

# SPECTRAL GAPS AND ERROR ESTIMATES FOR INFINITE-DIMENSIONAL METROPOLIS–HASTINGS WITH NON-GAUSSIAN PRIORS

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We study a class of Metropolis–Hastings algorithms for target measures that are absolutely continuous with respect to a large class of non-Gaussian prior measures on Banach spaces. The algorithm is shown to have a spectral gap in a Wasserstein-like semimetric weighted by a Lyapunov function. A number of error bounds are given for computationally tractable approximations of the algorithm including bounds on the closeness of Cesáro averages and other pathwise quantities via perturbation theory. Several applications illustrate the breadth of problems to which the results apply such as various likelihood approximations and perturbations of prior measures.

**1. Introduction.** The goal of this article is to study convergence rates and stability to perturbations of a class of Metropolis–Hastings (MH) algorithms for sampling target measures that are absolutely continuous with respect to an underlying non-Gaussian measure. Targets in this class naturally arise as posterior measures in Bayesian inverse problems with non-Gaussian priors. We show that under general conditions, the algorithms of interest to us have a dimension-independent spectral gap with respect to a transport semimetric on the space of probability measures. Furthermore, we present a general perturbation result stating that the invariant measure of the algorithm depends continuously on perturbations of the proposal kernel and acceptance ratio. We also give bounds on the closeness of Cesáro averages and other pathwise quantities from the perturbed transition kernel.

Let  $\mathcal{H}$  be a separable Banach space with norm  $\|\cdot\|$  and  $P(\mathcal{H})$  denote the space of Radon probability measures on  $\mathcal{H}$ , assigning measure 1 to the whole space. Consider  $\mu, \nu \in P(\mathcal{H})$  satisfying

$$(1) \quad \frac{d\nu}{d\mu}(u) = \frac{1}{Z} \exp(-\Psi(u)), \quad u \in \mathcal{H},$$

where  $\Psi : \mathcal{H} \mapsto \mathbb{R}$  is a measurable function and  $Z = \mu(\exp(-\Psi))$  is a normalizing constant, and for any measure  $\mu$  and function  $\varphi$ ,  $\mu(\varphi) := \int_{\mathcal{H}} \varphi(u) \mu(du)$ . In Bayesian inference, the measure  $\nu$  is precisely the posterior measure, which is absolutely continuous with respect to the prior measure  $\mu$ . In applications such as Bayesian inverse problems and uncertainty quantification, our goal is often to estimate integrals of the form  $\nu(\varphi)$  for a function of interest  $\varphi : \mathcal{H} \rightarrow \mathcal{X}$  where  $\mathcal{X}$  is a separable Hilbert space with norm  $\|\cdot\|_{\mathcal{X}}$ . Since this integral is often intractable we approximate it using  $n^{-1} \sum_{k=1}^n \varphi(U_k)$ , where the sequence  $\{U_k\}_{k=1}^n$  are distributed according to  $\nu$  as  $n \rightarrow \infty$ .

We are primarily interested in the setting where we cannot sample from  $\nu$  directly but we can sample from  $\mu$ . Then an algorithm is needed that can approximately sample  $\nu$ . A common approach constructs a Markov transition operator  $\mathcal{P}$  with invariant measure  $\nu$ , then collects

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Received May 2022.

*MSC2020 subject classifications.* 65C05, 60J05, 62G99, 62M40.

*Key words and phrases.* Markov chain Monte Carlo, spectral gap, non-Gaussian, Bayesian inverse problem, weak Harris' theorem.

paths  $U_k \sim \mathcal{P}^{k-1} \delta_{u_0}$  starting from a fixed initial condition  $U_0 = u_0$ . Because it is not possible to simulate paths numerically on an infinite-dimensional state space, in practice finite-dimensional approximations to the exact algorithm are used. A well-known example of such an algorithm is the preconditioned Crank–Nicholson (pCN) algorithm [6]; a MH algorithm for  $\mu$  that is absolutely continuous with respect to a Gaussian measure. In this article, we consider a generalization of the pCN algorithm, called the RCAR algorithm, that generalizes this assumption to non-Gaussian prior measures.

The remainder of this section is organized as follows: We recall the RCAR algorithm in Section 1.1 and give an overview of our main results in Section 1.2. Relevant literature to our work is discussed in Section 1.3 followed by a concrete running example in Section 1.4 which is used throughout the article to demonstrate our theoretical results and conditions in a practical setting. An outline of the article is given in Section 1.5.

**1.1. The RCAR algorithm.** The MH algorithm we study utilizes a proposal akin to a random coefficient autoregressive proposal (RCAR), defined as follows.

**DEFINITION 1 (RCAR-MH kernel).** Given a function  $\Psi : \mathcal{H} \mapsto \mathbb{R}$ , a transition kernel  $\mathcal{K}(u, \cdot)$ , and an innovation measure  $\lambda \in P(\mathcal{H})$ , the RCAR-MH transition kernel  $\mathcal{P}$  is defined as

$$(2) \quad \mathcal{P}(u, dv) := \mathcal{Q}(u, dv) \alpha(u, v) + \delta_u \int_{\mathcal{H}} (1 - \alpha(u, w)) \mathcal{Q}(u, dw), \quad u \in \mathcal{H},$$

with proposal transition kernel

$$(3) \quad \mathcal{Q}(u, \cdot) := \mathcal{K}(u, \cdot) * \lambda,$$

and acceptance ratio function

$$(4) \quad \alpha(u, v) := 1 \wedge \exp(\Psi(u) - \Psi(v)).$$

The RCAR-MH family of kernels defined above are commonly encountered in the design of MH algorithms. The form (2) is often referred to as the *lazy chain* representation of  $\mathcal{P}$ . The first term accounts for the proposal of a new point  $v \sim \mathcal{Q}(u, \cdot)$  that is then accepted with probability  $\alpha(u, v)$  and the chain moves from  $u$  to  $v$ . The second term accounts for the event where the proposed point  $v$  is rejected and the chain remains at  $u$ .

We summarize the RCAR algorithm in Algorithm 1 for reference. Many common MH algorithms such as the random walk (RW) algorithm [42] and pCN [6] fall within the RCAR-MH family. In both RW and pCN the measure  $\lambda$  is taken to be an appropriate Gaussian measure. The kernel  $\mathcal{K} = \delta_u$  for RW, while  $\mathcal{K} = \delta_{\beta u}$  for a constant  $\beta \in (0, 1)$  in the case of pCN.

Consider the measure  $\nu$  defined in (1) with  $\mu \in P(\mathcal{H})$ . It was shown in [23], Theorem 2.1, that under mild conditions on  $\Psi$ , the measure  $\nu$  is an invariant measure of  $\mathcal{P}$  provided that  $\mathcal{Q}$  is reversible with respect to  $\mu$ , that is,

$$(5) \quad \int_A \mathcal{Q}(u, B) \mu(du) = \int_B \mathcal{Q}(u, A) \mu(du),$$

for Borel sets  $A, B \in \mathcal{H}$ .

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**Algorithm 1** Generic RCAR-MH

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1. Set  $j = 0$  and choose  $U_0 \in \mathcal{H}$ .
  2. At iteration  $j$  propose  $W_{j+1} = \zeta_{j+1} + \xi_{j+1}$  where  $\zeta_{j+1} \sim \mathcal{K}(U_j, \cdot)$  and  $\xi_{j+1} \sim \lambda$ .
  3. Set  $U_{j+1} = W_{j+1}$  with probability  $\alpha(U_j, W_{j+1})$ .
  4. Otherwise set  $U_{j+1} = U_j$ .
  5. Set  $j \leftarrow j + 1$  and return to step 2.
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The article [23] presents multiple numerical experiments demonstrating the ability of RCAR to sample the target measures  $\nu$  that arise as posterior measures in Bayesian inverse problems with non-Gaussian priors. The RCAR algorithm is widely applicable since appropriate proposal kernels  $\mathcal{Q}$  can be identified for many commonly used probability distributions such as Gaussian, Laplace and Gamma (together with their extensions to infinite-dimensional measures). However, as yet no analysis of RCAR convergence rates and existence/uniqueness of invariant measures of  $\mathcal{P}$  has been performed. In general, ensuring that  $\mathcal{Q}$  is  $\mu$ -reversible depends on the choice of  $\lambda$  and  $\mathcal{K}$  in relation to  $\mu$  and is often the most difficult aspect of designing new MH algorithms, especially when  $\mathcal{H}$  is infinite-dimensional [6, 23]. However,  $\mu$ -reversibility of  $\mathcal{Q}$  only ensures that  $\nu$  is an invariant measure of  $\mathcal{P}$ . The primary goal of this article is to analyze the convergence properties of  $\mathcal{P}$ , showing the existence and uniqueness of an invariant measure to which convergence occurs at an exponential rate. We further justify the use of perturbed/finite-dimensional versions of the algorithm by providing general perturbation bounds.

**1.2. Overview of main results.** We now give a brief survey of our main results with simplified technical assumptions. Details of these results are presented in Sections 3 and 4. Let  $\mathcal{P}$  be an RCAR-MH kernel as in Definition 1 with a transition kernel  $\mathcal{K}$  and innovation measure  $\lambda$ . Our first result concerns the existence of a spectral gap for  $\mathcal{P}$  in certain semimetrics implying exponential convergence to a unique invariant measure; throughout we will use the term “spectral gap” in topologies other than  $L^2$ , consistent with [18, 19].

For  $q \geq 1$  and  $\eta, \omega, \theta > 0$  define the semimetric

$$(6) \quad \tilde{d}_q(u, v) := \left[ \left( 1 \wedge \frac{(1 + \eta\|u\| + \eta\|v\|)^q \|u - v\|}{\omega} \right) (2 + \theta\|u\|^q + \theta\|v\|^q) \right]^{1/2},$$

for points  $u, v \in \mathcal{H}$ . We refer to  $\tilde{d}_q$  as a semimetric since it does not satisfy the triangle inequality but satisfies other metric axioms. This semimetric further induces a transport semimetric

$$(7) \quad \tilde{d}_q(\nu_1, \nu_2) := \inf_{\pi \in \Upsilon(\nu_1, \nu_2)} \int_{\mathcal{H} \times \mathcal{H}} \tilde{d}_q(u, v) \pi(du, dv) \quad \forall \nu_1, \nu_2 \in P(\mathcal{H}),$$

where  $\Upsilon(\nu_1, \nu_2)$  denotes the space of all couplings between probability measures  $\nu_1, \nu_2$ . We let  $P^1(\mathcal{H}; \tilde{d}_q) \subset P(\mathcal{H})$  denote the subspace of probability measures on  $\mathcal{H}$  for which  $\tilde{d}_q(\cdot, 0)$  is integrable. Then our first main result states that the RCAR-MH kernel has a unique invariant measure to which exponential convergence occurs in  $\tilde{d}_q$ .

**MAIN RESULT 1.** *Suppose  $\lambda$  has bounded moments of degree  $p \geq 1$  and the Lipschitz constant of  $\Psi$  does not grow faster than  $\|\cdot\|^q$  for some integer  $q \leq p$ . Then under regularity conditions on  $\mathcal{K}$  and for an appropriate choice of the constants  $\eta, \omega, \theta$  it holds that:*

(a) *There exist constants  $(\gamma, n) \in (0, 1) \times \mathbb{N}$  so that*

$$(8) \quad \tilde{d}_q(\mathcal{P}^n \nu_1, \mathcal{P}^n \nu_2) \leq \gamma \tilde{d}_q(\nu_1, \nu_2) \quad \forall \nu_1, \nu_2 \in P^1(\mathcal{H}; \tilde{d}_q).$$

(b)  *$\mathcal{P}$  has a unique invariant measure  $\nu \in P^1(\mathcal{H}; \tilde{d}_q)$ .*

(c) *If  $\mathcal{Q}$  is  $\mu$ -reversible then  $\nu$  coincides with the target measure (1).*

Detailed statement and proof of this result is presented in Section 3 where we give a detailed statement of the underlying assumptions on  $\Psi$  and  $\mathcal{K}$  required to prove the three statements as well as the detailed versions of these results. The proofs are further postponed to Appendix A.

It was shown in [23] that the RCAR algorithm satisfies detailed balance whenever (5) holds and so has unique invariant measure  $\nu$  given by (1). In Section 3.2 we present an alternative proof of the fact that  $\mathcal{P}$  has a unique invariant measure by showing that  $\mathcal{P}$  is Feller, implying that  $\mathcal{P}$  has a unique invariant measure under more general conditions than (5). However, without (5) one cannot guarantee that the invariant measure is the target  $\nu$  in (1); see also Remark 3.

Our second main result concerns the perturbation properties of RCAR-MH kernels. In many applications, such as when  $\mathcal{H}$  is a function space, the RCAR algorithm cannot be implemented exactly since it is not possible to simulate  $\mathcal{K}$  and  $\lambda$ , and one resorts to numerical approximations by discretization or direct approximations of  $\Psi$ ,  $\mathcal{K}$  or  $\lambda$ . To this end, we provide bounds on the approximation error resulting from using perturbations  $\mathcal{P}_\varepsilon$  of an RCAR-MH kernel  $\mathcal{P}$ . We characterize closeness of the invariant measure(s) of  $\mathcal{P}_\varepsilon$  to  $\nu$ , as well as the similarity of the dynamics of  $U_k \sim \mathcal{P}^k \delta_{u_0}$  to  $U_k^\varepsilon \sim \mathcal{P}_\varepsilon^k \delta_{u_0}$  for  $u_0 \in \mathcal{H}$ . We emphasize that while the main result below is stated for RCAR-MH kernels, the results we prove in Section 4 are indeed more general and are applicable to any  $\mathcal{P}$  that satisfies the conditions of the weak Harris' theorem (see Proposition 1) below. Before proceeding further let us recall the Lipschitz seminorm with respect to the semimetric  $\tilde{d}_q$  on (Bochner) measurable functions  $\varphi : \mathcal{H} \rightarrow \mathcal{X}$  for a separable Hilbert space  $\mathcal{X}$ :

$$(9) \quad \|\varphi\|_{\tilde{d}_q} := \sup_{u \neq v} \frac{\|\varphi(u) - \varphi(v)\|_{\mathcal{X}}}{\tilde{d}_q(u, v)}.$$

**MAIN RESULT 2.** *Suppose that the conditions of Main Result 1 hold and let  $\mathcal{P}_\varepsilon$  be a Markov transition kernel on  $\mathcal{H}$ . Suppose that  $\|\cdot\|^q$  is a common Lyapunov function (see Definition 3) for  $\mathcal{P}$  and  $\mathcal{P}_\varepsilon$  for sufficiently small  $\varepsilon$  so that*

$$\mathcal{P}\|u\|^q \leq \kappa\|u\|^q + K, \quad \mathcal{P}_\varepsilon\|u\|^q \leq \kappa\|u\|^q + K,$$

for constants  $(\kappa, K) \in (0, 1) \times (0, +\infty)$ , and that there exists a bounded function  $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$  for which

$$\tilde{d}_q(\mathcal{P}_\varepsilon \delta_u, \mathcal{P} \delta_u) \leq \psi(\varepsilon)(1 + \|u\|^{q/2}).$$

(a) *Then there exists a constant  $C_1 > 0$  independent of  $\varepsilon > 0$  so that*

$$\tilde{d}_q(\nu, \nu_\varepsilon) \leq C_1 \psi(\varepsilon)[1 + \nu_\varepsilon(\|\cdot\|^{q/2})],$$

where  $\nu$  is the unique invariant measure of  $\mathcal{P}$  and  $\nu_\varepsilon$  is any invariant measure of  $\mathcal{P}_\varepsilon$ .

(b) *Let  $\mathcal{X}$  be a separable Hilbert space with norm  $\|\cdot\|_{\mathcal{X}}$  and  $\varphi : \mathcal{H} \mapsto \mathcal{X}$  be  $\nu$ -Bochner measurable and satisfy  $\|\varphi\|_{\tilde{d}_q} < +\infty$ . Then there exist constants  $C_j \geq 0$ ,  $j = 2, \dots, 4$ , independent of  $n \geq 2$  and  $\varepsilon > 0$ , such that*

$$\mathbb{E} \left\| \frac{1}{n} \sum_{k=0}^{n-1} \varphi(U_k^\varepsilon) - \nu(\varphi) \right\|_{\mathcal{X}} \leq \frac{\|\varphi\|_{\tilde{d}_q}}{1 - \gamma} \left( C_2 \psi(\varepsilon) + C_3 \frac{\psi(\varepsilon)}{n} + C_4 \frac{1}{\sqrt{n}} \right),$$

where  $U_k^\varepsilon \sim \mathcal{P}_\varepsilon^{k-1} \delta_{u_0}$  for any initial state  $u_0 \in \mathcal{H}$ .

We present detailed versions of the above statements together with our underlying assumptions on  $\mathcal{P}$ ,  $\mathcal{P}_\varepsilon$  in Section 4. Detailed proofs of those results are postponed to Appendix B.

1.3. *Relevant literature.* The convergence rate results we give here rely on the existence of Lyapunov functions of  $\mathcal{P}$  and  $\mathcal{P}_\varepsilon$  to control stochastic stability. The use of Lyapunov functions has been important at least since [21], and their application to convergence analysis of Markov chains is developed in great detail in the influential text of Meyn and Tweedie [36]; see also the more recent text of Douc et al. [14]. In the Markov chain Monte Carlo (MCMC) literature, convergence is often studied by showing a form of Harris' classic theorem [20], which states that a Markov chain is uniquely ergodic if there exists a set satisfying an analogue of Doeblin's condition, perhaps holding only for the  $n$ -step kernel  $\mathcal{P}^n$  for some  $n < +\infty$ , that is visited infinitely often. One typically proves Harris' result by showing a minorization condition for  $\mathcal{P}^n$  on sublevel sets of the Lyapunov function [44]; an elementary proof can be found in [17]. Example applications of such "drift and minorization" arguments to MH algorithms can be found in [25, 35, 43].

Proofs of Harris' theorem utilizing a Lyapunov condition typically guarantee exponential convergence toward the unique invariant measure in a total variation (TV) metric weighted by the Lyapunov function [17]. When the state space is high or infinite-dimensional, such TV metrics are a poor choice because probability measures on infinite-dimensional spaces have a tendency to become mutually singular after small perturbations [4]. Due to this phenomenon it is typically not possible to couple two copies of a Markov chain such that they move to exactly the same point with positive probability, even over multiple steps. However, for measures on Banach spaces one can typically show a topological irreducibility condition, that is, that the two copies draw together over time in an appropriate (semi)metric, at least when initialized inside of sublevel sets of a Lyapunov function.

We study convergence of the RCAR algorithm on infinite-dimensional Banach spaces using the "weak Harris" theorem of Hairer et al. [18]. This can be viewed as an extension of the ordinary Harris theorem to transport semimetrics. These semimetrics are designed to induce a topology on bounded sets such that the topological irreducibility condition holds. An application of the weak Harris' theorem to the pCN algorithm can be found in [19], wherein it is proved that pCN has a dimension-independent spectral gap. As the pCN algorithm is a special case of RCAR with a Gaussian innovation  $\lambda$  and a deterministic kernel  $\mathcal{K}(u, \cdot) = \delta_{\beta u}$ , our results for dimension-independent spectral gap of RCAR can be viewed as a generalization of [19].

Our approximation theory on the other hand is inherently different from [19]. Rather than showing analogous spectral gap results for discretizations of the algorithm and then showing that the invariant measures are close as in [8, 33], we instead utilize perturbation bounds as in [27] to bound the distance between the invariant measures by the  $n$ -step approximation error between the exact kernel  $\mathcal{P}$  and the approximation  $\mathcal{P}_\varepsilon$ . Perturbation theory for MCMC is also studied in [37, 38, 45], but these results are not well suited to our infinite-dimensional state space setting, since they require the triangle inequality which is typically not satisfied by the transport semimetrics that we work with. The perturbation bounds we obtain have the advantage of controlling all of the quantities of interest in terms of the spectral gap of the kernel  $\mathcal{P}$  and a pointwise bound on the approximation error of the approximate kernel  $\mathcal{P}_\varepsilon$ . We further apply these results to cases where the innovation  $\lambda$  or the kernel  $\mathcal{K}$  cannot be exactly simulated, using arguments similar in spirit to those used to prove convergence rates for discretization of MH proposal kernels in [27] but technically much more involved. Rather than changing norms to  $L_2(\nu)$  to obtain a central limit theorem, we proceed in the tradition of [16, 30, 31, 34] and give variation bounds using the Poisson equation. This gives approximation error bounds to  $\nu$ , as well as approximation error bounds for pathwise quantities for both  $\mathcal{P}$  and  $\mathcal{P}_\varepsilon$  for elements of the function space  $\{\varphi : \|\varphi\|_{\tilde{d}_q} < +\infty\}$ , without requiring any direct analysis of  $\mathcal{P}_\varepsilon$  or  $\nu_\varepsilon$ . This technique is related to classical Martingale and potential methods [41].

We highlight that our purpose here is to show error bounds that are independent of dimension and allow us to obtain rates for the error in quantities of interest, not to produce quantitative estimates of the number of steps necessary to achieve a particular accuracy. All of the bounds we give for approximate versions of the algorithm depend only on the spectral gap of the exact kernel  $\mathcal{P}$ , and the pointwise accuracy of the approximate kernel  $\mathcal{P}_\varepsilon$  as well as the constants in its Foster–Lyapunov condition. While the former is independent of dimension, the latter two quantities relating to  $\mathcal{P}_\varepsilon$  typically improve as  $\varepsilon \rightarrow 0$  and  $\mathcal{P}_\varepsilon$  draws closer to  $\mathcal{P}$ . This behavior contrasts with the typical performance of “drift and minorization” bounds in weighted TV norms for finite-dimensional problems where the spectral gap tends to vanish as dimension increases (see e.g., [40]). In a sense, the dimension-independence of our results can be attributed to choosing a semimetric that is better adapted to high or infinite-dimensional spaces than weighted TV.

It is worth noting that one can typically obtain sharper numerical estimates of mixing or relaxation times using geometric inequalities, such as log-Sobolev, Cheeger and Poincaré inequalities. A thorough review of these techniques and their application to MH algorithms is given in [13]. More recent work applying geometric inequalities to obtain sharp bounds for mixing times of MH on bounded subsets of  $\mathbb{R}^d$  can be found in [11, 12]. Geometric inequalities are combined with Lyapunov arguments to obtain sharper estimates of relaxation times for MH on  $\mathbb{R}$  in [26]. At the time of this writing, we are not aware of analogous results for infinite-dimensional MH.

**1.4. An illustrative example in nonlinear regression.** We now outline the details of a running example that is used throughout the article to give context to the main ideas and assumptions in our analysis. The motivation for this example is the semi-supervised regression (SSR) problem [2, 15]; the task of inferring a function on a graph from indirect and limited observations of its values on a subset of the nodes. In the large graph limit, as the number of vertices tend to infinity, the SSR problem converges to a nonlinear regression problem where a nonlinear transformation of a latent function is observed at a few points and the goal is to recover the latent function.

**EXAMPLE 1 (Nonlinear regression).** Let  $\mathbb{T}$  be the unit circle and  $\mathcal{H} = H^1(\mathbb{T})$  the Sobolev space of weakly differentiable functions on the unit circle with square integrable first derivatives. Suppose  $u^\dagger \in H^1(\mathbb{T})$  is the *ground truth* function from which the following data is measured:

$$y \in \mathbb{R}^m, \quad y_j = \tanh(u^\dagger(x_j)) + \epsilon_j.$$

Here  $\{x_j\}_{j=1}^m$  are fixed points in  $\mathbb{T}$  and the  $\epsilon_j \stackrel{\text{i.i.d.}}{\sim} N(0, \sigma^2)$  with variance  $\sigma > 0$ . Since  $H^1(\mathbb{T})$  is embedded in  $C(\mathbb{T})$  by the Sobolev embedding theorem [1], then the pointwise evaluation of  $u^\dagger$  is well defined.

Now consider the inverse problem of inferring the function  $u^\dagger$  from an instance of the data  $y$ . To solve the problem we write Bayes’ rule [47] in the following form:

$$(10) \quad \frac{dv}{d\mu}(u) = \frac{1}{Z(y)} \exp(-\Psi(u; y)), \quad Z(y) = \int_{L^2(\mathbb{T})} \exp(-\Psi(u; y)) \mu(du),$$

where  $\Psi$  is the likelihood potential,  $\mu$  is the prior probability measure and  $v$  is the posterior measure. Since the  $\epsilon_j$  are Gaussian we ascertain that the likelihood potential  $\Psi(u; y)$  is given by

$$\Psi(u; y) = \frac{1}{2\sigma^2} \sum_{j=1}^m |\tanh(u(x_j)) - y_j|^2.$$

As for the prior measure  $\mu$  we take

$$(11) \quad \mu = \text{Law} \left\{ \sum_{j=1}^{\infty} a_j \eta_j \phi_j \right\},$$

where  $\{\eta_j\}_{j=1}^{\infty} \stackrel{\text{i.i.d.}}{\sim} \text{Gamma}(1/5, 1)$  random variables with Lebesgue density  $f(t) = \frac{1}{\Gamma(1/5)} \times t^{1/5-1} \exp(-t) \mathbf{1}_{(0, \infty)}(t)$  for  $t \in \mathbb{R}$ . The  $\phi_j$  are the DB12 wavelet basis [10] normalized in  $L^2(\mathbb{T})$  with scaling function  $\phi_0$  and

$$\phi_{2^k+m}(t) = 2^{k/2} \phi(2^k t - m_k), \quad k = 1, 2, \dots, m_k = 0, 1, 2, \dots, 2^k - 1,$$

with  $\phi$  denoting the DB12 mother wavelet. Finally, the coefficients  $\{a_j\}_{j=1}^{\infty}$  are chosen as

$$a_1 = 1 \quad \text{and} \quad a_{2^k+m_k} = 2^{-2k}.$$

Our choice of the  $a_j$  and the laws of  $\eta_j$  together with the regularity of the DB12 wavelets ensure that  $\mu$  has full support on the subspace of  $H^1(\mathbb{T})$  consisting of functions with positive wavelet coefficients.

