The matrix Bochner problem

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THE MATRIX BOCHNER PROBLEM

By W. RILEY CASPER and MILEN YAKIMOV

Abstract. A long standing question in the theory of orthogonal matrix polynomials is the matrix Bochner problem, the classification of $N \times N$ weight matrices $W(x)$ whose associated orthogonal polynomials are eigenfunctions of a second order differential operator. Based on techniques from noncommutative algebra (semiprime PI algebras of Gelfand-Kirillov dimension one), we construct a framework for the systematic study of the structure of the algebra $\mathcal{D}(W)$ of matrix differential operators for which the orthogonal polynomials of the weight matrix $W(x)$ are eigenfunctions. The ingredients for this algebraic setting are derived from the analytic properties of the orthogonal matrix polynomials. We use the representation theory of the algebras $\mathcal{D}(W)$ to resolve the matrix Bochner problem under the two natural assumptions that the sum of the sizes of the matrix algebras in the central localization of $\mathcal{D}(W)$ equals N (fullness of $\mathcal{D}(W)$) and the leading coefficient of the second order differential operator multiplied by the weight $W(x)$ is positive definite. In the case of 2×2 weights, it is proved that fullness is satisfied as long as $\mathcal{D}(W)$ is noncommutative. The two conditions are natural in that without them the problem is equivalent to much more general ones by artificially increasing the size of the matrix $W(x)$.

Contents.

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1. Introduction.

1.1. An overview of the results in the paper. Orthogonal matrix polynomials are sequences of matrix-valued polynomials which are pairwise orthogonal with respect to a matrix-valued inner product defined by a weight matrix $W(x)$. They were defined seventy years ago by Krein [32] and since then have been shown to have a wide variety of applications in both pure and applied mathematics, including spectral theory, quasi-birth and death processes, signal processing, Gaussian quadrature, special functions, random matrices, integrable systems and representation theory. Of great specific interest are those orthogonal matrix polynomials which are simultaneously eigenfunctions of a matrix-valued differential operator. They in particular generalize the classical orthogonal polynomials of Hermite, Laguerre, and Jacobi whose utility in diverse research areas is difficult to understate.

The current and potential applications of matrix-valued orthogonal polynomials and the study of their analytic properties naturally motivate the problem of the classification of all orthogonal matrix polynomials which are eigenfunctions of a second order differential operator. This problem was posed and solved by Bochner [7] in the scalar case and later extended by Durán [17] to the matrix case.

Problem 1.1. (Matrix Bochner Problem) Classify all $N \times N$ weight matrices $W(x)$ whose associated sequence of orthogonal matrix polynomials are eigenfunctions of a second order matrix differential operator.

Bochner [7] proved that for $N = 1$ up to an affine change of coordinates the only weight matrices satisfying these properties are the classical weights e^{-x^2}, $x^b e^{-x} 1_{(0,\infty)}(x)$, and $(1-x)^a(1+x)^b 1_{(-1,1)}(x)$ of Hermite, Laguerre, and Jacobi respectively. For brevity, we hereafter refer to affine transformations of these weight functions as classical weights. However, for $N > 1$ the solution of the matrix Bochner problem has proved difficult.

Grünbaum, Pacharoni, and Tirao [24, 25] found the first nontrivial solutions of the matrix Bochner problem using Harish-Chandra modules for real simple groups and the associated matrix spherical functions. Further examples related to Lie groups are found in [31]. In the past twenty years numerous other examples have been found, not necessarily associated with Lie theory. More recent work has focused on the study of the algebra $\mathcal{D}(W)$ of all differential operators for which the matrix-valued polynomials are eigenfunctions. Alternatively, $\mathcal{D}(W)$ may be described as the algebra of all differential operators which are degree-preserving

and W-adjointable. This algebra is studied both from a general stand point and for specific weights $W(x)$, [10, 26, 37, 44, 48]. However, the previous general results on the structure of $\mathcal{D}(W)$ are very limited and general classification results on the matrix Bochner problem have remained illusive.

In this paper, we carry out a general study of the algebra $\mathcal{D}(W)$ using techniques from noncommutative algebra. Starting from the analytics properties of the related sequence of orthogonal matrix polynomials, we equip the algebra $\mathcal{D}(W)$ with the structure of a $*$-algebra with a positive involution. From here we obtain that $\mathcal{D}(W)$ is an affine semiprime algebra of Gelfand-Kirillov dimension at most one. Among the hardest of these properties to prove is that $\mathcal{D}(W)$ is affine. We show this by first proving that the center is affine by constructing an embedding into a product of Dedekind domains. We then show that the whole algebra $\mathcal{D}(W)$ is a large subalgebra of an order in a semisimple algebra. Once all the properties of $\mathcal{D}(W)$ are established, the Small-Stafford-Warfield theorem [43] tells us $\mathcal{D}(W)$ is a Noetherian algebra which is finitely generated module over its center $\mathcal{Z}(W)$. This is used to show that the extension of scalars of $\mathcal{D}(W)$ to the total ring of fractions of $\mathcal{Z}(W)$ is isomorphic to a product of matrix algebras over the function fields of the irreducible components of $\operatorname{Spec}\mathcal{Z}(W)$ (one matrix algebra for each irreducible component). This fact in turn allows us to define the notion of the *rank of the algebra* $\mathcal{D}(W)$ which equals to the sum of the sizes of the matrix algebras. This integer is shown to be between 1 and N and another characterizations of it is given as the maximal number of generalized orthogonal idempotents of $\mathcal{D}(W)$ which sum to a central element.

The above structural results allow us to define canonical submodules of the $\Omega(x), \mathcal{D}(W)$-bimodule $\Omega(x)^{\oplus N}$, where $\Omega(x)$ is the matrix Weyl algebra with rational coefficients. Using representation theory, we demonstrate that the algebraic structure of the algebra $\mathcal{D}(W)$ has a profound influence on the shape of the weight matrix $W(x)$ itself. Specifically, using $\mathcal{D}(W)$ modules defined from the maximal set of orthogonal idempotents, we prove that when the algebra $\mathcal{D}(W)$ is *full* in the sense that the rank is as large as possible (i.e., equals N), the matrix $W(x)$ is congruent to a diagonal weight matrix via a rational matrix $T(x)$,

$$W(x) = T(x)\operatorname{diag}\big(f_1(x), \ldots, f_N(x)\big)T(x)^* \tag{1.2}$$

where $f_i(x)$ is a classical weight for all i. Further arguments with these $\mathcal{D}(W)$ modules allow us to control the size of the *Fourier* algebra of the matrix weight $W(x)$, which is defined as the algebra of matrix differential operators that applied to the orthogonal matrix polynomials of $W(x)$ equal to a shift operator applied to the same sequence. This is a larger algebra than $\mathcal{D}(W)$, which in turn is used to show that under natural assumptions, solutions of the matrix Bochner problem come from bispectral Darboux transformations of a direct sum of classical weights.

Our main theorem is the following Bochner-type classification result:

THEOREM 1.3. (Classification Theorem) *Let $W(x)$ be a weight matrix and suppose that $\mathcal{D}(W)$ contains a W-symmetric second order differential operator*

$$\mathfrak{D} = \partial_x^2 D_2(x) + \partial_x D_1(x) + D_0(x) \tag{1.4}$$

with $D_2(x)W(x)$ positive-definite on the support of $W(x)$. Then the algebra $\mathcal{D}(W)$ is full if and only if $W(x)$ is a noncommutative bispectral Darboux transformation of a direct sum of classical weights. Furthermore, in this case (1.2) *holds.*

A matrix differential operator is called W-symmetric when it equals its formal adjoint with respect to $W(x)$, see Definition 2.14. The algebra $\mathcal{D}(W)$ contains a second order differential operator if and only if it contains a W-symmetric second order differential operator. For such an operator (1.4), the matrix $D_2(x)W(x)$ is necessarily Hermitian.

We pause here to briefly explain the assumptions in the theorem, arguing that they are both relatively weak and natural. The requirement that $\mathcal{D}(W)$ contains a second order W-symmetric operator $\mathfrak{D}$ with $D_2(x)W(x)$ positive definite on the support of $W(x)$ is a non-degeneracy condition. It implies that $\mathfrak{D}$ is not a zero divisor in $\mathcal{D}(W)$ and that $\mathfrak{D}$ does not annihilate an infinite dimensional space of matrix-valued polynomials. It fits well into the literature wherein the stronger condition that the leading coefficient of the second order differential operator is a scalar is often assumed.

Furthermore, without this positivity assumption, the problem is closely related to the much more general one for classifying matrix weights for which $\mathcal{D}(W)$ is nontrivial, i.e., contains a differential operator of an arbitrary nonzero order. This problem is not solved even in the scalar case. To see the stated relation, consider an $N \times N$ matrix weight $W(x)$ for which $\mathcal{D}(W)$ contains an operator of order $2k$. Let $\widetilde{W}(x)$ be a bispectral Darboux transformation of $W(x)$ obtained by factorizing the differential operator into a product of operators of orders k in the sense of Definition 2.33. Then the $2N \times 2N$ block diagonal weight matrix $\operatorname{diag}(W(x), \widetilde{W}(x))$ has the property that $\mathcal{D}(\operatorname{diag}(W, \widetilde{W}))$ contains a differential operator of order k, see Example 3.11 for details. In various common situations this can be iterated to bring the minimal order of a differential operator in $\mathcal{D}(W)$ down to 2 by increasing the size of $W(x)$. The positivity assumption on the leading terms of the differential operator $\mathfrak{D}$ avoids this vast expansion of the matrix Bochner problem. In a similar fashion, when one artificially increases the size of a matrix weight, the difference between N and the rank of $\mathcal{D}(W)$ increases too, leading to cases where the algebra $\mathcal{D}(W)$ is substantially smaller than the size of the matrix weight $W(x)$. The fullness condition ensures that the problem does not become wild in this way.

Additionally, under the assumptions of the Classification Theorem we obtain an explicit description of the algebra $\mathcal{D}(W)$ itself. In particular Theorem 5.26 provides us with a matrix differential operator conjugating $\mathcal{D}(W)$ into a subalgebra of

$\mathcal{D}(f_1 \oplus \cdots \oplus f_N)$ for some classical weights $f_1, \ldots, f_N$. When the $f_1, \ldots, f_N$ are suitably chosen, this latter algebra is a maximal order (which is in this context a direct product of matrix algebras over polynomial rings). In this way, we can think of our process as a noncommutative desingularization of the original algebra $\mathcal{D}(W)$.

We prove that our results take on a more explicit form in the case of matrix weights $W(x)$ of size 2×2. In this case we show that, if $\mathcal{D}(W)$ is noncommutative, then $\mathcal{D}(W)$ must be full. This leads us to the following classification result for 2×2 weight matrices.

THEOREM 1.5. (2×2 case) *Let $W(x)$ be a 2×2 weight matrix and suppose that $\mathcal{D}(W)$ contains a W-symmetric second order differential operator whose leading coefficient multiplied by $W(x)$ is positive definite on the support of $W(x)$. The algebra $\mathcal{D}(W)$ is noncommutative if and only if the weight $W(x)$ is a noncommutative bispectral Darboux transformation of $r(x)I$ for some classical weight $r(x)$.*

Thus aside from various degenerate cases when $\mathcal{D}(W)$ consists of polynomials of a single differential operator of order 2, this fully resolves the Bochner problem for the 2×2 case.

For a comprehensive background on noncommutative rings and polynomial identity (PI) algebras we refer the reader to the books [23, 35] and [13], respectively. A comprehensive treatment of orders in central simple algebras is given in [38]. A concise and illuminating treatment of PI rings can be found in [8, Sections I.13 and III.1].

We refer the reader to the arxiv version of the paper (arXiv:1803.0440, Section 6) for detailed examples of noncommutative bispectral Darboux transformations for 2×2 matrix weights of Hermite, Laguerre, and Jacobi type. These examples in particular illustrate how previous solutions of the matrix Bochner problem fit into our framework.

1.2. From algebraic geometry to noncommutative algebra. The idea that the algebraic properties of a commutative algebra of differential or difference operators can tell us about the operators themselves is not new. It is a powerful point of view that can be pointed to as the central concept in the unification of various phenomena in integrable systems and algebraic geometry during the 70s and 80s, [14, 33, 41, 46]. This resulted in a number of strong applications of algebraic geometry to the analysis of solutions of integrable systems, and in spectacular applications in the opposite direction, for instance the solution of the Schottky problem [42].

More formally, there is a natural correspondence between differential operator algebras, vector bundles on algebraic curves, and exact solutions of certain nonlinear partial differential equations. The most basic example of this is when a Schrödinger operator $\partial^2 + u(x)$ commutes with a differential operator of order 3.

In this case, $u(x)$ gives rise to a soliton solution of the Korteweg-de Vries equation [34]

$$\phi_{xxx}(x,t) + 6\phi(x,t)\phi_x(x,t) = \phi_t(x,t).$$

This concept has also been applied in the context of bispectral algebras of differential operators of low rank (here rank means the gcd of the orders of differential operators in the algebra). Most strikingly, Wilson provided a complete classification of bispectral differential operators of rank 1 in terms of rational projective curves with no nodal singularities [47]. General methods for constructing bispectral operators with $C(\mathfrak{d})$ of arbitrary rank have been developed [1, 2, 15, 29] and the bispectral algebras $C(\mathfrak{d})$ containing an operator of prime order have been classified [27]. Moving into a noncommutative direction, rank 1 bispectral solutions of the KP hierarchy and the related Calogero–Moser systems were analyzed in [3, 4, 6] using $\mathcal{D}$-modules, one-sided ideals of the first Weyl algebras and noncommutative projective surfaces.

The basis of the applications of algebraic geometry to integrable systems is the consideration of commutative algebras of differential and difference operators. New in this paper is the systematic study of the properties of orthogonal matrix polynomials for a weight matrix $W(x)$, based on structural results for the algebra $\mathcal{D}(W)$ which is generally noncommutative. The noncommutative case is quite a bit more challenging; for example, we no longer have the power of algebraic geometry to rely on, at least directly. Our methods use PI algebras, and more precisely, noncommutative algebras which are module-finite over their centers, their relation to central simple algebras, and their representation theory [8, 13].

Similarly, to Shiota's solution of the Schottky problem [42], we expect that our methods will have applications of orthogonal matrix polynomials to noncommutative algebra, in that interesting PI algebras can be realized as the algebras $\mathcal{D}(W)$ for some weight matrices $W(x)$. Such a relation, can be used to study fine properties of these algebras using orthogonal polynomials.

1.3. Notation. Throughout this paper, we will use capital letters to represent matrices and lower case letters to represent scalars. We will also use the Gothic font to represent differential operators and script to represent difference operators. For example, we will use $f(x)$ or $F(x)$ to represent a function of x, depending on whether it is a scalar or matrix-valued function. Similarly, we will use $\mathfrak{d}$ or $\mathfrak{D}$ to represent a differential operator, again depending on whether it is scalar or matrix-valued. Furthermore an expression like $\mathscr{M}$ will denote a matrix-valued discrete operator. Wherever feasible, we will use capitalized calligraphic font, such as $\mathcal{A}$, to denote various operator algebras, subalgebras, and ideals. Exceptions to this will include certain special algebras, such as the algebra of all differential operators, that will have their own special notation.

For reasons pertaining to compatibility with the matrix-valued inner product $\langle \cdot, \cdot \rangle_W$ defined below, our differential operators will act on the right. For example, the basic differential operator ∂_x acts on a function $f(x)$ by

$$f(x) \cdot \partial_x = f'(x).$$

An arbitrary matrix differential operator $\mathfrak{D} = \sum_{j=0}^{n} \partial_x^j A_j(x)$ acts on a matrix-valued function $F(x)$ by

$$F(x) \cdot \mathfrak{D} = \sum_{j=0}^{n} F^{(j)}(x) A_j(x).$$

Because this action is a right action, the algebra of all differential operators will satisfy the fundamental commutation relation

$$x\partial - \partial x = 1.$$

Note this is reversed from the typical identity for the Weyl algebra, since differential operators are most often taken to act on the left. Thus our algebra of differential operators will actually be the opposite algebra of usual Weyl algebra.

In addition, we will adopt the following notation:

- For any ring R, $R[x]$, $R(x)$, $R[[x]]$ and $R((x))$ will denote the rings of polynomials, rational functions, power series and Laurent series with coefficients in R, respectively.
- We will use $\Omega[x]$, $\Omega(x)$, $\Omega[[x]]$ and $\Omega((x))$ to denote the opposite algebras of the Weyl algebras with coefficients in $\mathbb{C}[x]$, $\mathbb{C}(x)$, $\mathbb{C}[[x]]$ and $\mathbb{C}((x))$ respectively (i.e., the rings of differential operators with right action).
- The symbol $\dagger$ will always denote the W-adjoint, as defined below.
- For any ring R, $M_N(R)$ will denote the ring of matrices with coefficients in R and E_{ij} will denote the element of this ring with a 1 in the i,j'th entry and zeros elsewhere.
- The symbols x, t, and n will represent indeterminants, unless specified otherwise.

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2. Background.

2.1. Orthogonal matrix polynomials and Bochner's problem. We begin with a brief review of the basic theory of orthogonal matrix polynomials and Bochner's problem.

Definition 2.1. A *weight matrix* $W(x)$ supported on a (possibly unbounded) interval (x_0, x_1) is Hermitian matrix-valued function $W : \mathbb{C} \to M_N(\mathbb{C})$ which is entrywise-smooth on $\mathbb{R}$, identically zero outside of (x_0, x_1), and positive definite on (x_0, x_1) with finite moments:

$$\int x^m W(x) dx < \infty, \quad \forall m \geq 0.$$

We call the interval (x_0, x_1) the *support of* $W(x)$. Note that in general x_0, x_1 may be extended real numbers.

Remark 2.2. We are restricting our attention to "smooth" weight matrices in order to avoid more delicate analytic considerations. In general one may consider matrix valued measures on $\mathbb{R}$ satisfying an appropriate generalization of the above definition. This is discussed further in [12].

A weight matrix $W(x)$ defines a matrix-valued inner product $\langle \cdot, \cdot \rangle_W$ on the vector space $M_N(\mathbb{C}[x])$ of all $N \times N$ complex matrix polynomials by

$$\langle P, Q \rangle_W := \int_{\mathbb{C}} P(x) W(x) Q(x)^* dx, \quad \forall P, Q \in M_N(\mathbb{C}[x]). \tag{2.3}$$

By applying Gram-Schmidt we may determine a sequence of matrix-valued polynomials $P(x,n) \in M_N(\mathbb{C}[x])$, $n \in \mathbb{N}$, with $P(x,n)$ degree n with nonsingular leading coefficient such that $\langle P(x,n), P(x,m) \rangle_W = 0$ for $m \neq n$. Moreover this sequence $P(x,n)$ is unique up to left multiplication by a nonsingular constant matrix. This leads to the following definition.

Definition 2.4. We call a sequence of matrix polynomials $P(x,n)$, $n \in \mathbb{N}$, a *sequence of orthogonal matrix polynomials for* $W(x)$ if for all n the polynomial $P(x,n)$ has degree n with nonsingular leading coefficient and

$$\langle P(x,n), P(x,m) \rangle_W = 0, \quad \forall m \neq n.$$

The sequence $P(x,n)$ will be called *monic* if the leading coefficient of each $P(x,n)$ is I.

The sequence of monic orthogonal polynomials of a weight matrix $W(x)$ must satisfy a three-term recursion relation and a converse statement also holds.

THEOREM 2.5 (Durán, Van Assche and López-Rodriguez [20, 19]). *Suppose that $P(x,n)$ is a sequence of monic orthogonal matrix polynomials for a weight matrix $W(x)$. Then for some sequences of complex matrices $B(n)$ and $C(n)$, we have*

$$xP(x,n) = P(x,n+1) + B(n)P(x,n) + C(n)P(x,n-1), \quad \forall n \geq 1. \tag{2.6}$$

Conversely, given sequences of matrices $B, C : \mathbb{N} \to \mathbb{C}$ satisfying natural mild assumptions, there exists a weight matrix $W(x)$ for which the sequence of polynomials $P(x,n)$ defined by (2.6) *is a sequence of monic orthogonal matrix polynomials.*

The focus of our paper is the Matrix Bochner Problem 1.1. Specifically, we wish to determine for which weights $W(x)$ the associated sequence $P(x,n)$ of monic orthogonal polynomials satisfy a second order matrix differential equation

$$P''(x,n)A_2(x) + P'(x,n)A_1(x) + P(x,n)A_0(x) = \Lambda(n)P(x,n) \tag{2.7}$$

for all n for some sequence of complex matrices $\Lambda(n)$. Equivalently, we want to know when the $P(x,n)$ are eigenfunctions of a second order matrix differential operator $\mathfrak{D} = \partial_x^2 A_2(x) + \partial_x A_1(x) + A_0(x)$ acting on the right with matrix-valued eigenvalues. More generally, we can consider the algebra of all differential operators for which the sequence $P(x,n)$ are eigenfunctions.

Definition 2.8. Let $W(x)$ be a weight matrix with sequence of monic orthogonal polynomials $P(x,n)$. We define $\mathcal{D}(W)$ to be the collection of all matrix-valued differential operators for which the $P(x,n)$ are eigenfunctions for all n. We call $\mathcal{D}(W)$ the *algebra of matrix differential operators associated to* $W(x)$. We use $\mathcal{Z}(W)$ to denote the center of $\mathcal{D}(W)$.

Remark 2.9. Later we will realize $\mathcal{D}(W)$ as the right bispectral algebra associated to the bispectral function $P(x,n)$. However we will retain the notation $\mathcal{D}(W)$ throughout the paper.

2.2. Adjoints of differential operators. Our methods extensively use $*$-algebras constructed using the adjoint of a matrix differential operator. We briefly recall some definitions and ideas here and refer the reader to the excellent reference [16] for a comprehensive treatment. A matrix differential operator $\mathfrak{D}$ may always be viewed as an unbounded linear operator on a suitably chosen Hilbert space. Therefore it makes sense to think about the adjoint of a differential operator strictly in terms of functional analysis.

Definition 2.10. An *unbounded linear operator* on a Hilbert space $\mathcal{H}$ is a linear function T defined on a dense subset $\mathcal{H}$ called the *domain of* T. The adjoint of T

is an unbounded linear operator T^* with domain

$$\{y \in \mathcal{H} : x \to \langle Tx, y\rangle \text{ is continuous}\}$$

defined by $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all x and y in the domain of T and T^*, respectively.

Remark 2.11. Some authors do not require an unbounded operator to be defined on a dense subspace. However it is necessary for Hahn-Banach to imply the existence of T^*, so we adopt it as part of our definition. Since we will be working with differential operators with rational coefficients, this definition will be sufficient for us.

The algebra of differential operators also have a natural adjoint operation $*$ called the formal adjoint.

Definition 2.12. The *formal adjoint* on $M_N(\Omega((x)))$ is the unique involution extending Hermitian conjugate on $\Omega((x))$ and sending $\partial_x I$ to $-\partial_x I$.

Now consider specifically the Hilbert space $\mathcal{H} = L^2([-1,1])$. The formal adjoint is defined in such a way that the adjoint of an unbounded operator and its formal adjoint will agree on a subset of $\mathcal{H}$ whose closure has finite codimension. However as the next example shows, the formal adjoint may not be equal to the adjoint of $\mathfrak{D}$ as an unbounded linear operator.

Example 2.13. Let $\mathcal{H} = L^2([-1,1])$ with the usual inner product and consider two polynomials $p(x), q(x)$. Then the formal adjoint of the differential operator ∂_x is $\partial_x^* = -\partial_x$. However we calculate

$$\begin{aligned}\langle p(x)\cdot\partial_x, q(x)\rangle &= \int_{-1}^{1} p'(x)\overline{q(x)}dx \\ &= p(1)\overline{q(1)} - p(0)\overline{q(0)} - \int_{-1}^{1} p(x)\overline{q'(x)}dx \\ &= \langle p(x), q(x)\cdot\partial_x^*\rangle + p(1)\overline{q(1)} - p(0)\overline{q(0)}.\end{aligned}$$

Since there is an extra term on the right-hand side, the formal adjoint of ∂_x does not agree with the adjoint as an unbounded linear operator on its domain.

More generally a weight matrix $W(x)$ defines a Hilbert space of matrix-valued functions $M_N(\mathcal{H})$, where $\mathcal{H}$ is the Hilbert space of complex-valued $L^2(\mathrm{tr}(W(x))dx)$ functions on the support (x_0, x_1) of $W(x)$. Any matrix-valued differential operator $\mathfrak{D} \in M_N(\Omega[[x]])$ with suitably nice coefficients will define an unbounded linear operator on $M_N(\mathcal{H})$. There is also a natural choice of formal adjoint here which takes into account the form of the inner product.

Definition 2.14. Let $W(x)$ be a weight matrix supported on an interval (x_0, x_1) containing 0 and let $\mathfrak{D} \in M_N(\Omega[[x]])$. A *formal adjoint of* $\mathfrak{D}$ *with respect to* $W(x)$, or *formal* $W(x)$*-adjoint of* $\mathfrak{D}$ is the unique differential operator $\mathfrak{D}^\dagger \in M_N(\Omega[[x]])$ defined on (x_0, x_1) by

$$\mathfrak{D}^\dagger := W(x)\mathfrak{D}^* W(x)^{-1}.$$

An operator $\mathfrak{D}$ is called *formally* $W(x)$*-symmetric* if $\mathfrak{D}^\dagger = \mathfrak{D}$.

