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A cluster structure on the coordinate ring of partial flag varieties ☆

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The main goal of this paper is to show that the (multi-homogeneous) coordinate ring of a partial flag variety $\mathbb{C}[G/P_K^-]$ contains a cluster algebra if G is any semisimple complex algebraic group. We use derivation properties and a special lifting map to prove that the cluster algebra structure $\mathcal{A}$ of the coordinate ring $\mathbb{C}[N_K]$ of a Schubert cell constructed by Goodearl and Yakimov can be lifted, in an explicit way, to a cluster structure $\widehat{\mathcal{A}}$ living in the coordinate ring of the corresponding partial flag variety. Then we use a minimality condition to prove that the cluster algebra $\widehat{\mathcal{A}}$ is equal to $\mathbb{C}[G/P_K^-]$ after localizing some special minors.

1. Introduction

Cluster algebras were introduced in 2002 by Fomin and Zelevinsky and they have rapidly become one of the active areas in mathematics. This is due to their deep relations to other areas of mathematics like representation theory, combinatorics, homological algebra, algebraic geometry, Poisson geometry, Teichmüller theory and mathematical

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physics. On the other hand, the study of partial flag varieties is significant in representation theory and algebraic geometry. The first connection between these two studies appeared in Scott's work on Grassmannians and cluster algebras [15] in 2003. In 2008, Geiß, Leclerc and Schröer [6] showed that, in some simply-laced cases, namely A_n and D_4, the localization of the (multi-homogeneous) coordinate ring of a partial flag variety by non-minuscule minors matches the localization of some cluster structure by the same minors. They conjectured that this is true in general, that is, when the type of G is arbitrary. This paper proves this conjecture with respect to another localization. The main ideas of the proof of [6] motivate our work here. Indeed, Geiß, Leclerc and Schröer proved that the coordinate ring of a partial flag variety contains a cluster structure by showing the following:

(1) The coordinate ring of a Schubert cell has a cluster algebra structure $\mathcal{A}$.
(2) The cluster algebra $\mathcal{A}$ of the previous step can be lifted to some special cluster algebra $\widehat{\mathcal{A}}$ that lives in the coordinate ring of the partial flag variety corresponding to the coordinate ring of the cell of the previous step.
(3) The cluster algebra $\widehat{\mathcal{A}}$ coincides with the coordinate ring of the partial flag variety after localization with respect to some special minors.

Although the first step was only conjectured in [6], it was fully proved in [8]. Moreover, despite the fact that we prove the second step independently, it was also generalized to the non-simply-laced ones by Demonet in [2].

Unfortunately, some essential tools of the proof of Geiß, Leclerc and Schröer were based on the fact that they work on the simply-laced case. In fact, they used some categorification in their work, which works in the simply-laced case only, to show the first and the second steps, while they treated the third step for types A_n and D_4 case by case. Because of that, the generalization we seek has to use some other results.

Goodearl and Yakimov [12,13] proved that the coordinate ring of any Schubert cell admits a cluster structure. Moreover, their construction matches the one of [6] in the simply-laced case, yet it gives an explicit cluster structure to the coordinate ring of a cell in the non-simply-laced as well. It is worth mentioning here that in spite of the fact that the theory of cluster algebras is a mix between combinatorics and algebra, the work of Goodearl and Yakimov was almost purely algebraic.

The work of [12] and [13] enables us to go back to the strategy of [6], that is, the three steps mentioned above, and follow them to prove that the coordinate ring of a partial flag variety contains a cluster algebra, no matter if we are in the simply-laced or the non-simply-laced case. Of course, we have to find different ways to treat steps 2 and 3, but thanks to Goodearl and Yakimov, the first step is already there.

In particular, to get step 2, we proved the following theorem:

Theorem 1.1. *Let* $\{(\mathbf{x}, B)\}$ *be the collection of seeds of the cluster algebra* $\mathcal{A}$ *of* $\mathbb{C}[N_K]$*. The corresponding collection of pairs* $\left\{(\widehat{\mathbf{x}}, \widehat{B})\right\}$ *constructed in Definition 5.7 forms a col-*

lection of seeds related by mutation. In other words, if $(\mathbf{x}, B)$ *and* $(\mathbf{x}', B')$ *are two seeds of the coordinate ring of the cell* $\mathbb{C}[N_K]$ *such that* $(\mathbf{x}', B') = \mu_k(\mathbf{x}, B)$*, then correspondingly* $(\widehat{\mathbf{x}'}, \widehat{B'}) = \mu_k(\widehat{\mathbf{x}}, \widehat{B})$*. In particular, if* $(\mathbf{x}_0, B_0)$ *is an initial seed of* $\mathcal{A} = \mathbb{C}[N_K]$ *then* $(\widehat{\mathbf{x}_0}, \widehat{B_0})$ *is an initial seed of a cluster algebra* $\widehat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$*.*

For step 3, we actually proved that:

Theorem 1.2. *The localization of the homogeneous coordinate ring of the flag variety* $\mathbb{C}[G/P_K^-]$ *by* $\Delta_{\varpi_j,\varpi_j}$*, where* $j \in J$*, equals the localization of the cluster algebra* $\widehat{\mathcal{A}}$ *by the same elements. In symbols,*

$$\mathbb{C}[G/P_K^-][\Delta^{-1}_{\varpi_j,\varpi_j}]_{j\in J} = \widehat{\mathcal{A}}[\Delta^{-1}_{\varpi_j,\varpi_j}]_{j\in J}.$$

Basically, we complete the second step of the strategy of Geiß, Leclerc and Schröer in the first theorem and then do the third step in the second theorem. It is worth to mention here that the localization in [6] is over the minors that are indexed by the set J and are not minuscule, while we localize by the minors that are indexed by J and omit the second condition.

Here is an outline of how the paper is organized: In the following section, we give the reader an overview of the structure of cluster algebras, while in Section 3 we go through the needed results from partial flag varieties. However, in Section 4, we focus on the highlights of the work of Goodearl and Yakimov. Indeed, we discuss the relation between Poisson geometry and cluster algebras and show how the cluster algebra $\mathcal{A}$ of the coordinate of a Schubert cell looks based on the structure of Goodearl and Yakimov. In fact, it is shown in Theorem 4.8 that the variables of their initial extended cluster are nothing but restrictions of some special homogeneous elements of the corresponding coordinate ring of a partial flag variety, called *generalized minors*. Also, the exchange matrix of their work is given explicitly in the same theorem. Using the intuition from the work of [6], we then assigned, in Definition 5.7, a pair $(\widehat{\mathbf{x}}, \widehat{B})$ to each seed $(\mathbf{x}, B)$ of $\mathcal{A}$. In this new pair, $\widehat{\mathbf{x}}$ consists of the lifting of the same elements of $\mathbf{x}$ plus the generalized minors $\Delta_{\varpi_j,\varpi_j}$ for which the restriction is 1 in the coordinate ring of the cell. Also, the matrix $\widehat{B}$ is the matrix B together with some additional rows given in some special form. After that, we show in Theorem 5.8 that these pairs are actually seeds of some cluster algebra $\widehat{\mathcal{A}}$ sitting inside the coordinate ring of the partial flag variety. Moreover, two pairs are related by a mutation if their corresponding original seeds of $\mathcal{A}$ are. This finishes step 2 of the strategy of [6]. Subsequently, we use a minimality property in Theorem 5.13 to show that the cluster algebra $\widehat{\mathcal{A}}$ is indeed equal to the coordinate ring of the partial flag variety, up to the aforementioned localization.

