Student Reasoning About the Least-Squares Problem in Inquiry-Oriented Linear Algebra

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The method of Least Square Approximation is an important topic in some linear algebra classes. Despite this, little is known about how students come to understand it, particularly in a Realistic Mathematics Education setting. Here, we report on how students used literal symbols and equations when solving a least squares problem in a travel scenario, as well as their reflections on the least squares equation in an open-ended written question. We found students used unknowns and parameters in a variety of ways. We highlight how their use of dot product equations can be helpful towards supporting their understanding of the least squares equation.

Keywords: linear algebra, least squares, dot product, student reasoning, inquiry

Linear algebra courses frequently include the topic of least squares approximation. A central focus of the least squares problem is when the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ has no solution. When this occurs but some kind of solution is needed, one tries to "find an $\boldsymbol{x}$ that makes $A\boldsymbol{x}$ as close as possible to $\boldsymbol{b}$" (Lay et al., 2016, p. 362). This is consistent with a subspace orientation to the least squares problem (i.e., finding the best approximation in a subspace to a vector not in a subspace), which we leverage in our design research, as compared to a statistical interpretation. The set of least squares solutions to $A\boldsymbol{x} = \boldsymbol{b}$ are the vector(s) $\hat{\boldsymbol{x}}$ that are solution(s) to $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$. How can we, as instructors, help students come to understand why that equation is relevant to solving the least squares problem? Or, how can we, as curriculum designers, engage students in the guided reinvention of that equation? In this paper we pursue the research question: How do students use literal symbols and equation types as they solve an experientially real least squares problem, and what reactions do they share about the least squares equation $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$?

The work presented in this paper is from a project (DUE-1915156/1914841/1914793) that aims to create research-based curricular materials for the guided reinvention of core concepts within an inquiry-oriented linear algebra class. This work is guided by making sense of student thinking and the complexity of mathematical ideas, which informs the refinement of the curricular materials. In pursuit of our research question, we examine written data from 13 students and analyze the variety of ways they use literal symbols and equations when solving a least squares problem in an experientially real (Gravemeijer, 1999) task setting. We also analyze the students' reactions to the least squares equation $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$. This analysis will inform a future conceptual analysis of the mathematics as well as refinements to the task sequence.

Theoretical Framing and Literature Review

Our approach to design research is informed by what we refer to as the Design Research Spiral (Wawro et al., 2022), which is based on the design research cycle (e.g., Cobb et al., 2003), is composed of five phases, with revisions occurring between each phase based on ongoing analyses of student thinking and reflections on the mathematics. Within the first phase, the Design phase, our work is based in Freudenthal's philosophy of mathematics as a human activity (1973) and the design principles that emerged from his work in Realistic Mathematics Education

(RME) (Gravemeijer, 2020; Gravemeijer & Terwel, 2000; Treffers, 1987). RME design principles include didactical phenomenology (the means for creating the task setting of the phenomena to be organized), emergent models (a process through which students can progress from a less formal understanding of the phenomena to a more mathematized organization of the phenomena), and guided reinvention (a mechanism by which students can reinvent mathematical ideas guided by the task structure and their interactions with the instructor and their peers).

There is little literature on the teaching or learning of least squares approximation within the context of linear algebra. Turgut (2013) designed a lesson to teach least squares as a line of best fit using Mathematica. His lesson invited students to use commands in Mathematica to plot dots, form matrices from points, transpose a matrix, and take an inverse of a matrix in finding the best possible solution. A topic we view as related to the teaching and learning of least squares is dot product. Donevska-Todorova (2015) identified three definitions of the dot product within three modes of description (arithmetic, geometric, abstract-axiomatic) and created an applet to promote students' geometric understanding of the dot product. Cooley et al. (2014) developed a module to teach dot product, focusing on the cosine definition. Their task included comparing frequency vectors and determining if an author wrote two different texts. Dray and Manogue (2006) found projection to be essential in understanding dot product. They claimed the geometric approach of the dot product benefits students in many applications of physics and engineering.