Remark 2.15. We require 0 to be in the interval (x_0, x_1) in order for $W(x)$ and $W(x)^{-1}$ to have series expansions at 0, and thereby representations in $M_N(\mathbb{C}[[x]])$. This is a technical necessity for $\dagger$ to be defined on $M_N(\Omega[[x]])$, but is not an important restriction since we can always affinely translate until the support contains 0.

Definition 2.16. Let $W(x)$ be a weight matrix and let $\mathfrak{D} \in M_N(\Omega[x])$. We say that $\mathfrak{D}$ is W*-adjointable* if there exists a differential operator $\widetilde{\mathfrak{D}} \in M_N(\Omega[x])$ satisfying

$$\langle P \cdot \mathfrak{D}, Q\rangle_W = \langle P, Q \cdot \widetilde{\mathfrak{D}}\rangle_W, \quad \forall P, Q \in M_N\big(\mathbb{C}[x]\big).$$

In this case, we call $\widetilde{\mathfrak{D}}$ the *adjoint of* $\mathfrak{D}$ *with respect to* $W(x)$, or alternatively the W*-adjoint of* $\mathfrak{D}$. If $\mathfrak{D} = \widetilde{\mathfrak{D}}$, then $\mathfrak{D}$ is called W*-symmetric*.

Even if a given differential operator $\mathfrak{D}$ is W-symmetric, it is not necessarily formally W-symmetric, as the following example from [45] shows.

Example 2.17. Let $r(x) = \frac{1}{x}\exp(-\ln(x)^2/2)$ defined for $x > 0$ and consider the weight matrix

$$W(x) = \begin{pmatrix} 4r(x) & r(x)\sin\big(2\pi\ln(x)\big) \\ r(x)\sin\big(2\pi\ln(x)\big) & 2r(x) \end{pmatrix} 1_{(0,\infty)}.$$

Then for any polynomials $P(x), Q(x) \in M_2(\mathbb{C}[x])$ and real numbers a, b we have

$$\big\langle P(x)\,\mathrm{diag}(a,b), Q(x)\big\rangle_W = \big\langle P(x), Q(x)\,\mathrm{diag}(a,b)\big\rangle_W,$$

due to the fact that $\int_0^\infty x^n r(x)\sin(2\pi\ln(x))dx = 0$ for all $n \geq 0$. Consequently the element $\mathrm{diag}(a,b)$ is W-symmetric. However, $\mathrm{diag}(a,b)$ is not formally W-symmetric since $\mathrm{diag}(a,b)^\dagger \neq \mathrm{diag}(a,b)$.

Remark 2.18. We are grateful to the anonymous referee for pointing out this example to us, greatly contributing to the clarity of the presentation of adjoints for weight matrices with unbounded support.

The situation described in the previous case is pathological, arising from the unbounded support of $W(x)$. Specifically it is due to the fact that in an unbounded

domain two different functions can have identical moments. However, if two different smooth functions have the same moments, and both decrease sufficiently rapidly for large values of $|x|$, then they will necessarily be the same. This prompts the following definition.

Definition 2.19. We say that a matrix-valued function $F(x) : \mathbb{R} \to M_N(\mathbb{C})$ *decreases exponentially at infinity* if there exist positive real numbers a, b such that

$$\lim_{x \to \pm\infty} e^{a|x|^b} F(x) = 0.$$

Remark 2.20. In particular, if the support of a weight matrix $W(x)$ is bounded, then by definition $W(x)$ decreases exponentially at infinity. If $F(x)$ decreases exponentially at infinity and all its moments exist and are zero, then $F(x)$ will itself be zero.

Under the following assumption about $W(x)$, the formal W-adjoint of a differential operator will always agree with the W-adjoint of a differential operator when the latter exists.

Assumption 2.21. For each integer $n \geq 0$ the n'th derivative $W^{(n)}(x)$ decreases exponentially at infinity and there exists a scalar polynomial $p_n(x)$ such that $W^{(n)}(x)p_n(x)$ has finite moments.

Here by finite moments, we mean that the entries of $W^{(n)}p_n(x)x^k$ are in $L^1(\mathbb{R})$ for all nonnegative integers k.

LEMMA 2.22. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ has finite moments and decreases exponentially at infinity. If $\int f(x)x^n dx = 0$ for all $n \geq 0$ then $f(x) = 0$ for a.e. x.*

Proof. The moment condition is a real constraint, so to prove our lemma it suffices to consider the case when $f(x)$ is real-valued. Write $f(x) = f_+(x) - f_-(x)$ for functions $f_\pm(x)$ which are positive with complementary support. Since $f(x)$ has finite moments and decreases exponentially at infinity, so too do $f_\pm(x)$. Therefore there exist positive constants $r_\pm, a_\pm, b_\pm$ such that $f_\pm(x) \leq \exp(-a_\pm x^{b_\pm})$ for $|x| > r_\pm$. We calculate that the n'th moment of $f_\pm(x)$ satisfies the inequality

$$\begin{aligned} m_{n,\pm} = \int_{\mathbb{R}} x^n f_\pm(x)dx &\leq \int_{-r_\pm}^{r_\pm} |x|^n f_\pm dx + \int_{\mathbb{R}} |x|^m e^{-a_\pm x^{b_\pm}} dx \\ &\leq r_\pm^n \|f_\pm\|_1 + \frac{2}{b_\pm} \frac{1}{a_\pm^{(n+1)/b_\pm}} \Gamma\left(\frac{n+1}{b_\pm}\right) \\ &\leq \frac{K}{b_\pm} \frac{1}{a_\pm^{(n+1)/b_\pm}} \Gamma\left(\frac{n+1}{b_\pm}\right), \end{aligned}$$

where K exists because $\Gamma(n)$ must (eventually) grow faster than $(a_\pm^{1/b_\pm} r_\pm)^n$. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{m_{2n,\pm}}} \geq \sum_{n=1}^{\infty} \left(\frac{b_\pm a_\pm^{1/b_\pm}}{K} \right)^{1/2n} a^{1/b_\pm} \frac{1}{\Gamma\left(\frac{2n+1}{b_\pm}\right)^{1/2n}} = \infty.$$

By Carleman's condition, it follows that $f_\pm$ is the only nonnegative function with the sequence of moments $m_{n,\pm}$. By assumption $\int f_+(x) x^n dx = \int f_-(x) x^n dx$ for all $n \geq 0$, so that $m_{n,+} = m_{n,-}$ for all n. Consequently $f_+(x) = f_-(x)$ a.e., and therefore $f(x) = 0$ a.e. □

PROPOSITION 2.23. *Let $W(x)$ be a weight matrix satisfying Assumption 2.21 and let $\mathfrak{D} \in M_N(\Omega[x])$ be a matrix differential operator. If $\mathfrak{D}$ is W-adjointable and $\mathfrak{D}^\dagger \in M_N(\Omega[x])$, then its W-adjoint is equal to the formal W-adjoint $\mathfrak{D}^\dagger$.*

Proof. Suppose that $\mathfrak{D}$ is W-adjointable and let $\widetilde{\mathfrak{D}} \in M_N(\Omega[x])$ be the W-adjoint of $\mathfrak{D}$. Additionally let n be the order of $\mathfrak{D}$ and choose $f(x) \in \mathbb{C}[x]$ such that $W^{(j)}(x) f(x) \in M_N(\mathcal{H})$ for all $0 \leq j \leq n$. Furthermore let $p(x) \in \mathbb{C}[x]$ be the unique monic polynomial vanishing only at the finite endpoints of the support of $W(x)$ and let $q(x) = p_1(x) p_2(x) \dots p_n(x) p(x)^n$, for n the order of $\mathfrak{D}$ and $p_1(x), \dots, p_n(x)$ defined as in Assumption 2.21. Then for any $Q(x) \in M_N(\mathbb{C}[x])$ there exist matrices $B_k(x), C_k(x) \in M_N(\mathbb{C}[x])$ (depending on the values of $Q(x)$ and $\mathfrak{D}$) such that

$$\big(\big(q(x)Q(x)\big) \cdot \big(W(x)\mathfrak{D}^* - \widetilde{\mathfrak{D}} W(x)\big)\big)^* = \sum_{k=0}^{n} B_k(x) p_k(x) W^{(k)}(x) C_k(x).$$

Therefore the expression in the left-hand side decreases exponentially at infinity and has finite moments. As a consequence, if all the moments are zero, then the expression will be identically zero.

Integration by parts tells us that for all $P(x), Q(x) \in M_N(\mathbb{C}[x])$

$$\begin{aligned} \big\langle P(x) \cdot \mathfrak{D}, q(x)Q(x) \big\rangle_W &= \int_{x_0}^{x_1} \big(P(x) \cdot \mathfrak{D}\big) W(x) q(x) Q(x)^* dx \\ &= \int_{x_0}^{x_1} P(x) \big(\big(q(x)Q(x)\big) \cdot W(x)\mathfrak{D}^*\big)^* dx. \end{aligned}$$

Combining this with the $W(x)$-adjoint definition, we obtain that for all $P(x)$, $Q(x) \in M_N(\mathbb{C}[x])$

$$\int_{x_0}^{x_1} P(x) \big(\big(q(x)Q(x)\big) \cdot \big(W(x)\mathfrak{D}^* - \widetilde{\mathfrak{D}} W(x)\big)\big)^* dx = 0.$$

This implies that the entries of $((q(x)Q(x)) \cdot (W(x)\mathfrak{D}^* - \widetilde{\mathfrak{D}} W(x)))^*$ have all moments equal to zero. It follows that $((q(x)Q(x)) \cdot (W(x)\mathfrak{D}^* - \widetilde{\mathfrak{D}} W(x))) = 0$ for

all $Q(x) \in M_N(\mathbb{C}[x])$ and therefore the kernel of $W(x)\mathfrak{D}^* - \widetilde{\mathfrak{D}}W(x)$ contains $q(x)M_N(\mathbb{C}[x])$. Hence $\mathfrak{D}^\dagger - \widetilde{\mathfrak{D}} = 0$, completing the proof. □

In the case that $\mathcal{D}(W)$ contains a W-symmetric second order differential operator of the form (1.4), then $W(x)$ satisfies the noncommutative Pearson equation

$$2\big(D_2(x)W(x)\big)' = D_1(x)W(x) + W(x)D_1(x)^*. \tag{2.24}$$

When $D_2(x)W(x)$ is positive-definite on the support of $W(x)$, one may use this to show that the weight matrix $W(x)$ will satisfy Assumption 2.21, as proved in the next lemma. Hence for the purposes of this paper, the formal W-adjoint will always be equal to the W-adjoint, whenever the latter exists.

LEMMA 2.25. *Suppose that $W(x)$ is a weight matrix and that $\mathcal{D}(W)$ contains a W-symmetric, second order differential operator $\mathfrak{D}$ of the form* (1.4) *with $D_2(x)W(x)$ positive-definite on the support of $W(x)$. Then $W(x)$ satisfies Assumption 2.21.*

Proof. Note that $\det(D_2(x))$ is not identically zero, so its inverse exists as a matrix-valued rational function. Therefore $D_2(x)$ is invertible in $M_N(\mathbb{C}(x))$. Let $p_1(x) = \det(D_2(x)) \in \mathbb{C}[x]$ so that $D_2(x)^{-1}p_1(x) \in M_N(\mathbb{C}[x])$. Then by using the noncommutative Pearson equation (2.24), one may show inductively that $p_1(x)^n W^{(n)}(x)$ is equal to a sum of matrix polynomials multiplying $W(x)$ on the left and right, and consequently has finite moments, and that if $W(x)$ decreases exponentially at infinity, then so too does $W^{(n)}(x)$ for each n. Thus it suffices to show that $W(x)$ decreases exponentially at infinity.

Set $Q(x) = D_1(x)D_2(x)^{-1} \in M_N(\mathbb{C}(x))$ and $H(x) = D_2(x)W(x)$. Since $W(x) = D_2(x)^{-1}H(x)$, to prove that $W(x)$ decreases exponentially at infinity it suffices to prove that $H(x)$ decreases exponentially at infinity. Note that the noncommutative Pearson equation (2.24) reads

$$2H'(x) = Q(x)H(x) + H(x)Q(x)^*.$$

This is a first-order, $N^2 \times N^2$ linear system of differential equations for the entries of $H(x)$, with coefficients in $\mathbb{C}(x)$. In particular by reindexing, we can write this equation as

$$\frac{d}{dx}\vec{y} = M(x)\vec{y}, \tag{2.26}$$

where $M(x)$ is an $N^2 \times N^2$ rational matrix whose entries are determined by the entries of $Q(x)$ and the entries of $\vec{y}$ are the entries $H_{ij}(x)$ of $H(x)$ arranged in lexicographical order.

We view (2.26) as a differential equation on $\mathbb{C}$. We consider two possibilities: either (2.26) has at most a singularity of the first kind at ∞, or else (2.26) has a singularity of the second kind at infinity.

In the first case, by [11, Theorem 5.1] the general solution $\Phi(z)$ of (2.26) is meromorphic in a neighborhood of ∞, and therefore the functions $H_{ij}(x)$ extend to meromorphic functions in a neighborhood of infinity. Consequently $1/H_{ij}(z)$ is meromorphic in a neighborhood of infinity also and so for each i,j there exists an integer $n_{ij} > 0$ with $1/(H_{ij}(z)z^{n_{ij}}) \to 0$ as $z \to \infty$. This implies that $H_{ij}(z) = |H_{ij}(z)| > |z|^{-n_{ij}}$ for $|z|$ large enough and since the $H_{ij}(z)$ have finite moments this implies that the support of $W(x)$ must be bounded.

In the second case, by [11, Theorem 6.1], (2.26) has a formal-solution matrix $\Phi(z)$ of (2.26) of the form

$$\Phi(z) = S(z)\text{diag}(e^{q_1(z)}, \dots, e^{q_{N^2}(z)}),$$

where here each $q_i(z)$ is a polynomial in $\mathbb{C}[z^{1/h}]$ for some fixed integer $h > 0$ and $S(z)$ is an $N^2 \times N^2$ matrix whose determinant does not vanish for large $|z|$ and

$$S_{ij}(z) = z^{r_{ij}} \sum_{m=0}^{m_{ij}} \sigma_{ijm}(z) \log^m(z), \quad \sigma_{ijm}(z) = \sum_{\ell=0}^{\infty} \sigma_{ijm\ell} z^{-\ell/h}$$

for some constants $\sigma_{ijm\ell}$. Appropriate linear combinations of the formal solution will agree with the values of $H_{ij}(x)$ at points in the support of $W(x)$ near infinity, but the coefficients in the expansion may change depending on which direction we approach ∞ on the real line, due to branching.

Suppose that the support of $W(x)$ is unbounded to the right. From the previous paragraph, we know that $H_{ij}(x)$ will be of the form

$$H_{ij}(x) = \sum_{k=1}^{N^2} s_{ijk}(x) e^{q_k(x)}$$

for some functions $s_{ijk}(x)$ expressible in terms of the entries of $S(z)$. In particular, the entries of $s_{ijk}(x) \sim x^{n_{ijk}/h} \log(x)^{m_{ijk}}$ as $x \to \infty$. Therefore for the $H_{ij}(x)$ to have finite moments, we must have that $\text{Re}(q_k(x))$ goes to zero as $x \to \infty$. Since $s_{ijk}(x) = \mathcal{O}(x^{t_{ijk}})$ for some integer $t_{ijk} \in \mathbb{Z}$ as $x \to \infty$, this shows that $H_{ij}(x)$ decreases exponentially as $x \to \infty$.

Similarly, if the support of $W(x)$ is unbounded to the left, then we may show that the $H_{ij}(x)$ decrease exponentially as $x \to -\infty$. Thus in any case, the function $H(x)$ decreases exponentially at infinity. □

2.3. Bispectrality. The notion of bispectrality arose from questions originally posed by Duistermaat and Grünbaum [15] on finding locally meromorphic functions $\psi(x,y)$ which define a family of eigenfunctions for a fixed differential operator in x *and* a family of eigenfunctions of another differential operator in y.

Examples of functions $\psi(x,y)$ satisfying this property are referred to in the literature as bispectral functions. The simplest example of a bispectral function is the exponential function $\psi(x,y) = e^{xy}$.

Example 2.27. The exponential function $\psi(x,y) = e^{xy}$ satisfies

$$\partial_x \cdot \psi(x,y) = y\psi(x,y), \quad \text{and} \quad \partial_y \cdot \psi(x,y) = x\psi(x,y).$$

Therefore $\psi(x,y)$ defines a bispectral function.

Since its inception, the concept of bispectrality has been generalized in various natural directions, particularly in directions which include certain noncommutative aspects. To begin, we present bispectrality in very general terms.

Remark 2.28. In our definition, we will use lower case for elements of algebras. Since the algebras themselves are completely abstract, this is not a reflection on whether the elements themselves are scalar or matrix-valued. Furthermore, the elements of each of the algebras are never assumed to be commutative. However, the algebras are assumed to be associative with identity.

Definition 2.29. We define an *operator algebra* to be an algebra $\mathcal{A}$ with a fixed subalgebra $\mathcal{M}(\mathcal{A})$, referred to as the subalgebra of *multiplicative operators* of $\mathcal{A}$. A *bispectral context* is a triple $(\mathcal{A},\mathcal{B},\mathcal{H})$ where $\mathcal{A}$ and $\mathcal{B}$ are operator algebras and $\mathcal{H}$ is an $\mathcal{A},\mathcal{B}$-bimodule. A *bispectral triple* is a triple (a,b,ψ) with $a \in \mathcal{A}, b \in \mathcal{B}$ nonconstant and $\psi \in \mathcal{H}$ satisfying the property that ψ has trivial left and right annihilator and

$$a \cdot \psi = \psi \cdot g, \quad \text{and} \quad \psi \cdot b = f \cdot \psi$$

for some $f \in \mathcal{M}(\mathcal{A})$ and $g \in \mathcal{M}(\mathcal{B})$. In the case that ψ forms part of a bispectral triple we call ψ *bispectral*.

In the classical case $\mathcal{A}$ and $\mathcal{B}$ are both algebras of differential operators with multiplicative operator subalgebras consisting of the functions on which the differential operators act (eg. polynomials, holomorphic functions, smooth function, etc). However, as is clear from the definition, bispectrality is a far more general construction. The generalization of specific relevance to this paper is the case when the multiplicative operator subalgebra $\mathcal{M}(\mathcal{A})$ and $\mathcal{M}(\mathcal{B})$ are themselves not commutative.

Given $\psi \in \mathcal{H}$ bispectral, we define certain natural subalgebras of $\mathcal{A}$ and $\mathcal{B}$.

Definition 2.30. Let $\psi \in \mathcal{H}$ be bispectral. We define the *left and right Fourier algebras* $\mathcal{F}_L(\psi)$ and $\mathcal{F}_R(\psi)$ to be

$$\mathcal{F}_L(\psi) = \{a \in \mathcal{A} : \exists b \in \mathcal{B},\ a \cdot \psi = \psi \cdot b\},$$
$$\mathcal{F}_R(\psi) = \{b \in \mathcal{B} : \exists a \in \mathcal{A},\ a \cdot \psi = \psi \cdot b\}.$$

We define the *left and right bispectral algebras* $\mathcal{B}_L(\psi)$ and $\mathcal{B}_R(\psi)$ to be

$$\mathcal{B}_L(\psi) = \{a \in \mathcal{A} : \exists g \in \mathcal{M}(\mathcal{B}),\ a \cdot \psi = \psi \cdot g\},$$
$$\mathcal{B}_R(\psi) = \{b \in \mathcal{B} : \exists f \in \mathcal{M}(\mathcal{A}),\ f \cdot \psi = \psi \cdot b\}.$$

Since the left and right annihilators of ψ are both trivial, there is a natural isomorphism $b_\psi : \mathcal{F}_L(\psi) \to \mathcal{F}_R(\psi)$ defined by the identity

$$a \cdot \psi = \psi \cdot b_\psi(a).$$

Definition 2.31. Let $\psi \in \mathcal{H}$ be bispectral. We call the natural isomorphism $b_\psi : \mathcal{F}_L(\psi) \to \mathcal{F}_R(\psi)$ defined above the *generalized Fourier map*.

Remark 2.32. The name generalized Fourier map comes from our first example. In that case $\mathcal{A} = \Omega(x)^{op}, \mathcal{B} = \Omega(y)$ and $\psi = e^{xy}$. Then $\mathcal{F}_L(\psi) = \Omega[x]^{op}$, $\mathcal{F}_R(\psi) = \Omega[y]$ and the generalized Fourier map is in fact the Fourier transform:

$$b_\psi : \sum_{i,j=0}^{r} a_{ij} x^i \partial_x^j \longmapsto \sum_{i,j=0}^{r} a_{ij} y^j \partial_y^i.$$

Here and below, for an algebra $\mathcal{A}$, $\mathcal{A}^{op}$ denotes the one with opposite product.

2.4. Bispectral Darboux transformations. In practice many of the important families of bispectral triples arise from simpler or more obvious examples through bispectral Darboux transformations.

Definition 2.33. Let $(\mathcal{A}, \mathcal{B}, \mathcal{H})$ be a bispectral context, and let $\psi, \widetilde{\psi} \in \mathcal{H}$. We say that $\widetilde{\psi}$ is a *bispectral Darboux transformation* of ψ if there exist $u, \widetilde{u} \in \mathcal{F}_R(\psi)$ and units $p, \widetilde{p} \in \mathcal{M}(\mathcal{A})$ and units $q, \widetilde{q} \in \mathcal{M}(\mathcal{B})$ with

$$\widetilde{\psi} = p^{-1} \cdot \psi \cdot u q^{-1} \quad \text{and} \quad \psi = \widetilde{p}^{-1} \cdot \widetilde{\psi} \cdot \widetilde{q}^{-1} \widetilde{u}. \tag{2.34}$$

In the case that $\mathcal{M}(\mathcal{A})$ or $\mathcal{M}(\mathcal{B})$ is noncommutative, this is also called a *noncommutative bispectral Darboux transformation* in [22].

It follows from the definition that

$$\psi \cdot (u q^{-1} \widetilde{q}^{-1} \widetilde{u}) = p \widetilde{p} \cdot \psi,$$
$$\left(b_\psi^{-1}(\widetilde{u}) \widetilde{p}^{-1} p^{-1} b_\psi^{-1}(u)\right) \cdot \psi = \psi \cdot \widetilde{q} q.$$

Furthermore, one works out that

$$\widetilde{p} p \cdot \widetilde{\psi} = \widetilde{\psi} \cdot (\widetilde{q}^{-1} \widetilde{u} u q^{-1}),$$
$$\left(p^{-1} b_{\widetilde{\psi}}^{-1}(u) b_{\widetilde{\psi}}^{-1}(\widetilde{u}) \widetilde{p}^{-1}\right) \cdot \widetilde{\psi} = \widetilde{\psi} \cdot q \widetilde{q},$$

see [22, Theorem 2.1]. Thus both ψ and $\widetilde{\psi}$ are bispectral.

The following symmetry property of bispectral Darboux transformations directly follows from the definition and its proof is left to the reader.

THEOREM 2.35. *If $\widetilde{\psi}$ is a bispectral Darboux transformation of ψ then ψ is a bispectral Darboux transformation of $\widetilde{\psi}$.*

The bispectral nature of this type of Darboux transformations is established by the next theorem.

THEOREM 2.36. (Geiger, Horozov, and Yakimov [22]) *Suppose that $\widetilde{\psi}$ is a bispectral Darboux transformation of ψ. Then for all $a \in \mathcal{F}_L(\psi)$*

$$\widetilde{p}ap \cdot \widetilde{\psi} = \widetilde{\psi} \cdot \widetilde{q}^{-1}\widetilde{u}b_{\psi}(a)uq^{-1},$$
$$p^{-1}b_{\psi}^{-1}(u)ab_{\psi}^{-1}(\widetilde{u})\widetilde{p}^{-1} \cdot \widetilde{\psi} = \widetilde{\psi} \cdot qb_{\psi}(a)\widetilde{q}.$$

2.5. Degree-filtration preserving and exceptional differential operators. Each differential operator $\mathfrak{D} \in \mathcal{D}(W)$ will have an entire $M_N(\mathbb{C})$-module basis of $M_N(\mathbb{C}[x])$ as eigenfunctions. As such for any polynomial $Q(x) \in M_N(\mathbb{C}[x])$ we know that the degree of $Q(x) \cdot \mathfrak{D}$ will be at most the degree of $Q(x)$. We call such operators degree-filtration preserving.

Definition 2.37. A matrix-valued differential operator $\mathfrak{D} \in M_N(\mathbb{C}(x))$ is called *degree-filtration preserving* if for all polynomials $Q(x) \in M_N(\mathbb{C}[x])$ the function $Q(x) \cdot \mathfrak{D}$ is a polynomial whose degree is no larger than the degree of $Q(x)$.

Degree-filtration preserving differential operators will necessarily have polynomial coefficients.

PROPOSITION 2.38. *An operator $\mathfrak{D} \in M_N(\mathbb{C}(x))$ is degree-filtration preserving if and only if*

$$\mathfrak{D} = \sum_{i=0}^{\ell} \partial_x^i A_i(x)$$

with $A_i(x) \in M_N(\mathbb{C}[x])$ a polynomial of degree at most i for all i.