In fact, it is an important problem to understand the relationship between the cluster structures of Demonet [2] and ours. We plan to return to this in a future publication.

2. Cluster algebras

This section gives an overview of the construction of cluster algebras and the main concepts. For more details about this, the reader is referred to [3], [5], [10], or [16].

Definition 2.1. In our setting, the term *ambient field* will be referring to a field $\mathcal{F}$ that is isomorphic to $\mathbb{C}(x_1, ..., x_n, ..., x_m)$, where $\{x_1, ..., x_n, ..., x_m\}$ is an algebraically independent generating set.

Remark 2.2. We usually write $\mathbb{C}(x_1, ..., x_n, ..., x_m)$ instead of writing $\mathbb{C}(x_1, ..., x_m)$ to emphasize that there is a distinction between the first n variables and the remaining $m - n$ ones. This distinction will become clear in the following sequence of definitions and remarks.

Definition 2.3. A (*labeled*) *seed* is a pair $(\widetilde{\mathbf{x}}, \widetilde{B})$ where $\widetilde{\mathbf{x}}$ is a tuple of algebraically independent variables $\widetilde{\mathbf{x}} = (x_1, ..., x_n, ..., x_m)$ generating an ambient field $\mathcal{F}$ and $\widetilde{B}$ is an $m \times n$ matrix whose northwestern $n \times n$ submatrix B is *skew-symmetrizable*, that is, can be transformed to a skew-symmetric matrix by multiplying each row r_i by some nonzero integer d_i. The tuple $\widetilde{\mathbf{x}}$ is called an *extended cluster*, where its first n variables are called the *cluster* (or *mutable*) variables and the next $m - n$ variables are called the *coefficient* (or *frozen*) variables. The tuple $\mathbf{x} = (x_1, ..., x_n)$ is called a *cluster*. In the same context, the northwestern $n \times n$ submatrix B of $\widetilde{B}$ is called the *exchange* matrix, while the matrix $\widetilde{B}$ is called the *extended exchange* matrix.

Remark 2.4. Sometimes the skew-symmetrizable matrix is replaced by a *quiver* Q, which is a directed graph with n *mutable* and $m - n$ *frozen* vertices such that it has no loops, no oriented 2-cycles and no edges between two frozen vertices. In fact, each quiver gives rise to an $m \times n$ skew-symmetrizable matrix $\widetilde{B}(Q)$, where its entries are given by

$$b_{ij} = \begin{cases} \#(i \to j), & \text{if } i > j, \\ 0, & \text{if } i = j, \\ -\#(i \leftarrow j), & \text{if } i < j; \end{cases}$$

where $\#(i \to j)$ is the number of arrows from i to j and $\#(i \leftarrow j)$ is the number of arrows from j to i.

Definition 2.5. Let $(\widetilde{\mathbf{x}}, \widetilde{B})$ be a seed. A *mutation* μ_k at $k \in [1, n]$ is a transformation to a new seed $\mu_k(\widetilde{\mathbf{x}}, \widetilde{B}) = (\widetilde{\mathbf{x}}', \widetilde{B}')$, where the entries of the matrix $\widetilde{B}'$ are given by

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise}; \end{cases} \tag{2.1}$$

and $\widetilde{\mathbf{x}}' = (x'_1, ..., x'_m)$, where $x'_i = x_i$ if $i \neq k$ and

$$x_k x'_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}.$$

Two seeds are said to be *mutation equivalent* if one of them can be obtained from the other one by a finite sequence of mutations.

Remark 2.6. It is not hard to verify that μ_k is an *involution*, that is,

$$\mu_k(\mu_k(\widetilde{\mathbf{x}}, \widetilde{B})) = (\widetilde{\mathbf{x}}, \widetilde{B}).$$

Remark 2.7. Let us start with an *initial seed* $(\widetilde{\mathbf{x}}, \widetilde{B})$. By definition any mutable variable can be obtained from $(\widetilde{\mathbf{x}}, \widetilde{B})$ by some sequence of mutations at some mutable indices.

Definition 2.8. Let $(\widetilde{\mathbf{x}}, \widetilde{B})$ be a seed. Let $\mathcal{X}$ be the set of all possible mutable variables, that is, the mutable ones of the initial seed or the mutable ones generated by any sequence of mutations applied to the initial seed. Let $\mathcal{R}$ be the polynomial ring $\mathcal{R} = \mathbb{C}[x_{n+1}, ..., x_m]$, where $x_{n+1}, ..., x_m$ are the frozen variables of the seed $(\widetilde{\mathbf{x}}, \widetilde{B})$. The *cluster algebra* (of *geometric type*) is the algebra $\mathcal{A} = \mathcal{R}[\mathcal{X}]$, the subalgebra of the ambient field generated by all variables (mutable or frozen). If $(\widetilde{\mathbf{x}}, \widetilde{B})$ is an initial seed of a cluster algebra $\mathcal{A}$, then we may denote $\mathcal{A}$ by $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B})$.

Definition 2.9. Let $(\widetilde{\mathbf{x}}, \widetilde{B})$ be a seed. The *rank* of the seed or its corresponding cluster algebra is the number of its mutable variables, while the number of all variables of the seed is referred to as the *cardinality* of the seed. Thus in our setting above, the rank of $(\widetilde{\mathbf{x}}, \widetilde{B})$ is n and the cardinality of it is m.

Definition 2.10. A cluster algebra $\mathcal{A}(\widetilde{\mathbf{x}}, \widetilde{B})$ is said to be *of finite type* if it has a finite number of mutable variables. Otherwise it is said to be *of infinite type.*

3. Partial flag varieties

This section captures the required overview from the partial flag varieties. We need first review the definition of a partial flag variety and look at some facts about its coordinate ring. Other useful overviews, with probably more details about this, can be found in [6], [7], [10], or [14].

Remark 3.1. It is known that each semisimple group induces a *Cartan matrix* whose information can be encoded in the corresponding *Dynkin diagram.* One of the significant consequences of this is that every semisimple complex Lie algebra is fully characterized, up to isomorphism, by its Dynkin diagram.

Remark 3.2. From now on, the set I denotes the vertex set of the Dynkin diagram Δ corresponding to G.

Definition 3.3. A *parabolic* subgroup P of G is a closed subgroup that lies between G and some Borel subgroup B.

Example 3.4.

(1) Any Borel subgroup B is parabolic.
(2) Fix a nonempty subset $J \subset I$ and let $K = I \setminus J$. Denote by $x_i(t)$ $(i \in I, t \in \mathbb{C})$ the simple root subgroups of the unipotent radical N of B and denote by $y_i(t)$ the simple root subgroups of the unipotent radical N^- of B^-. The subgroup P_K generated by B and the one-parameter subgroups $y_k(t)$ $(k \in K, t \in \mathbb{C})$ are parabolic. Similarly, the subgroup P_K^- generated by B^- and the one-parameter subgroups $x_k(t)$ $(k \in K, t \in \mathbb{C})$ is a parabolic subgroup.

Definition 3.5. A quotient G/P is called a (*partial*) *flag variety* if P is a parabolic subgroup of G.

Remark 3.6. It is known that any parabolic subgroup is conjugate to a parabolic subgroup of the form P_K. This reduces the study of partial flag varieties to the ones of the form G/P_K.