In order to sample the resulting posterior we employ [23], Algorithm 4, an instance of the RCAR algorithm for the prior  $\mu$ . For a fixed  $\beta \in (0, 1)$  we take

$$\mathcal{K}(u, \cdot) = \text{Law} \left\{ \sum_{j=1}^{\infty} \tau_j \langle u, \phi_j \rangle_{L^2(\mathbb{T})} \phi_j \right\}, \quad \{\tau_j\}_{j=1}^{\infty} \stackrel{\text{i.i.d.}}{\sim} \text{Beta}(\beta/5, (1-\beta)/5),$$

with  $\langle \cdot, \cdot \rangle_{L^2(\mathbb{T})}$  denoting the  $L^2(\mathbb{T})$ -inner product. We then define the innovation  $\lambda$  as

$$(12) \quad \lambda = \text{Law} \left\{ \sum_{j=1}^{\infty} a_j \xi_j \phi_j \right\}, \quad \xi_j \stackrel{\text{i.i.d.}}{\sim} \text{Gamma}((1-\beta)/5, 1).$$

It then follows from [23], Theorem 3.4, that the resulting proposal kernel  $\mathcal{Q}$  as in (3) is  $\mu$ -reversible, implying that the RCAR-MH kernel  $\mathcal{P}$  is  $\nu$ -reversible in this example.

Figure 1 depicts an example application of the RCAR-MH algorithm described above for recovering a function  $u^\dagger$  with sparse and positive wavelet coefficients; the details of this experiment are summarized in Section 5.1. Here we truncate the infinite sum in (11) up to  $N$  terms. Figure 1(a) shows the ground truth function  $u^\dagger$  together with the measurements  $y$  and the resulting posterior mean obtained from RCAR-MH samples with  $N = 128$  wavelet modes. Figure 1(b) shows the average MH acceptance ratio as a function of the step size

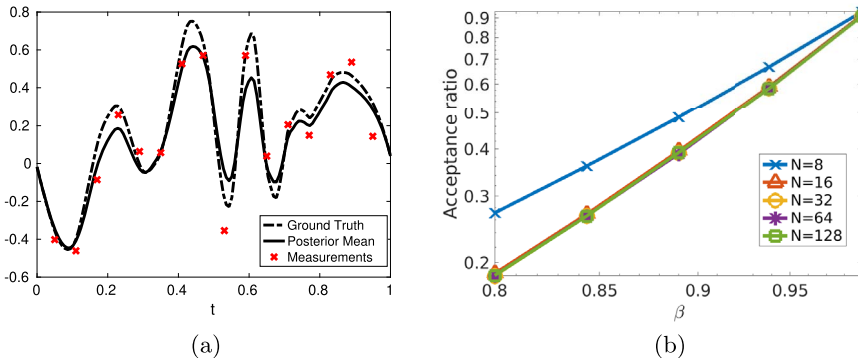


FIG. 1. Representative numerical results for Example 1. (a) The ground truth function  $u^\dagger$  together with the measurements  $y$  and the RCAR-MH estimate of the posterior mean. (b) The acceptance ratio of RCAR-MH as a function of the parameter  $\beta$  for different number of wavelet modes  $N$ .

parameter  $\beta$  for various choices of  $N$  (the dimension of the inference parameter). The independence of the acceptance ratio from the dimension  $N$  is a telltale sign of the dimension-independent convergence properties of RCAR-MH. Similar behavior was also observed in the numerical experiments considered in [23].

**1.5. Outline of the article.** We dedicate Section 2 to preliminary results and definitions that are used throughout the article and fix our notation. In Section 3 we give results pertaining to the convergence properties of RCAR-MH kernels which together constitute the detailed version of Main Result 1. Analogously Section 4 contains our main perturbation results for Markov kernels that satisfy the conditions of weak Harris’ theorem constituting the detailed statement of Main Result 2. We consider several applications of our results in Section 5, and dedicate Section 6 to conclusion and offer some thoughts on future directions. Appendices A–C contains the technical proofs of key results that are not included in the main text.

**2. Preliminaries.** We gather here some preliminary results on ergodic theorems and the weak Harris’ theorem as well as some notation and terminology that is used throughout the article. Results on ergodicity, most notably the weak Harris’ theorem are reviewed in Section 2.1 while further notation is outlined in Section 2.2.

**2.1. Results on ergodicity.** We study convergence in the context of the *weak Harris* theorem of [18], which is an extension of the classical Harris’ theorem to Wasserstein-type notions of distance defined in terms of lower semi-continuous semimetrics referred to as “distance-like” in [18].

**DEFINITION 2.** A function  $d : \mathcal{H} \times \mathcal{H} \mapsto \mathbb{R}$  is *distance-like* if it is positive, symmetric, lower semi-continuous and  $d(u, v) = 0$  iff  $u = v$ .

Nonnegative functions that satisfy all of the metric axioms save the triangle inequality are often referred to as *semimetrics*, and we also adopt this terminology. Thus, a distance-like function is a lower semi-continuous semimetric. Given a distance-like function  $d$ , we can extend it to a Wasserstein or transport-like positive function on  $P(\mathcal{H})$  via

$$(13) \quad d(\mu_1, \mu_2) := \inf_{\pi \in \Upsilon(\mu_1, \mu_2)} \int_{\mathcal{H} \times \mathcal{H}} d(u, v) \pi(du, dv),$$

where we recall  $\Upsilon(\mu_1, \mu_2)$  is the space of all couplings of  $\mu_1$  and  $\mu_2$ , i.e., the space of measures  $\pi \in P(\mathcal{H} \times \mathcal{H})$  whose marginals on the first and second variables coincide with  $\mu_1$  and  $\mu_2$  respectively. We also introduce the subspace  $P^1(\mathcal{H}; d) \subset P(\mathcal{H})$  as

$$(14) \quad P^1(\mathcal{H}, d) := \left\{ \mu \in P(\mathcal{H}) \mid \int_{\mathcal{H}} d(u, 0) \mu(du) < +\infty \right\},$$

following the standard notation for Wasserstein topologies [50].

Given a distance-like function  $d$  we also introduce the space  $\text{Lip}(d)$  consisting of functions that are Lipschitz continuous with respect to  $d$ . More precisely, for a separable Hilbert space  $\mathcal{X}$  with norm  $\|\cdot\|_{\mathcal{X}}$ , define

$$(15) \quad \text{Lip}(d) := \{ \varphi : \mathcal{H} \mapsto \mathcal{X} : \|\varphi\|_d < +\infty \},$$

where

$$(16) \quad \|\varphi\|_d := \sup_{u \neq v} \frac{\|\varphi(u) - \varphi(v)\|_{\mathcal{X}}}{d(u, v)}.$$

Observe that the definition in (9) is just a specific example of (16) with the choice of  $d = \tilde{d}_q$  defined in (6). We now define some properties of  $\mathcal{P}$  that together give the weak Harris theorem.

DEFINITION 3. A function  $V : \mathcal{H} \mapsto \mathbb{R}$  is a *Lyapunov* function for a Markov transition kernel  $\mathcal{P}$  if there exist  $(\kappa, K) \in (0, 1) \times (0, +\infty)$  so that

$$(\mathcal{P}V)(u) \leq \kappa V(u) + K \quad \forall u \in \mathcal{H}.$$

Lyapunov functions are a standard way to control tail behavior of  $\mathcal{P}$ . We further require that when initiated from two  $d$ -nearby points, we can couple two copies of the Markov chain evolving according to  $\mathcal{P}$  such that they draw together in one step.

DEFINITION 4. A distance-like function  $d : \mathcal{H} \times \mathcal{H} \rightarrow [0, 1]$  is contracting for a Markov operator  $\mathcal{P}$  if there exists  $\gamma_1 \in (0, 1)$  so that

$$d(\mathcal{P}\delta_u, \mathcal{P}\delta_v) \leq \gamma_1 d(u, v) \quad \text{whenever } d(u, v) < 1.$$

The assumption that  $d$  is capped at 1 is entirely innocuous; for details see [18], Remark 4.7. Finally, we will need a type of topological irreducibility on sublevel sets of  $V$  reminiscent of Doeblin's condition [36], Section 16.2.1, in the classical TV theory of convergence.

DEFINITION 5. For every  $R > 0$  the sublevel sets  $S(R) := \{u \mid V(u) < R\}$  of  $V$  are  $d$ -small for a distance-like function  $d : \mathcal{H} \times \mathcal{H} \rightarrow [0, 1]$  if there exists  $\gamma_2(R) \in (0, 1)$  and  $n \in \mathbb{N}$  so that,

$$\sup_{u, v \in S(R)} d(\mathcal{P}^n \delta_u, \mathcal{P}^n \delta_v) \leq \gamma_2(R).$$

In some cases it is possible to show that Definitions 4 and 5 hold with  $n = 1$ , but it is imperative for our technical results to use Definition 5 with a sufficiently large  $n$ . Given a distance-like function  $d$ , we define a new weighted distance-like function

$$(17) \quad \tilde{d}(u, v) := [d(u, v)(2 + \theta V(u) + \theta V(v))]^{\frac{1}{2}},$$

for a parameter  $\theta > 0$ . Observe that the  $\tilde{d}_q$  semimetric introduced in (6) is a particular example of (17) with  $V(u) = \|u\|^q$  and  $d(u, v) = 1 \wedge \omega^{-1}(1 + \eta\|u\| + \eta\|v\|)^q \|u - v\|$ . Once again we use the same notation to denote the induced semimetric  $\tilde{d}$  on  $P(\mathcal{H})$  and in turn the subspace  $P^1(\mathcal{H}; \tilde{d})$  as in (14). The following is a discrete-time version of [18], Theorem 4.8, that is more natural for our setting. Although [27] shows this condition in the case where  $n = 1$  in Definition 5, the extension to general  $n$  is a minor modification of their argument; and indeed, of the continuous-time result in [18].

PROPOSITION 1 ([18], Theorem 4.8 and [27], Theorem 3.9). *Suppose  $d$  is contracting for  $\mathcal{P}$ , and  $\mathcal{P}$  has a continuous Lyapunov function  $V$  with  $d$ -small level sets. Then there exist constants  $(\gamma, \theta, n) \in (0, 1) \times (0, +\infty) \times \mathbb{N}$  such that*

$$\tilde{d}(\mathcal{P}^n \delta_u, \mathcal{P}^n \delta_v) \leq \gamma \tilde{d}(u, v) \quad \forall u, v \in \mathcal{H}.$$

*We refer to the constant  $1 - \gamma$  as the  $\tilde{d}$ -spectral gap of  $\mathcal{P}$ .*

Next we recall the definition of a Feller Markov operator. Let  $C(\mathcal{H})$  and  $C_b(\mathcal{H})$  denote the spaces of continuous functions and continuous and bounded functions on  $\mathcal{H}$  respectively.

DEFINITION 6. A Markov operator  $\mathcal{P}$  is Feller if  $\mathcal{P}\varphi \in C(\mathcal{H})$  for every  $\varphi \in C_b(\mathcal{H})$ . That is,  $\mathcal{P}$  is Feller if and only if  $u \mapsto \mathcal{P}(u, \cdot)$  is continuous in the topology of weak convergence.

By [18], Corollary 4.11, if  $\mathcal{P}$  is Feller and the distance-like function  $d$  satisfies some mild conditions, Proposition 1 also implies the existence of a unique invariant measure for  $\mathcal{P}$ .

**2.2. Notation.** We gather some notation for future reference. Throughout Sections 3  $\mathcal{H}$  is a separable Banach space with norm  $\|\cdot\|$ . In the applications in Section 5, we take  $\mathcal{H}$  to be a separable Hilbert space with norm  $\|\cdot\|$  and inner product  $\langle \cdot, \cdot \rangle$ . In either case  $\mathcal{H}^*$  denotes the dual of  $\mathcal{H}$ . Throughout,  $B_R(v) \subset \mathcal{H}$  denotes the closed ball in the topology of  $\|\cdot\|$  of radius  $R > 0$  centered at  $v$ . We say that a measure  $\mu \in P(\mathcal{H})$  has bounded moments of degree  $p$  whenever  $\|\cdot\|^p \in L^1(\mu)$ . At various points we also consider a second separable Hilbert space  $\mathcal{X}$  with norm  $\|\cdot\|_{\mathcal{X}}$  and inner product  $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ . Furthermore, we often use the notation  $\mu(\varphi)$  to denote the integral  $\int_{\mathcal{H}} \varphi(u) \mu(du)$  which is understood in the Bochner sense whenever  $\varphi : \mathcal{H} \rightarrow \mathcal{X}$ . We also use the standard notation  $\mathbb{E}\xi$  to denote expectation of a random variable  $\xi$  and  $\mathbb{P}(A)$  to denote the probability of an event  $A$ . The measure with respect to which this probability is computed will be clear from context.

**3. Convergence theory for RCAR.** In this section we gather our main theoretical results pertaining to the convergence properties of RCAR-MH kernels as in Definition 1, constituting a detailed version of Main Result 1. We gather our main assumptions on the potential  $\Psi$  and the kernel  $\mathcal{K}$  in Section 3.1, using Example 1 throughout to give context to our assumptions. We then outline Theorems 1, 2 and 3 which, in turn, identify a Lyapunov function for  $\mathcal{P}$ , show that  $\mathcal{P}$  is contracting for an appropriate semimetric, and that  $\mathcal{P}$  has a unique invariant measure. We postpone the proofs of these theorems to Appendix A and only summarize the important details and implications of our results.

**3.1. Assumptions on the potential  $\Psi$  and the kernel  $\mathcal{K}$ .** Following Definition 1 the RCAR-MH kernel  $\mathcal{P}$  requires three main ingredients: the potential function  $\Psi$ , which is used to define the acceptance ratio function  $\alpha$ ; the kernel  $\mathcal{K}$ ; and, the innovation measure  $\lambda$ . In order to prove Main Result 1 we need the function  $\Psi$  and the kernel  $\mathcal{K}$  to satisfy certain regularity and growth assumptions. As we will discuss below and also in Section 5.3, some of these conditions are rather strict and technical. Our only requirement for the innovation measure  $\lambda$  is that it has moments of degree  $p \geq 1$ .

**ASSUMPTION 1.** The function  $\Psi : \mathcal{H} \mapsto \mathbb{R}$  satisfies one or more of the following conditions:

(a) (locally bounded from above) For every  $R > 0$  there exists a constant  $M_1(R) \geq 0$  so that

$$\Psi(u) \leq M_1 \quad \forall u \in B_R(0).$$

(b) (locally bounded from below) There exist constants  $q, M_2, M_3 \geq 0$  so that

$$\Psi(u) \geq M_3 - M_2 \log(1 + \|u\|^q) \quad \forall u \in \mathcal{H}.$$

(c) (increasing in the tail) For every  $\tilde{\beta}, \tilde{b} \in (0, 1)$  there exist strictly positive constants  $R_0(\tilde{\beta}, \tilde{b}), M_4(\tilde{\beta}, \tilde{b}) > 0$  so that  $\forall u \in B_{R_0}(0)^c$

$$\inf_{v \in B_{\tilde{b}(1-\tilde{\beta})\|u\|}(\tilde{\beta}u)} \exp(\Psi(u) - \Psi(v)) \geq M_4.$$

(d) (locally Lipschitz) There exist constants  $L > 0$  and  $q \geq 0$  so that

$$|\Psi(u) - \Psi(v)| \leq L(1 \vee \|u\|^q \vee \|v\|^q) \|u - v\| \quad \forall u, v \in \mathcal{H}.$$

These assumptions are morally equivalent to the assumptions on the potential in [19]. Assumption 1(a) and (b) ensure that whenever  $\mu$  has bounded moments of degree  $p \geq 1$

then the measure  $\nu$  as in (1) is well defined for any  $q \leq p$  [22], Theorem 4.3. In the context of Bayesian inverse problems the function  $\Psi$  is a negative log-likelihood function and often these assumptions are easily satisfied. For example, for additive noise models [22], Section 4.1, the constants  $M_3$  and  $M_2$  can be taken as zero and Assumption 1(a) can be verified so long as the forward map is bounded. Assumption 1(d) is a regularity condition controlling the rate at which the Lipschitz constant of  $\Psi$  can grow. This condition is also commonly encountered in the literature on Bayesian inverse problems and can be verified in many applications [22].

Assumption 1(c), however, is not common and amounts to  $\Psi$  being an increasing function in the tails which, as discussed in Section 5.3, may not hold for many benchmark problems in statistics and inverse problems. Although a simple workaround can be devised using our perturbation theory in Section 4. Hence, replacing Assumption 1(c) with a weaker assumption could be an interesting generalization of the current work that is highly relevant to applications. A slightly different version of this assumption also appears in [19], where the radius of the ball over which the infimum is computed is left as a general function  $r(\|u\|)$ . In the special case of the pCN algorithm one can simply choose the constant function  $r(\|u\|) = c$ . However, our analysis reveals that for RCAR algorithms with non-Gaussian priors one really needs the radius to grow with  $\|u\|$ . Thus we explicitly state the assumption in this way. Simply put, the reason is that RCAR proposals concentrate less strongly around the point  $\tilde{\beta}\|u\|$  than the Gaussian proposals in pCN, making it more difficult to prove contractive properties of  $\mathcal{P}$ .

Next we collect a set of assumptions on the kernel  $\mathcal{K}$ .

**ASSUMPTION 2.** Consider the Markov transition kernel  $\mathcal{K}$  and let  $\zeta_u \sim \mathcal{K}(u, \cdot)$ . Then one or more of the following conditions hold:

- (a) (almost sure contraction)  $\|\zeta_u\| < \|u\|$  a.s.  $\forall u \in \mathcal{H}$ .
- (b) (local concentration) There exist constants  $0 < b_0 < \beta_0 < 1$  and  $\epsilon_0 > 0$  so that

$$\mathbb{P}[\|\zeta_u - \beta_0 u\| \leq b_0(1 - \beta_0)\|u\|] \geq \epsilon_0 \quad \forall u \in \mathcal{H}.$$

- (c) (contracting couplings) For all  $u, v \in \mathcal{H}$  there exists a coupling  $\varpi_{u,v} \in \Upsilon(\mathcal{K}\delta_u, \mathcal{K}\delta_v)$  so that

$$\|\zeta_u - \zeta_v\| < \|u - v\| \quad \text{a.s. for } (\zeta_u, \zeta_v) \sim \varpi_{u,v},$$

and there exists a uniform constant  $\beta_c \in (0, 1)$  so that

$$\sup_{u, v \in \mathcal{H}, u \neq v} \frac{\int_{\mathcal{H} \times \mathcal{H}} \|\zeta_u - \zeta_v\| \varpi_{u,v}(d\zeta_u, d\zeta_v)}{\|u - v\|} \leq \beta_c.$$

Unlike the assumptions on  $\Psi$ , our assumptions on  $\mathcal{K}$  do not have an analogue in [19] since our class of algorithms is considerably more general than pCN. Assumption 2(a) requires  $\mathcal{K}(u, \cdot)$  to behave like a random linear operator (matrix) that shrinks and possibly rotates the vector  $u$ . Assumption 2(b) ensures that  $\mathcal{K}(u, \cdot)$  dedicates positive probability mass to a neighborhood of a point  $\beta_0 u$  for some  $\beta_0 < 1$ ; ensuring that  $\zeta_u$  can get sufficiently close to  $\beta_0 u$ . Assumption 2(c) is perhaps the most consequential due to the fact that our technical arguments rely on couplings between measures  $\mathcal{K}\delta_u$  and  $\mathcal{K}\delta_v$  such that they contract in one step. This contractive property is important for handling several technical difficulties that arise in proving the  $d$ -contraction and  $d$ -smallness conditions in the weak Harris' theorem (Proposition 1). In the pCN algorithm,  $\mathcal{K}_\beta(u, \cdot) = \delta_{\beta u}$  is just a delta measure at  $\beta u$  for  $\beta < 1$ , so the existence of this coupling is trivial, unlike the RCAR algorithm in general. Let us return to Example 1 and verify that Assumptions 1 and 2 hold for the RCAR algorithm presented in that example.

EXAMPLE 1 (Continued). As  $\tanh$  is smooth, globally Lipschitz and bounded, and  $\Psi$  is quadratic, we immediately have that Assumption 1(a), (b) and (d) hold; in fact,  $\Psi$  is globally Lipschitz. It remains to check condition (c). For a fixed  $u$  let  $v \in B_{\tilde{b}(1-\tilde{\beta})\|u\|_{H^1(\mathbb{T})}}(\tilde{\beta}u)$  for  $\tilde{\beta}, \tilde{b} \in (0, 1)$ , where we used  $\|\cdot\|_{H^1(\mathbb{T})}$  to denote the  $H^1(\mathbb{T})$  Sobolev norm. Using the fact that  $\tanh(\cdot) \in (-1, 1)$  we can write

$$\begin{aligned} & 2\sigma^2(\Psi(u) - \Psi(v)) \\ &= \sum_{j=1}^m |\tanh(hu(x_j))|^2 - |\tanh(hv(x_j))|^2 \\ &\quad - 2y_j[\tanh(hu(x_j)) - \tanh(hv(x_j))] \geq -4 \sum_{j=1}^m |y_j| - m. \end{aligned}$$

Thus, Assumption 1(c) holds with  $M_4 = \exp(-4\|y\|_1 - m)$ .

Now recall the kernel  $\mathcal{K}$  given by

$$\mathcal{K}(u, \cdot) = \text{Law} \left\{ \sum_{j=1}^{\infty} \tau_j \langle u, \phi_j \rangle_{L^2(\mathbb{T})} \phi_j, \tau_j \stackrel{\text{i.i.d.}}{\sim} \text{Beta}(\beta/5, (1-\beta)/5) \right\},$$

for  $\beta \in (0, 1)$ . Since  $\text{Beta}(\beta/5, (1-\beta)/5)$  is supported on  $(0, 1)$  and has bounded moments of all degrees we can directly verify, using Markov's inequality, that  $\mathcal{K}$  satisfies Assumption 2(a).

Next we check Assumption 2(b). Let  $\beta_0, b_0 \in (0, 1)$ . Then by Markov's inequality once more,

$$\mathbb{P}[\|\zeta_u - \beta_0 u\|_{H^1(\mathbb{T})}^2 > b_0^2(1-\beta_0)^2\|u\|_{H^1(\mathbb{T})}^2] \leq \frac{\mathbb{E}\|\zeta_u - \beta_0 u\|_{H^1(\mathbb{T})}^2}{b_0^2(1-\beta_0)^2\|u\|_{H^1(\mathbb{T})}^2}.$$

Following [7], Section 2, we characterize the  $H^1(\mathbb{T})$  norm via the DB12 wavelets

$$(18) \quad \|u\|_{H^1(\mathbb{T})}^2 = \sum_{j=1}^{\infty} j^2 \langle u, \phi_j \rangle_{L^2(\mathbb{T})}^2.$$

By this expression and Fubini we have

$$\mathbb{E}\|\zeta_u - \beta_0 u\|_{H^1(\mathbb{T})}^2 = \mathbb{E} \sum_{j=1}^{\infty} j^2 (\tau_j - \beta_0)^2 \langle u, \phi_j \rangle_{L^2(\mathbb{T})}^2 = \|u\|_{H^1(\mathbb{T})}^2 \mathbb{E}(\tau_1 - \beta_0)^2.$$

To this end,

$$\mathbb{P}[\|\zeta_u - \beta_0 u\|_{H^1(\mathbb{T})}^2 > b_0^2(1-\beta_0)^2\|u\|_{H^1(\mathbb{T})}^2] \leq \frac{\mathbb{E}(\tau_1 - \beta_0)^2}{b_0^2(1-\beta_0)^2}.$$

Since  $\tau_1$  is a Beta random variable we can always choose constants  $b_0, \beta_0 \in (0, 1)$  so that  $\mathbb{E}(\tau_1 - \beta_0)^2 < b_0^2(1-\beta_0)^2$ . For example, choosing  $\beta_0$  to be arbitrarily small we can simply choose  $b_0^2 > \mathbb{E}\tau_1^2$ . This ensures that

$$\mathbb{P}[\|\zeta_u - \beta_0 u\|_{H^1(\mathbb{T})} \leq b_0(1-\beta_0)\|u\|_{H^1(\mathbb{T})}] \geq \epsilon_0 > 0.$$

It remains to verify Assumption 2(c). We will construct the coupling  $\varpi_{u,v}$  explicitly. Take  $u, v \in H^1(\mathbb{T})$  and draw an i.i.d sequence  $\{\tau_j\}_{j=1}^{\infty}$  where  $\tau_j \sim \text{Beta}(\beta/5, (1-\beta)/5)$ . Then define  $\zeta_u$  and  $\zeta_v$  via

$$\zeta_u = \sum_{j=1}^{\infty} \tau_j \langle u, \phi_j \rangle_{L^2(\mathbb{T})} \phi_j, \quad \zeta_v = \sum_{j=1}^{\infty} \tau_j \langle v, \phi_j \rangle_{L^2(\mathbb{T})} \phi_j.$$

That is, the two chains use the same draw of the  $\tau_j$ 's. Then a straightforward calculation using (18), Jensen's inequality and Fubini gives

$$\mathbb{E} \|\zeta_u - \zeta_v\|_{H^1(\mathbb{T})} \leq (\mathbb{E} \|\zeta_u - \zeta_v\|_{H^1(\mathbb{T})}^2)^{1/2} = (\mathbb{E} \tau_1^2)^{1/2} \|u - v\|_{H^1(\mathbb{T})}.$$

Thus, Assumption 2(c) holds with  $\beta_c = \sqrt{\mathbb{E} \tau_1^2} = \sqrt{\frac{\beta(\beta+5)}{6}}$  where we used well-known expressions for the second raw moment of  $\tau_1$  [28].