Proof. Suppose $\mathfrak{D} \in M_N(\Omega(x))$. Then we may write

$$\mathfrak{D} = \sum_{i=0}^{\ell} \partial_x^i A_i(x)$$

for some matrix-valued rational functions $A_i(x) \in M_N(\mathbb{C}(x))$. It is clear that if $A_i(x) \in M_N(\mathbb{C}[x])$ is a polynomial of degree at most i for all i, then $\mathfrak{D}$ is degree-filtration preserving.

To prove the converse, suppose that $\mathfrak{D}$ is degree-filtration preserving and that for some i the matrix-valued rational function $A_i(x)$ is either not a polynomial or

has degree larger than i. Let j be the smallest nonnegative integer with $A_j(x)$ not a polynomial of degree $\leq j$. Then we calculate

$$\frac{1}{j!}x^j \cdot \mathfrak{D} = A_j(x) + \sum_{i=0}^{j-1} \frac{1}{(j-i)!}x^{j-i}A_i(x).$$

Since $\mathfrak{D}$ is degree-filtration preserving the left-hand side of the above equality is a polynomial of degree at most j. Furthermore the sum on the right-hand side is a polynomial of degree at most j, so we conclude

$$A_j(x) = \sum_{i=0}^{j-1} \frac{1}{(j-i)!}x^{j-i}A_i(x) - \frac{1}{j!}x^j \cdot \mathfrak{D}$$

is a polynomial of degree at most j. This is a contradiction and completes the proof. □

A Darboux conjugate $\widetilde{\mathfrak{D}}$ of $\mathfrak{D}$ will still have a great many polynomials which are eigenfunctions, but not necessarily a full basis for $M_N(\mathbb{C}[x])$. For this reason $\widetilde{\mathfrak{D}}$ may have non-polynomial rational coefficients. This leads us to consider the notion of an exceptional differential operator [21, Definition 4.1].

Definition 2.39. A differential operator $\mathfrak{d} \in \Omega(x)$ is called *exceptional* if it has polynomial eigenfunctions of all but finitely many degrees. More precisely $\mathfrak{d}$ is exceptional if there exists a finite subset $\{m_1, \ldots, m_k\} \subseteq \mathbb{N}$ satisfying the property that there exists a polynomial $p(n)$ of degree $n \in \mathbb{N}$ which is an eigenfunction of $\mathfrak{d}$ if and only if $n \notin \{m_1, \ldots, m_k\}$. The values $m_1, \ldots, m_k$ are called the *exceptional degrees* of $\mathfrak{d}$.

Example 2.40. The classical differential operators

$$\partial_x^2 - \partial_x 2x, \quad \partial_x^2 x + \partial_x(b+1-x), \quad \partial_x^2(1-x^2) + \partial_x\big(b-a+(b+a-2)x\big)$$

of Hermite, Laguerre, and Jacobi respectively have polynomial eigenfunctions of every degree and are therefore exceptional operators.

Example 2.41. The differential operator

$$\mathfrak{d} = \partial_x^2 - \partial_x\left(2x + \frac{4x}{1+2x^2}\right)$$

is an exceptional differential operator with exceptional degrees 1 and 2.

We can create new exceptional differential operators from old ones via Darboux conjugation.

Definition 2.42. Let $\mathfrak{d}, \widetilde{\mathfrak{d}} \in \Omega(x)$. We say that $\widetilde{\mathfrak{d}}$ is a *Darboux conjugate* of $\mathfrak{d}$ if there exists a differential operator $\mathfrak{h} \in \Omega(x)$ satisfying $\mathfrak{h}\mathfrak{d} = \widetilde{\mathfrak{d}}\mathfrak{h}$.

PROPOSITION 2.43. *if $\widetilde{\mathfrak{d}}$ is a Darboux conjugate of $\mathfrak{d}$, then $\mathfrak{d}$ is a Darboux conjugate of $\widetilde{\mathfrak{d}}$.*

Proof. Assume that $\widetilde{\mathfrak{d}}$ is a Darboux conjugate of $\mathfrak{d}$. Then there exists a differential operator $\mathfrak{h} \in \Omega(x)$ such that $\mathfrak{h}\mathfrak{d} = \widetilde{\mathfrak{d}}\mathfrak{h}$. This implies that $\ker(\mathfrak{h}) \cdot \widetilde{\mathfrak{d}} \subseteq \ker(\mathfrak{h})$. The kernel is finite dimensional, so we may choose a polynomial $q(\widetilde{\mathfrak{d}}) \in \mathbb{C}[\widetilde{\mathfrak{d}}]$ with $\ker(\mathfrak{h}) \cdot q(\widetilde{\mathfrak{d}}) = 0$. This implies that there exists $\widetilde{\mathfrak{h}}$ satisfying $q(\widetilde{\mathfrak{d}}) = \mathfrak{h}\widetilde{\mathfrak{h}}$. In particular $\widetilde{\mathfrak{d}}$ commutes with $\mathfrak{h}\widetilde{\mathfrak{h}}$ so that

$$\mathfrak{h}\mathfrak{d}\widetilde{\mathfrak{h}} = \widetilde{\mathfrak{d}}\mathfrak{h}\widetilde{\mathfrak{h}} = \mathfrak{h}\widetilde{\mathfrak{h}}\widetilde{\mathfrak{d}}.$$

Since $\Omega(x)$ is an integral domain, this implies $\mathfrak{d}\widetilde{\mathfrak{h}} = \widetilde{\mathfrak{h}}\widetilde{\mathfrak{d}}$. Therefore $\mathfrak{d}$ is a Darboux conjugate of $\widetilde{\mathfrak{d}}$. □

PROPOSITION 2.44. *Let $\mathfrak{d}$ be an exceptional differential operator and suppose that $\mathfrak{d}$ is a Darboux conjugate of $\widetilde{\mathfrak{d}}$. Additionally assume $\mathfrak{h}\widetilde{\mathfrak{d}} = \mathfrak{d}\mathfrak{h}$ for $\mathfrak{h} \in \Omega[x]$. Then $\widetilde{\mathfrak{d}}$ is also exceptional.*

Proof. For all but finitely many n, there exists a polynomial $p(x,n)$ of degree n which is an eigenfunction of $\mathfrak{d}$. This implies that $p(x,n) \cdot \mathfrak{h}$ is a polynomial eigenfunction of $\widetilde{\mathfrak{d}}$. One may easily show that there exists an integer $m \in \mathbb{Z}$ such that for all but finitely many n the degree of $p(x,n) \cdot \mathfrak{h}$ is $n+m$. Consequently $\widetilde{\mathfrak{d}}$ is exceptional. □

Definition 2.45. We call a differential operator $\mathfrak{d} \in \Omega(x)$ *degree-filtration preserving* if for all polynomials $p(x) \in \mathbb{C}[x]$ the function $p(x) \cdot \mathfrak{d}$ is a polynomial of degree at most the degree of $p(x)$.

The most interesting fact about exceptional operators is that for low orders exceptional operators always arise as Darboux conjugates of degree preserving differential operators. For order 0 or 1, one may in fact show that exceptional differential operators are necessarily degree-filtration preserving. The case of order 2 is substantially more difficult and is treated by the next theorem.

THEOREM 2.46. (García-Ferrero, Gómez-Ullate, and Milson [21]) *If $\mathfrak{d}$ is an exceptional, second order differential operator then $\mathfrak{d}$ is a Darboux conjugate to a degree-filtration preserving differential operator.*

3. Bispectral Darboux transformations of weight matrices.

3.1. W-adjoints of algebras of matrix differential and shift operators. In this paper, we will be restricting our attention to a single specific bispectral context. Specifically, we will consider the bispectral context $(M_N(\mathcal{S}), M_N(\Omega[x]), \mathcal{P})$ where $\mathcal{P}$ is the set of all functions $P : \mathbb{C} \times \mathbb{N} \to M_N(\mathbb{C})$ satisfying the property that for any fixed n, $P(x,n)$ is a matrix-valued rational function in x. Equivalently,

P is the set of all semi-infinite sequences of matrix-valued rational functions. The algebra $\mathcal{S}$ is the collection of discrete operators in variable n. The multiplicative subalgebra $\mathcal{M}(\mathcal{S})$ of $\mathcal{S}$ consists of all functions $\mathbb{N} \to \mathbb{C}$ and $\mathcal{S}$ itself is the set of operators $\mathscr{L}$ of the form

$$\mathscr{L} = \sum_{j=0}^{k} a_j(n)\mathscr{D}^j + \sum_{j=1}^{\ell} b_j(n)(\mathscr{D}^*)^j$$

for $a_j, b_j : \mathbb{N} \to \mathbb{C}$ and $\mathscr{D}, \mathscr{D}^*$ the operators acting on sequences by

$$(\mathscr{D} \cdot a)(n) = a(n+1), \quad \text{and} \quad (\mathscr{D}^* \cdot a)(n) = \begin{cases} a(n-1), & n \neq 0 \\ 0, & n = 0. \end{cases}$$

Remark 3.1. The operator $\mathscr{D}^*$ is adjoint to the operator $\mathscr{D}$ in the Hilbert space $\ell^2(\mathbb{N})$, and this explains the notation. Moreover $\mathscr{D}\mathscr{D}^* = 1$ so $\mathscr{D}^*$ is a right inverse of $\mathscr{D}$. However $\mathscr{D}^*\mathscr{D} = 1 - \delta_{0n}$, so $\mathscr{D}^*$ is not quite a left inverse of $\mathscr{D}$.

Remark 3.2. For us, sequences will always be indexed over $\mathbb{N}$. With this in mind, we will always take the value of a sequence at a negative integer to be 0, unless otherwise stated. Furthermore, n will be treated as an indeterminant.

Now let $W(x)$ be a weight matrix and let $P(x,n)$ be the associated sequence of monic orthogonal polynomials. Then the three-term recursion relation of $P(x,n)$ tells us that there exists $\mathscr{L} \in M_N(\mathcal{S})$ of the form

$$\mathscr{L} = \mathscr{D} + A(n) + B(n)\mathscr{D}^*$$

such that

$$\mathscr{L} \cdot P(x,n) = P(x,n)x.$$

Thus if $\mathcal{D}(W)$ contains a differential operator $\mathfrak{D}$ of positive order, then $P(x,n)$ will be bispectral with respect to this bispectral context.

As proved by Tirao and Grünbaum [26], the algebra $\mathcal{D}(W)$ is closed under the W-adjoint †. Since the left and right bispectral algebras of P are isomorphic, it makes sense that the involution † of $\mathcal{B}_R(P)$ induces an involution on $\mathcal{B}_L(P)$. Moreover, the involution † is actually the restriction of an involution on a much larger algebra. In fact, the right Fourier algebra $\mathcal{F}_R(P)$ is closed under †. Thus $\mathcal{F}_L(P)$ is closed under the involution induced by the generalized Fourier map also. To prove this, we first describe the induced involution. First note that $M_N(\mathcal{S})$ has a natural $*$ operation extending Hermitian conjugation on matrices and sending $\mathscr{D}$

to $\mathscr{D}^*$ (and vice-versa). Specifically it is defined by

(3.3)
$$\left(\sum_{j=0}^{\ell} A_j(n)\mathscr{D}^j + \sum_{j=1}^{m} \left(B_j(n)\mathscr{D}^*\right)^j\right)^* := \sum_{j=0}^{\ell} A_j(n-j)^*(\mathscr{D}^*)^j + \sum_{j=1}^{m} B_j(n+j)\mathscr{D}^j.$$

With this in mind, we define the W-adjoint on $M_N(\mathcal{S})$.

Definition 3.4. The *W-adjoint of a shift operator* $\mathscr{M} \in M_N(\mathcal{S})$ is defined to be

$$\mathscr{M}^\dagger := \big\|P(x,n)\big\|_W^2 \mathscr{M}^* \big\|P(x,n)\big\|_W^{-2},$$

where here we view $\|P(x,n)\|_W^2 = \langle P(x,n), P(x,n)\rangle_W$ as the sequence of Hermitian matrices in $M_N(\mathbb{C})$ defined by the W-norms of the monic orthogonal polynomials of the weight matrix $W(x)$.

The next lemma shows that this adjoint operation is well behaved with respect to the inner product defined by the weight matrix $W(x)$.

LEMMA 3.5. *Let $\mathscr{M} \in M_N(\mathcal{S})$. We write $\mathscr{M}_n$ or $\mathscr{M}_m$ to emphasize that the shift operator is acting on the discrete variable n or m, respectively. Then consider the two polynomial sequences $P(x,n), P(x,m)$ in variables n and m, respectively. We have that*

$$\big\langle \mathscr{M}_n \cdot P(x,n), P(x,m)\big\rangle_W = \big\langle P(x,n), \mathscr{M}_m^\dagger \cdot P(x,m)\big\rangle_W.$$

Proof. By linearity, it is enough to show that the above identity holds when $\mathscr{M}_n = A(n)\mathscr{D}^\ell$ or when $\mathscr{M}_n = A(n)(\mathscr{D}^*)^\ell$ for some sequence $A(n)$ and some integer $\ell \geq 0$. We will prove the first case, leaving the second to the reader. In this case $\mathscr{M}_n^* = A(n-\ell)^*(\mathscr{D}^*)^\ell$ so that

$$\mathscr{M}_n^\dagger = \big\|P(x,n)\big\|_W^2 A(n-\ell)^* \big\|P(x,n-\ell)\big\|^{-2}(\mathscr{D}^*)^\ell.$$

We calculate

$$\begin{aligned}
&\big\langle \mathscr{M}_n \cdot P(x,n), P(x,m)\big\rangle_W \\
&\quad = \big\langle A(n)\cdot P(x,n+\ell), P(x,m)\big\rangle_W \\
&\quad = A(n)\big\|P(m)\big\|_W^2 \delta_{n+\ell,m} \\
&\quad = \big\|P(x,n)\big\|_W^2 \big\|P(x,m-\ell)\big\|_W^{-2} A(m-\ell)\big\|P(x,m)\big\|_W^2 \delta_{n,m-\ell} \\
&\quad = \big\langle P(x,n), \big\|P(x,m)\big\|_W^2 A(m-\ell)^* \big\|P(x,m-\ell)\big\|_W^{-2} \cdot P(x,m-\ell)\big\rangle_W \\
&\quad = \big\langle P(x,n), \mathscr{M}_m^\dagger \cdot P(x,m)\big\rangle_W.
\end{aligned}$$

This proves the lemma. □

3.2. The Fourier algebras of orthogonal matrix polynomials. We wish to describe the left and right Fourier algebras of P. The next theorem describes the corresponding Fourier algebras in this case. To prove it, we first require a lemma characterizing the differential operators contained in the algebra of left $M_N(\mathbb{C})$-linear endomorphisms of $M_N(\mathbb{C}[x])$.

For every pair of elements a, b of an algebra $\mathcal{A}$, we denote

$$\mathrm{Ad}_a(b) := ab - ba \quad \text{and} \quad \mathrm{Ad}_a^{k+1}(b) := \mathrm{Ad}_a^k\big(\mathrm{Ad}_a(b)\big).$$

LEMMA 3.6. *Suppose that $\Phi : M_N(\mathbb{C}[x]) \to M_N(\mathbb{C}[x])$ is a left $M_N(\mathbb{C})$-module endomorphism. If $\mathrm{Ad}_{xI}^{\ell+1}(\Phi) = 0$ for some integer $\ell \geq 0$, then there exists a differential operator $\mathfrak{D} \in M_N(\Omega[x])$ of order at most ℓ satisfying $\Phi(P(x)) = P(x) \cdot \mathfrak{D}$ for all $P(x) \in M_N(\mathbb{C}[x])$.*

Proof. First recall that any left $M_N(\mathbb{C}[x])$-module endomorphism of $M_N(\mathbb{C}[x])$ is of the form $P(x) \mapsto P(x)A(x)$ for some $A(x) \in M_N(\mathbb{C}[x])$. With this in mind, we proceed by induction on ℓ. If $\ell = 0$, then $\mathrm{Ad}_{xI}(\Phi) = 0$ so that $\Phi(xP(x)) = x\Phi(P(x))$ for all $P(x) \in M_N(\mathbb{C}[x])$. Since $\Phi(x)$ is a left $M_N(\mathbb{C})$-module endomorphism of $M_N(\mathbb{C}[x])$ it follows that Φ is a left $M_N(\mathbb{C}[x])$-module endomorphism. Thus by the fact recalled at the start our lemma is true when $\ell = 0$.

As an inductive assumption, suppose that the statement of the lemma is true for all $\ell \leq m$. Now suppose that $\mathrm{Ad}_{xI}^{m+1}(\Phi) = 0$. Then $\mathrm{Ad}_{xI}(\Phi)$ is a left $M_N(\mathbb{C})$-module endomorphism of $M_N(\mathbb{C}[x])$ with $\mathrm{Ad}_{xI}^{m}(\mathrm{Ad}_{xI}(\Phi)) = 0$. Thus by our inductive assumption there exists a differential operator $\mathfrak{D} \in M_N(\mathbb{C}[x])$ of order at most m satisfying

$$\mathrm{Ad}_{xI}(\Phi)\big(P(x)\big) = P(x) \cdot \mathfrak{D}$$

for all $P(x) \in M_N(\mathbb{C}[x])$. We write $\mathfrak{D} = \sum_{j=0}^{m} \partial_x^j A_j(x)$ and define a new left $M_N(\mathbb{C})$-module endomorphism Ψ of $M_N(\mathbb{C}[x])$ by

$$\Psi(P(x)) = \Phi(P(x)) - P(x) \cdot \left[\sum_{j=0}^{m} \partial_x^{j+1} \frac{1}{j+1} A_j(x) \right].$$

We calculate

$$\begin{aligned} \mathrm{Ad}_{xI}(\Psi)\big(P(x)\big) &= \Psi\big(xP(x)\big) - x\Psi\big(P(x)\big) \\ &= \mathrm{Ad}_{xI}(\Phi)\big(P(x)\big) - P(x) \cdot \left[\sum_{j=0}^{m} \partial_x^j A_j(x) \right] = 0. \end{aligned}$$

Therefore there exists a matrix $A(x) \in M_N(\mathbb{C}[x])$ with $\Psi(P(x)) = P(x)A(x)$ for all $P(x) \in M_N(\mathbb{C}[x])$. Hence for all $P(x)$ we have

$$\Phi(P(x)) = P(x) \cdot \left[\sum_{j=0}^{m} \partial_x^{j+1} \frac{1}{j+1} A_j(x) + A(x) \right].$$

Hence by induction our lemma is true for all ℓ. □

THEOREM 3.7. *Let $W(x)$ be a weight matrix and let $P(x,n)$ be the associated sequence of monic orthogonal polynomials and let $\mathscr{L} \in M_N(\mathcal{S})$ with $\mathscr{L} \cdot P(x,n) = P(x,n)x$. Then the Fourier algebras of $P(x,n)$ are given by*

$$\mathcal{F}_L(P) = \{\mathscr{M} \in M_N(\mathcal{S}) : \mathrm{Ad}_{\mathscr{L}}^{k+1}(\mathscr{M}) = 0 \text{ for some } k \geq 0\},$$
$$\mathcal{F}_R(P) = \{\mathfrak{D} \in M_N(\Omega[x]) : \mathfrak{D} \text{ is } W\text{-adjointable and } \mathfrak{D}^\dagger \in M_N(\Omega[x])\}.$$

Proof. We will first prove our formula for $\mathcal{F}_L(P)$. If $\mathscr{M} \in \mathcal{F}_L(P)$, then $\mathscr{M} \cdot P(x,n) = P(x,n) \cdot \mathfrak{D}$. If $\mathfrak{D}$ has order ℓ, then $\mathrm{Ad}_{xI}^{\ell+1}(\mathfrak{D}) = 0$. Applying the generalized Fourier map, this implies that $\mathrm{Ad}_{\mathscr{L}}^{\ell+1}(\mathscr{M}) = 0$. Thus

$$\mathcal{F}_L(P) \subseteq \{\mathscr{M} \in M_N(\mathcal{S}) : \mathrm{Ad}_{\mathscr{L}}^{k+1}(M) = 0 \text{ for some } k \geq 0\}.$$

To prove the opposite containment, suppose that $\mathscr{M} \in M_N(\mathcal{S})$ with $\mathrm{Ad}_{\mathscr{L}}^{\ell+1}(M) = 0$ for some integer ℓ. Consider the left $M_N(\mathbb{C})$-module endomorphism Φ of $M_N(\mathbb{C}[x])$ induced by

$$\Phi : P(x,n) \longmapsto \mathscr{M} \cdot P(x,n).$$

For any $Q(x) \in M_N(\mathbb{C}[x])$, we write $Q(x) \cdot \Phi$ to mean $\Phi(Q)(x)$. Then for all n we see

$$P(x,n) \cdot \mathrm{Ad}_{xI}^{\ell+1}(\Phi) = \mathrm{Ad}_{\mathscr{L}}^{\ell+1}(\mathscr{M}) \cdot P(x,n) = 0.$$

Consequently $\mathrm{Ad}_{xI}^{\ell+1}(\Phi) = 0$ and by the previous lemma we know that

$$\mathscr{M} \cdot P(x,n) = \Phi\big(P(x,n)\big) = P(x,n) \cdot \mathfrak{D}$$

for some differential operator $\mathfrak{D} \in M_N(\Omega[x])$ for all n. In particular $\mathscr{M} \in \mathcal{F}_L(P)$. This proves

$$\mathcal{F}_L(P) = \{\mathscr{M} \in M_N(\mathcal{S}) : \mathrm{Ad}_{\mathscr{L}}^{k+1}(\mathscr{M}) = 0 \text{ for some } k \geq 0\}.$$

Suppose that $\mathfrak{D} \in M_N(\Omega[x])$ and $\mathfrak{D}$ is W-adjointable with $\mathfrak{D}^\dagger \in M_N(\Omega[x])$. Then we may write

$$\mathfrak{D} = \sum_{i=0}^{\ell} \partial_x^i A_i(x) \quad \text{and} \quad \mathfrak{D}^\dagger = \sum_{i=0}^{\ell} \partial_x^i B_i(x)$$

for some polynomials $A_i(x), B_i(x) \in M_N(\mathbb{C}[x])$. Let m be the maximum degree of all of these polynomials. Then we may write

$$P(x,n)\cdot\mathfrak{D} = \sum_{j=0}^{n+m} C(n,j)P(x,j)$$

for the matrices $C(n,j) \in M_N(\mathbb{C})$, defined by

$$C(n,j) = \big\langle P(x,n)\cdot\mathfrak{D}, P(x,j)\big\rangle_W \big\|P(x,j)\big\|_W^{-2}.$$

Then using the W-adjointability, we see that

$$C(n,j) = \big\langle P(x,n), P(x,j)\cdot\mathfrak{D}^\dagger\big\rangle_W \big\|P(x,j)\big\|_W^{-2}$$

and therefore $C(n,j)$ is zero if $n-j>m$. Thus

$$P(x,n)\cdot\mathfrak{D} = \sum_{j=n-m}^{n+m} C(n,j)P(x,j) = \mathscr{M}\cdot P(x,n)$$

for

$$\mathscr{M} = \sum_{j=0}^{m} C(n,n+j)\mathscr{D}^j + \sum_{j=1}^{m} C(n,n-j)(\mathscr{D}^*)^j.$$

Therefore $\mathfrak{D}\in\mathcal{F}_R(P)$. Moreover, since $\mathrm{Ad}_x^{\ell+1}(\mathfrak{D})=0$ we know that $\mathrm{Ad}_{\mathscr{L}}^{\ell+1}(\mathscr{M})=0$. This proves

$$\mathcal{F}_R(P) \supseteq \big\{\mathfrak{D}\in M_N\big(\Omega[x]\big) : \mathfrak{D} \text{ is } W\text{-adjointable and } \mathfrak{D}^\dagger \in M_N\big(\Omega[x]\big)\big\}.$$

Next, suppose instead that $\mathfrak{D}\in\mathcal{F}_R(P)$. Then there exists $\mathscr{M}\in M_N(\mathcal{S})$ with $\mathscr{M}\cdot P(x,n) = P(x,n)\cdot\mathfrak{D}$. Then $\mathrm{Ad}_{\mathscr{L}}^{\ell+1}(\mathscr{M})=0$ for ℓ the order of $\mathfrak{D}$. Since xI is $\dagger$-symmetric, so too is $\mathscr{L}$. It follows that $\mathrm{Ad}_{\mathscr{L}}^{\ell+1}(\mathscr{M}^\dagger)=0$ and therefore $\mathscr{M}^\dagger\cdot P(x,n) = P(x,n)\cdot\widetilde{\mathfrak{D}}$ for some $\widetilde{\mathfrak{D}}\in\mathcal{F}_R(P)$. Note that by the previous lemma for all $m\in\mathbb{Z}$

$$\begin{aligned}\big\langle P(x,n)\cdot\mathfrak{D}, P(x,m)\big\rangle_W &= \big\langle \mathscr{M}_n\cdot P(x,n), P(x,m)\big\rangle_W \\ &= \big\langle P(x,n), \mathscr{M}_m^\dagger\cdot P(x,m)\big\rangle_W \\ &= \big\langle P(x,n), P(x,m)\cdot\widetilde{\mathfrak{D}}\big\rangle_W.\end{aligned}$$

This implies that $\langle P\cdot\mathfrak{D}, Q\rangle_W = \langle P, Q\cdot\widetilde{\mathfrak{D}}\rangle_W$ for all polynomials $P,Q\in M_N(\mathbb{C}[x])$. Hence $\mathfrak{D}$ is W-adjointable and $\widetilde{\mathfrak{D}}=\mathfrak{D}^\dagger$. Thus $\mathfrak{D}^\dagger$ is W-symmetric with $\mathfrak{D}^\dagger\in M_N(\Omega[x])$. This proves

$$\mathcal{F}_R(P) = \big\{\mathfrak{D}\in M_N\big(\Omega[x]\big) : \mathfrak{D} \text{ is } W\text{-adjointable and } \mathfrak{D}^\dagger \in M_N\big(\Omega[x]\big)\big\}.$$

□

As a corollary of this, we see that the left and right Fourier algebras are closed under the adjoint operation. In this way, the previous theorem provides a more conceptual proof of the fact $\mathcal{D}(W)$ is closed under † than the proof found in [26], see Theorem 4.1 below.

