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We report on a variety of innovative projects that are at different stages of development and implementation. We start by presenting a project still in development to help address Klein's second discontinuity problem, that is, the perception of pre-college teachers that the advanced mathematics courses they took at the university are of little use in the practice of their profession. Then we briefly discuss the study and research paths (SRP). This is the proposal from the Anthropological Theory of the Didactic (ATD) to foment a move from the prevailing paradigm of visiting works to that of questioning the world. This is followed by the discussion of an online course for in-service teachers, designed to help them experience, adapt, and class-test a modeling intervention, as well as reflect on institutional issues that might constrain the future application of modeling in their teaching. We end with a discussion of a project based on the idea of guided reinvention, to design and study the implementation of inquiry-oriented linear algebra.

Keywords: Study and research paths, Klein's second discontinuity, modeling, inquiry-based mathematics education, linear algebra.

INTRODUCTION

What do we mean by innovation in university teaching? Century and Cassata (2016) define innovations as "programs, interventions, technologies, processes, approaches, methods, strategies, or policies that involve a change for the individual end-users enacting them." We add that innovation should help students learn a particular mathematical content better than traditional teaching, and the innovations considered in the panel must be based on mathematics education research. The first part of this definition underscores that the innovation does not have to be new to the field at large; rather, the practice should require that users change what they are doing, so what is emphasized is that the practice of interest is different from current practice. This way of viewing innovation stresses concern for change in teaching practices beyond the classroom of the individual researcher.

With this in mind, we have chosen four projects that propose teaching innovations at the university level and that are at different stages of implementation: one project, the Geometry Capstone course for pre-service teachers, has so far been implemented several semesters by the researcher in his own classroom; another project, the design and implementation of *study and research paths* (SRP), that has been implemented in different classrooms by different instructors and universities but that still is not widely adopted; a project dealing with an online MS program in mathematics education for in-service teachers that includes a modeling component that has been fully implemented in a university, and a project for an inquiry-oriented linear algebra course, that aims to attain national dissemination.

The following sections discuss each of the four highlighted innovations. Each is presented by the respective panelist Max Hoffmann, Ignasi Florensa, Avenilde Romo Vázquez, and Michelle Zandieh.

A CAPSTONE COURSE "GEOMETRY FOR STUDENT TEACHERS" AT PADERBORN UNIVERSITY

In this section, we present an innovation that we implemented in the context of a course named Geometry for Student Teachers at Paderborn University in Germany. The course is scheduled in the curriculum for upper secondary math teachers in the third year of study. Like other German universities, the subject-related part of this study program consists of courses on academic mathematics and on didactics of mathematics. While student teachers attend most of their mathematics courses jointly with mathematics major students, this course is taken exclusively by student teachers.

Innovation goals

In the project SiMpLe-Geo we develop and study innovations to increase professional orientation in the course Geometry for Student Teachers. In this way, we want to counteract the second discontinuity in teacher education. The course concept's theory-based development and initial research are part of the Ph.D. thesis of Hoffmann (2022). In addition, various other publications have been produced as part of the project (e.g., Biehler & Hoffmann, 2022; Hoffmann & Biehler, 2022), from which some text elements have been taken verbatim for this overview.

As a basis for the course concept, we have worked out the following three design principles for academic math courses for student teachers with a particular focus on professional orientation:

1. *Orientation to the scientific systematics of mathematics*: The course aims to treat an area of academic mathematics in a systematic and structured way. The course follows the usual scientific standards of mathematics. These can be prototypically described by the three steps: definition - theorem - proof. The necessary level of detail in the argumentation must be adapted to the students' level of knowledge and experience.
2. *Orientation to the math-specific presentation- and communication methods*: The study of mathematics uses methods common in scientific practice for gaining and exchanging knowledge. Accordingly, the three-step process described in 1. is supplemented by other elements, e.g., examples and non-examples, heuristics, and historical backgrounds.

3. *Implicit professional orientation*: The professional orientation should be considered in every decision to be made in the context of the course conception (e.g., selection of content). The basic credo should be: In any conceptual decision with several similarly suitable options, the one that can best be related to the future teaching profession should be chosen.
4. *Explicit professional orientation*: At appropriate points of the course, activities in which the mathematical knowledge and skills acquired are explicitly used functionally as a disposition for acting in profession-oriented situations. This use must also be explicitly reflected upon.

Overview of the innovations

We take a holistic approach to implementing professional orientation in the course, using innovation at both the content level and the level of teaching/learning methods.

