IEEE Control Systems Letters paper presented at
2023 American Control Conference (ACC)
San Diego, CA, USA. May 31 - June 2, 2023

Multi-robot-assisted human crowd control for emergency evacuation: A stabilization approach

Zhenyuan Yuan[1] Tongjia Zheng[2] Mollik Nayyar[3] Alan R. Wagner[3] Hai Lin[2] Minghui Zhu[1]

Abstract— **This paper studies the problem of controlling a group of mobile robots to drive a group of humans to an exit for emergency evacuation. The interactions between the robots and the humans are modeled by a social force model. A novel optimization problem is formulated to synthesize a controller with closed-form expression. Sufficient conditions for global asymptotic stability are established for the humans and the robots. A simulation is conducted to evaluate the proposed controller.**

I. Introduction

Human emergency evacuation can be a challenging situation due to the need to relocate a possibly large crowd of people safely without causing choke points that slow down the evacuation process [1]. Recent studies have shown that it can be promising to use mobile robots to guide human crowds for reasons such as safety and efficiency [2] [3]. Due to the chaotic nature of the process, ensuring the success of evacuation becomes critical.

A majority of work on robot-assisted human evacuation is focused on single-robot scenarios. For example, paper [4] uses a robot to guide evacuees to select an exit within the minimum escape time. A robot is deployed to redirect evacuees toward the least congested exit in [5]. An autonomous mobile robot acting as a dynamic obstacle around the exit is proposed in paper [6] to control the flow of the pedestrians. Interested readers are referred to the recent survey paper [7] for more discussion. These works typically consider small simplified environments where the evacuees only require a few instructions from the robots during the evacuation process.

Multi-robot systems are known to be able to increase the amount of interactions with human crowds and therefore evacuate more humans in more complicated scenarios. Existing work on multi-robot-assisted human evacuation can be categorized into model-free approaches and model-based approaches. When accurate human-robot interaction (HRI) models are unavailable, model-free approaches are deployed such that the robots can learn the end-to-end mapping between sensory inputs and the optimal actions. For example, pedestrian regulation is studied in [8] using deep reinforcement learning that maps a robots' image perception of the human crowd to the robots' control inputs.

When accurate HRI models are available, model-based approaches are desired for stronger performance analysis. The HRI models can be categorized into macroscopic models and microscopic models. Macroscopic approaches model human evacuees as a large number of identical entities and treat the crowd as a continuum flow, analogous to gas and fluid, driven by vector fields [9]. By modeling the humans as a swarm, guidance robots are deployed in [10] to control the humans by providing directions. By using a density function to describe the distribution of the states of the human crowd, control strategies in [11] [12] generate the desired velocity fields that aim to drive the crowd density to a target density. Robots moving back-and-forth at the entrances are deployed in [13] to regulate the inflow of humans, and a feedback control scheme is designed to achieve optimal traffic flow in the pedestrian corridor by adjusting the motion frequency of the robots.

As a complement, microscopic approaches treat each individual as a distinct entity, consider the motion of each human and aim to ensure the evacuation of each individual. Based on the models of human dynamics with respect to the robots, microscopic approaches can be further categorized into two classes [14]: i) Cellular automaton models [5] [15], where both the time and the state spaces of the humans and the robots are discrete. ii) Force-based models [16] [17] [18]: where the human dynamics are modeled using ordinary differential equations in terms of the positions of the robots. For cellular automaton models, the controllers derived are mostly logic-based, i.e., using if-else statements. For force-based models, the corresponding controllers derived are usually solutions to optimization problems. However, to the best of our knowledge, there is no theoretical guarantee established for the existing controllers on ensuring the convergence of humans to exits.

Contribution statement. This paper considers the problem of using multiple robots to drive a group of humans to a given exit for emergency evacuation. We explicitly consider human-robot interaction and human-human interaction using

*This work was supported by the National Science Foundation under Grant No. CNS-1830335 and CNS-1830390. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

[1]Zhenyuan Yuan and Minghui Zhu are with School of Electrical Engineering and Computer Science, Pennsylvania State University, University Park, PA 16802, USA (email: {zqy5086,muz16}@psu.edu).