COROLLARY 3.8. *Let $W(x)$ be a weight matrix and $P(x,n)$ be the associated sequence of monic orthogonal polynomials. The left and right Fourier algebras $\mathcal{F}_L(P)$ and $\mathcal{F}_R(P)$ are closed under †. Furthermore for all $\mathscr{M} \in \mathcal{F}_L(P)$ we have $b_P(\mathscr{M}^\dagger) = b_P(\mathscr{M})^\dagger$.*

Proof. It is clear from the statement of the previous theorem that $\mathcal{F}_R(P)$ is closed under †. We showed in the proof of the previous theorem that if $\mathfrak{b}_P(\mathscr{M}) = \mathfrak{D}$, then $b_P(\mathscr{M}^\dagger) = \mathfrak{D}^\dagger$. Therefore the identity in the statement of the corollary is true. Since b_P is an isomorphism, this implies $\mathcal{F}_L(P)$ is also closed under †. □

Definition 3.9. Let $W(x)$ and $\widetilde{W}(x)$ be weight matrices and let $P(x,n)$ and $\widetilde{P}(x,n)$ be their associated sequences of monic orthogonal polynomials. We say that $\widetilde{P}(x,n)$ is a bispectral Darboux transformation of $P(x,n)$ if there exist differential operators $\mathfrak{T}, \widetilde{\mathfrak{T}} \in \mathcal{F}_R(P)$, polynomials $F(x), \widetilde{F}(x)$ and sequences of matrices $C(n), \widetilde{C}(n)$ which are nonsingular for almost all n and satisfy

$$C(n)\widetilde{P}(x,n) = P(x,n)\cdot \mathfrak{T}F(x) \quad \text{and} \quad \widetilde{C}(n)P(x,n) = \widetilde{P}(x,n)\cdot \widetilde{F}(x)\widetilde{\mathfrak{T}}.$$

We say that $\widetilde{W}(x)$ is a *bispectral Darboux transformation* of $W(x)$ if $\widetilde{P}(x,n)$ is a bispectral Darboux transformation of $P(x,n)$.

Remark 3.10. We do not require the $C(n), \widetilde{C}(n)$'s to be nonsingular for all n, because doing so would eliminate many important bispectral Darboux transformations, including many trivial ones. In particular, the vanishing of $C(n)$ or $\widetilde{C}(n)$ corresponds to "poles" of the eigenvalues, which occur naturally. For example, we should expect any $\mathfrak{T}, \widetilde{T} \in \mathcal{B}_R(P)$ which are not zero divisors to define a bispectral Darboux transformation of P to itself. However, if we required the $C(n), \widetilde{C}(n)$'s to not have poles then these trivial transformations would have to be thrown out. This is a feature of the "discrete" bispectral case not featured in the continuous case, since in the latter we deal with meromorphic eigenvalues which can already have poles.

Example 3.11. Bispectral Darboux transformations can be used to artificially create weight matrices $W(x)$ for which the algebra $\mathcal{D}(W)$ contains differential operators of lower order at the expense of enlarging the size of the weight by the following procedure.

Let $W(x)$ and $\widetilde{W}(x)$ be a pair of weight matrices of size $N \times N$ which are bispectral Darboux transformations from each other. Denote by $P(x,n)$ and $\widetilde{P}(x,n)$

their associated sequences of monic orthogonal polynomials. Thus

$$\widetilde{P} = p^{-1} \cdot P \cdot \mathfrak{d} q^{-1} \quad \text{and} \quad P = \widetilde{p}^{-1} \cdot \widetilde{P} \cdot \widetilde{q}^{-1} \widetilde{\mathfrak{d}}$$

for some $\mathfrak{d}, \widetilde{\mathfrak{d}} \in \mathcal{F}_R(P)$ and some functions $q, \widetilde{q} \in \mathcal{F}_R(P)$, $p, \widetilde{p} \in \mathcal{F}_L(P)$.

The two sequences $P(x,n)$ and $\widetilde{P}(x,n)$ are eigenfunctions of differential operators of orders $\operatorname{ord}\mathfrak{d} + \operatorname{ord}\widetilde{\mathfrak{d}}$,

$$P \cdot (\mathfrak{d} q^{-1} \widetilde{q}^{-1} \widetilde{\mathfrak{d}}) = p \widetilde{p} \cdot P \quad \text{and} \quad \widetilde{p} p \cdot \widetilde{P} = \widetilde{P} \cdot (\widetilde{q}^{-1} \widetilde{\mathfrak{d}} \mathfrak{d} q^{-1}).$$

At the same time, the sequence of orthogonal polynomials for the block diagonal weight $\operatorname{diag}(W(x), \widetilde{W}(x))$ is $\operatorname{diag}(P(x,n), \widetilde{P}(x,n))$. It is an eigenfunction of a differential operator of order $\max\{\operatorname{ord}\mathfrak{d}, \operatorname{ord}\widetilde{\mathfrak{d}}\}$,

$$\operatorname{diag}(\widetilde{p}, p) \cdot \operatorname{diag}(P, \widetilde{P}) = \operatorname{diag}(P, \widetilde{P}) \cdot \begin{pmatrix} 0 & \mathfrak{d} q^{-1} \\ \widetilde{q}^{-1} \widetilde{\mathfrak{d}} & 0 \end{pmatrix}.$$

Note that the leading coefficient of the operator in the right-hand side does not have the positivity property from Theorem 1.3. Thus by doubling the size of the weight matrix $W(x)$ we were able to construct orthogonal matrix polynomials that are eigenfunctions of differential operators of order $\max\{\operatorname{ord}\mathfrak{d}, \operatorname{ord}\widetilde{\mathfrak{d}}\}$ using orthogonal matrix polynomials which were eigenfunctions of differential operators of order $\operatorname{ord}\mathfrak{d} + \operatorname{ord}\widetilde{\mathfrak{d}}$.

4. The algebraic structure of $\mathcal{D}(W)$. In this section, we describe the algebraic structure of the algebra $\mathcal{D}(W)$ in very specific terms.

4.1. The $*$-involution of $\mathcal{D}(W)$. The structural results featured in this subsection are dependent on the existence of a proper $*$-involution on $\mathcal{D}(W)$. Recall that a (complex) $*$-*algebra* is an algebra $\mathcal{A}$ over $\mathbb{C}$ with an involutive skew-linear anti-automorphism $\mathfrak{D} \mapsto \mathfrak{D}^\dagger$. One of the first general results regarding the algebra $\mathfrak{D}(W)$ is that it is a $*$-algebra under the W-adjoint $\dagger$.

THEOREM 4.1. (Grünbaum and Tirao [26]) *Let $W(x)$ be a weight matrix, and let $\mathfrak{D} \in \mathcal{D}(W)$. Then a W-adjoint of $\mathfrak{D}$ exists and is in $\mathcal{D}(W)$. The operator $\mathfrak{D} \mapsto \mathfrak{D}^\dagger$ is an involution on $\mathcal{D}(W)$ giving $\mathcal{D}(W)$ the structure of a $*$-algebra.*

This is also a consequence of Corollary 3.8 in the previous section. As a result, we have the following corollary proved in [26].

COROLLARY 4.2. *Let $W(x)$ be a weight matrix. Then $\mathcal{D}(W)$ contains a differential operator of order m if and only if $\mathcal{D}(W)$ contains a W-symmetric differential operator of order m.*

Definition 4.3. The involution $\mathfrak{D} \mapsto \mathfrak{D}^\dagger$ of a $*$-algebra $\mathcal{A}$ is called *positive* if for all $\mathfrak{D}_1, \ldots, \mathfrak{D}_n \in \mathcal{A}$,

$$\mathfrak{D}_1\mathfrak{D}_1^\dagger + \cdots + \mathfrak{D}_n\mathfrak{D}_n^\dagger = 0 \quad \text{if and only if } \mathfrak{D}_1 = \cdots = \mathfrak{D}_n = 0. \tag{4.4}$$

The involution is called *proper* if the above property holds for $n = 1$, see [5, Sect. 2].

THEOREM 4.5. (Casper [9]) *Let $W(x)$ be a weight matrix. The involution of the $*$-algebra $\mathcal{D}(W)$ is positive. As a consequence $\mathcal{D}(W)$ is a semiprime PI-algebra, and thus its center $\mathcal{Z}(W)$ is reduced.*

Proof. The generalized Fourier map defined in the next subsection embeds $\mathcal{D}(W)$ into the matrix ring $M_N(\mathbb{C}[n])$ of polynomials in variable n. Hence $\mathcal{D}(W)$ is a PI-algebra.

To prove that the involution of the $*$-algebra $\mathcal{D}(W)$ is positive, consider $\mathfrak{D}_1, \ldots, \mathfrak{D}_n \in \mathcal{D}(W)$ such that $\mathfrak{D}_1\mathfrak{D}_1^\dagger + \cdots + \mathfrak{D}_n\mathfrak{D}_n^\dagger = 0$. Then for all polynomials $P(x)$ we have that

$$\begin{aligned} 0 &= \left\langle P(x) \cdot (\mathfrak{D}_1\mathfrak{D}_1^\dagger + \cdots + \mathfrak{D}_n\mathfrak{D}_n^\dagger), P(x) \right\rangle_W \\ &= \sum_k \left\langle P(x) \cdot \mathfrak{D}_k, P(x) \cdot \mathfrak{D}_k \right\rangle_W = \sum_k \left\| P(x) \cdot \mathfrak{D}_k \right\|_W^2. \end{aligned}$$

Therefore $P(x) \cdot \mathfrak{D}_k = 0$ for all polynomials $P(x)$ and $1 \leq k \leq n$ which implies $\mathfrak{D}_k = 0$.

Every $*$-algebra $\mathcal{A}$ with a proper involution is semiprime. For this one needs to show that for any $\mathfrak{D} \in \mathcal{A}$, the condition $\mathfrak{D}\mathcal{A}\mathfrak{D} = 0$ implies $\mathfrak{D} = 0$. For consistency, denote the $*$-operation of $\mathcal{A}$ by $\dagger$. Suppose $\mathfrak{D}\mathcal{A}\mathfrak{D} = 0$. Then $\mathfrak{D}\mathfrak{D}^\dagger\mathfrak{D} = 0$ and therefore

$$0 = \mathfrak{D}\mathfrak{D}^\dagger\mathfrak{D}\mathfrak{D}^\dagger = (\mathfrak{D}\mathfrak{D}^\dagger)(\mathfrak{D}\mathfrak{D}^\dagger)^\dagger.$$

Applying (4.4) gives $\mathfrak{D}\mathfrak{D}^\dagger = 0$ and, applying (4.4) one more time, gives $\mathfrak{D} = 0$.

The center of every semiprime algebra $\mathcal{A}$ is reduced. If $\mathfrak{D} \in \mathcal{Z}(\mathcal{A})$ and $\mathfrak{D}^2 = 0$, then $\mathfrak{D}\mathcal{A}\mathfrak{D} = 0$, hence $\mathfrak{D} = 0$. □

4.2. The center of $\mathcal{D}(W)$. In this subsection we will use the involution of $\mathcal{D}(W)$ to prove that the center $\mathcal{Z}(W)$ of $\mathcal{D}(W)$ is a Noetherian algebra of Krull dimension at most 1 and that the minimal prime ideals of $\mathcal{Z}(W)$ are preserved by the W-adjoint of $\mathcal{D}(W)$. There are simple examples of commutative subalgebras of $M_2(\mathbb{C}[n])$ which are not finitely generated, such as $\mathbb{C}I \oplus \left(\begin{smallmatrix} 0 & \mathbb{C}[n] \\ 0 & 0 \end{smallmatrix}\right)$, so the presence of the involution is crucial to our arguments.

LEMMA 4.6. *Let $\mathcal{R}_1, \dots, \mathcal{R}_\ell$ be any collection of Dedekind domains whose fraction fields are finite extensions of $\mathbb{C}(t)$. Then any subalgebra of $\mathcal{R}_1 \times \cdots \times \mathcal{R}_\ell$ is finitely generated.*

Proof. Note that each $\mathcal{R}_i$ is isomorphic to the ring of functions of $X_i \backslash \{p\}$ for some smooth, projective algebraic curve X_i over $\mathbb{C}$ and some closed point $p_i \in X_i$ called the point at infinity. The local ring of X_i at p_i is isomorphic to a discrete valuation ring S_i with fraction field $\mathcal{L}_i$ and residue field $\mathbb{C}$. The completion of this ring at its maximal ideal is isomorphic to $\mathbb{C}[[t]]$, and thus we have an injection $\mathcal{L}_i \subseteq \mathbb{C}((t))$ into the ring of Laurent series with coefficients in $\mathbb{C}$. Restriction of functions on $X_i \backslash \{p_i\}$ to $(X_i \backslash \{p_i\}) \cap \operatorname{Spec}(S_i)$ defines an injection $\mathcal{R}_i \to \mathcal{L}_i$ and thereby an embedding $\phi_i : \mathcal{R}_i \to \mathbb{C}((t))$. Note that for any nonzero $r \in \mathcal{R}_i$ the image $\phi_i(r)$ has nonpositive degree.

Let $\mathcal{A}$ be a subalgebra of $\mathcal{R}_1 \times \cdots \times \mathcal{R}_\ell$. For any nonzero $A = (r_1, \dots, r_\ell) \in \mathcal{A}$, we define

$$\deg(A) = \max_{r_j \neq 0} \deg\big(\phi_j(r_j)\big) \quad \text{and} \quad \operatorname{supp}(A) = \big\{i : \deg(r_i) = \deg(A)\big\}.$$

Further, set

$$\mathcal{A}_m = \big\{A \in \mathcal{A} : A = 0 \text{ or } \deg(A) \geq -m\big\}.$$

We define $\chi : \mathcal{R}_1 \times \cdots \times \mathcal{R}_\ell \to \mathbb{C}^{\oplus \ell}$ by setting $\chi(r_1, \dots, r_\ell)$ to be the vector whose i'th entry is the leading coefficient of r_i if $\nu_i(r_i) = \nu(r_1, \dots, r_\ell)$ and 0 otherwise (with $\chi(0) = 0$). Note that χ induces a $\mathbb{C}$-linear injection $\mathcal{A}_{m+1}/\mathcal{A}_m \to \mathbb{C}^{\oplus \ell}$ so that in particular $\dim(\mathcal{A}_{m+1}/\mathcal{A}_m) \leq \ell$.

We have $\operatorname{supp}(A^j) = \operatorname{supp}(A)$ for all $A \in \mathcal{A}$. Now consider any two elements $A, B \in \mathcal{A}$ with $\nu(A) = m$ and $\nu(B) = n$. Then for some nonzero $\lambda \in \mathbb{C}$ we have

$$\operatorname{supp}(A^n + \lambda B^m) = \operatorname{supp}(A) \cup \operatorname{supp}(B).$$

Thus there exists $C \in \mathcal{A}$ such that $\operatorname{supp}(A) \subseteq \operatorname{supp}(C)$ for all $A \in \mathcal{A}$. Denote $k = \nu(C)$. Multiplication by C induces an injection $(\mathcal{A}_{m+1}/\mathcal{A}_m) \to (\mathcal{A}_{m+k+1}/\mathcal{A}_{m+k})$ for all m. Since the dimension of $(\mathcal{A}_{m+1}/\mathcal{A}_m)$ is bounded by ℓ, there must exist $n > 0$ such that for all $m > n$ the above map is an isomorphism. It follows that $\mathcal{A}$ is generated by a $\mathbb{C}$-linear basis for $\mathcal{A}_{n+k}$, keeping in mind that $C \in \mathcal{A}_{n+k}$. In particular $\mathcal{A}$ is finitely generated. □

LEMMA 4.7. *Let $\mathcal{A} \subseteq M_N(\mathbb{C}[t])$ be a subalgebra of the matrix algebra with coefficients in the polynomial ring $\mathbb{C}[t]$ in an indeterminant t. If all of the elements of $\mathcal{A}$ are simultaneously diagonalizable over the algebraic closure $\overline{\mathbb{C}(t)}$ of $\mathbb{C}(t)$, then $\mathcal{A}$ is finitely generated as an algebra over $\mathbb{C}$.*

Proof. Let $\mathcal{F} = \mathbb{C}(t)$ and suppose that $\mathcal{A}$ is simultaneously diagonalizable over the algebraic closure $\overline{\mathcal{F}}$. Let $\mathcal{B}$ be the $\mathcal{F}$-subalgebra of $M_N(\mathcal{F})$ spanned by $\mathcal{A}$.

Note in particular that all the elements of $\mathcal{B}$ are simultaneously diagonalizable. As a consequence, the nilradical of $\mathcal{B}$ is 0 and since $\mathcal{B}$ is finite dimensional over $\mathcal{F}$ we conclude that $\mathcal{B}$ is semisimple. Therefore by Artin-Wedderburn, $\mathcal{B}$ is isomorphic to a direct product of matrix algebras over skew-fields over $\mathcal{F}$. However, since $\mathcal{B}$ is commutative this in fact implies that $\mathcal{B}$ is isomorphic to a direct product of field extensions

$$\mathcal{B} \cong \mathcal{L}_1 \times \cdots \times \mathcal{L}_\ell$$

where each $\mathcal{L}_i$ is a finite extension of the field $\mathcal{F}$.

For each i, let $\mathcal{R}_i$ denote the integral closure of $\mathbb{C}[t]$ in $\mathcal{L}_i$. In particular it satisfies the assumptions of the previous lemma. The algebra $\mathcal{A}$ is contained in $M_N(\mathbb{C}[t])$, so each element of $\mathcal{A}$ is integral over $\mathbb{C}[t]$. Hence the image of $\mathcal{A}$ in $\mathcal{L}_1 \times \cdots \times \mathcal{L}_\ell$ is contained in $\mathcal{R}_1 \times \cdots \times \mathcal{R}_\ell$. By the previous lemma, $\mathcal{A}$ must be finitely generated. □

THEOREM 4.8. *The center $\mathcal{Z}(W)$ of $\mathcal{D}(W)$ is a Noetherian algebra of Krull dimension at most* 1.

Proof. First recall that the generalized Fourier map embeds $\mathcal{D}(W)$ as a subalgebra $\mathcal{A}$ of $M_N(\mathbb{C}[n])$. This latter ring has GK dimension 1, so the GK dimension of $\mathcal{A}$ must be at most 1. Therefore the GK dimension of the center $\mathcal{Z}$ of $\mathcal{A}$ is at most 1. For a commutative ring the GK dimension and the Krull dimension are the same, so the Krull dimension of $\mathcal{Z}$ is at most 1.

Let $\mathcal{A} \subseteq M_N(\mathbb{C}[n])$ denote the preimage of $\mathcal{D}(W)$ under the generalized Fourier map b_P and let $\mathcal{Z}$ denote the center of $\mathcal{A}$. The involution † on $\mathcal{A}$ is an anti-automorphism of $\mathcal{A}$ and therefore must restrict to an automorphism of $\mathcal{Z}$. Furthermore for any W-symmetric $A(n) \in \mathcal{A}$ we know that $\|P(x,n)\|_W^{-1} A(n) \|P(x,n)\|_W$ is Hermitian and therefore diagonalizable. Therefore $A(n)$ evaluates to a diagonalizable matrix for every integer value of n. Since $A(n)$ is polynomial in n, it follows that $A(n)$ is diagonalizable for all values of n. The algebra $\mathcal{Z}$ is commutative and spanned over $\mathbb{C}$ by its W-symmetric elements, so therefore all of the elements of $\mathcal{Z}$ are simultaneously diagonalizable. By the previous lemma this implies that $\mathcal{Z}$ is finitely generated as an algebra over $\mathbb{C}$. The statement of the theorem follows immediately. □

PROPOSITION 4.9. *The minimal prime ideals $\mathcal{P}_1(W), \ldots, \mathcal{P}_r(W)$ of $\mathcal{Z}(W)$ are fixed by the adjoint operation* †.

Proof. For brevity set $\mathcal{P}_i := \mathcal{P}_i(W)$ and $\mathcal{Z} := \mathcal{Z}(W)$. The involution † defines an anti-isomorphism of $\mathcal{D}(W)$ and therefore restricts to an isomorphism of $\mathcal{Z}$. Consequently the image $\mathcal{P}_i^\dagger$ of the minimal prime ideal $\mathcal{P}_i$ must be a minimal prime ideal also. If $\mathcal{Z}$ has only one minimal prime, then the result of the proposition is trivial. Therefore we assume otherwise, so that in particular each of the minimal prime is not the zero ideal.

For each i, consider the annihilator ideal of $\mathcal{P}_i$ defined by

$$\operatorname{Ann}(\mathcal{P}_i) = \{\mathfrak{D} \in \mathcal{Z}(W) : \mathfrak{D}\mathcal{P}_i = 0\}.$$

Since $\mathcal{Z}(W)$ is Noetherian, the ideal $\mathcal{P}_i$ is finitely generated by some elements $\mathfrak{D}_1, \ldots, \mathfrak{D}_m \in \mathcal{Z}(W)$. For each i there exists $\mathfrak{D}_i' \in \mathcal{Z}(W)\backslash\mathcal{P}_i$ with $\mathfrak{D}_i\mathfrak{D}_i' = 0$. The set $\mathcal{Z}(W)\backslash\mathcal{P}_i$ is multiplicatively closed and does not contain zero, so $\mathfrak{D}' = \mathfrak{D}_1' \ldots \mathfrak{D}_m'$ is nonzero. Since $\mathfrak{D}'$ annihilates the generators of $\mathcal{P}_i$ this proves $\operatorname{Ann}(\mathcal{P}_i)$ is nonzero.

For all $j \neq i$, by definition $\operatorname{Ann}(\mathcal{P}_i)\mathcal{P}_i = 0 \subseteq \mathcal{P}_j$. Since $\mathcal{P}_j$ is prime either $\operatorname{Ann}(\mathcal{P}_i) \subseteq \mathcal{P}_j$ or $\mathcal{P}_i \subseteq \mathcal{P}_j$. However since $\mathcal{P}_j$ is a minimal prime, it must be the former. Hence $\operatorname{Ann}(\mathcal{P}_i) \subseteq \bigcap_{j\neq i} \mathcal{P}_j$.

Choose a nonzero $\mathfrak{D} \in \operatorname{Ann}(\mathcal{P}_i)$. Since $\mathfrak{D}\mathfrak{D}^\dagger \neq 0$, this implies that $\mathfrak{D}^\dagger \notin \mathcal{P}_i$ and therefore $\mathfrak{D} \notin \mathcal{P}_i^\dagger$. By the previous paragraph $\mathfrak{D} \in \mathcal{P}_j$ for all $j \neq i$, so therefore $\mathcal{P}_i^\dagger \neq \mathcal{P}_j$ for all $j \neq i$. Thus $\mathcal{P}_i^\dagger = \mathcal{P}_i$. This completes the proof. □

4.3. $\mathcal{D}(W)$ **is finite over its center.** The results of the previous subsection tell us that $\mathcal{Z}(W)$ is reduced and Noetherian with Krull dimension 1. In this subsection we will prove that the algebra $\mathcal{D}(W)$ is affine (i.e., finitely generated as an algebra over $\mathbb{C}$), from which we conclude that it is Noetherian and finitely generated over its center.

The spectrum of the center will consist of a disjoint union of affine curves and points. As in the previous section, let $\mathcal{P}_1, \ldots, \mathcal{P}_r$ denote the minimal primes of $\mathcal{Z}(W)$. Let $I \subseteq \{1, \ldots, r\}$ be the set of indices of minimal primes corresponding to discrete points of $\operatorname{Spec}(\mathcal{Z}(W))$, i.e.

$$I = \big\{i : \mathcal{Z}(W)/\mathcal{P}_i \cong \mathbb{C}\big\}.$$

For each $i \in I$, there exists a W-symmetric idempotent $\mathfrak{E}_i \in \mathcal{Z}(W)$ which maps to 1 in $\mathcal{Z}(W)/\mathcal{P}_j$ if $j = i$ and to 0 otherwise. Using the system of orthogonal idempotents $\{\mathfrak{E}_i : i \in I\}$ the rings $\mathcal{D}(W)$ and $\mathcal{Z}(W)$ decompose as a direct sums

$$\begin{aligned}
\mathcal{D}(W) &= \mathcal{D}_{\text{disc}}(W) \oplus \mathcal{D}_{\text{cont}}(W);\\
&\qquad \mathcal{D}_{\text{disc}}(W) = \bigoplus_{i\in I} \mathfrak{E}_i \mathcal{Z}(W) \mathfrak{E}_i,\ \mathcal{D}_{\text{cont}}(W) = \mathfrak{E}\mathcal{D}(W)\mathfrak{E};\\
\mathcal{Z}(W) &= \mathcal{Z}_{\text{disc}}(W) \oplus \mathcal{Z}_{\text{cont}}(W);\\
&\qquad \mathcal{Z}_{\text{disc}}(W) = \bigoplus_{i\in I} \mathfrak{E}_i \mathcal{Z}(W) \mathfrak{E}_i,\ \mathcal{Z}_{\text{cont}}(W) = \mathfrak{E}\mathcal{Z}(W)\mathfrak{E}
\end{aligned}$$

for $\mathfrak{E} = I - \sum_{i\in I} \mathfrak{E}_i$. Each of the above summands is an algebra and the above identifications are algebra isomorphisms. Since $\mathcal{D}(W)$ is a semiprime PI-algebra, both $\mathcal{D}_{\text{disc}}(W)$ and $\mathcal{D}_{\text{cont}}(W)$ must be semiprime PI-algebras.