**3.2. Statement of main results: Convergence of RCAR-MH kernels.** In this section we present our main theoretical results pertaining to the convergence properties of RCAR-MH kernels  $\mathcal{P}$  that together constitute Main Result 1. Theorem 1 gives families of Lyapunov functions for  $\mathcal{P}$ . This Lyapunov function is then used to define the family of semimetrics  $\tilde{d}_q$  as in (6) with respect to which a uniform spectral gap exists according to Theorem 2. Finally, Theorem 3 shows that  $\mathcal{P}$  is Feller and hence has a unique invariant measure.

**THEOREM 1.** *Let  $\mathcal{P}$  be an RCAR-MH kernel as in Definition 1 and suppose Assumptions 1(c) and 2(a), (b) are satisfied by the function  $\Psi$  and the kernel  $\mathcal{K}$  respectively, and that the innovation measure  $\lambda \in P(\mathcal{H})$  has bounded moments of integer degree  $p \geq 1$ . Then  $V(u) = \|u\|^p$  is a Lyapunov function for  $\mathcal{P}$ .*

**REMARK 1.** One can easily check that the above theorem further implies that any polynomial of the form  $V(u) = \sum_{j=0}^p a_j \|u\|^j$  with coefficients  $a_j \geq 0$  is also a Lyapunov function of  $\mathcal{P}$ .

We present the proof of Theorem 1 in Appendix A.1 using a direct argument akin to the proof of Lyapunov functions in [19]. This result states that the choice of the Lyapunov function is tied to the tail decay of  $\lambda$  so long as  $\Psi$  is increasing in the tails following Assumption 1(c).

The choice of the Lyapunov function  $V$  is crucial since Proposition 1 gives the existence of a spectral gap in the  $\tilde{d}$  semimetrics which in turn depend on the choice of Lyapunov functions; recall (17). To this end, we introduce a family of semimetrics with respect to which  $\mathcal{P}$  has a uniform spectral gap.

For  $\omega, \eta > 0$  define

$$(19) \quad d_q(u, v) := 1 \wedge \frac{(1 + \eta\|u\| + \eta\|v\|)^q \|u - v\|}{\omega},$$

and in turn the  $V$ -weighted semimetric

$$(20) \quad \tilde{d}_q(u, v) := [d_q(u, v)(2 + \theta V(u) + \theta V(v))]^{\frac{1}{2}},$$

for any Lyapunov function  $V$  of  $\mathcal{P}$  following (17). Note,  $d_q(u, v)$  behaves similarly to  $1 \wedge \|u - v\|$ , except that nearby points in  $1 \wedge \|u - v\|$  become further away from each other in  $d_q$  as they get further away from the origin. Recall that  $d_q$  and  $\tilde{d}_q$  are just specific choices of the distance-like function  $d$  in Definition 2 and its  $V$ -weighted analogue in (17) that appear in the statement of the weak Harris theorem. In fact,  $\tilde{d}_q$  is the same metric introduced in (6) with the choice  $V(u) = \|u\|^q$ . With the  $\tilde{d}_q$  semimetric identified we can present our next result, showing that RCAR-MH kernels have uniform  $\tilde{d}_q$  spectral gaps.

**THEOREM 2.** *Let  $\mathcal{P}$  be an RCAR-MH kernel as in Definition 1 and suppose  $\Psi$  satisfies Assumption 1 with  $q \geq 0$ ,  $\mathcal{K}$  satisfies Assumption 2, and the innovation  $\lambda \in P(\mathcal{H})$  has*

bounded moments of integer degree  $p \geq 1 \vee \lceil q \rceil$ . Let  $\tilde{d}_q$  be as in (20) with a Lyapunov function  $V$  of the form

$$(21) \quad V(u) = \sum_{j=0}^p a_j \|u\|^j,$$

with coefficients  $a_p > 0$  and  $a_j \geq 0$  for  $0 \leq j < p$ . Then there exist constants  $(\theta, \omega, \eta, n, \gamma) \in (0, +\infty)^3 \times \mathbb{N} \times (0, 1)$  such that

$$(22) \quad \tilde{d}_q(\mathcal{P}^n \delta_u, \mathcal{P}^n \delta_v) \leq \gamma \tilde{d}_q(u, v).$$

This theorem is a detailed statement of Main Result 1(a) once we note that the bound (22) can be readily extended from point masses  $\delta_u, \delta_v$  to general measures  $\nu_1, \nu_2 \in P^1(\mathcal{H}; \tilde{d}_q)$  by the following remark.

REMARK 2. Observe that  $\tilde{d}_q : P^1(\mathcal{H}; \tilde{d}_q) \times P^1(\mathcal{H}; \tilde{d}_q) \mapsto \mathbb{R}_+$  is convex in both of its arguments and so,

$$\tilde{d}_q(\mathcal{P}^n \nu_1, \mathcal{P}^n \nu_2) \leq \int_{\mathcal{H} \times \mathcal{H}} \tilde{d}_q(\mathcal{P}^n \delta_u, \mathcal{P}^n \delta_v) \pi(du, dv),$$

for any coupling  $\pi \in \Upsilon(\nu_1, \nu_2)$ . In fact, this is true with  $\tilde{d}_q$  replaced with other transport distance-like functions [18], page 246.

The complete proof of Theorem 2 is given in Appendix A.2. Our method of proof relies on the weak Harris theorem (Proposition 1) which in turn requires us to show that  $d_q$  is contracting for  $\mathcal{P}$  and that the level sets of  $V$  are  $d_q$ -small; these are shown in Propositions 3 and 4 respectively. To prove that  $d_q$  is contracting for  $\mathcal{P}$  we need to choose the parameters  $\eta, \omega/\eta^q > 0$  to be sufficiently small depending on the constants appearing in Assumptions 1 and 2 as well as the tail decay of  $\lambda$ . The integer  $n$  then emerges in the proof of  $d_q$ -smallness of the  $V$  level sets and depends on the choice of  $V$  as well as the parameters  $\omega, \eta$  and the tail decay of  $\lambda$ .

For our third and final set of main theoretical results we show that  $\mathcal{P}$  has a unique invariant measure. By [18], Corollary 4.11, the weak Harris theorem guarantees the existence of a unique invariant measure if  $\mathcal{P}$  is Feller (recall Definition 6) and there exists a complete metric  $\bar{d}$  on  $\mathcal{H}$  such that  $\bar{d} \leq \sqrt{d_q}$ . Observe that for  $q \geq 0$ ,

$$\begin{aligned} d_q(u, v) &= 1 \wedge \omega^{-1}(1 + \eta\|u\| + \eta\|v\|)^q \|u - v\|, \\ \sqrt{d_q(u, v)} &\geq \sqrt{1 \wedge \omega^{-1}\|u - v\|}, \end{aligned}$$

and the right side is a complete metric since  $\|u - v\|$  is a complete metric. Thus to prove existence and uniqueness of invariant measures of  $\mathcal{P}$  we simply need to verify that it is indeed a Feller kernel, this result constitutes the claim in Main Result 1(b) which we summarize in Corollary 1 below.

THEOREM 3. Let  $\mathcal{P}$  be an RCAR-MH kernel as in Definition 1 and suppose Assumptions 1(a), (b), (d) and 2(a), (c) are satisfied by  $\Psi$  and  $\mathcal{K}$ , and  $\lambda \in P(\mathcal{H})$  has bounded moments of degree  $q$ . Then  $\mathcal{P}$  is Feller.

We prove this result in Appendix A.3 using direct arguments relying on the dominated convergence theorem and the assumptions on  $\mathcal{K}$  and  $\Psi$ .

**COROLLARY 1.** *Suppose the conditions of Theorems 1, 2 and 3 are satisfied. Then  $\mathcal{P}$  has a unique invariant measure.*

**REMARK 3.** It is important to note that existence of a unique invariant measure of  $\mathcal{P}$  does not guarantee that the invariant measure coincides with the target measure  $\nu$  defined in (1). To ensure that  $\nu$  is indeed the invariant measure we still require  $\mathcal{P}$  to be  $\nu$ -reversible, which in turn holds whenever  $\mathcal{Q}$  is  $\mu$ -reversible [23]. Unfortunately the latter is difficult to establish for general choices of  $\mu$  and requires explicit balancing of  $\mathcal{K}$  and  $\lambda$  to achieve  $\mu$ -reversibility of  $\mathcal{Q}$ .

**EXAMPLE 1 (Continued).** The measure  $\lambda$  defined in (12) has bounded moments of all degrees [23], Theorem 3.1, and so we may take  $V(u) = \|u\|_{H^1(\mathbb{T})}^p$  for any  $p \geq 2$  as the Lyapunov function. Furthermore, we already verified that  $\Psi$  satisfies Assumption 1(b) with  $q = 0$  and so we may choose the semimetrics,

$$(23) \quad \begin{aligned} d_0(u, v) &= 1 \wedge \frac{\|u - v\|_{H^1(\mathbb{T})}}{\omega}, \\ \tilde{d}_0(u, v) &= [d_0(u, v)(2 + \theta\|u\|_{H^1(\mathbb{T})}^2 + \theta\|v\|_{H^1(\mathbb{T})}^2)]^{\frac{1}{2}}. \end{aligned}$$

Application of Theorems 1 through 3 together with the weak Harris theorem (Proposition 1) then yield that  $\mathcal{P}$  has a uniform  $\tilde{d}_0$  spectral gap for appropriate choices of the constants  $\omega, \theta$ . Moreover,  $\mathcal{P}$  has a unique invariant measure and by [23], Theorem 3.4, that invariant measure is precisely the Bayesian posterior  $\nu$  defined in (10)

**4. Perturbation theory for MH kernels.** In this section we present our perturbation analysis of MH kernels that amounts to a detailed statement of Main Result 2. Our perturbation theory is much more general than the present application to RCAR-MH kernels, hence we present it for more general kernels  $\mathcal{P}_0$  and corresponding perturbations  $\mathcal{P}_\varepsilon$ . As the notation suggests, the approximation parameter  $\varepsilon$  controls the quality of the approximating kernel—akin to discretization resolution—so that  $\mathcal{P}_\varepsilon \rightarrow \mathcal{P}_0$  in an appropriate sense as  $\varepsilon \rightarrow 0$ .

Our results are of practical interest for two key reasons: First, simulation can only be done in finite dimensions and therefore approximations of the acceptance ratio  $\alpha(u, v)$  are unavoidable in practice when  $\mathcal{H}$  is a function space, as is the case in Example 1. Second, in some cases the innovation measure  $\lambda$  or the kernel  $\mathcal{K}$  may be intractable or costly to simulate.

The difficulty in obtaining approximation results using semimetrics such as  $\tilde{d}_q$  in (20) is the fact that these semimetric do not satisfy the triangle inequality. However, a “weak” triangle inequality is still satisfied; see Lemma 1 below. Fortunately, the weak triangle inequality allows us to bound the approximation error of  $\mathcal{P}_\varepsilon$  in a similar manner as if  $\tilde{d}_q$  were a metric. Throughout this section, we will often use the generic notation  $d$  and  $\tilde{d}$  for a distance-like function and its  $V$ -weighted version, since these perturbation results hold in general for Markov kernels  $\mathcal{P}_0$  that satisfy the weak Harris theorem. When we verify assumptions or apply results to RCAR-MH, we will make the specific choice of  $d_q$  and  $\tilde{d}_q$ .

In Section 4.1 we identify general assumptions on the kernels  $\mathcal{P}_0$  and  $\mathcal{P}_\varepsilon$  followed by our main perturbation theorems in Section 4.2, with the proofs postponed to Appendix B.

**4.1. Assumptions on the kernels  $\mathcal{P}_0$  and  $\mathcal{P}_\varepsilon$ .** Let us first collect our assumptions on the MH kernel  $\mathcal{P}_0$  and the distance-like function  $d$ .

**ASSUMPTION 3.** Let  $\mathcal{P}_0$  be a Markov transition kernel on  $P(\mathcal{H})$ . Then one or more of the following hold:

- (a)  $d : \mathcal{H} \times \mathcal{H} \mapsto [0, 1]$  is a distance-like function on  $\mathcal{H}$ .
  - (b)  $\mathcal{P}_0$  is contracting for  $d$ .
  - (c)  $\mathcal{P}_0$  has a continuous Lyapunov function  $V$ .
  - (d) For  $\theta > 0$  define  $\tilde{d}$  as in (17) using  $d$  and  $V$ . Then  $\tilde{d}$  satisfies a weak triangle inequality
- $$(24) \quad \tilde{d}(u, v) \leq G[\tilde{d}(u, w) + \tilde{d}(w, v)] \quad \forall u, v, w \in \mathcal{H},$$

where  $G > 0$  is a uniform constant.

- (e)  $\mathcal{P}_0$  has a unique invariant measure  $\nu_0 \in P^1(\mathcal{H}; \tilde{d})$ .
- (f) There exists an integer  $n \geq 1$  and a constant  $\gamma \in (0, 1)$  so that

$$\tilde{d}(\mathcal{P}_0^n \delta_u, \mathcal{P}_0^n \delta_v) \leq \gamma \tilde{d}(u, v) \quad \forall u, v \in \mathcal{H}.$$

Observe that conditions (b), (c), (f) are automatically satisfied if  $\mathcal{P}_0$  satisfies the weak Harris' theorem (Proposition 1) and so by proving that  $\mathcal{P}_0$  has a spectral gap one automatically verifies these assumptions. Moreover, the  $\tilde{d}_q$  semimetrics defined in (20) satisfy condition (d) by Lemma 1 below, and so the above assumptions on  $\mathcal{P}_0$  are naturally satisfied in the setting of Section 3.2 and for RCAR-MH kernels.

LEMMA 1. Define  $\tilde{d}_q$  as in (20) and let  $p, q \geq 0$  be integers and  $V(u) = \sum_{j=0}^p a_j \|u\|^p$  with  $a_p > 0$  and  $a_j \geq 0$  for  $j = 0, \dots, p-1$ . Then there exists  $G(\theta, p, q, \omega, \eta, a_j) > 0$  so that

$$(25) \quad \tilde{d}_q(u, v) \leq G(\tilde{d}_q(u, w) + \tilde{d}_q(w, v)) \quad \forall u, v, w \in \mathcal{H}.$$

See Appendix B.1 for the proof. Next we collect assumptions on the family of approximate kernels  $\mathcal{P}_\varepsilon$  following [27].

ASSUMPTION 4. Let  $\mathcal{P}_0$  be a Markov transition kernel on  $P(\mathcal{H})$  with Lyapunov function  $V$ , and let  $d$  be a distance-like function on  $P(\mathcal{H})$  and define  $\tilde{d}$  using  $d$  and  $V$  as in (17). Then there exists a constant  $\varepsilon_0 > 0$  so that the family of transition kernels  $\mathcal{P}_\varepsilon$  satisfy:

- (a) For every  $\varepsilon \in (0, \varepsilon_0)$  there exist constants  $(\kappa_\varepsilon, K_\varepsilon) \in (0, 1) \times (0, +\infty)$  so that

$$\mathcal{P}_\varepsilon V(u) \leq \kappa_\varepsilon V(u) + K_\varepsilon \quad \forall u \in \mathcal{H}.$$

- (b) There exists a bounded function  $\psi(\varepsilon) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  so that

$$(26) \quad \tilde{d}(\mathcal{P}_0 \delta_u, \mathcal{P}_\varepsilon \delta_u) \leq \psi(\varepsilon)(1 + \sqrt{V(u)}) \quad \forall u \in \mathcal{H}.$$

Note that our assumptions on  $\mathcal{P}_\varepsilon$  are far less stringent in comparison to  $\mathcal{P}_0$ . Simply put condition (a) requires  $\mathcal{P}_0$  and  $\mathcal{P}_\varepsilon$  to have the same Lyapunov function while (b) requires control on the one step error between  $\mathcal{P}_0$  and  $\mathcal{P}_\varepsilon$ . In fact,  $\mathcal{P}_\varepsilon$  is not required to have a unique invariant measure or satisfy any contractive properties directly. This flexibility allows access to larger classes of approximate kernels and makes our perturbation theory convenient to apply since there are fewer conditions to check. In comparison, more direct methods such as the perturbation analysis of [19] prove the convergence properties of  $\mathcal{P}_0$  and  $\mathcal{P}_\varepsilon$  separately and then show that the invariant measures are close to each other.

EXAMPLE 1 (Continued). We now consider an approximation of the kernel  $\mathcal{P}$  for the nonlinear regression problem and verify the above assumptions. Suppose the likelihood potential  $\Psi$  is replaced with the numerical approximation  $\Psi \circ \Pi_N$  where  $\Pi_N$  denotes the  $L^2$  projection on the span of the first  $N$  wavelets  $\phi_0, \dots, \phi_N$ . We let  $\varepsilon = 1/N$  and define the perturbed kernel  $\mathcal{P}_\varepsilon$  as the identical RCAR-MH kernel to  $\mathcal{P}$  with the accept/reject probability

$$\alpha_\varepsilon(u, v) := 1 \wedge \exp(\Psi(\Pi_N u) - \Psi(\Pi_N v)).$$

A straightforward application of Theorem 1 reveals that both  $\mathcal{P}$  and  $\mathcal{P}_\varepsilon$  share the same Lyapunov functions of the form  $V(u) = \sum_{j=1}^p a_j \|u\|_{H^1(\mathbb{T})}^p$  with positive coefficients  $a_j$  and constants  $\kappa_\varepsilon = \kappa$ ,  $K_\varepsilon = K$ , thus Assumption 4(a) is verified easily.

Assumption (b) can be verified via a coupling argument. Fix  $u \in H^1(\mathbb{T})$  and let  $\pi_u \in \Upsilon(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u)$  be an optimal coupling that achieves  $d_0(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u)$ . Then by Cauchy–Schwarz we have that

$$\begin{aligned}
 & \tilde{d}_0(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u)^2 \\
 & \leq \int_{H^1(\mathbb{T}) \times H^1(\mathbb{T})} d_0(v, w) (2 + \theta V(v) + \theta V(w)) \pi_u(dv, dw), \\
 (27) \quad & \leq d_0(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u) \int_{H^1(\mathbb{T}) \times H^1(\mathbb{T})} (2 + \theta V(v) + \theta V(w)) \pi_u(dv, dw), \\
 & \leq d_0(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u) [2 + 2\theta(\kappa V(u) + K)].
 \end{aligned}$$

It is thus sufficient to bound  $d_0(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u)$ . Now consider a coupling of  $\mathcal{P}\delta_u$  and  $\mathcal{P}_\varepsilon\delta_u$  where both chains propose a new point  $u^* = \zeta_u + \xi$  where  $\zeta_u \sim \mathcal{K}(u, \cdot)$  and  $\xi \sim \lambda$ . We then generate a uniform random variable  $\varsigma$  and one chain accepts  $u^*$  when  $\varsigma < \alpha(u, u^*)$  while the other chain accepts if  $\varsigma < \alpha_\varepsilon(u, u^*)$ . Since this coupling is not necessarily optimal we have

$$\begin{aligned}
 d_0(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u) & \leq \mathbb{E}[d_0(u^*, u^*) \mathbb{P}(\text{both chains accept})] \\
 & \quad + \mathbb{E}[d_0(u^*, u) \mathbb{P}(\text{only one chain accepts})] \\
 (28) \quad & \quad + \mathbb{E}[d_0(u, u) \mathbb{P}(\text{both chains reject})] \\
 & \leq \mathbb{E}[\mathbb{P}(\text{only one chain accepts})].
 \end{aligned}$$

Moreover, since  $(1 \wedge \exp)$  is Lipschitz and we already showed that  $\Psi$  is globally Lipschitz we have

$$\begin{aligned}
 \mathbb{P}(\text{only one chain accepts}) & \leq |\Psi(u^*) - \Psi(\Pi_N u^*)| \leq L \|u^* - \Pi_N u^*\|_{H^1(\mathbb{T})} \\
 & \leq L \|I - \Pi_N\|_{H^1(\mathbb{T})} \|u^*\|_{H^1(\mathbb{T})}.
 \end{aligned}$$

Now since  $\|\zeta_u\|_{H^1(\mathbb{T})} \leq \|u\|_{H^1(\mathbb{T})}$  by Assumption 2 and triangle inequality we have  $\|u^*\|_{H^1(\mathbb{T})} \leq \|u\|_{H^1(\mathbb{T})} + \|\xi\|_{H^1(\mathbb{T})}$ . Substituting back into (28) gives

$$\begin{aligned}
 d_0(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u) & \leq L \|I - \Pi_N\|_{H^1(\mathbb{T})} \mathbb{E}[\|u\|_{H^1(\mathbb{T})} + \|\xi\|_{H^1(\mathbb{T})}] \\
 & \leq C_1 \|I - \Pi_N\|_{H^1(\mathbb{T})} (1 + \|u\|_{H^1(\mathbb{T})}),
 \end{aligned}$$

for some constant  $C_1 > 0$ . To this end we have shown that

$$\tilde{d}_0(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u)^2 \leq C_2 \|I - \Pi_N\|_{H^1(\mathbb{T})} (1 + \tilde{V}(u)),$$

where  $C_2 > 0$  and  $\tilde{V}(u) = (1 + \|u\|_{H^1(\mathbb{T})} + \|u\|_{H^1(\mathbb{T})} V(u))$ . Noting that  $\tilde{V}$  is also a polynomial of  $\|u\|_{H^1(\mathbb{T})}$  with positive coefficients then verifies Assumption 4(b) with Lyapunov function  $\tilde{V}$  and  $\psi(\varepsilon) = C_3 \|I - \Pi_N\|_{H^1(\mathbb{T})}^{1/2}$  for a constant  $C_3 > 0$ .

#### 4.2. Statement of main results: Perturbation theory for MH kernels on Banach spaces.

We are now ready to present our main theoretical results pertaining to perturbations of Markov kernels on Banach spaces. Our first result is a generalization of [27], Theorem 3.13, to Banach spaces that allows us to bound the distance between the invariant measure(s) of  $\mathcal{P}_0$  and  $\mathcal{P}_\varepsilon$ . A similar argument is used to bound the distance between invariant measures of a Markov transition kernel and its perturbation in [18].

**THEOREM 4.** *Suppose Assumptions 3 and 4 are satisfied. Then there exists a function  $\vartheta(n) : \mathbb{N} \mapsto [0, \infty)$  so that for each  $k > 0$  such that  $\gamma^{[k/n]} < G^{-1}$  we have*

$$(29) \quad \tilde{d}(v_0, v_\varepsilon) \leq \frac{G\psi(\varepsilon)\vartheta(k)}{1 - G\gamma^{[k/n]}} (1 + v_0(\sqrt{V}) \wedge v_\varepsilon(\sqrt{V})) \quad \forall u, v \in \mathcal{H},$$

where  $v_0, v_\varepsilon \in P^1(\mathcal{H}; \tilde{d})$  are the invariant measure of  $\mathcal{P}_0$  and any invariant measure of  $\mathcal{P}_\varepsilon$ , respectively.

We prove this theorem in Appendix B.2. Note that the invariant measure  $v_\varepsilon$  need not be unique, though the Lyapunov condition for  $\mathcal{P}_\varepsilon$  in Assumption 4(a) guarantees there exists at least one. The function  $\vartheta(n)$  appears in our bound due to the fact that  $\tilde{d}$  satisfies the weak triangle inequality (25) with a constant  $G > 0$  that is possibly bigger than 1. To overcome this difficulty we choose  $k > n$  sufficiently large so that  $G\gamma^{[k/n]} < 1$ , where  $\gamma$  is the  $n$  step spectral gap of  $\mathcal{P}_0$  in Assumption 3(b). We then take  $\vartheta(k) = C \sum_{j=1}^k G^j (C^* \gamma)^{\lfloor j/n \rfloor} < \infty$  with constants  $C^*, C > 0$  depending on the Lipschitz constant of  $\mathcal{P}_0$  and the growth rate of  $V$ . Noting that  $G\vartheta(k)/(1 - G\gamma^{[k/n]}) < +\infty$  is a constant independent of  $\varepsilon$  and by taking  $\tilde{d}_q$  as our semimetric we obtain the detailed version of Main Result 2(a).

We can further extend the error bound (29) to a practical error bound between the expectation of  $\tilde{d}$ -Lipschitz functions under  $v_0$  and  $v_\varepsilon$ .

**COROLLARY 2.** *Suppose the conditions of Theorem 4 are satisfied. Let  $\mathcal{X}$  be a separable Hilbert space with norm  $\|\cdot\|_{\mathcal{X}}$  and consider a  $\tilde{d}$ -Lipschitz function  $f : \mathcal{H} \rightarrow \mathcal{X}$  satisfying  $\|f(u) - f(v)\|_{\mathcal{X}} \leq \tilde{d}(u, v) \forall u, v \in \mathcal{H}$ . Then we have,*

$$(30) \quad \|v_0(f) - v_\varepsilon(f)\|_{\mathcal{X}} \leq \frac{G\vartheta(k)}{1 - G\gamma^{[k/n]}} \psi(\varepsilon) (1 + v_0(\sqrt{V}) \wedge v_\varepsilon(\sqrt{V})).$$

**PROOF.** Since the argument is short we present it here. Let  $\pi$  be an optimal coupling between  $v_0$  and  $v_\varepsilon$  which exists following [50], Theorem 4.1. Then

$$\begin{aligned} \int_{\mathcal{H} \times \mathcal{H}} \tilde{d}(u, v) \pi(du, dv) &\geq \int_{\mathcal{H} \times \mathcal{H}} \|f(u) - f(v)\|_{\mathcal{X}} \pi(du, dv) \\ &\geq \left\| \int_{\mathcal{H}} f(u) dv - \int_{\mathcal{H}} f(v) v_\varepsilon(dv) \right\|_{\mathcal{X}}. \end{aligned}$$

The last step follows from Jensen's inequality.  $\square$

Note that the assumption that  $f$  is  $\tilde{d}$ -Lipschitz is not very restrictive given that such an  $f$  is continuous but  $\|f(u)\|_{\mathcal{X}}$  can grow as fast as the Lyapunov function  $V(u)$ .

