J. Farre (2021) "Borel and Volume Classes for Dense Representations
of Discrete Groups,"
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–66
doi:10.1093/imrn/rnab078

Borel and Volume Classes for Dense Representations of Discrete Groups

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We show that the bounded Borel class of any dense representation $\rho : G \to \mathrm{PSL}_n\mathbb{C}$ is non-zero in degree three bounded cohomology and has maximal semi-norm, for any discrete group G. When $n = 2$, the Borel class is equal to the three-dimensional hyperbolic volume class. Using tools from the theory of Kleinian groups, we show that the volume class of a dense representation $\rho : G \to \mathrm{PSL}_2\mathbb{C}$ is uniformly separated in semi-norm from any other representation $\rho' : G \to \mathrm{PSL}_2\mathbb{C}$ for which there is a subgroup $H \leq G$ on which ρ is still dense but ρ' is discrete or indiscrete but stabilizes a point, line, or plane in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. We exhibit a family of dense representations of a non-abelian free group on two letters and a family of discontinuous dense representations of $\mathrm{PSL}_2\mathbb{R}$, whose volume classes are linearly independent and satisfy some additional properties; the cardinality of these families is that of the continuum. We explain how the strategy employed may be used to produce non-trivial volume classes in higher dimensions, contingent on the existence of a family of hyperbolic manifolds with certain topological and geometric properties.

1 Introduction

The bounded cohomology of discrete groups admits an almost entirely algebraic description, but many bounded classes are most naturally understood in the context of the geometry of non-positively curved metric spaces. Isometric actions of discrete

Received May 19, 2019; Revised March 3, 2021; Accepted March 10, 2021
Communicated by Prof. Marc Burger

groups on quasi-trees and Gromov hyperbolic metric spaces are responsible for producing an abundance of non-trivial bounded classes in degree two. Conversely, bounded cohomology has been used as a tool to understand the space of isometric actions that a discrete group admits on, say, a non-compact symmetric space X. Broadly, we aim to understand to what extent bounded cohomology parameterizes such actions, including those that are not covering actions or that factor through some other group. We narrow our focus and consider the setting that $\mathrm{Isom}^+(X) = \mathrm{PSL}_n\mathbb{C}, n \geq 2$. In this paper, we will explain the extent to which bounded cohomology sees the different isometric actions of a discrete group G with a dense orbit in X. Our investigation relies heavily on the existence of certain *discrete* free subgroups of $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2\mathbb{C}$. Indeed, we will study the geometry of a complete hyperbolic *three*-manifold homeomorphic to a genus 2 handlebody in order to show that many non-conjugate dense representations of G yield different volume classes in $\mathrm{H}^3_{\mathrm{b}}(G;\mathbb{R})$.

Immersed locally geodesic tetrahedra in a complete hyperbolic *three*-manifold M lift to embedded geodesic tetrahedra in the universal cover $\mathbb{H}^3$. We can measure the (signed) hyperbolic volume of such a tetrahedron, which is bounded above by $v_3 = 1.01494....$ This rule defines a bounded cocycle, hence a class in the degree 3 bounded cohomology of the manifold.

The bounded cohomology ring is an invariant of the fundamental group $\pi_1(M)$ [29], and any (other) action $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ yields a bounded class $[\rho^*\mathrm{vol}_3] \in \mathrm{H}^3_{\mathrm{b}}(\pi_1(M);\mathbb{R})$, which is obtained from a cocycle that measures the volumes of geodesic tetrahedra with vertices in the orbit $\rho(\pi_1(M)).x$, where $x \in \mathbb{H}^3 \cup \partial\mathbb{H}^3$.

We say that a representation is *geometrically elementary* if its image stabilizes a line, a totally geodesic plane, or an (ideal) point in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$.

Theorem 1.1 (Theorem 4.6 and Theorem 4.11). If G is a discrete group and $\rho : G \to \mathrm{PSL}_2\mathbb{C}$ is a dense representation, i.e. $\overline{\rho(G)} = \mathrm{PSL}_2\mathbb{C}$, then

$$\|[\rho^*\mathrm{vol}_3]\|_\infty = v_3.$$

In particular $[\rho^*\mathrm{vol}_3] \neq 0 \in \mathrm{H}^3_{\mathrm{b}}(G;\mathbb{R})$. Moreover, if $\rho_0 : G \to \mathrm{PSL}_2\mathbb{C}$ is any other representation and there is a subgroup $H \leq G$ such that $\overline{\rho(H)} = \mathrm{PSL}_2\mathbb{C}$, but ρ_0 is geometrically elementary or discrete restricted to H, then

$$\|[\rho^*\mathrm{vol}_3] - [\rho_0^*\mathrm{vol}_3]\|_\infty \geq v_3,$$

and this bound is sharp.

We consider in Section 6.3, for a dense representation $\rho : F_2 \to \mathrm{PSL}_2\mathbb{C}$, the collection of *$\rho$-dense subgroups*

$$\mathcal{DS}(\rho) = \{H \leq_{f.g.} F_2 : \overline{\rho(H)} = \mathrm{PSL}_2\mathbb{C}\},$$

where $H \leq_{f.g.} F_2$ means that H is a finitely generated subgroup of F_2. If $\rho_1, \rho_2 \in \mathrm{Hom}(F_2, \mathrm{PSL}_2\mathbb{C})$ are conjugate, it is clear that $\mathcal{DS}(\rho_1) = \mathcal{DS}(\rho_2)$, while if ρ_1 and ρ_2 are dense and $\mathcal{DS}(\rho_1) \neq \mathcal{DS}(\rho_2)$, then Theorem 1.1 together with Lemma 2.2 implies that $\|[\rho_1^*\mathrm{vol}_3] - [\rho_2^*\mathrm{vol}_3]\|_\infty \geq v_3$. However, we have not been able to construct examples of non-conjugate dense representations ρ_1 and ρ_2 such that $\mathcal{DS}(\rho_1) = \mathcal{DS}(\rho_2)$. It is possible that $\mathcal{DS}(\rho)$ is a complete invariant of the $\mathrm{PSL}_2\mathbb{C}$ conjugacy class of a dense representation $\rho : F_2 \to \mathrm{PSL}_2\mathbb{C}$; by Theorem 1.1, $[\rho^*\mathrm{vol}_3]$ would then be a complete invariant of the $\mathrm{PSL}_2\mathbb{C}$ conjugacy class of a dense ρ; see Section 6.3.

Suppose M is a hyperbolic *three*-manifold of finite volume and $\rho : \pi_1(M) \to \mathrm{PSL}_2\mathbb{C}$; the *volume of* ρ is a numerical invariant that can be obtained by pairing the *bounded fundamental class* or *volume class* $[\rho^*\mathrm{vol}_3] \in \mathrm{H}^3_\mathrm{b}(\pi_1(M); \mathbb{R})$ of the representation with a (relative) fundamental cycle of M. This numerical invariant has been studied from this perspective in [5], where Bucher, Burger, and Iozzi show that when the maximal volume for a representation is achieved, the representation must be conjugated by an isometry to the lattice embedding $\pi_1(M) \hookrightarrow \mathrm{PSL}_2\mathbb{C}$. We refer to this kind of result informally as a *volume rigidity* result. Theorem 1.1 explains that, for a dense representation, while the volume of ρ is non-maximal, the *volume class* of that representation has maximal semi-norm.

Dunfield [19] proved a volume rigidity theorem for closed hyperbolic three-manifolds following an observation of Goldman [27] about Gromov and Thurston's proof of Mostow's famous rigidity theorem. Francaviglia [24] and Klaff [33] proved a volume rigidity theorem for finite volume hyperbolic manifolds, using the notion of *pseudo-developing maps* defined in [19]. While Dunfield's definition of the volume of a representation via pseudo-developing maps depends on choices, it turns out to be an invariant of the representation [24,Theorem 1.1]. The interested reader is directed toward [5] for more history of volume rigidity results.

One advantage of using bounded cohomology to define the volume of a representation is that no choices are made. We sometimes see that the pullback of a continuous bounded cohomology class *itself* can characterize a representation, up to conjugation. For example, [11,Theorem 3] states that the pullback of the bounded Kähler class is a complete invariant of continuous actions of a broad class of groups on an irreducible

Hermitian symmetric space that is not of tube type. This result fails miserably when the Hermitian symmetric space is of tube type. For example, $\mathbb{H}^2$ is such a space, and the bounded Kähler class in $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{PSL}_2\mathbb{R};\mathbb{R})$ is a multiple of the two dimensional hyperbolic volume class (also the Euler class). For any two discrete and faithful representations $\rho, \rho' : \pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ of a closed surface group, we have $[\rho^*\mathrm{vol}_2] = \pm[\rho'^*\mathrm{vol}_2]$; see, e.g. [9,Lemme 3.10]. The space of $\mathrm{PSL}_2\mathbb{R}$-conjugacy classes of discrete and faithful representations of a closed hyperbolic surface group is a union of two high-dimensional cells that can each be identified with the Teichmüller space of that surface.

We are most interested in hyperbolic manifolds with *infinite volume*, and we work directly with volume classes in bounded cohomology. For example, let S be a finite type oriented surface with negative Euler characteristic. We say that two representations $\rho_1, \rho_2 : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ are *quasi-isometric* if there is a (ρ_1, ρ_2)-equivariant quasi-isometry $\mathbb{H}^3 \to \mathbb{H}^3$. The following quasi-isometric volume rigidity theorem sees a combination of both of the phenomena in the previous paragraph.

Theorem 1.2 (Theorem 3.12). There exists a constant $\epsilon = \epsilon(S)$ such that the following holds. Suppose that $\rho_0 : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is a discrete and faithful representation without parabolic elements, and that $[\rho_0^*\mathrm{vol}_3] \neq 0$. If $\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is any other representation without parabolics satisfying

$$\|[\rho_0^*\mathrm{vol}_3] - [\rho^*\mathrm{vol}_3]\|_\infty < \epsilon,$$

then ρ is discrete and faithful, and ρ is quasi-isometric to ρ_0. If ρ_0 is totally degenerate, then ρ_0 and ρ are conjugate in $\mathrm{PSL}_2\mathbb{C}$.

Theorem 1.2 is proved by combining Theorem 1.1 with previous work of the author, the classification of finitely generated marked Kleinian groups, and a theorem of Soma (Theorem 2.1). We provide a detailed proof of Theorem 1.2 in Section 3.6 after establishing definitions and some background on the geometry and topology of tame hyperbolic *three*-manifolds of infinite volume. We also elaborate on the role of parabolic cusps and the geometric meaning of the quasi-isometric equivalence relation, therein. We note that quasi-isometric equivalence of discrete and faithful representations of finitely generated torsion free Kleinian groups is equivalent to the existence of a *volume preserving* bi-Lipschitz homeomorphism of quotient manifolds $\mathbb{H}^3/\mathrm{im}\rho \to \mathbb{H}^3/\mathrm{im}\rho_0$ inducing $\rho_0 \circ \rho^{-1}$ on fundamental groups.

Recently, Bucher, Burger, and Iozzi proved a volume rigidity result for representations of finite volume hyperbolic *three*-manifold groups into $\mathrm{PSL}_n\mathbb{C}$ with respect to the so-called *Borel invariant* of a representation, defined in [6]. To do so, they computed the continuous bounded cohomology $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{PSL}_n\mathbb{C};\mathbb{R})$, which is generated by a single class β_n, called the *bounded Borel class*. They also computed the semi-norm of the bounded Borel class and how it behaves under various natural inclusions $\mathrm{PSL}_k\mathbb{C} \hookrightarrow \mathrm{PSL}_n\mathbb{C}, k \leq n$ (see Section 2.5). The bounded Borel class generalizes hyperbolic volume in the sense that $\beta_2 = [\mathrm{vol}_3]$. Due to the work of [6], most of the argument that proves Theorem 1.1 generalizes to dense representations into $\mathrm{PSL}_n\mathbb{C}$.

Corollary 1.3 (Theorem 5.1 and Corollary 5.2). Let G be a discrete group and $\rho : G \to \mathrm{PSL}_n\mathbb{C}$ be dense. Then

$$\|\rho^*\beta_n\|_\infty = v_3 \frac{n(n^2-1)}{6}.$$

Suppose that $\rho_0 : G \to \mathrm{PSL}_2\mathbb{C}$ is such that there exists a subgroup $H \leq G$ such that $\overline{\rho(H)} = \mathrm{PSL}_n\mathbb{C}$ and ρ_0 is geometrically elementary or discrete and faithful restricted to H. Then

$$\|\rho^*\beta_n - (\iota_k \circ \rho_0)^*\beta_k\|_\infty \geq v_3 \frac{n(n^2-1)}{6},$$

for all $k \geq 2$, where $\iota_k : \mathrm{PSL}_2\mathbb{C} \to \mathrm{PSL}_n\mathbb{C}$ is the (unique up to conjugation) irreducible representation.

Remark 1.4. Compare the hypotheses of the second statements in Theorem 1.1 to those in the second statement of Corollary 1.3. In Corollary 1.3, we assume that ρ_0 is *faithful* in addition to being discrete. This is because it is easy to construct somewhat explicit dense representations of a free group F_2 on two letters to $\mathrm{PSL}_2\mathbb{C}$. From this, we obtain the additional control needed to remove the extra hypothesis and prove Theorem 1.1; see Proposition 4.9, Case 2.

We will consider the *reduced bounded cohomology* $\overline{\mathrm{H}}^3_{\mathrm{b}}(G;\mathbb{R}) = \mathrm{H}^3_{\mathrm{b}}(G;\mathbb{R})/\mathcal{Z}$, where $\mathcal{Z}$ is the subspace of zero-norm bounded cohomology. The reduced space $\overline{\mathrm{H}}^3_{\mathrm{b}}(G;\mathbb{R})$ is a Banach space with respect to the quotient norm. Our techniques apply to families of dense representations satisfying some technical condition on subgroups, as in Theorem 1.1.

Theorem 1.5 (Theorem 4.12). Suppose $\{\rho_i : G \to \mathrm{PSL}_2\mathbb{C}\}_{i=1}^N$ are dense representations. If there are subgroups $H_i \leq G$ such that $\overline{\rho_i(H_i)} = \mathrm{PSL}_2\mathbb{C}$ but $\rho_i|_{H_j}$ is geometrically elementary or discrete for $i \neq j$, then for any $a_1, ..., a_N \in \mathbb{R}$, we have

$$\left\| \sum_{i=1}^{N} a_i[\rho_i^*\mathrm{vol}_3] \right\|_\infty \geq \max\{|a_i|\} \cdot v_3.$$

Consequently, $\{[\rho_i^*\mathrm{vol}_3] : i = 1, 2, ..., N\} \subset \overline{\mathrm{H}}_{\mathrm{b}}^3(G; \mathbb{R})$ is a linearly independent set.

In Section 6.1, we construct families of representations that satisfy the hypotheses of Theorem 1.5. More specifically, for a non-abelian free group F_2 on two letters and every $\theta \in (0, 1)$, we construct a representation $\rho_\theta : F_2 \to \mathrm{PSL}_2\mathbb{C}$; any finite rationally independent set $\Lambda' \subset (0, 1)$ yields a family $\{\rho_\theta\}_{\theta \in \Lambda'}$ that satisfies the hypotheses of Theorem 1.5. In the first line of the following corollary, we invoke the axiom of choice.

Corollary 1.6 (Theorem 6.4). Let $\Lambda \subset (0, 1)$ be such that $\Lambda \cup \{1\}$ is a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space, and let $\{\rho_\theta\}_{\theta \in \Lambda}$ be the dense representations constructed in Section 6.1. The map

$$\Lambda \to \overline{\mathrm{H}}_{\mathrm{b}}^3(F_2)$$
$$\theta \mapsto [\rho_\theta^*\mathrm{vol}_3]$$

is injective with discrete image, and $\{[\rho_\theta^*\mathrm{vol}_3] : \theta \in \Lambda\}$ is a linearly independent set. In particular, $\dim_{\mathbb{R}}\langle[\rho_\theta^*\mathrm{vol}_3] : \theta \in \Lambda\rangle_{\mathbb{R}} = \#\mathbb{R}$.

As a further curiosity, we show that the spaces $\overline{\mathrm{H}}_{\mathrm{b}}^3(\mathrm{PSL}_2\mathbb{R}; \mathbb{R})$ and $\overline{\mathrm{H}}_{\mathrm{b}}^3(\mathrm{PSL}_2\mathbb{C}; \mathbb{R})$ are quite large when we endow $\mathrm{PSL}_2\mathbb{R}$ and $\mathrm{PSL}_2\mathbb{C}$ with the *discrete* topology. We construct 'wild' field maps $\sigma : \mathbb{C} \to \mathbb{C}$ that induce homomorphisms $\rho_\sigma : \mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_2\mathbb{C}$, which are continuous only if $\mathrm{PSL}_2\mathbb{R}$ is endowed with the discrete topology. If σ is not the identity or complex conjugation, then $\sigma(\mathbb{R})$ is dense in $\mathbb{C}$, hence $\rho_\sigma(\mathrm{PSL}_2\mathbb{R})$ is dense in $\mathrm{PSL}_2\mathbb{C}$. Indeed, we construct many such σ by extending bijections between transcendence bases of $\mathbb{C}$ over $\mathbb{Q}$. We carefully construct a family of dense representations $\{\rho_t' : \mathrm{PSL}_2\mathbb{C} \to \mathrm{PSL}_2\mathbb{C}\}_{t \in \mathbb{R}}$ that restrict to dense representations $\{\rho_t : \mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_2\mathbb{C}\}_{t \in \mathbb{R}}$. For many free subgroups $F_2 \leq \mathrm{PSL}_2\mathbb{R}$, $\rho_t(F_2)$ is Schottky, while other free subgroups are mapped densely by ρ_t.

Corollary 1.7 (Theorem 6.5). There are dense representations $\{\rho'_t : \mathrm{PSL}_2\mathbb{C} \to \mathrm{PSL}_2\mathbb{C}\}_{t\in\mathbb{R}}$ that restrict to dense representations $\{\rho_t : \mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_2\mathbb{C}\}_{t\in\mathbb{R}}$ such that $\{[\rho_t^*\mathrm{vol}_3] : t \in \mathbb{R}\}$ is a linearly independent set in $\overline{\mathrm{H}}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{R};\mathbb{R})$ and $\{[\rho_t'^*\mathrm{vol}_3] : t \in \mathbb{R}\}$ is a linearly independent set in $\overline{\mathrm{H}}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{C};\mathbb{R})$.

There seems to be quite a bit of flexibility in our construction of wild field maps $\sigma : \mathbb{C} \to \mathbb{C}$; the dimension of the vector space of bounded *three*-cochains on $\mathrm{PSL}_2\mathbb{C}$ or $\mathrm{PSL}_2\mathbb{R}$ is $2^{\#\mathbb{R}}$, which is an upper bound on the real dimension of degree 3 reduced bounded cohomology. It certainly seems possible that the dimension of reduced bounded cohomology for these groups is $2^{\#\mathbb{R}}$; see Question 6.6.

The main line of argument used to prove our theorems is as follows: a densely embedded group $G \leq \mathrm{PSL}_2\mathbb{C}$ can approximate the geometry of *any* finitely generated Kleinian group, up to a certain scale. More precisely, given a finitely generated Kleinian group $\Gamma = \langle \gamma_1, \dots, \gamma_k \rangle \leq \mathrm{PSL}_2\mathbb{C}$, a dense representation $\rho : G \to \mathrm{PSL}_2\mathbb{C}$, an $\epsilon > 0$, and a positive integer N, there are $g_1, \dots, g_k \in G$ such that all words of length at most N in the $\rho(g_i)$ are in the ϵ-neighborhood of corresponding words in the γ_i. Suppose $\mathbb{H}^3/\Gamma$ has a submanifold $K \subset M$ with large volume and small surface area. Assume that K is equipped with a straight triangulation by geodesic tetrahedra, and that the triangulation does not have too many triangles on the boundary. Then, we can homotope the triangulation, so that there is only one vertex and such that we do not lose too much volume during the homotopy, which is a small miracle of hyperbolic geometry. The edges of the tetrahedra are now labeled by elements of $\pi_1(M)$, because they are closed, based loops. This finite triangulation lifts to the universal cover. The idea is now to use our approximation of words of length at most N in the γ_i by words of length at most N in the $\rho(g_i)$ to build a chain on G that has almost the same shape as our lifted chain, via its ρ-action. In this way, we use the geometry of discrete groups to build chains on our abstract group G that have large volume and small boundary area, which is enough to show that $[\rho^*\mathrm{vol}_3] \neq 0$; if in addition, the chains on Γ are ϵ-*efficient*, we can show that $\|[\rho^*\mathrm{vol}_3]\| \geq \epsilon$. Adapting a construction of Soma [49,Lemma 3.2], we are able to construct $(v_3 - \epsilon)$-efficient chains, for all $\epsilon > 0$; see also Section 4.1 and, in particular, Lemma 4.1 where we recreate Soma's construction.

The preceding paragraph is made precise in Section 2.6 where we record a key observation of this paper, Proposition 2.8. Our main line of argument is very generally applicable in the sense that it can be adapted to work in all dimensions, contingent on the existence of a sequence of hyperbolic n-manifolds with prescribed topological properties. Proposition 7.2 formalizes this idea to give one of many sufficient conditions by which one can apply the techniques in this paper in higher dimensions. First, we need some terminology. The following definition is made in order to highlight the necessary algebraic and topological ingredients of the proof of Theorem 1.1.

Definition 1.8 (Definition 7.1). Let Γ be a discrete group, $\alpha \in \mathrm{H}_n(\Gamma;\mathbb{R})$, and $K > 0$. We say that α is *K-freely approximated* if there is an integer m, a homomorphism $\varphi : F_m \to \Gamma$, and a chain $Z \in \mathrm{C}_n(F_m;\mathbb{R})$ such that $\varphi_*(Z) \in \alpha$ and $\|\partial Z\|_1 \le K$.

The conclusion of the following proposition may seem surprising, at first.

Proposition 1.9 (Proposition 7.2). Suppose (M_i) is a sequence of oriented, closed hyperbolic n-manifolds with volume tending to infinity. Let $[M_i] \in \mathrm{H}_n(\pi_1(M_i);\mathbb{R})$ be the image of the fundamental class of M_i under the natural isomorphism $\mathrm{H}_n(M_i;\mathbb{R}) \to \mathrm{H}_n(\pi_1(M_i);\mathbb{R})$.

If there is a K such that $[M_i]$ is K-freely approximated for all i, then for any dense representation $\rho : F_2 \to \mathrm{Isom}^+(\mathbb{H}^n)$,

$$[\rho^*\mathrm{vol}_n] \neq 0 \in \mathrm{H}^n_{\mathrm{b}}(F_2;\mathbb{R}).$$

See also Remark 7.3 for a discussion of the (possible) dimension of $\mathrm{H}^n_{\mathrm{b}}(F_2;\mathbb{R})$, for even $n \ge 4$.

Remark 1.10. For many sequences (M_i) of closed hyperbolic surfaces and hyperbolic 3-manifolds with volume tending to infinity, one can produce a bound K depending on that sequence such that $[M_i]$ is K-freely approximated.

However, it is not at all clear if there can be any sequence of hyperbolic n-manifolds satisfying the hypotheses of the proposition, for $n \ge 4$. For example, if it were true that there is some acylindrically hyperbolic group Γ and $\mathrm{H}^n_{\mathrm{b}}(\Gamma;\mathbb{R}) = 0$, then $\mathrm{H}^n_{\mathrm{b}}(F_2;\mathbb{R}) = 0$, as well [23]. So if there is such an $n \ge 4$ and Γ, then no sequence of hyperbolic n-manifolds satisfying the topological constraints of the hypotheses of Proposition 7.2 can exist.

To obtain the uniform separation of certain volume classes, as in Theorem 1.1, we appeal to the classification theory and structure theory of finitely generated Kleinian groups, and make extensive use of the Covering Theorem 3.8 and the Tameness Theorem 3.3, which are now standard tools for studying hyperbolic three-manifolds. The Ending Lamination Theorem 3.10 also plays an important role and informs our understanding of the geometry of hyperbolic three-manifolds of infinite volume, generally.

The structure of the paper is as follows. In Section 2, we provide some background on bounded cohomology, volume classes, and the bounded Borel class. In Section 2.6, we give a general strategy for approximation and record our key insight Proposition 2.8.

In Section 3, we describe the geometry of ends of hyperbolic *three*-manifolds and give some context for the three major structural results we need in the sequel. In Section 3.6, we assume Theorem 1.1 and deduce Theorem 1.2 from previous work of the author, collected in Theorem 3.11.

In Section 4, we employ our resources coming from the classification theory of tame hyperbolic *three*-manifolds to prove Theorem 1.1. In Section 4.1, we recall Soma's construction of $(v_3 - \epsilon)$-efficient chains, collect some technical facts, and prove the first statement of Theorem 1.1. In Section 4.2, we take a rather technical foray into analyzing infinite index subgroups of discrete and dense representations of a free group F_2 on *two* letters (Lemma 4.7 and Proposition 4.9). The rest of Theorem 1.1 follows quickly, thereafter. Then, we generalize Theorem 1.1 to give Theorem 1.5.

In Section 5, we pull back a higher rank formulation of the volume class, known as the bounded Borel class of a dense representation $\rho : \Gamma \to \mathrm{PSL}_n\mathbb{C}$; the argument we present for $n = 2$ generalizes to the higher rank setting, after a technical detour, thanks to [6], and we record the necessary modifications to the proof of Theorem 1.1 to obtain Corollary 1.3.

Section 6 concerns constructions of certain families of dense representations, 'applications' of Theorem 1.5 that produce large subspaces of bounded cohomology, and questions. In Section 6.1, we construct an uncountable family of non-conjugate representations, every finite subset of which satisfies the hypotheses of Theorem 1.5. This shows that the dimension of (reduced) bounded cohomology spanned by the volume classes for dense representations of a free group is large. Later, we use wild field embeddings to produce many linearly independent volume classes in degree three reduced bounded cohomology of $\mathrm{PSL}_2\mathbb{C}$ and $\mathrm{PSL}_2\mathbb{R}$, endowed with the discrete topology. We pose some questions that presented themselves to us during this investigation.

Finally, in Section 7, we discuss the problem of understanding the (non-)triviality of volume classes of dense representations in dimensions $n \geq 4$. We also show that closed hyperbolic surfaces are *two*-freely approximated; in fact our proof shows that a closed hyperbolic surface of genus at least *two* can be ϵ-freely approximated for any $\epsilon > 0$, using covering space theory; note the similarity of Lemma 7.4 with the standard computation of simplicial volume of closed surfaces of negative Euler characteristic. The main result of Section 7, Proposition 7.2, is essentially independent of the more technical work done in Sections 3-6, and so can be read and understood directly after Section 2.6.

2 Continuous bounded cohomology of groups and spaces

In this section, we establish some preliminaries on bounded cohomology, the volume class, and the bounded Borel class. In Section 2.6, we introduce the main novel idea of the paper, Proposition 2.8.