As we examined students' work on the question shown in Figure 2, we were struck by the variety of ways literal symbols and equations were used as students completed the problem. In undergraduate mathematics, literal symbols are used in many ways. It is important to understand students' interpretation of literal symbols and how they are used when solving problems. For our analysis, we draw from Philipp (1992) and Drijvers (2003) for unpacking the nuance in literal symbol use. Drawing on works such as Keiran (1988), Küchemann (1978), and Usiskin (1988), Philipp uses literal symbol to describe the mathematical use of a letter; he provides seven literal symbol uses: labels, constants, unknowns, generalized numbers, varying quantities, parameters, and abstract symbols; most relevant to our work is unknown, varying quantity, and parameter. Philipp states that an "unknown involves the use of a literal symbol when the goal is to solve an equation" (p.558), such as in the role of x in $8x + 4 = 28$. Philipp's use of varying quantity is consistent with Knuth et al.'s (2005) definition of variable as "a literal symbol that represents, at once, a range of numbers" (p. 70), such as x and y in $y = 3x + 5$. Finally, Philipp describes parameters as generalized constants, such as m and b in the linear equation $y = mx + b$.

Drijvers (2003) focused on design research related to the concept of parameter, which he sees as "an 'extra variable' in a formula or function that makes it represent a class of formulas, a family of functions and a sheaf of graphs" (p. 60). Drijvers delineates four roles that a parameter can assume: placeholder, changing quantity, generalizer, and unknown. First, a *parameter as a placeholder* plays the role of a constant value that does not change; whether known or unknown, its value is fixed, and filling in different known values relate to different situations rather than variations of the same situation. Second, a *parameter as a changing quantity* represents a numerical value that takes on a dynamic character of systematic variation. It runs smoothly through a reference set, affecting the complete, global situation set rather than a single situation (e.g., p and q in $y = (x - p)^2 + q$). Third, a *parameter as a generalizer* does not stand for a specific number but rather for an exemplary number or set of numbers. It facilitates seeing the general in the particular, formulating solutions at a general level, and solving concrete cases at once by means of a parametric general solution (e.g., t in a parametric solution $x = t(1,2,3)$). Fourth, the *parameter as unknown* facilitates "selecting particular cases from the general representation on the basis of an extra condition or criterion. In such situations, the parameter acquires the role of unknown-to-be-found" (p. 69) (e.g., solve for t for a specific x in the above). Finally, these roles are not fixed and can change in the solution process.

The Task Setting

Our task sequence leverages the subspace-oriented version of the least squares problem (the best approximation to a vector not in a subspace is its orthogonal projection onto the subspace). We designed an experientially real task setting called *Delivering Mail to Gauss*, which is based on the Magic Carpet Ride sequence (Wawro, et al., 2012) in the Inquiry-Oriented Linear Algebra (IOLA) curriculum (Wawro et al., 2013). Even though it is a fantasy setting, we have found that students can immediately engage with the idea of different transportation modes, each traveling forward and backward in a single vector direction. The first Delivering Mail task asks students to use three specific travel vectors (the same vectors in Task 3 of the Magic Carpet Ride sequence) to travel to Gauss in $\mathbb{R}^3$ so that they can deliver his mail. This differs from the Magic Carpet ride task in that Gauss is now in a location outside of the span of the travel vectors. Using previous knowledge, students determine Gauss cannot be reached and the travel vectors span a plane in R^3. Students are then told their cousin has a drone they can use to deliver Gauss's mail, on the condition that they get as close as they can to Gauss using the travel vectors before they use the drone. They then determine where to travel to, how to get there with the travel vectors, along what vector the drone would travel, and what distance the drone's trip would be (Figure 1a).

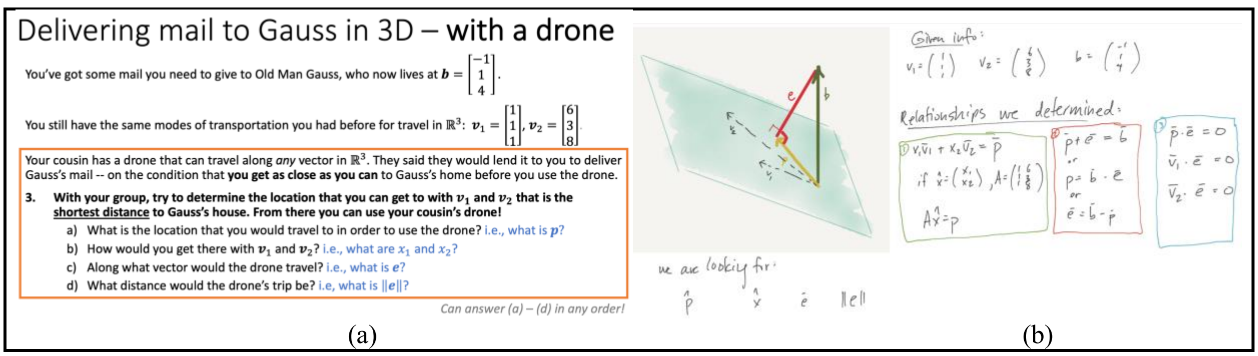

Figure 1. The setup of Task 3 and the class's shared development of known relationships to help solve the task.