We continue to derive practical error bounds pertaining to Markov kernels and their perturbations, turning our attention to pathwise properties of realizations of the Markov chains. More precisely we bound the error of finite-time Cesáro averages from  $\mathcal{P}_\varepsilon$  and expectations under  $v_0$  for real valued  $\tilde{d}$ -Lipschitz functions. Our bounds are desirable as they are a major improvement over standard arguments using the weak triangle inequality. This is a consequence of the fact that the  $\tilde{d}$ -Lipschitz seminorm  $\| \cdot \|_{\tilde{d}}$  obeys the triangle inequality even when  $\tilde{d}$  does not, indeed for functions  $f, g : \mathcal{H} \mapsto \mathcal{X}$ ,

$$(31) \quad \begin{aligned} \|f + g\|_{\tilde{d}} &= \sup_{u \neq v} \frac{\|f(u) + g(u) - f(v) - g(v)\|_{\mathcal{X}}}{\tilde{d}(u, v)} \\ &\leq \sup_{u \neq v} \frac{\|f(u) - f(v)\|_{\mathcal{X}} + \|g(u) - g(v)\|_{\mathcal{X}}}{\tilde{d}(u, v)} = \|f\|_{\tilde{d}} + \|g\|_{\tilde{d}}. \end{aligned}$$

Our next main result bounds the mean error of pathwise estimates from  $\mathcal{P}_\varepsilon$  with respect to expected values under the exact target  $v_0$ .

**THEOREM 5.** *Suppose Assumptions 3 and 4 are satisfied and let  $\mathcal{X}$  be a separable Hilbert space. Then for  $U_k^\varepsilon \sim \mathcal{P}_\varepsilon^{k-1} \delta_{u_0}$  and for any function  $\varphi : \mathcal{H} \mapsto \mathcal{X}$  with  $\|\varphi\|_{\tilde{d}} < +\infty$  we have:*

(a) *There exist positive constants  $C_1, C_2, C_3 > 0$  that are independent of  $n$  but depend on  $\theta, \kappa_\varepsilon, K_\varepsilon, \gamma$  and  $V(u_0)$  such that*

$$\mathbb{E} \left\| \frac{1}{n} \sum_{k=0}^{n-1} \varphi(U_k^\varepsilon) - v_0(\varphi) \right\|_{\mathcal{X}} \leq \frac{\|\varphi\|_{\tilde{d}}}{1-\gamma} \left( C_2 \psi(\varepsilon) + C_3 \frac{\psi(\varepsilon)}{n} + C_4 \frac{1}{\sqrt{n}} \right).$$

(b) *There exist constants  $C_4, C_5, C_6 > 0$  independent of  $n$  but depending on  $\theta, \kappa_\varepsilon$  and  $K_\varepsilon$  such that*

$$\mathbb{E} \left\| \frac{1}{n} \sum_{k=0}^{n-1} \varphi(U_k^\varepsilon) - v_0(\varphi) \right\|_{\mathcal{X}} \leq \frac{\|\varphi\|_{\tilde{d}}}{1-\gamma} \left( C_4 \psi(\varepsilon) + C_5 \frac{\psi(\varepsilon)}{n} + \frac{C_6}{n} \right).$$

The complete proof is given in Appendix B.3. To prove part (a) we utilize an approach similar to that of [16]. The complication is that because  $\varphi$  is  $\mathcal{X}$ -valued rather than  $\mathbb{R}$ -valued, we need to prove that the potential  $\sum_{k=0}^\infty \mathcal{P}^k \varphi$  is a solution to the Poisson equation; this result is of course well known for real-valued  $\varphi$ . The inner product structure on  $\mathcal{X}$  is used only once, to control the expected  $\|\cdot\|_{\mathcal{X}}$ -norm of a Martingale. While it is possible to control this term without the inner product structure, in most applications in statistics and Bayesian inverse problems the functions of interest are Hilbert-space valued, so the result above is sufficiently general. Let us now return to Example 1 once more to apply Theorems 4 and 5 to obtain error bounds on the approximate posteriors and the posterior Césaro average.

**EXAMPLE 1 (Continued).** We already verified that RCAR-MH kernel  $\mathcal{P}$  and the approximate kernel  $\mathcal{P}_\varepsilon$  obtained by discretizing the likelihood potential  $\Psi$  via projection onto the first  $N$  wavelet bases. Then a direct application of Theorem 4 yields a bound of the form

$$\tilde{d}_0(v, v_\varepsilon) \leq C_1 \|I - \Pi_N\|_{H^1(\mathbb{T})}^{1/2},$$

where  $C_1 > 0$  is a constant independent of  $N$  and we recall that we defined  $\varepsilon = 1/N$  and  $v$  denotes the true posterior measure. Let us also consider the function  $\varphi : u \mapsto u$  and apply Theorem 5(a) to obtain an error bound on the pathwise Césaro averages of the discretized RCAR-MH algorithm:

$$\mathbb{E} \left\| \frac{1}{n} \sum_{k=0}^{n-1} U_k^\varepsilon - v(\varphi) \right\|_{H^1(\mathbb{T})} \leq C_2 \left( \|I - \Pi_N\|_{H^1(\mathbb{T})}^{1/2} + \frac{1}{\sqrt{n}} \right),$$

where  $C_2 > 0$  is independent of  $N, n \geq 1$ . To this end, both the error between the invariant measures and the Césaro averages of the RCAR-MH algorithm are controlled by the square root of the  $H^1(\mathbb{T})$ -operator norm of  $I - \Pi_N$ . The Césaro average is also controlled by the standard Monte Carlo rate  $n^{-1/2}$ .

**5. Applications.** Here we discuss a number of applications of our main theoretical results from Sections 3 and 4 with a particular focus on approximations of the RCAR algorithm. We start in Section 5.1 by providing a detailed explanation of the numerical experiments presented in Figure 1 and pertaining to Example 1. In Section 5.2 we consider an application of the pCN algorithm where the Karhunen–Loève modes of the prior are perturbed. Finally, in Section 5.3 we discuss the practicality of Assumption 1(c).

5.1. *Details of numerical experiments in Figure 1.* To generate Figure 1 we utilized the RCAR algorithm of [23] tailored for gamma random variables (see in particular Algorithm 6 of that article). The function  $u^\dagger$  depicted in Figure 1(a) is the function obtained by setting the first, fourth, eighth and sixteenth DB12 wavelet coefficients equal to 2 while the rest of the coefficients are zero. That is, our  $u^\dagger$  is sparse in the DB12 wavelet basis and its nonzero coefficients are positive so that it is consistent with our choice of the prior  $\mu$  which constrains the wavelet coefficients to be positive. The posterior mean in Figure 1(a) was computed by truncating the prior at  $N = 128$  wavelet modes and running the chain for  $10^5$  iterations with parameter  $\beta = 0.9$  and with a burn-in of  $5 \times 10^4$ .

Figure 1(b) was generated by varying  $N$ ,  $\beta$  over the indicated ranges and running the RCAR algorithm for the same data  $y$  shown in Figure 1(a). Each data point on Figure 1(b) was generated by running the chain for  $10^5$  iterations with burn-in of  $5 \times 10^4$  over five restarts of the chain with random initial conditions from the prior. The acceptance ratios for the five restarts were then averaged to obtain a data point in the figure. The restarts were performed to reduce the effect of the initial condition of the chain and other random effects on the reported values.

5.2. *pCN with approximate Karhunen–Loève expansions.* We now consider a perturbation example where the pCN algorithm of [6] is applied with a perturbed prior covariance. More precisely, consider the same nonlinear regression problem as Example 1 but this time we wish to recover  $u^\dagger \in \mathcal{H}^s(\Omega)$  where  $\Omega \subset \mathbb{R}^d$  and  $\mathcal{H}^s(\Omega)$  is a Sobolev-type space. Let  $\phi_j$  be the Neumann eigenfunctions (normalized in  $L^2(\Omega)$ ) of the standard Laplacian operator on  $\Omega$ , that is, they solve the problems

$$\begin{aligned} -\Delta \phi_j &= c_j \phi_j \quad \text{in } \Omega, \\ \nabla \phi_j \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where  $\mathbf{n}$  is the unit outward pointing normal vector on  $\partial\Omega$  and the  $c_j \geq 0$  are the eigenvalues of  $\Delta$ . Indeed, one can verify that  $\Delta$  is positive semi-definite and self-adjoint and so  $c_0 = 0$ , with the corresponding eigenfunction  $\phi_0$  being a constant on  $\Omega$ , while the  $c_j > 0$  for  $j \geq 1$ . Now for integer  $s > 0$  consider the spaces  $\mathcal{H}^s(\Omega) \subset L^2(\Omega)$  defined as

$$\mathcal{H}^s(\Omega) := \left\{ u \in L^2(\Omega) : \|u\|_{\mathcal{H}^s(\Omega)}^2 := \sum_{j=0}^{\infty} (1 + c_j)^s \langle u, \phi_j \rangle_{L^2(\Omega)}^2 < +\infty \right\}.$$

It is known (see [15], Lemma 7.1) that for any  $s > 0$  it holds that  $\mathcal{H}^s(\Omega) \subset H^s(\Omega)$  where  $H^s(\Omega)$  denotes the standard Sobolev space of index  $s$  on  $\Omega$ . Then an application of the Sobolev embedding theorem [1] yields  $\mathcal{H}^s(\Omega) \subset C(\Omega)$  for  $s > d/2$ . Thus the nonlinear regression problem in Example 1 is well defined for functions  $u^\dagger \in \mathcal{H}^s(\Omega)$  for  $s > d/2$ ; which we assume holds henceforth. An identical reasoning to Example 1 then verifies Assumption 1.

Now define the prior measure  $\mu$  as

$$\mu = \text{Law} \left\{ \sum_{j=0}^{\infty} a_j \eta_j \phi_j \right\},$$

where  $\eta_j \stackrel{\text{i.i.d.}}{\sim} N(0, 1)$  and  $a_j = (1 + c_j)^{-k}$  with  $k > s$ . Let  $u \sim \mu$  and write

$$\|u\|_{\mathcal{H}^s(\Omega)}^2 = \sum_{j=0}^{\infty} (1 + c_j)^{-k} \eta_j^2.$$

Observe that  $\mathbb{E}(1 + c_j)^{-k} \eta_j^2 = (1 + c_j)^{-k}$  and  $\mathbb{E}(1 + c_j)^{-2k} \eta_j^4 = 3(1 + c_j)^{-2k}$ . By Weyl's law  $c_j \asymp j^{2/d}$  so that  $(1 + c_j)^{-k} \asymp j^{-2k/d}$ . Since we assumed that  $k > s > d/2$  we infer that

$\sum_{j=0}^{\infty} (1 + c_j)^{-k} < +\infty$  and  $\sum_{j=0}^{\infty} (1 + c_j)^{-2k} < +\infty$ . Kolmogorov's two series theorem then yields that  $\|u\|_{\mathcal{H}^s(\Omega)}^2 < +\infty$  a.s. So our prior is supported on  $\mathcal{H}^s(\Omega)$  as desired.

Since  $\mu$  is Gaussian our RCAR-MH algorithm reduces to the pCN algorithm, that is, for a step size  $\beta \in (0, 1)$  we take the kernel  $\mathcal{K}(u, \cdot) = \delta_{\beta u}$  and the innovation measure

$$\lambda = \text{Law} \left\{ \sum_{j=0}^{\infty} a_j \xi_j \phi_j \right\}, \quad \xi_j \stackrel{\text{i.i.d.}}{\sim} N(0, (1 - \beta^2)).$$

These choices yield the pCN kernel

$$(32) \quad \begin{aligned} \mathcal{P}(u, dv) &= \alpha(u, v)(\delta_{\beta u} * \lambda)(dv) \\ &+ \delta_u \int_{\mathcal{H}^1(\Omega)} (1 - \alpha(u, w))(\delta_{\beta u} * \lambda)(dw). \end{aligned}$$

Since pCN is a special case of RCAR we readily verify, by the same calculations presented for Example 1, that pCN satisfies the conditions of the weak Harris' theorem and so has a  $\tilde{d}_0$ -spectral gap, where we recall the semimetrics  $d_0, \tilde{d}_0$  defined in (23) with the  $H^1(\mathbb{T})$  norms replaced with  $\mathcal{H}^s(\Omega)$  norms and possibly different constants  $(\theta, \omega)$ .

Let us now consider a perturbation of pCN by replacing the eigenpairs  $(c_j, \phi_j)$  with perturbations  $(c_j^\varepsilon, \phi_j^\varepsilon)$  for a parameter  $\varepsilon > 0$ . We have in mind applications where we can only compute  $(c_j, \phi_j)$  numerically, using for example a finite element method, since the domain  $\Omega$  can have complicated geometry. We further assume for brevity that there exists a sufficiently small constant  $\varepsilon_0 > 0$  so that for all  $\varepsilon \in (0, \varepsilon_0)$  we have  $c_j^\varepsilon \asymp j^{2/d}$  and the  $\phi_j^\varepsilon$  are normalized in  $L^2(\Omega)$  and linearly independent such that  $\text{span}\{\phi_j^\varepsilon\} \subseteq \mathcal{H}^1(\Omega)$ .

Our goal is to obtain an error bound between the true posterior  $\nu$  and the limit distribution  $\nu_\varepsilon$  of the perturbation of pCN that utilizes the eigenpairs  $(c_j^\varepsilon, \phi_j^\varepsilon)$  rather than the exact pairs  $(c_j, \phi_j)$ . To this end, define the perturbed innovation measure

$$\lambda_\varepsilon = \text{Law} \left\{ \sum_{j=0}^{\infty} a_j^\varepsilon \xi_j \phi_j^\varepsilon \right\}, \quad \xi_j \stackrel{\text{i.i.d.}}{\sim} N(0, (1 - \beta^2)),$$

where  $a_j^\varepsilon = (1 + c_j^\varepsilon)^{-k}$  as well as the corresponding perturbed pCN kernel

$$(33) \quad \begin{aligned} \mathcal{P}_\varepsilon(u, dv) &= \alpha(u, v)(\delta_{\beta u} * \lambda_\varepsilon)(dv) \\ &+ \delta_u \int_{\mathcal{H}^1(\Omega)} (1 - \alpha(u, w))(\delta_{\beta u} * \lambda_\varepsilon)(dw). \end{aligned}$$

Repeating the same calculation we did for  $\mu$  in the above yields that  $\lambda_\varepsilon$  is a Gaussian measure supported on  $\mathcal{H}^1(\Omega)$  for all  $\varepsilon \in (0, \varepsilon_0)$  and so has bounded moments of all orders and so by Theorem 1 any function of the form  $V(u) = \|u\|_{\mathcal{H}^1(\mathbb{T})}^p$  for  $p \geq 1$  is a Lyapunov function for  $\mathcal{P}_\varepsilon$  and so Assumption 4(a) is satisfied. Thus it remains to verify Assumption 4(b) before we can apply Theorem 4 to bound  $\tilde{d}_0(\nu, \nu_\varepsilon)$ . Following this argument, we obtain the following proposition, the proof of which is postponed to Appendix C.

**PROPOSITION 2.** *Consider the above setting with the pCN kernel  $\mathcal{P}$  as in (32) and the perturbation  $\mathcal{P}_\varepsilon$  introduced in (33). Suppose that the following conditions hold:*

(a) *There exists a common Lyapunov function  $V$  for  $\mathcal{P}, \mathcal{P}_\varepsilon$  so that*

$$\mathcal{P}V(u) \leq \kappa V(u) + K, \quad \mathcal{P}_\varepsilon V(u) \leq \kappa_\varepsilon V(u) + K_\varepsilon \quad \forall u \in \mathcal{H}^1(\Omega),$$

*and furthermore*

$$\kappa \vee \sup_{\varepsilon \in (0, \varepsilon_0)} \kappa_\varepsilon \in (0, 1) \quad \text{and} \quad K \vee \sup_{\varepsilon \in (0, \varepsilon_0)} K_\varepsilon \in [0, +\infty).$$

(b) It holds that the sequences  $\{a_j \|\phi_j\|_{\mathcal{H}^s(\Omega)}^2\}$ ,  $\{a_j \|\phi_j - \phi_j^\varepsilon\|_{\mathcal{H}^s(\Omega)}\}$  and  $\{|a_j - a_j^\varepsilon| \|\phi_j\|_{\mathcal{H}^s(\Omega)}\}$  belong to  $\ell^1$ .

Then  $\forall \varepsilon \in (0, \varepsilon_0)$  and for any  $u \in \mathcal{H}^s(\Omega)$  it holds that

$$\tilde{d}_0(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u)^2 \leq C\psi(\varepsilon)[1 + V(u)] \quad \forall u \in H^1(\mathbb{T}),$$

where  $C > 0$  is a constant independent of  $\varepsilon$  and

$$\psi(\varepsilon) = \left[ \left( \sum_{j=0}^{\infty} a_j \frac{\|\phi_j - \phi_j^\varepsilon\|_{\mathcal{H}^s(\Omega)}^2}{\|\phi_j\|_{\mathcal{H}^s(\Omega)}^2} \right)^{1/2} + \left( \sum_{j=0}^{\infty} \frac{|a_j - a_j^\varepsilon|^2}{a_j^2} \|\phi_j\|_{\mathcal{H}^s(\Omega)}^2 \right)^{1/2} \right].$$

An application of Theorem 4 then yields the existence of a constant  $C > 0$  so that

$$\tilde{d}_0(v, v_\varepsilon) \leq C\sqrt{\psi(\varepsilon)}$$

which is the desired result.

**REMARK 4.** The above proposition identifies conditions on approximations schemes for the eigenpairs  $\{a_j, \phi_j\}$  in connection with our choice of the prior  $\mu$ . Most notably, the condition that  $\{|a_j - a_j^\varepsilon| \|\phi_j\|_{\mathcal{H}^s(\Omega)}\} \in \ell^1$  requires the absolute error in computing the eigenvalues  $c_j$  to decay rapidly since the  $\{\|\phi_j\|_{\mathcal{H}^s(\Omega)}\}$  is not summable (higher frequency eigenfunctions have larger Sobolev norms). However, one can get around this difficulty simply by prescribing a different sequence  $a_j$  that can be implemented exactly. For example, by taking  $a_j = (1 + b_j)^{-k}$  for another sequence of numbers  $b_j \asymp j^{-2/d}$ . It can be verified that the resulting prior will still be a Gaussian supported on  $\mathcal{H}^s(\Omega)$  but with a different covariance operator. The conditions  $(\{a_j \|\phi_j\|_{\mathcal{H}^s(\Omega)}^2\}, \{a_j \|\phi_j - \phi_j^\varepsilon\|_{\mathcal{H}^s(\Omega)}\}) \in \ell^1$  can be viewed as guidelines for choosing the index  $k > s$  according to the regularity of the  $\phi_j$  and accuracy of our numerical scheme for computing the  $\phi_j^\varepsilon$ . Note that we expect the sequence  $\|\phi_j - \phi_j^\varepsilon\|_{\mathcal{H}^s(\Omega)}$  to grow since the error of standard numerical schemes for computing eigenfunctions grows with their frequency due to their growing  $\mathcal{H}^s(\Omega)$  norms. Thus, choosing a larger index  $k$  allows us to control this approximation error.

**REMARK 5.** Note that the above bound can also be viewed as an error bound between two posteriors  $v, v_\varepsilon$  that arise from two Gaussian priors  $\mu$  and  $\mu_\varepsilon$ . Indeed, our calculations yield a method for controlling the distance between posterior measures in terms of prior perturbations, a contemporary topic in the theory of Bayesian inverse problems [46]. Admittedly, our method is inefficient as it goes through the construction of a Markov chain that converges to the two posteriors. Regardless, such posterior perturbation bounds are often difficult to achieve, essentially due to the Feldman–Hajek theorem [3], Section 2.7, which implies that perturbations of Gaussian prior measures can often lead to mutually singular priors and in turn mutually singular posteriors. Classic stability analyses of Bayesian inverse problems utilize TV or Hellinger distances [9, 22, 24, 48] and, due to the singularity of the posterior measures, one has no hope of obtaining a useful error bound in those topologies. Our calculations above, and similarly the results of [46], suggest that transport (semi-)metrics hold the key for stability analysis of posterior measures due to prior perturbations.

**5.3. Practicality of Assumption 1.** We dedicate this subsection to a discussion of the relevance of Assumption 1 in practical applications. The conditions (a), (b) and (d) are standard in the theory of Bayesian inverse problems and can be verified for large classes of inverse problems such as deconvolution, phase retrieval, porous medium flow, etc. [9, 22, 24, 47, 48]. The condition (c) however is not crucial to ensure the existence and uniqueness of the

target measure  $\nu$ , but it is central to Theorems 1 and 2. We need this condition to make sure that  $\Psi$  is uniformly increasing when sufficiently far from the origin. Intuitively this means that if the chain is far away then the probability of accepting a proposal that is even farther away decays uniformly. While we verified Assumption 1(c) for Example 1, it does not hold even in simple linear inverse problems, as we now demonstrate with an example in deconvolution [23, 51].

Let  $\mathcal{H} = H^1(\mathbb{T})$  once more and consider  $\mathcal{G} : H^1(\mathbb{T}) \mapsto \mathbb{R}^m$  a bounded linear operator of the form

$$(\mathcal{G}(u))_j := (g * u)(x_j),$$

with  $g \in C^\infty(\mathbb{T})$  a smooth kernel and distinct points  $x_j \in \mathbb{T}$ . Let  $u^\dagger \in H^1(\mathbb{T})$  be the ground truth function giving rise to the data  $y_j = \mathcal{G}(u^\dagger) + \epsilon_j$  where  $\epsilon_j \sim N(0, 1)$ . These assumptions induce the quadratic likelihood potential

$$\Psi(u; y) := \frac{1}{2} \|\mathcal{G}(u) - y\|_2^2.$$

In light of the smoothing effect of  $(g * \cdot)$  we can readily see that Assumption 1(c) cannot be verified: Let  $u$  be a point that has large  $H^1(\mathbb{T})$  norm and evaluate  $\Psi(u)$ . Then add to  $u$  a highly oscillatory function  $\delta u$  with small amplitude that will increase the Sobolev norm of  $u$  significantly. Since convolution is linear we have  $g * (u + \delta u) = g * u + g * \delta u$  and the perturbation  $(g * \delta u)(x_j)$  to the observed data  $y|u$  will be small; meaning that  $\Psi(u + \delta u)$  is close to  $\Psi(u)$ . Then the probability of accepting a move towards  $u + \delta u$  is not guaranteed to decrease uniformly.

This suggests that Assumption 1(c) is too restrictive. But we claim that a slight modification of the prior  $\mu$  or the likelihood  $\Psi$  can remedy this problem in many applications including deconvolution. For a choice of  $\tilde{b}$  and  $\tilde{\beta}$  let  $c = \tilde{b}(1 - \tilde{\beta})$ . Pick  $R_0 > 0$  and define the perturbed likelihood potential

$$\Psi_\varepsilon(u; y) := \Psi(u; y) + \max\{0, \varepsilon \|u\|^2 - R_0^2\},$$

where  $\varepsilon > 0$  is a fixed constant satisfying

$$\varepsilon > \frac{2c^2}{1 - c^2} \|\mathcal{G}\|^2,$$

with  $\|\mathcal{G}\|$  denoting the operator norm of  $\mathcal{G}$ . Then for any  $u \in B_{R_0}(0)^c$  and  $v \in B_{c\|u\|}(0)$  we have

$$\begin{aligned} 2\sigma^2(\Psi_\varepsilon(u; y) - \Psi_\varepsilon(v; y)) &= \|\mathcal{G}(u) - y\|_2^2 + \varepsilon \|u\|^2 - \|\mathcal{G}(v) - y\|_2^2 - \varepsilon \|v\|^2 \\ &\geq \varepsilon \|u\|^2 - (2\|\mathcal{G}\|^2 + \varepsilon) \|v\|^2 - 2\|y\|_2^2 \\ &\geq \varepsilon \|u\|^2 - c^2(2\|\mathcal{G}\|^2 + \varepsilon) \|u\|^2 - 2\|y\|_2^2 \\ &= (\varepsilon - c^2(2\|\mathcal{G}\|^2 + \varepsilon)) \|u\|^2 - 2\|y\|_2^2. \end{aligned}$$

The above lower bound is a second order polynomial of  $\|u\|$  with a positive leading coefficient—due to the lower bound on  $\varepsilon$ —and so  $\Psi_\varepsilon$  satisfies Assumption 1(c) for any choice of  $\tilde{b}, \tilde{\beta} \in (0, 1)$  and  $R_0 > 0$ .