Content structure of the course

A significant part of the course deals with axiomatic plane geometry. The careful selection of the axiom system represents an important aspect of implementing professional orientation. We use one based on the work of Iversen (1992), in which neutral plane geometry is built upon metric spaces and later is supplemented by the parallel axiom. Two major advantages of this approach are that it is productively interconnected with the fundamental analysis and linear algebra courses the students already have taken (e.g., we use the real numbers right from the beginning) and the fact that many definitions and proofs can be didactically reduced for school geometry.

Interface weeks

Interface-Weeks are one of the two main innovations on the level of teaching/learning methods. The idea is, to shift the course focus from a mathematical theory to discussing and reflecting on connections between the academic mathematics learned and the aspired profession. Therefore, lectures, exercise groups, and home assignments are designed according to the principle of *explicit professional orientation* and differ substantially from the other weeks. As the main focus of the interface weeks, we have chosen the central geometric concepts of *congruence* and *symmetry*. For both topics, first, essential characteristics of their rigorous mathematical treatment are detached from the particular axiomatic approach of the lecture. We do this by explicating so-called *interface aspects*, which result from inductive subject-specific-didactical analyses (Biehler & Hoffmann, 2022; Hoffmann & Biehler, 2022). Using such *interface aspects*, typical approaches to the concepts (e.g., from textbooks) are discussed from a professional perspective. In addition, various focal points of the instructional treatment of these concepts will be located from a mathematics perspective, with special consideration given to intellectual honesty. In the exercise groups, students work on corresponding, discussion-oriented tasks and collaboratively use the mathematical knowledge they have learned in contexts relevant to their profession. The homework consists exclusively of tasks for the *interface-ePortfolio*.

Interface-ePortfolio

The course-accompanying *interface-ePortfolio* is the second main innovation at the level of teaching/learning methods. In this learning activity, we combined the idea of a course-accompanying ePortfolio (see, e.g., the project *dikopost* (Siebenhaar et al., 2013)) with the use of profession-oriented tasks, so-called *interface-tasks* (e.g., Bauer, 2013). The use of this innovation is organized in such a way that in some weeks, ePortfolio-tasks replace some of the ordinary homework tasks. In addition, the students got feedback on their work from a student tutor. Those suggestions for improvement could be used for optional revision. The ePortfolios are technically realized so that only the student tutor can see the students' real names; the lecturer can only see them in pseudonymized form. This was done to keep the *interface-ePortfolio* as an ungraded learning opportunity with a high amount of (honest) reflection.

We used four different task formats to work on the ePortfolio:

- *Competence Grids for Self-Assessment*: Using these grids, students must self-assess their competencies in mathematical backgrounds of school geometry concepts and theorems and their skills in dealing with math-containing job-tasks. This activity is used at the beginning and the end of the semester, which allows the students to reflect on their competence growth during the course.
- *Interface Tasks*: In these tasks, students use their mathematical knowledge and skills as dispositions to look at and analyze profession-oriented situations (e.g., a real or fictional student contribution or a textbook page).
- *Reflection Tasks*: In the context of their *interface-ePorfolio*, students have to work on reflection tasks on different levels. This includes activities in which students reflect on how the competencies acquired in the course influence their work on interface tasks, occasions for reflection on their prior knowledge of the central geometric concepts, activities that generally refer to which sense students see geometry as relevant content in school mathematics, and self-perceptions about the ability to teach geometry.
- *Fact-Sheets for Geometric Mappings*: During the semester, students study different geometric mappings (orthographic projections, reflections, rotations, central dilations, reflections at circles) and their properties. In this fourth type of task, students summarize their stepwise growing knowledge of those geometric mappings in a pre-structured way. This consists of a formal definition, an explanation of all possible variants of the formalization (e.g., as a term or as a matrix), and a detailed written example calculation.

Current interim status of the project

We have already taught the course according to this concept four times and researched and further developed it within a design research approach. Initial results show that students substantially contribute to overcoming the second discontinuity (related to plane geometry), at least from a subjective perspective. We are currently evaluating further data to gain insights into the objective impact.

STUDY AND RESEARCH PATHS: THE ATD PROPOSAL

SRPs: an ATD-founded device

Study and research paths (SRP) are inquiry-based teaching formats framed in the Anthropological Theory of the Didactic (ATD). SRPs are long teaching and learning processes, lasting from some 8-10 sessions (2h) to an entire course, that start with the consideration of an open generating question that student(s) address under the guidance of the teacher(s). Describing and analyzing the SRP proposal cannot be done without explicitly mentioning some of the ATD principles and theoretical developments that are in the inner heart of the proposal.