The next lemma captures the crucial algebraic consequence of our involution.

LEMMA 4.10. *Consider a weight matrix $W(x)$ with sequence of monic orthogonal matrix polynomials $P(x,n)$. Let $\mathfrak{D} \in \mathcal{D}(W)$ and let $A(n) \in M_N(\mathbb{C}[n])$ be the preimage of $\mathfrak{D}$ under the generalized Fourier map b_P. If the eigenvalues of $A(n)A(n)^\dagger$ are all constant, then both $A(n)$ and $\mathfrak{D}$ are constant.*

Proof. Let $B(n) = A(n)A(n)^\dagger$ and assume all of the eigenvalues of $B(n)$ are constant. Then the characteristic polynomial of $B(n)$ has constant coefficients. In particular there exists a nonzero polynomial $p(t) \in \mathbb{C}[t]$ satisfying $p(B(n)) = 0$.

Assume that $\mathfrak{D}$ is not a constant differential operator, i.e., $\mathfrak{D} \notin M_N(\mathbb{C})$. Then we may write

$$\mathfrak{D} = \sum_{j=0}^{\ell} \partial_x^j A_j(x), \quad \text{for some } \ell > 0 \text{ and } A_j(x) \in M_N(\mathbb{C}[x]) \text{ with } A_\ell(x) \neq 0.$$

Since $W(x)$ is Hermitian and positive-definite on its support, the product $A_\ell(x)W(x)A_\ell(x)^*$ is not identically zero. Hence the operator $\mathfrak{D}\mathfrak{D}^\dagger$ is W-symmetric of order 2ℓ with leading coefficient $A_\ell(x)W(x)A_\ell(x)^*W(x)^{-1}$. By a short argument, one proves we may factor $W(x) = T(x)T(x)^*$ for some $T(x)$ smooth on the support of $W(x)$. Then the leading coefficient of $T(x)^{-1}\mathfrak{D}\mathfrak{D}^\dagger T(x)$ is $(T(x)^{-1}A_\ell(x)T(x))(T(x)^{-1}A_\ell(x)T(x))^*$. In particular it is nonzero and Hermitian and therefore not nilpotent. Hence for all j the operator $(T(x)^{-1}\mathfrak{D}\mathfrak{D}^\dagger T(x))^j$, has order $2\ell j$. Thus so too does the operator $\mathfrak{D}\mathfrak{D}^\dagger$. In particular $\mathfrak{D}\mathfrak{D}^\dagger$ cannot be a root of a nonzero polynomial $p(t) \in \mathbb{C}[t]$. This is a contradiction, so $\mathfrak{D}$ and $A(n)$ must be constant. □

As a consequence of the previous lemma, we may show that the discrete part $\mathcal{D}_{\text{disc}}(W)$ of $\mathcal{D}(W)$ is finite dimensional.

LEMMA 4.11. *For all i, the algebra $\mathfrak{E}_i\mathcal{D}(W)\mathfrak{E}_i$ is finite dimensional. Consequently $\mathcal{D}_{\text{disc}}(W)$ is finite dimensional.*

Proof. For brevity, let $\mathcal{A} \subseteq M_N(\mathbb{C}[n])$ denote the preimage of $\mathfrak{E}_i\mathcal{D}(W)\mathfrak{E}_i$ under the generalized Fourier map b_P and let $E(n)$ be the preimage of $\mathfrak{E}_i$. Note that $E(n)$ is the identity element of $\mathcal{A}$ and that the center of $\mathfrak{A}$ is $\mathbb{C}E(n)$.

The algebra $\mathcal{A}$ has an action on the $\mathbb{C}[n]$-submodule $E(n)\mathbb{C}[n]^{\oplus N}$ of $\mathbb{C}[n]^{\oplus N}$. Since $\mathbb{C}[n]$ is a PID, we know that $E(n)\mathbb{C}[n]^{\oplus N}$ is isomorphic to $\mathbb{C}[n]^{\oplus m}$ for some integer m. The action of $\mathcal{A}$ on $E(n)\mathbb{C}[n]^{\oplus N}$ is faithful, and therefore induces an algebra monomorphism $\pi : \mathcal{A} \to M_m(\mathbb{C}[n])$.

Since $\mathcal{D}(W)$ is a semiprime PI algebra, so too is $\mathcal{A}$. Now suppose that $A(n) \in \mathcal{A}$, let $\lambda \in \mathbb{C}$ and set $B(n) = A(n)A(n)^\dagger$. By [39, Theorem 2] the two-sided ideal of $\mathcal{A}$ generated by $\lambda E(n) - B(n)$ must intersect nontrivially with the center of $\mathcal{A}$. Therefore there exists $C(n), \widetilde{C}(n) \in \mathcal{A}$ such that $C(n)(B(n) - \lambda E(n))\widetilde{C}(n) =$

$E(n)$. This implies that for all λ the determinant $\det(\pi(B(n)-\lambda E(n)))$ lies in $\mathbb{C}^\times$. Therefore the characteristic polynomial of $B(n)$ satisfies

$$p_B(t,n) := \det\big(tI - B(n)\big) = t^{N-m}\det\big(\pi\big(tE(n)-B(n)\big)\big) \in \mathbb{C}[t].$$

In particular the matrix $B(n)$ has constant eigenvalues. By Lemma 4.10 this implies that $A(n)$ is constant. Thus we conclude all of the matrices in $\mathcal{A}$ are constant, and so $\mathcal{A}$ is finite dimensional. □

THEOREM 4.12. *The algebra $\mathcal{D}(W)$ is affine.*

Proof. From the previous lemma, it suffices to show that $\mathcal{D}_{\text{cont}}(W)$ is finitely generated. Thus without loss of generality, we may assume that the center of $\mathcal{D}(W)$ consists of a disjoint union of (possibly singular) algebraic curves over $\mathbb{C}$. Let $\mathcal{A}$ be the preimage of $\mathcal{D}(W)$ in $M_N(\mathbb{C}[n])$ under the generalized Fourier map and let $\mathcal{B} = \text{span}_{\mathbb{C}(n)}(\mathcal{A})$. Then $\mathcal{B}$ is finite dimensional over $\mathbb{C}(n)$ and has no nontrivial two-sided nilpotent ideals. Therefore $\mathcal{B}$ is semisimple and by the Artin-Wedderburn theorem $\mathcal{B}$ is isomorphic to a direct sum

$$\mathcal{B} \cong \bigoplus_{j=1}^{\ell} M_{n_j}(\mathcal{L}_j)$$

for some skew-field extensions $\mathcal{L}_j$ of $\mathbb{C}(n)$. Note that by Tsen's theorem, each of the $\mathcal{L}_j$ is actually simply a field extension of $\mathbb{C}(n)$. For each j let R_j denote the integral closure of $\mathbb{C}[n]$ in $\mathcal{L}_j$. Let $\mathcal{L}$ be a finite field extension of $\mathbb{C}(n)$ containing each of the $\mathcal{L}_j$'s and let $\mathcal{R}$ be the integral closure of $\mathbb{C}[n]$ in $\mathcal{L}$ and define $\Delta = \bigoplus_{j=1}^{\ell} M_{n_j}(\mathcal{L})$.

Let $\mathcal{O}$ be the $\mathcal{R}$-subalgebra of Δ generated by $\mathcal{A}$. Then $\text{span}_L(\mathcal{O}) \subseteq \Delta$ contains $\text{span}_L(\text{span}_F(\mathcal{A})) = \text{span}_L(\bigoplus_{j=1}^{\ell} M_{n_j}(\mathcal{L}_j)) = \Delta$. Therefore $\text{span}_L(\mathcal{O}) = \Delta$. Furthermore every element of $\mathcal{A}$ is integral over $\mathbb{C}[n]$ and therefore integral over $\mathcal{R}$. Therefore every element of $\mathcal{A}$ is integral over $\mathcal{R}$, and it follows that every element of $\mathcal{O}$ is integral over $\mathcal{R}$. Therefore by [38, Theorem 10.3] $\mathcal{O}$ is an $\mathcal{R}$-order in Δ, and in particular $\mathcal{O}$ is affine and Noetherian.

The center of $\mathcal{A}$ maps into the center of $\mathcal{B}$, the latter being $\oplus_{j=1}^{\ell}\mathcal{L}_j I_{n_j}$. Every element of $\mathcal{A}$ is actually integral over $\mathbb{C}[n]$, so the image of the center of $\mathcal{A}$ is isomorphic to a subalgebra $\mathcal{Z}$ of $\widetilde{Z} := \oplus_{j=1}^{\ell}\mathcal{R} I_{n_j}$. Since $\mathcal{Z}$ is Noetherian and $\mathcal{D} = \mathcal{D}_{\text{cont}}(W)$, the spectrum of $\mathcal{Z}$ consists of a collection of affine curves over $\mathbb{C}$. The inclusion $\mathcal{Z} \to \oplus_{j=1}^{\ell}\mathcal{R}_j I_{n_j}$ is precisely the normalization of $\mathcal{Z}$, while the inclusion maps $\mathcal{R}_j \to \mathcal{R}$ correspond finite morphisms of smooth curves. Therefore the inclusion $\mathcal{Z} \subseteq \mathcal{Z}'$ corresponds to the finite morphism of affine algebraic curves $\text{Spec}(\mathcal{Z}') \to \text{Spec}(\mathcal{Z})$, so in particular $\mathcal{Z}'$ is finitely generated as a module over $\mathcal{Z}$. As a consequence, $\mathcal{O}$ is finitely generated as a module over $\mathcal{A}$. The algebra $\mathcal{O}$ is an $\mathcal{A}$-centralizing extension, so by Montgomery and Small's extension of the Artin-Tate lemma [36, Proposition 2], $\mathcal{A}$ must be affine. □

THEOREM 4.13. (Casper [9]) *The algebra $\mathcal{D}(W)$ is Noetherian and is finitely generated as a module over its center $\mathcal{Z}(W)$.*

Proof. Consider the preimage $\mathcal{E}(W)$ of $\mathcal{D}(W)$ under the generalized Fourier map b_P. The algebra $\mathcal{E}(W)$ is a subalgebra of $M_N(\mathbb{C}[n])$, and thus has GK dimension at most the GK dimension of $M_N(\mathbb{C}[n])$. The algebras $M_N(\mathbb{C}[n])$ and $\mathbb{C}[n]$ have equal GK dimension by [35, Proposition 3.2.7], and $\mathbb{C}[n]$ has GK dimension 1. Thus $\mathcal{E}(W)$ has GK dimension at most 1. Since $\mathcal{D}(W)$ is affine and semiprime of GK dimension at most 1, we can apply the Small-Stafford-Warfield theorem [43, Theorem 1.6] to obtain that $\mathcal{D}(W)$ is Noetherian and finitely generated over its center. □

4.4. Localizing $\mathcal{D}(W)$ on its center. Recall that for a commutative ring with zero divisors, we can still define a *total ring of fractions* to be the localization of the ring at the multiplicative set of elements in $\mathcal{Z}(W)$ which are not zero divisors. Geometrically the total ring of fractions is isomorphic to the product of the ring of fractions of every irreducible component of the spectrum of the ring.

Definition 4.14. Let $W(x)$ be a weight matrix, let $P(x,n)$ be the associated sequence of monic orthogonal polynomials, and let $b_P : M_N(\mathbb{C}[n]) \to \mathcal{D}(W)$ denote the generalized Fourier map. We define $\mathcal{F}(W)$ to be the total ring of fractions of $\mathcal{Z}(W)$.

We will show that when we localize $\mathcal{D}(W)$ by extending scalars in the ring $\mathcal{D}(W)$ to the total ring of fractions $\mathcal{F}(W)$ of $\mathcal{Z}(W)$, we get a product of matrix algebras over the fraction fields of each of the irreducible components. Equivalently, this means that $\mathcal{D}(W)$ is generically isomorphic to a matrix algebra over each irreducible component of $\mathcal{Z}(W)$. This result is analogous to Posner's theorem for prime PI rings.

THEOREM 4.15. *Suppose that $W(x)$ is a weight matrix. Then there exist positive integers $n_1, \ldots, n_r$ with r the number of minimal primes of $\mathcal{Z}(W)$ such that*

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) \cong \bigoplus_{i=1}^{r} M_{n_i}\big(\mathcal{F}_i(W)\big), \tag{4.16}$$

where $\mathcal{F}_i(W)$ is the localization of $\mathcal{Z}(W)$ at the minimal prime $\mathcal{P}_i(W)$ of $\mathcal{Z}(W)$, i.e., the function field of the irreducible curve $\operatorname{Spec}(\mathcal{Z}(W)/\mathcal{P}_i(W))$. *Moreover the localization map $\mathcal{D}(W) \to \mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W)$ is injective.*

Remark 4.17. The proof of the theorem actually establishes the following stronger statement:

Any affine $*$-algebra with proper involution, which is finitely generated as a module over its center, localizes over its center to a product of central simple algebras.

The center of such an algebra is affine by the Artin-Tate Lemma [8, p. 116].

Rowen [40, Appendix] has previously obtained results which are similar in nature. Specifically he proved that quasi-Goldie, semiprime, PI-algebras localize over their centers to semisimple Artinian rings. Here quasi-Goldie means every set of independent ideals is finite and every set of left annihilators of ideals contains a maximal element. We do not know of the exact relation between these assumptions and ours.

Proof. First we show that the natural map $\mathcal{D}(W) \to \mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W)$ is injective. To see this, suppose that $\mathfrak{D} \in \mathcal{D}(W)$ maps to 0 in the localization. Then there must exist an element $\mathfrak{D}' \in \mathcal{Z}(W)$ which is not a zero divisor in $\mathcal{Z}(W)$ and which satisfies $\mathfrak{D}\mathfrak{D}' = 0$. It follows that $\mathfrak{D}'$ annihilates the two-sided ideal of $\mathcal{D}(W)$ generated by $\mathfrak{D}$. However, since $\mathcal{D}(W)$ is a semiprime PI algebra [39, Theorem 2] tells us any nonzero two-sided ideal of $\mathcal{D}(W)$ must intersect nontrivially with $\mathcal{Z}(W)$. This contradicts the fact that $\mathfrak{D}'$ is not a zero divisor in $\mathcal{Z}(W)$, and thus the localization map is injective.

Next note that

$$\mathcal{F}(W) = \mathcal{F}_1(W) \times \cdots \times \mathcal{F}_r(W).$$

Using this, we have

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) = \prod_{i=1}^{r} \mathcal{R}_i, \quad \text{for } \mathcal{R}_i := \mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}_i(W).$$

We know that $\mathcal{D}(W)$ is finitely generated as a module over its center, so the localizations $\mathcal{R}_i$ will also be finitely generated over $\mathcal{F}_i(W)$. The latter is a field, so each of the algebras $\mathcal{R}_i$ is a finite dimensional $\mathcal{F}_i(W)$ vector space. Note that since $\mathcal{R}_i$ is a localization, all elements in $\mathcal{R}_i$ are of the form $\mathfrak{D} \otimes \mathfrak{A}$ for some $\mathfrak{D} \in \mathcal{D}(W)$ and $\mathfrak{A} \in F_i(W)$.

The adjoint operation $\dagger$ fixes $\mathcal{Z}(W)$ and extends to an adjoint operation on $\mathcal{F}(W)$. Since the adjoint operation fixes each minimal prime ideal $\mathcal{P}_i(W)$, it also preserves the complement $\mathcal{Z}(W)\backslash\mathcal{P}_i(W)$, and thus induces an adjoint operation on $\mathcal{F}_i(W)$. This in turn induces an adjoint operation on $\mathcal{R}_i$, defined by $(\mathfrak{D} \otimes \mathfrak{A})^\dagger = \mathfrak{D}^\dagger \otimes \mathfrak{A}^\dagger$ for $\mathfrak{D} \in \mathcal{D}(W)$ and $\mathfrak{A} \in \mathcal{F}_i(W)$.

We claim that the property (4.4) is maintained under this extension. To see this, suppose that $(\mathfrak{D} \otimes \mathfrak{A})(\mathfrak{D} \otimes \mathfrak{A})^\dagger = 0$ for some $\mathfrak{D} \in \mathcal{D}(W)$ and $\mathfrak{A} \in \mathcal{F}_i(W)$. Note that $(\mathfrak{D} \otimes \mathfrak{A})(\mathfrak{D} \otimes \mathfrak{A})^\dagger = (\mathfrak{D}\mathfrak{D}^\dagger) \otimes (\mathfrak{A}\mathfrak{A}^\dagger)$, and this latter element is zero if and only if there exists $\widetilde{\mathfrak{D}} \in \mathcal{Z}(W)\backslash\mathcal{P}_i$ with $(\mathfrak{D}\mathfrak{D}^\dagger)\widetilde{\mathfrak{D}} = 0$. This then would imply that $\mathfrak{D}\widetilde{\mathfrak{D}}(\mathfrak{D}\widetilde{\mathfrak{D}})^\dagger = 0$ and therefore that $\mathfrak{D}\widetilde{\mathfrak{D}} = 0$. Hence $\mathfrak{D} \otimes \mathfrak{A} = 0$. Thus the desired property holds.

Each $\mathcal{R}_i$ is a $*$-algebra (over $\mathcal{F}_i(W)$) with proper involution, therefore it is a semiprime PI-algebra. In particular, $\mathcal{R}_i$ has no nonzero two-sided nilpotent ideals. Since it is finite dimensional over $\mathcal{F}_i(W)$, this implies the Jacobson radical of $\mathcal{R}_i$

is trivial, so that $\mathcal{R}_i$ is a finite dimensional semisimple algebra over $\mathcal{F}_i(W)$. Furthermore, by virtue of the construction of the $\mathcal{R}_i$ and the injectivity of the localization map $\mathcal{D}(W) \to \mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W)$, the center of $\mathcal{R}_i$ is $\mathcal{F}_i(W)$. Thus $\mathcal{R}_i$ is a central simple algebra with center $\mathcal{F}_i(W)$, and by Tsen's theorem is therefore isomorphic to $M_{n_i}(\mathcal{F}_i(W))$ for some integer n_i. This completes the proof. □

Definition 4.18. Suppose that $W(x)$ is a weight matrix with $\mathcal{D}(W)$ containing an operator of positive order. Using the notation of Theorem 4.15, we call $n_1 + n_2 + \cdots + n_r$ the *module rank* of $\mathcal{D}(W)$ over $\mathcal{Z}(W)$.

PROPOSITION 4.19. *Suppose that $W(x)$ is a weight matrix with $\mathbb{C}I \subsetneq \mathcal{Z}(W)$. Then the module rank of $\mathcal{D}(W)$ is at most N.*

Proof. Let m be the rank of $\mathcal{D}(W)$. Then we may choose m non-nilpotent elements $\mathfrak{V}_1, \ldots, \mathfrak{V}_m \in \mathcal{D}(W)$ such that $\mathfrak{V}_i \mathfrak{V}_j = 0$ for $i \neq j$. Let $A_j(n) \in M_N(\mathbb{C}[n])$ be the image of $\mathfrak{V}_j$ under the generalized Fourier map. Then each $A_j(n)$ is nonzero and satisfies $A_j(n)A_k(n) = 0$ for $j \neq k$. Since each $\mathfrak{V}_j$ is non-nilpotent, each $A_j(n)$ is also non-nilpotent. Therefore we may choose a column vector $\vec{v}_j(n)$ of $\vec{A}_j(n)$ with $\vec{A}_j(n)\vec{v}_j(n) \neq \vec{0}$. With this choice, $\vec{A}_j(n)\vec{v}_k(n) = \vec{0}$ if and only if $j \neq k$ and thus the m vectors $\vec{v}_1(n), \ldots, \vec{v}_m(n)$ in $\mathbb{C}[n]^{\oplus N}$ are $\mathbb{C}[n]$-linearly independent. We conclude that $m \leq N$. □

Definition 4.20. We say $\mathcal{D}(W)$ is *full* if the module rank of $\mathcal{D}(W)$ is N.

Fullness is a relatively weak condition, especially for N small as we see in the next theorem.

THEOREM 4.21. *Suppose that $W(x)$ is a 2×2 matrix and that $\mathcal{D}(W)$ is noncommutative. Then the associated algebra of differential operators $\mathcal{D}(W)$ is full. In fact in this case $\mathcal{F}(W)$ is a field and $\mathcal{D}(W)$ is a $\mathcal{Z}(W)$-order in $M_2(\mathcal{F}(W))$.*

Proof. Since $\mathcal{D}(W)$ is noncommutative, so too is its localization over its center. Using the notation of Theorem 4.15 and referring to (4.16) the noncommutativity implies that $n_i > 1$ for some i. However, since the rank is bounded by 2, we get $n_1 + \cdots + n_r \leq 2$. Thus $r = 1$ and $n_1 = 2$, so that (4.16) says

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) \cong M_2\big(\mathcal{F}(W)\big).$$

Thus $\mathcal{D}(W)$ is a $\mathcal{Z}(W)$-order in $M_2(\mathcal{F}(W))$ and in particular $\mathcal{D}(W)$ is full. □

More generally, fullness can be characterized in terms of the existence of enough orthonormal idempotents in the localization of $\mathcal{D}(W)$ over $\mathcal{Z}(W)$. This is summarized by the next theorem.

THEOREM 4.22. *The algebra $\mathcal{D}(W)$ is full if and only if there exist nonzero W-symmetric elements $\mathfrak{V}_1, \mathfrak{V}_2, \ldots, \mathfrak{V}_N$ in $\mathcal{D}(W)$ satisfying $\mathfrak{V}_i \mathfrak{V}_j = 0$ for $i \neq j$ with $\mathfrak{V}_1 + \cdots + \mathfrak{V}_N$ a central element of $\mathcal{D}(W)$ which is not a zero divisor.*

Proof. Suppose that $\mathcal{D}(W)$ is full. Then we have an isomorphism

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) \cong \bigoplus_{i=1}^{r} M_{n_i}\big(\mathcal{F}_i(W)\big),$$

with $n_1 + \cdots + n_r = N$.

Fix an integer $1 \leq i \leq n$. By the above, we know that the adjoint operation $\dagger$ of $\mathcal{D}(W)$ restricts to an adjoint operation on both $\mathcal{F}_i(W)$ and on $M_{n_i}(\mathcal{F}_i(W))$. However, the adjoint operation on $M_{n_i}(\mathcal{F}_i(W))$ may not be the one naturally induced by the adjoint operation on $\mathcal{F}_i(W)$. In fact we in general have two adjoints on $M_{n_i}(\mathcal{F}_i(W))$, the adjoint coming from the restriction of $\dagger$ (which we also denote by $\dagger$), and the one obtained by applying $\dagger$ to each of the matrix entries of an element of $M_{n_i}(\mathcal{F}_i(W))$ and then taking the transpose. We denote this latter adjoint operation by $\star$. The composition of these two adjoint operations $(\cdot)^{\dagger\star}$ defines an automorphism of $M_{n_i}(\mathcal{F}_i(W))$ which fixes the base field $\mathcal{F}_i(W)$. By the Skolem-Noether theorem, this automorphism must be inner. Thus there exists an invertible matrix $U \in M_{n_i}(\mathcal{F}_i(W))$ such that

$$A^{\dagger} = (U^{\star})^{-1} A^{\star} U^{\star}, \quad \forall A \in M_{n_i}\big(\mathcal{F}_i(W)\big).$$

We can apply the same argument to the opposite composition $(\cdot)^{\star\dagger}$, and if we use the fact that $\dagger$ and $\star$ are order two, we find

$$A^{\dagger} = U^{-1} A^{\star} U, \quad \forall A \in M_{n_i}\big(\mathcal{F}_i(W)\big).$$

Putting this together, we find that $U^{\star} = U$. Therefore U has a Cholesky-type factorization of the form $U = PDP^{\star}$ for some invertible matrices $P, D \in M_N(\mathcal{F}_i(W))$ with D diagonal and $\star$-symmetric. Then for $1 \leq j \leq N$, we have

$$(P^{-1\star} E_{jj} P^{\star})^{\dagger} = U^{-1} P E_{jj} P^{-1} U = U^{-1} P E_{jj} D P^{\star} = P^{-1\star} D^{-1} E_{jj} D P^{\star}.$$

For each $1 \leq j \leq n_i$, choose $\mathfrak{D}'_{ij} \in \mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W)$ corresponding to the matrix $(P^{\star})^{-1} E_{ii} P^{\star}$. Then by definition $\mathfrak{D}'_{ij}$ is $\dagger$-symmetric and satisfy $\mathfrak{D}'_{ij}\mathfrak{D}'_{ab} = 0$ for $(a,b) \neq (i,j)$.