It can be verified that this modification of  $\Psi$  will result in a perturbation to the posterior  $\nu$  that is controlled by the parameter  $\varepsilon$ , the radius  $R_0$  and the tails of  $\mu$ . Define the perturbed posterior

$$\frac{d\nu_\varepsilon}{d\mu}(u) = \frac{1}{Z_\varepsilon(y)} \exp(-\Psi_\varepsilon(u)).$$

Using direct computations akin to the proof of [22], Theorem 5.2, we can then show  $\exists C > 0$  such that

$$d_{\text{TV}}(v_\varepsilon, \nu) \leq C \int_{\{\|u\|_{H^1} \geq R_0\}} (\varepsilon \|u\|_{H^1}^2 - R_0) \mu(du),$$

where  $d_{\text{TV}}$  denotes the usual TV metric on  $\mathcal{P}(H^1(\mathbb{T}))$ . In other words, so long as  $\mu$  has bounded moments of degree at least two the TV distance between  $v_\varepsilon$  and  $\nu$  can be made arbitrarily small by choosing a large  $R_0$ .

This perturbation of the likelihood  $\Psi_\varepsilon$  can also be viewed as a modification of the prior  $\mu$ , which results in including the term  $\min\{0, \varepsilon \|u\|_{H^1}^2 - R_0^2\}$  in the MH acceptance ratio. In other words, because we use a proposal kernel that preserves the original prior, an additional factor (not involving the likelihood potential  $\Psi$ ) shows up in the MH acceptance probability. Regardless of the interpretation, this example illustrates that while Assumption 1(c) may be difficult to verify in some examples, it often holds for a small perturbation of the problem. Since the term  $\min\{0, \varepsilon \|u\|^2 - R_0^2\}$  is zero near the origin, the dynamics of the Markov chain are entirely unchanged in a ball around the origin. Since we can take  $R_0$  as large as we like, in practice, this means that the RCAR algorithm corresponding to this perturbation is virtually identical to the original algorithm and the modification is needed only to control tail behavior necessary to prove exponential rates of convergence. These observations may also be taken as a sign that Assumption 1(c) is an artifact of our method of proof and can be relaxed to a more realistic assumption. This would be an interesting direction for future research.

**6. Conclusion.** In this article we analyzed the convergence properties of a class of MH algorithms on infinite-dimensional Banach spaces that use an RCAR type proposal kernel  $\mathcal{Q}$  with a likelihood ratio acceptance probability. We showed that under very general conditions on the likelihood potential  $\Psi$  and the proposal kernel  $\mathcal{Q}$  the algorithms have a spectral gap with respect to an appropriate Wasserstein-type semimetric  $\tilde{d}_q$  which implied exponential convergence to the target measure  $\nu$  in (1). Our results generalize the dimension-independent spectral gaps of [19] to a larger class of algorithms applicable to non-Gaussian prior measures.

Results showing spectral gaps in infinite dimensions are of particular interest in studying the computational complexity of MCMC. Often, a spectral gap on the infinite-dimensional space ensures that the variance of time-averaging estimators for finite-dimensional—and therefore computationally tractable—approximations of the Markov kernel is uniformly bounded as a function of dimension. Thus the computational complexity of the algorithm is simply a function of its per-step simulation cost. The results given here and those of [19] thus imply that RCAR algorithms are among the simplest MCMC algorithms whose computational complexity depends on dimension only through the per-step computational cost. This is of course a special feature of the Ornstein–Uhlenbeck-like proposal, as the random walk MH algorithm is known to have dimension-dependent spectral gap. It remains to be seen whether more sophisticated algorithms can also be designed to have similarly attractive dimensional scaling properties.

We further developed a general perturbation theory for approximations of MH algorithms; showing error bounds for computationally tractable approximations of the algorithm that is arguably more direct than previous works while offering similar error estimates. Our main result here was that given an exact MH kernel  $\mathcal{P}_0$ , an approximation  $\mathcal{P}_\varepsilon$  and an appropriate semimetric  $d$ , the distance between the invariant measures of  $\mathcal{P}_0$  and  $\mathcal{P}_\varepsilon$  can be bounded in terms of the one-step error  $d(\mathcal{P}_0\delta_u, \mathcal{P}_\varepsilon\delta_u)$ —an error bound that can often be shown using coupling arguments. We further applied our perturbation theory to the RCAR algorithm and obtained error bounds for various approximations including discretization of the likelihood potential  $\Psi$  by Galerkin projections as well as approximation of the prior  $\mu$ .

Our success in applying the weak Harris' theorem and perturbation theory to this large collection of Markov chains suggests the broad utility of this approach to studying Markov chains on infinite-dimensional state spaces and computationally tractable approximations thereof. The tendency of probability measures on infinite-dimensional spaces to be mutually singular limits the utility of traditional weighted TV norms in these settings. This suggests at the usefulness of alternative metrics such as the Wasserstein-type semimetrics employed here for studying sequences of problems of increasing dimension, which describes many applications of interest in modern statistics and stochastic dynamics.

## APPENDIX A: PROOF OF CONVERGENCE RESULTS FROM SECTION 3

**A.1. Proof of Theorem 1.** We recall two technical lemmata that are useful in the proof of Theorem 1 as well as the rest of the appendix.

**LEMMA 2.** *For every  $\varrho \in P(\mathcal{H})$ , there exists sufficiently large  $R > 0$  so that  $\varrho(B_R(0)) > 0$ .*

**LEMMA 3.** *Let  $w, v \in \mathcal{H}$ . Then for  $s \geq 0$*

$$\|w + v\|^s \leq 2^s (\|w\|^s + \|v\|^s).$$

**PROOF.** When  $s \in (0, 1)$  the inequality follows from the identity  $(a + b)^s \leq a^s + b^s$  for positive real numbers  $a$  and  $b$ . The case with  $s > 1$  follows from [49], Corollary 3.1. In fact, the constant  $2^s$  is not optimal and can be replaced by  $1 \vee 2^{s-1}$ , but it makes for convenient notation.  $\square$

Let us outline a roadmap of the proof that follows the proof strategy of [19]. Two cases are considered:  $u \in B_R(0)$  and the complement of this event. The first case is dispensed with using moment conditions on  $\lambda$  and  $\mathcal{K}$  to bound  $\sup_{u \in B_R(0)} (\mathcal{P}V)(u)$ . The second case is more difficult. Here we pick an event  $A$  such that  $\mathbb{P}(A) > 0$  uniformly for all  $u \in B_R(0)^c$  and prove the existence of a uniform constant  $\tilde{\kappa} \in (0, 1)$  so that

$$(\mathcal{P}V)(u) \leq \tilde{\kappa} \mathbb{P}(A) V(u) + \int_{A^c} \{V(u) \vee V(\zeta_u + \xi)\} \mathcal{K}(u, d\zeta_u) \lambda(d\xi),$$

and show that the integral term on the right hand side is uniformly bounded as well. In [19], conditional on  $u \in B_R(0)^c$ , the event  $A = \{\xi \in B_r(\beta u) : \xi \sim \mu_\beta\}$  is considered. Because the potential is (eventually) increasing in the tails, the probability of accepting conditional on  $A$  can be uniformly bounded away from 0. Further, because the pCN proposal is centered at  $\beta u$  for some constant  $\beta \in (0, 1)$ , this event always has positive probability when  $\lambda$  is Gaussian. This, combined with control of the moments of  $\lambda$  and the fact that when rejection occurs,  $V$  does not increase, is enough to prove that  $V$  contracts far from the origin for pCN. Our proof uses the event

$$A = \{\|\zeta_u - \beta_0 u\| < b_0(1 - \beta_0)\|u\| \cap \|\xi\| < b_1\|\zeta\| : \xi \sim \lambda, \zeta_u \sim \mathcal{K}(u, d\zeta_u)\}.$$

The key difference is that we must now consider the behavior of  $\xi$  and  $\zeta_u$  together, and control them simultaneously to ensure that the acceptance probabilities conditional on  $A$  can be uniformly bounded from below when  $u$  is far from the origin. This introduces complications in the second part of the argument.

PROOF OF THEOREM 1. Fix  $R > 0$  and  $u \in B_R(0)$  then, using Assumption 2(a) and Lemma 3 we have

$$\begin{aligned}
 \sup_{u \in B_R(0)} (\mathcal{P}V)(u) &= \sup_{u \in B_R(0)} \int_{\mathcal{H}} V(v) \alpha(u, v) \mathcal{Q}(u, dv) \\
 &\quad + V(u) \left( 1 - \int_{\mathcal{H}} \alpha(u, v) \mathcal{Q}(u, dv) \right) \\
 &\leq \sup_{u \in B_R(0)} V(u) + \int_{\mathcal{H}} V(v) \mathcal{Q}(u, dv) \\
 &\leq R^p + \sup_{u \in B_R(0)} \int_{\mathcal{H} \times \mathcal{H}} \|\zeta_u + \xi\|^p \mathcal{K}(u, d\zeta_u) \lambda(d\xi) \\
 &\leq R^p + 2^p \sup_{u \in B_R(0)} \int_{\mathcal{H} \times \mathcal{H}} (\|\zeta_u\|^p + \|\xi\|^p) \mathcal{K}(u, d\zeta_u) \lambda(d\xi) \\
 &\leq R^p + 2^p \sup_{u \in B_R(0)} \|u\|^p + \int_{\mathcal{H}} \|\xi\|^p \lambda(d\xi) \\
 &\leq R^p + 2^p R^p + C_0 \equiv K_1.
 \end{aligned}$$

Now consider  $u \in B_R(0)^c$ ,  $1 > \beta_0 > b_0 > 0$ , as in Assumption 2(b) and define  $A$  to be the event

$$A := \{ \|\zeta_u - \beta_0 u\| \leq b_0(1 - \beta_0)\|u\|, \|\xi\| < b_1\|\zeta_u\| : \zeta_u \sim \mathcal{K}(u, d\zeta_u), \xi \sim \lambda \},$$

where  $b_1 \in (0, 1)$  is a constant to be specified. Observe that in the event of  $A$  we have that  $\|\zeta_u\| \geq (\beta_0 - b_0(1 - \beta_0))\|u\| > 0$ . Thus, by the independence of  $\zeta_u$  and  $\xi$  we have

$$\mathbb{P}[A|u] \geq \mathbb{P}[\|\zeta_u - \beta_0 u\| \leq b_0(1 - \beta_0)\|u\|] \mathbb{P}[\|\xi\| < b_1(\beta_0 - b_0(1 - \beta_0))R].$$

By Assumption 2(b) we have

$$\mathbb{P}[\|\zeta_u - \beta_0 u\| \leq b_0(1 - \beta_0)\|u\|] \geq \epsilon_0.$$

On the other hand, for fixed  $1 > \beta_0 > b_0 > 0$  and  $b_1$  it follows from Lemma 2 that if  $R$  is sufficiently large then  $\mathbb{P}[\|\xi\| < b_1(\beta_0 - b_0(1 - \beta_0))R] \geq \epsilon_1 > 0$ . To this end,  $\mathbb{P}[A|u] \geq \epsilon_0 \epsilon_1 > 0$ . Furthermore, we have that in the event of  $A$

$$\begin{aligned}
 \|\zeta_u + \xi\|^p &\leq [(1 + b_1(\beta_0 - b_0(1 - \beta_0)))\|\zeta_u\|]^p \\
 &\leq [(1 + b_1(\beta_0 - b_0(1 - \beta_0)))(\beta_0 + b_0(1 - \beta_0))]^p \|u\|^p \\
 &\leq \kappa_1 \|u\|^p.
 \end{aligned}$$

Now if  $b_1$  is sufficiently small then  $\kappa_1 < 1$ . In summary given  $b_0, \beta_0 \in (0, 1)$ , which depend on the kernel  $\mathcal{K}$ , we choose  $b_1$  so that  $\kappa_1 < 1$  and then we choose  $R$  large enough so that  $\epsilon_0 \epsilon_1 > 0$ . It then follows that in the event of  $A$  we have  $V(\zeta_u + \xi) \leq \kappa_1 V(u)$ . Now we have

$$\begin{aligned}
 (\mathcal{P}V)(u) &\leq \mathbb{P}(A) [\mathbb{P}(\text{accept}|A) \kappa_1 V(u) + \mathbb{P}(\text{reject}|A) V(u)] \\
 &\quad + \int_{A^c} \{V(\zeta_u + \xi) \vee V(u)\} \mathcal{K}(u, d\zeta_u) d\lambda(\xi) \\
 &= \mathbb{P}(A) [(1 - (1 - \kappa_1)) \mathbb{P}(\text{accept}|A)] V(u) \\
 &\quad + \int_{A^c} \{V(\zeta_u + \xi) \vee V(u)\} \mathcal{K}(u, d\zeta_u) d\lambda(\xi) \\
 &\leq \kappa_2 \mathbb{P}(A) V(u) + \int_{A^c} \{V(\zeta_u + \xi) \vee V(u)\} \mathcal{K}(u, d\zeta_u) d\lambda(\xi),
 \end{aligned}$$

where  $\kappa_2 = (1 - (1 - \kappa_1))\mathbb{P}(\text{accept}|A)$ . Since  $\Psi$  satisfies Assumption 1(c), given  $\tilde{\beta} \in (0, 1)$  we can take  $R > R_0$  which implies  $\mathbb{P}(\text{accept} | A) > 0$  uniformly for all  $u \in B_R(0)^c$  and so it follows that  $\kappa_2 < 1$  uniformly over  $B_R(0)^c$ . It remains to bound the last integral:

$$\begin{aligned}
 & \int_{A^c} \{V(\zeta_u + \xi) \vee V(u)\} \mathcal{K}(u, d\zeta_u) d\lambda(\xi) \\
 &= \int_{A^c} \|\zeta_u + \xi\|^p \vee \|u\|^p \mathcal{K}(u, d\zeta_u) \lambda(d\xi) \\
 &\leq \int_{A^c} (\|\zeta_u\| + \|\xi\|)^p \vee \|u\|^p \mathcal{K}(u, d\zeta_u) \lambda(d\xi) \\
 &\leq \int_{A^c} (\|u\| + \|\xi\|)^p \mathcal{K}(u, d\zeta_u) \lambda(d\xi) \\
 &= \int_{A^c} \sum_{k=0}^p \binom{p}{k} \|u\|^{p-k} \|\xi\|^k \lambda(d\xi) \\
 &= \|u\|^p \mathbb{P}(A^c) + \sum_{k=1}^p \binom{p}{k} \|u\|^{p-k} \int_{A^c} \|\xi\|^k \lambda(d\xi) \\
 &\leq \|u\|^p \mathbb{P}(A^c) + \frac{(1 - \kappa_2)\epsilon_0\epsilon_1}{2} \|u\|^p + K_2,
 \end{aligned}$$

where we used Assumption 2(a) to bound  $\|\zeta_u\|$  by  $\|u\|$ , and the last step followed because the second term in the penultimate line is a polynomial in  $\|u\|$  of order  $p - 1$ . Since  $R > 1$ , this term can be bounded by  $c\|u\|^p + K_2$  for any  $c > 0$ , where  $K_2$  depends on  $c$  but not  $u$ . Substituting the above result back into the bound on  $(\mathcal{P}V)(u)$  gives

$$\begin{aligned}
 (\mathcal{P}V)(u) &\leq \left( \kappa_2 \mathbb{P}(A) + \mathbb{P}(A^c) + \frac{(1 - \kappa_2)\epsilon_0\epsilon_1}{2} \right) V(u) + K_2 \\
 &\leq \left[ 1 - (1 - \kappa_2)\epsilon_0\epsilon_1 + \frac{(1 - \kappa_2)\epsilon_0\epsilon_1}{2} \right] V(u) + K_2 \\
 &\leq \kappa V(u) + K_2,
 \end{aligned}$$

for  $\kappa = 1 - \frac{(1 - \kappa_2)\epsilon_0\epsilon_1}{2} \in (0, 1)$ , which does not depend on  $u$ . Setting  $K = K_1 + K_2$  we obtain the desired result

$$(\mathcal{P}V)(u) \leq \kappa V(u) + K \quad \forall u \in \mathcal{H}. \quad \square$$

**A.2. Proof of Theorem 2.** The proof of this theorem follows from Theorem 1 and Propositions 1, 3 and 4, which together establish that the  $n$ -step kernel  $\mathcal{P}^n$  is contracting for  $d_q$  and the level sets of the Lyapunov functions  $V(u) = \sum_{j=0}^p a_j \|u\|^j$  are  $d_q$ -small.

First, let us define the notation

$$(34) \quad d^*(u, v) := \frac{(1 + \eta\|u\| + \eta\|v\|)^q \|u - v\|}{\omega}, \quad u, v \in \mathcal{H},$$

for  $q, \eta, \omega > 0$ . Recall that by (19) we simply have  $d_q(u, v) = 1 \wedge d^*(u, v)$ . We then have the following auxiliary lemma concerning  $d^*$  and  $d_q$ .

**LEMMA 4.** *Let  $q \geq 0$ . Then  $d_q$  and  $d^*$  satisfy the following properties:*

(a) *If  $\eta, d_q(u, v) < 1$  then*

$$\eta^q (1 + \|u\| + \|v\|)^q \|u - v\| < \omega \quad \text{and} \quad \frac{\eta^q}{\omega} \|u - v\| \leq d_q(u, v).$$

(b) Let  $u, v, \zeta_u, \zeta_v \in \mathcal{H}$  such that

$$d_q(u, v) < 1 \quad \text{and} \quad \|\zeta_u\| \leq \|u\| \quad \text{and} \quad \|\zeta_v\| \leq \|v\|.$$

Define proposals  $u^*, v^*$  as in (35). Then

$$\frac{d^*(u^*, v^*)}{d^*(u, v)} \leq (1 + 2\eta\|\xi\|^q) \frac{\|\zeta_u - \zeta_v\|}{\|u - v\|}.$$

PROOF. Statement (a) follows from the fact that  $d_q(u, v) = d^*(u, v)$  whenever  $d_q(u, v) < 1$ . Then assuming  $\eta < 1$  we have the series of inequalities

$$\begin{aligned} \frac{\eta^q}{\omega} \|u - v\| &\leq \frac{\eta^q(1 + \|u\| + \|v\|)^q}{\omega} \|u - v\| \\ &\leq \frac{(1 + \eta\|u\| + \eta\|v\|)^q}{\omega} \|u - v\| = d_q(u, v) < 1, \end{aligned}$$

from which the statements follow. Now (b) can be proven directly by the following calculation:

$$\begin{aligned} d^*(u^*, v^*) &= \frac{1}{\omega} (1 + \eta\|\zeta_u + \xi\| + \eta\|\zeta_v + \xi\|)^q \|\zeta_u - \zeta_v\| \\ &\leq \frac{1}{\omega} (1 + \eta(\|\zeta_u\| + \|\xi\|) + \eta(\|\zeta_v\| + \|\xi\|))^q \|\zeta_u - \zeta_v\| \\ &= \left( \frac{1 + \eta(\|\zeta_u\| + \|\xi\|) + \eta(\|\zeta_v\| + \|\xi\|)}{1 + \eta\|u\| + \eta\|v\|} \right)^q \frac{\|\zeta_u - \zeta_v\|}{\|u - v\|} d^*(u, v) \\ &\leq (1 + 2\eta\|\xi\|)^q \frac{\|\zeta_u - \zeta_v\|}{\|u - v\|} d^*(u, v). \end{aligned} \quad \square$$

We are now ready to show that  $\mathcal{P}$  is  $d_q$ -contracting for appropriate choices of  $\omega, \eta$  in (19).

**PROPOSITION 3** (Contracting for  $d_q$ ). *Suppose conditions of Theorem 2 are satisfied. Then  $\mathcal{P}$  is contracting for  $d_q$  if  $\eta, \omega/\eta^q > 0$  are sufficiently small.*

Our proof strategy is as follows: Since the  $d_q$  semimetric for measures is defined as the infimum over couplings, naturally our argument relies on showing that there exists a coupling for which the desired contraction property holds when the two chains start close to each other. Our approach shares some similar features with [19], as well as with earlier work. The proof in [19], Sections 3.1.2 and 3.2.2, uses a “basic” or “same-noise” coupling of pCN proposals that is well known in coupling of diffusion processes (see, e.g., [32]), along with utilizing the same uniform random variable to make the accept-reject decision for the two coupled chains. This coupling has also appeared in the statistics literature, where it is used for convergence diagnosis [29], page 164. We use a different coupling in our proof. More precisely, we consider two chains starting at  $(u, v)$  and propose

$$u^* = \zeta_u + \xi, \quad v^* = \zeta_v + \xi,$$

where  $(\zeta_u, \zeta_v) \sim \varpi_{u,v}$ , the coupling in Assumption 2(c). and  $\xi \sim \lambda$ . we then utilize the same uniform random number  $\varsigma$  to make the accept/reject decision; recall Algorithm 1. The existence of a  $\|\cdot\|$ -contractive coupling  $\varpi_{u,v}$  is necessary at this point as without this condition the  $d_q$ -contraction condition can easily fail.

With the above coupling at hand our proof then proceeds by considering three possible outcomes to show the desired contractility results: either both chains accept, both reject, or

one rejects and one accepts. As is the case in proving these properties for the MH algorithms such as pCN, the last case is the hardest, since in principle the two components can land far from one another in  $d_q$ . In our setting the argument is somewhat lengthy since we need to control both  $\zeta_u$  and  $\zeta_v$  as well as the innovation  $\xi$  at the same time.

PROOF. Pick  $u, v \in \mathcal{H}$  so that  $d_q(u, v) < 1$  implying that  $(1 + \eta\|u\| + \eta\|v\|)^q \|u - v\| < \omega$  and fix  $\beta \in (0, 1)$ . Let  $\varpi_{u,v} \in \Upsilon(\mathcal{K}\delta_u, \mathcal{K}\delta_v)$  be the coupling in Assumption 2(c). We then define  $\pi_0 \in \Upsilon(\mathcal{P}\delta_u, \mathcal{P}\delta_v)$  the *basic coupling* between the  $u$  and  $v$  chains by the following procedure. Draw  $(\zeta_u, \zeta_v) \sim \varpi_{u,v}$ ,  $\xi \sim \lambda$ , and consider proposals

$$(35) \quad u^* = \zeta_u + \xi, \quad v^* = \zeta_v + \xi.$$

Then draw  $\varsigma \sim \mathcal{U}([0, 1])$  and accept  $u^*$  if  $\varsigma \leq \alpha(u, u^*)$  and accept  $v^*$  if  $\varsigma \leq \alpha(v, v^*)$ . That is, the two chains use the same innovation  $\xi$  and uniform random variable  $\varsigma$  for the accept-reject step.

Now pick  $R > 0$  sufficiently large so that  $R - 1 > R_0$  where  $R_0$  is as in Assumption 1(c). We will present the proof for two cases where  $u, v \in B_R(0)$  and  $u, v \in B_{R-1}(0)^c$ . Note that if  $\omega/\eta^q < 1$  we can guarantee

$$(36) \quad \begin{aligned} & \{u, v \in \mathcal{H} : d_q(u, v) < 1\} \\ &= \{u, v \in B_R(0) : d_q(u, v) < 1\} \cup \{u, v \in B_{R-1}(0)^c : d_q(u, v) < 1\}. \end{aligned}$$

To see this take  $u, v \in \mathcal{H}$  such that  $d_q(u, v) < 1$  and consider the nontrivial case where  $u, v$  do not belong to the same set  $B_R(0)$ . Without loss of generality let  $u \in B_R(0)$  and  $v \in B_R(0)^c \subset B_{R-1}(0)^c$ . By Lemma 4(a) we have  $\|u - v\| < \frac{\omega}{\eta^q} < 1$  and so  $u \in B_1(v)$ . But  $\{u : v \in B_R(0)^c \text{ and } u \in B_1(v)\} = B_{R-1}(0)^c$  and so  $u, v \in B_{R-1}(0)^c$ .