Remark 3.7. The partial flag variety G/P_K^- can be naturally embedded as a closed subset of the product of projective spaces

$$\prod_{j \in J} \mathbb{P}(L(\varpi_j)^*),$$

where ϖ_j is a fundamental weight of G, and for a dominant weight λ, the corresponding $L(\lambda)$ is the finite-dimensional irreducible G-module with highest weight λ; and $L(\lambda)^*$ denotes the right G-module obtained by twisting the action of G. The $L(\varpi_i)$'s are called the *fundamental representations.*

Remark 3.8. Let $\Pi_J \cong \mathbb{N}^J$ denote the monoid of dominant integral weights of the form $\lambda = \sum_{j \in J} a_j \varpi_j$, where $a_j \in \mathbb{N}$. The multi-homogeneous coordinate ring $\mathbb{C}[G/P_K^-]$ is a Π_J-graded algebra. In particular,

$$\mathbb{C}[G/P_K^-] = \bigoplus_{\lambda \in \Pi_J} L(\lambda).$$

One of the significant results is that $\mathbb{C}[G/P_K^-]$ can be identified with the subalgebra of $\mathbb{C}[G/N^-]$ generated by the homogeneous elements of degree ϖ_j, where $j \in J$.

Remark 3.9. For a Weyl group W of G, the longest element in this paper will always be denoted by w_0 and the Coxetor generators will be denoted by s_i where i runs in I.

The notation of the length of some $w \in W$ will be $\ell(w)$. The Chevalley generators of the Lie algebra $\mathfrak{g}$ of G are denoted e_i, f_i, h_i, where again $i \in I$. The e_i's here generate $\mathrm{Lie}(N) = \mathfrak{n}$. An important consequence of this is that N acts naturally from the left and right on $\mathbb{C}[N]$ by the following left and right actions respectively:

$$(x \cdot f)(n) = f(nx), \quad (f \in \mathbb{C}[N] \text{ and } x, n \in N),$$
$$(f \cdot x)(n) = f(xn), \quad (f \in \mathbb{C}[N] \text{ and } x, n \in N).$$

One might differentiate these two actions to get left and right actions of $\mathfrak{n}$ on $\mathbb{C}[N]$, respectively.

Notation. The right action of e_i on $f \in \mathbb{C}[N]$ will be denoted by $e_i^\dagger(f) := f \cdot e_i$.

Remark 3.10. For each simple reflection $s_i \in W$, let $\overline{s_i} := \exp(f_i)\exp(e_i)\exp(f_i)$. If $w = s_{i_1}...s_{i_r}$ with r being the length of w, then define $\overline{w} = \overline{s_{i_1}}...\overline{s_{i_r}}$. Let $G_0 = N^- HN$ be the open set of G consisting of elements having Gaussian decomposition. Indeed, each $x \in G_0$ can be uniquely represented as

$$x = [x]_-[x]_0[x]_+,$$

where $[x]_- \in N^-$, $[x]_0 \in H$, $[x]_+ \in N$. Let V_i^+ be the irreducible representation whose highest weight is ϖ_i and highest weight vector is v_i^+. For any $h \in H$ one has that v_i^+ is an eigenvector, that is, $hv_i^+ = [h]^{\varpi_i} v_i^+$ and $[h]^{\varpi_i} \in \mathbb{C} \setminus \{0\}$. This gives the following definition introduced by Fomin and Zelevinsky in [4].

Definition 3.11. For $u, v \in W$ and $i \in I$ define the *generalized minor* to be the regular function on G given by

$$\Delta_{u\varpi_i, v\varpi_i}(x) = [\overline{u}^{-1} x \overline{v}]_0^{\varpi_i}.$$

Remark 3.12. The distinguished elements $\Delta_{\varpi_j, w(\varpi_j)}$, $(w \in W)$, are of degree ϖ_j (see 2.3 in [1] or section 2 and 6 in [6] for more details). They make the coordinate ring of the cell and the coordinate ring of the corresponding flag variety related by the following:

$$\mathbb{C}[N_K] = \mathbb{C}[G/P_K^-] \Big/ (\Delta_{\varpi_j, \varpi_j} - 1)_{j \in J}.$$

The generalized minors are nothing but a generalization of the flag minors of SL_n. Their significance in the cluster structure of the coordinate ring of partial flag varieties will be seen in section 5.

4. Preliminaries from Poisson algebras

In [12] and [13], Goodearl and Yakimov made the relationship between the coordinate ring of Schubert cells and cluster algebras clear and explicit. They proved that each such coordinate ring admits a cluster structure. Thus, since the coordinate ring of any cell is the quotient of the coordinate ring of some flag variety modded out by some generalized minors, it is clear that the result of Goodearl and Yakimov should play an important role in this paper. Their results were based on Poisson geometry and so we capture here the main elements that we need from their work. More details about the relation between Poisson geometry and cluster algebras can be found in [10] and [12].

Definition 4.1.

(1) A *Poisson bracket* $\{-,-\}$ is a Lie bracket that is a derivation also in each variable for the associative products.
(2) A *Poisson algebra* is a commutative algebra R together with a Poisson bracket.
(3) For $a \in R$ the *Hamiltonian associated* with a is the derivation $\{a,-\}$.
(4) A *Poisson ideal* of R is an ideal I such that $\{R,I\} \subset I$.

Remark 4.2. The Poisson bracket of a Poisson algebra R induces a Poisson bracket on any quotient of R by a Poisson ideal.

Definition 4.3. Define the *Poisson-Ore extensions* to be $B[x;\sigma,\delta]_p$ where B is a Poisson algebra, $B[x;\sigma,\delta]_p = B[x]$ is a polynomial ring and σ, δ are suitable Poisson derivations on B such that for any $b \in B$ we have

$$\{x,b\} = \sigma(b) + \delta(x).$$

Let $\mathbb{K}$ be a base field of characteristic 0. For an iterated Poisson-Ore extension

$$R = \mathbb{K}[x_1]_p[x_2;\sigma_2,\delta_2]_p \cdots [x_m;\sigma_m,\delta_m]_p$$

and $k \in [0,m]$, define

$$R_k = \mathbb{K}[x_1,...,x_k] = \mathbb{K}[x_1]_p[x_2;\sigma_2,\delta_2]_p \cdots [x_k;\sigma_k,\delta_k]_p,$$

where $R_0 = \mathbb{K}$.

Definition 4.4. A *Poisson-CGL extension* is an iterated Poisson-Ore extension R as above that is endowed with a rational Poisson action of a torus $\mathcal{H}$ such that

(1) The elements $x_1,...,x_k$ are $\mathcal{H}$-eigenvectors;
(2) The map δ_k is locally nilpotent on R_{k-1} for any $k \in [2,m]$;

(3) For any $k \in [1, m]$ there is an $h_k \in \mathrm{Lie}\mathcal{H}$ such that $\sigma_k = h_k|_{R_{k-1}}$ and the h_k-eigenvalue of x_k nonzero and denoted by λ_k.

Definition 4.5. Let R be a Noetherian Poisson domain. An element $p \in R$ is called a *Poisson-prime element* if any of the following equivalent conditions hold:

(1) The ideal (p) is a prime ideal and it is a Poisson ideal.
(2) The element p is a prime element of R such that $p|\{p, -\}$, that is, p divides $\{p, x\}$ for all $x \in R$.
(3) [In the case $\mathbb{K} = \mathbb{C}$]: The element p is a prime element of R and the zero locus $V(p)$ is a union of symplectic leaves of the maximal spectrum of R.