We can repeat the above process for each i, obtaining N elements $\mathfrak{D}'_{ij}$. For each i, j, we can clear denominators using W-symmetric, central elements to obtain an element $\mathfrak{D}_{ij} \in \mathcal{D}(W)$. By definition, the elements $\mathfrak{D}_{ij}$ are W-symmetric. Furthermore, since the localization map is injective the N elements $\mathfrak{D}_{ij}$ satisfy $\mathfrak{D}_{ij}\mathfrak{D}_{k\ell} = 0$ if $i \neq k$ or $j \neq \ell$, and the sum is a central element of $\mathcal{D}(W)$ which is not a zero divisor.

Conversely, suppose that the $\mathfrak{D}_1, \ldots, \mathfrak{D}_N$ exist and set $\mathfrak{D} = \mathfrak{D}_1 + \cdots + \mathfrak{D}_N$. Then the N elements

$$\mathfrak{D}^{-1} \otimes \mathfrak{D}_i, \quad i = 1, 2, \ldots, N$$

define a family of N orthogonal idempotents in $\mathcal{F}(W) \otimes_{\mathcal{Z}(W)} \mathcal{D}(W)$. From the isomorphism

$$\mathcal{D}(W) \otimes_{\mathcal{Z}(W)} \mathcal{F}(W) \cong \bigoplus_{i=1}^{r} M_{n_i}\big(\mathcal{F}_i(W)\big),$$

we see that $\bigoplus_{i=1}^{r} M_{n_i}(\mathcal{F}_i(W))$ must have a collection of N orthogonal idempotents. This implies that $n_1 + \cdots + n_r$ is at least N and therefore exactly N. □

PROPOSITION 4.23. *Let $W(x)$ and $\widetilde{W}(x)$ be weight matrices. Suppose that $\mathcal{D}(W)$ is full and that $\widetilde{W}(x)$ is a bispectral Darboux transformation of $W(x)$. Then $\mathcal{D}(\widetilde{W})$ is also full.*

Proof. This follows from Theorem 2.36, combined with the previous theorem. □

Definition 4.24. Let $W(x)$ be a weight matrix. We call a collection of nonzero W-symmetric elements $\mathfrak{V}_1, \ldots, \mathfrak{V}_N \in \mathcal{D}(W)$ satisfying the condition that $\mathfrak{V}_i \mathfrak{V}_j = 0$ for $i \neq j$ an *orthogonal system* for $\mathcal{D}(W)$.

Since $\mathcal{D}(W)$ is finitely generated over its center $\mathcal{Z}(W)$, it also makes sense to consider the support of a given element $\mathfrak{D} \in \mathcal{D}(W)$ over the center.

Definition 4.25. Let $W(x)$ be a weight matrix. The *support* $\operatorname{supp}(\mathfrak{D})$ of an element $\mathfrak{D} \in \mathcal{D}(W)$ is the set of prime ideals of $\mathcal{D}(W)$ containing the annihilator $\operatorname{Ann}(\mathfrak{D})$ of $\mathfrak{D}$ over its center, i.e.,

$$\operatorname{Ann}(\mathfrak{D}) := \big\{\mathfrak{V} \in \mathcal{Z}(W) : \mathfrak{V}\mathfrak{D} = 0\big\}.$$

Let $\mathcal{Z}_1(W), \ldots, \mathcal{Z}_r(W)$ be the irreducible components of $\mathcal{Z}(W)$ and let $\mathcal{P}_1, \ldots, \mathcal{P}_r$ be the corresponding minimal primes. We say that an element $\mathfrak{D} \in \mathcal{D}(W)$ is *supported on $\mathcal{Z}_i(W)$* if $\mathcal{P}_i$ is contained in one of the primes in $\operatorname{supp}(\mathfrak{D})$. We say that two elements $\mathfrak{D}_1, \mathfrak{D}_2 \in \mathcal{D}(W)$ are *supported on the same irreducible component* if for some i both $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are supported on $\mathcal{Z}_i(W)$.

Alternatively, one may define the support of $\mathfrak{D} \in \mathcal{D}(W)$ to be the set of prime ideals $\mathcal{P} \subseteq \mathcal{Z}(W)$ such that $\mathfrak{D}$ is nonzero in the localization of $\mathcal{D}(W)$ at $\mathcal{P}$. The support of any element in $\mathcal{D}(W)$ defines a closed subscheme of $\mathcal{Z}(W)$. Clearly any element of $\mathcal{D}(W)$ must be supported on at least one irreducible component of $\mathcal{Z}(W)$.

There is a simple algebraic criterion for two elements $\mathfrak{D}_1, \mathfrak{D}_2 \in \mathcal{D}(W)$ to be supported on the same irreducible component. The proof is a consequence of (4.16) and is left to the reader.

PROPOSITION 4.26. *Let $W(x)$ be a weight matrix with $\mathcal{D}(W)$ full and suppose $\mathfrak{D}_1, \mathfrak{D}_2 \in \mathcal{D}(W)$. Then $\mathfrak{D}_1 \mathcal{D}(W) \mathfrak{D}_2 \neq 0$ if and only if both share an irreducible component of $\mathcal{Z}(W)$ on which they are supported.*

5. Consequences of the algebraic structure of $\mathcal{D}(W)$. We demonstrate in this section that the algebraic structure of $\mathcal{D}(W)$ has strong consequences for the form of the weight matrix $W(x)$. Most importantly, we prove the Classification Theorem 1.3. Throughout this section, we will always suppose Assumption 2.21. Note again, that this assumption always holds in the case that $\mathcal{D}(W)$ contains a W-symmetric, second-order differential operator of the form (1.4) with $A_2(x)W(x)$ positive definite on the support of $W(x)$.

We will focus exclusively on the case when $\mathcal{D}(W)$ is full. With this in mind, throughout this section the elements $\mathfrak{V}_1, \dots, \mathfrak{V}_N$ will denote an orthogonal system for $\mathcal{D}(W)$, i.e., a collection of nonzero, W-symmetric matrix differential operators in $\mathcal{D}(W)$ with $\mathfrak{V}_i \mathfrak{V}_j = 0$ for $i \neq j$ and with $\mathfrak{V}_1 + \cdots + \mathfrak{V}_N \in \mathcal{Z}(W)$ a nonzero divisor. Of course, the existence of an orthogonal system is guaranteed by Theorem 4.22.

5.1. Cyclic modules and diagonalization of matrix weights. To understand the application of the orthogonal system to our understanding of the structure of $\mathcal{D}(W)$, we first consider a certain left $\Omega(x)$-module and its right counterpart. For each $\mathfrak{V}_i$, we consider the left $\Omega(x)$-module

$$\mathcal{M}_i := \left\{ \vec{\mathfrak{u}} \in \Omega(x)^{\oplus N} : \vec{\mathfrak{u}}^T \mathfrak{V}_j = \vec{0}^T,\ \forall i \neq j \right\}, \tag{5.1}$$

as well as the right $\Omega(x)$-module

$$\mathcal{N}_i := \left\{ \vec{\mathfrak{w}} \in \Omega(x)^{\oplus N} : \mathfrak{V}_j \vec{\mathfrak{w}} = \vec{0},\ \forall i \neq j \right\}. \tag{5.2}$$

The next result proves that the modules $\mathcal{M}_i$ and $\mathcal{N}_i$ are cyclic.

THEOREM 5.3. *For each i, there exists $\vec{\mathfrak{u}}_i, \vec{\mathfrak{w}}_i \in \Omega(x)^{\oplus N}$ with*

$$\mathcal{M}_i = \Omega(x) \vec{\mathfrak{u}}_i \quad \textit{and} \quad \mathcal{N}_i = \vec{\mathfrak{w}}_i \Omega(x). \tag{5.4}$$

Proof. The ring $\Omega(x)$ is a left and right principal ideal domain. The module $\mathcal{M}_i$ is a submodule of the free left module $\Omega(x)^{\oplus N}$ in this PID, and is therefore free. Furthermore, $\mathcal{M}_i$ contains the transposes of all the row vectors of $\mathfrak{V}_i$. Since $\mathfrak{V}_i$ is nonzero, this implies that $\mathcal{M}_i$ is nonzero. In particular, each of the $\mathcal{M}_i$ has rank at least 1 as a left $\Omega(x)$-module.

Now if $\vec{\mathfrak{u}} \in \mathfrak{V}_i \cap \mathfrak{V}_j$ for some $i \neq j$, then $\vec{\mathfrak{u}}^T \mathfrak{V}_k = \vec{0}^T$ for all $1 \leq k \leq N$. This in turn would imply that $\vec{\mathfrak{u}}^T (\mathfrak{V}_1 + \cdots + \mathfrak{V}_N) = \vec{0}^T$. Since $\mathfrak{V}_1 + \cdots + \mathfrak{V}_N$ is not a zero divisor, this implies that $\vec{\mathfrak{u}}^T = \vec{0}^T$. Thus $\mathcal{M}_i \cap \mathcal{M}_j = \vec{0}$ for $i \neq j$. Since each of the $\mathcal{M}_i$ is a submodule of $\Omega(x)^{\oplus N}$, it follows immediately that $\mathcal{M}_i$ is free of rank 1 for all i. The same proof works for $\mathcal{N}_i$. □

We fix some choice of generators $\vec{\mathfrak{u}_i}$ of $\mathcal{M}_i$ for $i = 1, \dots, N$ and define $\mathfrak{U}$ to be the matrix differential operator whose rows are $\vec{\mathfrak{u}_1}^T, \dots, \vec{\mathfrak{u}_N}^T$

$$\mathfrak{U} = \begin{bmatrix} \vec{\mathfrak{u}_1} & \vec{\mathfrak{u}_2} & \cdots & \vec{\mathfrak{u}_N} \end{bmatrix}^T. \tag{5.5}$$

Each of the $\vec{\mathfrak{u}_i}$ may be written as

$$\vec{\mathfrak{u}_i} = \sum_{j=0}^{\ell_i} \partial_x^j \vec{u}_{ji}(x) \tag{5.6}$$

for some $\vec{u}_{0i}(x), \dots, \vec{u}_{\ell_i i}(x) \in \mathbb{C}(x)^{\oplus N}$ with $\vec{u}_{\ell_i i}(x)$ not identically zero. We define $U(x)$ to be the matrix whose rows are $\vec{u}_{\ell_1 1}(x), \dots, \vec{u}_{\ell_N N}(x)$

$$U(x) = \begin{bmatrix} \vec{u}_{\ell_1 1}(x) & \vec{u}_{\ell_2 2}(x) & \cdots & \vec{u}_{\ell_N N}(x) \end{bmatrix}^T. \tag{5.7}$$

We will see below that $U(x)$ is a unit in $M_N(\mathbb{C}(x))$.

For all i, we know that $\vec{\mathfrak{u}_i}^T \mathfrak{V}_i \mathfrak{V}_j = \vec{0}^T$ for $i \neq j$. Moreover, by definition $\vec{\mathfrak{u}_i}^T \mathfrak{V}_j = \vec{0}^T$ for $i \neq j$. This means that $(\vec{\mathfrak{u}_i}^T \mathfrak{V}_i)^T \in \mathcal{M}_i$ for all i, and consequently there exists $\mathfrak{v}_i \in \Omega(x)$ such that

$$\mathfrak{v}_i \vec{\mathfrak{u}_i}^T = \vec{\mathfrak{u}_i}^T \mathfrak{V}_i \quad \text{and} \quad \vec{\mathfrak{u}_i}^T \mathfrak{V}_j = \vec{0}^T, \quad \forall i \neq j. \tag{5.8}$$

We will let m_i be the order of $\mathfrak{v}_i$ and write

$$\mathfrak{v}_i = \sum_{j=0}^{m_i} \partial_x^j v_{ji}(x) \tag{5.9}$$

for some rational functions $v_{ji}(x)$.

Remark 5.10. The principality of certain annihilator ideals is reminiscent of Rickart $*$-rings [5, p. 12, Definition 2]. While $\mathcal{D}(W)$ itself is not a Rickart $*$-ring, as it contains (left) annihilators not generated by idempotents, it would be interesting to know what properties $\mathcal{D}(W)$ might share with more general families of $*$-algebras. More specifically, it would be interesting to know more about the ideal structure of $\mathcal{D}(W)$ in general.

One of the more immediate consequences of the fullness of $\mathcal{D}(W)$ is that there exists a *rational* matrix which diagonalizes the bilinear form associated to $W(x)$.

THEOREM 5.11. *Let $W(x)$ be a weight matrix with $\mathcal{D}(W)$ full and let $\mathfrak{V}_1, \dots, \mathfrak{V}_N$, $\mathfrak{U}$ and $U(x)$ be defined as above. The matrix differential operator $\mathfrak{U}W(x)\mathfrak{U}^*$ is diagonal with leading coefficient*

$$\begin{aligned} &\operatorname{diag}\big(r_1(x), \dots, r_N(x)\big) \\ &\quad = U(x)W(x)U(x)^* \quad \textit{where } r_i(x) = \vec{u}_{\ell_i i}(x)^T W(x) \vec{u}_{\ell_i i}(x)^{T*}. \end{aligned} \tag{5.12}$$

Proof. Applying (5.8), we see

$$\mathfrak{U}\mathfrak{V}_i = E_{ii}\mathfrak{v}_i\mathfrak{U}.$$

Now since $\mathfrak{V}_i$ is W-symmetric, this also tells us

$$\mathfrak{V}_i W(x)\mathfrak{U}^* = W(x)\mathfrak{U}^* E_{ii}\mathfrak{v}_i^*.$$

Multiplying both sides of the equality on the left by $\mathfrak{U}$, this says that for all i

$$\mathfrak{v}_i E_{ii}\mathfrak{U}W(x)\mathfrak{U}^* = \mathfrak{U}W(x)\mathfrak{U}^* E_{ii}\mathfrak{v}_i^*.$$

In particular, $\mathfrak{U}W(x)\mathfrak{U}^*$ must be a diagonal matrix

$$\mathfrak{U}W(x)\mathfrak{U}^* = \operatorname{diag}(\mathfrak{b}_1,\dots,\mathfrak{b}_N), \quad \text{where } \mathfrak{b}_i = \vec{\mathfrak{u}}_i^T W(x)\vec{\mathfrak{u}}_i^{T*}.$$

Since $W(x)$ is Hermitian and positive-definite on (x_0,x_1), we know that the function $r_i(x) := \vec{u}_{\ell_i i}(x)^T W(x)\vec{u}_{\ell_i i}(x)^{T*}$ is not identically zero. Consequently the differential operator $\mathfrak{b}_i$ is order $2\ell_i$ with leading coefficient $r_i(x)$ for all i. In particular

$$U(x)W(x)U(x)^* = R(x) := \operatorname{diag}\big(r_1(x),\dots,r_N(x)\big). \qquad \square$$

Remark 5.13. The $U(x)$ and $r_i(x)$ are not quite the same as the $T(x)$ and $f_i(x)$ from (1.2). In particular the $r_i(x)$ do not need to be classical weights. However, as we will see below the $r_i(x)$ will be classical weights up to multiplication by some rational functions.

In the case of generalized Gegenbauer polynomials, the decomposition (5.12) was previously obtained in [30] and was interpreted in terms of an LDU decomposition of $W(x)$.

Note that in the proof of the previous theorem, we also showed the following symmetry-type result for the $\mathfrak{v}_i$'s:

$$\mathfrak{v}_i\mathfrak{b}_i = \mathfrak{b}_i\mathfrak{v}_i^*. \tag{5.14}$$

We also showed above that $\vec{\mathfrak{u}}_i^T W(x)\vec{\mathfrak{u}}_j^{T*} = 0$ for $i \neq j$, so that

$$\vec{u}_{\ell_i i}(x)^T W(x)\vec{u}_{\ell_j j}(x)^{T*} = 0 \quad \text{for } i \neq j.$$

Since $W(x)$ is Hermitian and positive-definite on (x_0,x_1), it follows that $\vec{u}_{\ell_1 1}(x)$, $\dots,\vec{u}_{\ell_N N}(x)$ are $\mathbb{C}(x)$-linearly independent. In particular $\det(U(x))$ is not identically zero so that $U(x)$ is a unit in $M_N(\mathbb{C}(x))$.

5.2. Control of the size of Fourier algebras. As a consequence of the previous theorem, we find that the involution † preserves a large subalgebra of $M_N(\Omega(x))$. Moreover, we can deduce that the right Fourier algebra of the orthogonal polynomials of $W(x)$ is very large.

To see this, first note that the operators $\mathfrak{v}_1, \dots, \mathfrak{v}_N$ may be related to one another, and these relationships are encoded by the center $\mathcal{Z}(W)$ of $\mathcal{D}(W)$.

PROPOSITION 5.15. *Let $W(x)$ be a weight matrix with $\mathcal{D}(W)$ full and let $\mathfrak{V}_1, \dots, \mathfrak{V}_N$, $\mathfrak{U}$, $\mathfrak{v}_1, \dots, \mathfrak{v}_N$ and $U(x)$ be defined as above. If $\mathfrak{V}_i$ and $\mathfrak{V}_j$ are supported on the same irreducible component of $\mathcal{Z}(W)$, then $\mathfrak{v}_i$ and $\mathfrak{v}_j$ are Darboux conjugates and $r_i(x)/r_j(x)$ is rational.*

Proof. Suppose $\mathfrak{V}_i$ and $\mathfrak{V}_j$ are supported on the same irreducible component of $\mathcal{Z}(W)$. Then there exists $\mathfrak{D} \in \mathcal{D}(W)$ such that $\mathfrak{V}_i \mathfrak{D} \mathfrak{V}_j \neq 0$. Without loss of generality, we may take $\mathfrak{D}$ to be W-symmetric. Moreover $\vec{u_k}^T \mathfrak{V}_i \mathfrak{D} \mathfrak{V}_j \mathfrak{V}_\ell \neq \vec{0}^T$ only if $k = i$ and $j = \ell$, and therefore

$$\mathfrak{U}\mathfrak{V}_i \mathfrak{D} \mathfrak{V}_j = E_{ij} \mathfrak{d} \mathfrak{U}$$

for some $\mathfrak{d} \in \Omega(x)$. Similarly

$$\mathfrak{U}\mathfrak{V}_j \mathfrak{D} \mathfrak{V}_i = E_{ji} \widetilde{\mathfrak{d}} \mathfrak{U}$$

for some $\widetilde{\mathfrak{d}} \in \Omega(x)$.

Since $\mathfrak{V}_i \mathfrak{D} \mathfrak{V}_j \neq 0$, we know that $\mathfrak{d} \neq 0$. Now since $\mathfrak{V}_1 + \cdots + \mathfrak{V}_N$ is in the center of $\mathcal{Z}(W)$ we know that $\mathfrak{V}_i \mathfrak{D} \mathfrak{V}_j$ commutes with $\mathfrak{V}_1 + \cdots + \mathfrak{V}_N$ and therefore $\mathfrak{v}_1 E_{11} + \cdots + \mathfrak{v}_N E_{NN}$ commutes with $E_{ij}\mathfrak{d}$. Consequently $\mathfrak{v}_i \mathfrak{d} = \mathfrak{d} \mathfrak{v}_j$ and this shows that $\mathfrak{v}_i$ and $\mathfrak{v}_j$ are Darboux conjugates.

Next, using the fact that $\mathfrak{D}$, $\mathfrak{V}_i$ and $\mathfrak{V}_j$ are all W-symmetric we calculate

$$\widetilde{\mathfrak{d}} \mathfrak{b}_i = \mathfrak{b}_j \mathfrak{d}^*.$$

This in particular implies that $\widetilde{\mathfrak{d}}$ is nonzero. Comparing leading coefficients, we see that $r_i(x)/r_j(x)$ must be rational. This completes the proof. □

As the previous proposition shows, $r_i(x)/r_j(x)$ will be rational for some values of i, j. We can actually say more than this in the case that $W(x)$ is irreducible.

Definition 5.16. A weight matrix $W(x)$ is *reducible* if there exists a nonsingular constant matrix $A \in M_N(\mathbb{C})$ such that $AW(x)A^*$ is a direct sum of matrix weights of smaller size.

As shown in [45, Theorem 2.8] a weight matrix $W(x)$ is reducible if and only if there exists a formally W-symmetric constant idempotent matrix different from 0 and I. Since our interest is in the classification of weight matrices, it makes sense to focus specifically on irreducible weight matrices.

PROPOSITION 5.17. *Suppose that $W(x)$ is an irreducible weight matrix with $\mathcal{D}(W)$ full. Then for all i,j the ratio $r_i(x)/r_j(x)$ is a rational function.*

Proof. Suppose that for some i there exists a j such that the ratio $r_i(x)/r_j(x)$ is not a rational function. Define $S \subseteq \{1,\dots,N\}$ by

$$S = \big\{ j : r_i(x)/r_j(x) \text{ is not rational} \big\}.$$

Then by assumption both S and $S' := \{1,\dots,N\}\backslash S$ are nonempty. Furthermore if $j \in S$ and $k \in S'$ then $r_j(x)/r_k(x)$ is not rational. Define $R(x) = \operatorname{diag}(r_1(x),\dots,r_N(x))$ and note that $W(x) = U(x)^{-1}R(x)(U(x)^{-1})^*$. Then for any W-symmetric $\mathfrak{D} \in \mathcal{D}(W)$ we calculate

$$R(x)\big(U(x)\mathfrak{D}U(x)^{-1}\big)^* = \big(U(x)\mathfrak{D}U(x)^{-1}\big)R(x).$$

Clearly $U(x)\mathfrak{D}U(x)^{-1} \in M_N(\Omega(x))$. Comparing leading coefficients, we see that if the j,k'th entry of $U(x)\mathfrak{D}U(x)^{-1}$ is nonzero, then $r_j(x)/r_k(x)$ must be rational. Hence for all $j \in S$ and $k \in S'$ the j,k'th entry of $U(x)\mathfrak{D}U(x)^{-1}$ must be zero. It follows immediately that the nonzero idempontent matrix-valued rational function

$$G(x) := U(x)^{-1}\left(\sum_{j\in S} E_{jj}\right)U(x)$$

commutes with $\mathfrak{D}$ for all $\mathfrak{D} \in \mathcal{D}(W)$. Consider the Laurent series for $G(x)$ at ∞

$$G(x) = \sum_{j=-\ell}^{\infty} G_j x^{-j}.$$

for some integer $\ell \in \mathbb{Z}$ and matrices $G_j \in M_N(\mathbb{C})$ with $G_\ell \neq 0$. Note that since $G(x)$ is idempotent, we must have $\ell \geq 0$.

Let $P(x,n)$ be the sequence of monic orthogonal polynomials for the weight matrix $W(x)$ and let $\mathcal{E}(W) \subseteq M_N(\mathbb{C}[n])$ be the preimage of $\mathcal{D}(W)$ under b_P. For any $n \geq 0$ and any $\Lambda(n) \in \mathcal{E}(W)$ we calculate

$$\Lambda(n)P(x,n)G(x) = P(x,n)\cdot b_P(\Lambda(n))G(x) = P(x,n)G(x)\cdot b_P\big(\Lambda(n)\big).$$

Since $b_P(\Lambda(n))$ is degree-filtration preserving it will have the form $b_P(\Lambda(n)) = \sum_{j=0}^{d}\partial_x^j D_j(\mathfrak{s})$ for some $D_j(\mathfrak{s}) \in M_N(\mathbb{C}[\mathfrak{s}])$ with $\mathfrak{s} = \partial_x x$ and both D_0 and Λ are equal polynomials. Comparing leading coefficients, this shows that $G_\ell\Lambda(n+\ell) = \Lambda(n)G_\ell$ for all $n \geq 0$. Since $\mathcal{D}(W)$ is full, there exists a nonconstant $f(n) \in \mathbb{C}[n]$ such that $f(n)I \in \mathcal{E}(W)$. Letting $\Lambda(n) = f(n)I$, the above implies that $G_\ell(f(n+\ell) - f(n)) = 0$ for all $n \geq 0$ and therefore that $\ell = 0$ since $G_\ell \neq 0$. Thus $G(x)$ is of

the form

$$G(x) = G_0 + \sum_{j=1}^{\infty} G_j x^{-j}. \tag{5.18}$$

and G_0 is a nonzero, idempotent matrix commuting with everything in $\mathcal{E}(W)$.

We claim that $G(x) = G_0$. Set $\mathfrak{D} = b_P(f(n)I)$ so that for all integers $n \geq 0$

$$\big(G_0 P(x,n) - P(x,n)G(x)\big) \cdot \mathfrak{D} = f(n)\big(G_0 P(x,n) - P(x,n)G(x)\big)$$

The degree of $G_0P(x,n) - P(x,n)G(x)$ is strictly less than n, and so if it is not zero for some value of n then we may write $G_0P(x,n) - P(x,n)G(x) = \sum_{j=-\infty}^{\ell_n} Q_j x^j$ for some matrices $Q_j \in M_N(\mathbb{C})$ with $Q_{\ell_n} \neq 0$ for some $\ell_n < \ell$. Therefore by comparing leading coefficients, we see that $f(n)Q_{\ell_n} = f(\ell_n)Q_{\ell_n}$. Since $f(n)$ is either strictly increasing or strictly decreasing for $n \gg 0$, this is a contradiction. Hence $G_0P(x,n) = P(x,n)G(x)$ for $n \gg 0$. Because of (5.18), if $G(x) \neq G_0$, then $G(x)$ has a pole at some $a \in \mathbb{C}$ and therefore there exists an integer $\ell > 0$ such that $\widetilde{G}(x) = G(x)(x-a)^{\ell}$ has no pole at a and satisfies $\widetilde{G}(a) \neq 0$. Therefore for all $n \gg 0$ we have $P(a,n)\widetilde{G}(a) = G_0P(a,n)(a-a)^{\ell} = 0$, so that the nonzero column vectors of $\widetilde{G}(a)$ are in the kernel of $P(a,n)$ for all $n \gg 0$. However, this contradicts [18, Theorem 1.1], so we must conclude $G(x) = G_0$.