2.1 Continuous bounded cohomology of groups

Let $\mathcal{G}$ be a topological group. Then, $\mathcal{G}$ acts on the space of continuous functions $\{\mathcal{G}^k \to \mathbb{R}\}$ as follows. If $g \in \mathcal{G}$ and $f : \mathcal{G}^k \to \mathbb{R}$ is continuous, then

$$g.f(g_1, ..., g_k) := f(g^{-1}g_1, ..., g^{-1}g_k).$$

We define a cochain complex for $\mathcal{G}$ by considering the collection of continuous, $\mathcal{G}$-invariant functions

$$\mathrm{C}^n(\mathcal{G}; \mathbb{R}) = \{f : \mathcal{G}^{n+1} \to \mathbb{R} : g.f = f,\ \forall g \in \mathcal{G}\}.$$

The homogeneous co-boundary operator δ for the trivial $\mathcal{G}$-action on $\mathbb{R}$ is, for $f \in \mathrm{C}^n(\mathcal{G}; \mathbb{R})$,

$$\delta f(g_0, ..., g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, ..., \hat{g}_i, ..., g_{n+1}),$$

where $\hat{g}_i$ means to omit g_i, as usual. The co-boundary operator gives the collection $\mathrm{C}^\bullet(\mathcal{G}; \mathbb{R})$ the structure of a cochain complex. An n-cochain f is *bounded* if

$$\|f\|_\infty = \sup |f(g_0, ..., g_n)| < \infty,$$

where the supremum is taken over all $(n+1)$-tuples $(g_0, ..., g_n) \in \mathcal{G}^{n+1}$. The subspace of continuous bounded n-cochains is denoted by $\mathrm{C}^n_{\mathrm{b}}(\mathcal{G}; \mathbb{R})$.

The operator $\delta : \mathrm{C}^n_{\mathrm{b}}(\mathcal{G}; \mathbb{R}) \to \mathrm{C}^{n+1}_{\mathrm{b}}(\mathcal{G}; \mathbb{R})$ is a bounded linear, hence continuous, operator between Banach spaces with operator norm at most $n+2$, so the collection of bounded cochains $\mathrm{C}^\bullet_\mathrm{b}(\mathcal{G};\mathbb{R})$ forms a subcomplex of the ordinary cochain complex. The cohomology of $(\mathrm{C}^\bullet_\mathrm{b}(\mathcal{G};\mathbb{R}),\delta)$ is called the *continuous bounded cohomology* of $\mathcal{G}$, and we denote it by $\mathrm{H}^\bullet_\mathrm{cb}(\mathcal{G};\mathbb{R})$. The ∞-norm $\|\cdot\|_\infty$ descends to a semi-norm on bounded cohomology, so that if $\alpha\in\mathrm{H}^\bullet_\mathrm{cb}(\mathcal{G};\mathbb{R})$,

$$\|\alpha\|_\infty=\inf_{a\in\alpha}\|a\|_\infty.$$

A continuous group homomorphism $\varphi: H\to\mathcal{G}$ induces a map $\varphi^*:\mathrm{H}^\bullet_\mathrm{cb}(\mathcal{G};\mathbb{R})\to\mathrm{H}^\bullet_\mathrm{cb}(H;\mathbb{R})$ that is norm non-increasing.

When G is a discrete group, the continuity assumption on cochains is vacuous, and we write $\mathrm{H}^\bullet_\mathrm{b}(G;\mathbb{R})=\mathrm{H}^\bullet_\mathrm{cb}(G;\mathbb{R})$ to denote its bounded cohomology. Soma has shown [50] that the pseudo-norm is in general not a norm in degree ≥ 3. We will consider the quotient $\overline{\mathrm{H}}^3_\mathrm{b}(G;\mathbb{R})=\mathrm{H}^3_\mathrm{b}(G;\mathbb{R})/\mathcal{Z}$, where $\mathcal{Z}\subset\mathrm{H}^3_\mathrm{b}(G;\mathbb{R})$ is the subspace of zero-semi-norm elements. Then, $\overline{\mathrm{H}}^3_\mathrm{b}(G;\mathbb{R})$ is a Banach space with the quotient norm $\|\cdot\|_\infty$.

2.2 Norms on chain complexes

Given a connected countable CW-complex X, we define a norm on the singular chain complex of X as follows. Let $\Sigma_n=\{\sigma:\Delta_n\to X\}$ be the collection of singular n-simplices. Write a singular chain $A\in\mathrm{C}_n(X;\mathbb{R})$ as an $\mathbb{R}$-linear combination

$$A=\sum\alpha_\sigma\sigma,$$

where each $\sigma\in\Sigma_n$. The ℓ_1-norm of A is defined as

$$\|A\|_1=\sum|\alpha_\sigma|.$$

This norm promotes the algebraic chain complex $\mathrm{C}_\bullet(X;\mathbb{R})$ to a chain complex of normed linear spaces; the boundary operator is a bounded linear operator. Keeping track of this additional structure, we can take the topological dual chain complex

$$\left(\mathrm{C}_\bullet(X;\mathbb{R}),\partial,\|\cdot\|_1\right)^*=\left(\mathrm{C}^\bullet_\mathrm{b}(X;\mathbb{R}),\delta,\|\cdot\|_\infty\right).$$

The ∞-norm is naturally dual to the ℓ_1-norm, so the dual chain complex consists of *bounded* cochains. Define the *bounded cohomology* $\mathrm{H}^\bullet_\mathrm{b}(X;\mathbb{R})$ as the cohomology of this complex. Gromov [29] gave an argument, using the theory of multicomplexes, showing that for reasonable spaces M, the homotopy class of a classifying map $M\to K(\pi_1(M),1)$

induces an isometric isomorphism $\mathrm{H}^\bullet_{\mathrm{b}}(\pi_1(M);\mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{b}}(M;\mathbb{R})$. Recently, [22] provided a thorough treatment of the theory of multicomplexes and gave self contained account of Gromov's theorem; they also give other applications to bounded cohomology and simplicial volume. See also [30] for an approach using normed homological algebra.

For a discrete group G, we will consider the normed chain complex $(\mathrm{C}_\bullet(G;\mathbb{R}), \partial, \|\cdot\|_1)$ that defines the homology of G. The collection of *n-co-invariants* $\mathrm{C}_n(G;\mathbb{R})$ of G is the $\mathbb{R}$-linear span of $\Sigma_n(G) = \{(g_0, \dots, g_n) : g_i \in G\}/\sim$, where $\sim$ is the equivalence relation generated by $(g_0, \dots, g_n) \sim (gg_0, \dots, gg_n)$; we denote an equivalence class by $[g_0, \dots, g_n] \in \Sigma_n(G)$, and we think of $[g_0, \dots, g_n]$ as an n-simplex in the universal cover of a $K(G,1)$ for G, defined up to covering transformations, thus defining a simplex in the quotient $K(G,1)$. A group chain or n-co-invariant $Z \in \mathrm{C}_n(G;\mathbb{R})$ is then a sum

$$Z = \sum_{i=1}^{k} a_i [g_0^i, \dots, g_n^i],$$

where $[g_0^i, \dots, g_n^i] \neq [g_0^j, \dots, g_n^j]$ for $i \neq j$. The ℓ_1-norm is defined by $\|Z\|_1 = \sum_{i=1}^k |a_i|$. The boundary operator $\partial : \mathrm{C}_n(G;\mathbb{R}) \to \mathrm{C}_{n-1}(G;\mathbb{R})$ is the pre-dual of the co-boundary operator δ. One thinks of ∂ as the alternating sum of face maps on n-simplices. If $f \in \mathrm{C}^n_{\mathrm{b}}(G;\mathbb{R})$ and $Z \in \mathrm{C}_n(G;\mathbb{R})$, then we have a trivial inequality $|f(Z)| \leq \|f\|_\infty \|Z\|_1$.

2.3 Isometric chain maps

We will be interested in free marked Kleinian groups $\rho : F_d \to \mathrm{PSL}_2\mathbb{C}$, i.e. F_d is a free group of rank d and ρ is a discrete and faithful representation. Thus, $\mathrm{im}\rho = \Gamma$ acts properly discontinuously by orientation preserving isometries on $\mathbb{H}^3$, and the space $M_\rho = \mathbb{H}^3/\Gamma$ of orbits is a complete hyperbolic *three*-manifold of infinite volume. Call the orbit projection $\pi : \mathbb{H}^3 \to M_\rho$. There is a natural subspace of the singular chain complex $\mathrm{C}_\bullet(M_\rho)$ obtained by *straightening*. We have an ℓ_1-norm non-increasing chain map [56]

$$\mathrm{str} : \mathrm{C}_\bullet(M_\rho;\mathbb{R}) \to \mathrm{C}_\bullet(M_\rho;\mathbb{R})$$

defined by homotoping a singular n-simplex $\sigma : \Delta_n \to M_\rho$, relative to its vertex set, to the unique locally geodesic hyperbolic tetrahedron $\mathrm{str}\sigma$. We ignore issues of parameterization, though Thurston provides a natural one in [56,Chapter 6.1]. Then, $\mathrm{C}^{\mathrm{str}}_\bullet(M_\rho;\mathbb{R})$ denotes the image of str, and if $\bar{x} \in M_\rho$, we denote by $\mathrm{C}^{\mathrm{str}}_\bullet(M_\rho, \{\bar{x}\};\mathbb{R})$ the subcomplex spanned by the straight simplices whose vertices all map to $\bar{x}$. We will now construct a chain map $\mathrm{str}_{\bar{x}} : \mathrm{C}_\bullet(M_\rho;\mathbb{R}) \to \mathrm{C}^{\mathrm{str}}_\bullet(M_\rho, \{\bar{x}\};\mathbb{R})$.

Fix $x \in \pi^{-1}(\bar{x})$, and let $\mathcal{D} = \{y \in \mathbb{H}^3 : d(x,y) \leq d(\gamma(x),y) \text{ for all} \gamma \in \Gamma\}$ be the Dirichlet fundamental domain for Γ centered at x; delete a face of $\mathcal{D}$ in each face-pair $(F, \gamma F)$ to obtain a connected Borel set of representatives for the action of Γ on $\mathbb{H}^3$, which we still call $\mathcal{D}$. Let $\sigma : \Delta_n \to M_\rho$ and choose a lift $\widetilde{\sigma} : \Delta_n \to \mathbb{H}^3$. The vertices $v_0, \dots, v_n$ of $\widetilde{\sigma}$ are uniquely labeled by group elements $v_i = \gamma_i y_i$ where $\gamma_i \in \Gamma, y_i \in \mathcal{D}$. Define $\mathrm{str}_{\bar{x}}\sigma = \pi(\sigma_x(\gamma_0, \dots, \gamma_n))$, where $\sigma_x(\gamma_0, \dots, \gamma_n)$ is the straightening of any simplex whose ordered vertex set is $(\gamma_0 x, \dots, \gamma_n x)$. The definition is independent of the choice of lift, because any other lift of σ has vertex set equal to $(\gamma\gamma_0 y_0, \dots, \gamma\gamma_n y_n)$ for some $\gamma \in \Gamma$. All maps are chain maps and the operator norm satisfies $\|\mathrm{str}_{\bar{x}}\| \leq 1$. This is just because some simplices in a chain may collapse or cancel after applying $\mathrm{str}_{\bar{x}}$.

Let $\tau : \Delta_k \to M_\rho$ be a straight simplex. We can apply the prism operator to the straight line homotopy between lifts of τ and $\mathrm{str}_{\bar{x}}\tau$ to $\mathbb{H}^3$. We obtain a chain homotopy $H^\bullet_{\bar{x}} : \mathrm{C}^{\mathrm{str}}_\bullet(M_\rho;\mathbb{R}) \to \mathrm{C}^{\mathrm{str}}_{\bullet+1}(M_\rho;\mathbb{R})$ between $\mathrm{str}_{\bar{x}}$ and id. That is,

$$H^{\bullet-1}_{\bar{x}}\partial + \partial H^\bullet_{\bar{x}} = \mathrm{str}_{\bar{x}} - id.$$

Compare with [56,Chapter 6.1]. The homotopy space $\Delta_k \times I$ is triangulated by the prism operator using $k+1$ simplices of dimension $k+1$, so

$$\|H^k_{\bar{x}}\| = k+1. \tag{1}$$

If $Z \in \mathrm{C}^{\mathrm{str}}_n(M_\rho, \{\bar{x}\};\mathbb{R})$, then Z defines a chain in $\mathrm{C}_n(\Gamma;\mathbb{R})$ by linear extension of the rule

$$\pi(\sigma_x(\gamma_0, \dots, \gamma_n)) \mapsto [\gamma_0, \dots, \gamma_n].$$

To see that this map is well-defined, observe that $\pi(\sigma_x(\gamma_0, \dots, \gamma_n)) = \pi(\sigma_x(\gamma'_0, \dots, \gamma'_n))$ means that $\sigma_x(\gamma_0, \dots, \gamma_n)$ differs from $\sigma_x(\gamma'_0, \dots, \gamma'_n)$ by a deck transformation $\gamma \in \Gamma$; hence, $[\gamma'_0, \dots, \gamma'_n] = [\gamma\gamma_0, \dots, \gamma\gamma_n] = [\gamma_0, \dots, \gamma_n]$. We denote this map by $\iota_x : \mathrm{C}^{\mathrm{str}}_\bullet(M_\rho, \{\bar{x}\};\mathbb{R}) \to \mathrm{C}_\bullet(\Gamma;\mathbb{R})$.

One checks easily that ι_x is an isometric isomorphism of normed chain complexes with their ℓ_1-norms. Thus, if $Z \in \mathrm{C}^{\mathrm{str}}_n(M_\rho, \{\bar{x}\};\mathbb{R})$, then we have $\|Z\|_1 = \|\iota_x(Z)\|_1$ and $\|\partial\iota_x(Z)\|_1 = \|\iota_x(\partial Z)\|_1 = \|\partial Z\|_1$. Conversely, if we have a chain $Z \in \mathrm{C}^{\mathrm{str}}_n(\Gamma;\mathbb{R})$, one sees that $\pi_*(Z.x) = \iota_x^{-1}(Z) \in \mathrm{C}^{\mathrm{str}}_n(M_\rho, \{\bar{x}\};\mathbb{R})$.

2.4 The volume class

Let $x \in \mathbb{H}^3 \cup \partial\mathbb{H}^3$ and consider the function $\mathrm{vol}_3^x : (\mathrm{PSL}_2\mathbb{C})^4 \to \mathbb{R}$ which assigns to $(g_0, \ldots, g_3)$ the signed hyperbolic volume of the convex hull of the points $g_0x, \ldots, g_3x$. Any geodesic tetrahedron in $\mathbb{H}^3$ is contained in an ideal geodesic tetrahedron, and there is an upper bound v_3 on the volume that is attained by a *regular* ideal geodesic tetrahedron. That is, $\|\mathrm{vol}_3^x\|_\infty = v_3$. One checks using Stokes' Theorem that $\delta\mathrm{vol}_3^x = 0$, so that $[\mathrm{vol}_3^x] \in \mathrm{H}^3_{\mathrm{cb}}(\mathrm{PSL}_2\mathbb{C}; \mathbb{R})$. Moreover, for any $x, y \in \mathbb{H}^3 \cup \partial\mathbb{H}^3$, we have $[\mathrm{vol}_3^x] = [\mathrm{vol}_3^y]$. This is because the straight line homotopy between geodesic triangles can be triangulated by *three* (partially ideal) tetrahedra using the prism operator, and so

$$\mathrm{vol}_3^x - \mathrm{vol}_3^y = \delta H_{x,y}, \tag{2}$$

where $H_{x,y} \in \mathrm{C}^2_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{C}; \mathbb{R})$ measures the volume of the straight line homotopy between the geodesic triangles $(g_0, g_1, g_2).x$ and $(g_0, g_1, g_2).y$. In particular, $\|H_{x,y}\|_\infty \le 3v_3$, so that $[\mathrm{vol}_3^x] = [\mathrm{vol}_3^y]$, as claimed; the previous section gives a dual discussion. If basepoints are not relevant, we write $[\mathrm{vol}_3]$ to denote the class of $[\mathrm{vol}_3^x]$.

The continuous bounded cohomology of $\mathrm{PSL}_2\mathbb{C}$ is generated by $[\mathrm{vol}_3]$, i.e. $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{PSL}_2\mathbb{C}; \mathbb{R}) = \langle [\mathrm{vol}_3] \rangle_{\mathbb{R}}$ [12]. In fact, $\|[\mathrm{vol}_3]\|_\infty = v_3$; see e.g. [5] for a discussion of the hyperbolic volume class in dimensions $n \ge 3$. If Γ is a discrete group and $\rho : \Gamma \to \mathrm{PSL}_2\mathbb{C}$ is a group homomorphism, then $[\rho^*\mathrm{vol}_3] \in \mathrm{H}^3_{\mathrm{b}}(\Gamma; \mathbb{R})$ is called the *bounded fundamental class of* ρ or the *volume class of* ρ. Observe that for any $g \in \mathrm{PSL}_2\mathbb{C}$, we have an equality at the level of cochains

$$(g\rho g^{-1})^*\mathrm{vol}_3^{gx} = \rho^*\mathrm{vol}_3^x,$$

so that the volume class is an invariant of the $\mathrm{PSL}_2\mathbb{C}$-conjugacy class of ρ. It is also true that $[\rho^*\mathrm{vol}_3]$ is an invariant of the *closure* of the action of $\mathrm{PSL}_2\mathbb{C}$ by conjugation, if Γ is finitely generated. Indeed, Thurston [57,proof of Proposition 1.1] pointed out that for non-conjugate representations ρ and ρ' of a finitely generated, discrete group Γ into $\mathrm{Isom}^+(\mathbb{H}^3)$, $\rho \in \overline{\mathrm{PSL}_2\mathbb{C}.\rho'}$ implies that both $\mathrm{im}\rho$ and $\mathrm{im}\rho'$ are virtually abelian. By Theorem 2.1 or Lemma 2.2 below, the volume classes of such representations are zero. Hence if Γ is finitely generated, we obtain a well-defined, equivariant function on the character variety

$$\mathrm{Hom}(\Gamma, \mathrm{PSL}_2\mathbb{C}) /\!\!/ \mathrm{PSL}_2\mathbb{C} \to \mathrm{H}^3_{\mathrm{b}}(\Gamma; \mathbb{R})$$

with respect to the natural actions of $\mathrm{Out}(\Gamma)$ on each space.

The following theorem of Soma characterizes when a finitely generated Kleinian representation has non-trivial volume class. See Section 3 for definitions of geometrically finite hyperbolic manifolds and characterizations of ends of geometrically infinite hyperbolic manifolds.

Theorem 2.1 ([48,Theorem 1]). If $\Gamma \le \mathrm{PSL}_2\mathbb{C}$ is a finitely generated Kleinian group of infinite co-volume without elliptic elements and $\rho : \Gamma \to \mathrm{PSL}_2\mathbb{C}$ is any discrete and faithful representation, then the following are equivalent:

- $[\rho^*\mathrm{vol}_3] = 0 \in \mathrm{H}^3_{\mathrm{b}}(\Gamma; \mathbb{R})$
- $\|[\rho^*\mathrm{vol}_3]\|_\infty < v_3$
- M_ρ is geometrically finite or Γ is virtually abelian.

Hence, if M_ρ has a geometrically infinite relative end, then $\|[\rho^*\mathrm{vol}_3]\|_\infty = v_3$. See [49, Lemma 3.2 and Proposition 3.3] for the proof of this fact. We record here an observation.

Lemma 2.2. Let G be a discrete group. If $\rho : G \to \mathrm{PSL}_2\mathbb{C}$ is indiscrete but not dense, then ρ is geometrically elementary and $[\rho^*\mathrm{vol}_3] = 0 \in \mathrm{H}^3_{\mathrm{b}}(G; \mathbb{R})$.

Proof. Since ρ is indiscrete and not dense, $H = \overline{\rho(G)} \le \mathrm{PSL}_2\mathbb{C}$ is a proper, closed Lie subgroup. According to [55, Proposition, p. 246], H fixes a point, an ideal point, or stabilizes a geodesic plane, i.e. H is geometrically elementary.

We claim that $\rho^*\mathrm{vol}_3^y = 0$ for some $y \in \mathbb{H}^3 \cup \partial\mathbb{H}^3$. We just need to choose y to be contained in the invariant point or plane so that every tetrahedron has zero volume. Since $\rho^*\mathrm{vol}_3^y = 0 \in [\rho^*\mathrm{vol}_3]$, it follows that $[\rho^*\mathrm{vol}_3] = 0$. ■

2.5 The bounded Borel class

We will consider a generalization of the hyperbolic volume class $\beta_n \in \mathrm{H}^3_{\mathrm{cb}}(\mathrm{PSL}_n\mathbb{C}; \mathbb{R})$ called the *bounded Borel class*, which coincides with the *three*-dimensional hyperbolic volume class for $n = 2$. Using a stability theorem of Monod [43], Bucher, Buger, and Iozzi [6,Theorem 2] show that $\langle \beta_n \rangle_\mathbb{R} = \mathrm{H}^3_{\mathrm{cb}}(\mathrm{PSL}_n\mathbb{C}; \mathbb{R})$, for all $n \ge 3$, and they compute the ℓ_1-norm of β_n; see Theorem 2.5.

Let $\mathscr{F}(\mathbb{C}^n)$ be the space of complete flags of $\mathbb{C}^n$. That is, $F \in \mathscr{F}(\mathbb{C}^n)$ is a sequence of vector subspaces $\{0\} \le F^1 \le \ldots \le F^n = \mathbb{C}^n$ such that $\dim_\mathbb{C}(F^j) = j$. We fix $F \in \mathscr{F}(\mathbb{C}^n)$

and consider a Borel measurable $\mathrm{PSL}_n\mathbb{C}$-invariant function

$$B_n^F : (\mathrm{PSL}_n\mathbb{C})^4 \to \mathbb{R} \tag{3}$$
$$(g_0, ..., g_3) \mapsto B_n(g_0.F, ..., g_3.F)$$

that satisfies the cocycle condition everywhere, but is not everywhere continuous. Equation (5), below, provides a useful definition of $B_n : \mathscr{F}(\mathbb{C}^n)^4 \to \mathbb{R}$ on generic configurations of flags. The cocycle B_n was defined on generic configurations of quadruples of flags in [28,Section 2] and extended to non-generic configurations in [6,Equation (6)].

Continuity. Since B_n^F is not everywhere continuous, $B_n^F \notin \mathrm{C}_\mathrm{b}^3(\mathrm{PSL}_n\mathbb{C}; \mathbb{R})$, the continuous cochain group from Section 2.1. However, in the appropriate cochain complex that computes the continuous bounded cohomology of $\mathrm{PSL}_n\mathbb{C}$, B_n^F represents β_n; we prefer to omit the technical details of the construction of the *strong resolution of* $\mathbb{R}$ *by relatively injective* $\mathrm{PSL}_n\mathbb{C}$*-Banach modules* in which B_n^F is a cocycle (see [6,Sections 3-7]). Instead it is convenient to think of $[B_n^F] \in \mathrm{H}_\mathrm{b}^3(\mathrm{PSL}_n\mathbb{C}; \mathbb{R})$, where $\mathrm{PSL}_n\mathbb{C}$ is given the *discrete* topology.

More precisely, let $\mathrm{PSL}_n\mathbb{C}^\delta$ denote $\mathrm{PSL}_n\mathbb{C}$ with the discrete topology and let $id : \mathrm{PSL}_n\mathbb{C}^\delta \to \mathrm{PSL}_n\mathbb{C}$ be the identity. Then id is continuous, hence induces a map $id^* : \mathrm{H}_\mathrm{cb}^\bullet(\mathrm{PSL}_n\mathbb{C}; \mathbb{R}) \to \mathrm{H}_\mathrm{cb}^\bullet(\mathrm{PSL}_n\mathbb{C}^\delta; \mathbb{R})$. We can work with the class

$$id^*\beta_n = [B_n^F] \in \mathrm{H}_\mathrm{cb}^3(\mathrm{PSL}_n\mathbb{C}^\delta; \mathbb{R}) = \mathrm{H}_\mathrm{b}^3(\mathrm{PSL}_n\mathbb{C}; \mathbb{R}).$$

Our goal in this section is to describe the main results from [6] and to extract a continuity property of the cocycle B_n^F on non-degenerate generic configurations of complete flags.

The case $n = 2$ and the Bloch-Wigner function. Note that $\mathscr{F}(\mathbb{C}^2) = \mathbb{CP}^1 = \partial\mathbb{H}^3$. The *Bloch–Wigner function* $D : \mathbb{CP}^1 \to \mathbb{R}$ is a variant of the di-logarithm function that computes the volume of the ideal tetrahedron with ordered vertex set $(\infty, 0, 1, z) \in (\mathbb{CP}^1)^4$. If $z_0, z_1, z_2 \in \mathbb{CP}^1$ are distinct and $z_3 \in \mathbb{CP}^1$ is arbitrary, then there exists a unique $g \in \mathrm{PSL}_2\mathbb{C}$ such that $g.(z_0, ..., z_3) = (\infty, 0, 1, g.z_3)$. Then, $[z_0 : ... : z_3] := g.z_3$ is the *cross ratio* of the *four*-tuple, and

$$\mathrm{vol}(z_0, ..., z_3) := D([z_0 : ... : z_3])$$

is the oriented volume of the ideal geodesic tetrahedron spanned by $(z_0, ..., z_3)$. When at least two of $z_0, ..., z_3$ coincide, $\mathrm{vol}(z_0, ..., z_3) = 0$.

It is well known that D is real analytic on $\mathbb{C}\setminus\{0,1,\infty\}$, attains its extreme values at $\zeta_3 = e^{i\pi/3}$ and $\bar{\zeta}_3$, and extends continuously to $\mathbb{CP}^1$. However, $\mathrm{vol} : (\mathbb{CP}^1)^4 \to \mathbb{R}$ is not continuous everywhere. Indeed, consider the sequence $\underline{z}_k = (\infty, 0, 2^{-k}, 2^{-k}\zeta_3)$, for $k = 0, 1, \ldots$; the Möbius transformation $z \mapsto 2^k z$ takes $\underline{z}_k$ to $\underline{z}_0$, hence $\mathrm{vol}(\underline{z}_k)$ is constant (and positive), but the limiting configuration $(\infty, 0, 0, 0)$ has 0 volume. For fixed $z \in \mathbb{CP}^1$, it is not difficult to see that the map

$$\begin{aligned} B_2^z : (\mathrm{PSL}_2\mathbb{C})^4 &\to \mathbb{R} \\ (g_0, \ldots, g_3) &\mapsto \mathrm{vol}(g_0.z, \ldots, g_3.z) \end{aligned}$$

is a $\mathrm{PSL}_2\mathbb{C}$-invariant cocycle. The bounded Borel class is just $\beta_2 := [\mathrm{vol}_3^x] \in \mathrm{H}^3_{\mathrm{cb}}(\mathrm{PSL}_2\mathbb{C};\mathbb{R})$, where $x \in \mathbb{H}^3$ is *not* an ideal point, and it is clear that $id^*[\mathrm{vol}_3^x] = [B_2^z] \in \mathrm{H}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{C};\mathbb{R})$ for any $x \in \mathbb{H}^3 \cup \partial\mathbb{H}^3$; see Section 2.4. In Section 6, we show that $\mathrm{H}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{C};\mathbb{R})$ is quite large.

Generic configurations of flags. For convenience, we fix a Hermitian inner product $\mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}$, thus identifying a maximal subgroup $K \leq \mathrm{PSL}_n\mathbb{C}$ preserving this inner product. Then, K is a maximal compact subgroup isomorphic to $\mathrm{PSU}(n)$. Note that $\mathrm{PSL}_n\mathbb{C}$ acts transitively on $\mathscr{F}(\mathbb{C}^n)$, so that if we choose also an orthonormal ordered basis $(e_1, e_2, \ldots, e_n)$, we may identify the stabilizer of the standard flag $\langle e_1\rangle \leq \langle e_1, e_2\rangle \leq \ldots \leq \langle e_1, \ldots, e_n\rangle$ with the upper triangular group $P \leq \mathrm{PSL}_n\mathbb{C}$, and $\mathscr{F}(\mathbb{C}^n) \cong \mathrm{PSL}_n\mathbb{C}/P$.