One of the great successes is due to the work of Goodearl, Yakimov when they proved the following:

Theorem 4.6. *Every symmetric Poisson-CGL extension R such that $\lambda_l/\lambda_j \in \mathbb{Q}_{>0}$ for all l, j has a canonical cluster algebra structure that coincides with its upper cluster algebra.*

Remark 4.7. The cluster variables in the construction of Goodearl and Yakimov are the unique homogeneous Poisson-prime elements of Poisson-CGL (sub)extensions not belonging to smaller subextensions. The mutation matrices of their seeds can be computed using linear systems of equations that come from the Poisson structure.

A significant consequence of the work of Goodearl and Yakimov is:

Theorem 4.8. *The coordinate ring $\mathbb{C}[N_K]$ has a canonical cluster algebra structure.*

Proof. The notation of this proof follows [13]. Throughout, the kth vector of the standard basis of $\mathbb{Z}^m$ is denoted by e_k, the notation $a[j, k]$ is given by

$$a[j,k] := \|(w_{[j,k]} - 1)\varpi_{i_k}\|^2/4 \in \frac{1}{2}\mathbb{Z},$$

and the notation $S(w)$ is the *support of* w and is given by

$$S(w) := \{i \in I \mid s_i \le w\} = \{i \in I \mid i = i_k \text{ for some } k \in [1, m]\}.$$

Also, set

$$p(k) := \begin{cases} \max\{j < k \mid i_j = i_k\}, & \text{if such } j \text{ exists;} \\ -\infty, & \text{otherwise.} \end{cases}$$

$$s(k) := \begin{cases} \min\{j > k \mid i_j = i_k\}, & \text{if such } j \text{ exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

Let A_q, $\mathbf{A}$, $\mathbf{U}$, $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$ and $U_q(\mathfrak{n}_-(w))^{\vee}_{\mathcal{A}^{1/2}}$ be as in [13]. From Theorem 10.1 in [11] and Theorem 7.3 in [13] we know that the quantum Schubert cell, denoted by $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$, has the quantum cluster structure given by the equation

$$A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}} = \mathbf{A}(M^w, \widetilde{B}^w, \varnothing)_{\mathcal{A}^{1/2}} = \mathbf{U}(M^w, \widetilde{B}^w, \varnothing)_{\mathcal{A}^{1/2}},$$

where the extended cluster variables are given by

$$M^w(e_j) = q^{a[1,j]} D_{\varpi_{i_j}, w_{\leq j}\varpi_{i_j}},$$

for all $j \in [1, m]$, where

$$D_{\varpi_j, w(\varpi_j)} = \mathrm{proj}(\Delta_{\varpi_j, w(\varpi_j)}),$$

where the frozen variables are the ones indexed by $j \in [1, m]$ such that $s(j) = \infty$. The map

$$\mathrm{proj} : \mathbb{C}[G/P_K^-] \to \mathbb{C}[N_K]$$

denotes the standard projection from $\mathbb{C}[G/P_K^-]$ to $\mathbb{C}[N_K]$. The exchange matrix $\widetilde{B}^w$ is of size $m \times (m - |S(w)|)$ and its $j \times k$ entry is given by

$$(\widetilde{B}^w)_{jk} = \begin{cases} 1, & \text{if } j = p(k), \\ -1, & \text{if } j = s(k), \\ a_{i_j i_k}, & \text{if } j < k < s(j) < s(k), \\ -a_{i_j i_k}, & \text{if } k < j < s(k) < s(j), \\ 0, & \text{otherwise;} \end{cases}$$

where the entry $a_{i_j i_k}$ is the same $i_j \times i_k$ entry of the Cartan matrix of the same type. By [13] we have that $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}} \cong U_q(\mathfrak{n}_-(w))^{\vee}_{\mathcal{A}^{1/2}}$. Thus, by corollary 3.7 in [9], it follows that

$$\mathbb{C} \otimes A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}} \cong A(\widetilde{B}^w).$$

On the other hand, by (4.7) in [17], we know that the left-hand side is isomorphic to the quotient of $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$ by $(q-1)$. Consequently, we get the desired cluster structure in the classical case whose exchange matrix is $\widetilde{B}^w$ and cluster variables are $D_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}$. $\square$

5. Cluster algebra structure on $\mathbb{C}[G/P_K^-]$

In the work of Geiß, Leclerc and Schröer [6], they proved that $\mathbb{C}[G/P_K^-]$ up to localization admits a cluster structure if G is simply-laced of type A_n or D_4. Their work motivates our construction here. The idea is to translate their work, which was in terms of categorification, to another language that works in the general case.

Notation. The cluster structure on $\mathbb{C}[N_K]$ constructed by the work of Goodearl and Yakimov will be denoted by $\mathcal{A}_J$, where J and K are as defined before. We may write $\mathcal{A}$ instead of $\mathcal{A}_J$ if the context is clear.

Lemma 5.1. *For every $f \in \mathbb{C}[N_K]$ there exists a unique homogeneous element $\widetilde{f} \in \mathbb{C}[G/P_K^-]$ such that its projection to $\mathbb{C}[N_K]$ is f and whose multi-degree is minimal with respect to the usual partial ordering obtained by the usual ordering of weights, that is, $\mu \preceq \lambda$ iff $\lambda - \mu$ is an $\mathbb{N}$-linear combination of weights ϖ_j $(j \in J)$.*

Proof. This is Lemma 2.4 in [6]. Despite the fact that the main results of that paper are for the simply-laced case, this one is general and works for any type. $\square$

Remark 5.2. The proof of the preceding lemma in [6] involves the following important points:

(1) Set

$$a_j(f) = \max\left\{ s \mid (e_j^\dagger)^s f \neq 0 \right\}.$$

(2) Set

$$\lambda(f) = \sum_{j \in J} a_j(f)\varpi_j.$$

(3) The minimality in the previous lemma means that $\lambda(f)$ is minimal in the following sense: if $\tilde{\tilde{f}} \in L(\lambda)$ and $\mathrm{proj}(\tilde{\tilde{f}}) = f$ then $\lambda(f) \preceq \lambda$. On the other hand, the projection of each piece $L(\lambda)$ to $\mathbb{C}[N_K]$ is injective and so if there is an element there whose projection is f, then it is unique in $L(\lambda)$. These two pieces of information together are the main ingredients in proving the existence and uniqueness of $\lambda(f)$.

Remark 5.3. The endomorphisms $e_j^\dagger$ are derivations of $\mathbb{C}[N_K]$. Thus, for all $f, g \in \mathbb{C}[N_K]$ we have the following:

(1) The image of fg under $e_j^\dagger$ is

$$e_j^\dagger(fg) = e_j^\dagger(f)g + fe_j^\dagger(g);$$

(2) By Leibniz formula,

$$(e_j^\dagger)^{a_j(f)+a_j(g)}(fg) = (e_j^\dagger)^{a_j(f)}(e_j^\dagger)^{a_j(g)}(g) \neq 0;$$