[2]Tongjia Zheng and Hai Lin are with the Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA (email: {tzheng1,hlin1}@nd.edu).

[3]Mollik Nayyar and Alan R. Wagner are with the Department of Aerospace Engineering, The Pennsylvania State University, University Park, PA 16802, USA (email: {mxn244, azw78}@psu.edu).

a social force model. A novel optimal control problem is formulated to synthesize a state-feedback controller with closed-form expression. Specifically, by using Lyapunov stability theory, the convergence of the humans to the exit is imposed as hard constraints, and the convergence of the robots to the exit is imposed as soft constraints. The proposed controller ensures the asymptotic convergence of both the humans and the robots. Our contribution is summarized as follows:

- A novel optimal controller is proposed for multi-robot-assisted human evacuation.
- Theoretical guarantees are derived for the convergence of the humans and the robots to the exit.

Simulation is conducted to evaluate the proposed controller and the theoretical guarantees.

Notations. In this paper, we use lower-case letters, e.g., a, to denote scalars, bold letters, e.g., $\boldsymbol{a}$, to denote vectors; we use upper-case letters, e.g., A, to denote matrices, calligraphic letters, e.g., $\mathcal{A}$, to denote sets. Denote $I_n \in \mathbb{R}^{n\times n}$ the n-by-n-dimensional identity matrix, $\mathbf{0}_n \in \mathbb{R}^n$ the column vector with n zeros and $\mathbf{0}_{n\times m} \in \mathbb{R}^{n\times m}$ the matrix with $n \times m$ zeros. We use lowerscript $(\cdot)_i$ to distinguish the local values of agent (human or robot) i.

II. Problem formulation

In this section, we introduce the dynamic models of the robots and the humans and state the objective of the human crowd control problem.

Dynamic models. Consider a group of robot $\mathcal{R} \triangleq \{1, \cdots, n_r\}$ with single integrator dynamics

$$\dot{\boldsymbol{z}}_j(t) = \boldsymbol{u}_j(t). \tag{1}$$

where $\boldsymbol{z}_j \in \mathcal{Z} \subset \mathbb{R}^2$ is the location of robot $j \in \mathcal{R}$ and $\boldsymbol{u}_j \in \mathcal{U} \subset \mathbb{R}^2$ is the corresponding control inputs.

Consider a group of humans $\mathcal{H} \triangleq \{1, \cdots, n_h\}$. Denote $\boldsymbol{x}_i \in \mathcal{X} \subset \mathbb{R}^2$ the location of human $i \in \mathcal{H}$, and $\boldsymbol{v}_i \in \mathcal{V} \subset \mathbb{R}^2$ the corresponding velocity. Denote $d_{ij} \triangleq \|\boldsymbol{x}_i - \boldsymbol{z}_j\|$ if $j \in \mathcal{R}$ and $d_{ij} \triangleq \|\boldsymbol{x}_i - \boldsymbol{x}_j\|$ if $j \in \mathcal{H}$ the distance between the center of mass of human i and robot/human j, $r_i^{\mathcal{R}} > 0$ the comfort distance of human i with respect to robot and $r_i^{\mathcal{H}} > 0$ the comfort distance of human i with respect to other humans. In this paper, we assume that each human i is assigned to and only affected by a fixed subset of robots $\mathcal{R}_i \subset \mathcal{R}$, e.g., each robot raises a specific sign and the humans only follow the robots with the specific signs. The social force dynamics [19] of human i is given by

$$\begin{bmatrix}\dot{\boldsymbol{x}}_i(t)\\ \dot{\boldsymbol{v}}_i(t)\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{2\times2} & I_2\\ \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_i(t)\\ \boldsymbol{v}_i(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{2\times2}\\ I_2\end{bmatrix}\boldsymbol{f}_i^{soc}(t) \tag{2}$$

where $\boldsymbol{f}_i^{soc}(t) \triangleq \sum_{j\in\mathcal{R}_i}\boldsymbol{f}_{ij}^{HR}(t) + \sum_{j\in\mathcal{H}\setminus i}\boldsymbol{f}_{ij}^{HH}(t)$,