Given a flag $F \in \mathscr{F}(\mathbb{C}^n)$, using the Gram–Schmidt process and our chosen inner product, we may find an orthonormal basis $(f^1, \ldots, f^n)$ such that

$$F^j = \langle f^j\rangle \perp \langle f^1, \ldots, f^{j-1}\rangle,$$

and each f^j is uniquely determined up to multiplication by a complex number of norm 1. Call $(f^1, \ldots, f^n)$ an *affine representative* of F.

Say that $(F_0, \ldots, F_3) \in \mathscr{F}(\mathbb{C}^n)^4$ is a *generic* configuration of flags if whenever $0 \leq j_0, \ldots, j_3 \leq n-1$ satisfy $j_0 + \ldots + j_3 = k$, then $\dim\langle F_0^{j_0}, \ldots, F_3^{j_3}\rangle = \min\{n, k\}$; genericity is an open condition among 4-tuples of flags. Let

$$\mathbb{M} = \{(j_0, \ldots, j_3) \in \{0, 1, \ldots, n-2\}^4 : j_0 + \ldots + j_3 = n-2\},$$

so that if $(F_0, \ldots, F_3)$ is a generic configuration of flags and $(j_0, \ldots, j_3) \in \mathbb{M}$, then

$$\dim_{\mathbb{C}} \frac{\langle F_0^{j_0+1}, \ldots, F_3^{j_3+1} \rangle}{\langle F_0^{j_0}, \ldots, F_3^{j_3} \rangle} = 2. \tag{4}$$

Moreover, $\#\mathbb{M} = \frac{1}{6} n(n-1)(n+1)$, and if $(j_0, \ldots, j_3) \in \{0, \ldots, n-1\}^4 \setminus \mathbb{M}$, then equality (4) does not hold [6, p. 3147].

Definition of the cocycle on generic configurations. Let $\mathscr{F}(\mathbb{C}^n)^{(4)} \subset \mathscr{F}(\mathbb{C}^n)^4$ be the subspace of generic configurations of flags. Let $\underline{F} = (F_0, \ldots, F_3) \in \mathscr{F}(\mathbb{C}^n)^{(4)}$ and $\mathbb{J} = (j_0, \ldots, j_3) \in \mathbb{M}$. Find also an affine representative $(f_i^1, \ldots, f_i^n)$ of F_i, for $i = 0, \ldots, 3$. The functions

$$\underline{F} \mapsto V_{\mathbb{J}}(\underline{F}) := \langle f_0^1, \ldots, f_0^{j_0}, \ldots, f_3^1, \ldots, f_3^{j_3} \rangle \in \mathrm{Gr}_{n-2}(\mathbb{C}^n)$$

and

$$\underline{F} \mapsto V_{\mathbb{J}}(\underline{F})^{\perp} \in \mathrm{Gr}_2(\mathbb{C}^n)$$

vary continuously in generic configurations $\underline{F}$. The orthogonal projection

$$\mathbb{C}^n = V_{\mathbb{J}}(\underline{F}) \oplus V_{\mathbb{J}}(\underline{F})^{\perp} \to V_{\mathbb{J}}(\underline{F})^{\perp}.$$

coincides with the quotient

$$\mathbb{C}^n \to \frac{\langle F_0^{j_0+1}, \ldots, F_3^{j_3+1} \rangle}{\langle F_0^{j_0}, \ldots, F_3^{j_3} \rangle} = \frac{\mathbb{C}^n}{V_{\mathbb{J}}(\underline{F})} \cong V_{\mathbb{J}}(\underline{F})^{\perp}.$$

Using genericity again, the orthogonal projections of $f_0^{j_0+1}, \ldots, f_3^{j_3+1}$ to $V_{\mathbb{J}}(\underline{F})^{\perp}$ are non-zero and pairwise linearly independent. Let

$$t_{\mathbb{J}}(\underline{F}) \in \mathbb{P}(V_{\mathbb{J}}(\underline{F})^{\perp})^4 \cong (\mathbb{CP}^1)^4$$

be the *four*-tuple consisting of the projectivizations of the orthogonal projections of $f_i^{j_i+1}$ to $V_{\mathbb{J}}(\underline{F})^{\perp}$. Observe that $t_{\mathbb{J}}$ is independent of our choice of affine representatives of $F_0, \ldots, F_3$, because the projectivization map only depends on the complex line spanned by f_i^j.

Following [28] and [6], we define the $\mathrm{PSL}_n\mathbb{C}$-invariant function

$$
\begin{aligned}
B_n : \mathscr{F}(\mathbb{C}^n)^{(4)} &\to \mathbb{R} \\
\underline{F} &\mapsto \sum_{\mathbb{J}\in\mathbb{M}} \mathrm{vol}(t_{\mathbb{J}}(\underline{F})).
\end{aligned} \tag{5}
$$

From the definition, it is clear that

$$\sup_{\underline{F}\in\mathscr{F}(\mathbb{C}^n)^{(4)}} |B_n(\underline{F})| \leq \#\mathbb{M}\cdot v_3 = \frac{(n^3-n)}{6}\cdot v_3.$$

By avoiding non-generic configurations of flags, for each $\mathbb{J}\in\mathbb{M}$, the function $\underline{F}\mapsto t_{\mathbb{J}}(\underline{F})$ varies continuously and has image contained in the locus of *distinct four*-tuples in $(\mathbb{CP}^1)^4$. Since vol varies continuously on distinct *four*-tuples, all summands in (5) are continuous, and so B_n is continuous with respect to the subspace topology on $\mathscr{F}(\mathbb{C}^n)^{(4)}$. We refer the reader to [6, Section 3], which explains how to extend B_n to a bounded Borel measurable cocycle $\mathscr{F}(\mathbb{C}^n)^4 \to \mathbb{R}$. After defining this extension, for a fixed flag $F\in\mathscr{F}(\mathbb{C}^n)$, equation (3) gives a $\mathrm{PSL}_n\mathbb{C}$-invariant alternating cocycle B_n^F [6, Corollary 13].

The Veronese embedding. There is a unique irreducible representation $\iota_n : \mathrm{PSL}_2\mathbb{C}\to\mathrm{PSL}_n\mathbb{C}$, up to conjugation, and it induces an equivariant map $\hat{\iota}_n : \partial\mathbb{H}^3 \to \mathscr{F}(\mathbb{C}^n)$ called the *Veronese embedding*.

Let $x\in\partial\mathbb{H}^3$; the cocycle $B_n^{\hat{\iota}_n(x)}$ pulls back to a multiple of the volume cocycle vol_3^x, which allows us to push constructions in *three*-dimensional hyperbolic geometry to the higher rank setting.

Proposition 2.3 ([6, Proposition 21]). For all $g_0,\dots,g_3\in\mathrm{PSL}_2\mathbb{C}$ and $\mathbb{J}\in\mathbb{M}$,

$$\mathrm{vol}(t_{\mathbb{J}}(\hat{\iota}_n(g_0.x),\dots,\hat{\iota}_n(g_3.x))) = \mathrm{vol}(g_0.x,\dots,g_3.x),$$

so that

$$B_n^{\hat{\iota}_n(x)}(\iota_n(g_0),\dots,\iota_n(g_3)) = \frac{n(n^2-1)}{6}\mathrm{vol}_3^x(g_0,\dots,g_3).$$

In other words, $\iota_n^* B_n^{\hat{\iota}_n(x)} = \frac{n(n^2-1)}{6}\mathrm{vol}_3^x$.

The function B_n^F is not continuous near non-generic configurations of flags. However, B_n^F satisfies the following continuity property, which is all we need in the sequel.

Lemma 2.4. Let $x \in \partial\mathbb{H}^3$ and suppose $g_0, \ldots, g_3 \in \mathrm{PSL}_2\mathbb{C}$ are such that $g_0.x, \ldots, g_3.x$ are pairwise distinct. Then, there exist open neighborhoods $U_i \subset \mathrm{PSL}_n\mathbb{C}$ of $\iota_n(g_i)$ such that the map

$$U_0 \times \ldots \times U_3 \subset (\mathrm{PSL}_n\mathbb{C})^4 \to \mathbb{R}$$
$$(h_0, \ldots, h_3) \mapsto B_n^{\hat{\iota}_n(x)}(h_0, \ldots, h_3)$$

is continuous.

Proof. Since *four*-tuples of distinct flags in the image of $\hat{\iota}_n$ are in generic position, we have $(\hat{\iota}_n(g_0.x), \ldots, \hat{\iota}_n(g_3.x)) \in \mathscr{F}(\mathbb{C}^n)^{(4)}$. Let $V \subset \mathscr{F}(\mathbb{C}^n)^{(4)}$ be an open neighborhood of $(\hat{\iota}_n(g_0.x), \ldots, \hat{\iota}_n(g_3.x))$, so that V is open in $\mathscr{F}(\mathbb{C}^n)^4$, as well. Consider the stabilizer P_x of $\hat{\iota}_n(x)$, the quotient projection $\pi : \mathrm{PSL}_n\mathbb{C} \to \mathrm{PSL}_n\mathbb{C}/P_x \cong \mathscr{F}(\mathbb{C}^n)$, and the product $\pi^4 : (\mathrm{PSL}_n\mathbb{C})^4 \to \mathscr{F}(\mathbb{C}^n)^4$. Then, $(\pi^4)^{-1}(V) \subset (\mathrm{PSL}_n\mathbb{C})^4$ is an open neighborhood of $(\iota_n(g_0), \ldots, \iota_n(g_3))$. Products of open sets form a basis for the topology of the product of spaces, and B_n is continuous on $\mathscr{F}(\mathbb{C}^n)^{(4)}$. This completes the proof of the lemma. ■

We will need one more important result about the bounded Borel class and its semi-norm.

Theorem 2.5 ([6, Theorem 2]). For each $n \geq 2$, the Borel class β_n generates $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{PSL}_n\mathbb{C}; \mathbb{R})$, and its ℓ_∞-norm is

$$\|\beta_n\|_\infty = v_3 \frac{n(n^2-1)}{6}.$$

For the irreducible representation $\iota_n : \mathrm{PSL}_2\mathbb{C} \to \mathrm{PSL}_n\mathbb{C}$, the pullback satisfies

$$\iota_n^*\beta_n = \frac{n(n^2-1)}{6}[\mathrm{vol}_3] \in \mathrm{H}^3_{\mathrm{cb}}(\mathrm{PSL}_2\mathbb{C}; \mathbb{R}).$$

For a discrete group G and representation $\rho : G \to \mathrm{PSL}_n\mathbb{C}$, the *bounded Borel class of* ρ or the *Borel class of* ρ is $\rho^*\beta_n \in \mathrm{H}^3_{\mathrm{cb}}(G; \mathbb{R})$. Note that by Theorem 2.5, if

$\rho : F_2 \to \mathrm{PSL}_2\mathbb{C}$ is discrete, faithful, and geometrically infinite, then $\|(\iota_n \circ \rho)^*\beta_n\|_\infty = \frac{n(n^2-1)}{6}\|[\rho^*\mathrm{vol}_3]\|_\infty = v_3\frac{n(n^2-1)}{6}$, where the last equality was by Theorem 2.1.

2.6 An approximation scheme

We now give a criterion for the pullback of a continuous bounded class to be non-zero and have positive semi-norm in bounded cohomology. We claim no originality for the following lemma; it is a distillation and abstraction of a standard argument. See [59, Section 3] and [49, Proposition 3.3].

Lemma 2.6. Let G be a discrete group, $\mathcal{G}$ a group, $\rho : G \to \mathcal{G}$ a homomorphism, and $[B] \in \mathrm{H}^n_\mathrm{b}(\mathcal{G};\mathbb{R})$. Suppose there exist $\epsilon > 0$ and chains $Z_k \in \mathrm{C}_n(G;\mathbb{R})$ for $k = 1, 2, \ldots$ such that

1. $\frac{|\rho^*B(Z_k)|}{\|Z_k\|_1} > \epsilon$ for all k,
2. $\liminf_{k\to\infty} \frac{\|\partial Z_k\|_1}{\|Z_k\|_1} = 0$.

Then, $[\rho^*B] \neq 0 \in \mathrm{H}^n_\mathrm{b}(G;\mathbb{R})$ and $\|[\rho^*B]\|_\infty \geq \epsilon$.

Proof. Given $b \in \mathrm{C}^{n-1}_\mathrm{b}(G;\mathbb{R})$, we need to show that $\|\rho^*B + \delta b\|_\infty > \epsilon$. We have the trivial inequality

$$|(\rho^*B + \delta b)(Z_k)| \leq \|\rho^*B + \delta b\|_\infty \|Z_k\|_1.$$

By the triangle inequality, we have

$$|(\rho^*B + \delta b)(Z_k)| \geq |\rho^*B(Z_k)| - |\delta b(Z_k)|.$$

Another application of the trivial inequality yields

$$|\delta b(Z_k)| = |b(\partial Z_k)| \leq \|b\|_\infty \|\partial Z_k\|_1.$$

Stringing together the inequalities and dividing through by $\|Z_k\|_1$, we obtain

$$\frac{|(\rho^*B + \delta b)(Z_k)|}{\|Z_k\|_1} \geq \frac{|\rho^*B(Z_k)|}{\|Z_k\|_1} - \|b\|_\infty \frac{\|\partial Z_k\|_1}{\|Z_k\|_1}.$$

By passing to a subsequence, we assume that $\lim_{k\to\infty} \frac{\|\partial Z_k\|_1}{\|Z_k\|_1} = 0$, and we see that for any $e > 0$, there is a K such that for $k \geq K$, we have $\frac{|(\rho^*B+\delta b)(Z_k)|}{\|Z_k\|_1} > \epsilon - e$; thus, $\|\rho^*B + \delta b\|_\infty > \epsilon - e$. Since b and e were arbitrary, this implies that $\|[\rho^*B]\| \geq \epsilon$. ∎

The following lemma is an easy consequence of the continuity of multiplication and inversion in a topological group $\mathcal{G}$.

Lemma 2.7. Let $\rho_0 : F_d \to \mathcal{G}$ be a homomorphism, $W \subset F_d$ a finite set, and $\{z^1, \dots, z^d\}$ a free basis for F_d. Given neighborhoods V_w of $\rho_0(w)$ for $w \in W$, there are neighborhoods U_i of $\rho_0(z^i)$ such that if $\rho : F_d \to \mathcal{G}$ is any homomorphism satisfying $\rho(z^i) \in U_i$ for each $i = 1, \dots, d$, then $\rho(w) \in V_w$, for all $w \in W$.

Note that in the above lemma, we do not require ρ to be faithful. For example, given a homomorphism $\rho' : G \to \mathcal{G}$, and a set $\{g_1, \dots, g_d\} \subset G$, the rule $z^i \mapsto \rho'(g_i)$ defines a homomorphism $\rho : F_d \to \mathcal{G}$. The following proposition is the key insight in this paper.

Proposition 2.8. Let G be a discrete group, $\mathcal{G}$ be a topological group, and $\rho : G \to \mathcal{G}$ a homomorphism with dense image. Consider a homomorphism $\rho_0 : F_d \to \mathcal{G}$ and a continuous cocycle $B \in \mathrm{C}_\mathrm{b}^n(\mathcal{G}; \mathbb{R})$. For any $Z \in \mathrm{C}_n(F_d; \mathbb{R})$ and for any $\epsilon > 0$, there exists $Z(\epsilon) \in \mathrm{C}_n(G; \mathbb{R})$ such that

$$|\rho_0^* B(Z) - \rho^* B(Z(\epsilon))| < \epsilon.$$

Moreover, $\|Z(\epsilon)\|_1 \le \|Z\|_1$ and $\|\partial Z(\epsilon)\|_1 \le \|\partial Z\|_1$.

Proof. Write $Z = \sum_{j=1}^M a_j [v_0^j, \dots, v_n^j]$, and choose the representative $(id, w_1^j, \dots, w_n^j) \in [v_0^j, \dots, v_n^j]$ so that $w_i^j = (v_0^j)^{-1} v_i^j$ for each $i = 1, \dots, n$. Take $W = \{w_i^j : i = 1, \dots, n \text{ and} j = 1, \dots, M\} \subset F_d$, and let $\{z^1, \dots, z^d\}$ be a free basis for F_d.

The cocycle $B : \mathcal{G}^{n+1} \to \mathbb{R}$ is continuous, so for each $i = 1, \dots, n$ and $j = 1, \dots, M$, there are neighborhoods $V_i^j \subset \mathcal{G}$ of $\rho_0(w_i^j)$ such that if $\gamma_i^j \in V_i^j$, then

$$|B(id, \rho_0(w_1^j), \dots, \rho_0(w_n^j)) - B(id, \gamma_1^j, \dots, \gamma_n^j)| < \frac{\epsilon}{\|Z\|_1}, \ \forall j = 1, \dots, M. \tag{6}$$

Given the data ρ_0, $W \subset F_d$, $\{z^1, \dots, z^d\}$, and V_i^j, find neighborhoods $U_i \subset \mathcal{G}$ of $\rho_0(z^i)$ guaranteed to us by Lemma 2.7. Now, ρ has dense image, so we can find $z_\epsilon^i \in G$ such that $\rho(z_\epsilon^i) \in U_i$ for each $i = 1, \dots, d$. Define $\iota_\epsilon : F_d \to G$ by $z^i \mapsto z_\epsilon^i$. Then, $\rho(\iota_\epsilon(w_i^j)) \in V_i^j$ for all $w_i^j \in W$.

We write $Z(\epsilon) = \iota_{\epsilon *}(Z) \in \mathrm{C}_n(G; \mathbb{R})$ for the chain corresponding to Z under this identification. The map $i_{\epsilon *}$ is an ℓ_1-norm non-increasing chain map, so $\|Z(\epsilon)\|_1 \le \|Z\|_1$ and $\|\partial Z(\epsilon)\|_1 \le \|\partial Z\|_1$.

Repeated applications of the triangle inequality and (6) give

$$|\rho_0^* B(Z) - \rho^* B(Z(\epsilon))| < \frac{\epsilon}{\|Z\|_1} \|Z\|_1 = \epsilon,$$

which is what we wanted to show. ■

Remark 2.9. Note that if $Z_k \in \mathrm{C}_n(F_d; \mathbb{R})$ satisfy $\|Z_k\|_1 \to \infty$ and $Z_k(\epsilon) \in \mathrm{C}_n(G; \mathbb{R})$ are obtained as in Proposition 2.8, then $\|Z_k(\epsilon)\|_1 \to \infty$ as well.

We will apply Lemma 2.6 and Proposition 2.8 in two settings. In Section 4, we set $\mathcal{G} = \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2\mathbb{C}$ and $[B] = [\mathrm{vol}_3^x] \in \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Isom}^+(\mathbb{H}^3); \mathbb{R})$, where $x \in \mathbb{H}^3$. In Section 5, we set $\mathcal{G} = \mathrm{PSL}_n\mathbb{C}$, for $n \geq 2$ and consider the cocycle $B = B_n^F : (\mathrm{PSL}_n\mathbb{C})^4 \to \mathbb{R}$ from Section 2.5 representing the Borel class $\beta_n \in \mathrm{H}^3_{\mathrm{cb}}(\mathrm{PSL}_n\mathbb{C}; \mathbb{R})$. Later, we consider hyperbolic volume classes in dimensions *four* and higher and establish a criterion that would guarantee that the volume class of a dense representation does not vanish. Unfortunately, we do not know if the criterion is ever satisfied.

We have already encountered a technical issue: the cocycle B_n^F is *not* everywhere continuous. However, Lemma 2.4 provides us with *enough* continuity. It is straightforward to modify the proof of Proposition 2.8, using Lemma 2.4, to obtain the following corollary. Recall that the Veronese embedding $\hat{\iota}_n : \partial\mathbb{H}^3 \to \mathscr{F}(\mathbb{C}^n)$ is a topological embedding that is equivariant with respect to the irreducible representation $\iota_n : \mathrm{PSL}_2\mathbb{C} \to \mathrm{PSL}_n\mathbb{C}$.

Corollary 2.10. Let $\rho_0 : F_d \to \mathrm{PSL}_2\mathbb{C}$ be a homomorphism, $\rho : G \to \mathrm{PSL}_n\mathbb{C}$ be dense, and $Z = \sum_{j=1}^M a_i[w_0^j, \ldots, w_3^j] \in \mathrm{C}_3(F_d; \mathbb{R})$.

If there is a point $x \in \mathbb{H}^3$ such that, for each $j = 1, \ldots, M$, the four points $\rho_0(w_0^j).x, \ldots., \rho_0(w_3^j).x \subset \partial\mathbb{H}^3$ are pairwise distinct, then for any $\epsilon > 0$, there exists $Z(\epsilon) \in \mathrm{C}_3(G; \mathbb{R})$ such that

$$|(\iota_n \circ \rho_0)^* B_n^{\hat{\iota}_n(x)}(Z) - \rho^* B_n^{\hat{\iota}_n(x)}(Z(\epsilon))| < \epsilon.$$

Moreover, $\|Z(\epsilon)\|_1 \leq \|Z\|_1$ and $\|\partial Z(\epsilon)\|_1 \leq \|\partial Z\|_1$.

The reader who is interested only in the question of (non-)vanishing of higher dimensional volume classes can skip directly to Section 7, and in particular Proposition 7.2, where the ideas from this section are applied.

3 The structure of tame hyperbolic *three*-manifolds

In this section, we review the classification theory of finitely generated Kleinian groups. We use this classification to provide a detailed argument to deduce Theorem 1.2 from Theorem 1.1 and previous work of the author, building on work of Soma. The presence of parabolic elements in Kleinian groups significantly complicates the discussion of ends and end invariants. The only part of the paper in which we need to understand influence of parabolic cusps is in Lemma 4.7, and is essentially independent of the main line of argument.

We begin by defining some of the basic objects associated to a marked Kleinian group, turn to an observation about groups of isometries of non-positively curved symmetric spaces generated by 'small' elements, and discuss the structure of the ends of complete hyperbolic *three*-manifolds with infinite volume and finitely generated fundamental group. We end with the proof of Theorem 1.2, once we have established these preliminary notions.

A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}_2\mathbb{C}$. Let Γ be an abstract discrete group, and suppose $\rho : \Gamma \to \mathrm{PSL}_2\mathbb{C}$ is an injective group homomorphism with discrete image, i.e. ρ is discrete and faithful. We call ρ a *Kleinian representation* or a *marked Kleinian group*. We will be most interested in the case that Γ is a non-abelian free group or the fundamental group of a closed oriented surface of genus at least two; we will always assume that Γ is torsion free, however, if Γ is finitely generated and admits a Kleinian representation, then Γ is virtually torsion free, by Selberg's Lemma.

We spend some time providing statements and context for the structure and classification theorems that we use in the sequel. Namely, the Tameness Theorem (Theorem 3.3), the Covering Theorem (Theorem 3.8), and the Ending Lamination Theorem (Theorem 3.10) for finitely generated marked Kleinian groups. In order to state these theorems, we discuss families of hyperbolic surfaces, called *simplicial hyperbolic surfaces*, that exit the geometrically infinite ends of hyperbolic *three*-manifolds. We use properties of simplicial hyperbolic surfaces again in Section 4.1.

Given $\rho : \Gamma \to \mathrm{PSL}_2\mathbb{C}$ as above, the space $M_\rho = \mathbb{H}^3/\mathrm{im}\rho$ of orbits is a complete hyperbolic *three*-manifold. Since $\mathbb{H}^3$ is contractible, M_ρ is a classifying space for Γ, and ρ induces a homotopy class of maps $K(\Gamma, 1) \to M_\rho$ that labels the homotopy classes of loops in M_ρ. If Γ is a closed surface group or a free group, then we can take $K(\Gamma, 1)$ to be a closed surface or a *three*-dimensional handlebody, respectively.

3.1 The Margulis Lemma

Let $\mathcal{G}$ be a semi-simple Lie group of non-compact type, let K be a maximal compact subgroup, and $X = \mathcal{G}/K$ the associated Riemannian symmetric space. In this paper, we will be most interested in $\mathcal{G} = \mathrm{PSL}_n\mathbb{C}$ and $\mathcal{G} = \mathrm{Isom}^+(\mathbb{H}^d)$ for $n, d \geq 2$. Then, $X = \mathbb{H}^3$ when $n = 2$ or $d = 3$. Following Thurston [56,Lemma 5.10.1], the following is known to hyperbolic geometers as "The Margulis Lemma," although it is perhaps more accurate to refer to it as the "Kazhdan–Margulis Theorem" [31,Theorem 4.53].

Lemma 3.1 (The Margulis Lemma). Given $\mathcal{G}$ and X as above, there is a number $\mu = \mu(X) > 0$ such that the following holds. If $\Gamma = \langle \gamma_1, \ldots, \gamma_k \rangle \leq \mathcal{G}$ is a discrete group, and there is a point $x \in X$ such that $d(x, \gamma_i.x) \leq \mu$ for each $i = 1, \ldots, k$, then Γ is virtually nilpotent.

For the rest of this section, we will only be interested in the case that $G = \mathrm{PSL}_2\mathbb{C}$ and $X = \mathbb{H}^3$. In this setting, the largest number $\mu_3 = \mu(\mathbb{H}^3)$ for which Lemma 3.1 holds is called the *three-dimensional Margulis constant*. Any discrete virtually nilpotent subgroup $\Gamma \leq \mathrm{PSL}_2\mathbb{C}$ is virtually isomorphic to $1, \mathbb{Z}$ or $\mathbb{Z}^2$.

Thurston [56, Corollary 5.10.2] pointed out that a hyperbolic manifold admits a *thick–thin* decomposition as a consequence of Lemma 3.1. Given a complete hyperbolic n-manifold M and $\epsilon > 0$, let

$$M^{\geq\epsilon} = \overline{\{x \in M : \text{the ball of radius } \epsilon \text{ centered at } x \text{ is embedded in } M\}},$$

be the *thick part* of M and $M^{<\epsilon} = M \setminus M^{\geq\epsilon}$ the *thin part*. In general, the *injectivity radius* $\mathrm{inj}_x(M)$ is the supremum of the radii of balls centered at x that embed into M. A hyperbolic manifold M is said to have *bounded geometry* if there is a positive lower bound to $\mathrm{inj}_x(M)$, which is independent of x.

Corollary 3.2 (Thick–Thin Decomposition). Every component of $M^{<\mu_3}$ is either

- the quotient of a horoball by a group of parabolic isometries that is free abelian of rank 1 or 2, or
- the r neighborhood of a closed geodesic in M, where $r \geq 0$.

The pre-compact components of $M^{<\mu_3}$ are called *Margulis tubes*. The horoball quotients are called *rank*-1 or *rank*-2 *parabolic cusps*, accordingly.