(3) For any integer $k \geq 1$,

$$(e_j^\dagger)^{a_j(f)+a_j(g)+k}(fg) = 0;$$

(4) Consequently,

$$a_j(fg) = a_j(f) + a_j(g).$$

Lemma 5.4. *For any two elements $f, g \in \mathbb{C}[N_K]$, we have $\widetilde{f \cdot g} = \widetilde{f} \cdot \widetilde{g}$. If for any $j \in J$ we have $a_j(f+g) = \max\{a_j(f), a_j(g)\}$, then there are some relatively prime monomials μ, ν in the generalized minors $\Delta_{\varpi_j,\varpi_j}$ such that*

$$\widetilde{f+g} = \mu\widetilde{f} + \nu\widetilde{g}.$$

Proof. Lemma 2.5 in [6]. □

Remark 5.5. Let $(\widetilde{\mathbf{x}}, \widetilde{B})$ be a seed of the cluster algebra $\mathcal{A} = \mathbb{C}[N_K]$. Then the mutation formula tells us that

$$x_k x_k' = M(k) + L(k),$$

where $M(k), L(k)$ are monomials in the variables $x_1, ..., x_{k-1}, x_{k+1}, ..., x_n$. As a consequence of the previous lemma (cf. [6]) we get that

$$\widetilde{x_k}\widetilde{x_k'} = \mu(k)\widetilde{M(k)} + \nu(k)\widetilde{L(k)},$$

where $\mu(k)$ and $\nu(k)$ are relatively prime monomials in $\Delta_{\varpi_j,\varpi_j}$ $(j \in J)$. This means that we can write $\mu(k)$ and $\nu(k)$ as

$$\mu(k) = \prod_{j \in J} \Delta_{\varpi_j,\varpi_j}^{\alpha_j} \quad \text{and} \quad \nu(k) = \prod_{j \in J} \Delta_{\varpi_j,\varpi_j}^{\beta_j}.$$

Consequently, it is reasonable to expect that the variables $\widetilde{x_i}$ form the cluster variables of some cluster algebra contained in $\mathbb{C}[G/P_K^-]$. This was proved in type A_n and D_4 by Geiß, Leclerc and Schröer.

Definition 5.6. A *lift* of a cluster algebra $\mathscr{A}$ is a cluster algebra $\widetilde{\mathscr{A}}$ such that $\mathscr{A}$ is a quotient algebra of it. Alternatively, we may say that $\mathscr{A}$ *can be lifted* to $\widetilde{\mathscr{A}}$.

Definition 5.7. For any seed $(\mathbf{x}, B)$ of the cluster algebra $\mathcal{A}_J = \mathbb{C}[N_K]$ constructed in [12] define a new pair $(\widehat{\mathbf{x}}, \widehat{B})$ of $\mathbb{C}[G/P_K^-]$ by raising each variable x of $(\mathbf{x}, B)$ to the variable $\widetilde{x}$ (see Lemma 5.4) preserving the same type (mutable or frozen) and by adding the generalized minors $\Delta_{\varpi_j,\varpi_j}$ modded out in $\mathbb{C}[N_K]$ as frozen variables. The matrix $\widehat{B}$

of this lift is obtained as follows: Extend the matrix B of the construction of Goodearl and Yakimov [12] by $|J|$ rows labeled by the elements of J such that the entries are

$$\widehat{b}_{jk} = \begin{cases} \beta_j, & \text{if } \beta_j \neq 0; \\ -\alpha_j, & \text{else,} \end{cases}$$

where α_j and β_j are as in Remark 5.5. This process is called the *(seed) homogenization.*

Theorem 5.8. *Let $\{(\mathbf{x}, B)\}$ be the collection of seeds of the cluster algebra $\mathcal{A}_J$ of $\mathbb{C}[N_K]$. The corresponding collection $\left\{(\widehat{\mathbf{x}}, \widehat{B})\right\}$ constructed above forms a valid collection of seeds. In other words, if $(\mathbf{x}, B)$ and $(\mathbf{x}', B')$ are two seeds of the coordinate ring of the cell $\mathbb{C}[N_K]$ such that $(\mathbf{x}', B') = \mu_k(\mathbf{x}, B)$, then correspondingly $(\widehat{\mathbf{x}'}, \widehat{B'}) = \mu_k(\widehat{\mathbf{x}}, \widehat{B})$.*

Proof. First, we start with a proof overview. Let $(\mathbf{x}, B)$ and $(\mathbf{x}', B')$ be two seeds of the cluster algebra $\mathcal{A}_J = \mathbb{C}[N_K]$ such that $(\mathbf{x}', B') = \mu_k(\mathbf{x}, B)$. Since $\{(\mathbf{x}, B)\}$ is a collection of seeds, it suffices to show that $(\widehat{\mathbf{x}'}, \widehat{B'}) = \mu_k(\widehat{\mathbf{x}}, \widehat{B})$ to get that the corresponding collection $\left\{(\widehat{\mathbf{x}}, \widehat{B})\right\}$ is indeed a collection of seeds. In other words, the aim is to show that the homogenization of the mutation, at some index of a seed, is the same as the mutation, at the same index, of the homogenization of the same seed. In symbols, to reach our goal, it is enough to show that

$$\widehat{\mu_k(B)} = \mu_k(\widehat{B}),$$

where on the left-hand side we mutate and then homogenize, while we do the reverse on the right-hand side. In homogenization, one needs to mutate to get all the needed entries. Thus, since the equation we seek to get involves homogenization and mutation, we will need to deal with two steps of mutation, one follows the other. As the variables of the tuple $\widehat{\mathbf{x}}$ live in $\mathbb{C}[G/P_K^-]$, the success of proving the aimed result will prove that $\mathbb{C}[G/P_K^-]$ contains a cluster algebra.

Second, we begin the actual proof. Let k be a mutable index in the construction of [12]. We need to show that $\mu_k(\widehat{B}) = \widehat{B'}$. In other words, we need to show that the matrix entries of the mutation of $\widehat{B}$ match the ones coming from the homogenization of B'. Note that the entries of the homogenization of B' are the same as B' together with additional rows whose entries are extracted from the mutation equations of the mutated seed $(\mathbf{x}', B')$. Fixing a mutable index t, these equations are of the form

$$\widetilde{x'(t)}\widetilde{x''(t)} = \mu'(t)\widetilde{M'(t)} + \nu'(t)\widetilde{L'(t)},$$

where $x'(t) = x'_t$ denotes the tth variable in the mutated extended cluster in a direction k and $x''(t) = x''_t$ denotes the tth variable coming from a second mutation in a direction t. Let $\widehat{b'}_{st}$ denote the entry of position $s \times t$ in $\mu_k(\widehat{B})$. Obviously, if $s \notin J$ then $\widehat{b'}_{st}$ equals the $s \times t$ entry of $\mu_k(B)$, as the entries of $\widehat{B}$ and B match when $s \notin J$. Consequently, the entries of the mutation of both coincide again when $s \notin J$. Assume now that $s \in J$. If $t = k$, then by the fact that the construction of [12] is indeed a cluster algebra, we get that $M'(k) = L(k)$ and $L'(k) = M(k)$. This clearly makes $\alpha'_j = \beta_j$ and $\beta'_j = \alpha_j$. Since $\mu'(t)$ and $\nu'(t)$ are relatively prime, we see easily from the construction that the entry we get is $-\widehat{b}_{st}$ which equals $\widehat{b'}_{st}$ by the mutation formula.