$$\boldsymbol{f}_{ij}^{HR}(t) \triangleq a_i^{\mathcal{R}}\exp[(r_i^{\mathcal{R}} - d_{ij}(t))/b_i^{\mathcal{R}}]\boldsymbol{n}_{ij}(t),$$
$$\boldsymbol{f}_{ij}^{HH}(t) \triangleq a_i^{\mathcal{H}}\exp[(r_i^{\mathcal{H}} - d_{ij}(t))/b_i^{\mathcal{H}}]\boldsymbol{n}_{ij}(t),$$

$\boldsymbol{n}_{ij}(t) \triangleq \frac{\boldsymbol{x}_i(t)-\boldsymbol{z}_j(t)}{d_{ij}(t)}$ if $j \in \mathcal{R}$ and $\boldsymbol{n}_{ij}(t) \triangleq \frac{\boldsymbol{x}_i(t)-\boldsymbol{x}_j(t)}{d_{ij}(t)}$ if $j \in \mathcal{H}$, and $a_i^{\mathcal{R}}, a_i^{\mathcal{H}}, b_i^{\mathcal{R}}, b_i^{\mathcal{H}} \in \mathbb{R}$. Furthermore, we assume that each robot only affects one human. This can be achieved by assigning each human a unique sign to follow.

Problem statement. The objective of this paper is to design a controller $\pi : \mathcal{Z}^{n_r} \times (\mathcal{X}\times\mathcal{V})^{n_h} \to \mathcal{U}$ to drive the humans and the robots to an exit. The problem is challenged by the fact that the coupled dynamics of (1) and (2) is nonlinear respect to the robot control inputs $\boldsymbol{u}_j(t)$.

III. Controller design

Denote $A \triangleq \begin{bmatrix}\mathbf{0}_{2\times2} & I_2\\ \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2}\end{bmatrix}$, $B \triangleq [\mathbf{0}_{2\times2}, I_2]^T$, and $\boldsymbol{q}_i \triangleq [\boldsymbol{x}_i^T, \boldsymbol{v}_i^T]^T$. Then the human dynamics in (2) can be compactly written as

$$\dot{\boldsymbol{q}}_i(t) = A\boldsymbol{q}_i(t) + B\boldsymbol{f}_i^{soc}(t) \tag{3}$$

which is a standard linear time-invariant system with respect to input $\boldsymbol{f}_i^{soc}(t)$ [20]. Without loss of generality, we let the origin $\boldsymbol{x}_i = \mathbf{0}_2$ be the exit the humans need to reach. Then the problem of driving all the humans and the robots to an exit can be treated as a stabilization problem in control theory. Consider Lyapunov function $V^{\mathcal{H}}(\boldsymbol{q}_i) \triangleq \frac{1}{2}\boldsymbol{q}_i^T Q\boldsymbol{q}_i$, where Q is positive definite. Then the following lemma summarizes the sufficient condition to stabilize system (3).

Lemma III.1. Let a desired social force be given by $\boldsymbol{f}_i^*(t) = -K(t)\boldsymbol{q}_i(t)$ such that the derivative of $V^{\mathcal{H}}$ along the trajectories of $\dot{\boldsymbol{q}}_i(t) = A\boldsymbol{q}_i(t) + B\boldsymbol{f}_i^*(t)$ given by

$$\begin{aligned}\dot{V}^{\mathcal{H}}(\boldsymbol{q}_i(t)) &= \boldsymbol{q}_i^T(t)Q(A\boldsymbol{q}_i(t) + B\boldsymbol{f}_i^*(t))\\ &= \boldsymbol{q}_i^T(t)(QA - QBK(t))\boldsymbol{q}_i(t)\end{aligned}$$

satisfies $\dot{V}^{\mathcal{H}}(\mathbf{0}_4) = 0$ and $\dot{V}^{\mathcal{H}}(\boldsymbol{q}_i(t)) < 0$ for all $\boldsymbol{q}_i(t) \neq 0$. Then the system $\dot{\boldsymbol{q}}_i(t) = A\boldsymbol{q}_i(t) + B\boldsymbol{f}_i^*(t)$ is globally asymptotically stable.