3.2 Relative ends and tameness

Fix a Kleinian representation $\rho : \Gamma \to \mathrm{PSL}_2\mathbb{C}$, where Γ is finitely generated and torsion free. Let Q_ρ denote the union of parabolic cusps in $M_\rho^{<\mu_3}$, and let $M_\rho^\circ = M_\rho \setminus Q_\rho$. The set $P_\rho = \partial M_\rho^\circ$ is called the *parabolic locus* and consists of a finite number of tori and open annuli corresponding to the frontiers of the rank-2 and rank-1 parabolic cusps, respectively. McCullough has shown [37] that there is a *relative compact core* $K_\rho \subset M_\rho^\circ$, a co-dimension 0 compact submanifold such that the inclusion $K_\rho \hookrightarrow M_\rho$ is a homotopy equivalence, ∂K_ρ contains all toroidal components of P_ρ, and ∂K_ρ meets each annular component in a compact annulus. We let $P_\rho^\circ = K_\rho \cap P_\rho$, so that $P_\rho^\circ \hookrightarrow \partial M_\rho^\circ$ is a homotopy equivalence.

Given a relative compact core $K_\rho \subset M_\rho$, the ends $\mathcal{E}_\rho$ of M_ρ° are in one-to-one correspondence with the components of $\partial K_\rho \setminus P_\rho^\circ$, which are oriented surfaces of finite type and negative Euler characteristic. The elements of $\mathcal{E}_\rho$ are called *relative ends* of M_ρ; for each component $R \subset \partial K_\rho \setminus P_\rho^\circ$, the component E_R of $M_\rho^\circ \setminus K_\rho$ whose closure contains R is a neighborhood of the relative end $[E_R] \in \mathcal{E}_\rho$.

The following theorem is known as the Tameness Theorem, and has been proved independently by Agol [1, Section 6 and Theorem 10.2] and Calegari–Gabai [18, Theorem 7.3] building on important partial results of Marden, Thurston, Bonahon, Canary, Souto, and many others. The statement of the tameness theorem that we include here can be found in [47, Section 4.2].

Theorem 3.3 (The Tameness Theorem). There is a relative compact core $K_\rho \subset M_\rho^\circ$ such that for each component R of $\partial K_\rho \setminus P_\rho^\circ$, the closure of E_R is homeomorphic to $R \times [0, \infty)$. Moreover, M_ρ is homeomorphic to $\mathrm{int} K_\rho$, and M_ρ° is homeomorphic to the complement in K_ρ of $\partial K_\rho \setminus P_\rho^\circ$.

If $K_\rho \subset M_\rho$ is a relative compact core such that the complement of K_ρ in M_ρ° consists of product neighborhoods of ends, as in Theorem 3.3, say that K_ρ is a *standard* relative compact core. A *compression body* B is an oriented, compact, irreducible *three*-manifold which has a distinguished boundary component $\partial_{ext} B$ whose fundamental group surjects onto $\pi_1(B)$; $\partial_{ext} B$ is called the *exterior boundary* of B. If the exterior boundary of B is incompressible, then B is homeomorphic to the trivial interval bundle over $\partial_{ext} B$. If B is not a trivial interval bundle, the exterior boundary is the unique compressible component of ∂B. The compression body B is a handlebody if and only if $\partial B = \partial_{ext} B$, which is true if and only if $\pi_1(B)$ is a free group; see [52, Section 2.5].

Theorem 3.4 ([13,Theorem 2.1]). Let S be a component of ∂K_ρ. There is a compression body $B_S \subset K_\rho$ whose exterior boundary is S such that inclusion $B_S \hookrightarrow K_\rho$ induces an injection $\pi_1(B_S) \hookrightarrow \pi_1(K_\rho) = \pi_1(M_\rho)$, and B_S is unique up to isotopy.

The submanifold B_S is called the *characteristic compression body neighborhood* of S for K_ρ; see also [40,Theorem 1.1.1], where characteristic compression body neighborhoods are called *incompressible neighborhoods*. If a component of ∂K_ρ is incompressible, then the characteristic compression body neighborhood corresponding to that component is just a collar neighborhood. Furthermore, the covering space corresponding to that surface group is a trivial $\mathbb{R}$ bundle over that surface by Bonahon's Tameness Theorem [14,Théorème A]. The following corollary is our main application of the Tameness Theorem 3.3.

Corollary 3.5. If Γ is a free group of rank $k < \infty$, $\rho : \Gamma \to \mathrm{PSL}_2\mathbb{C}$ is discrete and faithful and K_ρ is a standard compact core for M_ρ, then there is a homeomorphism $f : \mathcal{H}_k \to K_\rho$, where $\mathcal{H}_k$ is a closed handlebody of genus k and $f_* = \rho$ on fundamental groups. There are homotopically essential and distinct simple closed curves $\nu_1, \ldots, \nu_m \subset \partial\mathcal{H}_k$ with disjoint representatives such that $f(\{\nu_1, \ldots, \nu_m\})$ are the core curves of the annuli $P_\rho^\circ \subset \partial K_\rho$, and $m \leq 3(k-1)$. Finally, M_ρ is homeomorphic to the interior of $\mathcal{H}_k$.

Proof. We are guaranteed the existence of a standard compact core K_ρ from Theorem 3.3. Then $\pi_1(K_\rho) \cong F_k$, because K_ρ is homotopy equivalent to M_ρ, which has fundamental group isomorphic to $\Gamma = F_k$. So the characteristic compression body neighborhood of ∂K_ρ in K_ρ is a handlebody of genus k, which has only one boundary component; hence, K_ρ is a handlebody of genus k. We can thus find $f : \mathcal{H}_k \to K_\rho$ inducing ρ on fundamental groups. There are at most $3(k-1)$ homotopically essential, distinct, simple closed curves in a surface of genus k bounding a closed handlebody of genus k. The identification of $f(\{\nu_1, .., \nu_m\})$ with the core curves of P_ρ° is immediate from Theorem 3.3, as is the statement that M_ρ is homeomorphic to the interior of $\mathcal{H}_k$. ■

Remark 3.6. The homeomorphism f is not unique, nor are the curves $\nu_1, \ldots, \nu_m$, even up to isotopy. However, given any two homeomorphisms $f : \mathcal{H}_k \to K_\rho$ and $f' : \mathcal{H}_k \to K_\rho$ inducing ρ on fundamental groups, there is a homeomorphism $\phi : \partial\mathcal{H}_k \to \partial\mathcal{H}_k$ that extends to a homeomorphism $\Phi : \mathcal{H}_k \to \mathcal{H}_k$ such that $f \circ \Phi$ is homotopic to f'. If $\nu_1', \ldots, \nu_m'$ are the curves corresponding to $f' : \mathcal{H}_k \to M_\rho$ from Corollary 3.5, then $\phi(\{\nu_i\})$ is isotopic in $\partial\mathcal{H}_k$ to $\{\nu_i'\}$. See [47, Section 4.2].

Given a component R of $\partial K_\rho \setminus P_\rho^\circ$, let $\mathrm{Mod}_0(R, K_\rho \setminus P_\rho^\circ) \leq \mathrm{Mod}(R)$ be the group of homotopy classes of orientation preserving self homeomorphisms of R which extend to homeomorphisms of K_ρ homotopic to the identity on K_ρ. If R is incompressible, then $\mathrm{Mod}_0(R, K_\rho \setminus P_\rho^\circ)$ is trivial. For a component S of ∂K_ρ, define $\mathrm{Mod}_0(S, K_\rho)$ similarly.

3.3 Structure of ends and invariants

Let $\rho : \Gamma \to \mathrm{PSL}_2\mathbb{C}$ be a discrete and faithful representation of a finitely generated group Γ without torsion, as above. The *limit set* $\Lambda_\rho \subset \partial\mathbb{H}^3$ is the set of accumulation points of $(\mathrm{im}\rho).x$ for some (any) $x \in \partial\mathbb{H}^3$. The complement $\Omega_\rho = \partial\mathbb{H}^3 \setminus \Lambda_\rho$ is called the *domain of discontinuity*, and $\mathrm{im}\rho$ acts properly discontinuously as a group of conformal automorphisms of $\partial\mathbb{H}^3$ preserving Ω_ρ. Then, $\overline{M}_\rho = \mathbb{H}^3 \cup \Omega_\rho/\mathrm{im}\rho$ is a *three-*manifold with boundary $\Omega_\rho/\mathrm{im}\rho$ that is a disjoint union of Riemann surfaces with finite hyperbolic area. The *convex core* $\mathcal{CC}(M_\rho) \subset M_\rho$ is the quotient of the convex hull of Λ_ρ by $\mathrm{im}\rho$.

Let K_ρ be a standard relative compact core for M_ρ. Say that a relative end $[E_R] \in \mathcal{E}_\rho$ is *geometrically finite* if E_R meets $\mathcal{CC}(M_\rho)$ in a set of finite volume; we also say that the corresponding boundary component $R \subset \partial K_\rho \setminus P_\rho^\circ$ is geometrically finite. Call $[E_R]$ (or R) *geometrically infinite* otherwise. If R is geometrically finite, then the complement $E_R \setminus \mathcal{CC}(M_\rho)$ has *flaring geometry*; the metric on $E_R \setminus \mathcal{CC}(M_\rho) \cong R \times (0, \infty)$ is isotopic to a metric that is bi-Lipschitz equivalent to the metric $\cosh^2(t)dx^2 + dt^2$, where dx^2 is the intrinsic path metric on the component of $\partial\mathcal{CC}(M_\rho)$ corresponding to R.

Suppose R is geometrically finite; then the inclusion $R \hookrightarrow E_R$ is isotopic, through level surfaces of the product E_R, into $\partial\overline{M}_\rho$ and defines a point in a quotient of the Teichmüller space $\nu(R) \in \mathscr{T}(R)/\mathrm{Mod}_0(R, K_\rho \setminus P_\rho^\circ)$. The equivalence class $\nu(R)$ is the *end invariant* of $[E_R]$.

We would like to give a description of geometrically infinite ends; to motivate Definition 3.7, we start with an example. Let S be a closed, oriented surface of negative Euler characteristic, and $\varphi : S \to S$ be a pseudo-Anosov homeomorphism. The equivalence relation generated by $(x, t) \sim (\varphi(x), t + 1)$ defines a normal covering projection $\pi : S \times \mathbb{R} \to N_\varphi$ onto the mapping torus N_φ of φ with infinite cyclic deck group. Thurston's Hyperbolization Theorem [57, Theorem 0.2] states that the mapping torus N_φ has a complete hyperbolic metric; thus, so does the Riemannian covering space $\pi : S \times \mathbb{R} \to N_\varphi$, which gives rise to a discrete and faithful representation $\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. We assume that the Riemannian covering $\pi : M_\rho \to N_\varphi$ is a local isometry. Let γ_0 be a simple closed curve in $S \times \{0\} \subset M_\rho$. Then $\pi(\gamma_0)$ has a unique geodesic representative $\pi(\gamma_0)^*$ in its homotopy class. Each component of the preimage $\pi^{-1}(\pi(\gamma_0)^*) = \{\gamma_i^*\}_{i\in\mathbb{Z}}$ corresponds to a translate of the geodesic representative γ_0^* of γ_0 in M_ρ by an element of the deck group of π. Since the action of a generator of the deck group induces φ on level surfaces, the curve γ_i^* is homotopic to $\gamma_i = \varphi^i(\gamma_0) \subset S \times \{0\}$. In particular, the two ends of M_ρ in this example are geometrically infinite, because $\mathcal{CC}(M_\rho)$ contains all closed geodesics, and $\{\gamma_i^*\}_{i\in\mathbb{Z}} \subset \mathcal{CC}(M_\rho)$ is not a compact set, but $\{\gamma_i^*\}$ exits both ends of M_ρ.

Although we will not discuss measured geodesic laminations in detail, we note also that the curves $\{\gamma_i\}_{i>0}$ limit, as projective measured laminations, to the projective class of a measured geodesic lamination that is fixed by φ, and similarly in the opposite direction.

Definition 3.7. With notation as above, a relative end E_R of M_ρ is called *simply degenerate* or *degenerate* if there is a sequence of closed geodesics $\{\gamma_i^*\}_{i\in\mathbb{N}} \subset E_R$ that exit compact subsets of E_R and which are homotopic in E_R to *simple* curves $\gamma_i \subset R$.

By [18, Theorem 7.2], if R is not geometrically finite, then E_R is *simply degenerate*. Given a Riemannian metric g of finite area and pinched negative curvature, a *geodesic lamination* $\lambda \subset R$ is a closed set foliated by complete g-geodesics. Bonahon [14] showed that if $\{\gamma_i\}$ and $\{\gamma_i'\}$ are any two sequences of closed geodesics in M_ρ exiting E_R, then the *geometric intersection*

$$i\left(\frac{\gamma_i}{\ell_g(\gamma_i)}, \frac{\gamma_i'}{\ell_g(\gamma_i')}\right) \to 0, \text{ as} i \to \infty.$$

Here, $\ell_g(\gamma)$ denotes the length on (R, g) of the unique closed geodesic in the homotopy class of γ. By Thurston's theory of *measured geodesic laminations* [56, Chapter 8], there is a geodesic lamination λ_R such that $\gamma_i/\ell_g(\gamma_i)$ converges in measure to a measured geodesic lamination whose support is λ_R. It is known ([14, 16, 56]) that λ_R is compactly supported, *minimal*, and *filling*, i.e. the complement of λ_R in R consists of a collection of ideal polygons with geodesic boundary and once punctured polygons, and no leaf of λ_R is isolated. A compactly supported, minimal and filling geodesic lamination is called an *ending lamination* for R, and the set of ending laminations, given a topology by *unmeasuring*, defines a space $\mathcal{EL}(R)$ of ending laminations supported by R. See [34] to see why the topology on $\mathcal{EL}(R)$ given by unmeasuring is the natural one, from the perspective of Kleinian groups. See [25] for a precise definition of the topology of $\mathcal{EL}(R)$ and an investigation of its connectivity properties. For any two negatively curved

Riemannian structures on R, the spaces of geodesic ending laminations are canonically homeomorphic. The ending lamination $\lambda_R \in \mathcal{EL}(R)/\mathrm{Mod}_0(R, K_\rho \setminus P_\rho^\circ)$ is the *end invariant* $\nu(R)$ of the degenerate end $[E_R]$.

The following theorem of Thurston [56] and Canary [17, Corollary B] states that the only way that 'new' geometrically infinite relative ends can appear in the total space of a Riemannian covering of complete hyperbolic *three*-manifolds of infinite volume are as finite covers of 'old' geometrically infinite relative ends. All other ends are geometrically finite.

Theorem 3.8 (The Covering Theorem). Let $\rho : \Gamma \to \mathrm{PSL}_2\mathbb{C}$ be a torsion free finitely generated marked Kleinian group such that M_ρ has infinite volume, and let $i : \hat{\Gamma} \to \Gamma$ be inclusion of a finitely generated subgroup. Then either

(a) $M_{\rho\circ i}$ is geometrically finite, i.e. all relative ends of $M_{\rho\circ i}$ are geometrically finite, or

(b) For every simply degenerate end of $M_{\rho\circ i}^\circ$, there is a neighborhood $E_{\hat{R}}$ of that end and a neighborhood E_R of a simply degenerate end of M_ρ° such that the covering projection $M_{\rho\circ i} \to M_\rho$ restricts to a finite sheeted covering $E_{\hat{R}} \to E_R$.

We may assume that $E_{\hat{R}}$ and E_R are both products, so that the covering $E_{\hat{R}} \to E_R$ induces a finite sheeted covering $\hat{R} \to R$ of surfaces.

3.4 Simplicial hyperbolic surfaces

One way to study the geometry of simply degenerate ends of hyperbolic 3three-manifolds is to 'probe' them with negatively curved surfaces. A *triangulation* $\mathcal{T}$ of a closed surface S, is a three-tuple $\mathcal{T} = (V, A, T)$; V is a finite set of vertices, A is a maximal simple arc system, and T is a union of oriented *two*-simplices with embedded interiors in S that are compatible with V and A; the sum of simplices in T is required to represent the fundamental class of S. With this definition, a triangulation $\mathcal{T}$ is not necessarily associated to the geometric realization of a simplicial complex, e.g. $\mathcal{T}$ may only have the structure of a Δ-complex.

The following definitions are essentially due to Bonahon [14]; we make some modifications to allow for the possibility that a relative compact core for M_ρ has compressible boundary components. Namely, we will be interested in the case that $\rho : F_k \to \mathrm{PSL}_2\mathbb{C}$ is discrete and faithful and without parabolics, so that M_ρ is the interior of a genus k handlebody, by Corollary 3.5. There is only one component S of the boundary of a standard relative compact core for M_ρ. In what follows, we encourage the reader to keep in mind this example with $E = E_S$, a product neighborhood of the end of M_ρ. See also [17,Section 4].

Let $E \subset M_\rho$ be a codimension *zero* submanifold homeomorphic to a product $S \times \mathbb{R}$ where S is a closed oriented surface of genus at least *two*. A *simplicial pre-hyperbolic surface* is a pair $(f, \mathcal{T})$ where $f : S \to E$ is a π_1-injective continuous map; $\mathcal{T}$ is a triangulation of S, so that for each $\alpha \in A$, im$f \circ \alpha \subset E$ is an M_ρ-local geodesic segment, and for each 2two-simplex τ of $\mathcal{T}$, im$f \circ \tau \subset E$ is M_ρ-geodesically immersed. A simplicial pre-hyperbolic surface is a *simplicial hyperbolic surface* if the cone angle about each vertex in the intrinsic metric g_f on S induced by f is at least 2π. By a lemma of Ahlfors [2], there is a unique hyperbolic metric g on S in the conformal class of g_f such that the identity $(S, g) \to (S, g_f)$ is 1-Lipschitz. Thus, geometric bounds on hyperbolic surfaces translate to geometric bounds for simplicial hyperbolic surfaces, further giving us geometric estimates on E, since the composition mapping $S \to E \to M_\rho$ is 1-Lipschitz.

From a pre-simplicial hyperbolic surface $(f : S \to E, \mathcal{T})$ with one vertex, we can often construct a simplicial hyperbolic surface $(g : S \to E, \mathcal{T})$ homotopic within E to f as follows. Since $\mathcal{T}$ has only one vertex v, every arc $\alpha \in A$ maps to a locally geodesic loop based at $f(v)$; the image is not necessarily smooth at $f(v)$. Suppose there is an arc $\alpha \in A$ such that the geodesic representative $f(\alpha)^*$ of $f(\alpha)$ in M_ρ is contained in E, as is (the projection of) the straight line homopty between (appropriate lifts of) $f(\alpha)$ and $f(\alpha)^*$. There is a new map $g : S \to E$ homotopic to f obtained by 'dragging' all arcs along the image of v under the straight line homotopy between $f(\alpha)$ and $f(\alpha)^*$ and re-straightening all 1- and 2-simplices in the image relative to $g(v)$. As long as the straight line homotopy between f and g is contained in E, then $(g, \mathcal{T})$ is a simplicial hyperbolic surface. Indeed, we now just need to check that the cone angle about v is at least 2π. But, this follows from the fact that $g(v)$ lies on a smooth geodesic subsegment of $g(\alpha) = f(\alpha)^*$; the intersection of $g(S)$ with a small sphere about $g(v)$ is a piecewise spherical geodesic loop passing through antipodal points, hence the cone angle in the metric induced by g about v is at least 2π [17, Lemma 4.2].

We will make use of the following result for simplicial hyperbolic surfaces, due to Bonahon [14].

Lemma 3.9 (Bounded Diameter Lemma). For any compact set $K \subset M_\rho$, there is a compact set $K' \subset M_\rho$ such that if $f : S \to E \subset M_\rho$ is a simplicial hyperbolic surface and im$f \cap K \neq \emptyset$, then im$f \subset K'$. The diameter of $f(S)$ is bounded above by a constant that depends only on the topology of S and the injectivity radius of M_ρ in K'.

Suppose $\rho : F_k \to \mathrm{PSL}_2\mathbb{C}$ is discrete and faithful without parabolics and M_ρ has a geometrically infinite end E_S, where S is closed. Then using also our characterization of geometrically infinite ends of a hyperbolic *three*-manifold as those that have a sequence $\{\gamma_i^*\}$ of closed geodesics exiting the end E_S of M_ρ homotopic to simple curves in $S \times \{0\}$, we can construct an infinite sequence of 'well spaced' simplicial hyperbolic surfaces exiting E_S. Choose points $v_i \in \gamma_i^*$, and let A_i be a maximal system of based loops containing $\gamma_i \subset S \times \{0\}$, cutting S into triangles. For large enough i, the straight line homotopy between A_i and the corresponding locally geodesic loops based at v_i is contained in an open neighborhood of E_S [16], thus we can 'hang' simplicial hyperbolic surfaces $f_i : S \to E_S$ from the geodesics γ_i^*, as above. Since $\{\gamma_i^*\}$ exit all compact subsets of E_S to $[E_S]$, by Lemma 3.9, we can pass to a subsequence so that $\mathrm{im} f_i$ has empty intersection with $\mathrm{im} f_j$, if $i \neq j$.

3.5 The Ending Lamination Theorem

We continue with our notation from before; $\rho : \Gamma \to \mathrm{PSL}_2\mathbb{C}$ is a torsion free finitely generated marked Kleinian group, and K_ρ is a standard relative compact core for M_ρ. We summarize how to collect the end invariants ν. Let $S_1, \ldots, S_k$ be the components of ∂K_ρ with negative Euler characteristic; they are closed surfaces that inherit the boundary orientation from K_ρ. The annular components of P_ρ° have core curves that are identified with homotopically distinct, essential simple closed curves $\nu(P_\rho^\circ) \subset \sqcup S_i$. Then for each component $R \subset \partial K_\rho \setminus P_\rho^\circ$ we record a piece of data; if R is geometrically finite, then $\nu(R) \in \mathscr{T}(R)/\mathrm{Mod}_0(R, K_\rho \setminus P_\rho^\circ)$ is an equivalence class of conformal structure at infinity. If R is geometrically infinite, then $\nu(R) \in \mathcal{EL}(R)/\mathrm{Mod}_0(R, K_\rho \setminus P_\rho^\circ)$ is the corresponding equivalence class of ending lamination. To ρ, we associate all of these data $\nu(\rho)$.

The motto of the Ending Lamination Theorem is, 'the topology and geometry at infinity determine the metric.' The Ending Lamination Theorem is a classification theorem for finitely generated Kleinian groups; it helps answer many of the questions about the behavior of hyperbolic *three*-manifolds, their deformations spaces, and limiting behavior.

Theorem 3.10 (The Ending Lamination Theorem). Let Γ be a finitely generated non-abelian group without torsion, let $\rho, \rho' : \Gamma \to \mathrm{PSL}_2\mathbb{C}$ be marked Kleinian groups, and let $K_\rho \subset M_\rho^\circ$ and $K_{\rho'} \subset M_{\rho'}^\circ$ be standard relative compact cores.

If there is a homeomorphism $\phi : K_\rho \setminus P_\rho^\circ \to K_{\rho'} \setminus P_{\rho'}^\circ$ such that $\phi_* = \rho' \circ \rho^{-1}$ and $\phi(\nu(\rho)) = \nu(\rho')$, then ϕ extends to a homeomorphism $\Phi : M_\rho \to M_{\rho'}$ that is isotopic to an isometry inducing $\rho' \circ \rho^{-1}$ on fundamental groups. Equivalently, there is a $g \in \mathrm{PSL}_2\mathbb{C}$ such that $g\rho g^{-1} = \rho'$.

In other words, if the topological type of M_ρ and $M_{\rho'}$ agree and so do the relative end invariants, then M_ρ is isometric to $M_{\rho'}$ in the homotopy class of the classifying map determined by $\rho' \circ \rho^{-1}$.

We owe the reader attributions and references. The final ingredients to prove Theorem 3.10 were given by Brock–Canary–Minsky [8, Ending Lamination Theorem for Incompressible Ends], and in a follow up paper [7] that carries out the details needed to modify the proof when M_ρ has compressible ends, using Canary's Branched Cover Trick [16], as outlined in [8, Section 1.2]. For Kleinian surface groups, Minsky built a *model manifold* M_ν from a list of end invariants ν out of *blocks* and *tubes*. He also constructed a Lipschitz map (with Lipschitz constant depending only on the topology of the surface) $M_\nu \to M$, where M is a hyperbolic manifold with end invariants ν [39, Extended Model Theorem]. The geometry and arrangement of the tubes in the model manifold M_ν is extracted from a *hierarchy of tight geodesics* and careful analysis of the geometry of Harvey's *complex of curves* carried out by Masur–Minsky in [41, 42]. The Lipschitz model map was then promoted to a bi-Lipschitz model map that extends to a conformal map at infinity [8,Bi-Lipschitz Model Theorem]. So, if M_ρ and $M_{\rho'}$ have the homotopy type of a finite type surface (with a parabolicity condition on the boundary curves of the surface) and they have the same end invariants, as seen from some reference surface, then there is a bi-Lipschitz homeomorphism between them, compatible with markings, that extends to a conformal map at infinity; by Sullivan's Rigidity Theorem [54], the bi-Lipschitz mapping is homotopic to an isometry.

If all of the relative ends of M_ρ are simply degenerate, we say that M_ρ is *totally degenerate*. The Ending Lamination Theorem 3.10 implies that the marked isometry type of a totally degenerate manifold M_ρ is completely determined by the topology of a standard relative compact core and list of ending laminations.

The general statement for the Ending Lamination Theorem (including the case that there are compressible relative ends) is made possible by the Tameness Theorem 3.3. One studies the covering spaces associated to the relative ends of a given manifold M, builds models of the ends from the end invariants, and assembles the pieces to obtain a bi-Lipschitz model for M. All of this builds on the important contributions of Thurston, Ahlfors, Bers, Marden, Maskit, Sullivan, Bonahon, Otal, O'shika, and many others. Soma [51] has recently given an alternate strategy using methods from bounded cohomology and volume rigidity.

3.6 Quasi-isometric classification of marked Kleinian surfaces and free groups

For constants $M \geq 1$ and $A \geq 0$, an (M, A)-quasi-isometry $f : X \to Y$ between metric spaces is a map that satisfies

$$\frac{1}{M} d_X(x, x') - A \leq d_Y(f(x), f(x')) \leq M d_X(x, x') + A$$

for all $x, x' \in X$. Thurston defined the *quasi-isometric topology* on the space of hyperbolic manifolds with a given homotopy type in [57, Section 1]; say that two discrete and faithful representations $\rho_1, \rho_2 : \Gamma \to \mathrm{PSL}_2\mathbb{C}$ of a finitely generated non-elementary group Γ are *quasi-isometric*, and write $\rho_1 \sim_{q.i} \rho_2$, if there are M and A as above and a (ρ_1, ρ_2)-equivariant (M, A)-quasi-isometry $\mathbb{H}^3 \to \mathbb{H}^3$. An equivariant quasi-isometry of representations extends to an equivariant quasi-conformal map at infinity, where the quasi-isometry constants give control on the quasi-conformal constant of the map at infinity.

A weaker version of the implication '(1) $\Rightarrow$ (3)' in the following theorem was given by Soma [49, Theorem A]; in that paper, all manifolds are homotopy equivalent to a closed surface, have bounded geometry, and two degenerate ends. The number ϵ in Soma's [49, Theorem A] depends on the lower bound ϵ' for the injectivity radius of the manifolds involved and ϵ goes to zero with ϵ'. In particular, the techniques that prove [49, Theorem A] do not apply to manifolds with unbounded geometry. Part of the novelty of Theorem 3.11 is that our techniques are equally amenable to manifolds with unbounded geometry, a topologically generic set at the boundary of the deformation space of hyperbolic metrics, and our constant ϵ is uniform over all metrics; it only depends on the homotopy type of the manifolds M_{ρ_i} but not their geometry.