The first aspect that is undetachable from the SRP proposal is the notion of didactic paradigm. SRPs are conceived as didactic devices fostering a shift in the prevailing didactic paradigm in our societies, from the *paradigm of visiting works* (PVW) to the *paradigm of questioning the world* (PQW) (Chevallard, 2015). Teaching and learning processes in undergraduate mathematics courses are particularly experiencing this crisis of the old PVW, where content organizations are presented for students to "visit" them, which contrasts with the emergence of the PQW, where the study of questions becomes the center of the study process. The implementation of an SRP is a way to analyze the conditions needed for the paradigm shift. Diverse experiences at the undergraduate level show relevant results of this evolution (e.g., Barquero et al., 2018; Florensa et al., 2018a).

A second point that is necessary to consider when describing the SRP proposal is its link to the didactic engineering methodology (Barquero & Bosch, 2015; García et al., 2019). In other words, SRPs are also research artifacts allowing the research community to generate empirical material to conduct didactic and epistemological analyses. In fact, the strong relationship between the conception of the knowledge to be taught and the didactic phenomena emerging in the teaching and learning processes assumed as a founding principle of the ATD, turns SRPs into key elements of didactic research. An illustrative example of the role of the SRPs as artifacts allowing researchers to modify and study specific didactic phenomena are the works of Berta Barquero when describing, analyzing, and modifying the phenomenon of "applicationism" in mathematics courses in applied sciences degrees (Barquero, Bosch, & Gascón, 2014). An important aspect of SRPs is this twofold nature: a teaching proposal and a research artifact that inevitably fosters a change on the activity existing in school (or university) institutions.

The institutional approach of the ATD is the third aspect that defines the SRP proposal. On the one hand, as mentioned before, SRPs have the capacity to modify teaching and activating activities in school institutions. On the other hand, the implementation and viability of SRPs are undetachable from the study of the institutional *ecology*, that is, the set of conditions needed and restrictions hindering their viability in different institutions. The ecology of SRPs has been studied in some research works at the university level (Barquero, Bosch, & Gascón, 2013; Barquero, 2018) using the scale of levels of didactic codeterminacy (Chevallard, 2015). It allows researchers to identify restrictions that appear outside the level of the classroom and, at the same time, make explicit the changes that the change of paradigm (and in particular the SRP implementation) would cause in the organization of subjects and contents.

Finally, the study and analysis of SRPs cannot be detached from the notion of Herbartian schema. The Herbartian schema is a model of inquiry processes. It considers didactic system as formed around a question (and not a specific work as usual in school institutions):

$$[S(X;Y;Q_0) \rightarrow M] \hookrightarrow A^{\heartsuit}$$

In this context, the group of students X with the help of a group of teachers Y must provide an answer to Q_0: $A^{\heartsuit}$. The process of inquiry of Q_0 leads the community of study (X, Y) to meet different pre-existing answers $A^{\diamond}$, derived questions Q_i, other works W_n needed to interpret $A^{\diamond}$, and empirical data D_j. The set of these elements constitutes the *milieu* M of the inquiry:

$$M = \{A_1^{\diamond}, A_2^{\diamond}, \dots, A_m^{\diamond}, W_{m+1}, \dots W_n, Q_{n+1}, \dots Q_p, D_{p+1}, \dots D_q\}$$

The Herbartian schema pinpoints the fact that putting the questions at the center of the inquiry process fosters the transition between didactic paradigms but not in terms of "substitution": the works and pre-existent answers are still relevant and are studied. However, its new role is subordinated to the generation of an answer $A^{\heartsuit}$ to a question Q_0, which remains in the heart of the study process. This capacity of SRPs to enable moments of study of previously existing works and moments of research during the same inquiry process contrasts with other proposals where the study activity is not present. The Herbartian schema highlights another commonality in the different SRPs implementations: the responsibility of enriching the milieu during the inquiry is shared by X and Y. While in other proposals the teachers often assume the role of enrichers and validators, in an SRP the evolution of the process is taken by the whole community of study.

SRPs: from the first implementations to the transposition to lecturers

From the first implementations of SRP at undergraduate level in 2005 with the thesis of Berta Barquero, the way SRPs have been integrated and their role has very much evolved. We can describe this evolution in terms of integration to the courses, role of the teacher or inquiry guide, domains of intervention, and dissemination to teachers.

Regarding the integration to the courses, the first SRPs were implemented as *modelling workshops* running in parallel to the mathematics courses. This evolution is closely related to the ecological conditions in the institutions where the SRPs were implemented. According to Barquero et al. (2021), the different settings require different levels of change in the previous organization. This flexibility in the SRP organization facilitates its implementation in very different institutions with different pedagogical conditions and constraints.