Proof: This is a direct results of Theorem 4.2 in [21]. ■.

An example pair of selection is $Q = \begin{bmatrix}I_2 & I_2\\ \mathbf{0}_{2\times2} & I_2\end{bmatrix}$ and $K(t) = 4\begin{bmatrix}I_2 & I_2\end{bmatrix}$ for all $t \geqslant 0$. Nevertheless, $\boldsymbol{f}_i^{soc}(t)$, as opposite to $\boldsymbol{u}_j(t)$, cannot be directly controlled. To stabilize (3) using control inputs $\boldsymbol{u}_j(t)$, $j \in \mathcal{R}$, we propose a novel optimal control formulation that takes stabilizing the humans as hard constraints and stabilizing the robots as soft constraints. More detailed description is given as follows.

A. Human-robot stabilization.

Human stabilization. Recall that the complete system of using the robots to drive the humans is given by the combination of (3) and (1). We employ a backstepping approach by introducing a new state $\tilde{\boldsymbol{f}}_i(t) \triangleq \boldsymbol{f}_i^{soc}(t) - \boldsymbol{f}_i^*(t)$, the error between the desired social force and the actual social force human i is experiencing. Next we derive the relation of how control inputs $\boldsymbol{u}_j(t)$, $j \in \mathcal{R}$, from the robots change the error of social force $\tilde{\boldsymbol{f}}_i(t)$ acting on the humans. Denote

$$H_{ij}^{\mathcal{R}}(t) \triangleq (\frac{1}{b_i^{\mathcal{R}}} + \frac{1}{d_{ij}(t)})\hat{\boldsymbol{a}}_{ij}^{\mathcal{R}}(t)(\boldsymbol{x}_i(t) - \boldsymbol{z}_j(t))^T - a_{ij}^{\mathcal{R}}(t)I_2, \tag{4}$$

$$H_{ij}^{\mathcal{H}}(t) \triangleq (\frac{1}{b_i^{\mathcal{H}}} + \frac{1}{d_{ij}(t)})\hat{\boldsymbol{a}}_{ij}^{\mathcal{H}}(t)(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t))^T - a_{ij}^{\mathcal{H}}(t)I_2,$$

$$\hat{\boldsymbol{a}}_{ij}^{\mathcal{R}}(t) \triangleq a_i^{\mathcal{R}} \exp[\frac{r_i - d_{ij}(t)}{b_i^{\mathcal{R}}}] \frac{\boldsymbol{x}_i(t) - \boldsymbol{z}_j(t)}{d_{ij}^2(t)}, j \in \mathcal{R},$$

$$a_{ij}^{\mathcal{R}}(t) \triangleq \frac{a_i^{\mathcal{R}}}{d_{ij}(t)} \exp[\frac{r_i - d_{ij}(t)}{b_i^{\mathcal{R}}}],$$

$$\hat{\boldsymbol{a}}_{ij}^{\mathcal{H}}(t) \triangleq a_i^{\mathcal{H}} \exp[\frac{r_i - d_{ij}(t)}{b_i^{\mathcal{H}}}] \frac{\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)}{d_{ij}^2(t)}, j \in \mathcal{H},$$

$$a_{ij}^{\mathcal{H}}(t) \triangleq \frac{a_i^{\mathcal{H}}}{d_{ij}(t)} \exp[\frac{r_i - d_{ij}(t)}{b_i^{\mathcal{H}}}].$$

The dynamics of $\tilde{\boldsymbol{f}}_i(t)$ is given as

$$\begin{aligned}\dot{\tilde{\boldsymbol{f}}}_i(t) =& K(t)\dot{\boldsymbol{q}}_i(t) + \dot{K}(t)\boldsymbol{q}_i(t) \\ &+ \sum_{j \in \mathcal{R}_i} \Big(H_{ij}^{\mathcal{R}}(t)\boldsymbol{u}_j(t) - H_{ij}^{\mathcal{R}}(t)\boldsymbol{v}_i(t) \Big) \\ &- \sum_{j \in \mathcal{H}\backslash\{i\}} \Big(H_{ij}^{\mathcal{H}}(t)\big(\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)\big) \Big). \qquad (5)\end{aligned}$$

The derivation of (5) can be found in Appendix V-A.