Theorem 3.11 ([20, 21]). Let S be an orientable hyperbolic surface of finite type. There is a constant $\epsilon = \epsilon(S) > 0$ such that if $\rho_1, \rho_2 : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ are discrete and faithful with no parabolic elements, then the following are equivalent.

1. $\|[\rho_1^*\mathrm{vol}_3] - [\rho_2^*\mathrm{vol}_3]\|_\infty < \epsilon$
2. $[\rho_1^*\mathrm{vol}_3] = [\rho_2^*\mathrm{vol}_3]$
3. $\rho_1 \sim_{q.i.} \rho_2$.

Moreover, if M_{ρ_1} has a geometrically infinite end and $\rho_3 : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is an arbitrary representation satisfying $\|[\rho_1^*\mathrm{vol}_3] - [\rho_3^*\mathrm{vol}_3]\|_\infty < \epsilon$, then ρ_3 is faithful.

Proof of Theorem 3.11. Let $\epsilon < \min\{\epsilon'/2, v_3\}$, where ϵ' is as in [20, Theorem 1.2] and only depends on the topology of S. The last statement is [20, Theorem 1.3], which states that ρ_3 is faithful.

Condition (2) implies condition (1), trivially. Now we show that (3) implies (2). We need the classical fact that an equivariant quasi-isometry extends to an equivariant quasi-conformal homeomorphism at infinity (see, e.g. [56, Corollary 5.9.6]). This quasi-conformal homeomorphism extends to a *volume preserving* bi-Lipschitz diffeomorphism $M_{\rho_1} \to M_{\rho_2}$ inducing $\rho_2 \circ \rho_1^{-1}$ on fundamental groups; see [38, Appendix B], [4, Theorem 5.6], or [21, Theorem 3.1], where those results are summarized. We can apply [21, Corollary 3.6] or the proof of [21, Theorem 3.2] to see that $[\rho_1^*\mathrm{vol}_3] = [\rho_2^*\mathrm{vol}_3]$.

Finally, we show that (1) implies (3). With ϵ as in the first paragraph, [20, Theorem 1.2] states that the geometrically infinite end invariants of M_{ρ_1} must be the same as the geometrically infinite end invariants of M_{ρ_2}. If M_{ρ_1} is totally degenerate, then $\nu(\rho_1) = \nu(\rho_2)$. By the Ending Lamination Theorem 3.10, ρ_1 is conjugate, hence quasi-isometric, to ρ_2.

So, we now assume that at least one end of M_{ρ_1} is geometrically finite. If $\pi_1(S)$ is free, then by Corollary 3.5, M_{ρ_1} and M_{ρ_2} are handlebodies of the same genus, and since $\mathrm{im}\rho_i$ has no parabolic elements, there is only one relative end of M_{ρ_i}, for $i = 1, 2$. For geometrically finite hyperbolic manifolds with no parabolic cusps, $\mathcal{CC}(M_{\rho_i})$ is a standard compact core for M_{ρ_i}; we may find a bi-Lipschitz homeomorphism $\phi : \mathrm{int}\mathcal{CC}(M_{\rho_1}) \to \mathrm{int}\mathcal{CC}(M_{\rho_2})$ inducing $\rho_2 \circ \rho_1^{-1}$ on fundamental groups. The neighborhood $E_i = M_{\rho_i} \setminus \mathcal{CC}(M_{\rho_i})$ of the end of M_{ρ_i} is a product $S' \times (0, \infty)$, where S' is a closed surface of genus equal to the rank of $\pi_1(S)$, and its metric is bi-Lipschitz equivalent to $\cosh^2(t)dx_i^2 + dt^2$, where dx_i is the induced path metric on $\partial\mathcal{CC}(M_{\rho_i})$. Thus, ϕ extends to a homeomorphism mapping level surfaces of E_1 to level surfaces of E_2 with respect to the aforementioned product structure on E_i. Since the path metrics on $\partial\mathcal{CC}(M_{\rho_i})$ are bi-Lipschitz equivalent to each other and $\mathcal{CC}(M_{\rho_1})$ and $\mathcal{CC}(M_{\rho_2})$ are compact, the map $\phi : M_{\rho_1} \to M_{\rho_2}$ is a bi-Lipschitz homeomorphism that lifts to an equivariant bi-Lipschitz homeomorphism $\mathbb{H}^3 \to \mathbb{H}^3$. Thus, $\rho_1 \sim_{q.i.} \rho_2$.

If S is closed then $M_{\rho_i} \cong S \times \mathbb{R}$ by Bonahon's Tameness Theorem [14, Théorèm A] and since $\mathrm{im}\rho_i$ contains no parabolics, standard compact cores for M_{ρ_i} are of the form $S \times [0, 1]$, for each $i = 1, 2$. There are two possibilities for the geometry of the ends of M_{ρ_i}. If M_{ρ_1} has no geometrically infinite ends, then the same is true for M_{ρ_2}. A similar argument to the previous paragraph produces an equivariant bi-Lipschitz mapping $\mathbb{H}^3 \to \mathbb{H}^3$, and so $\rho_1 \sim_{q.i} \rho_2$. The last possibility is that M_{ρ_i} each has one degenerate end and one geometrically finite end. In this case, the convex cores of M_{ρ_i} are neighborhoods of their degenerate ends, and [8, Bi-Lipschitz Model Theorem] supplies us with a bi-Lipschitz homeomorphism $\phi : \mathcal{CC}(M_{\rho_1}) \to \mathcal{CC}(M_{\rho_2})$ inducing $\rho_2 \circ \rho_1^{-1}$ on fundamental groups. Again, we extend ϕ to a bi-Lipschitz homeomorphism $M_{\rho_1} \to M_{\rho_2}$ mapping flaring level surfaces to flaring level surfaces in the complement of the convex core. The conclusion of the theorem follows, as in the previous cases. ■

There are geometrically finite representations of a closed surface S of genus $g \geq 2$ that are not quasi-isometric to each other. For example, take pants decompositions α and β of a closed surface S with no common curves. There is a unique conjugacy class $\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ such that the quotient manifold M_ρ has $6g - 6$ rank-1 cusps. K_ρ is a trivial interval bundle over S, with two boundary components S^+ and S^-. The components of P_ρ° are annular subsurfaces of S^+ and S^- with core curves $\alpha \subset S^+$ and $\beta \subset S^-$; each complementary component of $\partial K_\rho \setminus P_\rho^\circ$ is homeomorphic to a *three*-times punctured sphere. The only simple closed curves on three times punctured spheres are parallel to the punctures and do not have geodesic representatives in a finite area metric of non-positive curvature, so there are no ending laminations, being limits of geodesic simple closed curves, on a three times punctured sphere. Every relative end of this example is geometrically finite. In fact, since the Teichmüller space of a three times punctured sphere is a point, every relative end is isometric to every other relative end in this example. The volume class vanishes by Theorem 2.1, but ρ is not quasi-isometric to any Fuchsian representation. Thus, the assumption that representations have no parabolic elements cannot be dropped in Theorem 3.11.

Now we prove Theorem 1.2, assuming Theorem 1.1, which is the last ingredient for our quasi-isometric volume rigidity result.

Theorem 3.12. There exists a constant $\epsilon = \epsilon(S)$ such that the following holds. Suppose that $\rho_0 : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is a discrete and faithful representation without parabolic elements, and that $[\rho_0^*\mathrm{vol}_3] \neq 0$. If $\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is any other representation without parabolics satisfying

$$\|[\rho_0^*\mathrm{vol}_3] - [\rho^*\mathrm{vol}_3]\|_\infty < \epsilon,$$

then ρ is discrete and faithful, and ρ is quasi-isometric to ρ_0. If ρ_0 is totally degenerate, then ρ_0 and ρ are conjugate in $\mathrm{PSL}_2\mathbb{C}$.

Proof. Choose ϵ as in Theorem 3.11. Then ρ_0 is discrete, faithful, and $[\rho_0^*\mathrm{vol}_3] \neq 0$, so we can apply Theorem 2.1 to see that $\|[\rho_0^*\mathrm{vol}_3]\|_\infty = v_3$, and M_{ρ_0} has a geometrically infinite end. We can apply the last statement of Theorem 3.11 to deduce that ρ is faithful.

By Theorem 1.1, since ρ_0 is discrete and $\|[\rho_0^*\mathrm{vol}_3] - [\rho^*\mathrm{vol}_3]\|_\infty < \epsilon < v_3$, ρ cannot have dense image. The triangle inequality gives $\|[\rho^*\mathrm{vol}_3]\| > v_3 - \epsilon > 0$. Thus, $[\rho^*\mathrm{vol}_3] \neq 0$ and so ρ has discrete image by Lemma 2.2. Now that we know that ρ is discrete and faithful, we can apply the main body of Theorem 3.11 to see that $\rho \sim_{q.i.} \rho_0$. This concludes the proof of the first statement of the theorem.

If M_{ρ_0} is totally degenerate, then all of the end invariants of M_ρ and M_{ρ_0} coincide. By the Ending Lamination Theorem 3.10, ρ_0 is conjugate to ρ in $\mathrm{PSL}_2\mathbb{C}$, as in the proof of Theorem 3.11. ■

4 Volume classes of dense representations

We will prove our main theorems for the *three*-dimensional volume class in this section by building efficient chains from a geometrically infinite Kleinian free group $\Gamma = \langle a, b\rangle \leq \mathrm{PSL}_2\mathbb{C}$ that contains no parabolics. Then, we will approximate the shape of those chains using a dense representation. Call a n-chain Z ϵ-*efficient* or just *efficient* if $\frac{|\mathrm{vol}_n(Z)|}{\|Z\|_1} > \epsilon$. We recall Soma's construction of efficient chains in a hyperbolic manifold $\mathbb{H}^3/\Gamma$, where Γ is any finitely generated, torsion free, infinite co-volume, geometrically infinite Kleinian group $\Gamma \leq \mathrm{PSL}_2\mathbb{C}$. Then, we use tools from Section 2.3 to turn these efficient chains into efficient *three*-chains with only one vertex, which then define chains on Γ. If we have a dense representation $\rho : G \to \mathrm{PSL}_2\mathbb{C}$, we can approximate a and b by sequences $a_n = \rho(x_n)$ and $b_n = \rho(y_n)$ for suitably chosen $x_n, y_n \in G$. The shape of the chain on Γ with large volume and small boundary area can be approximated by chains on the groups $\langle a_n, b_n\rangle$. The chains in the approximates have essentially the same volume as the chains which came from the manifold $\mathbb{H}^3/\Gamma$, as in Section 2.6.

We have a natural way to collapse a straightened singular three-chain in $\mathbb{H}^3\Gamma$ with many vertices onto a chain with only one vertex. We lose some volume during this collapsing process, but the loss of volume is controlled in terms of the ℓ_1-norm of the boundary of the original chain, which is uniformly bounded. We need to prove a number of technical facts about these one-vertex chains that rely on geometrical facts concerning hyperbolic three-manifolds. This allows us to show that the volume class of any dense representation has maximal semi-norm.

Then, we prove some auxiliary results about certain subgroups of Kleinian groups to show that certain volume classes are separated in semi-norm, as announced in Theorem 1.1 and Theorem 1.5.

4.1 Constructing efficient chains

There are several chain complexes in which a three-chain Z could live, e.g. in a group Γ or a quotient manifold $\mathbb{H}^3/\Gamma$, and vol_3 should be interpreted in whichever context it makes sense. In what follows, we show that for chains Z whose boundary has small ℓ_1-norm, it is not so important where we compute the volume; this is most of the content of Proposition 4.2.

In [20], we constructed sequences of three-chains on geometrically infinite genus g handlebodies with large volume and small boundary area. These chains were ϵ_g-efficient, where $\epsilon_g > 0$ depends only on the topology of the handlebody (but not on its geometry), and for which the boundary surfaces of the chains were well controlled. In fact, they were simplicial hyperbolic surfaces; see Section 3.4. Soma [48, 49] constructed $(v_3 - \epsilon)$-efficient chains, for any $\epsilon > 0$, but the boundaries of his chains are not well controlled and grow wilder as the efficiency constant gets closer to v_3.

We will revisit Soma's construction of efficient chains for a geometrically infinite manifold with free fundamental group to extract a technical feature that we require in the proof of Lemma 5.4. Soma's chains are constructed via 'smearing,' as in Thurston's construction of efficient (measure) cycles representing the fundamental class of a closed hyperbolic manifold [56, Chapter 6].

We will follow [49, Section 3]; note that although Soma requires that Γ be isomorphic to a closed surface group, [49, Lemma 3.2] only depends on the topological and geometrical structure of a geometrically infinite end of a topologically tame hyperbolic *three*-manifold. See the proof of [48, Theorem 1] for further comments.

For the rest of this section, fix a discrete, faithful, and geometrically infinite representation $\rho_0 : F_d \to \mathrm{PSL}_2\mathbb{C}$ with no parabolic elements, and let Γ_0 denote $\mathrm{im}\rho_0$. Let $\pi : \mathbb{H}^3 \to \mathbb{H}^3/\Gamma_0 = M_{\rho_0}$ be the covering projection. It will be convenient to assume that $M_{\rho_0} = \mathbb{H}^3/\Gamma_0$ has bounded geometry[1] . By Corollary 3.5, M_{ρ_0} is homeomorphic to a handlebody $\mathcal{H}_d$ of genus d. There is a standard compact core K_{ρ_0} that is also a

[1] Generically, a simply degenerate end of a complete hyperbolic *three*-manifold homeomorphic to a handlebody without parabolics has a sequence of closed geodesics with length tending to *zero*. However, bounded geometry geometrically infinite handlebodies arise, for example, as geometric limits of closed hyperbolic *three*-manifolds with bounded Heegaard genus and gluing data represented by powers of a pseudo-Anosov homeomorphism going to infinity; see [46].

handlebody, and $E = M_{\rho_0} \setminus K_{\rho_0}$ is a neighborhood of the end of M_{ρ_0} that is simply degenerate; its closure is homeomorphic to $S \times [0, \infty)$, where S is an oriented surface of genus d.

Find a sequence $\{(f_n : S \to E, \mathcal{T}_n)\}$ of simplicial hyperbolic surfaces, homotopic to the inclusion $S \hookrightarrow S \times \{0\}$, with $X_n = \mathrm{im}(f_n)$ exiting E toward $[E]$. The maps f_n are not embeddings, but $E \setminus X_n$ consists of some compact components and a non-compact component $E_n \in [E]$. Pass to a subsequence such that $\{X_n\}$ are pairwise disjoint (see the discussion following the Bounded Diameter Lemma 3.9) and $E_m \supset E_n$ if $m < n$. Also let $L_{m,n}$ denote the closure of $E_m \setminus E_n$, which is a compact set.

Let $\sigma : \Delta_3 \to \mathbb{H}^3$ be a non-degenerate straight simplex; abusing notation we ignore the parameterization of the map and identify σ with its image in $\mathbb{H}^3$ or even its ordered vertex set. In each straight simplex, there exists a unique inscribed ball, meeting each face in a point. Let $\mathrm{center}(\sigma) \in \mathbb{H}^3$ denote the center of this inscribed ball. We describe a Borel measure $\mathrm{smear}\sigma$ on the space of locally straight *three*-simplices

$$\mathscr{S}(M_{\rho_0}) = (\mathbb{H}^3)^4 / \Gamma_0$$

in M_{ρ_0}. If σ is a straight simplex in $\mathbb{H}^3$, then $\pi(\sigma)$ denotes the corresponding element of $\mathscr{S}(M_{\rho_0})$.

Consider the Haar measure μ on $\mathrm{PSL}_2\mathbb{C}$ normalized so that, for any $x \in \mathbb{H}^3$ and any Borel measurable set $K \subset \mathbb{H}^3$,

$$\mu(\{g \in \mathrm{PSL}_2\mathbb{C} : g.x \in K \subset \mathbb{H}^3\}) = \mathrm{Vol}(K),$$

the hyperbolic volume of K. Then, μ descends to a measure (with the same name) on $\Gamma_0 \backslash \mathrm{PSL}_2\mathbb{C}$. Finally, we define $\mathrm{smear}\sigma$ as follows: for a Borel measurable set $K \subset \mathscr{S}(M_{\rho_0})$, we define

$$\mathrm{smear}\sigma(K) = \mu(\{\bar{g} \in \Gamma_0 \backslash \mathrm{PSL}_2\mathbb{C} : \pi(g.\sigma) \in K\}).$$

Also to $K \subset M_{\rho_0}$ we associate the set of simplices

$$\mathscr{S}_\sigma(K) = \{\pi(g.\sigma) \in \mathscr{S}(M_{\rho_0}) : \pi(\mathrm{center}(g.\sigma)) \in K\},$$

so that

$$\mathrm{smear}\sigma(\mathscr{S}_\sigma(K)) = \mathrm{Vol}(K).$$

Fix a geodesic plane $\mathbb{H}^2 \subset \mathbb{H}^3$ and let $r : \mathbb{H}^3 \to \mathbb{H}^3$ be reflection in $\mathbb{H}^2$, so that the simplex $r \circ \sigma$ is isometric to σ with the opposite orientation. Define

$$z(\sigma) = \frac{1}{2}(\mathrm{smear}\sigma - \mathrm{smear}r \circ \sigma).$$

By restricting the support of $z(\sigma)$ to $\mathscr{S}_\sigma(L_{0,k})$, we obtain a family of Borel measures $z_k(\sigma)$ with total variation $\|z_k(\sigma)\| = \mathrm{Vol}(L_{0,k})$ and

$$\int_{\mathscr{S}(M_{\rho_0})} \mathrm{vol}_3(\sigma')\, dz_k(\sigma)(\sigma') = \mathrm{vol}_3(\sigma)\mathrm{Vol}(L_{0,k}). \tag{7}$$

In Thurston's smooth measure homology theory, each $z_k(\sigma)$ defines a *measure chain*, i.e. a signed measure on the space of smooth singular simplices satisfying a local finiteness condition. A straight singular chain $Z = \sum a_i\sigma_i \in \mathrm{C}_k^{\mathrm{str}}(M_{\rho_0})$ defines a smooth measure chain $\sum a_i\delta_{\sigma_i}$, where δ_τ is the Dirac measure supported on τ. This correspondence induces a continuous chain map with respect to the topology induced by the ℓ_1- and the total variation norms, respectively. In fact, singular homology and smooth measure homology are isometrically isomorphic with respect to the ℓ_1- and total variation semi-norms [36, Theorem 1.2].

Lemma 4.1 ([49, Lemma 3.2]). Given d, there exists a constant $K_d > 0$ depending only the topology of the surface $S_d \cong \partial K_{\rho_0}$, such that for every $\epsilon > 0$, there is a sequence $V_k \in \mathrm{C}_3^{\mathrm{str}}(M_{\rho_0}; \mathbb{R})$ such that the following properties hold:

1. $\frac{|\mathrm{vol}_3(V_k)|}{\|V_k\|_1} > v_3 - \epsilon$ for all k,
2. $\|\partial V_k\|_1 \leq K_d$, for all k, and
3. $|\mathrm{vol}_3(V_k)| \to \infty$.

Proof. Sketch of proof We consider the regular simplex σ_t with edge lengths $t \gg 0$ and the Borel measures $\{z_k(\sigma_t)\}_{k=1}^{\infty}$ on $\mathscr{S}(M_{\rho_0})$. We know that $|\mathrm{vol}_3(\sigma_t) - v_3|$ decreases exponentially in t [56, Theorem 6.4.1], but we only need the fact that for t large enough, $\mathrm{vol}_3(\sigma_t) > v_3 - \epsilon/2$.

For the appropriate definition of the boundary of a measure chain, we have $\|\partial z_k(\sigma_t)\| \leq K_d$, where K_d only depends on the topology of the surface $S_d \cong \partial K_{\rho_0}$. Indeed, consider a simplex $\pi(g.\sigma_t)$ with $\mathrm{center}(\pi(g.\sigma_t)) \in L_{0,k}$ but far from $\partial L_{0,k}$. Let τ be a face of σ_t and let r_τ be reflection through the plane containing τ; r_τ is conjugate to r in $\mathrm{PSL}_2\mathbb{C}$. A computation shows that

$$d_{\mathbb{H}^3}(\mathrm{center}(g.r_\tau \circ \sigma_t), \mathrm{center}(g.\sigma_t)) < 2. \tag{8}$$

Thus $\mathrm{center}(\pi(g.r_\tau \circ \sigma_t))$ is far from $\partial L_{0,k}$. The faces corresponding to τ coming from $\pi(g.\sigma_t)$ and the reflected copy match and cancel after applying ∂ to $z_k(\sigma_t)$. If $\mathrm{center}(\pi(g.\sigma_t))$ is close to $\partial L_{0,k}$, then $\pi(g.r_\tau \circ \sigma_t)$ may not be in the support of $z_k(\sigma_t)$. However, $\pi(g.r_\tau \circ \sigma_t) \in \mathscr{S}_{r\circ\sigma_t}(\mathcal{N}_2(\partial L_{0,k}))$. Thus $\mathrm{supp}(\partial z_k(\sigma_t))$ is contained in the set of locally straight triangles which are faces of tetrahedra with centers in $\mathcal{N}_2(\partial L_{0,k})$; we conclude that $\|\partial z_k(\sigma_t)\| \leq \mathrm{Vol}(\mathcal{N}_2(\partial L_{0,k}))$ [49, Proof of Lemma 3.2]. Using the fact that the induced metric on X_k has curvature everywhere at most -1, we can find a universal constant $V > 0$ such that $\mathrm{Vol}(\mathcal{N}_2(\partial L_{0,k})) \leq V \cdot |\chi(S_d)| = K_d$ (see, for example, [48, Proof of Theorem 1]).

To each measure chain $z_k(\sigma_t)$, we will associate a straight 3-chain $V_k^t \in \mathrm{C}_3^{\mathrm{str}}(M_{\rho_0}; \mathbb{R})$. For t large enough, every $1/t$-ball in M_{ρ_0} is embedded, because we have assumed that M_{ρ_0} has bounded geometry. Find a maximal $(1/t)$-separated collection of points $\{p_i^t\}_{i=1}^\infty \subset M_{\rho_0}$, and let $\{U_i^t\}$ be the Voronoi cells generated by $\{p_i^t\}$. The boundary of the closure of each cell has zero *three*-dimensional Lebesgue measure, and each cell is connected, simply connected, precompact, and has a distinguished point $p_i^t \in U_i^t$. For each k, there is a finite subset of $\{U_i^t\}$ that meet any simplex in the support of $z_k(\sigma_t)$, because $L_{0,k}$ is compact, so that for each $\pi(g.\sigma_t)$ with center in $L_{0,k}$, we have $\pi(g.\sigma_t) \subset \mathcal{N}_t(L_{0,k})$. So, only finitely many terms in the sum

$$V_k^t := \sum z_k(\sigma_t)(\{\pi(g.\sigma_t) : g.\sigma_t^{(0)} \in \tilde{U}_{i_0}^t \times \ldots \times \tilde{U}_{i_3}^t\}) \cdot \pi(\sigma(\tilde{p}_{i_0}^t, \ldots, \tilde{p}_{i_3}^t)) \in \mathrm{C}_3^{\mathrm{str}}(M_{\rho_0}; \mathbb{R}), \tag{9}$$

are non-zero. The sum (9) ranges over all sets of the form $\tilde{U}_{i_0}^t \times \ldots \times \tilde{U}_{i_3}^t$ where $\tilde{U}_{i_j}^t$ is a lift of $U_{i_j}^t$, and $\sigma(x, y, z, w)$ denotes the straight simplex in $\mathbb{H}^3$ with ordered vertex set (x, y, z, w). The correspondence given by (9) actually defines a norm non-increasing chain map between smooth measure chains and straight chains. In particular, $\|\partial V_k^t\|_1 \leq \|\partial z_k(\sigma_t)\| \leq K_d$.

For large t, the cells U_i^t have very small diameter, so we can ensure that

$$|\mathrm{vol}_3(\sigma(\tilde{p}_{i_0}^t, \ldots, \tilde{p}_{i_3}^t)) - \mathrm{vol}_3(\sigma_t))| < \epsilon/2 \ \text{ and } \mathrm{vol}_3(\sigma_t) \geq v_3 - \epsilon/2. \tag{10}$$

For μ-almost every $g \in \mathrm{PSL}_2\mathbb{C}$, the vertices of $\pi(g.\sigma_t) \in \mathrm{supp} z_k(\sigma_t)$ all lie in the interior of cells $U^t_{i_0}, \ldots, U^t_{i_3}$. The Voronoi cells $\{U^t_i\}$ form a measurable partition of M_{ρ_0}, from which it follows that

$$\|V^t_k\|_1 = \|z_k(\sigma_t)\|. \tag{11}$$

By construction, $\|z_k(\sigma_t)\| = \mathrm{Vol}(L_{0,k})$, and $\mathrm{Vol}(L_{0,k}) \to \infty$ as $k \to \infty$, because $\cup_k L_{0,k} = E_0 \in [E]$.

Finally, from (10), (11), and (7), we obtain

$$\mathrm{vol}_3(V^t_k) \geq \|V^t_k\|_1(v_3 - \epsilon),$$

if t is large enough; set $V_k = V^t_k$. ■

Proposition 4.2. For every positive integer $d \geq 2$, there is a constant $K_d > 0$ depending only on the topology of $S_d \cong \partial K_{\rho_0}$ with the following properties. For every $\epsilon > 0$, there exists a sequence of chains $Z_k \in C_3(F_d;\mathbb{R})$ such that for any $x \in \mathbb{H}^3$:

(i) $\frac{|\rho_0^* \mathrm{vol}_3^x(Z_k)|}{\|Z_k\|_1} > v_3 - \epsilon$, for all k, and
(ii) $\|\partial Z_k\|_1 \leq K_d$, for all k, and
(iii) $\|Z_k\|_1 \to \infty$ and $\lim_{k\to\infty} \frac{\|\partial Z_k\|_1}{\|Z_k\|_1} = 0$.

Proof. Choose a point $\bar{x} \in M_{\rho_0}$, and construct the chain map $\mathrm{str}_{\bar{x}} : C_\bullet(M_{\rho_0};\mathbb{R}) \to C^{\mathrm{str}}_\bullet(M_{\rho_0}, \{\bar{x}\};\mathbb{R})$ from Section 2.3; recall that the operator norm satisfies $\|\mathrm{str}_{\bar{x}}\| \leq 1$ and we have a chain homotopy $H^\bullet_{\bar{x}}$ between $\mathrm{str}_{\bar{x}}$ and id, such that $\|H^k_{\bar{x}}\| = k+1$; see (1). Since vol_3 is a bounded cocycle on the straight chains $V_k \in C^{\mathrm{str}}_3(M_{\rho_0};\mathbb{R})$ provided by Lemma 4.1, we have

$$|\mathrm{vol}_3(\mathrm{str}_{\bar{x}} V_k - V_k)| = |\mathrm{vol}_3(H^2_{\bar{x}} \partial V_k)| \leq \|\mathrm{vol}_3\|_\infty \|H^2_{\bar{x}} \partial V_k\|_1.$$

Using property (2) and the fact that $\|H^2_{\bar{x}}\| = 3$, we get

$$|\mathrm{vol}_3(\mathrm{str}_{\bar{x}} V_k - V_k)| \leq 3v_3 \cdot K_d.$$

Apply the map $(\rho_{0*})^{-1} \circ \iota_x : \mathrm{C}^{\mathrm{str}}_\bullet(M_{\rho_0}, \{\bar{x}\}; \mathbb{R}) \to \mathrm{C}_\bullet(\Gamma_0; \mathbb{R}) \to \mathrm{C}_\bullet(F_d; \mathbb{R})$ to obtain a sequence

$$Z_k = (\rho_{0*})^{-1} \circ \iota_x(\mathrm{str}_{\bar{x}} V_k)$$

so that $\rho_0^* \mathrm{vol}_3^x(Z_k) = \mathrm{vol}_3^x \iota_x(\mathrm{str}_{\bar{x}}(V_k))$. Since $(\rho_{0*})^{-1} \circ \iota_x$ is an isometric chain map and $\|\mathrm{str}_{\bar{x}}\| \le 1$, we have $\|Z_k\|_1 \le \|V_k\|_1$ and $\|\partial Z_k\|_1 \le \|\partial V_k\|_1 \le K_d$, establishing property (2).