Another relevant aspect is the SRP's teacher or guide (*Y*, in the didactic system). In the first experiences, researchers were those in charge of the design and management of the SRPs. This situation led to very fragile environments. In other words, the first SRPs lasted while the researcher kept the position of guide of the study. Once the researcher left the institution, the SPRs tended to disappear or significantly reduce their time dedication.

The first SRPs implemented at the undergraduate level were implemented in mathematics courses for applied sciences and in business administration degrees. However, this past decade, SRPs have spread in different domains. One of the first domains that adopted SRPs outside mathematics education was mechanical engineering (with subjects such as elasticity and strength of materials) (Florensa et al., 2018a, Bartolomé et al., 2019) and applied statistics (Markulin et al., 2021). In the last two years SRPs have also been adopted in Chemical and ICT courses for engineers and accounting courses in Business Administration degrees.

This spreading of SRPs cannot be understood without two factors that foster SRP dissemination as a research-based teaching innovation device. First, is the explicit training of university teachers (Florensa et al., 2018b). The implementation of diverse teacher development courses has enabled teachers to start collaborating with researchers to design and implement SRPs, overcoming the fragility of the researcher-teacher positions concentrated in a sole person.

Second, the diffusion of SRPs has been done in parallel with the transposition of different tools and devices that have helped both teachers and students to deal with the new organization and conception of knowledge around questions. The incorporation of questions-answers maps, logbooks or weekly reports and final reports addressed to the receiver of the answer seem to facilitate the inquiry management and assessment.

A final aspect that remains open is the (inquiry) contract that needs to be established around the generating question. What characteristics does it need to fulfill? Even if there is still a lot of research to do in this field, some of our last analyses seem to indicate that the existence of an external instance receiving the answer to the generating question facilitates the implementation of a rich inquiry process and a shared assumption of responsibilities within the community of study.

MATHEMATICAL MODELLING COURSES IN AN ONLINE PROFESSIONAL DEVELOPMENT PROGRAM

In 2000, Mexico's National Polytechnic Institute created a master's program for in-service mathematics teachers in the virtual modality. The groups formed could include teachers from different educational levels: secondary school, high school, and university, and from different geographical locations in Mexico and Latin America. This heterogeneity made it necessary to design courses that could contribute to the analysis, innovation, and regulation of diverse teaching practices. In 2010, some courses were designed to focus on designing mathematical modelling activities specifically for the study of non-mathematical contexts, such as engineering. One objective was to offer tools to aid in designing didactic proposals for training non-specialists at the university level. In the framework of these courses, professors implemented mathematical modelling activities that related math to other disciplines and encouraged reflection on the minimum conditions necessary for integrating mathematical modelling into teaching. Some examples of these courses and the work carried out by the teachers will illustrate this professional development proposal, its scope, and its limitations.

A Mexican professional development program for in-service mathematics teachers: ProME

Currently in Mexico, there is training for future teachers for elementary and secondary school, but no specific training for high school and university mathematics teachers. Most mathematics teachers and professors at these levels are mathematicians, engineers, or professionals with a four-year undergraduate degree in an area with a specific mathematical-scientific orientation who have a vocation and interest in teaching. Many in-service teachers feel a significant need for specific training. Several master's programs have been created in Mexico to meet the professional and didactic needs of high school and university mathematics teachers and professors. These are two-year programs that include several courses and the elaboration of a master's thesis. Most are offered at universities in the in-person modality. Some are full-time and research oriented. Most students in those programs have scholarships. Other programs are part-time and oriented more towards professional development. However, teachers who live far from universities cannot register in these programs. For this reason, the program for the professional development of mathematics teachers (ProME) was created in 2000 at the National Polytechnic Institute in the online modality with two goals, one academic, the other social:

Academic: To introduce groups of mathematics teachers into the practices, theories, and languages of Mathematics Education by connecting research with practice.

Social: To modify, as far as possible, the scenario of social exclusion that many in-service teachers experience because the opportunities for training in Mathematics Education do not provide them with any space.

The Study and Research Path: a theoretical tool for analyzing ProME's educational model

In general, the courses in this master's program can be analyzed by considering a didactical system composed of students (X), educators (Y), and courses (Q):

- *The teacher-students* (X) are in-service mathematics teachers and professors from Mexico and other Latin American countries with diverse professional backgrounds, and different teaching experiences who were working at distinct educational levels: secondary school, high school, and university.

