective value $F \bullet X^*$, where X^* is an optimal solution to (27)-(31), attains the optimal objective value for (3).

Proof: We can express the social welfare objective in (1) in the form $F \bullet X$ with $F = \begin{bmatrix} -H & I \\ I & O \end{bmatrix}$—see [23]. We start by writing the social welfare objective as a constraint $F \bullet X \geq t$ under ellipsoid uncertainty:

$$F \bullet X = F_0 \bullet X + \sum_{i=1}^{n}\sum_{j=1}^{n} v_{i,j}\epsilon_{i,j}X_{i,j} \geq t \tag{32}$$

where t represents the designer's objective value. In the above summation, all elements of the perturbation matrix v are involved. Given the assumption of ellipsoid perturbations in (25), it is guaranteed that v is within a ball of radius $\frac{n^2\rho}{2n-1}$, i.e., $v \in \text{Ball}_{\frac{n^2\rho}{2n-1}}$. We can write (32) as a minimization problem that aims to find the worst case scenario:

$$\min_{||v|| \leq \frac{n^2\rho}{2n-1}} \sum_{i=1}^{n}\sum_{j=1}^{n} v_{i,j}\epsilon_{i,j}X_{i,j} \leq F_0 \bullet X - t \tag{33}$$

Solution to (33) is the tractable robust constraint given in (28) [22, Sec. 1.3]. Next, we substitute H in (7) into (26),

$$\left| \sum_{j\in\mathcal{N}} ([H_0]_{ij} + v_{ij}\epsilon_{ij})X_{i,j} - X_{i,n+i} \right| \leq \alpha \quad \forall i \in \mathcal{N}, v \in \mathcal{V}_l. \tag{34}$$

We can rewrite (34) in terms of matrices $R_{0,l}, \forall l \in \mathcal{N}$ and X as in (10):

$$\left| R_{0,l} \bullet X + \sum_{(i,j)\in\mathcal{Y}_l} v_{i,j}\epsilon_{i,j}X_{i,j} \right| \leq \alpha, \quad \forall l \in \mathcal{N}. \tag{35}$$

We split the absolute value into two linear constraints (positive and negative sides). When we write the maximization problem over the uncertain constraint (35) for the positive side, we have

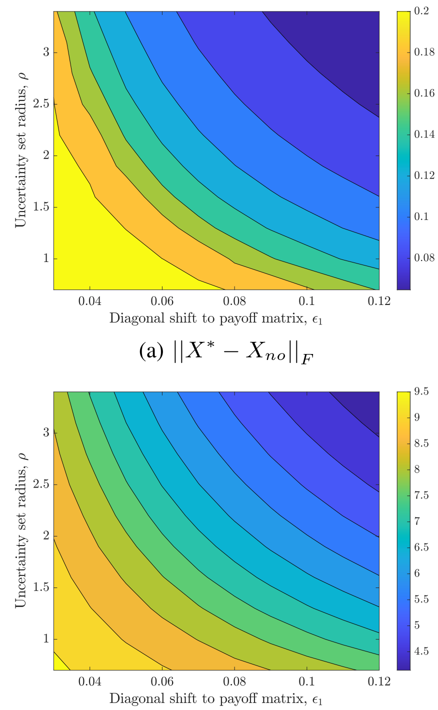
$$\max_{||v_l|| \leq \rho} \sum_{(i,j)\in\mathcal{Y}_l} v_{i,j}\epsilon_{i,j}X_{i,j} \leq \alpha - R_{0,l} \bullet X, \quad \forall l \in \mathcal{N} \tag{36}$$

where $\mathcal{Y}_l$ is given in (18). Solution to (36) gives the tractable constraint (29) [22, Sec. 1.3]. Repeating the same steps for the negative side yields (30). Constraint (31) enforces assignment of known covariance matrix of payoff states, $cov(\gamma)$ to the corresponding parts of X.

When $\alpha = 0$ and H is such that $||v||_2 = \frac{n^2\rho}{2n-1}$, the designer's objective is equal to the solution to (3) with H in (7). ■

It is easy to check that no information disclosure $X_{no} = \begin{bmatrix} O & O \\ O & \text{var}(\gamma) \end{bmatrix}$ is a feasible solution even when $\alpha = 0$—see [1] for the derivation of X_{no}. As noted in Remark 1, formulation with $\alpha = 0$ may be too restrictive. When $\alpha > 0$, the incentive compatibility of the solution X^* is compromised, but the set of feasible solutions increases. We also note that the BCE conditions in (11) for the scenarios that are not the worst case are not satisfied, as was the case in the finite scenarios setting.

Given an optimal solution X^*, the designer can draw actions from a Gaussian distribution with mean 0 and covariance matrix X^*, and send these values to the players as signals.



(a) $||X^* - X_{no}||_F$

(b) Optimal objective value

Fig. 2. Contour plots of (a) normalized Frobenius matrix norm distance $||X^* - X_{no}||_F$ between the optimal covariance matrix (X^*) and no information disclosure covariance matrix (X_{no}), and (b) optimal objective value with respect to uncertainty ball radius ρ and diagonal shift ϵ_1 to coefficient matrix H under a symmetric supermodular game with social welfare objective. Optimal solution X^* approaches to no information disclosure as ρ and ϵ_1 increase.

V. Numerical Experiments

We consider a designer that wants to maximize the social welfare of $n = 5$ players. The designer knows the nominal payoff matrix given as follows: $[H_0]_{i,i} = 5$ for $i \in \{1, \ldots, 5\}$, and $[H_0]_{i,j} = -1$ for $i \neq j$, $i, j \in \{1, 2, .., 5\}$. The variance of the unknown payoff state γ is given as follows: $var(\gamma)_{i,i} = 5$ for $i = \{1, \ldots, 5\}$, and $var(\gamma)_{i,j} = 0.5$ for $i \neq j$, $i, j \in \{1, 2, ., 5\}$. We consider ellipsoid perturbations with $\rho \in \{0.7, 1, 1.3, .., 3.4\}$ and let $\alpha = 0.1$. Given the setup, we solve (27)-(31) in order to obtain the optimal information design X^*.