Let us proceed with the proof starting with the case where  $u, v \in B_R(0)$ . Let  $D$  be the event where  $\|\xi\| \leq R$  and  $\|\zeta_u - \zeta_v\| < \tilde{\beta}\|u - v\|$  where  $\tilde{\beta} \in [\beta_c, 1)$  and  $\beta_c$  is the constant in Assumption 2(c). Due to independence of  $(\zeta_u, \zeta_v)$  and  $\xi$  we have

$$\mathbb{P}(D) = \mathbb{P}(\|\xi\| \leq R) \mathbb{P}(\|\zeta_u - \zeta_v\| < \tilde{\beta}\|u - v\|).$$

By Lemma 2  $\mathbb{P}(\|\xi\| \leq R) > 0$  if  $R$  is sufficiently large. Furthermore, by Markov's inequality

$$\mathbb{P}(\|\zeta_u - \zeta_v\| \geq \tilde{\beta}\|u - v\|) \leq \frac{\mathbb{E}\|\zeta_u - \zeta_v\|}{\tilde{\beta}\|u - v\|} \leq \frac{\beta_c}{\tilde{\beta}} < 1.$$

Thus,  $\mathbb{P}(D) \geq \epsilon_1 > 0$  uniformly for all  $u, v \in B_R(0)$ . Recalling that  $d_q \leq 1$ , we have

$$\begin{aligned} d_q(\mathcal{P}\delta_u, \mathcal{P}\delta_v) &\leq \int_{\mathcal{H} \times \mathcal{H}} d_q(s, t) \pi_0(ds, dt) \\ &\leq \int_D [\mathbb{P}(\text{both accept} | \zeta_u, \zeta_v, \xi) d_q(u^*, v^*) \\ &\quad + \mathbb{P}(\text{both reject} | \zeta_u, \zeta_v, \xi) d_q(u, v)] \varpi_{u,v}(d\zeta_u, d\zeta_v) \lambda(d\xi) \\ &\quad + \int_{D^c} [d_q(u^*, v^*) \vee d_q(u, v)] \varpi_{u,v}(d\zeta_u, d\zeta_v) \lambda(d\xi) \\ &\quad + \mathbb{P}(\text{only one is accepted}) \\ &=: T_1 + T_2 + T_3. \end{aligned}$$

*Bound on  $T_1$ .* Since  $\mathbb{P}(\text{both accept} | \zeta_u, \zeta_v, \xi) \leq 1 - \mathbb{P}(\text{both reject} | \zeta_u, \zeta_v, \xi)$  then

$$\begin{aligned} T_1 &\leq \int_D [\mathbb{P}(\text{both accept} | \zeta_u, \zeta_v, \xi) d_q(u^*, v^*) \\ &\quad + [1 - \mathbb{P}(\text{both accept} | \zeta_u, \zeta_v, \xi)] d_q(u, v)] \varpi_{u,v}(d\zeta_u, d\zeta_v) \lambda(d\xi) \end{aligned}$$

$$\begin{aligned}
&= \int_D \mathbb{P}(\text{both accept} | \zeta_u, \zeta_v, \xi) [d_q(u^*, v^*) - d_q(u, v)] \varpi_{u,v}(d\zeta_u, d\zeta_v) \lambda(d\xi) \\
&\quad + \mathbb{P}(D) d_q(u, v).
\end{aligned}$$

By the definition of the set  $D$  and Lemma 4(b) we can write

$$\begin{aligned}
&T_1 - \mathbb{P}(D) d_q(u, v) \\
&\leq \int_D \mathbb{P}(\text{both accept} | \zeta_u, \zeta_v, \xi) [(1 + 2\eta R)^q \tilde{\beta} - 1] d_q(u, v) \varpi_{u,v}(d\zeta_u, d\zeta_v) \lambda(d\xi).
\end{aligned}$$

By Assumption 1(a) and (b)  $\mathbb{P}(\text{both accept} | \zeta_u, \zeta_v, \xi) \geq \epsilon_2 \geq 0$  uniformly for all  $u, v \in B_R(0)$  and so

$$\begin{aligned}
T_1 &\leq \mathbb{P}(D) [1 + \epsilon_2 ((1 + 2\eta R)^q \tilde{\beta} - 1)] d_q(u, v) \\
&= \mathbb{P}(D) (1 - \kappa).
\end{aligned}$$

*Bound on  $T_2$ .* Using Lemma 4(b) and Assumption 2(c) we can write

$$\begin{aligned}
T_2 &= \mathbb{P}(D^c) \mathbb{E}(d_q(u^*, v^*) \vee d_q(u, v) | D^c) \\
&\leq d_q(u, v) \mathbb{E}\left(1 \vee \frac{d_q(u^*, v^*)}{d_q(u, v)} \middle| D^c\right) \\
&\leq d_q(u, v) \mathbb{E}\left(1 \vee (1 + 2\eta \|\xi\|)^q \frac{\|\zeta_u - \zeta_v\|}{\|u - v\|} \middle| D^c\right) \\
&\leq d_q(u, v) \mathbb{E}((1 + 2\eta \|\xi\|)^{\lceil q \rceil} | D^c) \\
&\leq d_q(u, v) \mathbb{E}(1 + C(2\eta \|\xi\|)^{\lceil q \rceil} | D^c) \\
&\leq d_q(u, v) (\mathbb{P}(A^c) + \eta^{\lceil q \rceil} C_R),
\end{aligned}$$

where  $C > 0$  depends on  $\lceil q \rceil$  and is bounded following the binomial expansion formula. Then  $C_R > 0$  is a bounded constant due to the fact that  $\lambda$  has bounded moments of degree  $p \geq \lceil q \rceil$ .

*Bound on  $T_3$ .* Using Lemma 3, Assumption 2(c), and Lemma 4(a) we have

$$\begin{aligned}
T_3 &= \int_{\mathcal{H}} \int_{\mathcal{H} \times \mathcal{H}} \mathbb{P}[\text{only one is accepted} | \zeta_u, \zeta_v, \xi] \varpi_{u,v}(d\zeta_u, d\zeta_v) \lambda(d\xi) \\
&= \int_{\mathcal{H}} \int_{\mathcal{H} \times \mathcal{H}} \mathbb{P}[\zeta \text{ between } \alpha(u, u^*) \text{ and } \alpha(v, v^*) | \zeta_u, \zeta_v, \xi] \varpi_{u,v}(d\zeta_u, d\zeta_v) \lambda(d\xi) \\
&\leq \int_{\mathcal{H}} \int_{\mathcal{H} \times \mathcal{H}} |\Psi(u) - \Psi(v)| + |\Psi(\zeta_u + \xi) - \Psi(\zeta_v + \xi)| \varpi_{u,v}(d\zeta_u, d\zeta_v) \lambda(d\xi) \\
&\leq L(1 \vee \|u\|^q \vee \|v\|^q) \|u - v\| \\
&\quad + L \int_{\mathcal{H}} \int_{\mathcal{H} \times \mathcal{H}} (1 \vee \|\zeta_u + \xi\|^q \vee \|\zeta_v + \xi\|^q) \|\zeta_u - \zeta_v\| \varpi_{u,v}(d\zeta_u, d\zeta_v) \lambda(d\xi) \\
&\leq LR^q \|u - v\| + L \int_{\mathcal{H}} \int_{\mathcal{H} \times \mathcal{H}} [1 \vee 2^q (\|\zeta_u\|^q + \|\xi\|^q) \vee 2^q (\|\zeta_v\|^q + \|\xi\|^q)] \\
&\quad \times \|\zeta_u - \zeta_v\| \varpi_{u,v}(d\zeta_u, d\zeta_v) \lambda(d\xi) \\
&\leq LR^q \|u - v\| \\
&\quad + L \int_{\mathcal{H}} \int_{\mathcal{H} \times \mathcal{H}} (1 + 2^q R^q + 2^q \|\xi\|^q) \|\zeta_u - \zeta_v\| \varpi_{u,v}(d\zeta_u, d\zeta_v) \lambda(d\xi)
\end{aligned}$$

$$\begin{aligned}
&\leq LR^q \|u - v\| + L \int_{\mathcal{H}} (1 + 2^q R^q + 2^q \|\xi\|^q) \beta_c \|u - v\| \lambda(d\xi) \\
&\leq LR^q \|u - v\| + L\beta_c \|u - v\| \int_{\mathcal{H}} 1 + 2^q R^q + 2^q \|\xi\|^q \lambda(d\xi) \\
&\leq L \frac{\omega}{\eta^q} C'_R d_q(u, v).
\end{aligned}$$

Here  $C'_R > 0$  is a uniform constant independent of  $u, v$  which is bounded since  $\lambda$  has bounded moments of degree  $p \geq [q]$ . Putting together the bounds for  $T_1, T_2$  and  $T_3$  we finally have

$$\begin{aligned}
&d_q(\mathcal{P}\delta_u, \mathcal{P}\delta_v) \\
&\leq [\mathbb{P}(D)[1 + \epsilon_2((1 + 2\eta R)^q \tilde{\beta} - 1)] + \mathbb{P}(D^c) + \eta^{[q]} C_R + \eta^{-q} \omega L C'_R] d_q(u, v).
\end{aligned}$$

Since  $\epsilon_2 \in (0, 1)$ ,  $\tilde{\beta} < 1$ , and  $R > 0$  are uniform constants, we can choose  $\eta$  sufficiently small so that  $1 + \epsilon_2((1 + 2\eta R)^q \tilde{\beta} - 1) < 1$ . The constants  $C_R, C'_R > 0$  are also uniform and so we can choose  $\eta$  and  $\omega/\eta^q$  sufficiently small so that the term inside the square brackets is less than one which gives the desired result

$$d_q(\mathcal{P}\delta_u, \mathcal{P}\delta_v) \leq \gamma_1 d_q(u, v),$$

for some  $\gamma_1 \in (0, 1)$ .

Let us now consider the case where  $u, v \in B_{R-1}(0)^c$ . The method of proof is very similar to the first case where  $u, v \in B_R(0)$  and so we only highlight the differences. Let  $\tilde{D}$  be the event where  $\|\xi\| \leq R - 1$  and  $\|\zeta_u - \zeta_v\| \leq \tilde{\beta} \|u - v\|$  where as before  $\tilde{\beta} \in [\beta_c, 1)$ . The same argument as before yields that  $\mathbb{P}(\tilde{D}) \geq \epsilon_3 > 0$ . Furthermore, using the same argument as before we can write

$$\begin{aligned}
d_q(\mathcal{P}\delta_u, \mathcal{P}\delta_v) &\leq \int_{\tilde{D}} [\mathbb{P}(\text{both accept} | \zeta_u, \zeta_v, \xi) d_q(u^*, v^*) \\
&\quad + \mathbb{P}(\text{both reject} | \zeta_u, \zeta_v, \xi) d_q(u, v)] \varpi_{u,v}(d\zeta_u, d\zeta_v) \lambda(d\xi) \\
&\quad + \int_{\tilde{D}^c} [d_q(u^*, v^*) \vee d_q(u, v)] \varpi_{u,v}(d\zeta_u, d\zeta_v) \lambda(d\xi) \\
&\quad + \mathbb{P}(\text{only one is accepted}) \\
&=: T'_1 + T'_2 + T'_3.
\end{aligned}$$

By the same argument used to bound  $T_1$  we have

$$T'_1 \leq \mathbb{P}(\tilde{D}) [1 + \epsilon_4((1 + 2\eta(R - 1))^q \tilde{\beta} - 1)] d_q(u, v),$$

where here  $\epsilon_4$  is a constant so that  $\mathbb{P}(\text{both accept} | \zeta_u, \zeta_v, \xi) \geq \epsilon_4$  uniformly over  $\tilde{D}$ . By Assumption 1(c)  $\epsilon_4 > 0$  uniformly for all  $u, v \in B_{R-1}(0)^c$ . Furthermore, we bound  $T'_2$  identically to  $T_2$ ,

$$T'_2 \leq d_q(u, v) (\mathbb{P}(\tilde{D}^c) + \eta^{[q]} C_{R-1}),$$

using Lemma 4(b) and Assumption 2(b); and bound  $T'_3$  identically to  $T_3$ ,

$$T'_3 \leq \frac{L\omega}{\eta^q} C'_{R-1} d_q(u, v),$$

using Lemmata 3 and 4(a) as well as Assumption 2(c). In the above bounds  $C_{R-1}, C'_{R-1} > 0$  are uniform constants since  $\lambda$  has bounded moments of degree  $p \geq [q]$ . Thus, we have the bound

$$\begin{aligned}
d_q(\mathcal{P}\delta_u, \mathcal{P}\delta_v) &\leq [\mathbb{P}(\tilde{D}) [1 + \epsilon_4((1 + 2\eta(R - 1))^q \tilde{\beta} - 1)] \\
&\quad + \mathbb{P}(\tilde{D}^c) + \eta^{[q]} C_{R-1} + \eta^{-q} \omega L C'_{R-1}] d_q(u, v).
\end{aligned}$$

Once again choosing  $\eta$  and  $\omega/\eta^q$  sufficiently small we obtain

$$d_q(\mathcal{P}\delta_u, \mathcal{P}\delta_v) \leq \gamma_2 d_q(u, v),$$

for some constant  $\gamma_2 \in (0, 1)$ . Combining our results for the two cases of  $u, v \in B_R(0)$  and  $u, v \in B_{R-1}(0)^c$  we have the desired bound

$$d_q(\mathcal{P}\delta_u, \mathcal{P}\delta_v) \leq (\gamma_1 \vee \gamma_2) d_q(u, v). \quad \square$$

**PROPOSITION 4** ( $d_q$ -small  $V$  level-sets). *Suppose the conditions of Theorem 2 are satisfied and let  $S(R) = \{u \mid V(u) \leq R\}$  for some  $R > 0$ . Then there exists an integer  $n \geq 1$  and a constant  $\tilde{\gamma}_2 \in (0, 1)$  so that*

$$d_q(\mathcal{P}^n \delta_u, \mathcal{P}^n \delta_v) \leq \tilde{\gamma}_2 \quad \forall u, v \in S(R).$$

Following a similar approach to [19], we prove this proposition using the coupling introduced in the proof of Proposition 3 and conditioning on the event that the coupled proposals are accepted  $n$  times in a row. The probability of this event is uniformly bounded away from zero on sublevel sets of  $V$  following Assumption 1(a), (b), which is critical in making the argument. Using the fact that the sublevel sets of  $V$  have finite diameter we then show that if  $n$  is sufficiently large then eventually the coupled chains draw within  $d_q$ -distance one.

**PROOF.** Fix  $R > a_0$  and let  $R_* > 0$  be the solution of the equation  $\sum_{j=0}^p a_j R_*^j - R = 0$ ; note that  $R_*$  is unique so long as  $a_j \geq 0$  and there is at least one coefficient  $a_j > 0$  as it is the root of a monotone polynomial on  $(0, +\infty)$ . Let  $\pi_0$  be the basic coupling used in the proof of Proposition 3. We use  $(u_k, v_k)$  to denote the chain after step  $k$  with initial points  $u_0 = u, v_0 = v \in S$  and denote the innovation at each step with  $\xi_k$ . Fix  $\tilde{\beta} \in [\beta_c, 1)$  and consider the events  $D_k$  for  $k = 1, \dots, n$  where  $\|\zeta_{u_{k-1}} - \zeta_{v_{k-1}}\| < \tilde{\beta} \|u_{k-1} - v_{k-1}\|$  and  $\|\xi_k\| < r/n$  for some constant  $r > 0$  to be specified. The events  $D_k$  are similar to the event  $D$  from the proof of Proposition 3.

Let  $E$  be the event that the the proposals  $u_k^* = \zeta_{u_{k-1}}^* + \xi_k$  and  $v_k^* = \zeta_{v_{k-1}}^* + \xi_k$  are accepted  $n$  times in a row conditioned on the intersection of the events  $D_k$ . Thus, conditional on  $E$  we have

$$\begin{aligned} d_q(u_n, v_n) &\leq \frac{(1 + \eta \|u_n\| + \eta \|v_n\|)^q}{\omega} \|u_n - v_n\| \\ (37) \quad &\leq \frac{\tilde{\beta}^n (1 + \eta \|u\| + \eta \|v\| + 2\eta r)^q}{\omega} \|u - v\| \\ &\leq (1 + 2\eta R_* + 2\eta r)^q \frac{\tilde{\beta}^n}{\omega} \text{diam } S, \end{aligned}$$

where we used  $\text{diam } S := \sup_{u, v \in S} \|u - v\|$  to denote the diameter of  $S$ . Choosing

$$n = \left\lceil \frac{1}{\log \tilde{\beta}} \log \left( \frac{\omega}{2(1 + 2\eta R_* + 2\eta r)^q \text{diam } S} \right) \right\rceil$$

conditional on  $E$ , we have  $d(u_n, v_n) < 1/2$  and so

$$\sup_{u, v \in S(R)} d_q(\mathcal{P}^n \delta_u, \mathcal{P}^n \delta_v) \leq \mathbb{P}(E) \frac{1}{2} + (1 - \mathbb{P}(E)) < 1.$$

It remains to show that  $\mathbb{P}(E) > 0$ . By Lemma 2 we can choose  $r$  large enough that  $\mathbb{P}(\|\xi_k\| \leq r/n) > 0$  uniformly for all  $k$ . Furthermore, using an identical argument as in the

proof of Proposition 3 to show  $\mathbb{P}(D) > 0$ , we can use Assumption 2(b) and Markov's inequality to show that  $\mathbb{P}(\|\zeta_{u_{k-1}} - \zeta_{v_{k-1}}\| \leq \tilde{\beta} \|u_{k-1} - v_{k-1}\|) > 0$  uniformly for all  $k = 1, \dots, n$ . This follows because all pairs  $(u_k, v_k)$  are contained within  $B_{R_*+r}(0)$ . Thus there exists  $\epsilon > 0$  so that

$$\inf_{u_0, v_0 \in S(R)} \inf_{k \in \{1, \dots, n\}} \mathbb{P}\left(D_k \mid \bigcap_{j=1}^{k-1} D_j\right) \geq \epsilon > 0.$$

Let  $I = \bigcap_{k=1}^n D_k$ . Then by the law of total probability

$$\inf_{u_0, v_0 \in S(R)} \mathbb{P}(I) \geq \epsilon^n > 0.$$

On the other hand, by Assumption 1(a) and (b),  $\Psi$  is bounded above and below on bounded sets and so

$$\inf_{u_0, v_0 \in S(R)} \mathbb{P}(E \mid I) > 0.$$

Putting together the above lower bounds we obtain the desired result:

$$\inf_{u_0, v_0 \in S(R)} \mathbb{P}(E) = \inf_{u_0, v_0 \in S(R)} \mathbb{P}(I) \mathbb{P}(E \mid I) > 0. \quad \square$$

**PROOF OF THEOREM 2.** Propositions 3 and 4 show that  $d_q$  is contracting for  $\mathcal{P}$  and that the sublevel sets of the  $V$  are  $d_q$ -small; recall Definitions 4 and 5. Furthermore by Theorem 1 and Remark 1 we have that the function  $V$  as in (21) is a continuous Lyapunov function for  $\mathcal{P}$ . An application of Proposition 1 then completes the proof.  $\square$

**A.3. Proof of Theorem 3.** We present a direct proof of the Feller property showing that for any sequence  $u_j \rightarrow u$  and any function  $\varphi \in C_b(\mathcal{H})$  we have that  $\mathcal{P}\varphi(u_j) \rightarrow \mathcal{P}\varphi(u)$ . The main difficulty in the proof is the fact that the kernel  $\mathcal{K}(u, \cdot)$  depends on the point  $u$  in a nontrivial manner. To make matters more complicated we have to deal with integrals of the form  $\int_{\mathcal{H}} \varphi(x) \alpha(u_j, x) \mathcal{K}(u_j, dx)$  that we wish to show converge to  $\int_{\mathcal{H}} \varphi(x) \alpha(u, x) \mathcal{K}(u, dx)$ ; that is, both the integrand and the measure depend on the sequence  $u_j$  and so the dominated convergence theorem cannot be applied directly. However, by Assumption 2(c) we know that as  $u_j \rightarrow u$  we can construct a coupling  $\varpi_{u_j, u}$  of the random variables  $\zeta_j \sim \mathcal{K}(u_j, \cdot)$  and  $\zeta \sim \mathcal{K}(u, \cdot)$  in such way that  $\zeta_j \rightarrow \zeta$  a.s. This yields the weak convergence of  $\varpi_{u_j, u}$  to the trivial coupling  $(\text{Id} \times \text{Id})_{\#} \mathcal{K}(u, \cdot)$ . Using this property, the boundedness of  $\varphi$ , Lipschitz continuity of  $\alpha$  due to Assumption 1(d) and more standard applications of dominated convergence theorem we can then prove the desired result.

**PROOF.** Let  $\varphi \in C_b(\mathcal{H})$ , our goal is to show that  $\mathcal{P}\varphi \in C(\mathcal{H})$ . By (2) we have that

$$\begin{aligned} \mathcal{P}\varphi(u) &= \int_{\mathcal{H}} \int_{\mathcal{H}} \varphi(\zeta + \xi) \alpha(u, \zeta + \xi) \mathcal{K}(u, d\zeta) \lambda(d\xi) \\ &\quad + \varphi(u) \int_{\mathcal{H}} (1 - \alpha(u, \zeta + \xi)) \mathcal{K}(u, d\zeta) \lambda(d\xi) \\ &=: T_1(u) + T_2(u). \end{aligned}$$

In order to prove that  $\mathcal{P}$  is Feller we need to show that  $T_1, T_2$  are continuous. We establish this for  $T_1$ , as it is the more complicated of the two functions, the argument for  $T_2$  will follow from very similar steps but simpler since the function  $\varphi$  appears outside of the integral.

Let  $\{u_j\}$  be a sequence of points in  $\mathcal{H}$  converging to  $u$  and let  $\varpi_{u_j, u}$  be the coupling in Assumption 2(c) between  $\mathcal{K}(u_j, \cdot)$  and  $\mathcal{K}(u, \cdot)$ . We then have, for fixed  $\xi \in \mathcal{H}$  that

$$\begin{aligned}
 & \left| \int_{\mathcal{H}} \varphi(\zeta_j + \xi) \alpha(u_j, \zeta_j + \xi) \mathcal{K}(u_j, d\zeta_j) - \int_{\mathcal{H}} \varphi(\zeta + \xi) \alpha(u, \zeta + \xi) \mathcal{K}(u, d\zeta) \right| \\
 &= \left| \int_{\mathcal{H} \times \mathcal{H}} \varphi(\zeta_j + \xi) \alpha(u_j, \zeta_j + \xi) - \varphi(\zeta + \xi) \alpha(u, \zeta + \xi) \varpi_{u_j, u}(d\zeta_j, d\zeta) \right| \\
 &\leq \int_{\mathcal{H} \times \mathcal{H}} |\varphi(\zeta_j + \xi)| |\alpha(u_j, \zeta_j + \xi) - \alpha(u, \zeta + \xi)| \varpi_{u_j, u}(d\zeta_j, d\zeta) \\
 (38) \quad &+ \left| \int_{\mathcal{H} \times \mathcal{H}} [\varphi(\zeta_j + \xi) - \varphi(\zeta + \xi)] \alpha(u, \zeta + \xi) \varpi_{u_j, u}(d\zeta_j, d\zeta) \right| \\
 &\leq \|\varphi\|_{\infty} \int_{\mathcal{H} \times \mathcal{H}} |\alpha(u_j, \zeta_j + \xi) - \alpha(u, \zeta + \xi)| \varpi_{u_j, u}(d\zeta_j, d\zeta) \\
 &+ \int_{\mathcal{H} \times \mathcal{H}} |\varphi(\zeta_j + \xi) - \varphi(\zeta + \xi)| \varpi_{u_j, u}(d\zeta_j, d\zeta) \\
 &=: T'_j(\xi) + T''_j(\xi),
 \end{aligned}$$

where in the last inequality we used  $\|\varphi\|_{\infty} := \sup_{u \in \mathcal{H}} |\varphi(u)| < +\infty$  since  $\varphi \in C_b(\mathcal{H})$  in the first integral and also the fact that  $\alpha$  is positive and bounded by 1 by definition, in the second integral. We now aim to show that  $\int_{\mathcal{H}} T'_j(\xi) + T''_j(\xi) \lambda(d\xi) \rightarrow 0$  as  $u_j \rightarrow u$  implying that  $|T_1(u_j) - T_1(u)| \rightarrow 0$ .