- *The educators* (*Y*) are researchers in Mathematics Education with experience as math teachers.
- *Courses/SRP-TE* (*Q*). Three types of courses are offered: theoretical, theoretical-practical, and seminars. The first focus on specific theoretical frameworks. The second analyze elements of research in Mathematics Education in relation to teaching practices in mathematics, while the seminars guide the students in writing up their theses.

The design of the theoretical-practical courses identified two types of questions:

- Professional questions that arise in practice, such as how to integrate technology into mathematics teaching and how to design mathematical modelling activities.
- Research questions analyzed in the context of math education, such as how to identify the nature of obstacles –didactical, epistemological, etc.– in teaching mathematics.

In other words, we identify objects of study and outcomes of Mathematics Education related to professional issues that math teachers and professors may not be aware of.

Mathematical modelling courses

There are two kinds of mathematics modelling courses, discussed here as a Study and Research Path for teacher education (SRP-TE). They were designed after 2013. The generating questions that motivated these courses were Q_0-*TE* (professional questions):

- How can a learning process related to *mathematical modelling* be *analyzed, adapted, developed*, and *integrated* into our teaching practice?
- How can *long-term learning processes* based on modelling *be sustained institutionally*? What *difficulties* need to be overcome? What didactic tools are needed? What new questions arise and how can they be addressed?

In general, these generating questions are integrated using the methodology proposed by Ruiz-Olarría (2015) and adapted to the online modality by our team of educators (see, for example, Barquero et al. (2018)). The strategy developed has four steps:

1. Allow teachers to experience an SRP like mathematicians or apprentice mathematicians.
2. Analyze the SRP using didactic tools:

   - Mathematical analysis (reference epistemological model)
   - Didactic analysis: changes in didactics (and pedagogy) contracts, dialectic media-milieu, questions and answers, etc.
   - Ecology and sustainability of the SRP: institutional conditions

3. Adapt the SRP experienced (in step 1) so it can be implemented with a given group of students
4. Implement an a posteriori analysis of the SRP experienced with their students.

The small difference between the two types of SRP-TE is the way in which the SRP proposed in step 1 is designed. For the first type, the design of the SRP does not require an analysis of a non-mathematical context, but in the second type this is necessary. The first SRP-TE proposed, for example, analyzing and solving 'Forecasting sales for *Desigual* (a Spanish fashion brand)'. An epistemological dimension is considered by addressing several questions, such as What is modelling? How can the modelling process be described? and What is inquiry? These SRP-TE have been implemented in several editions by a large team of educators from Mexico and Spain to make the institutional conditions that drive –or constrain– the integration of mathematical modelling activities in the classroom visible to math teachers and professors (see Barquero et al., 2018; Romo et al., 2016). The second type of SRP-TE integrates the SRP that originated in non-mathematical contexts; for example, the Blind Source Separation method (BSS) used in acoustics, geophysics, and biosignal analysis. The BSS is an exciting method as it constitutes a case of inverse modelling that makes it possible to separate mixes without knowing the components or how they were mixed. One of the algorithms involved is based on the matrix model, $Ax=b$ (Vázquez et al., 2016). Using this approach, Camilo Ramírez designed an SRP in his Ph.D. thesis–in progress– that was implemented in an SRP-TE, as discussed below.

An example of a mathematical modelling SRP-TE: the case of the BSS method

The SRP-TE lasted four weeks (September 28-October 23, 2020) and was composed of three activities. Six members of the group (two secondary school teachers, two high school teachers, two university professors) and three educators participated (an engineer-researcher who was an expert in the BSS method and two researchers in Mathematics Education). In activity 1, two teams of students develop an SRP using the BSS method and then analyzed the process followed to answer the generating question: what is the mathematical technique that makes it possible to separate a mixture of sounds? The main media for this activity was an online resource that showed three different mixes of the same sounds. The mixes differed in terms of the distance between the sources (sound instrument) and the observations (recorders). Various elements were provided to analyze these mixes, including a geometric representation of the sources (sounds) and observations (recordings) and the hearing and tabular representations. In addition, we proposed identifying the derived questions and their answers to analyze the modelling process followed in this activity.