Assume now that t is a mutable index other than k. It suffices to show that in

$$\widetilde{x'(t)}\widetilde{x''(t)} = \mu'(t)\widetilde{M'(t)} + \nu'(t)\widetilde{L'(t)},$$

the exponents of the minors of the monomials $\mu'(t)$ and $\nu'(t)$ match the formula of the matrix mutation. Equivalently, we may assume that $\mu'(t)$ and $\nu'(t)$ are as we desire and then show that $\mu'(t)\widetilde{M'(t)} + \nu'(t)\widetilde{L'(t)}$, is an element whose proj is $M'(t) + L'(t)$ and whose order is minimal with respect to $\preceq$. The first property is straightforward. Now,

$$\lambda\big(M'(t) + L'(t)\big) = \sum_{j\in J} a_j\big(M'(t) + L'(t)\big)\varpi_j = \sum_{j\in J} a_j\varpi_j \tag{5.1}$$

where

$$\begin{aligned}
a_j &= a_j\big(M'(t) + L'(t)\big)\\
&= \max\{s \mid (e_j^\dagger)^s\big(M'(t) + L'(t)\big) \neq 0\}\\
&= \max\left\{ s \;\middle|\; (e_j^\dagger)^s \left(\begin{array}{c} \displaystyle\prod_{b_{it}+\frac{|b_{ik}|b_{kt}+b_{ik}|b_{kt}|}{2}>0} {x'}_i^{b_{it}+\frac{|b_{ik}|b_{kt}+b_{ik}|b_{kt}|}{2}} \\ + \\ \displaystyle\prod_{b_{it}+\frac{|b_{ik}|b_{kt}+b_{ik}|b_{kt}|}{2}<0} {x'}_i^{-\left(b_{it}+\frac{|b_{ik}|b_{kt}+b_{ik}|b_{kt}|}{2}\right)} \end{array} \right) \neq 0 \right\}\\
&= \max\left\{ \begin{array}{c} \displaystyle\sum_{b_{it}+\frac{|b_{ik}|b_{kt}+b_{ik}|b_{kt}|}{2}>0} a_j\left({x'}_i^{b_{it}+\frac{|b_{ik}|b_{kt}+b_{ik}|b_{kt}|}{2}}\right), \\ \displaystyle\sum_{b_{it}+\frac{|b_{ik}|b_{kt}+b_{ik}|b_{kt}|}{2}<0} a_j\left({x'}_i^{-\left(b_{it}+\frac{|b_{ik}|b_{kt}+b_{ik}|b_{kt}|}{2}\right)}\right) \end{array} \right\}.
\end{aligned}$$

Note here that the last equality is obtained by the fact that $a_j(fg) = a_j(f) + a_j(g)$. Using the same fact once again, we clearly get that

$$a_j = \sum_i \left| b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} \right| a_j({x'}_i),$$

where, depending on j, the range of the sum is either the set of indices satisfying the inequality

$$b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} > 0$$

or the set of indices satisfying the inequality

$$b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} < 0.$$

Recall also that $x'_i = x_i$ for $i \notin \{k, t\}$. So,

$$a_j = \sum_i \left| b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} \right| a_j(x_i).$$

But by equation (5.1), it follows that

$$\begin{aligned}\lambda\big(M'(t) + L'(t)\big) &= \sum_{j\in J} a_j \varpi_j \\ &= \sum_{j\in J}\sum_i \left(\left| b_{it} + \frac{|b_{ik}|b_{kt} + b_{jk}|b_{kt}|}{2} \right| a_j(x_i) \right) \varpi_j.\end{aligned}$$

Now, since $\mathbb{C}[G/P_K^-]$ is graded by the lattice spanned by the ϖ_i's, the last equation implies that

$$\begin{aligned} L\left(\lambda\big(M'(t) + L'(t)\big)\right) &= L\left(\sum_{j\in J}\sum_i \left(\left| b_{it} + \frac{|b_{ik}|b_{kt} + b_{jk}|b_{kt}|}{2} \right| a_j(x_i) \right) \varpi_j \right) \\ &\supset \prod_{j\in J}\prod_i L\left(\left| b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} \right| a_j(x_i) \varpi_j \right) \\ &= \prod_{j\in J}\prod_i L\left(d_i a_j(x_i)\varpi_j\right), \end{aligned}$$

where

$$d_i := \left| b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} \right|.$$

So, we get that

$$L\left(\lambda\big(M'(t) + L'(t)\big)\right) \supset \prod_{j\in J}\prod_i \underbrace{L\left(a_j(x_i)\varpi_j\right)\cdots L\left(a_j(x_i)\varpi_j\right)}_{d_i \text{ times}}. \tag{5.2}$$

Note here that the sum over i is the sum over the d_i's where the absolute value is taken over the positive values only or the negative values only. A similar work with $L\left(\lambda\big(M(k) + L(k)\big)\right)$ shows that

$$L\left(\lambda\big(M(k)+L(k)\big)\right) = L\left(\sum_{j\in J}\sum_{i}\left(|b_{ik}|a_j\left(x_i\right)\right)\varpi_j\right)$$
$$\supset \prod_{j\in J}\prod_{i} L\left(|b_{ik}|a_j\left(x_i\right)\varpi_j\right).$$

This implies that

$$L\left(\lambda\big(M(k)+L(k)\big)\right) \supset \prod_{j\in J}\prod_{i}\underbrace{L\left(\varpi_j\right)\cdots L\left(\varpi_j\right)}_{|b_{ik}|a_j(x_i)\text{ times}}.$$

But since $\Delta_{\varpi_j,\varpi_j}$ is of degree ϖ_j, it follows that the possible occurrences of the exponents of $\Delta_{\varpi_j,\varpi_j}$ are the integers

$$0,1,2,...,\sum_{i} a_j(x_i)|b_{ik}|.$$

However, the minimality of

$$a_j(\lambda(M(k)+L(k))) = \sum_{j\in J}\sum_{i}|b_{ik}|a_j(x_i)$$

shows that the only possibility is $\sum_i a_j(x_i)|b_{ik}|$, because the rest are still available in some $L(\lambda)$'s in which λ is less than $\lambda(M(k)+L(k))$. Consequently, the only possibilities for α_j and β_j are 0 or $\sum_i a_j(x_i)|b_{ik}|$. But, thank to the homogeneity of the construction of [12], there exists a unique j in which $a_j(x_i)\neq 0$. Hence, one of α_{jk} and β_{jk} is $a_j(x_i)|b_{ik}|$ and the other is 0.

Therefore, for every i there is a unique j in which one of the following must be true:

$$a_j(x_i)b_{it} = \pm\alpha_{jt}, \quad a_j(x_i)b_{ik} = \pm\alpha_{jk} \quad \text{and} \quad a_j(x_i)|b_{ik}| = \alpha_{jk},$$

or

$$a_j(x_i)b_{it} = \pm\beta_{jt}, \quad a_j(x_i)b_{ik} = \pm\beta_{jk} \quad \text{and} \quad a_j(x_i)|b_{ik}| = \beta_{jk}.$$

Now,

$$\left(\Delta^{a_j(x_i)}_{\varpi_j,\varpi_j}\widetilde{x}_i\right)^{d_i} \in L\left(d_i a_j(x_i)\varpi_j\right)$$
$$\left(\Delta^{a_j(x_i)}_{\varpi_j,\varpi_j}\widetilde{x}_i\right)^{\left|b_{it}+\frac{|b_{ik}|b_{kt}+b_{ik}|b_{kt}|}{2}\right|} \in L\left(d_i a_j(x_i)\varpi_j\right)$$

It is not hard now to see that $\left(\Delta^{a_j(x_i)}_{\varpi_j,\varpi_j}\widetilde{x}_i\right)^{d_i}$ forms one factor of the monomial $\mu'(t)\widetilde{M'}_t$ or the monomial $\nu'(t)\widetilde{L'}_t$. The rest is similarly there. Since $L\left(\lambda\big(M'(t)+L'(t)\big)\right)$ and $L\left(\lambda\big(M(k)+L(k)\big)\right)$ are homogeneous ideals and since

$$\widetilde{x_k x'_k} = \widetilde{x}_k \widetilde{x'}_k = \mu(k)\widetilde{M(k)} + \nu(k)\widetilde{L(k)} \in L\left(\lambda\big(M(k) + L(k)\big)\right),$$

it is again not difficult to combine these pieces of information to see that

$$\widetilde{x'_t x''_t} = \widetilde{x'}_t \widetilde{x''}_t = \mu'(t)\widetilde{M'(t)} + \nu'(t)\widetilde{L'(t)} \in L\left(\lambda\big(M'(t) + L'(t)\big)\right).$$

This completes the proof. □

Notation. The cluster algebra contained in $\mathbb{C}[G/P_K^-]$ and obtained from the preceding theorem will be denoted by $\widehat{\mathcal{A}_J}$ or simply $\widehat{\mathcal{A}}$ if the context is clear.