Since $\det(G(x)) = 0$, $G(x) \neq I$. One easily checks that $G(x)^{\dagger} = G(x)$, so this contradicts the assumption that $W(x)$ was irreducible. This completes the proof. □

The next theorem shows that the right Fourier algebra is significantly large.

THEOREM 5.19. *Let $W(x)$ be an $N \times N$ irreducible weight matrix with $N > 1$ and $\mathcal{D}(W)$ full, and let $P(x,n)$ be the associated sequence of monic orthogonal polynomials. Then for all $\mathfrak{D} \in M_N(\Omega(x))$ we have $\mathfrak{D}^{\dagger} \in M_N(\Omega(x))$ and there exists a polynomial $q(x) \in \mathbb{C}[x]$ with $q(x)\mathfrak{D}, \mathfrak{D}q(x) \in \mathcal{F}_R(P)$.*

Proof. Let $\mathfrak{V}_1, \ldots, \mathfrak{V}_N, \mathfrak{U}, U(x)$, and $\mathfrak{v}_1, \ldots, \mathfrak{v}_N$ be defined as above. Note that if $\vec{u}_{\ell_i i}(x)^T$ times the leading coefficient of $\mathfrak{V}_i$ is identically zero, then $U(x)$ times the leading coefficient of $\mathfrak{V}_i$ is identically zero. Since $U(x)$ is a unit in $M_N(\mathbb{C}(x))$, this is impossible. Thus the order m_i of $\mathfrak{v}_i$ must agree with the order of $\mathfrak{V}_i$. Furthermore since $W(x)$ is irreducible the order of $\mathfrak{V}_i$ must be greater than 0 for all i, as otherwise $\mathfrak{V}_i$ would define a constant idempotent matrix in $\mathcal{D}(W)$ different from 0 and I. Therefore $m_i > 0$ for all i.

Then by the previous theorem, we know that

$$U(x)W(x)U(x)^* = \operatorname{diag}\big(r_1(x), \ldots, r_N(x)\big).$$

By definition each of the $\mathfrak{v}_i$ is a differential operator with rational coefficients. By (5.9) we have

$$\mathfrak{v}_i = \sum_{j=0}^{m_i} \partial_x^j v_{ji}(x)$$

for some rational functions $v_{0i}(x), \dots, v_{m_i i}(x) \in \mathbb{C}(x)$ for all i, with $v_{m_i i}(x) \neq 0$. Additionally, (5.14) tells us

$$\mathfrak{b}_i \mathfrak{v}_i^* = \mathfrak{v}_i \mathfrak{b}_i$$

where here $\mathfrak{b}_i = \vec{\mathfrak{u}}_i^T W(x) \vec{\mathfrak{u}}_i^{T*}$ is a differential operator of order $2\ell_i$ with leading coefficient $r_i(x)$. Let $s_i(x)$ represent the subleading coefficient of $\mathfrak{b}_i$. Comparing coefficients, we see that

$$(-1)^{m_i} r_i(x) v_{m_i i}(x)^* = v_{m_i i}(x) r_i(x)$$

and also that

$$\begin{aligned} &v_{m_i i}(x) s_i(x) + v_{(m_i-1)i}(x) r_i(x) + 2\ell_i v'_{m_i i}(x) r_i(x) \\ &\quad = (-1)^{m_i} r_i(x) (m_i v'_{m_i i}(x)^* - v_{(m_i-1)i}(x)^*) \\ &\qquad + (-1)^{m_i} s_i(x) v_{m_i i}(x)^* + (-1)^{m_i} m_i r_i'(x) v_{m_i i}(x)^*. \end{aligned}$$

Combined with the first equation, this second equation simplifies to

$$\begin{aligned} &\big(v_{(m_i-1)i}(x) + (-1)^{m_i} v_{(m_i-1)i}(x)\big)^* r_i(x) + (2\ell_i - m_i) v'_{m_i i}(x) r_i(x) \\ &\quad = m_i r_i'(x) v_{m_i i}(x). \end{aligned}$$

Solving this differential equation for $r_i(x)$, we obtain for m_i even

$$r_i(x) = v_{m_i i}(x)^{2\ell_i/m_i - 1} \exp \int \frac{\operatorname{Re}\big(v_{(m_i-1)i}(x)\big)}{(m_i/2) v_{m_i i}(x)} dx, \tag{5.20}$$

and for m_i odd

$$r_i(x) = v_{m_i i}(x)^{2\ell_i/m_i - 1} \exp \int \frac{\operatorname{Im}\big(v_{(m_i-1)i}(x)\big)}{(m_i/2) v_{m_i i}(x)} dx. \tag{5.21}$$

In either case, conjugation by $r_i(x)$ sends $\Omega(x)$ to $\Omega(x)$.

Now suppose that $\mathfrak{D} \in M_N(\Omega(x))$. Set $R(x) = \operatorname{diag}(r_1(x), \dots, r_N(x))$. Then we know that $W(x) = U(x)^{-1} R(x) (U(x)^{-1})^*$. Therefore

$$\mathfrak{D}^\dagger = U(x)^{-1} R(x) (U(x)^{-1})^* \mathfrak{D}^* U(x)^* R(x)^{-1} U(x)$$

has rational entries if and only if $R(x)^{-1} U(x) \mathfrak{D} U(x)^{-1} R(x)$ has rational entries. By the previous proposition combined with the fact that conjugation by $r_i(x)$ preserves $\Omega(x)$, we see that $\mathfrak{D}^\dagger$ has rational entries.

Now suppose that $\mathfrak{D} \in M_N(\Omega(x))$. Choose a polynomial $q_0(x)$ such that both $q_0(x)\mathfrak{D}$ and $\mathfrak{D}^\dagger q_0(x)$ have polynomial coefficients. Let $p(x)$ be the unique monic polynomial vanishing only at the finite endpoints of the support of $W(x)$. If the order of $\mathfrak{D}$ is ℓ set $q_1(x) = q_0(x)p(x)^{2\ell}$. Then both $q_1(x)\mathfrak{D}$ and $(q_1(x)\mathfrak{D})^\dagger$ are in $M_N(\Omega[x])$, and the coefficients of $q_1(x)\mathfrak{D}$ vanish to sufficiently high order at the endpoints of the support of $W(x)$ so $q_1(x)\mathfrak{D}$ is W-adjointable. Hence $q_1(x)\mathfrak{D} \in \mathcal{F}_R(P)$. Similarly, we can choose $q_2(x) \in \mathbb{C}[x]$ with $\mathfrak{D}q_2(x) \in \mathcal{F}_R(P)$. Taking $q(x) = q_1(x)q_2(x)$, we get $q(x)\mathfrak{D}, \mathfrak{D}q(x) \in \mathcal{F}_R(P)$. □

5.3. First part of the proof of the Classification Theorem. We next prove a theorem that comprises *most* of the Classification Theorem from the introduction. It tells us that $\mathfrak{V}_1, \dots, \mathfrak{V}_N$ are Darboux conjugate to degree-filtration preserving differential operators of order at most two. To do so, we first recall a result for algebras of commuting differential operators with rational spectra.

LEMMA 5.22. (Kasman [28]) *Let $\mathcal{A} \subseteq \Omega(x)$ be a commutative subalgebra with* $\mathrm{Spec}(\mathcal{A})$ *a rational curve. Then there exist differential operators $\mathfrak{h}, \mathfrak{d} \in \Omega(x)$ with*

$$\mathfrak{h}\mathcal{A}\mathfrak{h}^{-1} \subseteq \mathbb{C}[\mathfrak{d}]$$

where $\mathfrak{d}$ has order equal to the greatest common divisor of the orders of operators in $\mathcal{A}$.

With this in mind, we have the following theorem.

THEOREM 5.23. *Suppose that $W(x)$ is a weight matrix with $\mathcal{D}(W)$ full and that $\mathcal{D}(W)$ contains a W-symmetric second order differential operator whose leading coefficient multiplied by $W(x)$ is positive definite on the support of $W(x)$. Then there exist rational matrix differential operators $\mathfrak{T}, \widetilde{\mathfrak{T}} \in M_N(\Omega(x))$ with*

$$\mathfrak{T}\widetilde{\mathfrak{T}} = \mathrm{diag}\left(p_i(\mathfrak{d}_1), \dots, p_i(\mathfrak{d}_N)\right) \quad \textit{and} \quad \widetilde{\mathfrak{T}}E_{ii}\mathfrak{T} = q(\mathfrak{V}_i) \tag{5.24}$$

for all i where for each i $\mathfrak{d}_i \in \Omega(x)$ is a differential operator of order 1 *or* 2 *and p_i, q are nonzero polynomials. If any nonconstant polynomials in $\mathfrak{v}_i$ and $\mathfrak{v}_j$ are Darboux conjugates (for example if $\mathfrak{V}_i$ and $\mathfrak{V}_j$ are supported on the same irreducible component of $\mathcal{Z}(W)$) then we may take $\mathfrak{d}_i = \mathfrak{d}_j$.*

Proof. Suppose that $\mathfrak{D} \in \mathcal{D}(W)$. Then we calculate for all i, j, k that $\vec{\mathfrak{u}}_i^T \mathfrak{V}_j \mathfrak{D} \mathfrak{V}_j \mathfrak{V}_k$ is $\vec{0}^T$ if $i \neq j$ or $j \neq k$. Therefore for all j there exists $\mathfrak{d}_j \in \Omega(x)$ satisfying

$$\mathfrak{U}\mathfrak{V}_j \mathfrak{D} \mathfrak{V}_j = \mathfrak{d}_j E_{jj} \mathfrak{U}.$$

Since $\mathfrak{V}_1 + \cdots + \mathfrak{V}_N \in \mathcal{Z}(W)$ we also know that $\mathfrak{d}_j$ commutes with $\mathfrak{v}_j$. Thus for all j the algebra $\mathfrak{V}_j \mathcal{D}(W) \mathfrak{V}_j$ is isomorphic to a subalgebra $\mathcal{A}_j \subseteq \Omega(x)$ defined by

$$\mathcal{A}_j = \left\{ \mathfrak{d} \in \Omega(x) : \mathfrak{d} E_{jj} \mathfrak{U} = \mathfrak{U} \mathfrak{V}_j \mathfrak{D} \mathfrak{V}_j,\ \mathfrak{D} \in \mathcal{D}(W) \right\}.$$

Each element of $\mathcal{A}_j$ commutes with the operator $\mathfrak{v}_j$, so by a result of Schur the algebra $\mathcal{A}_j$ is a commutative subalgebra of $\Omega(x)$.

For all i let m_i be the order of $\mathfrak{v}_i$. Let $\mathfrak{D}$ be a second order differential operator in $\mathcal{D}(W)$ whose leading coefficient multiplied by $W(x)$ is positive definite on the support of $W(x)$. For all i, the differential operator $\mathfrak{V}_i \mathfrak{D} \mathfrak{V}_i$ has order $2m_i + 2$. To see this, write $\mathfrak{V}_i = \sum_{j=0}^{m_i} \partial_x^j V_{ji}(x)$ and $\mathfrak{D} = \partial_x^2 D_2(x) + \partial_x D_1(x) + D_0(x)$ and note that since $\mathfrak{V}_i$ is W-symmetric we must have

$$V_{m_i i}(x) D_2(x) V_{m_i i}(x) = V_{m_i i}(x) D_2(x) W(x) V_{m_i i}(x)^* W(x)^{-1}.$$

By assumption $D_2(x)W(x)$ is positive-definite and Hermitian on the support of $W(x)$. Therefore $V_{m_i i}(x) D_2(x) V_{m_i i}(x)^*$ is not identically zero on the support of $W(x)$. Hence $V_{m_i i}(x) D_2(x) W(x) V_{m_i i}(x)^* W(x)^{-1}$ is not identically zero so the product $\mathfrak{V}_i \mathfrak{D} \mathfrak{V}_i$ has the desired order.

As mentioned in the previous paragraph, there must exist $\mathfrak{a}_i \in \Omega(x)$ with

$$\mathfrak{U} \mathfrak{V}_i \mathfrak{D} \mathfrak{V}_i = \mathfrak{a}_i E_{ii} \mathfrak{U}.$$

In other words, for all i and j

$$\vec{\mathfrak{u}_j}^T \mathfrak{V}_i \mathfrak{D} \mathfrak{V}_i = \delta_{ji} \mathfrak{a}_i \vec{\mathfrak{u}_i}^T.$$

If the order of $\mathfrak{a}_i$ is less than the order of $\mathfrak{V}_i \mathfrak{D} \mathfrak{V}_i$, then this would imply that $U(x)$ would be a left zero divisor of the leading coefficient of $\mathfrak{V}_i \mathfrak{D} \mathfrak{V}_i$. The determinant of $U(x)$ is not identically zero, so this is impossible. Thus the order of $\mathfrak{a}_i$ is $2m_i + 2$. In particular the greatest common divisor of the orders of elements in $\mathcal{A}_i$ is either 1 or 2.

We claim that $\mathrm{Spec}(\mathcal{A}_j)$ is a rational curve. To see this, for all i let $\Lambda_i(n)$ be the sequence of matrices satisfying $\Lambda_i(n) P(x,n) = P(x,n) \cdot \mathfrak{V}_i$. In terms of the generalized Fourier map $b_P(\Lambda_i(n)) = \mathfrak{V}_i$ for all i. Since $\mathfrak{V}_i$ is W-symmetric, we also know that $\|P(x,n)\|_W^2 \Lambda_i(n)^* \|P(x,n)\|_W^{-2} = \Lambda_i(n)$ for all n. Therefore $\|P(x,n)\|_W^{-1} \Lambda_i(n) \|P(x,n)\|_W$ defines a sequence of Hermitian matrices for all i. The spectral theorem tells us they are unitarily diagonalizable and since they commute, they are *simultaneously* unitarily diagonalizable. Thus there exists a sequence of unitary matrices $U(n)$ and sequences of real numbers $\lambda_1(n), \ldots, \lambda_N(n)$ such that for all i

$$C(n)^{-1} \Lambda_i(n) C(n) = \lambda_i(n) E_{ii},$$

where here $C(n) := \|P(x,n)\|_W U(n)$.

Now again take $\mathfrak{D} \in \mathcal{D}(W)$. Let $\Theta_i(n) = b_P^{-1}(\mathfrak{V}_i \mathfrak{D} \mathfrak{V}_i)$. Since $\mathfrak{V}_i \mathfrak{D} \mathfrak{V}_i$ commutes with $\mathfrak{V}_i$ for all i and is annihilated by $\mathfrak{V}_j$ for $i \neq j$, we know that there exist sequences of complex numbers $\theta_1(n), \ldots, \theta_N(n)$ with $C(n)^{-1}\Theta_i(n)C(n) = \theta_i(n)E_{ii}$. We also know that for all i the sequence $\Theta_i(n) \in M_N(\mathbb{C}[n])$. Since $\theta_i(n) = \mathrm{tr}(\Theta_i(n))$, we see that $\theta_i(n) \in \mathbb{C}[n]$ for all i. Thus for all i, the generalized Fourier map defines an injection $\mathcal{A}_i \to \mathbb{C}[n]E_{ii}$ via

$$\mathcal{A}_i \xleftarrow{\cong} \mathfrak{V}_i \mathcal{D}(W) \mathfrak{V}_i \xrightarrow{C(n)^{-1} b_P^{-1}(\cdot) C(n)} \mathbb{C}[n]E_{ii}.$$

Thus $\mathrm{Spec}(\mathcal{A}_i)$ is unirational and by Lüroth's theorem $\mathrm{Spec}(\mathcal{A}_i)$ is rational.

Since $\mathrm{Spec}(\mathcal{A}_i)$ is rational, the previous lemma tells us that there exist differential operators $\mathfrak{h}_i, \mathfrak{d}_i \in \Omega(x)$ with the order of $\mathfrak{d}_i$ equal to the greatest common divisor of the orders of elements in $\mathcal{A}$ (either 1 or 2) such that $\mathfrak{h}_i \mathcal{A} \mathfrak{h}_i^{-1} \subseteq \mathbb{C}[\mathfrak{d}_i]$. If any nonconstant polynomials in $\mathfrak{v}_i$ and $\mathfrak{v}_j$ are Darboux conjugates, then $\mathfrak{d}_i$ and $\mathfrak{d}_j$ are Darboux conjugates and by modifying our choice of the $\mathfrak{h}_i$ we can take $\mathfrak{d}_i = \mathfrak{d}_j$.

We can also obtain a revised version of (5.14). To see this, define $\widetilde{\mathfrak{b}_i} = \mathfrak{h}_i \mathfrak{b}_i \mathfrak{h}_i^*$. Then for any $\mathfrak{a} \in \mathcal{A}_i$ coming from a W-symmetric $\mathfrak{D} \in \mathcal{D}(W)$, we have $\mathfrak{b}_i \mathfrak{a}^* = \mathfrak{a} \mathfrak{b}_i$. Therefore $\widetilde{\mathfrak{b}_i}(\mathfrak{h}\mathfrak{a}\mathfrak{h}^{-1})^* \widetilde{\mathfrak{b}} = \mathfrak{h}\mathfrak{a}\mathfrak{h}^{-1}$. Since $\mathcal{A}_i$ is spanned by such elements, this defines an involution of $\mathcal{A}_i$ and of $\mathfrak{h}\mathcal{A}_i\mathfrak{h}^{-1}$ and therefore of their fraction fields. Since the fraction field of the subalgebra $\mathfrak{h}_i \mathcal{A}_i \mathfrak{h}_i^{-1}$ of $\mathbb{C}[\mathfrak{d}_i]$ is $\mathbb{C}(\mathfrak{d}_i)$ (by rationality plus order arguments), this means that $p(\mathfrak{d}_i) \mapsto \widetilde{\mathfrak{b}_i} p(\mathfrak{d}_i)^* \widetilde{\mathfrak{b}_i}^{-1}$ defines a sesquilinear automorphism of $\mathbb{C}(\mathfrak{d}_i)$. Such an automorphism must map $\mathfrak{d}_i$ to a generator of $\mathbb{C}(\mathfrak{d}_i)$, i.e., to an element of the form $\frac{a\mathfrak{d}_i + b}{c\mathfrak{d}_i + d}$ for some $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. The automorphism fixes the polynomial $\mathfrak{h}_i \mathfrak{v}_i \mathfrak{h}_i^{-1} \in \mathbb{C}[\mathfrak{d}_i]$, so we must have $c = 0$ and we can take $d = 1$. Also by comparing leading coefficients of the polynomial fixed by the automorphism, we see a must be a root of unity. Therefore the automorphism sends $a^{1/2}\mathfrak{d}_i - (a^{1/2}\mathfrak{d}_i)^* = -a^{-1/2}b$ to $a^{-1/2}b$. Since the automorphism is sesquilinear, it follows that $a^{-1/2}b$ must be purely imaginary. Thus replacing $\mathfrak{d}_i$ with $a^{1/2}\mathfrak{d}_i + a^{-1/2}b/2$ we obtain

$$\widetilde{\mathfrak{b}_i} \mathfrak{d}_i^* = \mathfrak{d}_i \widetilde{\mathfrak{b}_i}. \tag{5.25}$$

For each i there exists a polynomial $p_i(\mathfrak{d}_i) \in \mathbb{C}[\mathfrak{d}_i]$ with

$$\mathfrak{h}_i \mathfrak{v}_i = p_i(\mathfrak{d}_i) \mathfrak{h}_i.$$

This implies that

$$\ker(\mathfrak{h}_i^*) \cdot \mathfrak{v}_i \subseteq \ker(\mathfrak{h}_i^*).$$

Choose a polynomial $q(\mathfrak{v}_i) \in \mathbb{C}[\mathfrak{v}_i]$ such that $\ker(\mathfrak{h}_i^*) \cdot q(\mathfrak{v}_i) = 0$ for all i. Then there exists a rational differential operator $\mathfrak{t}_i$ with

$$q(\mathfrak{v}_i) = \mathfrak{t}_i \mathfrak{h}_i.$$

Let $\mathfrak{c}_i = \mathfrak{b}_i^{-1}\mathfrak{v}_i$, viewed as a pseudo-differential operator. Note that $\mathfrak{c}_i$ may actually be shown to be a differential operator, but this is not necessary for our arguments. It is clear from (5.14) that $\mathfrak{c}_i$ is $*$-symmetric. Furthermore an easy calculation shows

$$\mathfrak{V}_i = W(x)\mathfrak{U}^*\mathfrak{c}_i E_{ii}\mathfrak{U}.$$

This means that

$$\begin{aligned}\mathfrak{V}_i q(\mathfrak{V}_i) &= W(x)\mathfrak{U}^*\mathfrak{c}_i q(\mathfrak{v}_i)E_{ii}\mathfrak{U}\\ &= W(x)\mathfrak{U}^*\mathfrak{c}_i\mathfrak{t}_i\mathfrak{h}_i E_{ii}\mathfrak{U}\end{aligned}$$

and also that

$$\begin{aligned}p_i(\mathfrak{d}_i)q\big(p_i(\mathfrak{d}_i)\big)E_{ii} &= \mathfrak{h}\mathfrak{v}_i q(\mathfrak{v}_i)\mathfrak{h}_i^{-1}E_{ii}\\ &= \mathfrak{h}\mathfrak{b}_i\mathfrak{c}_i q(\mathfrak{v}_i)\mathfrak{h}_i^{-1}E_{ii}\\ &= \mathfrak{h}\mathfrak{b}_i\mathfrak{c}_i\mathfrak{t}_i E_{ii}\\ &= \mathfrak{h}\mathfrak{U}W(x)\mathfrak{U}^*\mathfrak{c}_i\mathfrak{t}_i E_{ii}.\end{aligned}$$

Take

$$\mathfrak{T} = \operatorname{diag}(\mathfrak{h}_1,\dots,\mathfrak{h}_N)\mathfrak{U}$$

and also

$$\widetilde{\mathfrak{T}} = W(x)\mathfrak{U}^*\operatorname{diag}(\mathfrak{c}_1\mathfrak{t}_1,\dots,\mathfrak{c}_N\mathfrak{t}_N) = \sum_i W(x)\vec{\mathfrak{u}}_i^{\,*}\mathfrak{c}_i\mathfrak{t}_i\vec{e}_i^{\,T}.$$

It is clear from the construction that $\mathfrak{T}\in M_N(\Omega(x))$. Note that for all i the row vectors of $\mathfrak{V}_i$ belong to $\mathcal{M}_i$, so that $\mathfrak{V}_i = \vec{\mathfrak{a}}_i\vec{\mathfrak{u}}_i^{\,T*}$ for some $\vec{\mathfrak{a}}_i\in\Omega(x)^{\oplus N}$. Since the ring of pseudo-differential operators is an integral domain and $W(x)\vec{\mathfrak{u}}_i^{\,T*}\mathfrak{c}_i\vec{\mathfrak{u}}_i^{\,T} = \mathfrak{V}_i$, we find $W(x)\vec{\mathfrak{u}}_i^{\,T*}\mathfrak{c}_i = \vec{\mathfrak{a}}_i$. Hence $\widetilde{\mathfrak{T}}$ is rational. Furthermore for all i

$$\mathfrak{T}\widetilde{\mathfrak{T}} = \operatorname{diag}\big(p_1(\mathfrak{d}_1)q\big(p_1(\mathfrak{d}_1)\big),\dots,p_N(\mathfrak{d}_N)q\big(p_N(\mathfrak{d}_N)\big)\big)$$

and also

$$\widetilde{\mathfrak{T}}E_{ii}\mathfrak{T} = \mathfrak{V}_i q(\mathfrak{V}_i).$$ □

THEOREM 5.26. *Suppose that $W(x)$ satisfies the assumptions of the previous theorem and let p_i, q, $\mathfrak{T}$, $\widetilde{\mathfrak{T}}$ and $\mathfrak{d}_i$ be defined as in the above theorem, with $\mathfrak{d}_i = \mathfrak{d}_j$ if and only if some nonconstant polynomials in $\mathfrak{d}_i$ and $\mathfrak{d}_j$ are Darboux conjugates. Let $\mathfrak{d}'_1,\dots,\mathfrak{d}'_r$ be the* distinct *values of $\mathfrak{d}_1,\dots,\mathfrak{d}_N$, and let n_i be the multiplicity of $\mathfrak{d}'_i$, i.e., $n_i = \#\{i : \mathfrak{d}_i = \mathfrak{d}'_i\}$. Then the centralizer of $\mathfrak{d}_1\oplus\cdots\oplus\mathfrak{d}_N$ in $M_N(\Omega[x])$ is*

given by

$$\mathcal{C}(\mathfrak{d}_1 \oplus \cdots \oplus \mathfrak{d}_N) \cong \bigoplus_{i=1}^{r} M_{n_i}\big(\mathbb{C}[\mathfrak{d}_i']\big).$$

Moreover conjugation by $\mathfrak{T}$ defines an embedding of $\mathcal{D}(W)$ into $\mathcal{C}(\mathfrak{d}_1 \oplus \cdots \oplus \mathfrak{d}_N)$.