Activity 2 consisted in adapting the SRP developed in Activity 1 so that it could be implemented with students in an online modality (due to the conditions of the Covid-19 pandemic). To this end, three elements were given: 1) a BSS-praxeology ; 2) a school BSS-praxeology obtained from a didactic transposition performed on the BSS praxeology; and 3) an SRP designed for first-year university students that included four activities and elements to integrate a milieu: two free online resources (designed by one of the educators) that allowed them to listen to two mixtures of pure tones and explore different geometrical configurations between the sources and observations, such that they could identify the distance between them, which represented the coefficients of the system of linear equations; that is, a mathematical model of the mixtures. The other variables considered were frequency and amplitude. The student-teachers could use two of these four activities in their adapted SRP and had to modify the other two activities. Likewise, they had to perform an *a priori* analysis that showed the questions and answers that the students proposed. Activity 3 consisted in carrying out the *a posteriori* analysis. One of the most exciting adaptations of the SRP was made by a university professor with a background in engineering who adapted it for a group of volunteer high school students who had begun their first year of university. He modified the online resource proposed in Activity 1 and proposed quadratic signals and several activities to study three variables– distance, frequency, and amplitude– and the relations among them. His analysis of the students' activities showed the elements of the milieu associated with his SRP and affirmed that managing the SRP had proven to be: "Students had a clear difficulty in identifying that the modelling of the system is performed through a system of equations. Here, a series of activities that ask for different configurations to lead to the conclusion is probably required because giving them freedom to modify the scenario [online resource] was ineffective during implementation."

The other adaptions revealed the need to modify Activity 2 to analyze the milieu more deeply and determine how it can be extended or adapted with respect to the characteristics of the math class where the SRP will be implemented. Despite these issues, the student-teachers recognized that the SRP made it possible to perform a modelling activity in math class that allowed them to resolve challenging tasks.

PROJECT IOLA: INQUIRY ORIENTED LINEAR ALGEBRA

The Inquiry-Oriented Linear Algebra (IOLA) curriculum has been developed over the past 15 years and is continuing to evolve. The materials have been developed based on a set of design principles taken from Realistic Mathematics Education (RME; Freudenthal, 1991; Gravemeijer, 2020) and the design process is implemented through a series of teaching experiments and other mechanisms as described by our design research spiral (Wawro et al., 2022). The project began with a National Science Foundation (NSF) grant on student learning during which the initial tasks were developed (Rasmussen & Zandieh, 2007). This work continued with a grant focused specifically on the IOLA curriculum (Wawro, Zandieh, & Rasmussen, 2013). An additional grant (still in progress) is extending the IOLA materials (Wawro, Zandieh, Andrews-Larson, & Plaxco, 2019).

By the end of the 2013-2018 grant period, we had completed three Units, each with teacher support materials posted to our IOLA website (http://iola.math.vt.edu; Wawro, Zandieh et al., 2013). Each unit consists of a series of activities on a specific topic that typically takes 3-5 class periods. Figure 1 lists the units developed for the 2013 grant as well as the units that we are developing currently as part of the 2019 grant. The title of each unit refers to the experientially real setting (Gravemeijer & Doorman, 1999) in which the task sequence takes place. Below the title is a short description of the mathematical emphasis of the unit.

Created during previous grant	Created during the current grant
Unit 1: Magic Carpet Ride Span and linear independence	
	Unit 2: Meal Plans Solutions to systems of linear equations
Unit 3: Italicizing N Matrices as linear transformations	
	Unit 4: Distortion Determinants
	Unit 5: Hallways Subspaces
Unit 6: Blue to Black Change of basis, diagonalization, and eigentheory	
	Unit 7: Mail Delivery Least squares approximation and projection

Figure 1. IOLA curriculum units completed and under development.

We begin Unit 1 with vectors because we see vectors themselves, linear combinations of vectors, and vector equations as the most fundamental aspects of a beginning linear algebra course. We have also found that starting with a travel metaphor for exploring initial vector ideas works well as a starting point for students (Wawro et al., 2012). Research on the completed units and initial results regarding the new units can be found in various publications: Unit 2 (Smith et al., 2022), Unit 3 (Andrews-Larson et al., 2017), Unit 4 (Wawro et al., in press), Unit 5 (Andrews-Larson et al., 2021), Unit 6 (Zandieh et al., 2017; Plaxco et al., 2018), Unit 7 (Lee et al., 2022).

How do we design the units?

The development of all seven units has followed a design research cycle (Cobb et al., 2003) in which we engaged students with the activities, documented this process and used the results of this research to rewrite or refine the activities. In developing the four new units we have been particularly intentional in following the design research spiral shown in Figure 2 (Wawro et al., in press).