By Assumption 1(d) the function  $\alpha$  is Lipschitz in both of its arguments. In fact,

$$\begin{aligned}
 |\alpha(u, v) - \alpha(w, z)| &\leq |\alpha(u, v) - \alpha(w, v)| + |\alpha(w, v) - \alpha(w, z)| \\
 &\leq |\Psi(u) - \Psi(w)| + |\Psi(v) - \Psi(z)| \\
 &\leq L(1 \vee \|u\|^q \vee \|w\|^q \vee \|v\|^q \vee \|z\|^q)(\|u - w\| + \|v - z\|).
 \end{aligned}$$

Using the above bound together with Assumption 2(a), (d) and Lemma 3 we can write

$$\begin{aligned}
 (39) \quad \frac{1}{\|\varphi\|_{\infty}} T'_j(\xi) &\leq L \int_{\mathcal{H} \times \mathcal{H}} (1 \vee \|u\|^q \vee \|u_j\|^q \vee \|\zeta + \xi\|^q \vee \|\zeta_j + \xi\|^q) \\
 &\quad \times [\|u_j - u\| + \|\zeta_j - \zeta\|] \varpi_{u_j, u}(d\zeta_j, d\zeta) \\
 &< 2^{q+2} L(1 + \|u\|^q + \|u_j\|^q + \|\xi\|^q) \|u_j - u\|.
 \end{aligned}$$

Thus, integrating with respect to  $\lambda$  and using the hypothesis that  $\lambda$  has bounded moments of degree  $q$  we obtain the bound

$$\begin{aligned}
 \frac{1}{\|\varphi\|_{\infty}} \int_{\mathcal{H}} T'_j(\xi) \lambda(d\xi) &< 2^{q+2} L \|u_j - u\| \left( 1 + \|u\|^q + \|u_j\|^q + \int_{\mathcal{H}} \|\xi\|^q \lambda(d\xi) \right) \\
 &\leq 2^{q+2} L(C + \|u\|^q + \|u_j\|^q) \|u_j - u\|,
 \end{aligned}$$

for some constant  $C > 0$ . From this bound we deduce that

$$\int_{\mathcal{H}} T'_j(\xi) \lambda(d\xi) \rightarrow 0 \quad \text{as } u_j \rightarrow u.$$

Now consider  $T''_j(\xi)$ . Let  $\varpi_u^* = (\text{Id} \times \text{Id})_{\#} \mathcal{K}(u, \cdot)$  be the trivial coupling obtained by drawing  $\zeta' \sim \mathcal{K}(u, \cdot)$ , setting  $\zeta'' = \zeta'$  and  $\varpi_u^* = \text{Law}\{(\zeta', \zeta'')\}$ . Our first step is to show that  $\varpi_{u_j, u} \rightharpoonup \varpi_u^*$ ; converges in the weak sense. Let us equip the product space  $\mathcal{H} \times \mathcal{H}$  with the

norm  $\|(u, v)\| := \|u\| \vee \|v\|$  and let  $\varphi' \in \text{Lip}_1(\mathcal{H} \times \mathcal{H}) \cap C_b(\mathcal{H} \times \mathcal{H})$ . We then have that

$$\begin{aligned}
 & \left| \int_{\mathcal{H} \times \mathcal{H}} \varphi'(\zeta_j, \zeta) \varpi_{u_j, u}(\mathrm{d}\zeta_j, \mathrm{d}\zeta) - \int_{\mathcal{H} \times \mathcal{H}} \varphi'(\zeta', \zeta'') \varpi_u^*(\mathrm{d}\zeta', \mathrm{d}\zeta'') \right| \\
 & \leq \int_{\mathcal{H} \times \mathcal{H}} |\varphi'(\zeta_j, \zeta) - \varphi'(\zeta, \zeta)| \varpi_{u_j, u}(\mathrm{d}\zeta_j, \mathrm{d}\zeta) \\
 & \quad + \left| \int_{\mathcal{H} \times \mathcal{H}} \varphi'(\zeta, \zeta) \varpi_{u_j, u}(\mathrm{d}\zeta_j, \mathrm{d}\zeta) - \int_{\mathcal{H} \times \mathcal{H}} \varphi'(\zeta', \zeta'') \varpi_u^*(\mathrm{d}\zeta', \mathrm{d}\zeta'') \right| \\
 & = \int_{\mathcal{H} \times \mathcal{H}} |\varphi'(\zeta_j, \zeta) - \varphi'(\zeta, \zeta)| \varpi_{u_j, u}(\mathrm{d}\zeta_j, \mathrm{d}\zeta) \\
 & \quad + \left| \int_{\mathcal{H} \times \mathcal{H}} \varphi'(\zeta, \zeta) \mathcal{K}(u, \mathrm{d}\zeta) - \int_{\mathcal{H} \times \mathcal{H}} \varphi'(\zeta', \zeta') \mathcal{K}(u, \mathrm{d}\zeta') \right| \\
 & = \int_{\mathcal{H} \times \mathcal{H}} |\varphi'(\zeta_j, \zeta) - \varphi'(\zeta, \zeta)| \varpi_{u_j, u}(\mathrm{d}\zeta_j, \mathrm{d}\zeta).
 \end{aligned}$$

Since  $\varphi'$  has Lipschitz constant 1 we further have

$$\begin{aligned}
 \int_{\mathcal{H} \times \mathcal{H}} |\varphi'(\zeta_j, \zeta) - \varphi'(\zeta, \zeta)| \varpi_{u_j, u}(\mathrm{d}\zeta_j, \mathrm{d}\zeta) & \leq \int_{\mathcal{H} \times \mathcal{H}} \|\zeta_j - \zeta\| \varpi_{u_j, u}(\mathrm{d}\zeta_j, \mathrm{d}\zeta) \\
 & \leq \|u_j - u\|,
 \end{aligned}$$

where the last inequality follows from Assumption 2(d). Since  $\varphi'$  was arbitrary an application of Portmantheau theorem (see e.g., [5], 2.2.6) yields the weak convergence of  $\varpi_{u_j, u}$  to  $\varpi_u^*$ . Returning to the definition of  $T_j''(\xi)$  and recalling that  $\varphi \in C_b(\mathcal{H})$ , we have for any fixed  $\xi \in \mathcal{H}$  that  $T_j''(\xi) \rightarrow \int_{\mathcal{H} \times \mathcal{H}} |\varphi(\zeta' + \xi) - \varphi(\zeta + \xi)| \varpi_u^*(\mathrm{d}\zeta', \mathrm{d}\zeta) = 0$  as  $u_j \rightarrow u$ , that is, the  $T_j''$  converge to 0 pointwise. The boundedness of  $\varphi$  also yields the boundedness of  $T_j''$ . An application of the dominated convergence theorem then yields  $\int_{\mathcal{H}} T_j''(\xi) \lambda(\mathrm{d}\xi) \rightarrow 0$  as desired.  $\square$

## APPENDIX B: PROOF OF PERTURBATION RESULTS FROM SECTION 4

### B.1. Proof of Lemma 1.

PROOF. Observe that by Jensen's inequality it is sufficient to show there exists  $G' > 0$  so that

$$\begin{aligned}
 d_q(u, v)(2 + \theta V(u) + \theta V(v)) & \leq G'(d_q(u, w)(2 + \theta V(u) + \theta V(w)) \\
 & \quad + d_q(w, v)(2 + \theta V(w) + \theta V(v))).
 \end{aligned}$$

Furthermore, by the hypothesis on  $V$  we have that  $d_q(u, v)(2 + \theta V(u) + \theta V(v))$  is equivalent to  $d_{p+q}(u, v)$ . In fact,

$$\begin{aligned}
 & d_q(u, v)(2 + \theta V(u) + \theta V(v)) \\
 & = \frac{1}{\omega} \left( 2 + \theta \sum_{j=0}^p a_j (\|u\|^j + \|v\|^j) \right) (1 + \eta \|u\| + \eta \|v\|)^q \|u - v\| \\
 & \leq \frac{C}{\omega} (1 + \eta \|u\| + \eta \|v\|)^{p+q} \|u - v\| = C d_{p+q}(u, v),
 \end{aligned}$$

with  $C(\theta, \eta, p, a_j) > 0$ . Conversely, by Lemma 3 and the assumption that  $a_p > 0$  we have

$$\begin{aligned} d_{p+q}(u, v) &= \frac{1}{\omega} (1 + \eta\|u\| + \eta\|v\|)^{p+q} \|u - v\| \\ &\leq \frac{2^{2p}}{\omega} (1 + \eta^p\|u\|^p + \eta^p\|v\|^p) (1 + \eta\|u\| + \eta\|v\|)^q \|u - v\| \\ &\leq \frac{2^{2p}C'}{\omega} \left( 1 + \theta \sum_{j=0}^p a_j (\|u\|^j + \|v\|^j) \right) (1 + \eta\|u\| + \eta\|v\|)^q \|u - v\| \\ &= cd_q(u, v) (2 + \theta V(u) + \theta V(v)), \end{aligned}$$

where once again  $c(\theta, \eta, p, a_j) > 0$ . Thus it suffices if we prove the generalized triangle inequality for the  $d_s(u, v)$  semimetrics with  $s \in \mathbb{N}$ .

Let  $u, v, w \in \mathcal{H}$  and observe that if either  $d_q(u, w)$  or  $d_q(w, v)$  is equal to one then

$$d_q(u, v) \leq d_q(u, w) + d_q(w, v),$$

since  $d_q(u, v)$  is capped at one, so the standard triangle inequality holds. Now suppose both  $d_q(u, w)$  and  $d_q(w, v)$  are less than one, implying that  $\|u - w\| < \omega$  and  $\|w - v\| < \omega$ . Then Lemma 3 and multiple applications of the triangle inequality we can write

$$\begin{aligned} \omega d_q(u, v) &= (1 + \eta\|u\| + \eta\|v\|)^q \|u - v\| \\ &\leq (1 + \eta\|u\| + \eta\|w\| + \eta\|v - w\|)^q \|u - w\| \\ &\quad + (1 + \eta\|u - w\| + \eta\|w\| + \eta\|v\|)^q \|w - v\| \\ &\leq (1 + \eta\|u\| + \eta\|w\| + \eta\omega)^q \|u - w\| \\ &\quad + (1 + \eta\omega + \eta\|w\| + \eta\|v\|)^q \|w - v\| \\ &\leq (1 + \eta\omega)^q ((1 + \eta\|u\| + \eta\|w\|)^q \|u - w\| \\ &\quad + (1 + \eta\|w\| + \eta\|v\|)^q \|w - v\|). \end{aligned}$$

Thus, we have

$$d_q(u, v) \leq (1 + \eta\omega)^q (d_q(u, w) + d_q(w, v)). \quad \square$$

**B.2. Proof of Theorem 4.** Our strategy is to take  $k > n$  sufficiently large that  $G\gamma^{\lfloor k/n \rfloor} < 1$ , where  $\gamma$  is the  $n$  step spectral gap of  $\mathcal{P}_0$  in Assumption 3(b) and  $C > 0$  is the Lipschitz constant of  $\mathcal{P}^n$ . With  $k$  as above we then take  $\vartheta(k) = C \sum_{j=1}^k G^j (C^* \gamma)^{\lfloor j/n \rfloor} < \infty$  with  $C^*, C > 0$  constants depending on the Lipschitz constant of  $\mathcal{P}_0$  and growth of the Lyapunov function  $V$  and show that  $\tilde{d}(\mathcal{P}_\varepsilon^k \delta_u, \mathcal{P}_0^k \delta_u) \leq \vartheta(k) \psi(\varepsilon) (1 + \sqrt{V(u)})$ . The remainder of the argument then generalizes this bound for point masses  $\delta_u$  to a bound on  $\tilde{d}(\mathcal{P}_\varepsilon^k \mu_1, \mathcal{P}_0^k \mu_2)$  for general measures  $\mu_1, \mu_2$  and in turn for the invariant measures  $\nu_0, \nu_\varepsilon$ .

Before presenting the main proof we need an auxiliary lemma stating that  $\mathcal{P}_0^k$  is  $\tilde{d}$ -Lipschitz in the initial condition of the chain.

**LEMMA 5.** *Suppose Assumption 3(a), (b), (c) hold. Then for any integer  $k > 0$  there exists a constant  $C^*(k) > 0$ , so that*

$$\tilde{d}(\mathcal{P}_0^k \delta_u, \mathcal{P}_0^k \delta_v) \leq C^* \tilde{d}(u, v) \quad \forall u, v \in \mathcal{H}.$$

PROOF. We consider two cases where  $d(u, v) = 1$  and  $d(u, v) < 1$ .

Case 1: Suppose  $d(u, v) = 1$ . Then, letting  $\pi_{u,v} \in \Upsilon(\mathcal{P}_0^k \delta_u, \mathcal{P}_0^k \delta_v)$ , we have, using the Lyapunov condition and Jensen's inequality

$$\begin{aligned} \tilde{d}(\mathcal{P}_0^k \delta_u, \mathcal{P}_0^k \delta_v)^2 &\leq \int_{\mathcal{H} \times \mathcal{H}} \tilde{d}^2(x, y) \pi_{u,v}(dx, dy) \\ &= \int_{\mathcal{H} \times \mathcal{H}} (2 + \theta V(x) + \theta V(y)) \pi_{u,v}(dx, dy) \\ &\leq 2 + \theta \kappa^k (V(u) + V(v)) + \frac{\theta K}{1 - \kappa}, \\ \tilde{d}(\mathcal{P}_0^k \delta_u, \mathcal{P}_0^k \delta_v) &\leq \sqrt{d(u, v)} \sqrt{2 + \theta V(u) + \theta V(v)} + \sqrt{d(u, v)} \sqrt{\frac{\theta K}{1 - \kappa}} \\ &\leq C^* \tilde{d}(u, v), \end{aligned}$$

where  $C^*(k)$  is a universal constant that does not depend on  $u, v$ .

Case 2: If  $d(u, v) < 1$ , then because  $\mathcal{P}_0$  is contracting for  $d$  it follows that  $\mathcal{P}_0^k$  is contracting for  $d$ , and so

$$\begin{aligned} \tilde{d}(\mathcal{P}_0^k \delta_u, \mathcal{P}_0^k \delta_v)^2 &\leq \inf_{\pi_{u,v}} \int_{\mathcal{H} \times \mathcal{H}} d(x, y) \pi_{u,v}(dx, dy) \int_{\mathcal{H} \times \mathcal{H}} (2 + \theta V(x) + \theta V(y)) \pi_{u,v}(dx, dy) \\ &\leq \gamma_1 d(u, v) \left[ 2 + \theta \kappa^k (V(u) + V(v)) + \frac{\theta K}{1 - \kappa} \right], \\ \tilde{d}(\mathcal{P}_0^k \delta_u, \mathcal{P}_0^k \delta_v) &\leq \sqrt{d(u, v)} \left[ \sqrt{2 + \theta V(u) + \theta V(v)} + \sqrt{\frac{\theta K}{1 - \kappa}} \right] \\ &\leq C^* \tilde{d}(u, v), \end{aligned}$$

where  $C^*(k) > 0$  is the same constant as in Case 1.  $\square$

PROOF OF THEOREM 4. Suppose initially that there exists a positive function  $\vartheta(k)$  such that for every  $k > 0$

$$\tilde{d}(\mathcal{P}_\varepsilon^k \delta_u, \mathcal{P}_0^k \delta_u) \leq \vartheta(k) \psi(\varepsilon) (1 + \sqrt{V(u)}).$$

We will show below that this is implied by the one-step error control and Lemma 5. For any  $k > n$  we have by the weak triangle inequality

$$(40) \quad \tilde{d}(\mathcal{P}_\varepsilon^k \delta_u, \mathcal{P}_0^k \delta_v) \leq G \gamma^{[k/n]} \tilde{d}(u, v) + G \psi(\varepsilon) \vartheta(n) (1 + \sqrt{V(u)}).$$

Choose  $k$  large enough that  $\gamma^{[k/n]} < G^{-1}$  and put  $\gamma^* = \gamma^{[k/n]} G < 1$ . By Remark 2 the bound in (40) can be generalized to any two probability measures  $\mu_1, \mu_2 \in P^1(\mathcal{H}; \tilde{d})$  and so

$$\tilde{d}(\mathcal{P}_\varepsilon^k \mu_1, \mathcal{P}_0^k \mu_2) \leq \gamma^* \tilde{d}(\mu_1, \mu_2) + G \psi(\varepsilon) \vartheta(n) \left( 1 + \int_{\mathcal{H}} \sqrt{V} d\mu_1 \right),$$

for some  $\gamma^* < 1$ . The integral in the last term appears after integrating the right hand side of (40) with respect to the optimal coupling of  $\mu_1, \mu_2$ . Using the symmetry of  $\tilde{d}$  and by putting  $\mu_1 = \nu_\varepsilon$  and  $\mu_2 = \nu_0$  and vice versa we have

$$\tilde{d}(\nu_0, \nu_\varepsilon) \leq \frac{G \psi(\varepsilon) \vartheta(n)}{1 - \gamma^*} (1 + \nu_0(\sqrt{V}) \wedge \nu_\varepsilon(\sqrt{V})).$$

It remains to show the existence of  $\vartheta(n)$ . By Lemma 5 and (26) we have

$$\begin{aligned} & \tilde{d}(\mathcal{P}_\varepsilon^k \delta_u, \mathcal{P}_0^k \delta_u) \\ & \leq G\{C^* \gamma^{\lfloor k/n \rfloor} \tilde{d}(\mathcal{P}_0^{k-1} \delta_u, \mathcal{P}_\varepsilon^{k-1} \delta_u) + \psi(\varepsilon)[1 + (\mathcal{P}_\varepsilon^{k-1} \delta_u)(\sqrt{V})]\} \\ & \leq \psi(\varepsilon) \sum_{j=1}^k G^j(C^* \gamma)^{\lfloor j/n \rfloor} [1 + (\mathcal{P}_\varepsilon^{k-1} \delta_u)(\sqrt{V})] \\ & \leq \psi(\varepsilon) \sum_{j=1}^k G^j(C^* \gamma)^{\lfloor j/n \rfloor} \left(1 + \kappa_\varepsilon^{(k-j)/2} \sqrt{V(u)} + \frac{\sqrt{K_\varepsilon}}{1 - \sqrt{\kappa_\varepsilon}}\right) \\ & \equiv \psi(\varepsilon) \vartheta(k)(1 + \sqrt{V(u)}), \end{aligned}$$

where  $\vartheta(k) < \frac{\sqrt{K_\varepsilon}+1}{1-\sqrt{\kappa_\varepsilon}} \sum_{j=1}^k G^j(C^* \gamma)^{\lfloor j/n \rfloor}$  and  $C^*$  is the constant in Lemma 5.  $\square$

**B.3. Proof of Theorem 5.** Our strategy employs the Poisson equation and Martingale/potential methods. The argument is complicated by the fact that  $\tilde{d}$  is not a metric, and we seek to prove bounds for  $\varphi : \mathcal{H} \rightarrow \mathcal{X}$  for a separable Hilbert space  $\mathcal{X}$ . This requires us to first show that the potential  $\sum_{k=0}^\infty \mathcal{P}^k \varphi$  solves the Poisson equation, by checking that the potential converges to a well-defined limit and that  $\mathcal{P}$  is a bounded linear operator in an appropriate operator norm. We then are able to use the inner product and norm on  $\mathcal{H}$  to make a Martingale argument reminiscent of that in [16].

We prove three preparatory Lemmas that are used in the main proof. In what follows we let  $\mathcal{X}$  be a separable Hilbert space with norm  $\|\cdot\|_{\mathcal{X}}$ .

LEMMA 6. *Suppose there exists a  $C < \infty$  and  $k \in \mathbb{N}$  such that*

$$\tilde{d}(\mathcal{P}_0^k \delta_u, \mathcal{P}_0^k \delta_v) \leq C \tilde{d}(u, v).$$

*Then for any  $\varphi : \mathcal{H} \rightarrow \mathcal{X}$  with  $\|\varphi\|_{\tilde{d}} < \infty$ ,*

$$\|\mathcal{P}_0^k \varphi(u) - \mathcal{P}_0^k \varphi(v)\|_{\mathcal{X}} \leq C \|\varphi\|_{\tilde{d}} \tilde{d}(u, v).$$

PROOF. Since  $\mathcal{X}$  is a Hilbert space,

$$\|\varphi(u) - \varphi(v)\|_{\mathcal{X}} \geq |\|\varphi(u)\|_{\mathcal{X}} - \|\varphi(v)\|_{\mathcal{X}}|$$

and so

$$(41) \quad \|\varphi\|_{\tilde{d}} = \sup_{u \neq v} \frac{\|\varphi(u) - \varphi(v)\|_{\mathcal{X}}}{\tilde{d}(u, v)} \geq \sup_{u \neq v} \frac{|\|\varphi(u)\|_{\mathcal{X}} - \|\varphi(v)\|_{\mathcal{X}}|}{\tilde{d}(u, v)} = \|\|\varphi\|_{\mathcal{X}}\|_{\tilde{d}}.$$

So then

$$\begin{aligned} & \left\| \int_{\mathcal{H}} \varphi(x) (\mathcal{P}_0^k \delta_u - \mathcal{P}_0^k \delta_v)(dx) \right\|_{\mathcal{X}} \\ & \leq \int_{\mathcal{H}} \|\varphi(x)\|_{\mathcal{X}} (\mathcal{P}_0^k \delta_u - \mathcal{P}_0^k \delta_v)(dx) \\ & \leq \|\varphi\|_{\tilde{d}} \inf_{\pi_{u,v} \in \Upsilon(\mathcal{P}_0^k \delta_u, \mathcal{P}_0^k \delta_v)} \int_{\mathcal{H} \times \mathcal{H}} \tilde{d}(x, y) \pi_{u,v}(dx, dy) \\ & \leq \|\varphi\|_{\tilde{d}} C \tilde{d}(u, v), \end{aligned}$$

where the last inequality follows from the hypothesis of the lemma.  $\square$

This implies immediately that  $\mathcal{P}_0^k$  is a  $\|\cdot\|_{\tilde{d}}$  contraction.

COROLLARY 3. *Let  $k$  be the smallest integer for which  $\tilde{d}(\mathcal{P}_0^k \delta_u, \mathcal{P}_0^k \delta_v) < \gamma \tilde{d}(\delta_u, \delta_v)$ , then Lemma 6 immediately implies that*

$$\|\mathcal{P}_0^{kn} v_1 \varphi - \mathcal{P}_0^{kn} v_2 \varphi\|_{\mathcal{X}} \leq \gamma^n \|\varphi\|_{\tilde{d}} \tilde{d}(v_1, v_2).$$

Furthermore, for  $j < k$ , combining Lemmas 6 and 5, we obtain

$$\|\mathcal{P}_0^{kn} v_1 \varphi - \mathcal{P}_0^{kn} v_2 \varphi\|_{\mathcal{X}} \leq C^* \|\varphi\|_{\tilde{d}} \tilde{d}(v_1, v_2).$$

Finally, we show that for  $\|\cdot\|_{\tilde{d}}$ -Lipschitz  $\varphi$ , the potential  $\Theta^* : \mathcal{H} \rightarrow \mathcal{X}$  of  $\varphi$  is well defined, has bounded  $\|\cdot\|_{\tilde{d}}$  seminorm, and is a solution to the Poisson equation for  $\varphi$ .

LEMMA 7. *Consider  $\varphi : \mathcal{H} \mapsto \mathcal{X}$  which is  $v_0$ -Bochner-measurable and with  $\|\varphi\|_{\tilde{d}} < +\infty$ . Define  $\tilde{\varphi} = \varphi - v_0(\varphi)$  and the potential function*

$$(42) \quad \Theta^* := \sum_{j=0}^{\infty} \mathcal{P}_0^j \tilde{\varphi}.$$

If  $\mathcal{P}_0$  satisfies Assumption 3 it holds true that:

(a) *There exists a uniform constant  $C_0 > 0$  so that*

$$\|\Theta^*\|_{\tilde{d}} < \frac{C_0 \|\varphi\|_{\tilde{d}}}{1 - \gamma}.$$

(b)  *$\Theta^*$  is a solution to the Poisson equation*

$$(\mathcal{P}_0 - I)\hat{\Theta} = -\tilde{\varphi}.$$

(c)  *$\Theta^* : \mathcal{H} \rightarrow \mathcal{X}$  is well defined pointwise.*

PROOF. Let  $k$  be the smallest integer such that for all  $u, v \in \mathcal{X}$ ,  $\tilde{d}(\mathcal{P}_0^k \delta_u, \mathcal{P}_0^k \delta_v) < \gamma \tilde{d}(u, v)$  for some  $\gamma < 1$ , which is finite because  $\mathcal{P}_0$  satisfies Assumption 3(f).

$$(43) \quad \begin{aligned} \sum_{j=0}^{\infty} \mathcal{P}_0^j \tilde{\varphi} &= \sum_{j=0}^{k-1} \mathcal{P}_0^j \tilde{\varphi} + \sum_{j=0}^{k-1} \sum_{i=1}^{\infty} \mathcal{P}_0^{ik+j} \tilde{\varphi}, \\ \left\| \sum_{j=0}^{\infty} \mathcal{P}_0^j \tilde{\varphi} \right\|_{\tilde{d}} &\leq \sum_{j=0}^{k-1} \|\mathcal{P}_0^j \tilde{\varphi}\|_{\tilde{d}} + \sum_{j=0}^{k-1} \sum_{i=1}^{\infty} \|\mathcal{P}_0^{ik+j} \tilde{\varphi}\|_{\tilde{d}} \\ &\leq kC^* \|\varphi\|_{\tilde{d}} + \frac{k \|\varphi\|_{\tilde{d}}}{1 - \gamma} = k \left( C^* + \frac{1}{1 - \gamma} \right) \|\varphi\|_{\tilde{d}}, \end{aligned}$$

where the last line followed by observing  $\|\tilde{\varphi}\|_{\tilde{d}} = \|\varphi\|_{\tilde{d}}$  and applying Lemmas 6 and 5. This concludes the proof of (a).

To prove (b) consider the space  $L_1(\mathcal{X}, v_0; \mathcal{X})$  of  $v_0$ -Bochner-measurable functions  $f : \mathcal{H} \rightarrow \mathcal{X}$  satisfying  $v_0(\|f(\cdot)\|_{\mathcal{X}}) < \infty$ , equipped with the norm

$$(44) \quad \|f\|_{L_1(v_0)} = \int_{\mathcal{H}} \|f(u)\|_{\mathcal{X}} v_0(du).$$

We now show that the series in (42) converges in  $L_1(\mathcal{X}, v_0; \mathcal{X})$ . By (41),  $\|\varphi\|_{\tilde{d}} > \|\varphi\|_{\mathcal{X}}$ . Notice that since  $\|\varphi\|_{\tilde{d}} = \|\varphi - \varphi(0)\|_{\tilde{d}}$ , it follows that

$$\|\varphi(u) - \varphi(0)\|_{\mathcal{X}} = \|\varphi(u)\|_{\mathcal{X}} \leq \|\varphi\|_{\tilde{d}} \sqrt{2 + \theta V(u)},$$

since  $\tilde{d}(u, v) < \sqrt{2 + \theta V(u) + \theta V(v)}$  and  $V(0) = 0$ . Since  $v_0(V) < \infty$ , we have that for any  $\varphi$  with  $\|\varphi\|_{\tilde{d}} < \infty$

$$(45) \quad \|\varphi\|_{L_1(v_0)} = v_0(\|\varphi(\cdot)\|_{\mathcal{X}}) \leq \|\varphi\|_{\tilde{d}}(\sqrt{2} + \theta v_0\sqrt{V}).$$

Since  $L_1(\mathcal{H}, v_0; \mathcal{X})$  is a Banach space [39], page 2, (45) means it is enough to show that the sequence of partial sums  $\Theta_m = \sum_{j=0}^m \mathcal{P}_0^j \varphi$  is  $\|\cdot\|_{\tilde{d}}$ -Cauchy, since this also implies it is  $L_1(v_0)$ -Cauchy. Define  $\ell = \lfloor m/k \rfloor$ , and  $\tilde{n} = \lceil (n-m)/k \rceil$ , so for  $n > m$

$$\|\Theta_n - \Theta_m\|_{\tilde{d}} = \left\| \sum_{j=m+1}^n \mathcal{P}_0^j \tilde{\varphi} \right\|_{\tilde{d}} \leq \sum_{j=m+1}^n \|\mathcal{P}_0^j \tilde{\varphi}\|_{\tilde{d}} \leq k \|\varphi\|_{\tilde{d}} \sum_{j=0}^{\tilde{n}} \gamma^{\ell+j} \leq \frac{k\gamma^{\lfloor m/k \rfloor}}{1-\gamma} \|\varphi\|_{\tilde{d}}.$$

Therefore, the series in (42) converges in  $L_1(v_0)$ . Since  $v_0$  is the unique invariant measure of  $\mathcal{P}_0$ , and

$$\begin{aligned} \|\mathcal{P}_0\|_{\tilde{d}}^* &:= \sup_{\|\varphi\|_{\tilde{d}} < 1} \left\| \int_{\mathcal{H}} (\mathcal{P}_0 \varphi)(u) v_0(du) \right\|_{\mathcal{X}} \\ &\leq \sup_{\|\varphi\|_{\tilde{d}} < 1} \int_{\mathcal{H}} \|(\mathcal{P}_0 \varphi)(u)\|_{\mathcal{X}} v_0(du) \\ &< \int_{\mathcal{H}} (\mathcal{P}_0 \sqrt{2 + \theta V})(u) v_0(du) < \infty, \end{aligned}$$

so that  $\mathcal{P}_0$  is a bounded linear operator on the space of  $\text{Lip}(\tilde{d})$  functions from  $\mathcal{H}$  to  $\mathcal{X}$  equipped with the operator norm  $\|\cdot\|_{\tilde{d}}^*$ . Thus we conclude that  $\Theta^*$  is a solution of the Poisson equation for  $v_0$ -Bochner-measurable functions  $\varphi \in \text{Lip}(\tilde{d})$ .