We analyze the effects of shifts $\epsilon_{i,j}$ defined in (7) by assuming the diagonal elements and off-diagonal elements of shift matrix are homogeneous, i.e., $\epsilon_{i,i} = \epsilon_1$ and $\epsilon_{i,j} = \epsilon_2$ for all $i, j = 1, \ldots, n$ for constants ϵ_1 and ϵ_2.

In order to systematically analyze the effects of the shifts, we fix the off-diagonal shifts to a small value $\epsilon_2 = 0.001$, and vary the diagonal shift $\epsilon_1 \in \{0.03, 0.04, 0.05, .., 0.12\}$. Fig. 2(a) shows that as the uncertainty ball radius ρ and diagonal shift ϵ_1 increases, the optimal information structure remains a partial information disclosure but gets closer to the no information disclosure. Fig. 2(b) shows that social welfare decreases under increasing uncertainty.

We can discuss Fig. 2 in terms of the beauty contest game, which is a supermodular game. If we consider the common goods in the beauty contest game as a stock, we see that a social welfare maximizing information designer, i.e., the company whose stock is traded releases less information about the stock value γ, when the uncertainty about its shareholders' payoffs H increase.

VI. Conclusion

This letter considered the problem of designing information structures in incomplete information games when the designer does not know the game payoffs exactly. This is a common situation in many real-world settings, where the game payoffs are often uncertain due to various factors such as imperfect modeling, or unknown parameters. Specifically, we considered the information design for the setting when the unknown payoff parameters are adversarially chosen. For the robust information design problem, we developed a SDP formulation given quadratic payoffs, Gaussian signal distributions, ellipsoid perturbations to the unknown payoff parameters, and social welfare as the design objective. Numerical experiments show that the designer would choose to reveal less information about the payoff states to the players as its uncertainty about the players' payoffs grow. This suggests that in situations where the game payoffs are highly uncertain, it may be preferable to not disclose any information, rather than risk providing misleading information.

Appendix

We start with writing the first order condition equivalent to (2) for a given $\theta \equiv H$:

$$E_\zeta\left[\frac{\partial}{\partial a_i} u_i^\theta(s(\omega), \gamma)|\omega_i\right] = 0 \quad \Leftrightarrow$$
$$-2H_{i,i}s_i(\omega_i) - 2\sum_{i\neq j} H_{i,j}E_\zeta[s_j|\omega_i] + 2E_\zeta[\gamma_i|\omega_i] = 0 \qquad (37)$$

We solve (37) for the best response $s_i(\omega_i), \forall i \in \mathcal{N}$:

$$H_{i,i}s_i(\omega_i) = -\sum_{i\neq j} H_{i,j}E_\zeta[s_j|\omega_i] + E_\zeta[\gamma_i|\omega_i], \quad i \in \mathcal{N} \qquad (38)$$

We look for an equilibrium strategy of the form given below:

$$s_i(\omega_i) = \bar{a}_i + b_i(\omega_i - E_\zeta[\omega_i]), \quad \forall i \in \mathcal{N}, \qquad (39)$$

where $\bar{a}_i$ and $b_i, \forall i \in \mathcal{N}$ are constants. We plug (39) into the first order condition (38):

$$\sum_{j\in\mathcal{N}} H_{i,j}E[\bar{a}_j + b_j(\omega_j - E_\zeta[\omega_j])|\omega_i = \bar{\omega}_i] = E[\gamma_i|\omega_i = \bar{\omega}_i],$$

$\forall \bar{\omega}_i \in \mathbb{R}, i \in \mathcal{N}$. Via conditional expectation rule over multivariate normal distribution, we obtain following:

$$\sum_{j\in\mathcal{N}} H_{i,j}(b_j cov(\omega_j, \omega_i) var(\omega_i)^{-1}(\bar{\omega}_i - E_\zeta[\omega_i]) + \bar{a}_j)$$
$$= E[\gamma_i] + cov(\omega_i, \gamma_i) var(\omega_i)^{-1}(\bar{\omega}_i - E_\zeta[\omega_i]), \qquad (40)$$

$\forall \bar{\omega}_i \in \mathbb{R}, i \in \mathcal{N}$. Vectors $b_i, i \in \mathcal{N}$ and constants $\bar{a}_i, i \in \mathcal{N}$ are determined by following set equations when we separate (40) into respective parts: For $i \in \mathcal{N}$

$$\sum_{j\in\mathcal{N}} H_{i,j}b_j cov(\omega_j, \omega_i) var(\omega_i)^{-1} = cov(\omega_i, \gamma_i) var(\omega_i)^{-1}, \qquad (41)$$

$$\sum_{j\in\mathcal{N}} H_{i,j}\bar{a}_j = E[\gamma_i], \quad i \in \mathcal{N}. \qquad (42)$$

We divide both sides of (41) by $var(\omega_i)^{-1}$ and obtain the following set of equations:

$$\sum_{j\in\mathcal{N}} H_{i,j}b_j cov(\omega_j, \omega_i) = cov(\omega_i, \gamma_i), \quad i \in \mathcal{N}. \qquad (43)$$

For scalar signals $\omega_i \in \mathbb{R}$, if we let $b_i = 1$ and $\bar{a}_i = E_\zeta[\omega_i]$ for $i \in \mathcal{N}$, then we have $a_i = \omega_i$ by (39). Moreover, the set of equations in (43) is equivalent to (11).

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