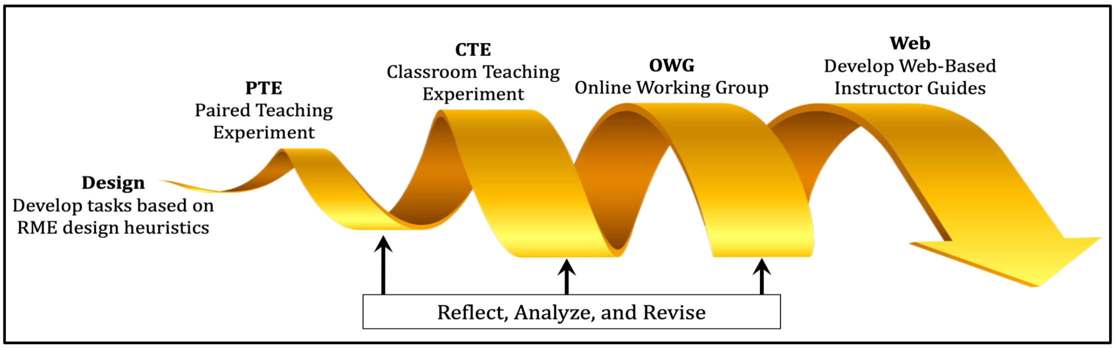

Figure 2. Design Research Spiral as shown in Wawro et al. (in press).

The initial task design is developed by a subgroup of the project team with other project team members working through the tasks in front of the development team to give initial feedback. The developers then conduct a PTE, paired teaching experiment, (similar to Steffe & Thompson, 2000) that allows for a detailed focus on the progression of the reasoning of the two students as they work through the tasks. We typically use pairs of students instead of individual students as this allows for students to learn from interacting with each other's ideas, much like we intend students to engage in a classroom setting. An analysis of the PTE allows for refinement that feeds into the CTE, classroom teaching experiment (Cobb, 2000).

The CTE is conducted by an IOLA project team member in an introductory level university linear algebra course that is part of his or her regular teaching load. Data is collected about how the students interacted with the activity in class, the role of the teacher in the classroom and student written work regarding the task. This data is analyzed and revisions to the unit are completed in preparation for the (OWG) online working group.

For the purpose of the development of the four new units, our online groups were designed to be composed of experienced IOLA instructors, i.e., instructors who had used the initial three units multiple times in the classroom. This included project team members but was intended to focus on getting feedback from outside of the project team. These groups met once per week for 6-8 weeks to prepare to implement the new unit, discuss reactions, questions, and feedback during the implementation, and then finally to reflect back on student and instructor interaction with the task sequence and how the unit may be improved.

Design Heuristics

The units are designed using three RME heuristics: didactical phenomenology, emergent models, and guided reinvention (Gravemeijer & Terwel, 2000; Gravemeijer, 2020). *Didactical phenomenology* is a way of determining a context that is well suited for the learning of a particular set of mathematical ideas. The context should be experientially real for the students; in other words, it is a setting that the students can immediately interact with and engage in. Given an appropriate task, students organize and structure aspects of that context in ways that create the mathematical ideas intended by the curriculum designers. The *emergent models heuristic* highlights how instructional designers can support students in transitioning from less formal to more formal ways of reasoning with and about these mathematical ideas.

Gravemeijer (1999) elaborated the development of emergent models as a progression through four levels of activity: situational activity, referential activity, general activity, and formal activity. Wawro, Rasmussen, et al. (2013) describe the transition across these levels of activity that occurs in Unit 1 of the IOLA curriculum. Students start working with two modes of transportation (a magic carpet and a hoverboard) each given by vectors in two dimensions. Initial exploration about what locations can be reached lead students to create ideas that the instructor can label as span, with further tasks (in three dimensions) leading to a formal definition of linear independence, and theorems about when a set of vectors will be linearly independent or dependent. Student initial activity in the task setting is organized or mathematized by the students in ways that the instructor can notate in terms of standard mathematical definitions and theorems. The process of students reinventing these ideas through their organizing activity, combined with the role of the instructor in guiding this process, is called *guided reinvention*.

Role of the Instructor

Given the important role of guided reinvention in the RME design heuristics, it is necessary to reflect on what instructional strategies can be implemented to support students in this process. Our project, IOLA, is called inquiry-oriented because we believe in both the importance of student inquiry into mathematical ideas and the importance of instructor inquiry into students' emerging mathematics (Rasmussen & Kwon, 2007). Johnson et al. (2015) created the TIMES (Teaching Inquiry-Oriented Mathematics: Establishing Supports) project to provide instructors with opportunities to implement inquiry-oriented curricula. As part of that process, they studied what is involved in inquiry-oriented instruction (IOI).