There are many aspects of the problem to symbolize. Gauss's location is denoted as $\boldsymbol{b}$. The three travel vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are in the first task statement, but once students realize the three span a plane, they work only with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. The sequence is designed to foster students' exploration of what location on the plane would be closest to Gauss, and students consistently suggest the location that creates a path *orthogonal* from the plane to Gauss (Lee et al., 2022). With an instructor's suggestions for which literal symbols to choose, the class uses vector $\boldsymbol{p}$ to denote where on the plane they should travel to and use the drone, **e** as the drone's path to Gauss, and $||\boldsymbol{e}||$ as the distance of the drone's trip. The class symbolizes the relationship between these as $\boldsymbol{p} + \boldsymbol{e} = \boldsymbol{b}$. To denote how to get to $\boldsymbol{p}$ using the travel vectors, the scalars x_1 and x_2 are used to mean how much and in what direction to travel on $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ so that $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 = \boldsymbol{p}$, which can be written as a matrix equation $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$ where A is the matrix with columns $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and $\widehat{\boldsymbol{x}}$ is the vector with components x_1 and x_2. Finally, the instructor leads the class in a derivation that the dot product of two orthogonal vectors is zero, which results in the class denoting $\boldsymbol{v}_1 \cdot \boldsymbol{e} = 0$, $\boldsymbol{v}_2 \cdot \boldsymbol{e} = 0$, and $\boldsymbol{p} \cdot \boldsymbol{e} = 0$. All of these relationships are summarized in Figure 1b, which shows one instructor's written work that recorded the relationships during class.

Students then use these known relationships to solve for $\boldsymbol{p}$, $\widehat{\boldsymbol{x}}$, $\boldsymbol{e}$, and $||\boldsymbol{e}||$. Students make progress in a variety of ways. If students combine $\boldsymbol{v}_1 \cdot \boldsymbol{e} = 0$ and $\boldsymbol{v}_2 \cdot \boldsymbol{e} = 0$ into a system of equations and write it as an augmented matrix, the instructor could notate the coefficient matrix as A^T and leverage the student work towards the matrix equation $A^T\boldsymbol{e} = \boldsymbol{0}$. The final task prompts students to "Combine three of our main equations, $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$, $\boldsymbol{p} + \boldsymbol{e} = \boldsymbol{b}$, and $A^T\boldsymbol{e} = \boldsymbol{0}$, to come up with one *general equation* (symbols only, no specific numbers) that would help us determine $\widehat{\boldsymbol{x}}$ for any A and $\boldsymbol{b}$. The only unknown in your *general equation* should be $\widehat{\boldsymbol{x}}$ (i.e., no $\boldsymbol{p}$ or $\boldsymbol{e}$)." Student work on this task allowed for the guided reinvention of the least squares equation $A^TA\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$ (abbreviated LSE), where $\widehat{\boldsymbol{x}}$ is the least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$.

Methods

The data for this paper come from in an in-person introductory linear algebra class at a large, public, research university in the Mid-Atlantic US. The course had 27 students, of which 13 both gave consent and completed the assignments analyzed in this paper. In the university system, 8 of these students chose he/him/his pronouns (pseudonyms begin with "M"), 4 chose she/her/hers pronouns (pseudonyms begin with "W"), and 1 did not choose pronouns (pseudonym P1). Most were second-year students by credit hours and were general engineering majors. The prerequisite was a B or higher in Calculus I or a passing grade in Calculus II. The data analyzed in this paper come from student responses to two written reflections. After most class sessions, students were asked to complete a reflection by the end of the day and submit their work via an online learning management system. Students were asked to spend 5-10 minutes on a reflection, for which full credit was awarded based on effort rather than correctness. Reflection #1 (Figure 2) was given the day that the class completed their solutions to the Delivering Mail to Gauss tasks and the reinvention of $A^TA\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$ (the LSE). The purpose of this reflection was to learn more about how students were making sense of the various aspects of the least squares problem and what solution approaches they would use. Reflection #2 (Figure 3) was given the following day to learn more about how students were making sense of the least squares equation.