Finally, we prove (c) by showing that  $\Theta^*(u)$  converges in  $\|\cdot\|_{\mathcal{X}}$ . We have

$$\begin{aligned} \|\Theta_n(u) - \Theta_m(u)\|_{\mathcal{X}} &= \left\| \sum_{j=m+1}^n \mathcal{P}_0^j \tilde{\varphi}(u) \right\|_{\mathcal{X}} = \left\| \sum_{j=m+1}^n \mathcal{P}_0^j \varphi(u) - v_0(\varphi) \right\|_{\mathcal{X}} \\ &\leq \sum_{j=m+1}^n \|\mathcal{P}_0^j \varphi(u) - v_0(\varphi)\|_{\mathcal{X}} \leq \|\varphi\|_{\tilde{d}} \frac{k\gamma^{\lfloor m/k \rfloor}}{1-\gamma} \tilde{d}(\delta_u, v_0). \end{aligned}$$

So the sequence is  $\|\cdot\|_{\mathcal{X}}$ -Cauchy for any  $u \in \mathcal{H}$ .  $\square$

We are now ready to present the complete proof of Theorem 5. We primarily focus on part (a) as part (b) follows as a corollary of the calculations in the proof of (a).

**PROOF OF THEOREM 5.** (a) Define  $\Theta = \Theta^* - \Theta^*(0)$  for any  $\varphi : \mathcal{H} \rightarrow \mathcal{X}$  with  $\|\varphi\|_{\tilde{d}} < \infty$ . By Lemma 7,  $\|\Theta^*\|_{\tilde{d}} < \frac{C_0 \|\varphi\|_{\tilde{d}}}{1-\gamma}$  for some  $C_0 < +\infty$ , and  $\Theta(u)$  is a well-defined element of  $\mathcal{X}$  for any  $u \in \mathcal{H}$ . So with  $C = C_0 \|\varphi\|_{\tilde{d}} (1-\gamma)^{-1}$ , we have

$$(46) \quad \begin{aligned} \|\Theta^*(u) - \Theta^*(v)\|_{\mathcal{X}} &\leq C \sqrt{2 + \theta V(u) + \theta V(v)}, \\ \|\Theta^*(u) - \Theta^*(0)\|_{\mathcal{X}} &= \|\Theta(u)\|_{\mathcal{X}} \leq C \sqrt{2 + \theta V(u)}, \end{aligned}$$

and furthermore  $\|\Theta\|_{\tilde{d}} = \|\Theta^*\|_{\tilde{d}}$ . Note that

$$(47) \quad (\mathcal{P}_0 - I)\Theta^*(u) = (\mathcal{P}_0 - I)\Theta(u) = -\tilde{\varphi}(u),$$

thus,

$$\begin{aligned}
 \Theta^*(U_n^\varepsilon) - \Theta^*(U_0^\varepsilon) &= \sum_{k=0}^{n-1} \Theta^*(U_{k+1}^\varepsilon) - \Theta^*(U_k^\varepsilon) = \sum_{k=0}^{n-1} \Theta(U_{k+1}^\varepsilon) - \Theta(U_k^\varepsilon) \\
 &= \sum_{k=0}^{n-1} [\Theta(U_{k+1}^\varepsilon) - \mathcal{P}_\varepsilon \Theta(U_k^\varepsilon)] + \sum_{k=0}^{n-1} (\mathcal{P}_\varepsilon - I) \Theta(U_k^\varepsilon) \\
 &= \sum_{k=0}^{n-1} [\Theta(U_{k+1}^\varepsilon) - \mathcal{P}_\varepsilon \Theta(U_k^\varepsilon)] \\
 &\quad + \sum_{k=0}^{n-1} (\mathcal{P}_0 - I) \Theta(U_k^\varepsilon) + \sum_{k=0}^{n-1} (\mathcal{P}_\varepsilon - \mathcal{P}_0) \Theta(U_k^\varepsilon).
 \end{aligned}$$

Using (47) and that  $\Theta^*(U_n^\varepsilon) - \Theta^*(U_0^\varepsilon) = \Theta(U_n^\varepsilon) - \Theta(U_0^\varepsilon)$  and defining the Martingale increments  $m_{k+1}^\varepsilon = \Theta(U_{k+1}^\varepsilon) - \mathcal{P}_\varepsilon \Theta(U_k^\varepsilon)$  and the Martingale  $M_n^\varepsilon = \sum_{k=1}^n m_k^\varepsilon$ , we have

$$\begin{aligned}
 (48) \quad \frac{1}{n} \sum_{k=0}^{n-1} \varphi(U_k^\varepsilon) - v_0(\varphi) &= \frac{\Theta(U_0^\varepsilon) - \Theta(U_n^\varepsilon)}{n} + \frac{1}{n} M_n^\varepsilon + \frac{1}{n} \sum_{k=0}^{n-1} (\mathcal{P}_\varepsilon - \mathcal{P}_0) \Theta(U_k^\varepsilon) \\
 &=: T_1 + T_2 + T_3.
 \end{aligned}$$

Note that the quantity we now care about is

$$\mathbb{E} \left\| \frac{1}{n} \sum_{k=0}^{n-1} \varphi(U_k^\varepsilon) - v_0(\varphi) \right\|_{\mathcal{X}} \leq \mathbb{E} [\|T_1\|_{\mathcal{X}} + \|T_2\|_{\mathcal{X}} + \|T_3\|_{\mathcal{X}}].$$

Let  $\mathcal{F}_k$  be the filtration indexed by time  $k$ . We have with  $\langle \cdot, \cdot \rangle_{\mathcal{X}}$  the  $\mathcal{X}$ -inner product,

$$\begin{aligned}
 \|M_n^\varepsilon\|_{\mathcal{X}}^2 &= \langle M_n^\varepsilon, M_n^\varepsilon \rangle_{\mathcal{X}} = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \langle m_k^\varepsilon, m_j^\varepsilon \rangle_{\mathcal{X}}, \\
 \mathbb{E} \|M_n^\varepsilon\|_{\mathcal{X}}^2 &= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \mathbb{E} [\mathbb{E} [\langle m_k^\varepsilon, m_j^\varepsilon \rangle_{\mathcal{X}} \mid \mathcal{F}_{j \wedge k}]] = \sum_{k=0}^{n-1} \mathbb{E} [\mathbb{E} [\langle m_k^\varepsilon, m_k^\varepsilon \rangle_{\mathcal{X}} \mid \mathcal{F}_k]] \\
 &= \sum_{k=0}^{n-1} \mathbb{E} [\mathbb{E} [\|m_k^\varepsilon\|_{\mathcal{X}}^2 \mid \mathcal{F}_k]] \leq \sum_{k=0}^{n-1} \mathbb{E} [\mathbb{E} [C^2 (2 + \theta V(U_{k+1}^\varepsilon)) \mid \mathcal{F}_k]] \\
 &\leq C^2 \left( 2n + \theta \sum_{k=0}^{n-1} \mathbb{E} [\kappa_\varepsilon V(U_k) + K_\varepsilon] \right) \\
 &\leq C^2 \left( 2n + \theta \sum_{k=0}^{n-1} \kappa_\varepsilon^k V(u_0) + \frac{K_\varepsilon}{1 - \kappa_\varepsilon} \right) \leq C^2 \left( 2n + \theta \frac{V(u_0) + nK_\varepsilon}{1 - \kappa_\varepsilon} \right),
 \end{aligned}$$

which in turn implies that

$$\begin{aligned}
 \mathbb{E} \|T_2\|_{\mathcal{X}} &= n^{-1} \mathbb{E} \|M_n^\varepsilon\|_{\mathcal{X}} \leq n^{-1} (\mathbb{E} \|M_n^\varepsilon\|_{\mathcal{X}}^2)^{1/2} \\
 &\leq n^{-1} \sqrt{n} C \left( 2 + \theta \frac{V(u_0)/n + K_\varepsilon}{1 - \kappa_\varepsilon} \right)^{1/2} \\
 &= \frac{C}{\sqrt{n}} \left( 2 + \theta \frac{V(u_0)/n + K_\varepsilon}{1 - \kappa_\varepsilon} \right)^{1/2}.
 \end{aligned}$$

Next we have

$$\begin{aligned}
 & \left\| \sum_{k=0}^{n-1} (\mathcal{P}_\varepsilon - \mathcal{P}_0) \Theta(U_k^\varepsilon) \right\|_{\mathcal{X}} \leq \sum_{k=0}^{n-1} \|(\mathcal{P}_\varepsilon - \mathcal{P}_0) \Theta(U_k^\varepsilon)\|_{\mathcal{X}} \\
 & \leq \sum_{k=0}^{n-1} C \psi(\varepsilon) (1 + \sqrt{V(U_k^\varepsilon)}), \\
 (49) \quad & \mathbb{E} \left\| \sum_{k=0}^{n-1} (\mathcal{P}_\varepsilon - \mathcal{P}_0) \Theta(U_k^\varepsilon) \right\|_{\mathcal{X}} \leq \sum_{k=0}^{n-1} C \psi(\varepsilon) (1 + \mathbb{E} \sqrt{V(U_k^\varepsilon)}) \\
 & \leq \sum_{k=0}^{n-1} C \psi(\varepsilon) \left( 1 + \kappa_\varepsilon^{k/2} V(u_0) + \frac{\sqrt{K_\varepsilon}}{1 - \sqrt{\kappa_\varepsilon}} \right) \\
 & = C \psi(\varepsilon) n \left( 1 + \frac{\sqrt{\kappa_\varepsilon}/n V(u_0) + \sqrt{K_\varepsilon}}{1 - \sqrt{\kappa_\varepsilon}} \right), \\
 & \mathbb{E} \|T_3\|_{\mathcal{X}} \leq C \psi(\varepsilon) \left( 1 + \frac{\sqrt{\kappa_\varepsilon}/n V(u_0) + \sqrt{K_\varepsilon}}{1 - \sqrt{\kappa_\varepsilon}} \right).
 \end{aligned}$$

Finally we have

$$\begin{aligned}
 & \|\Theta(U_0^\varepsilon) - \Theta(U_n^\varepsilon)\|_{\mathcal{X}} \leq C \sqrt{2 + \theta V(U_0^\varepsilon) + \theta V(U_n^\varepsilon)} \\
 & \leq C(\sqrt{2} + \sqrt{\theta}(\sqrt{V(U_0^\varepsilon)} + \sqrt{V(U_n^\varepsilon)})), \\
 (50) \quad & \mathbb{E} \|T_1\|_{\mathcal{X}} \leq \frac{C}{n} \left[ \sqrt{2} + \sqrt{\theta} \left( \sqrt{V(U_0^\varepsilon)} + \kappa_\varepsilon^{n/2} \sqrt{V(U_0^\varepsilon)} + \frac{\sqrt{K_\varepsilon}}{1 - \sqrt{\kappa_\varepsilon}} \right) \right] \\
 & \leq \frac{C}{n} \left[ \sqrt{2} + \sqrt{\theta} \left( \sqrt{V(U_0^\varepsilon)} (1 + \kappa_\varepsilon^{n/2}) + \frac{\sqrt{K_\varepsilon}}{1 - \sqrt{\kappa_\varepsilon}} \right) \right].
 \end{aligned}$$

Putting together the bounds for  $\mathbb{E}|T_1|_{\mathcal{X}}$ ,  $\mathbb{E}|T_2|_{\mathcal{X}}$ ,  $\mathbb{E}|T_3|_{\mathcal{X}}$  we arrive at

$$\begin{aligned}
 & \mathbb{E} \left\| \frac{1}{n} \sum_{k=0}^{n-1} \varphi(U_k^\varepsilon) - v_0(\varphi) \right\|_{\mathcal{X}} \\
 & \leq \frac{C}{n} \left[ \sqrt{2} + \sqrt{\theta} \left( \sqrt{V(U_0^\varepsilon)} (1 + \kappa_\varepsilon^{n/2}) + \frac{\sqrt{K_\varepsilon}}{1 - \sqrt{\kappa_\varepsilon}} \right) \right] \\
 & \quad + \frac{C}{\sqrt{n}} \left( 2 + \theta \frac{V(u_0)/n + K_\varepsilon}{1 - \kappa_\varepsilon} \right)^{1/2} + C \psi(\varepsilon) \left( 1 + \frac{\sqrt{\kappa_\varepsilon}/n V(u_0) + \sqrt{K_\varepsilon}}{1 - \sqrt{\kappa_\varepsilon}} \right) \\
 & = \frac{C_0 \|\varphi\|_{\tilde{d}}}{1 - \gamma} \left( C_1 \psi(\varepsilon) \left( 1 + \frac{1}{n} \right) + \frac{C_2}{\sqrt{n}} + \frac{C_3}{n} \right),
 \end{aligned}$$

completing the proof of part (a).

We now consider statement (b). We have from (48)

$$\begin{aligned}
 \left\| \mathbb{E} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(U_k^\varepsilon) - v_0(\varphi) \right\|_{\mathcal{X}} &= \left\| \mathbb{E} \left( \frac{\Theta(U_0^\varepsilon) - \Theta(U_n^\varepsilon)}{n} \right) \right. \\
 &\quad \left. + \frac{1}{n} \mathbb{E} M_n^\varepsilon + \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E} (\mathcal{P}_\varepsilon - \mathcal{P}) \Theta(U_k^\varepsilon) \right\|_{\mathcal{X}} \\
 &\leq \frac{1}{n} \mathbb{E} \|\Theta(U_0^\varepsilon) - \Theta(U_n^\varepsilon)\|_{\mathcal{X}} + \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E} \|(\mathcal{P}_\varepsilon - \mathcal{P}) \Theta(U_k^\varepsilon)\|_{\mathcal{X}}.
 \end{aligned}$$

Now we can bound the first term using (50)

$$\begin{aligned}
 \frac{1}{n} \mathbb{E} \|\Theta(U_0^\varepsilon) - \Theta(U_n^\varepsilon)\|_{\mathcal{X}} &\leq \frac{C}{n} \left[ \sqrt{2} + \sqrt{\theta} \left( \sqrt{V(U_0^\varepsilon)} (1 + \kappa_\varepsilon^{n/2}) + \frac{\sqrt{K_\varepsilon}}{1 - \sqrt{\kappa_\varepsilon}} \right) \right] \\
 &\equiv \frac{1}{n} \frac{\|\varphi\|_{\tilde{d}}}{1 - \gamma} C_4,
 \end{aligned}$$

where once again  $C = \frac{C_0}{1-\gamma} \|\varphi\|_{\tilde{d}}$ , and now using (49)

$$\begin{aligned}
 \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E} \|(\mathcal{P}_\varepsilon - \mathcal{P}_0) \Theta(U_k^\varepsilon)\|_{\mathcal{X}} &\leq C \psi(\varepsilon) \left( 1 + \frac{\sqrt{\kappa_\varepsilon}/n V(u_0) + \sqrt{K_\varepsilon}}{1 - \sqrt{\kappa_\varepsilon}} \right) \\
 &\equiv \frac{\|\varphi\|_{\tilde{d}}}{1 - \gamma} \psi(\varepsilon) \left( C_5 + \frac{C_6}{n} \right).
 \end{aligned}$$

□

## APPENDIX C: PROOF OF RESULTS IN SECTION 5

**PROOF OF PROPOSITION 2.** Repeating the same calculation as in (27) we have for any  $u \in \mathcal{H}^1(\Omega)$ ,

$$\tilde{d}_0(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u)^2 \leq d_0(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u) [2 + \theta((\kappa + \kappa_\varepsilon)V(u) + (K + K_\varepsilon))].$$

Now let  $\pi_u \in \Upsilon(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u)$  obtained as follows: draw  $\xi_j \stackrel{\text{i.i.d.}}{\sim} N(0, (1 - \beta^2))$  and set  $v = \sum_{j=0}^\infty a_j \xi_j \phi_j$  and  $v_\varepsilon = \sum_{j=0}^\infty a_j^\varepsilon \xi_j \phi_j^\varepsilon$  and propose

$$u^* = \beta u + v, \quad u_\varepsilon^* = \beta u + v_\varepsilon.$$

Next draw a uniform random variable  $\varsigma$ , then the first (exact) chain accepts the proposal  $u^*$  if  $\varsigma < \alpha(u, u^*)$  while the second (perturbed) chain accepts  $u_\varepsilon^*$  if  $\varsigma < \alpha(u, u_\varepsilon^*)$ . Since this coupling is not necessarily optimal we have that

$$\begin{aligned}
 d_0(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u) &\leq \mathbb{E}[d_0(u^*, u_\varepsilon^*) \mathbb{P}(\text{both chains accept})] \\
 &\quad + \mathbb{E}[d_0(u^*, u) \mathbb{P}(\text{only first chain accepts})] \\
 (51) \quad &\quad + \mathbb{E}[d_0(u, u_\varepsilon^*) \mathbb{P}(\text{only second chain accepts})] \\
 &\leq \mathbb{E}[d_0(u^*, u_\varepsilon^*) \mathbb{P}(\text{both chains accept})] \\
 &\quad + \mathbb{E}[\mathbb{P}(\text{only one chain accepts})].
 \end{aligned}$$

Since  $\Psi$  is globally Lipschitz and  $1 \wedge \exp$  is also 1-Lipschitz it follows that

$$\begin{aligned}
 \mathbb{P}(\text{only one chain accepts}) &\leq |\Psi(u^*) - \Psi(u_\varepsilon^*)| \\
 &\leq L \|u^* - u_\varepsilon^*\|_{\mathcal{H}^1(\Omega)} = L \|v - v_\varepsilon\|_{\mathcal{H}^s(\Omega)} \\
 &\leq L \left( \left\| \sum_{j=0}^{\infty} a_j \xi_j (\phi_j - \phi_j^\varepsilon) \right\|_{\mathcal{H}^s(\Omega)} + \left\| \sum_{j=0}^{\infty} (a_j - a_j^\varepsilon) \xi_j \phi_j \right\|_{\mathcal{H}^s(\Omega)} \right) \\
 &\leq L \sum_{j=0}^{\infty} a_j |\xi_j| \|\phi_j - \phi_j^\varepsilon\|_{\mathcal{H}^s(\Omega)} + |a_j - a_j^\varepsilon| |\xi_j| \|\phi_j\|_{\mathcal{H}^s(\Omega)}.
 \end{aligned}$$

Now by the hypothesis that the sequences  $\{a_j \|\phi_j - \phi_j^\varepsilon\|_{\mathcal{H}^s(\Omega)}\}$  and  $\{|a_j - a_j^\varepsilon| \|\phi_j\|_{\mathcal{H}^s(\Omega)}\}$  belong to  $\ell^1$ , Kolmogorov's two series theorem yields that the above sum converges a.s. Applying Cauchy-Schwarz we can write

$$\begin{aligned}
 \mathbb{P}(\text{only one chain accepts}) &\leq L \left( \sum_{j=0}^{\infty} a_j |\xi_j|^2 \|\phi_j\|_{\mathcal{H}^s(\Omega)}^2 \right)^{1/2} \left( \sum_{j=0}^{\infty} a_j \frac{\|\phi_j - \phi_j^\varepsilon\|_{\mathcal{H}^s(\Omega)}^2}{\|\phi_j\|_{\mathcal{H}^s(\Omega)}^2} \right)^{1/2} \\
 &\quad + L \left( \sum_{j=0}^{\infty} a_j^2 |\xi_j|^2 \right)^{1/2} \left( \sum_{j=0}^{\infty} \frac{|a_j - a_j^\varepsilon|^2}{a_j^2} \|\phi_j\|_{\mathcal{H}^s(\Omega)}^2 \right)^{1/2},
 \end{aligned}$$

from which it follows that

$$\begin{aligned}
 \mathbb{E} \mathbb{P}(\text{only one chain accepts}) &\leq L \left( \sum_{j=0}^{\infty} a_j \|\phi_j\|_{\mathcal{H}^s(\Omega)}^2 \mathbb{E} |\xi_j|^2 \right)^{1/2} \left( \sum_{j=0}^{\infty} a_j \frac{\|\phi_j - \phi_j^\varepsilon\|_{\mathcal{H}^s(\Omega)}^2}{\|\phi_j\|_{\mathcal{H}^s(\Omega)}^2} \right)^{1/2} \\
 &\quad + L \left( \sum_{j=0}^{\infty} a_j^2 \mathbb{E} |\xi_j|^2 \right)^{1/2} \left( \sum_{j=0}^{\infty} \frac{|a_j - a_j^\varepsilon|^2}{a_j^2} \|\phi_j\|_{\mathcal{H}^s(\Omega)}^2 \right)^{1/2} \\
 &\leq C_1 \left[ \left( \sum_{j=0}^{\infty} a_j \frac{\|\phi_j - \phi_j^\varepsilon\|_{\mathcal{H}^s(\Omega)}^2}{\|\phi_j\|_{\mathcal{H}^s(\Omega)}^2} \right)^{1/2} + \left( \sum_{j=0}^{\infty} \frac{|a_j - a_j^\varepsilon|^2}{a_j^2} \|\phi_j\|_{\mathcal{H}^s(\Omega)}^2 \right)^{1/2} \right].
 \end{aligned}$$

Since  $\{a_j\} \in \ell^2$  and  $\{a_j \|\phi_j\|_{\mathcal{H}^s(\Omega)}^2\} \in \ell^1$ . We further have, by a similar calculation as above, that

$$\begin{aligned}
 d_0(u^*, u_\varepsilon^*) &= 1 \wedge \frac{\|u^* - u_\varepsilon^*\|_{\mathcal{H}^1(\Omega)}}{\omega} = 1 \wedge \frac{\|v^* - v_\varepsilon^*\|_{\mathcal{H}^1(\Omega)}}{\omega} \\
 &\leq 1 \wedge \frac{1}{\omega} \left( \sum_{j=0}^{\infty} a_j |\xi_j|^2 \|\phi_j\|_{\mathcal{H}^s(\Omega)}^2 \right)^{1/2} \left( \sum_{j=0}^{\infty} a_j \frac{\|\phi_j - \phi_j^\varepsilon\|_{\mathcal{H}^s(\Omega)}^2}{\|\phi_j\|_{\mathcal{H}^s(\Omega)}^2} \right)^{1/2} \\
 &\quad + \frac{1}{\omega} \left( \sum_{j=0}^{\infty} a_j^2 |\xi_j|^2 \right)^{1/2} \left( \sum_{j=0}^{\infty} \frac{|a_j - a_j^\varepsilon|^2}{a_j^2} \|\phi_j\|_{\mathcal{H}^s(\Omega)}^2 \right)^{1/2} \\
 &=: 1 \wedge T_1 + T_2.
 \end{aligned}$$

Thus it follows by Markov's inequality that

$$\begin{aligned}\mathbb{E} d_0(u^*, u_\varepsilon^*) &\leq \mathbb{P}(T_1 + T_2 \geq 1) + \mathbb{E} T_1 + T_2 \\ &\leq 2 \mathbb{E}(T_1 + T_2) \\ &\leq \frac{2C_2}{\omega} \left[ \left( \sum_{j=0}^{\infty} a_j \frac{\|\phi_j - \phi_j^\varepsilon\|_{\mathcal{H}^s(\Omega)}^2}{\|\phi_j\|_{\mathcal{H}^s(\Omega)}^2} \right)^{1/2} + \left( \sum_{j=0}^{\infty} \frac{|a_j - a_j^\varepsilon|^2}{a_j^2} \|\phi_j\|_{\mathcal{H}^s(\Omega)}^2 \right)^{1/2} \right].\end{aligned}$$

Substituting the above bounds back into (51) we obtain

$$\begin{aligned}d_0(\mathcal{P}\delta_u, \mathcal{P}_\varepsilon\delta_u) \\ \leq C \left[ \left( \sum_{j=0}^{\infty} a_j \frac{\|\phi_j - \phi_j^\varepsilon\|_{\mathcal{H}^s(\Omega)}^2}{\|\phi_j\|_{\mathcal{H}^s(\Omega)}^2} \right)^{1/2} + \left( \sum_{j=0}^{\infty} \frac{|a_j - a_j^\varepsilon|^2}{a_j^2} \|\phi_j\|_{\mathcal{H}^s(\Omega)}^2 \right)^{1/2} \right].\end{aligned} \quad \square$$

**Acknowledgements.** The authors are thankful to Prof. Andrew Stuart for interesting conversations regarding convergence properties of MCMC algorithms, and to Prof. Jonathan Mattingly for pointing out an error in an early draft of the manuscript. The authors are also grateful to the anonymous reviewers whose comments and suggestions helped us improve the article immensely.

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