Since $|\mathrm{vol}_3(V_k)| \to \infty$ as $k \to \infty$, it follows that $|\rho_0^* \mathrm{vol}_3^x(Z_k)| \to \infty$, as well. From this, we see that $\|V_k\|_1$ and $\|Z_k\|_1$ tend to ∞, from which we obtain property (3). Using the triangle inequality, for k large enough, we have

$$v_3 - \epsilon < \frac{|\rho_0^* \mathrm{vol}_3^x(Z_k)|}{\|Z_k\|_1},$$

which establishes property (1) after reindexing. ■

Qualitatively, σ_t has extremely long and thin spikes; the spikes of $\pi(\sigma_t)$ wander circuitously around M_{ρ_0} and are generically recurrent to any compact set in the limit as $t \to \infty$.

Lemma 4.3. In the statement of Proposition 4.2, we may take $x \in \partial\mathbb{H}^3$ such that

- (1) holds for all k;
- if for each k, we write

$$Z_k = \sum_{j=1}^{M_k} a^{j,k}[w_0^{j,k}, \ldots, w_3^{j,k}] \in \mathrm{C}_3(F_d; \mathbb{R}),$$

  then, for each $j = 1, \ldots, M_k$, the points $\rho_0(w_0^{j,k}).x, \ldots, \rho_0(w_3^{j,k}).x \in \partial\mathbb{H}^3$ are pairwise distinct.

Proof. From Section 2.4, for any point $y \in \mathbb{H}^3 \cup \partial\mathbb{H}^3$ and for any $Z \in \mathrm{C}_3(F_d; \mathbb{R})$,

$$|\rho_0^* \mathrm{vol}_3^x(Z) - \rho_0^* \mathrm{vol}_3^y(Z)| \le \|\partial Z\|_1 3v_3.$$

Using the triangle inequality and the fact that ∂Z_k is bounded for all k, we can choose $x \in \partial\mathbb{H}^3$ so that (1) holds, after reindexing.

The group F_d is countable, so there are countably many attracting and repelling fixed points of non-trivial elements of $\mathrm{im}\rho_0$. Choose $x \in \partial\mathbb{H}^3$ that is not one of these fixed points. Then for each $k \geq 1$ and $j = 1, \ldots, M_k$, if $w_0^{j,k}, \ldots, w_3^{j,k}$ are pairwise distinct, then $\rho_0(w_0^{j,k}).x, \ldots, \rho_0(w_3^{j,k}).x \in \partial\mathbb{H}^3$ are pairwise distinct. Thus, we just need to show no simplex in Z_k is degenerate. For this, we revisit the construction of Soma's chains from Lemma 4.1. Fix $x \in \mathbb{H}^3$ and let $\mathcal{D} \subset \mathbb{H}^3$ be the Dirichlet fundamental domain for Γ_0 centered at x. By definition of $\iota_x : C_\bullet^{\mathrm{str}}(M_{\rho_0}, \{\bar{x}\}; \mathbb{R}) \to \mathrm{C}_\bullet(\Gamma_0; \mathbb{R})$ from Section 2.3, it is enough to show, given k, that if t is large enough, then every pair of vertices of any simplex $\pi(\sigma(\tilde{p}_{i_0}, \ldots, \tilde{p}_{i_3}))$ appearing in equation (10) defining V_k^t lie in distinct translates of $\mathcal{D}$. By invariance and without loss of generality, it is enough to show that if t is large enough, then no pair of vertices are both in $\mathcal{D}$.

Suppose not. Recall that $\mathcal{D}$ is convex being the intersection of countably many half-spaces, so that if two vertices of a straight simplex σ lie in $\mathcal{D}$, then an edge of σ lies in $\mathcal{D}$. Thus, for a fixed positive integer k, and any n, there is a $t_n \geq n$ and simplex σ_n appearing in the sum $V_k^{t_n}$ with a lift $\tilde{\sigma_n}$ whose geodesic edge $\gamma_n = \tilde{\sigma}^n([0,1])$ is contained in $\mathcal{D}$. By construction, the center of σ_n is contained in $L_{0,k} \subset E$, which is a compact set. Pass to a subsequence so that center$(\sigma_n) \to y \in M_{\rho_0}$. Then, the lift $\tilde{y}$ of y in $\mathcal{D}$ is distance at most 2 from γ_n, since the midpoint of a geodesic edge of a straight regular simplex passes close to its center (see Equation (8)). Passing to a further subsequence, the Arzelà–Ascoli Theorem guarantees that the geodesic maps γ_n converge to a bi-infinite geodesic γ. Since γ_n is contained in $\mathcal{D}$, $\gamma \subset \overline{\mathcal{D}}$.

The locally geodesic projection $\bar{\gamma}$ of γ to M_{ρ_0} cannot return to any compact set in M_{ρ_0} in forward or reverse time, by definition of $\mathcal{D}$. Thus, each end of $\bar{\gamma}$ must exits the only end $[E]$ of M_{ρ_0}. In fact, the two ends of γ are necessarily asymptotic in $\mathbb{H}^3$. Indeed, by Lemma 3.9 every surface in the sequence $\{X_n\}$ of simplicial hyperbolic surfaces exiting E has uniformly bounded diameter in M_{ρ_0}, because we assumed that M_{ρ_0} has bounded geometry. This means that $\pi^{-1}(X_n) \cap \mathcal{D}$ has uniformly bounded diameter. Since each X_n is separating in M_{ρ_0}, $\pi^{-1}(X_n) \cap \mathcal{D}$ separates $\mathcal{D}$. From this we conclude, that there is an N such that for all $n \geq N$, each end of γ must meet $\pi^{-1}(X_n) \cap \mathcal{D}$. Thus, each end of γ meets the same collection of uniformly bounded diameter subsets of $\mathbb{H}^3$ as they tend to infinity. But bi-infinite geodesics have distinct endpoints at infinity. We have reached a contradiction, and we conclude that for each k, there is a t_k large enough such that no tetrahedron in the Z_k that we construct from $V_k^{t_k}$ via Proposition 4.2 is degenerate. This completes the proof of the lemma. ∎

Remark 4.4. One of the anonymous referees suggested that we could avoid the proof of Lemma 4.3 by instead appealing to [35, Lemma 2.5].

We would like to use our approximation scheme from Section 2.6 to transfer this information to our dense representation.

Proposition 4.5. Let G be a discrete group and fix $x \in \mathbb{H}^3 \cup \partial\mathbb{H}^3$. If $\rho : G \to \mathrm{PSL}_2\mathbb{C}$ is dense, then for every $\epsilon > 0$, there is a sequence of chains $D_k \in \mathrm{C}_3(G;\mathbb{R})$ satisfying

(I) $\frac{|\rho^*\mathrm{vol}_3^x(D_k)|}{\|D_k\|_1} > v_3 - \epsilon$, for all k, and

(II) $\lim_{k\to\infty} \frac{\|\partial D_k\|_1}{\|D_k\|_1} = 0$.

Proof. Apply Proposition 4.2 to obtain $Z_k \in \mathrm{C}_3(F_d;\mathbb{R})$ that satisfy the conclusions (1), (2), and (3). For each k, we can now apply Proposition 2.8 to obtain $Z_k(1) \in \mathrm{C}_3(G;\mathbb{R})$ such that

$$|\rho^*\mathrm{vol}_3^x(Z_k(1)) - \rho_0^*\mathrm{vol}_3^x(Z_k)| < 1,$$

$\|Z_k(1)\|_1 \le \|Z_k\|$, and $\|\partial Z_k(1)\|_1 \le \|\partial Z_k\|_1 \le K_d$. Note that $\|Z_k(1)\|$ tends to ∞, because $\|Z_k\|$ does; see Remark 2.9. By property (1) and the above approximation, we have

$$\frac{|\rho^*\mathrm{vol}_3^x(Z_k(1))|}{\|Z_k(1)\|_1} \ge \frac{|\rho_0^*\mathrm{vol}_3^x(Z_k)|}{\|Z_k\|} - \frac{1}{\|Z_k(1)\|} > v_3 - \epsilon,$$

for large enough k, because $\|Z_k(1)\|_1$ tends to ∞. Since $\|\partial Z_k(1)\|_1$ stays bounded, $\lim_{k\to\infty} \frac{\|\partial Z_k(1)\|_1}{\|Z_k(1)\|_1} = 0$. Take $D_k = Z_k(1)$. ■

We can now prove the first part of Theorem 1.1.

Theorem 4.6. If G is a discrete group and $\rho : G \to \mathrm{PSL}_2\mathbb{C}$ is dense, then $[\rho^*\mathrm{vol}_3] \neq 0 \in \mathrm{H}_{\mathrm{b}}^3(G;\mathbb{R})$ and $\|[\rho^*\mathrm{vol}_3]\|_\infty = v_3$.

Proof. The chains D_k from Proposition 4.5 satisfy the hypotheses of Lemma 2.6, so that $\|[\rho^*\mathrm{vol}_3]\|_\infty \ge v_3 - \epsilon$. But $\epsilon > 0$ was arbitrary, so $\|[\rho^*\mathrm{vol}_3]\|_\infty \ge v_3$. On the other hand, $\|[\rho^*\mathrm{vol}_3]\|_\infty \le \|[\mathrm{vol}_3]\|_\infty = v_3$. ■

4.2 Separation of volume classes in semi-norm

For a finitely generated Kleinian group, the Covering Theorem 3.8 suggests that 'most' infinite index subgroups are geometrically finite. We know that the volume classes for geometrically finite classes are trivial by Soma's Theorem 2.1. The following technical lemma makes repeated use of the Tameness Theorem 3.3 and the Covering Theorem 3.8.

Let X be a space and $p : \hat{X} \to X$ be a covering space. A map $\hat{f} : \hat{Y} \to \hat{X}$ is an *elevation* of a map $f : Y \to X$ if $\hat{Y}$ is connected and $q : \hat{Y} \to Y$ is a minimal covering such that $p \circ \hat{f} = f \circ q$. We sometimes identify an elevation with its image in $\hat{X}$.

Lemma 4.7. Suppose $\rho : F_2 \to \mathrm{PSL}_2\mathbb{C}$ is discrete and faithful. There is a finite index subgroup $H \le F_2$ such that for any $\hat{H} \le H$, isomorphic to a free group of rank 2, $\rho \circ i$ is geometrically finite with infinite co-volume, where $i : \hat{H} \to F_2$ denotes inclusion.

Proof. By Corollary 3.5, there is a standard compact core K_ρ of M_ρ and a homeomorphism $f : \mathcal{H}_2 \to K_\rho$ inducing ρ on fundamental groups. The collection of core curves $\nu = \{\nu_1, \dots, \nu_m\} \subset \partial\mathcal{H}_2$ of $f^{-1}(P_\rho^\circ)$ consists of at most 3 homotopically essential distinct disjoint simple closed curves. The inclusion $\partial\mathcal{H}_2 \to \mathcal{H}_2$ induces a surjection $\iota : \pi_1(\partial\mathcal{H}_2) \to \pi_1(\mathcal{H}_2) = F_2$. We abuse notation and write $\langle\nu\rangle \le \pi_1(\partial\mathcal{H}_2)$ to denote the conjugacy classes of the cyclic subgroups corresponding to the components of ν. The F_2-conjugacy classes of $\iota(\langle\nu\rangle)$ are non-trivial and pairwise distinct, as they correspond to the distinct parabolic cusps of M_ρ.

Note that if P_ρ° has *three* components, then ν is a pants decomposition of $\partial\mathcal{H}_2$, hence ρ is maximally cusped hence every relative end is geometrically finite. Therefore, since M_ρ has no geometrically infinite relative ends, the Covering Theorem 3.8 guarantees that ρ restricts to a geometrically finite representation on *any* subgroup $\hat{H} \le F_2$.

Now, we assume that M_ρ does have at least one geometrically infinite relative end. If P_ρ° is empty, then M_ρ has exactly one geometrically infinite end $E = M_\rho \setminus K_\rho$. We claim that for any proper subgroup $\hat{H} \le F_2$ of rank 2, $\rho \circ i : \hat{H} \to \mathrm{PSL}_2\mathbb{C}$ is geometrically finite. As above, $K_{\rho\circ i}$ is homeomorphic to a closed handlebody $\hat{\mathcal{H}}_2$ of genus 2 with boundary $\partial\hat{\mathcal{H}}_2$. If $M_{\rho\circ i}$ does have a geometrically infinite relative end $[\hat{E}]$, the Covering Theorem 3.8 supplies us with a finite sheeted cover $\hat{E} \to E$ that defines a finite sheeted cover $\hat{Y} \to \partial\mathcal{H}_2$, where $\hat{Y} \subset \partial\hat{\mathcal{H}}_2$ is a homotopically essential subsurface. The only possibility is that $\hat{Y} = \partial\hat{\mathcal{H}}_2$, and that $\hat{Y} \to \partial\mathcal{H}_2$ is degree one, hence $\hat{H}$ maps onto F_2. This is a contradiction to the assumption that $\hat{H}$ is a proper subgroup of F_2, and so $\rho \circ i$ is geometrically finite. Thus, we may take H to be any proper, finite index subgroup of F_2, so that any rank 2 free subgroup $\hat{H} \leq H$ has infinite index in H, by Euler characteristic considerations. In this case, $\rho \circ i$ is geometrically finite, as desired.

The remaining case to consider is that P_ρ° has either 1 or 2 components. For a based loop $\alpha \in \pi_1(\partial\mathcal{H}_2)$ representing a component of ν, write $\iota(\alpha) = w_\alpha \in F_2$. Since free groups are residually finite [53], there is a finite index normal subgroup $H \triangleleft F_2$ such that $\{w_\alpha : \alpha \subset \nu\} \subset F_2 \setminus H$. Since H is normal, no conjugate of w_α lies in H. In terms of our notation from before, $\iota(\langle\nu\rangle) \subset F_2 \setminus H$. The covering $M_{\rho|_H} \to M_\rho$ has finite degree $[F_2 : H] < \infty$ and $M_{\rho|_H}$ is homeomorphic to the interior of a closed handlebody $\mathcal{H}_k$ of genus $k = [F_2 : H] + 1$. Moreover, $M_{\rho|_H} \to M_\rho$ extends to a covering $\mathcal{H}_k \to \mathcal{H}_2$ of closed handlebodies restricting to a finite cover $\partial\mathcal{H}_k \to \partial\mathcal{H}_2$; the geometrically infinite relative ends of $M_{\rho|_H}$ are elevations of the geometrically infinite relative ends of M_ρ.

By construction of H, no conjugate of α lifts to $\partial\mathcal{H}_k$, so any elevation $\tilde{\nu} \subset p^{-1}(\nu)$ covers a component of ν with degree at least two. Thus, if Y is any component of $\partial\mathcal{H}_2 \setminus \nu$, and $\tilde{Y} \subset p^{-1}(Y)$ is an elevation, then p restricts to a cover $\tilde{Y} \to Y$ and further restricts to a cover $\partial\tilde{Y} \to \partial Y$ of degree at least two on every component of $\partial\tilde{Y}$. If Y is a one-holed torus or three-holed sphere, and $\tilde{Y} \to Y$ is a degree 2 covering, then $\tilde{Y}$ can only be a four-holed sphere or a torus with two holes. In either case, there is a boundary component of $\tilde{Y}$ which maps homeomorphically onto a boundary component of Y. Thus, our construction requires that the degree of $\tilde{Y} \to Y$ must be at least three; equivalently, $|\chi(\tilde{Y})| \geq 3$. If Y is a two-holed torus or four-holed sphere, then $|\chi(\tilde{Y})| \geq 2|\chi(Y)| \geq 4$. In particular, $|\chi(\tilde{Y})| \geq 3$, for any component $\tilde{Y} \subset \partial\mathcal{H}_k \setminus p^{-1}(\nu)$.

Finally, if $\hat{H} \leq H$ is free of rank 2, then $\hat{H}$ is a proper subgroup of H. We apply Corollary 3.5 once more to see that $M_{\rho\circ i}$ is the interior of a closed genus 2 handlebody $\hat{\mathcal{H}}_2$. If $M_{\rho\circ i}$ has a geometrically infinite relative end $\hat{E}$, then $\hat{E} \cong \hat{Y} \times [0, \infty)$, where $\hat{Y}$ is an essential subsurface of a genus 2 surface. In particular, either $\hat{Y}$ is closed or $|\chi(\hat{Y})| \leq 2$. By Euler characteristic considerations, $\hat{Y}$ cannot cover any component $\tilde{Y} \subset \partial\mathcal{H}_k \setminus p^{-1}(\nu)$. Therefore, by the Covering Theorem 3.8, $M_{\rho\circ i}$ has no geometrically infinite ends. This completes the proof of the lemma. ■

We would like to showcase the utility of Lemma 4.7 by finding rank 2 free subgroups of F_2 on which one representation is dense and another is geometrically finite or geometrically elementary. We are thankful to one of the anonymous referees for pointing out a useful observation.

Observation 4.8. Let $\mathcal{G}$ be a connected topological group, $G \leq \mathcal{G}$ be a dense subgroup, and suppose $H \leq G$ has finite index. Then, H is dense in $\mathcal{G}$.

Proof. Since $\overline{H}$ has finite index in $\overline{G} = \mathcal{G}$, the left cosets $\{g\overline{H} : g \in \mathcal{G}\}$ form a finite partition of $\mathcal{G}$, so that $\overline{H}$ is open, being the complement of a finite union of closed sets. Then $\overline{H}$ is an open, closed, and non-empty subset of a connected space $\mathcal{G}$; thus, $\overline{H} = \mathcal{G}$. ■

Proposition 4.9. Let $\rho : F_2 \to \mathrm{PSL}_2\mathbb{C}$ be dense and faithful and suppose $\rho_0 : F_2 \to \mathrm{PSL}_2\mathbb{C}$ is discrete. Then, there is a free rank 2 subgroup $i : \hat{H} \to F_2$ such that $\rho \circ i$ is a dense representation and $\rho_0 \circ i$ is either geometrically elementary or discrete, faithful, and geometrically finite. In particular,

$$\|[(\rho \circ i)^* \mathrm{vol}_3]\|_\infty = v_3 \text{ and} [(\rho_0 \circ i)^* \mathrm{vol}_3] = 0 \in \mathrm{H}^3_\mathrm{b}(\hat{H}; \mathbb{R}).$$

Proof. We break the proof into two cases:

Case 1. Assume that ρ_0 is faithful. By Lemma 4.7, there is a finite index subgroup $H < F_2$ such that for any subgroup $\hat{H} \leq H$, which is free of rank 2, $\rho_0 \circ i$ is geometrically finite, where $i : \hat{H} \to H \to F_2$ is inclusion. By Observation 1, $\overline{\rho(H)} = \mathrm{PSL}_2\mathbb{C}$. According to [10, Theorem 1.1], there is a free subgroup $\hat{H} \leq H$ of rank 2 such that $\overline{\rho(\hat{H})} = \mathrm{PSL}_2\mathbb{C}$. Applying Lemma 4.7, $\rho_0 \circ i$ is geometrically finite. Thus $[(\rho_0 \circ i)^* \mathrm{vol}_3] = 0$, by Soma's Theorem 2.1, while Theorem 4.6 gives $\|[(\rho \circ i)^* \mathrm{vol}_3]\|_\infty = v_3$.

Case 2. Assume that ρ_0 is not faithful, and let $K = \ker \rho_0$. Then $K \triangleleft F_2$ is a free group of rank at least 2 (perhaps infinite rank). Since ρ is faithful, $\rho(K)$ is not virtually abelian. It follows, e.g. from [45, Lemma 2.3], that either there is an $a \in K$ such that $\rho(a)$ is loxodromic or $\rho(K)$ fixes a point in $\mathbb{H}^3$.

For sake of contradiction, suppose $\rho(K)$ fixes a point in $\mathbb{H}^3$. Since K is normal in F_2, the set of points fixed by $\rho(K)$ is a non-empty $\rho(F_2)$-invariant subset of $\mathbb{H}^3$. But ρ is dense, so the closure is all of $\mathbb{H}^3$. This can only happen if $\rho(K) = \{1\}$. However, ρ is faithful and K is non-trivial, which is a contradiction. Thus $\rho(K)$ does not have a global fixed point.

A loxodromic element $\gamma \in \mathrm{PSL}_2\mathbb{C}$ stabilizes a geodesic $\mathrm{axis}(\gamma) \subset \mathbb{H}^3$, and acts as a 'screw motion' of complex length $\tau(\gamma) = \ell + i\theta$ along $\mathrm{axis}(\gamma)$. That is, $\ell \in \mathbb{R}$ is the translation length of γ, which is realized along $\mathrm{axis}(\gamma)$, and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ represents rotation around $\mathrm{axis}(\gamma)$.

We have an $a \in K$ such that $\rho(a)$ is loxodromic. Let $y \in \mathbb{H}^3$ be a point on $\mathrm{axis}(\rho(a)) \subset \mathbb{H}^3$, and let $\epsilon > 0$ be much smaller than the *three*-dimensional Margulis constant μ_3. Using density of ρ, we can find $b \in F_2$ such that

- $\rho(b)$ is loxodromic,
- $|\tau(\rho(a)) - \tau(\rho(b))| < \epsilon$,

- $\tau(\rho(b))$ is not real,
- the attracting and repelling fixed points of $\rho(a)$ and $\rho(b)$ at infinity are pairwise distinct, and
- $\mathrm{axis}(\rho(b))$ is within distance $\epsilon/10$ of $\mathrm{axis}(\rho(a))$ in a $2|\tau(\rho(a))|$-neighborhood of y.

Let $\hat{H} = \langle a, b\rangle$; our goal is to show that $\rho(\hat{H})$ is dense in $\mathrm{PSL}_2\mathbb{C}$. Taking $z = \rho(a).y$, the above conditions imply that $\rho(ab^{-1}).z$ and $\rho(ab^{-2}a).z$ are both closer than μ_3 to z. Let $H' = \langle ab^{-1}, ab^{-2}a\rangle$; by the Margulis Lemma 3.1, $\rho(H') \leq \mathrm{PSL}_2\mathbb{C}$ is either indiscrete or virtually abelian. Since ρ is faithful and H' is free, $\rho(H')$ is free, hence not virtually abelian. So $\overline{\rho(H')}$ is a closed Lie subgroup of $\mathrm{PSL}_2\mathbb{C}$ with positive dimension that is not virtually abelian, and since $\hat{H} \geq H'$, $\overline{\rho(\hat{H})}$ has the same property. Moreover, the endpoints of the axes of $\rho(a)$ and $\rho(b)$ at infinity are pairwise distinct, so $\overline{\rho(\hat{H})}$ does not stabilize an ideal point, and $\overline{\rho(\hat{H})}$ is not conjugate into $\mathrm{PSL}_2\mathbb{R}$, since the complex translation length of $\rho(b)$ is not real. By Lemma 2.2, $\rho|_{\hat{H}}$ is dense.

By construction, $\rho_0(\hat{H})$ is cyclic, hence geometrically elementary. We conclude as in Case 1, supplementing Soma's Theorem 2.1 with the proof of Lemma 2.2 in case $\rho_0(\hat{H})$ has torsion, say. ■

Remark 4.10. The argument given in Case 2 also proves the proposition in Case 1; we will use the argument from Case 1 in the next section. Proposition 4.9 is an improvement of a result in a previous draft of this manuscript, and strengthens our main theorems from that version.

Our first main theorem now follows quite easily. We only need to observe that restrictions to subgroups induce semi-norm non-increasing maps in bounded cohomology. The first step in the proof of the following is a reduction; we use [10, Theorem 1.1] to find a free subgroup of rank 2 of H, densely embedding into $\mathrm{PSL}_2\mathbb{C}$ via ρ.

Theorem 4.11. *Suppose $\rho : G \to \mathrm{PSL}_2\mathbb{C}$ is a dense representation of a discrete group G. If $\rho_0 : G \to \mathrm{PSL}_2\mathbb{C}$ is any other representation and there is a subgroup $H \leq G$ such that $\overline{\rho(H)} = \mathrm{PSL}_2\mathbb{C}$, but ρ_0 is geometrically elementary or discrete restricted to H, then*

$$\|[\rho^*\mathrm{vol}_3] - [\rho_0^*\mathrm{vol}_3]\|_\infty \geq v_3.$$

Proof. By [10, Theorem 1.1], there is a free subgroup $F_2 \leq H$ such that ρ is dense and faithful on F_2. Moreover, ρ_0 is still geometrically elementary or discrete on $F_2 \leq H$, because these properties pass to subgroups, as we have seen. If ρ_0 is geometrically elementary, the volume class of ρ_0 vanishes when restricted to this F_2, by the proof of Lemma 2.2, so we set $\hat{H} = F_2$ and let $i : \hat{H} \to F_2 \to G$ denote inclusion. Otherwise, we apply Proposition 4.9 to obtain a free rank 2 subgroup $i : \hat{H} \to F_2 \to G$ where $\|[(\rho \circ i)^*\mathrm{vol}_3]\|_\infty = v_3$ and $[(\rho_0 \circ i)^*\mathrm{vol}_3] = 0$.

Since $i^* : \mathrm{H}^\bullet_\mathrm{b}(G; \mathbb{R}) \to \mathrm{H}^\bullet_\mathrm{b}(\hat{H}; \mathbb{R})$ is norm non-increasing, we have

$$\begin{aligned} v_3 &= \|[(\rho \circ i)^*\mathrm{vol}_3]\|_\infty \\ &= \|[(\rho \circ i)^*\mathrm{vol}_3] - [(\rho_0 \circ i)^*\mathrm{vol}_3]\|_\infty \\ &= \|[i^*([\rho^*\mathrm{vol}_3] - [\rho_0^*\mathrm{vol}_3])\|_\infty \\ &\leq \|[\rho^*\mathrm{vol}_3] - [\rho_0^*\mathrm{vol}_3]\|_\infty. \end{aligned}$$

This is what we wanted to show. ■

If we are more careful, we can obtain the following generalization without too much extra work.

Theorem 4.12. Suppose $\{\rho_i : G \to \mathrm{PSL}_2\mathbb{C}\}_{i=1}^N$ is a collection of dense representations of a discrete group G such that for every $i = 1, 2, \ldots, N$ there is a subgroup $\iota_i : H_i \hookrightarrow G$ such that $\overline{\rho_i(H_i)} = \mathrm{PSL}_2\mathbb{C}$, and $\rho_i \circ \iota_j : H_j \to \mathrm{PSL}_2\mathbb{C}$ is geometrically elementary or discrete for $i \neq j$. Then for any $a_1, \ldots, a_N \in \mathbb{R}$, we have

$$\left\| \sum_{i=1}^N a_i [\rho_i^* \mathrm{vol}_3] \right\|_\infty \geq \max\{|a_i|\} \cdot v_3.$$

Consequently, $\{[\rho_i^*\mathrm{vol}_3]\} \subset \overline{\mathrm{H}}^3_\mathrm{b}(G; \mathbb{R})$ is a linearly independent discrete set.