Corollary 5.9. *Let B be the matrix $\widetilde{B}^w$ of Theorem 4.8. The pair*

$$\left(\{\widetilde{D}_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}\} \sqcup \{\Delta_{\varpi_j,\varpi_j} \mid j \in J\}, \widehat{B}\right)$$

is an initial seed of the cluster algebra $\widehat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$.

Proof. Apply the construction of Theorem 5.8 to the initial seed $(D_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}, \widetilde{B}^w)$ of $\mathbb{C}[N_K]$ (see Theorem 4.8). The mutable and frozen variables are described in Definition 5.7. □

Remark 5.10. One might think naively that the lift $\widetilde{D}_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}$ is equal to a generalized minor. But this is not the case in general; see Example 10.3 of [6].

Remark 5.11. In the simply-laced case, it is obvious that the construction of $\widehat{\mathcal{A}_J}$ matches the one of [6].

Remark 5.12. By construction, it is clear that the extended clusters of $\mathcal{A}$ and $\widehat{\mathcal{A}}$ are in one-to-one correspondence. So, $\mathcal{A}$ and $\widehat{\mathcal{A}}$ must be of the same type (either both finite or both infinite).

Theorem 5.13. *The localization of the homogeneous coordinate ring of the flag variety $\mathbb{C}[G/P_K^-]$ by $\Delta_{\varpi_j,\varpi_j}$, $(j \in J)$ equals the localization of the cluster algebra $\widehat{\mathcal{A}}$ by the same elements. Namely,*

$$\mathbb{C}[G/P_K^-][\Delta^{-1}_{\varpi_j,\varpi_j}]_{j\in J} = \widehat{\mathcal{A}}[\Delta^{-1}_{\varpi_j,\varpi_j}]_{j\in J}.$$

Proof. Throughout the proof, for any element in $\mathbb{C}[G/P^-_{I\setminus\{j\}}]$ the term degree will be used to refer to its homogeneous degree (see Remark 3.8). Take $J' \subset J$ and $K' = I \setminus J'$. This gives us a reduced word w_0 of the form $i = (i', i'', i''')$, with i' and (i'', i''') being reduced words for w_0^K and $w_0^{K'}$ respectively. Therefore, the initial seed for the cluster algebra $\widehat{\mathcal{A}_J}$ associated with i contains the one of $\widehat{\mathcal{A}_{J'}}$ associated with i'. This means that the latter is a subalgebra of the first, that is, $\widehat{\mathcal{A}_{J'}} \subset \widehat{\mathcal{A}_J}$. Consequently, $\widehat{\mathcal{A}_{\{j\}}} \subset \widehat{\mathcal{A}_J}$ for any $j \in J$. Recall that $\mathbb{C}[G/P_K^-] = \bigoplus_{\lambda \in \Pi_J} L(\lambda)$ is generated as a ring by the subspaces $L(\varpi_j) \subset \mathbb{C}[G/P_{I\setminus\{j\}}^-]$. Thus, it is generated by $\mathbb{C}[G/P_{I\setminus\{j\}}^-]$, where $j \in J$. Therefore, the result follows if the localization of $\mathbb{C}[G/P_{I\setminus\{j\}}^-]$ by $\Delta_{\varpi_j,\varpi_j}$ is contained in the localization of $\widehat{\mathcal{A}_{\{j\}}}$, by the same element.

We proceed by contradiction. Let $f \in \mathbb{C}[G/P_{I\setminus\{j\}}^-]$ such that $f \notin \widehat{\mathcal{A}_{\{j\}}}$ and its degree is minimal. Let $g = \operatorname{proj}(f) \in \mathbb{C}[N_{I\setminus\{j\}}]$. Then $\operatorname{proj}(\widetilde{g} - f) = 0$. Thus, we have that $\widetilde{g} - f$ belongs to the principal ideal $(\Delta_{\varpi_j,\varpi_j} - 1)$, since

$$\mathbb{C}[N_{I\setminus\{j\}}] = \mathbb{C}[G/P_{I\setminus\{j\}}^-] \Big/ (\Delta_{\varpi_j,\varpi_j} - 1)\cdot$$

Consequently, there is some $h \in \mathbb{C}[G/P_{I\setminus\{j\}}^-]$ such that

$$\begin{aligned} \widetilde{g} - f &= h\left(\Delta_{\varpi_j,\varpi_j} - 1\right) \\ \widetilde{g} - f &= h\Delta_{\varpi_j,\varpi_j} - h. \end{aligned}$$

But note that the definition of $\widetilde{g}$ and the choice of f imply that the degree of the whole left-hand side is less than or equal to degree of f. On the other hand, it is obvious that the degree of the right-hand side is the degree of h plus ϖ_j. It follows that the degree of h is less than the one of f. Therefore, by minimality, we get that $h \in \widehat{\mathcal{A}_{\{j\}}}$. Also, since $\Delta_{\varpi_j,\varpi_j} \in \widehat{\mathcal{A}_{\{j\}}}$, it follows that $h\Delta_{\varpi_j,\varpi_j} \in \widehat{\mathcal{A}_{\{j\}}}$. Now, if the lifting $\widetilde{g} \in \widehat{\mathcal{A}_{\{j\}}}$, we have

$$f = \underbrace{\widetilde{g}}_{\in \widehat{\mathcal{A}_{\{j\}}}} - \underbrace{h\Delta_{\varpi_j,\varpi_j}}_{\in \widehat{\mathcal{A}_{\{j\}}}} + \underbrace{h}_{\in \widehat{\mathcal{A}_{\{j\}}}} \in \widehat{\mathcal{A}_{\{j\}}} \subset \widehat{\mathcal{A}_{\{j\}}}[\Delta_{\varpi_j,\varpi_j}^{-1}], \tag{5.3}$$

which is a contradiction to f being outside $\widehat{\mathcal{A}_{\{j\}}}$. Therefore, $\widetilde{g} \notin \widehat{\mathcal{A}_{\{j\}}}$. Now, write $g \in \mathbb{C}[N_{I\setminus\{j\}}]$ as $g = \sum_{i=1}^r c_i m_i$, where each m_i is a product of cluster variables (might not be from the same seed) and each c_i is a scalar. We may do this in such a way that the m_i's are distinct. Recall that $\mathbb{C}[N_K]$ can be identified with

$$\mathbb{C}[N_K] = \left\{ \frac{f}{\prod_{j\in J} \Delta_{\varpi_j,\varpi_j}^{a_j}} \mid f \in L\left(\sum_{j\in J} a_j\varpi_j\right) \right\}.$$

By the uniqueness and minimality of the tilde map in Lemma 5.1, this can be refined to

$$\mathbb{C}[N_K] = \left\{ \frac{f}{\prod_{j\in J} \Delta_{\varpi_j,\varpi_j}^{a_j}} \mid f \in L\left(\sum_{j\in J} a_j\varpi_j\right) \text{ and the } a_j\text{'s are minimal} \right\}.$$