Proof. Suppose that $\mathfrak{A} \in M_N(\Omega[x])$ commutes with $\mathfrak{d}_1 \oplus \cdots \oplus \mathfrak{d}_N$, and let $\mathfrak{a}_{ij}$ be the entries of $\mathfrak{A}$. For all i and j we must have $\mathfrak{d}_i\mathfrak{a}_{ij} = \mathfrak{a}_{ij}\mathfrak{d}_j$. Thus by the choice of the $\mathfrak{d}_i$'s we see that $\mathfrak{a}_{ij} = 0$ unless $\mathfrak{d}_i = \mathfrak{d}_j$, in which case $\mathfrak{a}_{ij}$ belongs to the centralizer of $\mathfrak{d}_j$ in $\Omega[x]$, the latter being $\mathbb{C}[\mathfrak{d}_j]$. From this the above isomorphism is clear.

Fix i, j. We define a map $\phi_{ij} : \mathcal{D}(W) \to \mathbb{C}(\mathfrak{d}_i)$ as follows. By arguments already applied in previous proofs, the element $\mathfrak{V}_i\mathfrak{D}\mathfrak{V}_j$ satisfies $\mathfrak{T}\mathfrak{V}_i\mathfrak{D}\mathfrak{V}_j = \mathfrak{a}E_{ij}\mathfrak{T}$ for some $\mathfrak{a}$ satisfying $\mathfrak{a}p_j(\mathfrak{d}_j) = p_i(\mathfrak{d}_i)\mathfrak{a}$. If $\mathfrak{a}$ is nonzero then again by the choice of the $\mathfrak{d}_i$ we know that $\mathfrak{d}_i = \mathfrak{d}_j$ so that $\mathfrak{a}$ commutes with $\mathfrak{p}_i(\mathfrak{d}_i)$ and therefore with $\mathfrak{d}_i$. We define $\phi_{ij}(\mathfrak{D}) = \mathfrak{a}p_i(\mathfrak{d}_i)^{-1}p_j(\mathfrak{d}_j)^{-1}$. More generally for any i, j and $\mathfrak{D} \in \mathcal{D}(W)$ we define $\phi_{ij}(\mathfrak{D}) \in \mathbb{C}(\mathfrak{d}_i)$ to be the unique element satisfying

$$p_i(\mathfrak{d}_i)\phi_{ij}(\mathfrak{D})p_j(\mathfrak{d}_j)E_{ij}\mathfrak{T} = \mathfrak{T}\mathfrak{D}.$$

We further define

$$\Phi : \mathcal{D}(W) \to \bigoplus_{i=1}^{r} M_{n_i}\big(\mathbb{C}(\mathfrak{d}_i')\big)$$

by setting $\Phi(\mathfrak{D})$ to be the matrix whose entries are $\phi_{ij}(\mathfrak{D})$ for all i, j. It is easy to see that Φ is an algebra monomorphism. In fact, if we view $\bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}(\mathfrak{d}_i'))$ as a subalgebra of the algebra of $N \times N$ matrices with pseudo-differential operator entries, then Φ is simply the conjugation map $\Phi : \mathfrak{D} \mapsto \mathfrak{T}\mathfrak{D}\mathfrak{T}^{-1}$.

The center $\mathcal{Z}(W)$ of $\mathcal{D}(W)$ is mapped under Φ to $\bigoplus_{i=1}^{r} \mathbb{C}[\mathfrak{d}_i']I_{n_i}$ (for I_{n_i} the $n_i \times n_i$ identity matrix). In particular, the image of the center $\mathcal{Z}(W)$ is integral over $\bigoplus_{i=1}^{r} \mathbb{C}[\mathfrak{d}_i']I_{n_i}$ (i.e., each element is the root of a monic polynomial with coefficients n this ring). The algebra $\mathcal{D}(W)$ is a finite module over its center, so we conclude that the image of $\mathcal{D}(W)$ is integral over $\bigoplus_{i=1}^{r} \mathbb{C}[\mathfrak{d}_i']I_{n_i}$.

We claim that Φ actually maps $\mathcal{D}(W)$ into $\bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}[\mathfrak{d}_i'])$. To see this, first note that as a consequence of (5.25), we have $\Phi(\mathfrak{D}^\dagger) = \Phi(\mathfrak{D})^\star$, where $\star$ represents the unique involution of $\bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}(\mathfrak{d}_i'))$ extending Hermitian conjugate on $M_{n_i}(\mathbb{C})$ and sending $\mathfrak{d}_i'$ to $\mathfrak{d}_i'$. Thus if $\mathcal{A}_j$ is the image of $\mathcal{D}(W) \to \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}(\mathfrak{d}_i')) \to M_{n_j}(\mathbb{C}(\mathfrak{d}_j'))$ then $\mathcal{A}_j$ may be identified with a subalgebra of $M_{n_j}(\mathbb{C}(t))$ closed under Hermitian conjugation, with each element satisfying a monic polynomial identity with coefficients in $\mathbb{C}[t]$. However if $F(t) \in M_{n_j}(\mathbb{C}(t))$ is Hermitian and integral over $\mathbb{C}[t]$, then $F(t)^2$ is integral over $\mathbb{C}[t]$. This means that $\mathrm{tr}(F(t)^2)$ is in $\mathbb{C}[t]$. However $\mathrm{tr}(F(t)^2) = \sum_{ij} |f_{ij}(t)|^2$, where $f_{ij}(t)$ are the

entries of $F(t)$, so this means $F(t)$ is in $M_{n_j}(\mathbb{C}[t])$. Since $\mathcal{A}_j$ is spanned by its Hermitian matrices, this shows that $\mathcal{A}_j \subseteq M_{n_j}(\mathbb{C}[\mathfrak{d}'_j])$. This proves the claim and the theorem. □

5.4. Second part of the proof of the Classification Theorem. We now complete the proof of the Classification Theorem stated in the introduction. Before doing so, we require a couple of lemmas.

LEMMA 5.27. *For all i the function $v_{m_i i}(x)$ is a polynomial in x which vanishes at the finite endpoints of the support (x_0, x_1) of $W(x)$.*

Proof. For each i we may write $\mathfrak{V}_i = \sum_{j=0}^{m_i} \partial_x^j V_{ji}(x)$ for some polynomials $V_{ji}(x) \in M_N(\mathbb{C}[x])$ with $V_{m_i i}(x)$ not identically zero and m_i the order of $\mathfrak{v}_i$, as above. Since $\mathfrak{V}_i$ is W-symmetric, the leading coefficient $V_{m_i i}(x)$ of $\mathfrak{V}_i$ must evaluate to a nilpotent matrix at the finite endpoints of the support of $W(x)$. In particular, its eigenvalues at these points must be zero. To see this, note that since $\mathfrak{V}_i$ is W-adjointable, so too is $\frac{1}{m_i!}\mathrm{Ad}_{xI}^{m_i-1}(\mathfrak{V}_i) = \partial_x V_{m_i i}(x) + \frac{1}{m_i} V_{(m_i-1)i}(x)$. Therefore so is $(\partial_x V_{m_i i}(x))^k$ for any integer $k > 0$. By taking appropriate linear combinations for various k, it follows that $V_{m_i i}(x)^k \partial_x^k$ is W-adjointable for all k. Hence for all k we must have the expression $V_{m_i i}(x)^k W(x)^{(k)}$ vanish as x approaches x_0 or x_1 from within (x_0, x_1). For each finite endpoint there must exist a value of k such that the derivative $W(x)^{(k)}$ evaluates to a nonzero matrix at the endpoint. Consequently $V_{m_i i}(x)^k$ must evaluate to the zero there.

Next note that the matrix $U(x)$ as defined above satisfies $U(x)V_{m_i i}(x) = v_{m_i i}(x)U(x)$ and since $U(x)$ is a unit in $M_N(\mathbb{C}(x))$, this implies that $v_{m_i i}(x)$ is equal to the trace of $V_{m_i i}(x)$. Hence it is a polynomial and it evaluates to 0 at the finite endpoints of the support of $W(x)$. □

LEMMA 5.28. *For all $1 \le i \le N$, let $h_i(x,n)$ be a sequence of classical orthogonal polynomials and let $P(x,n)$ be a sequence of matrix polynomials satisfying the property that*

$$P(x,n) = \mathrm{diag}\left(h_1(x,n-m), \ldots, h_N(x,n-m)\right)$$

for all integers $n \ge \ell$ for some fixed integers ℓ, m with $\ell \ge 0$. Then for all i, there exists a sequence of classical orthogonal polynomials $\widetilde{p}_i(x,n)$ such that

$$\widetilde{P}(x,n) := \mathrm{diag}\left(\widetilde{p}_1(x,n), \ldots, \widetilde{p}_N(x,n)\right)$$

is a bispectral Darboux transformation of $P(x,n)$.

Proof. The sequences of classical orthogonal polynomials come with intertwining operators which relate polynomials of various degrees. For example, consider the Jacobi polynomials $j_{\alpha,\beta}(x,n)$ for the weight $(1-x)^\alpha(1+x)^\beta 1_{(-1,1)}(x)$.

They are eigenfunctions of the Jacobi operator $\mathfrak{e}_{\alpha,\beta} = \partial_x^2(1-x^2) + \partial_x(\beta - \alpha - (\beta+\alpha+2)x)$. The intertwining operator for $j_{\alpha,\beta}(x,n)$ is defined as

$$\mathfrak{t}_{\alpha,\beta} := \partial_x(1-x^2) + \beta - \alpha - (\beta+\alpha+2)x.$$

Note that

$$\mathfrak{e}_{\alpha,\beta} = \partial_x \mathfrak{t}_{\alpha,\beta} \quad \text{and} \quad \mathfrak{e}_{\alpha+1,\beta+1} - (\beta+\alpha+2) = \mathfrak{t}_{\alpha,\beta}\partial_x$$

so $j_{\alpha+1,\beta+1}(x,n) \cdot \mathfrak{t}_{\alpha,\beta}$ is an polynomial eigenfunction of $\mathfrak{e}_{\alpha,\beta}$ of degree $n+1$ and therefore equal to a constant multiple of $j_{\alpha,\beta}(x,n+1)$. Similarly ∂_x sends $j_{\alpha,\beta}(x,n)$ to $j_{\alpha+1,\beta+1}(x,n-1)$. It follows that

$$j_{\alpha,\beta}(x,n-m) \cdot \partial_x^\ell \mathfrak{t}_{\alpha+\ell-1,\beta+\ell-1} \dots \mathfrak{t}_{\alpha-m,\beta-m} = c_n j_{\alpha-m,\beta-m}(x,n)$$

for some sequence of constants c_n. Furthermore ∂_x^m sends $j_{\alpha-m,\beta-m}(x,n)$ to a constant multiple of $j_{\alpha,\beta}(x,n-m)$ and therefore the $j_{\alpha,\beta}$ must be eigenfunctions of the product. Thus

$$(\partial_x^\ell \mathfrak{t}_{\alpha+\ell-1,\beta+\ell-1} \dots \mathfrak{t}_{\alpha-m,\beta-m})\partial_x^m \in \mathbb{C}[\mathfrak{e}_{\alpha,\beta}].$$

By similar arguments to those in the previous paragraph for the Hermite and Jacobi case, we see that taking appropriate products of ∂_x and the associated intertwining operators we can obtain differential operators $\mathfrak{t}_i, \widetilde{\mathfrak{t}_i} \in \Omega[x]$ such that the kernel of $\mathfrak{t}_i$ contains all polynomials of degree less than ℓ and such that

$$\mathfrak{t}_i \widetilde{\mathfrak{t}_i} \in \mathbb{C}[\mathfrak{d}_i] \quad \text{and} \quad h_i(x,n-m) \cdot \mathfrak{t}_i = c_i \widetilde{h}_i(x,n)$$

for some sequence of monic orthogonal polynomials $\widetilde{h}_i(x,n)$. Note that all but finitely many of the c_n's must be nonzero, since otherwise the kernel of the differential operator will be too large. Setting $\mathfrak{T} = \operatorname{diag}(\mathfrak{t}_1, \dots, \mathfrak{t}_N)$ and $\widetilde{\mathfrak{T}} = \operatorname{diag}(\widetilde{\mathfrak{t}_1}, \dots, \widetilde{\mathfrak{t}_N})$ it follows that

$$P(x,n) \cdot \mathfrak{T} = C(n)\widetilde{P}(x,n) \text{ and } \widetilde{P}(x,n) \cdot \widetilde{\mathfrak{T}} = \widetilde{C}(n)P(x,n)$$

for some sequences of matrices $C(n), \widetilde{C}(n)$ nonsingular for almost every n and for $\widetilde{P}(x,n)$ defined as in the statement of the lemma. By Theorem 5.19, it follows that $\widetilde{P}(x,n)$ is a bispectral Darboux transformation of $P(x,n)$. □

Proof of the Classification Theorem 1.3. We assume $W(x)$ is a weight matrix and that $\mathcal{D}(W)$ contains a W-symmetric second order differential operator whose leading coefficient multiplied by $W(x)$ is positive definite on the support of $W(x)$. Without loss of generality, we may assume that $W(x)$ is unitarily irreducible $N \times N$ weight matrix with $N > 1$.

The first direction is easy. If $W(x)$ is a bispectral Darboux transformation of a direct sum of classical weights $f_1(x) \oplus \cdots \oplus f_N(x)$, then the fact that $\mathcal{D}(f_1(x) \oplus$

$\cdots \oplus f_N(x))$ is full automatically implies $\mathcal{D}(W)$ is full. The difficult part is proving the converse. Assume $\mathcal{D}(W)$ is full.

By Theorem 5.23, we may choose $\mathfrak{T}, \widetilde{\mathfrak{T}} \in M_N(\Omega(x))$ with $\widetilde{\mathfrak{T}} E_{ii} \mathfrak{T} = q(\mathfrak{V}_i)$ and with $\mathfrak{T}\widetilde{\mathfrak{T}} = \operatorname{diag}(p_1(\mathfrak{d}_1), \dots, p_N(\mathfrak{d}_N))$ for some differential operators $\mathfrak{d}_1, \dots, \mathfrak{d}_N$ of order 1 or 2 and polynomials p_i, q. Note that by rescaling $\mathfrak{T}$ on the left and $\widetilde{\mathfrak{T}}$ on the right, we may without loss of generality assume that $\widetilde{\mathfrak{T}}$ has polynomial coefficients. Also from the proof of Theorem 5.23, we know that there exist $\mathfrak{h}_1, \dots, \mathfrak{h}_N \in \Omega(x)$ and polynomials $p_1(\mathfrak{d}_1), \dots, p_N(\mathfrak{d}_N)$ with $\mathfrak{h}_i \mathfrak{v}_i = p_i(\mathfrak{d}_i)\mathfrak{h}_i$ for all i. Let n_i be the degree of the polynomial p_i and write

$$\mathfrak{h}_i = \sum_{j=0}^{k_i} \partial_x^j h_{ji}(x)$$

for some rational functions $h_{0i}(x), \dots, h_{k_i i}(x) \in \mathbb{C}(x)$ with $h_{k_i i}(x) \neq 0$.

We also assumed that there exists a second order differential operator $\mathfrak{D} \in \mathcal{D}(W)$ whose leading coefficient multiplied by $W(x)$ is Hermitian and positive-definite on the support of $W(x)$. By Theorem 5.26 $\mathfrak{T}\mathfrak{D} = \widetilde{\mathfrak{D}}\mathfrak{T}$ for some second order differential operator $\widetilde{\mathfrak{D}} \in \mathcal{D}(W)$. Furthermore by the calculation in the proof of Theorem 5.23 we know that $E_{ii}\widetilde{\mathfrak{D}}E_{ii} = \mathfrak{a}_i E_{ii}$ for some second order differential operator $\mathfrak{a}_i \in \mathbb{C}[\mathfrak{d}_i]$. Let $C(n)$ be the sequence of matrices from the proof Theorem 5.23 and define a sequence of matrix polynomials $\widetilde{P}(x,n)$ and sequences of scalar polynomials $g_i(x,n)$ by

$$\widetilde{P}(x,n) = C(n)P(x,n) \cdot \widetilde{\mathfrak{T}} \quad \text{and} \quad g_i(x,n)E_{ii} = E_{ii}\widetilde{P}(x,n)E_{ii}.$$

Then for all i, we calculate

$$\widetilde{\mathfrak{T}} q(\mathfrak{d}_i) E_{ii} = q(\mathfrak{V}_i)\widetilde{\mathfrak{T}} \quad \text{and} \quad \widetilde{\mathfrak{T}} q(\mathfrak{d}_i)^2 \mathfrak{a}_i E_{ii} = \mathfrak{V}_i \mathfrak{D} \mathfrak{V}_i \widetilde{\mathfrak{T}}.$$

Therefore

$$\widetilde{P}(x,n) \cdot q(\mathfrak{d}_i) E_{ii} = q(\lambda_i(n)) E_{ii} \widetilde{P}(x,n)$$

and

$$\widetilde{P}(x,n) \cdot q(\mathfrak{d}_i)^2 \mathfrak{a}_i E_{ii} = q(\lambda_i(n))^2 \theta_i(n) E_{ii} \widetilde{P}(x,n)$$

for some polynomial $\theta_i(n)$. Putting this together

$$\widetilde{P}(x,n) \cdot \mathfrak{a}_i E_{ii} = \theta_i(n) E_{ii} \widetilde{P}(x,n).$$

In particular it follows that for all i

$$g_i(x,n) \cdot \mathfrak{a}_i = \theta_i(n) g_i(x,n).$$

Note also that $\widetilde{P}(x,n)E_{ii} \cdot \mathfrak{T} = p_i(\lambda_i(n))E_{ii}P(x,n)$. Defining $\mathfrak{t}_i$ by $\mathfrak{t}_i E_{ii} = E_{ii}\mathfrak{T}E_{ii}$, this says that for all but possibly finitely many n (corresponding to roots

of $p_i(\lambda_i(n))$) the function $g_i(x,n)\cdot \mathfrak{t}_i$ is a polynomial of degree n. Hence for each i there exists an integer m_i (possibly negative) such that for all but finitely many n the degree of $g_i(x,n)$ is $n-m_i$. In particular $\mathfrak{a}_i$ is exceptional and by Theorem 2.46 we know that $\mathfrak{a}_i$ is Darboux conjugate to a degree-filtration preserving differential operator of order 2.

The previous paragraph shows that we may have originally chosen $\mathfrak{h}_i$, $\mathfrak{T}, \widetilde{\mathfrak{T}}$ so that the $\mathfrak{a}_i$ (or equivalently the $\mathfrak{d}_i$) are degree-filtration preserving. Thus without loss of generality we take $\mathfrak{d}_i$ to be degree-filtration preserving for all i. In particular for each $\mathfrak{d}_i$ we may write

$$\mathfrak{d}_i = \partial_x^2 a_{2i}(x) + \partial_x a_{1i}(x) + a_{0i}(x)$$

where $a_{ji}(x)$ is a polynomial with $\deg(a_{ji}(x)) \leq j$ for all $1 \leq i \leq N$ and $0 \leq j \leq 2$.

We next show that $\mathfrak{d}_i$ must be order 2 and equal to one of the classical second order differential operators of Hermite, Laguerre and Jacobi (hereafter referred to simply as classical operators). Let t_i be the order of $\mathfrak{d}_i$, which is either 1 or 2. Comparing leading coefficients in the expression $\mathfrak{h}_i\mathfrak{v}_i = p_i(\mathfrak{d}_i)\mathfrak{h}_i$ we see

$$v_{m_i i}(x) = a_{t_i i}(x)^{n_i}.$$

Furthermore by comparing subleading coefficients we find

$$\begin{aligned} &h_{k_i i}(x)v_{(m_i-1)i}(x) + h_{(k_i-1)i}(x)v_{m_i i}(x) + m_i h'_{k_i i}(x)v_{m_i i}(x) \\ &\quad = a_{t_i i}(x)^{n_i} h_{(k_i-1)i}(x) + (n_i a_{t_i i}(x)^{n_i-1} a_{(t_i-1)i}(x) \\ &\qquad + (t_i n_i - t_i + m_i n_i) a_{t_i i}(x)^{n_i-1} a'_{t_i i}(x)) h_{k_i i}(x). \end{aligned}$$

Combining this with the first equation, this simplifies to

$$v_{(m_i-1)i}(x) = a_{t_i i}(x)^{n_i}\left[-m_i \frac{h'_{k_i i}(x)}{h_{k_i i}(x)} + \frac{n_i a_{(t_i-1)i}(x) + (n_i(m_i+t_i)-t_i)a'_{t_i i}(x)}{a_{t_i i}(x)}\right].$$

Therefore by (5.20)-(5.21) we calculate that up to a constant multiple the function $r_i(x)$ is given by

(5.29)
$$r_i(x) = \mathrm{Re}\left[a_{t_i i}(x)^{(\frac{2\ell_i n_i}{m_i} - n_i + \frac{2n_i(m_i+t_i)-2t_i}{m_i})} h_{k_i i}(x)^{-2} \exp \int \frac{2n_i}{m_i}\frac{a_{(t_i-1)i}(x)}{a_{t_i i}(x)}dx\right]$$

for m_i even and

(5.30)
$$r_i(x) = \mathrm{Im}\left[a_{t_i i}(x)^{(\frac{2\ell_i n_i}{m_i} - n_i + \frac{2n_i(m_i+t_i)-2t_i}{m_i})} h_{k_i i}(x)^{-2} \exp \int \frac{2n_i}{m_i}\frac{a_{(t_i-1)i}(x)}{a_{t_i i}(x)}dx\right]$$

for m_i odd.

The leading coefficient $v_{m_i i}(x)$ of $\mathfrak{v}_i$ must vanish at the finite endpoints of the support of $W(x)$. Since $v_{m_i i}(x) = a_{t_i i}(x)^{n_i}$, this implies that $a_{t_i i}(x)$ must vanish at the finite endpoints of the support of $W(x)$. If t_i is 1, then (5.29)-(5.30) imply that $r_i(x)$ is rational. Thus in this case $W(x)$ must be rational, and for $W(x)$ to have finite moments the support of $W(x)$ must then be a finite interval. However, $a_{t_i i}(x)$ can only have one root in this case, so $a_{t_i i}(x)$ does not vanish at both the endpoints of the support of $W(x)$. Thus we see that $t_i = 2$ for all i and that $\mathfrak{d}_i$ is an operator of order 2.

Suppose that for some i the polynomial $a_{2i}(x)$ has degree two. Then (5.29)-(5.30) tell us that for $W(x)$ to have finite moments the support of $W(x)$ must lie on a finite interval. Therefore in this case the roots of $a_{2i}(x)$ must be real and distinct and equal to the finite endpoints of the support of $W(x)$. Up to an affine translation, they can be taken to be ± 1 in which case $\mathfrak{d}_i$ is equal to a Jacobi operator. Note that in this case since $a_{2j}(x)$ must vanish on the support of $W(x)$, they must all be Jacobi operators.

Alternatively suppose that for some i the polynomial $a_{2i}(x)$ has degree 1. Then again to have finite moments and for $a_{2i}(x)$ to vanish at the finite endpoints of the support of $W(x)$, the support of $W(x)$ must be semi-infinite. Up to an affine translation, we can take the support to be $(0,\infty)$ in which case $\mathfrak{d}_i$ is a Laguerre operator. Again, this implies that all of the $\mathfrak{d}_j$ are Laguerre operators.

Finally, suppose that for some i the polynomial $a_{2i}(x)$ has degree 0. Then the weight matrix $W(x)$ must be supported on the whole real line. Also since $\mathfrak{d}_i$ is exceptional the polynomial $a_{1i}(x)$ must have degree 1 in this case (since otherwise $\mathfrak{d}_i$ would not have polynomial eigenfunctions of all but finitely many degrees). Therefore up to an appropriate translation we may take $\mathfrak{d}_i$ to be the Hermite operator.

In any case, $\mathfrak{d}_i$ is a classical operator. For all i let $f_i(x)$ be the weight function associated to the classical operator $\mathfrak{d}_i$, and let $h_i(x,n)$ be the associated sequence of monic orthogonal polynomials. From the above calculation (replacing $\mathfrak{d}_i$ with $\mathfrak{a}_i$), we know that the polynomials $\widetilde{P}(x,n) := C(n)P(x,n)\cdot\widetilde{\mathfrak{T}}$ are eigenfunctions of $\mathfrak{d}_i E_{ii}$ for all i with

$$\widetilde{P}(x,n)\cdot\mathfrak{d}_i E_{ii} = \theta_i(n)E_{ii}\widetilde{P}(x,n)$$

for some polynomials $\theta_1(n),\dots,\theta_N(n)$. If $\theta_i(n)$ is nonzero, the above implies that the i,j'th entry $\widetilde{P}(x,n)_{ij}$ of $\widetilde{P}(x,n)$ is zero for all $j \neq i$. Thus for all but finitely many n, the polynomial $\widetilde{P}(x,n)$ is diagonal. Furthermore, for all n sufficiently large the degree of the diagonal entry of $\widetilde{P}(x,n)$ is $n-m$. Thus for all but finitely many n, we know that $\widetilde{P}(x,n)$ is a diagonal polynomial matrix whose i'th diagonal piece is $\alpha_i h_i(x,n-m)$ for some nonzero constant α_i. Therefore by the previous lemma we know that $\widetilde{P}(x,n)$ is a bispectral Darboux transformation of a sequence of orthogonal polynomials for a direct sum of classical weights. □

Proof of Theorem 1.5. We know in this case that $\mathcal{D}(W)$ is full and that the center of $\mathcal{D}(W)$ is irreducible. The result follows immediately. □

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