> Create your own example of two travel vectors in $\mathbb{R}^3$ and a location for Gauss in $\mathbb{R}^3$that you cannot reach with your travel vectors. Then solve for at least two of the following: the vector closest to Gauss that you can reach, how you would get there with the travel vectors, and the distance from that location to Gauss. Show your work and/or explain your thinking

Figure 2. Reflection prompt #1.

> We started Least Squares Approximation with the "delivering mail to Gauss" scenario, eventually deriving the equation $A^TA\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$ as a way to directly solve for $\widehat{\boldsymbol{x}}$. We are really curious about your reaction to the equation. In 2-3 sentences, please share with us how you are making sense of it, your thoughts, or any questions you may have.

Figure 3. Reflection prompt #2.

To analyze the data, we began by creating thick descriptions for student responses to Reflection Prompt #1. In doing so, we were struck by the variety of ways in which most students leveraged the known relationships from Figure 1 to reach a solution for the travel scenario that they created, rather than using the LSE formalized in class that day. In order to capture the nuance of students' solution processes, we decided to focus on the diversity in their use of literal symbols and equation types. We studied the related literature and found Philipp's (1992) characterization of *literal symbols* and Drijver's (2003) characterization of *parameter* to be particularly appropriate; thus, we coded the data within their frameworks (summarized in the Theory section). We also studied the literature related to students' use of various types of equations in linear algebra, and we found Zandieh and Andrews-Larson (2019) to be most helpful. Their work, which is grounded in their prior research on three interpretations of $A\boldsymbol{x} = \boldsymbol{b}$ (Larson & Zandieh, 2013), characterizes students' symbolizing while solving linear systems. We coded the data in a way compatible with their approach, analyzing the various *equations types* (e.g., vector equation, matrix equation) students brought to bear in their solution process. For Reflection #2, we engaged in open coding to make sense of the variety of student responses. The first two authors independently coded all the data, conferred with each other to resolve any differences, discussed the data with the author team, and further refined as needed.

Results

Overall, in Reflection Prompt #1, we found that students use literal symbols as unknowns in three ways: as vectors, vector components, and scalars. We found that students used literal symbols as parameters in three ways: as placeholder, generalizer, and unknown. Students also leveraged six equation types in their problem solving: matrix equation, vector equation, system of linear equations, augmented matrix, dot product equation, and quadratic equation. Because of space, we focus on a vignette from one student. We chose M7 as a paradigmatic example because of the broad range of literal symbols and equation types that he used. A limitation of our data is that it is written data only; we cannot know how the students were thinking about the various symbols they wrote. Instead, we focus on how the literal symbols seemed to function in use; for this reason, our analysis makes claims such as "$\boldsymbol{e}$ and $\boldsymbol{p}$ are unknowns" rather than "the student reasoned about $\boldsymbol{e}$ and $\boldsymbol{p}$ as unknowns." To help the reader follow the analysis, we use italics for literal symbol use and underlining for equation type within the vignette. We do not label the equations that are of the type *unknown* = *determined value,* which communicate when a student completes a solution process for the unknown vector, component, parameter, or scalar.

M7's Work on Reflection Prompt #1

At the top of his page (see Figure 4), M7 wrote the relationships $\boldsymbol{e} \cdot \boldsymbol{p} = 0$, $\boldsymbol{e} \cdot \boldsymbol{v}_1 = 0$, and $\boldsymbol{e} \cdot \boldsymbol{v}_2 = 0$, which had been established in class (Figure 1). These three equations are <u>dot product equations</u>; within them are four literal symbols: $\boldsymbol{e}$ and $\boldsymbol{p}$ are *unknown vectors*, and vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are each a *parameter-as-placeholder*. M7 shifts to a <u>system of equations</u> $e_1 + 2e_2 + 3e_3 = 0$, $e_1 + e_2 + e_3 = 0$ created from the latter two dot product equations. This introduces three new literal symbols– e_1, e_2, and e_3 –which are each *unknown components.* We note that e_1, e_2, and e_3 are the components of the $\boldsymbol{e}$; to eventually solve for unknown vector $\boldsymbol{e}$, M7 decomposed it into unknown components. M7 transitions to an <u>augmented matrix equation</u>, using it four times as he carries out row reduction. M7 expresses the solution resulting from the row reduction as a <u>system of linear equations</u> $e_1 = e_3$ and $e_2 = -2e_3$, which again use e_1, e_2, and e_3; in this instance, however, the three literal symbols are now used as *varying quantities* because the system expresses how they are related and change together. It seems that M7 next compacted this information into $\boldsymbol{e} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$. M7 did not explicitly write the "$\boldsymbol{e} =$", but he substituted the vector in for $\boldsymbol{e}$ twice in his subsequent work. Thus, at this point in his work, M7 used the literal symbol t as a *parameter-as-generalizer* to represent all possible solutions for $\boldsymbol{e}$.