As $J = \{j\}$, we can use the second identification to write each m_i as $\dfrac{f_i}{\Delta_{\varpi_j,\varpi_j}^{a_{i,j}}}$, where $f_i \in L(a_{i,j}\varpi_j)$ and $a_{i,j}$ is minimal with this property. Clearly, the degree d_i of $\widetilde{m_i}$ is

$a_{i,j}\varpi_j$. It is not hard to see that $\widetilde{m_i} = f_i$ for all $i = 1, ..., r$. As distinct elements lift by the tilde map to distinct elements by Lemma 5.1, we get that the $\widetilde{m_i}$'s are distinct. Consequently, the f_i's are distinct. Let $a_j := \max\{a_{i,j} \mid i = 1, ..., r\}$. Now, if

$$\sum_{i=1}^{r} \left(c_i \Delta_{\varpi_j,\varpi_j}^{a_j - a_{i,j}} f_i \right) \tag{5.4}$$

is a lift of $g = \sum_{i=1}^{r} c_i m_i$ in the minimal way, then we get that $\widetilde{g} \in \widehat{\mathcal{A}_{\{j\}}}$, which is a contradiction. Otherwise, there is an element of lower degree in which $\widetilde{g}$ is equal to that element. Note that the multiplication of $\widetilde{g}$ by $\Delta_{\varpi_j,\varpi_j}^{s_j}$ for some positive integer s_j gives an element whose degree is equal to the degree of the element in (5.4) and whose projection is equal to g. Since the projection of each homogeneous piece $L(\lambda)$ to $\mathbb{C}[N_K]$ is injective, we get that

$$\widetilde{g} = \frac{\sum_{i=1}^{r} \left(c_i \Delta_{\varpi_j,\varpi_j}^{a_j - a_{i,j}} f_i \right)}{\Delta_{\varpi_j,\varpi_j}^{s_j}}.$$

Clearly, by (5.3) this implies again that $f \in \widehat{\mathcal{A}_{\{j\}}}[\Delta_{\varpi_j,\varpi_j}^{-1}]$. Consequently, the result follows, that is,

$$\mathbb{C}[G/P_K^-][\Delta_{\varpi_j,\varpi_j}^{-1}]_{j \in J} = \widehat{\mathcal{A}}[\Delta_{\varpi_j,\varpi_j}^{-1}]_{j \in J}. \quad \square$$

Conjecture 5.14. *The homogeneous coordinate ring of the flag variety $\mathbb{C}[G/P_K^-]$ equals the cluster algebra $\widehat{\mathcal{A}}$. In particular, $\mathbb{C}[G/P_K^-]$ is a cluster algebra whose initial seed is*

$$\left(\{\widetilde{D}_{\varpi_{i_k}, w_{\leq k}\varpi_{i_k}}\} \sqcup \{\Delta_{\varpi_j,\varpi_j} \mid j \in J\}, \widehat{B} \right).$$

Remark 5.15. Using the proof of the previous theorem, the conjecture is equivalent to proving that if $g = \sum_{i=1}^{r} c_i m_i$ is written where the number of terms is minimal, then

$$\widetilde{g} = \sum_{i=1}^{r} \left(c_i \Delta_{\varpi_j,\varpi_j}^{a_j - a_{i,j}} \widetilde{m_i} \right).$$

Example 5.16. Let G be a semisimple algebraic group of type B_3, say $G = SO_{2(3)+1} = SO_7$, $J = \{3\}$ and $K = I \setminus J = \{1, 2\}$. Consider the longest word

$$w_0 = s_1 s_2 s_1 \boldsymbol{s_3 s_2 s_1 s_3 s_2 s_3}.$$

The subword $w = s_3 s_2 s_1 s_3 s_2 s_3$ generates N_K. Since $s(3) = s(5) = s(6) = \infty$ and $s(k) \neq \infty$ for $k \in \{1, 2, 4\}$, we get that the mutable variables are indexed by $1, 2, 4$ and the frozen ones are indexed by $3, 5, 6$ using the function s of Theorem 4.8. Therefore, by the same theorem, the exchange matrix of the cluster algebra structure of $\mathbb{C}[N_K]$ is

$$\begin{array}{cc} \begin{array}{ccc} 1 & 2 & 4 \end{array} & \\ \begin{pmatrix} 0 & a_{i_1 i_2} & 1 \\ -a_{i_2 i_1} & 0 & a_{i_2 i_4} \\ -1 & -a_{i_4 i_2} & 0 \\ -a_{i_3 i_1} & -a_{i_3 i_2} & 0 \\ 0 & -1 & -a_{i_5 i_4} \\ 0 & 0 & -1 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 4 \\ 3 \\ 5 \\ 6 \end{array} \end{array}$$

$$= \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 4 \end{array} & \\ \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 4 \\ 3 \\ 5 \\ 6 \end{array} \end{array}$$

where the column labels denote the cluster variables and the row labels denote the extended cluster variables, as usual. Also, the extended cluster variables $D_{\varpi_{i_j}, w_{\leq j}\varpi_{i_j}}$ are

$$\begin{aligned}
j = 1 &\implies D_{\varpi_3, s_3\varpi_3}; && \text{(mutable)} \\
j = 2 &\implies D_{\varpi_2, s_3s_2\varpi_2}; && \text{(mutable)} \\
j = 3 &\implies D_{\varpi_1, s_3s_2s_1\varpi_1}; && \text{(frozen)} \\
j = 4 &\implies D_{\varpi_3, s_3s_2s_1s_3\varpi_3}; && \text{(mutable)} \\
j = 5 &\implies D_{\varpi_2, s_3s_2s_1s_3s_2\varpi_2}; && \text{(frozen)} \\
j = 6 &\implies D_{\varpi_3, s_3s_2s_1s_3s_2s_3\varpi_3}. && \text{(frozen)}
\end{aligned}$$

Therefore, by Theorem 5.8, the following list of variables forms an initial extended cluster of $\widehat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$

$$\begin{aligned}
&\widetilde{D}_{\varpi_3, s_3\varpi_3}; && \text{(mutable)} \\
&\widetilde{D}_{\varpi_2, s_3s_2\varpi_2}; && \text{(mutable)} \\
&\widetilde{D}_{\varpi_3, s_3s_2s_1s_3\varpi_3}; && \text{(mutable)} \\
&\widetilde{D}_{\varpi_1, s_3s_2s_1\varpi_1}; && \text{(frozen)} \\
&\widetilde{D}_{\varpi_2, s_3s_2s_1s_3s_2\varpi_2}; && \text{(frozen)} \\
&\widetilde{D}_{\varpi_3, s_3s_2s_1s_3s_2s_3\varpi_3}; && \text{(frozen)} \\
&\Delta_{\varpi_3, \varpi_3}. && \text{(frozen)}
\end{aligned}$$

Consequently, the extended exchange matrix $\widehat{B}$ of $\widehat{\mathcal{A}}$ attached to this extended cluster is

$$\widehat{B} = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 4 \end{array} & \\ \left(\begin{array}{ccc} 0 & -2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ \hline -1 & 0 & 0 \end{array}\right) & \begin{array}{c} 1 \\ 2 \\ 4 \\ 3 \\ 5 \\ 6 \\ j \in J \end{array} \end{array}.$$

Data availability

No data was used for the research described in the article.

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