Proof. For convenience, we assume that $|a_1| = \max\{|a_i|\}$. First of all, by [10] we may assume that $H_1 \cong F_2$ and that $\rho_1|_{H_1}$ is faithful. We inductively define a nested family of rank 2 subgroups

$$H_1^{(N)} \leq \ldots \leq H_1^{(2)} \leq H_1^{(1)} = H_1$$

by successively applying Proposition 4.9. The result is that $\rho_1|_{H_1^{(i)}}$ is dense and faithful for all i. Additionally, for each $j \geq 2$, denote inclusion by $\iota_1^{(j)} : H_1^{(j)} \to G$ so that for

$i \in \{2, \ldots, N\}$,

$$[(\rho_i \circ \iota_1^{(j)})^* \mathrm{vol}_3] = 0 \in \mathrm{H}^3_{\mathrm{b}}(H_1^{(j)}; \mathbb{R}).$$

As in Theorem 4.11, the operator $\iota_1^{(N)*} : \mathrm{H}^3_{\mathrm{b}}(H_1^{(N)}; \mathbb{R}) \to \mathrm{H}^3_{\mathrm{b}}(G; \mathbb{R})$ is norm non-increasing. Applying Theorem 4.6, we see

$$\begin{aligned}
|a_1| \cdot v_3 &= \left\| a_1[(\rho_1 \circ \iota_1^{(N)})^* \mathrm{vol}_3] \right\|_\infty \\
&= \left\| [a_1(\rho_1 \circ \iota_1^{(N)})^* \mathrm{vol}_3] + \sum_{i=2}^{N} a_i[(\rho_i \circ \iota_1^{(N)})^* \mathrm{vol}_3] \right\|_\infty \\
&= \left\| \iota_1^{(N)*} \left(\sum_{i=1}^{N} a_i[\rho_i^* \mathrm{vol}_3] \right) \right\|_\infty \\
&\leq \left\| \sum_{i=1}^{N} a_i[\rho_i^* \mathrm{vol}_3] \right\|_\infty ,
\end{aligned}$$

as promised. ■

5 Borel classes of dense representations

In this section, we show that pullbacks of the Borel class under dense representations have maximal semi-norm. Since the structure of discrete subgroups $\Gamma \leq \mathrm{PSL}_n\mathbb{C}$ is not well understood, we cannot give a simple criterion for the differences of pullbacks of Borel classes to be separated in semi-norm for arbitrary representations. Recall that there is a unique conjugacy class of irreducible representations $\iota_n : \mathrm{PSL}_2\mathbb{C} \to \mathrm{PSL}_n\mathbb{C}$. We will work with a dense representation $\rho : G \to \mathrm{PSL}_n\mathbb{C}$ and another representation $\rho_0' : G \to \mathrm{PSL}_n\mathbb{C}$ that factors through $\mathrm{PSL}_2\mathbb{C}$ via ι_n. We can then use tools developed in previous sections and 3three-dimensional hyperbolic geometry to give some criteria for which pullbacks to G of the bounded Borel class are separated in semi-norm.

It would be interesting to investigate whether maximality of the semi-norm of the pullback of the Borel class under discrete and faithful representations $\rho : \pi_1(S) \to \mathrm{PSL}_n\mathbb{C}$ can serve as a profitable definition for 'geometrical infiniteness,' where S is an orientable surface of finite type. In fact, this paper grew out of an attempt to initiate such an investigation.

Theorem 5.1. Let G be a discrete group and $\rho : G \to \mathrm{PSL}_n\mathbb{C}$ be dense. Then,

$$\|\rho^*\beta_n\|_\infty = v_3 \frac{n(n^2-1)}{6}.$$

The following will be immediate from the proof of Theorem 5.1 and Theorem 4.11.

Corollary 5.2. Let G be a discrete group and $\rho : G \to \mathrm{PSL}_n\mathbb{C}$ be dense. Suppose also that $\rho_0 : G \to \mathrm{PSL}_2\mathbb{C}$ is such that there exists a subgroup $H \leq G$ such that $\overline{\rho(H)} = \mathrm{PSL}_n\mathbb{C}$ and ρ_0 is geometrically elementary or discrete and faithful restricted to H. Then,

$$\|\rho^*\beta_n - (\iota_k \circ \rho_0)^*\beta_k\|_\infty \geq v_3 \frac{n(n^2-1)}{6},$$

for all $k \geq 2$.

Proof. We apply [10, Theorem 1.1] to obtain $F_2 \leq H$ such that $\rho|_{F_2}$ is faithful and dense. The argument in Proposition 4.9 (Case 1) provides us with a rank 2 subgroup $i : \hat{H} \to F_2$ such that $\rho \circ i$ is dense and faithful, while $[(\rho_0 \circ i)^*\mathrm{vol}_3] = 0$. Then, $(\iota_k \circ \rho_0 \circ i)^*\beta_k = 0$ for all $k \geq 2$, by Theorem 2.5, but $\|(\rho \circ i)^*\beta_n\|_\infty = v_3 \frac{n(n^2-1)}{6}$ by Theorem 5.1. Since i^* is semi-norm non-increasing on bounded cohomology, the corollary follows (as in the proof of Theorem 4.11). ■

Note that in the previous section, we obtain stronger results; in Corollary 5.2, we have made the additional assumption that ρ_0 is faithful, in addition to being discrete on H. This is because the structure of positive dimensional Lie subgroups of $\mathrm{PSL}_2\mathbb{C}$ is particularly simple (see the proof of Case 2, Proposition 4.9). We believe that with more work, it should be possible to upgrade the results in this section.

Remark 5.3. Another easy consequence of Theorem 5.1 and the triangle inequality is that if $\rho_1 : G \to \mathrm{PSL}_n\mathbb{C}$ and $\rho_2 : G \to \mathrm{PSL}_k\mathbb{C}$ are both dense, then

$$\|\rho_1^*\beta_n - \rho_2^*\beta_k\|_\infty \geq v_3 \left| \frac{n(n^2-1)}{6} - \frac{k(k^2-1)}{6} \right|.$$

We begin proving Theorem 5.1 by finding a sequence of efficient chains on which the Borel class evaluates to a large number with controlled boundary. These chains come from hyperbolic geometry, together with the explicit description of the cocycle B_n^F given by [6] (see Proposition 2.3) and the explicit formula for the semi-norm of the pullback under the irreducible representation $\iota_n : \mathrm{PSL}_2\mathbb{C} \to \mathrm{PSL}_n\mathbb{C}$; see Theorem 2.5.

Lemma 5.4. If G is a discrete group and $\rho : G \to \mathrm{PSL}_n\mathbb{C}$ is dense, then for every $\epsilon > 0$, there is a $y \in \mathscr{F}(\mathbb{C}^n)$ and a sequence of chains $D_k \in \mathrm{C}_3(G;\mathbb{R})$ such that

1. $\frac{|\rho^* B_n^y(D_k)|}{\|D_k\|_1} > (v_3 - \epsilon)\frac{n(n^2-1)}{6}$, for all k, and
2. $\lim_{k\to\infty} \frac{\|\partial D_k\|_1}{\|D_k\|_1} = 0$.

Proof. As in the proof of Proposition 4.5, fix a geometrically infinite discrete and faithful representation $\rho_0 : F_2 \to \mathrm{PSL}_2\mathbb{C}$ with no parabolic elements, take $\epsilon > 0$ and apply Proposition 4.2. We now have $K_2 > 0$ and $Z_k \in \mathrm{C}_3(F_2;\mathbb{R})$ that satisfy conditions (1), (2), and (3) of the conclusion of Proposition 4.2. By Lemma 4.3, we may assume that $x \in \partial\mathbb{H}^3$ for the conclusion (1), and no ideal simplex in $(\rho_{0*}Z_k).x$ is degenerate. Recall that the irreducible representation ι_n induces an equivariant continuous map of boundaries $\hat{\iota}_n : \partial\mathbb{H}^3 \to \mathscr{F}(\mathbb{C}^n)$. Take $y = \hat{\iota}_n(x)$, so that by equivariance of $\hat{\iota}_n$, we have $\hat{\iota}_n(\rho_{0*}(Z_k).x) = (\iota_n \circ \rho_0)_*(Z_k).y$. By Proposition 2.3, $\iota_n^* B_n^y = \frac{n(n^2-1)}{6}\mathrm{vol}_3^x$. Thus,

$$(\iota_n \circ \rho_0)^* B_n^y(Z_k) = \frac{n(n^2-1)}{6}\rho_0^*\mathrm{vol}_3^x(Z_k).$$

Since no simplex is degenerate, thanks to Lemma 4.3, we can apply Corollary 2.10 for each k to obtain $Z_k(1) \in \mathrm{C}_3(G;\mathbb{R})$ such that

$$|\rho^* B_n^y(Z_k(1)) - (\iota_n \circ \rho_0)^* B_n^y(Z_k)| < 1.$$

As in the proof of Proposition 4.5, the two conclusions now follow with $D_k = Z_k(1)$. ■

Proof of Theorem 5.1. The chains from Lemma 5.4 satisfy the hypotheses of Lemma 2.6. Thus, $\|\rho^*\beta_n\|_\infty \geq (v_3 - \epsilon)\frac{n(n^2-1)}{6}$, for all $\epsilon > 0$. By Theorem 2.5, $\|\rho^*\beta_n\|_\infty \leq v_3\frac{n(n^2-1)}{6}$, which yields the desired equality. ■

6 Constructions, examples, and questions

We now give some applications of the work that we have done to show that subspaces of bounded cohomology spanned by the pullback of volume classes can be quite large. We also pose some questions.

6.1 Constructing incompatible representations

We will construct a family of representations $\{\rho_\theta : F_2 \to \mathrm{PSL}_2\mathbb{C} : \theta \in \Lambda\}$ such that, given any finite subset $\{\rho_{\theta_1}, ..., \rho_{\theta_N}\}$, there are subgroups $H_i \leq F_2$ such that $\overline{\rho_{\theta_i}(H_i)} = \mathrm{PSL}_2\mathbb{C}$, but $\rho_{\theta_i}|_{H_j}$ is discrete, faithful, and convex co-compact, i.e. $\rho_{\theta_i}|_{H_j}$ is a *marked Schottky group* group for $i \neq j$. Marked Schottky groups are, in particular, geometrically finite, so we can apply Theorem 4.12 to show that the volume classes of these representations are linearly independent in reduced bounded cohomology. Furthermore, the set Λ has cardinality that of the continuum. The construction of dense representations in this section shares some features with the construction found in [26,Appendix A].

We start with a loxodromic element $a \in \mathrm{PSL}_2\mathbb{C}$ such that the complex translation length $\tau(a)$ of a has translational part r and non-zero rotational part strictly between 0 and $\pi/8$. In particular, a is not conjugate into $\mathrm{PSL}_2\mathbb{R}$. Find an elliptic element $b(\theta)$ with rotation angle $2\pi\theta$, where $\theta \in \mathbb{R}/\mathbb{Z}$ is irrational and with fixed line meeting axis(a) orthogonally in a point x. For infinitely many values n, axis$(b(\theta)^n ab(\theta)^{-n})$ makes a very small angle with axis(a) at x.

Lemma 6.1. Let $r > 0$ be given. There is a threshold $\tau_0 \in (0, 1/2)$ such that if $n\theta \in (-\tau_0, \tau_0) \mod 1$, then $\langle a, b(\theta)^n ab(\theta)^{-n}\rangle$ is dense in $\mathrm{PSL}_2\mathbb{C}$.

Proof. For any $\tau \in (0, 1)$, axis$(b(\tau)ab(\tau)^{-1}) = b(\tau)$axis$(a)$, and axis$(a) \cap b(\tau)$axis$(a) = \{x\}$, since the fixed line of $b(\tau)$ meets axis(a) at x. Using hyperbolic trigonometry, there is a $\tau_0 = \tau_0(r)$ such that if $-\tau_0 < \tau < \tau_0 \mod 1$, then in an $(r + 1)$-neighborhood of x, the Hausdorff distance between axis(a) and $b(\tau)$axis(a) is at most μ_3. Specifically, take $\tau_0 \in (0, 1)$ such that $2 \sinh^{-1}(\sin(2\pi\tau_0) \sinh(r + 1)) < \mu_3$. Suppose $-\tau_0 < \tau < \tau_0 \mod 1$, and for convenience, write b for $b(\tau)$.

Then $d(ax, bab^{-1}x) < \mu_3$, because $d(x, ax) = d(x, bab^{-1}x) = r < r + 1$. This means that $d(ba^{-1}b^{-1}ax, x) < \mu_3$. Similarly, $d(bab^{-1}a^{-1}x, x) < \mu_3$. Set $c = bab^{-1}$; by the Margulis Lemma 3.1, the group $\langle c^{-1}a, ca^{-1}\rangle$ is indiscrete if it is not virtually abelian. We will show that $\langle c^{-1}a, ca^{-1}\rangle$ contains a free subgroup, so it is not virtually abelian.

An element $\gamma \in \mathrm{PSL}_2\mathbb{C}$ is loxodromic if and only if the trace of a lift of γ to $\mathrm{SL}_2\mathbb{C}$ is not in the interval $[-2, 2]$. Trace identities give

$$\mathrm{tr}(c^{-1}a) = \mathrm{tr}(ac^{-1}) = \mathrm{tr}(ca^{-1}).$$

By construction, a and c can be represented by matrices

$$a = \begin{pmatrix} e^{\tau(a)/2} & 0 \\ 0 & e^{-\tau(a)/2} \end{pmatrix} \text{ and } c = \begin{pmatrix} \cos(2\pi\tau) & -\sin(2\pi\tau) \\ \sin(2\pi\tau) & \cos(2\pi\tau) \end{pmatrix} \begin{pmatrix} e^{\tau(a)/2} & 0 \\ 0 & e^{-\tau(a)/2} \end{pmatrix}$$
$$\begin{pmatrix} \cos(2\pi\tau) & \sin(2\pi\tau) \\ -\sin(2\pi\tau) & \cos(2\pi\tau) \end{pmatrix}.$$

Explicit computation with matrices shows that as long as the rotational part of $\tau(a)$ is not an integer multiple of π and b is not rotation by 0 or π (i.e. $\tau \neq 0, 1/2$), then the imaginary part of $\mathrm{tr}(c^{-1}a) = \mathrm{tr}(ca^{-1})$ is different from 0. Thus, our assumptions guarantee that both $c^{-1}a$ and ca^{-1} are loxodromic. The traces are equal, so the complex translation lengths are equal. This means that if $\mathrm{axis}(c^{-1}a) = \mathrm{axis}(ca^{-1})$, then $c^{-1}a$ and ca^{-1} are either equal or inverse to one another. Inspection shows that $c^{-1}a.x \neq ca^{-1}.x$, if the rotational part of $\tau(a)$ is small enough (less than $\pi/8$, for example). If $c^{-1}a = (ca^{-1})^{-1}$, then c commutes with a, which only happens if b is rotation by 0 or π.

So, the axes of $c^{-1}a$ and ca^{-1} are different. If the axes are asymptotic, then again since $\tau(c^{-1}a) = \tau(ca^{-1})$, the product $c^{-1}a(ca^{-1})^{-1} = c^{-1}a^2c^{-1}$ stabilizes a horosphere, hence has trace equal to ± 2. Another explicit computation shows that the imaginary part of the $\mathrm{tr}(c^{-1}a^2c^{-1})$ vanishes only when b is rotation by 0 or π or the rotational part of $\tau(a)$ is an integer multiple of $\pi/8$. We have assumed that the rotational part of $\tau(a)$ is non-zero and less than $\pi/8$, so the set of fixed points of $c^{-1}a$ and ca^{-1} at infinity are distinct. By the Ping-Pong Lemma, for large enough k, $\langle (c^{-1}a)^k, (ca^{-1})^k \rangle$ is a Schottky group. In particular, $\langle c^{-1}a, ca^{-1} \rangle$ contains a free subgroup, and so it is indiscrete. Thus, $\langle a, bab^{-1} \rangle$ is indiscrete, because $\langle c^{-1}a, ca^{-1} \rangle \leq \langle a, bab^{-1} \rangle$.

Finally, $\langle a, bab^{-1} \rangle$ does not fix an ideal point, nor does it stabilize a plane, since the rotational part of $\tau(a)$ is non-trivial. By the proof of Lemma 2.2, $\langle a, bab^{-1} \rangle$ is dense. ■

There are also infinitely many values of n such that $\mathrm{axis}(a)$ is nearly orthogonal to $b(\theta)^n \mathrm{axis}(a)$. The following is immediate from the Ping-Pong Lemma and another direct computation in $\mathbb{H}^2 \subset \mathbb{H}^3$.

Lemma 6.2. For $r > \frac{1+\cos(\pi/8)}{1-\cos(\pi/8)}$, if the translational part of $\tau(a)$ is at least r and $n\theta \in (1/8, 3/8) \mod 1$, then $\langle a, b(\theta)^n ab(\theta)^{-n} \rangle$ is a Schottky group of rank 2.

Now fix $a \in \mathrm{PSL}_2\mathbb{C}$ with translation length $r + it$, with $r > \frac{1+\cos(\pi/8)}{1-\cos(\pi/8)}$ and $t \in (0, \pi/8)$.

Lemma 6.3. Let $\{\theta_1, \ldots, \theta_N\} \subset (0,1)$ be a rationally independent set of irrational numbers, $F_2 = \langle z_1, z_2 \rangle$, and let $\rho_{\theta_i} : F_2 \to \mathrm{PSL}_2\mathbb{C}$ be defined by $\rho_{\theta_i}(z_1) = a$ and $\rho_{\theta_i}(z_2) = b(\theta_i)$. There are integers $n_1, \ldots, n_N$ such that $H_i := \langle z_1, z_2^{n_i} z_1 z_2^{-n_i} \rangle$ satisfies $\rho_{\theta_i}(H_i) \leq \mathrm{PSL}_2\mathbb{C}$ is dense but $\rho_{\theta_i}|_{H_j}$ is a marked Schottky group.

Proof. Since $\{\theta_i\}$ is a rationally independent set of irrational numbers, the self homeomorphism of the N-torus $\Theta : (\mathbb{R}/\mathbb{Z})^N \to (\mathbb{R}/\mathbb{Z})^N$ defined by $(x_1, \ldots, x_N) \mapsto (x_1 + \theta_1, \ldots, x_N + \theta_N)$ is topologically minimal, i.e. every orbit is dense. Let τ_0 be the threshold from Lemma 6.1, and let $D = (-\tau_0, \tau_0) \subset \mathbb{R}/\mathbb{Z}$ and $F = (1/8, 3/8) \subset \mathbb{R}/\mathbb{Z}$. For each i, let $p_i : (\mathbb{R}/\mathbb{Z})^N \to \mathbb{R}/\mathbb{Z}$ be projection onto the ith factor and take $U_i \subset (\mathbb{R}/\mathbb{Z})^N$ to be the product of D's and F's where $p_i(U_i) = D$ and $p_j(U_i) = F$, if $i \neq j$. By minimality of Θ, there is an n_i such that $\Theta^{n_i}(0) \in U_i$. By Lemma 6.1, $\overline{\rho_{\theta_i}(H_i)} = \mathrm{PSL}_2\mathbb{C}$, and $\rho_{\theta_i}|_{H_j} : H_j \to \mathrm{PSL}_2\mathbb{C}$ is discrete, faithful, and convex co-compact for $i \neq j$, by Lemma 6.2. ■

Using the axiom of choice, we can find a basis $\Lambda \sqcup \{1\}$ for $\mathbb{R}$ as a $\mathbb{Q}$-vector space. We may assume that $\Lambda \subset (0,1)$. For each $\theta \in \Lambda$ we have the representation $\rho_\theta : F_2 \to \mathrm{PSL}_2\mathbb{C}$ defined as in Lemma 6.3.

Theorem 6.4. The map

$$\Lambda \to \overline{\mathrm{H}}^3_{\mathrm{b}}(F_2)$$

$$\theta \mapsto [\rho_\theta^* \mathrm{vol}_3]$$

is injective with discrete image. Moreover, $\{[\rho_\theta^* \mathrm{vol}_3] : \theta \in \Lambda\}$ is a linearly independent set, and $\#\Lambda = \#\mathbb{R}$.

Proof. Any finite subset $\{\theta_1, \ldots, \theta_N\} \subset \Lambda$ is rationally independent. By Lemma 6.3, the collection $\{\rho_{\theta_i} : i = 1, \ldots, N\}$ satisfies the hypotheses of Theorem 4.12. This shows that $\{[\rho_\theta^* \mathrm{vol}_3] : \theta \in \Lambda\} \subset \overline{\mathrm{H}}^3_{\mathrm{b}}(F_2)$ is linearly independent and discrete. Injectivity follows from linear independence. ■

6.2 On the spaces $\overline{\mathrm{H}}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{R}; \mathbb{R})$ and $\overline{\mathrm{H}}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{C}; \mathbb{R})$

As a further application of Theorem 4.12, we will show that both $\overline{\mathrm{H}}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{R}; \mathbb{R})$ and $\overline{\mathrm{H}}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{C}; \mathbb{R})$ have dimension at least $\#\mathbb{R}$. We remind the reader that we are computing bounded cohomology of discrete groups, i.e. $\mathrm{PSL}_2\mathbb{R}$ and $\mathrm{PSL}_2\mathbb{C}$ are endowed with the discrete topology. The key is to use the axiom of choice to find 'wild' field automorphisms of $\mathbb{C}$ that induce embeddings $\mathrm{PSL}_2\mathbb{R} \hookrightarrow \mathrm{PSL}_2\mathbb{C}$ and $\mathrm{PSL}_2\mathbb{C} \hookrightarrow \mathrm{PSL}_2\mathbb{C}$; these embeddings would be highly discontinuous when these groups are considered with their standard smooth topologies.

Theorem 6.5. The real dimension of the degree three reduced bounded cohomology of $\mathrm{PSL}_2\mathbb{R}$ and $\mathrm{PSL}_2\mathbb{C}$ is at least $\#\mathbb{R}$. More specifically, there are dense representations $\{\rho_t : \mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_2\mathbb{C}\}_{t\in\mathbb{R}}$ such that $\{[\rho_t^*\mathrm{vol}_3] : t \in \mathbb{R}\}$ is a linearly independent set in $\overline{\mathrm{H}}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{R};\mathbb{R})$. The family $\{\rho_t\}$ are the restrictions of representations $\{\rho_t' : \mathrm{PSL}_2\mathbb{C} \to \mathrm{PSL}_2\mathbb{C}\}$, thus $\{[\rho_t'^*\mathrm{vol}_3] : t \in \mathbb{R}\}$ is a linearly independent set in $\overline{\mathrm{H}}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{C};\mathbb{R})$.

Proof. Let $\mathrm{Hom}(\mathbb{R},\mathbb{C})$ denote the set of injective homomorphisms of fields $\sigma : \mathbb{R} \to \mathbb{C}$. Every field mapping takes polynomial identities with rational coefficients to polynomial identities with rational coefficients, hence induces a group homomorphism $\rho_\sigma : \mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_2\mathbb{C}$. It is not hard to see that if $\sigma(\mathbb{R}) \subset \mathbb{R}$, then σ is order preserving, hence σ restricts to the identity mapping on $\mathbb{R}$, i.e. σ is trivial [58,Theorem 3]. In fact, if $\sigma \in \mathrm{Hom}(\mathbb{R},\mathbb{C})$ is not trivial, then $\sigma(\mathbb{R}\setminus\mathbb{Q})$ is dense in $\mathbb{C}$ [58,Theorem 4]. Hence $\rho_\sigma(\mathrm{PSL}_2\mathbb{R})$ is dense in $\mathrm{PSL}_2\mathbb{C}$ for non-trivial σ. By Theorem 4.6, $\|[\rho_\sigma^*\mathrm{vol}_3]\|_\infty = v_3$ for all non-trivial $\sigma \in \mathrm{Hom}(\mathbb{R},\mathbb{C})$.

Next, we will produce a family of non-trivial field maps $\sigma(t) \in \mathrm{Hom}(\mathbb{R},\mathbb{C})$ indexed by a set with cardinality that of the continuum, which we assume is $\mathbb{R}$ for simplicity. The field maps $\sigma(t)$ induce dense representations $\rho_t = \rho_{\sigma(t)} : \mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_2\mathbb{C}$. Finite subsets will satisfy the hypotheses of Theorem 4.12, but in fact any subset will satisfy the hypotheses of Theorem 4.12.

Let $\alpha,\beta \in \mathbb{C}$, and consider

$$x(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \text{ and} y(\beta) = \begin{pmatrix} (\beta+\beta^{-1})/2 & (\beta-\beta^{-1})/2 \\ (\beta-\beta^{-1})/2 & (\beta+\beta^{-1})/2 \end{pmatrix} \in \mathrm{PSL}_2\mathbb{C}.$$

We consider the group

$$H_{\alpha,\beta} = \langle x(\alpha), y(\beta) \rangle \le \mathrm{PSL}_2\mathbb{C}.$$

Geometrically, as long as $|\alpha|, |\beta| \neq 1$, then the two generators are loxodromic with axes that meet orthogonally in a point, hence are contained in a hyperbolic plane. If $|\alpha|$ and $|\beta|$ are large enough, by playing Ping-Pong, we see that $H_{\alpha,\beta}$ is Schottky, hence geometrically finite. If $|\alpha|, |\beta| < \mu_3$, then $H_{\alpha,\beta}$ is dense in $\mathrm{PSL}_2\mathbb{C}$ as long as at least one of α or β has argument not a multiple of $\pi/2$, essentially by an easier argument than found in the proof of Lemma 6.1.

Since $\mathbb{C}$ is algebraic over $\mathbb{Q}(\mathbb{R}) = \mathbb{R}$, there is a transcendence basis $T \subset \mathbb{R}$ for $\mathbb{C}$ over $\mathbb{Q}$ [44, Theorem 19.14]. Let $a, b \in \mathbb{C} \setminus \mathbb{R} \cup i\mathbb{R}$ be $\mathbb{Q}$-linearly independent, algebraic numbers with magnitude smaller than μ_3. By the Lindemann-Weierstrass Theorem [3,Theorem 1.4], $\{e^a, e^b\}$ is an algebraically independent set. Since $\mathbb{Q}(\{z \in \mathbb{C} : |z| > 100\}) = \mathbb{C}$ is algebraic over $\mathbb{C}$, we may find a transcendence basis T' for $\mathbb{C}$ over $\mathbb{Q}$ such that $\{e^a, e^b\} \subset T' \subset \{|z| > 100\} \cup \{e^a, e^b\}$ [44, Theorem 19.14]. Two transcendence bases for $\mathbb{C}$ over $\mathbb{Q}$ have the same cardinality, that of the continuum. The algebraic closures of $\mathbb{Q}(T)$ and $\mathbb{Q}(T')$ are both $\mathbb{C}$, so if $\pi : T \to T'$ is a bijection, then there is a field isomorphism $\sigma_\pi : \mathbb{C} \to \mathbb{C}$ extending π, by the Isomorphism Extension Theorem.