M7 then writes the <u>vector equation</u> $\boldsymbol{b} - \boldsymbol{e} = \boldsymbol{p}$. Here, $\boldsymbol{b}$ is a *parameter-as-placeholder* and $\boldsymbol{p}$ is still an *unknown vector.* However, we see a shift in $\boldsymbol{e}$ from vector unknown to *parameter-as-placeholder*; this is evidenced by the subsequent <u>vector equation</u> in which M7 substitutes in component-wise versions of both $\boldsymbol{b}$ and $\boldsymbol{e}$, using the parameterized version of $\boldsymbol{e}$, where again t functions as a *parameter-as-generalizer*. M7 then simplifies that vector equation into $\boldsymbol{p}$ in terms of t. Next, M7 next brings in the very first known relationship he had written, $\boldsymbol{e} \cdot \boldsymbol{p} = 0$, but now that dot product equation is written with component-wise expressions in terms of t for both $\boldsymbol{e}$ and $\boldsymbol{p}$. Thus, we see a shift in the role of t to that of *parameter-as-unknown*. This use of t continues in M7's simplification of the dot product equation into a quadratic equation in t; two additional quadratic equations are written as M7 simplifies in order to solve for t. This leads to M7's solution $t = 0, 1/6$, where t as a literal symbol is a *determined value* (i.e., the values of t that make the equation true, determined through a solution process). Choosing $1/6$ as an *assigned value* for t (Alaee et al., 2002), M7 gets exact solutions for $\boldsymbol{e}$ and $\boldsymbol{p}$ via substitution. M7 boxes the $\boldsymbol{p}$ vector and writes "closest vector we can reach." M7 completes his work by using a vector equation to write $\boldsymbol{p}$ as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, with the literal symbols s and t as *unknown scalars*. Although M7 again uses t as a literal symbol, we see no evidence that the two uses of t were connected in any way. M7 transitions to a system of linear equations in s and t and solves for the unknown scalars $s = -1/2$ and $t = 4/3$; M7 concludes by explaining how to use the travel vectors, presumably to reach $\boldsymbol{p}$, which is consistent with his algebraic solution.

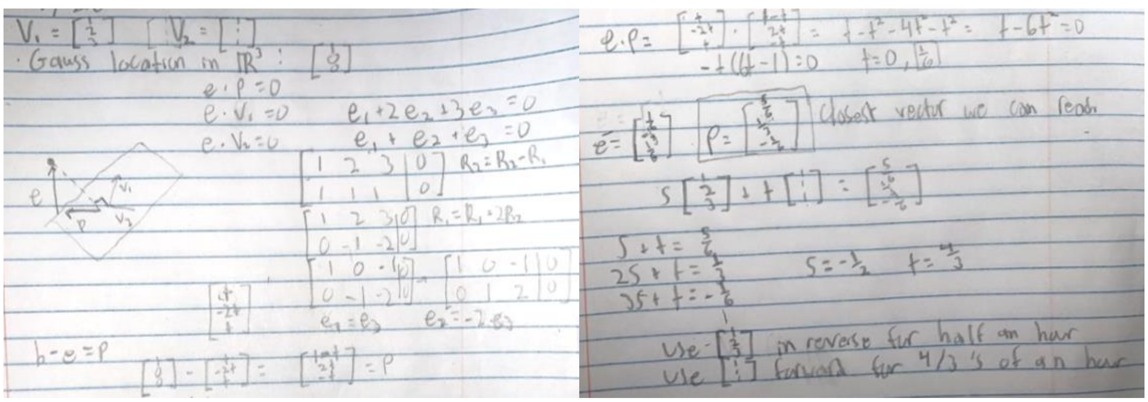

Figure 4. Student M7's written solution for Reflection 1.

We emphasize that the first augmented matrix M7 wrote corresponds to the matrix equation $A^T\boldsymbol{e} = \boldsymbol{0}$; he did not use the literal symbol A^T to notate his work, and we have no evidence he recognized the coefficient matrix as A^T. We point this out as an implicit use of A^T that grows out of the student's own problem-solving, which is important in terms of RME-inspired curriculum design and the class's use and understanding of the least squares equation $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$.