Kuster et al. (2017) characterize inquiry-oriented instruction around "four instructional principles: generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation," (p. 14). The instructor *generates students' ways of reasoning* by engaging them in goal-oriented activity with their classmates, usually in small group work. As student reasoning is generated, the instructor finds ways to *build on student contributions* with the goal of guiding students toward a reinvention of the mathematical ideas. To *develop a shared understanding* across students, the instructor acts as a broker between small groups and between small groups and the whole class (Rasmussen et al., 2009). The instructor also acts as a broker between the local classroom community and the broader mathematics community by helping the students *connect* their emerging mathematics *to standard mathematical language and notation*.

More specific ways of accomplishing these principles include what Rasmussen and Marrongelle (2006) refer to as pedagogical content tools. *Generative alternatives* are examples given by the instructor to elicit student reactions to possible alternative solutions or strategies. A transformational record is a way of notating student thinking that a student agrees captures their idea, but that the instructor knows is also a steppingstone to the standard mathematical notation. In these ways, an instructor may support guided reinvention by encouraging students to make explicit their ways of reasoning and by building on these through a transformational record toward a shared understanding that uses standard mathematical language and notation.

Implementation of IOLA

There are various ways that IOLA is being currently implemented in classrooms in varied instructional settings. Most recently there were over 700 accounts on the IOLA website (http://iola.math.vt.edu). Of course, not all accounts represent people who teach with the materials. In 2018 we conducted a survey of the then 328 faculty with accounts and found that of the 94 who responded to the survey 61 (65%) had adopted and integrated at least some of the existing IOLA materials in their classrooms. In addition, there is anecdotal evidence that some instructors who do not have accounts on the website use versions of the materials adopted from published sources like journal articles.

Over the years we have provided a variety of types of support to instructors who would like to use the IOLA materials. For account holders, the website has the full set of activities (for the initial three units) as well as instructor resources and examples of student thinking when using the materials. We have written articles for researchers and practitioners highlighting the progression of student thinking possible with the tasks as well as papers that explore the role of the instructor (e.g., Andrews-Larson et al., 2017; Zandieh et al., 2017). We have presented at research conferences and have provided workshops for instructors. Of particular note, the TIMES project (Johnson et al., 2015) recruited and worked with instructors using three inquiry-oriented curriculum materials, including IOLA. They leveraged the web-based instructional support materials provided by IOLA, provided summer workshops, and instituted Online Working Groups (OWG) that met to discuss implementation on a weekly basis. These OWG functioned to allow instructors new to IOLA, and perhaps new to any inquiry-oriented instruction, to have a place to get feedback, support, and exchange ideas with other instructors as they implemented something new to them.

To summarize, over the past 15 years the IOLA project has benefitted from a growing network of researchers and instructors contributing to this work. The project is centered around principles for curriculum design (RME) and research-based feedback on the design process (design research spiral). Implementation strategies include online instruction support materials as well as workshops and OWGs to aid instructors in implementing inquiry-oriented instruction (IOI).

CONCLUSION

The projects presented in the panel offer different views of inquiry in mathematics education. These projects can be positioned in different places on the continuum from open to directed inquiry; The more radical and open proposal is that of the SRPs. It can be expected to face institutional constraints in its quest to challenge the didactical paradigm that is prevalent at universities. This is followed by the inquiry fostered by the online modeling projects for in-service teachers, which can also be viewed as a special type of SRP (for teacher education). The openness of these modeling projects varies depending on the type of problem and the resources made available to students. Then, the guided reinvention of project IOLA may be thought to be within the paradigm of visiting works as it does not depart from a standard curriculum while following its instructional principle of connecting students' productions and ways of thinking to standard mathematical language and notation. The more directed modality of inquiry is that of the geometry capstone course for pre-service teachers. Students here inquire while working on interface tasks to relate the advanced viewpoint of the geometry course to their future careers. Nevertheless, it is in large part a lecture-based course that follows the definition-theorem-proof format.

Klein's second discontinuity problem, study and research paths, modeling, and inquiry-based mathematics education are all well-known approaches in the mathematics education community. They are actively researched, and the implementation and dissemination of their different proposals present a challenge. The projects discussed in the panel propose different ways to attend to this challenge. In the SRPs, this is the focus of their research; the modeling projects for in-service teachers include their adaptation and implementation at different educational levels thus providing a rich ground for the study of institutional constraints as well as for reflection on what it may take to implement modeling in these different contexts; and project IOLA facilitates its dissemination with their web page, articles, workshops for instructors, and online working groups. The geometry for pre-service teachers' project is in its development phase and can only start to envision what its approach will be to implementation and dissemination, a challenge we all share in the mathematics education community.

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