Find a partition $T = A \sqcup B$, with $\#A = \#B$ and bijections $\mathbb{R} \to A$ and $\mathbb{R} \to B$ denoted $t \mapsto \alpha_t$ and $t \mapsto \beta_t$, respectively. Now, for each real number t, we extend the assignment $\alpha_t \mapsto e^a$ and $\beta_t \mapsto e^b$ to a bijection $\pi(t) : T \to T'$. We thus induce a field isomorphism $\sigma(t) = \sigma_{\pi(t)} : \mathbb{C} \to \mathbb{C}$ that restricts to a field map $\mathbb{R} \to \mathbb{C}$ with the same name. We obtain, for each $t \in \mathbb{R}$, a homomorphism $\rho_t = \rho_{\sigma(t)} : \mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_2\mathbb{C}$ and also $\rho_t' : \mathrm{PSL}_2\mathbb{C} \to \mathrm{PSL}_2\mathbb{C}$.

By construction, $\rho_t(x(\alpha_t)) = x(e^a)$ and $\rho_t(y(\beta_t)) = y(e^b)$, so that $\rho_t(H_{\alpha_t,\beta_t}) = H_{e^a,e^b}$ is dense in $\mathrm{PSL}_2\mathbb{C}$, by our choice of a and b. For $s \neq t$, we see that the group $\rho_t(\langle x(\alpha_s), y(\beta_s)\rangle)$ is Schottky, because $\sigma_t(\alpha_s), \sigma_t(\beta_s) \in \{|z| > 100\}$. Thus *every* subset of representations of $\{\rho_t : \mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_2\mathbb{C} : t \in \mathbb{R}\}$ satisfies the hypotheses of Theorem 4.12. As in the proof of Theorem 6.4, $\{[\rho_t^*\mathrm{vol}_3] : t \in \mathbb{R}\}$ is a linearly independent subset of $\overline{\mathrm{H}}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{R}; \mathbb{R})$. ■

The cardinality of the set of functions from a continuum to itself is $2^{\#\mathbb{R}}$, as is the set of bounded functions. Thus, $\dim_{\mathbb{R}} \mathrm{H}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{K}; \mathbb{R}) \le 2^{\#\mathbb{R}}$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. The set of field mappings $\mathbb{R} \to \mathbb{C}$ or $\mathbb{C} \to \mathbb{C}$ has cardinality $2^{\#\mathbb{R}}$.

Question 6.6. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Is $\dim_{\mathbb{R}} \overline{\mathrm{H}}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{K}; \mathbb{R}) = 2^{\#\mathbb{R}}$? Assuming the axiom of choice, does the set of field embeddings $\mathbb{C} \to \mathbb{C}$ define an injective map with discrete and linearly independent image to $\mathrm{H}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{K}; \mathbb{R})$ by pulling back the standard volume class? Without the axiom of choice, what is the dimension of $\dim_{\mathbb{R}} \mathrm{H}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{K}; \mathbb{R})$? Do pullbacks of volume classes span all of $\mathrm{H}^3_{\mathrm{b}}(\mathrm{PSL}_2\mathbb{K}; \mathbb{R})$?

6.3 On the collection of ρ-dense subgroups

Let G be a discrete group and suppose $\rho : G \to \mathrm{PSL}_2\mathbb{R}$ is dense. Define the set $\mathcal{DS}(\rho)$ of *ρ-dense subgroups* by

$$\mathcal{DS}(\rho) = \{H \leq_{f.g.} G : \overline{\rho(H)} = \mathrm{PSL}_2\mathbb{R}\},$$

where $H \leq_{f.g.} G$ means that H is a finitely generated subgroup of G. The following fact may seem surprising, at first. An outline of the proof was communicated to the author by Yair Minsky.

Fact 6.7. Let $\rho_1, \rho_2 : F_2 \to \mathrm{PSL}_2\mathbb{R}$ be dense. If $\mathcal{DS}(\rho_1) = \mathcal{DS}(\rho_2)$, then ρ_1 is conjugate to ρ_2 in $\mathrm{PSL}_2\mathbb{R}$.

The proof of Fact 6.7 uses the fact that dense representations into $\mathrm{PSL}_2\mathbb{R}$ contain elliptic elements, since $(0, 2) \subset \mathbb{R}$ is open in the image of the absolute value of the trace function. Generically, dense representations $F_2 \to \mathrm{PSL}_2\mathbb{C}$ do not contain any elliptics. Due to the fractal nature of the boundary of Schottky space, one might expect an answer to the following to be more involved.

Question 6.8. Let $\rho_1, \rho_2 : F_2 \to \mathrm{PSL}_2\mathbb{C}$ be dense. If $\mathcal{DS}(\rho_1) = \mathcal{DS}(\rho_2)$, then is ρ_1 necessarily conjugate to ρ_2 in $\mathrm{PSL}_2\mathbb{C}$?

If Question 6.8 has an affirmative answer, then the volume class of (the conjugacy class of) a dense representation is distinguished and separated in semi-norm from every other such class, by Theorem 1.1 and Lemma 2.2. We note however, that Question 6.8 may have a negative answer, and all conjugacy classes of dense representations can still be separated in semi-norm, because there was a tremendous amount of freedom in our choice of discrete and faithful representation used to define chains on F_2 in Lemma 4.1 and Proposition 4.5. Otherwise, given $\rho_0 : F_2 \to \mathrm{PSL}_2\mathbb{C}$, it would be interesting to understand the set of representations that are not conjugate to ρ_0 but that have the same volume class in bounded cohomology. We expect that nothing interesting happens, however.

Conjecture 6.9. For a dense representation $\rho : F_2 \to \mathrm{PSL}_2\mathbb{C}$, if $\rho' : F_2 \to \mathrm{PSL}_2\mathbb{C}$ is any other representation such that

$$\|[\rho^*\mathrm{vol}] - [\rho'^*\mathrm{vol}]\| < v_3,$$

then ρ' is conjugate to ρ.

7 Higher and lower dimensional volume classes

For even $n \geq 4$, it is known that there is an $\epsilon_n > 0$ such that the Cheeger constant of $\mathbb{H}^n/\rho(H)$ is at least ϵ_n when H is a free group of finite rank, a closed surface group of genus at least *two*, or a finite volume hyperbolic 3three-manifold group and ρ is discrete and faithful [15]. Vanishing of the Cheeger constant is equivalent to the non-vanishing of the n-dimensional volume class of a discrete and faithful representation $\rho : H \to \mathrm{Isom}^+(\mathbb{H}^n)$ [32], so we cannot hope to prove that a dense representation $\rho : F_2 \to \mathrm{Isom}^+(\mathbb{H}^n)$ has non-zero volume class by approximating chains built from a discrete and faithful representation $\rho_0 : F_2 \to \mathrm{Isom}^+(\mathbb{H}^n)$.

In this section, we give a criterion to ensure that a dense representation $\rho : F_2 \to \mathrm{Isom}^+(\mathbb{H}^n)$ has non-vanishing n-dimensional volume class. We stress, however, that we do not know if our criterion is ever satisfied for $n \geq 4$. We recall a definition from the introduction.

Definition 7.1. Let Γ be a discrete group, $\alpha \in \mathrm{H}_n(\Gamma; \mathbb{R})$, and $K > 0$. We say that α is *K-freely approximated* if there is an integer $m \geq 2$, a homomorphism $\varphi : F_m \to \Gamma$, and a chain $Z \in \mathrm{C}_n(F_m; \mathbb{R})$ such that $\varphi_*(Z) \in \alpha$ and $\|\partial Z\|_1 \leq K$.

The conclusions of the following proposition should now feel somewhat believable. The proof more or less follows directly from the definitions and an application of the approximation scheme introduced in Section 2.6.

Proposition 7.2. Let $n \geq 2$ and suppose (M_i) is a sequence of closed and oriented hyperbolic n-manifolds with volume tending to infinity. Let $[M_i] \in \mathrm{H}_n(\pi_1(M_i); \mathbb{R})$ be the image of the fundamental class of M_i under the natural isomorphism $\mathrm{H}_n(M_i; \mathbb{R}) \to \mathrm{H}_n(\pi_1(M_i); \mathbb{R})$.

If there is a $K > 0$ such that $[M_i]$ is K-freely approximated for all i, then for any dense representation $\rho : F_2 \to \mathrm{Isom}^+(\mathbb{H}^n)$,

$$[\rho^*\mathrm{vol}_n] \neq 0 \in \mathrm{H}^n_{\mathrm{b}}(F_2; \mathbb{R}).$$

Proof. By definition of K-free approximation, there are maps $\varphi_i : F_{m_i} \to \pi_1(M_i)$ and chains $Z_i \in \mathrm{C}_n(F_{m_i}; \mathbb{R})$ such that $\varphi_{i*}(Z_i) \in [M_i]$ and $\|\partial Z_i\|_1 < K$. Let $\rho_i : \pi_1(M_i) \to \mathrm{Isom}^+(\mathbb{H}^n)$ be a hyperbolization of M_i; consider $\rho_i \circ \varphi_i : F_{m_i} \to \mathrm{Isom}^+(\mathbb{H}^n)$. Since

$\varphi_{i*}(Z_i) \in [M_i]$, for any $x \in \mathbb{H}^n$, we have

$$\langle [\varphi_{i*}(Z_i)], [\rho_i^* \mathrm{vol}_n^x] \rangle = \int_{M_i} d\mathrm{vol} = \mathrm{vol}(M_i) \to \infty.$$

Indeed,

$$\mathrm{vol}(M_i) = \langle \varphi_{i*}(Z_i), \rho_i^* \mathrm{vol}_n^x \rangle = \langle Z_i, (\rho_i \circ \varphi_i)^* \mathrm{vol}_n^x \rangle.$$

By Proposition 2.8, there is a chain $Z_i(1) \in \mathrm{C}_n(F_2; \mathbb{R})$ such that

$$|\rho^* \mathrm{vol}_n^x(Z_i(1))| > \mathrm{vol}(M_i) - 1,$$

$\|Z_i(1)\|_1 \leq \|Z_i\|_1$, and $\|\partial Z_i(1)\|_1 \leq \|\partial Z_i\|_1 < K$. If $b \in \mathrm{C}^{n-1}(F_2; \mathbb{R})$ is such that $\delta b = \rho^* \mathrm{vol}_n^x$, then

$$|\rho^* \mathrm{vol}_n^x(Z_i(1))| = |b(\partial Z_i(1))|.$$

But then

$$\|b\|_\infty > \frac{\mathrm{vol}(M_i) - 1}{K}$$

for every i. The right hand side goes to infinity with i, so b is not bounded. Thus, $[\rho^* \mathrm{vol}_n^x] \neq 0 \in \mathrm{H}_\mathrm{b}^n(F_2; \mathbb{R})$. ■

Remark 7.3. Suppose that one can find a sequence of n-manifolds (M_i) satisfying the hypotheses of Proposition 7.2. If one is able to construct dense representations $\{\rho_\theta : F_2 \to \mathrm{Isom}^+(\mathbb{H}^n)\}_{\theta \in \Lambda}$ by analogy with those defined in Section 6.1, then for even integers $n \geq 4$, $\dim_\mathbb{R} \mathrm{H}_\mathrm{b}^n(F_2; \mathbb{R}) = \#\mathbb{R}$. Indeed, assume we have a finite collection $\{\rho_{\theta_1}, \ldots, \rho_{\theta_m}\}$ and subgroups $\iota_j : H_{\theta_j} \to F_2$ such that $\rho_{\theta_k}(H_{\theta_j})$ is dense if $k = j$, and a Schottky group otherwise. Then, $[(\rho_{\theta_k} \circ \iota_j)^* \mathrm{vol}_n] = 0 \in \mathrm{H}_\mathrm{b}^n(H_{\theta_j}; \mathbb{R})$ by the combination of [32] and [15], as long as $k \neq j$. We can then apply a slight modification of the proof of Theorem 6.4; we would only be able to prove linear independence of volume classes of these dense representations in $\mathrm{H}_\mathrm{b}^n(F_2; \mathbb{R})$ but not necessarily in the reduced space. It seems that such representations $\{\rho_\theta : F_2 \to \mathrm{Isom}^+(\mathbb{H}^n)\}_{\theta \in \Lambda}$ should not be too difficult to construct by hand, as we have done for $n = 3$.

The above criterion is just an example of how to apply Proposition 2.8 to show that some bounded classes may be non-trivial, with the necessary auxiliary information. There are notions of straight chains with bounded volume in complex hyperbolic space; the statement of the proposition can be modified appropriately. Also, the manifolds (M_i) may be chosen to be non-compact with finite volume; the relevant feature is that the fundamental group of a cross section of a cusp is amenable. Many additional modifications can be made, and the reader is encouraged to make them.

We would like to convince the reader that free approximation of fundamental classes of manifolds is not an entirely contrived concept; although we admit that it may be a low dimensional phenomenon.

Lemma 7.4. Let X be a closed and oriented hyperbolic surface of genus $g \geq 2$. Then $[X]$ is 2two-freely approximated.

Proof. Sketch of proof There is a nice *one*-vertex triangulation of a hyperbolic surface; see Figure 1. We can find a base point $\bar{x}$ in X and a standard generating set $\{a_1, b_1, \ldots, a_g, b_g\}$ for $\pi_1(X, \bar{x}) \leq \mathrm{PSL}_2\mathbb{R}$ such that the union of the geodesic representatives of the a_i's and b_i's based at $\bar{x}$ is embedded. Cutting X open along these arcs produces a convex identification $4g$-gon with geodesic sides that can be embedded in the hyperbolic plane with vertex set contained in the preimage of $\bar{x}$ under the covering projection. By choosing a vertex $x \in \mathbb{H}^2$ of this $4g$-gon, we can join every vertex not adjacent to x with a geodesic segment. This process triangulates the identification polygon and defines a chain $V \in \mathrm{C}_2(\pi_1(X); \mathbb{R})$ that represents $[X]$; we have $\|V\|_1 = 4g - 2$. Let $x_1, y_1, \ldots, x_g, y_g$ be a free basis for F_{2g}. We define a homomorphsim $\varphi : F_{2g} \to \pi_1(X)$ given by $\varphi(x_j) = a_j$ and $\varphi(y_j) = b_j$.

There is a chain $Z \in \mathrm{C}_2(F_{2g}; \mathbb{R})$ such that $\varphi_*(Z) = V$ and such that $\|Z\|_1 = \|V\|_1$ and $\|\partial Z\|_1 = 2$; Z is obtained by replacing words in $\{a_i, b_i\}$ with the corresponding words spelled with $\{x_i, y_i\}$. Topologically, the quotient of Z by the identifications induced by F_{2g} is a *two*-dimensional CW-complex that is obtained by taking the metric completion of X after removing the interior of the based geodesic b_g. The boundary of this complex consists of two *one*-cells. ■

Take any sequence (X_g) of closed hyperbolic surfaces with genus g tending to ∞. By Lemma 7.4, $[X_g]$ is *two*-freely approximated for all g. In the sketch of the proof of Lemma 7.4, the chains V_g are $\frac{\mathrm{Area}(X_g)}{\|V_g\|_1} = \frac{4\pi(g-1)}{4g-2}$-efficient. Combined with Lemma 2.6, we obtain the following.

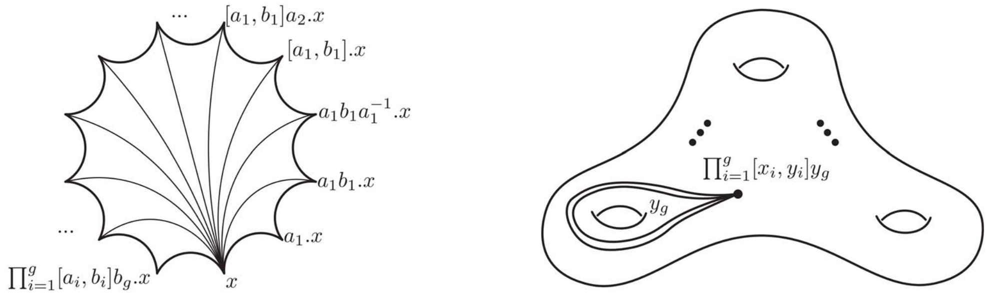

Fig. 1. Left: X cut along a standard generating set and a nice *one*-vertex locally geodesic triangulation of X. Right: Using F_{2g}, the polygon on the left does not quite close up to form a closed surface, because $y_g \neq \prod_{i=1}^{g}[x_i, y_i]y_g \in F_{2g}$.

Corollary 7.5. If $\rho : F_2 \to \mathrm{PSL}_2\mathbb{R}$ is a dense representation, then $\|[\rho^*\mathrm{vol}_2]\|_\infty = \pi$.

Corollary 7.5 is obtained easily from the sophisticated theory of computing bounded cohomology via boundary maps; see, e.g. [11,Section 4.3]. However, we emphasize that our hands on approach leads to a completely elementary and geometric proof.

Remark 7.6. For any closed hyperbolic *three*-manifold M, there is a sequence of covers $M_d \to M$ such that $[M_d]$ is K-freely approximated for all d; the constant K is a function of the genus of a certain immersed π_1-injective surface in M, that of a virtual fiber. The proof, which we omit, relies on the positive resolution of the Virtual Fibering Conjecture.

Funding

The author recognizes the support of the NSF, in particular, grants DMS-1246989 and DMS-1509171.

Acknowledgments

The author would like to thank the anonymous referees for thorough readings of this manuscript, insightful mathematical comments, and suggestions that improved the quality and correctness of this article. We would also like to thank Maria Beatrice Pozzetti for many very enjoyable conversations, her sustained interest in this work, and for suggesting that we consider the bounded Borel class. We thank Kenneth Bromberg for his guidance and support and Michael Landry for producing the figure.

References

[1] Agol, I. *Tameness of hyperbolic 3-manifolds*. 2004. arXiv:math.GT/0405568.

[2] Lars, V. "Ahlfors, *An extension of Schwarz's lemma*." *Trans. Amer. Math. Soc.* 43, no. 3 (1938): 359–64.

[3] Baker, A. *Transcendental Number Theory*. London, New York: Cambridge University Press, 1975.

[4] Brock, J. and K. Bromberg. "Geometric inflexibility and 3-manifolds that fiber over the circle." *J. Topol.* 4, no. 1 (2011): 1–38.

[5] Bucher, M., M. Burger, and A. Iozzi. "*A dual interpretation of the Gromov-Thurston proof of Mostow rigidity and volume rigidity for representations of hyperbolic lattices*." *Trends in harmonic analysis, Springer INdAM Ser*, Vol. 3. 47–76. Milan: Springer, 2013.

[6] Bucher, M., M. Burger, and A. Iozzi. "The bounded Borel class and 3-manifold groups." *Duke Math. J.* 167, no. 17 (2018): 3129–69.

[7] Brock, J. F., R. D. Canary, and Y. N. Minsky. *The classification of finitely generated Kleinian groups, in preparation*.

[8] Brock, J. F., R. D. Canary, and Y. N. Minsky. "The classification of Kleinian surface groups, II: the ending lamination conjecture." *Annals of Mathematics. Second Series* 176, no. 1 (2012): 1–149.

[9] Barge, J. and É. Ghys. "Surfaces et cohomologie bornée." *Invent. Math.* 92, no. 3 (1988): 509–26.

[10] Breuillard, E. and T. Gelander. "On dense free subgroups of lie groups." *J. Algebra* 261, no. 2 (2003): 448–67.

[11] Burger, M., A. Iozzi, and A. Wienhard. "Hermitian symmetric spaces and Kähler rigidity." *Transform. Groups* 12, no. 1 (2007): 5–32.

[12] Bloch, S. J. "*Higher regulators, algebraic K-theory, and zeta functions of elliptic curves*." *CRM Monograph Series*, Vol. 11. Providence, RI: American Mathematical Society, 2000.

[13] Bonahon, F. "Cobordism of automorphisms of surfaces." *Ann. Sci. École Norm. Sup. (4)* 16, no. 2 (1983): 237–70.

[14] Bonahon, F. "Bouts des variétés hyperboliques de dimension 3." *Annals of Mathematics. Second Series* 124, no. 1 (1986): 71–158.

[15] Bowen, L. "Cheeger constants and L^2-Betti numbers." *Duke Math. J.* 164, no. 3 (2015): 569–615.

[16] Richard, D. "Canary, *Ends of hyperbolic 3-manifolds*." *J. Amer. Math. Soc.* 6, no. 1 (1993): 1–35.

[17] Richard, D. "Canary, *A covering theorem for hyperbolic 3-manifolds and its applications*." *Topology* 35, no. 3 (1996): 751–78.

[18] Calegari, D. and D. Gabai. "Shrinkwrapping and the taming of hyperbolic 3-manifolds." *J. Amer. Math. Soc.* 19, no. 2 (2006): 385–446.

[19] Dunfield, N. M. "Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds." *Invent. Math.* 136, no. 3 (1999): 623–57.

[20] Farre, J. “Bounded cohomology of finitely generated Kleinian groups.” *Geom. Funct. Anal.* 28, no. 6 (2018): 1597–640.

[21] Farre, J. “Relations in bounded cohomology.” *J. Topol.* 13, no. 3 (2020): 1084–118.

[22] Frigerio, R. and M. Moraschini. “Gromov’s theory of multicomplexes with applications to bounded cohomology and simplicial volume.” To appear in *Mem. Amer. Math. Soc.* 2018.

[23] Frigerio, R., M. B. Pozzetti, and A. Sisto. “Extending higher-dimensional quasi-cocycles.” *J. Topol.* 8, no. 4 (2015): 1123–55.

[24] Francaviglia, S. “Hyperbolic volume of representations of fundamental groups of cusped 3-manifolds.” *Internat. Math. Res. Notices* 9 (2004): 425–59.

[25] Gabai, D. “On the topology of ending lamination space.” *Geom. Topol.* 18, no. 5 (2014): 2683–745.

[26] Guéritaud, F., O. Guichard, F. Kassel, and A. Wienhard. “Anosov representations and proper actions.” *Geom. Topol.* 21, no. 1 (2017): 485–584.

[27] Goldman, W. M. “Characteristic classes and representations of discrete subgroups of lie groups.” *Bull. Amer. Math. Soc. (N.S.)* 6, no. 1 (1982): 91–4.

[28] Alexander, B. “Goncharov, *Geometry of configurations, polylogarithms, and motivic cohomology*.” *Adv. Math.* 114, no. 2 (1995): 197–318.

[29] Gromov, M. “Volume and bounded cohomology.” *Inst. Hautes Études Sci. Publ. Math. (1982)* 56 (1983): 5–99.

[30] Ivanov, N. V. “Foundations of the Theory of Bounded Cohomology.” *J. Sov. Math.* 37 (1987): 1090–115.

[31] Kapovich, M. “*Hyperbolic Manifolds and Discrete Groups*.” *Modern Birkhäuser Classics*. Boston, MA: Birkhäuser Boston, Ltd, 2009 Reprint of the 2001 edition.

[32] Kim, S. and I. Kim. “Bounded cohomology and the Cheeger isoperimetric constant.” *Geom. Dedicata* 179 (2015): 1–20.

[33] Klaff, Benjamin G. *Boundary Slopes of Knots in Closed 3 Manifolds with Cyclic Fundamental Group*, *ProQuest LLC, Ann Arbor*, MI, 2003, PhD Thesis, University of Illinois at Chicago.

[34] Klarreich, E. *The boundary at infinity of the curve complex and the relative Teichmüller space*, 2018.

[35] Kuessner, T. “Efficient fundamental cycles of cusped hyperbolic manifolds.” *Pacific J. Math.* 211, no. 2 (2003): 283–313.

[36] Löh, C. “Measure homology and singular homology are isometrically isomorphic.” *Math. Z.* 253, no. 1 (2006): 197–218.

[37] McCullough, D. “Compact submanifolds of 3-manifolds with boundary.” *Quart. J. Math. Oxford Ser. (2)* 37, no. 147 (1986): 299–307.

[38] McMullen, Curtis T. Renormalization and 3-Manifolds which Fiber over the circle, *Annals of Mathematics Studies*, vol. 142, Princeton University Press, Princeton, NJ, 1996.

[39] Minsky, Y. N. “The classification of kleinian surface groups, I: models and bounds.” *Annals of Mathematics. Second Series* 171, no. 1 (2010): 1–107.

[40] McCullough, D. and A. Miller. “Homeomorphisms of 3-manifolds with compressible boundary.” *Mem. Amer. Math. Soc.* 61, no. 344 (1986) xii+100.

[41] Masur, H. A. and Y. N. Minsky. "Geometry of the complex of curves. I. Hyperbolicity." *Invent. Math.* 138, no. 1 (1999): 103–49.

[42] Masur, H. A. and Y. N. Minsky. "Geometry of the complex of curves. II. Hierarchical structure." *Geom. Funct. Anal.* 10, no. 4 (2000): 902–74.

[43] Monod, N. "*Stabilization for SL_n in Bounded Cohomology*." *Discrete Geometric Analysis, Contemp. Math*, Vol. 347. 191–202. Providence, RI: Amer. Math. Soc., 2004.

[44] Morandi, P. "*Field and Galois Theory*." *Graduate Texts in Mathematics*, Vol. 167. New York: Springer-Verlag, 1996.

[45] Matsuzaki, K., M. Taniguchi, H. Manifolds, and K. Groups. *Oxford Mathematical Monographs*. New York: The Clarendon Press, Oxford University Press, 1998 Oxford Science Publications.

[46] Namazi, H. and J. Souto. "Heegaard splittings and pseudo-Anosov maps." *Geom. Funct. Anal.* 19, no. 4 (2009): 1195–228.

[47] Namazi, H. and J. Souto. "Non-realizability and ending laminations: proof of the density conjecture." *Acta Math.* 209, no. 2 (2012): 323–95.

[48] Soma, T. "Bounded cohomology and topologically tame kleinian groups." *Duke Math. J.* 88, no. 2 (1997): 357–70.

[49] Soma, T. "Bounded cohomology of closed surfaces." *Topology* 36, no. 6 (1997): 1221–46.

[50] Soma, T. "Existence of non-banach bounded cohomology." *Topology* 37, no. 1 (1998): 179–93.

[51] Soma, T. *Volume and structure of hyperbolic 3-manifolds*, 2019. arXiv:math.GT/1906.00595.

[52] Souto, J. "A note on the tameness of hyperbolic 3-manifolds." *Topology* 44, no. 2 (2005): 459–74.

[53] John, R. "Stallings, *Topology of finite graphs*." *Invent. Math.* 71, no. 3 (1983): 551–65.

[54] Sullivan, D. "*On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*." *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), Ann. Math. Stud*, Vol. 97. 465–96. Princeton, N.J: Princeton Univ. Press, 1981.

[55] Sullivan, D. "Quasiconformal homeomorphisms and dynamics. II. Structural stability implies hyperbolicity for Kleinian groups." *Acta Math.* 155, no. 3–4 (1985): 243–60.

[56] William, P. *Thurston, The Geometry and Topology of 3-Manifolds*. Princeton University Lecture Notes, 1982.

[57] Thurston, W. P. "Hyperbolic structures on 3-manifolds. I: deformation of acylindrical manifolds." *Ann. of Math.* 124, no. 2 (1986): 203–46.

[58] Yale, P. B. "Automorphisms of the complex numbers." *Math. Mag.* 39, no. 3 (1966): 135–41.

[59] Yoshida, T. *On 3-Dimensional Bounded Cohomology of Surfaces* Homotopy Theory and Related Topics (Kyoto, 1984), Adv. Stud. Pure Math, Vol. 9. 173–6. Amsterdam: North-Holland, 1987.