Student work on Reflection Prompt #2

When asked about their sense-making, thoughts, or remaining questions they may have about $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$ (Figure 3) after the following class, students shared a range of reactions about both individual parts of the equation and it as a whole.

Reactions to individual parts of the equation were typically related to the interpretation of its components. Among these, the most common topic invoked by students was that of A^T, both its properties and its function in the LSE. For example, W4 shared, "I'm wondering how $A\boldsymbol{x} = \boldsymbol{b}$ and $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$ don't have the same solutions since it's just multiplication A^T to both sides." We note that W4 wrote $\boldsymbol{x}$ rather than $\hat{\boldsymbol{x}}$ in her LSE, helping us understand that the nuance between $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ may not be straightforward for students. W7 wrote, "It may be because I'm still a little confused as to how transposing a matrix effects [sic] the original image, but I still don't understand how it adds possible solutions to the equation." We interpret the first part as W7 trying to make sense of what A^T means as a linear transformation, which we see as a valuable curiosity. We interpret the second part as W7 grappling with how the two equations are related and what it means to be a solution, which seems related to W4's response. We do have evidence that some students understood the utility of the transpose matrix; for example, M6 wrote "$A^T\boldsymbol{e}$ does the job of dot product each column of A by $\boldsymbol{e}$, that's why $A^T\boldsymbol{e}$ would yield the zero vector."

Other students commented on the efficiency of the LSE or contextualized their understanding on how to use the LSE within the class's work together. For example, M2 wrote "It condensed a mess of variables and diagrams into a single expression," and M8 said "The formula is quite simple to use" (M8). It appears that some students' comfort in using the equation related to their understanding of its derivation. For example, W5 wrote: "I would not know what to do with [the equation] if I didn't understand the derivation." It is unclear what W5 means by "what to do," possibly meaning use the equation to solve a least square problems or knowing what each literal symbols means in the context of least square problems. W6 wrote: "The equation makes sense to me based on how we derived it based on what we knew. However, I don't really understand how it all works together/why it all works." It is unclear what the distinction is for W6 between the LSE making sense and understanding why it works.

Discussion

The method of Least Squares is an important topic in linear algebra, although it is not always discussed in a first course possibly because of the background needed to understand all aspects involved in the SLE solution method. The first task in the least squares sequence, *Getting Mail to Gauss,* is straightforward enough to introduce in the first or second week of an introductory linear algebra course (such as after the second task of the *Magic Carpet Ride* IOLA sequence, Wawro et al., 2012). As explored in this paper and in our previous work (Lee et al., 2021), however, the solution process may bring to bear equation types and solution strategies learned across the entire introductory course. M7's work above illustrates the range of equation types and literal symbol use that this student has knowledge of and can flexibly move between in reconstructing a successful solution method.

One of the key features of the LSE is the presence of the matrix A^T. The student work in this study (not all of which we could share in limited space) suggests some connections the students were making between the LSE and A^T. While some students used A^T immediately in Reflection #1, such as M8, others such as M7 derived it through their solution process without labeling with the literal symbol A^T, such as M3, M7, and P1. The use of the array of numbers that experts think of as A^T was not problematic for students such as M7 when using these as coefficients in a system of equations or within the related augmented matrix equation. Students in this class were familiar with converting between systems of equations (or augmented matrices) and matrix equations of the form $A\boldsymbol{x} = \boldsymbol{b}$. So, converting equations such as M7's initial work into the expression $A^T\boldsymbol{e} = \boldsymbol{0}$ was not itself problematic for students; however, thinking of this array as a matrix that A, $A\widehat{\boldsymbol{x}}$, or $\boldsymbol{b}$ can be multiplied by seemed to be something students wondered about. The questions some students such as W7 and W6 wondered about on Reflection #2 seemed to be about the meaning or role of A^T when multiplied by the other expressions in the LSE. Given that students in an IOLA classroom tend to be familiar with reasoning about a matrix times a vector as a transformation of that vector (Andrews-Larson et al., 2017), these students may have wondered what transformation A^T imparts on input vectors.

Our future work involves further analyzing the nuances in student solution strategies and their conceptual understanding of the LSE, using this to make adjustments to the task sequence, and developing a conceptual analysis for the mathematics in least squares approximation.

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