Consider Lyapunov function

$$\hat{V}^{\mathcal{H}}(\boldsymbol{q}_i(t), \tilde{\boldsymbol{f}}(t)) \triangleq \frac{1}{2}\boldsymbol{q}_i^T(t)Q\boldsymbol{q}_i(t) + \frac{1}{2}\tilde{\boldsymbol{f}}_i^T(t)\tilde{\boldsymbol{f}}_i(t).$$

Then the derivative of $\hat{V}^{\mathcal{H}}$ along the trajectories of the augmented system of (3) and (5) is given as

$$\begin{aligned}&\dot{\hat{V}}^{\mathcal{H}}(\boldsymbol{q}_i(t), \tilde{\boldsymbol{f}}_i(t)) = \boldsymbol{q}_i^T(t)Q\dot{\boldsymbol{q}}_i(t) + \tilde{\boldsymbol{f}}_i^T(t)\dot{\tilde{\boldsymbol{f}}}_i(t) \\ &= \boldsymbol{q}_i^T(t)Q(A\boldsymbol{q}_i(t) + B\boldsymbol{f}_i^*(t) + B\tilde{\boldsymbol{f}}_i(t)) + \tilde{\boldsymbol{f}}_i^T(t)\dot{\tilde{\boldsymbol{f}}}_i(t) \\ &= \boldsymbol{q}_i^T(t)Q(A\boldsymbol{q}_i(t) + B\boldsymbol{f}_i^*(t)) \\ &\quad + \tilde{\boldsymbol{f}}_i^T(t)(\dot{\tilde{\boldsymbol{f}}}_i(t) + B^TQ^T\boldsymbol{q}_i(t))\end{aligned}$$

By (5), we can compactly write

$$\dot{\tilde{\boldsymbol{f}}}_i(t) + B^TQ^T\boldsymbol{q}_i(t) = \sum_{j \in \mathcal{R}_i} H_{ij}^{\mathcal{R}}(t)\boldsymbol{u}_j(t) + \boldsymbol{h}_i(t) \qquad (6)$$

where

$$\begin{aligned}\boldsymbol{h}_i(t) \triangleq& K(t)\dot{\boldsymbol{q}}_i(t) + \dot{K}(t)\boldsymbol{q}_i(t) + B^TQ^T\boldsymbol{q}_i(t) \\ &- \sum_{j \in \mathcal{R}_i} H_{ij}^{\mathcal{R}}(t)\boldsymbol{v}_i(t) - \sum_{j \in \mathcal{H}\backslash\{i\}} H_{ij}^{\mathcal{H}}(t)(\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)).\end{aligned}$$

Recall that $\boldsymbol{q}_i^T(t)Q(A\boldsymbol{q}_i(t) + B\boldsymbol{f}_i^*(t)) \leqslant 0$ according to the selection of Q and $\boldsymbol{f}_i^*(t) = -K(t)\boldsymbol{q}_i(t)$ in Lemma III.1. Denote $H_i^{\mathcal{R}}(t) \triangleq [H_{i,1}^{\mathcal{R}}(t), \cdots, H_{i,n_i}^{\mathcal{R}}(t)]$. Then a sufficient condition for stabilizing the augmented system of (3) and (5) for all $i \in \mathcal{H}$ is can be characterized by the following lemma.

Lemma III.2. Suppose $\boldsymbol{f}_i^*(t)$ is chosen according to Lemma III.1 and it holds that

$$\begin{aligned}&\tilde{\boldsymbol{f}}_i^T(t)\sum_{j \in \mathcal{R}_i}\Big(H_{ij}^{\mathcal{R}}(t)\boldsymbol{u}_j(t)\Big) + \tilde{\boldsymbol{f}}_i^T(t)\boldsymbol{h}_i(t) \\ &\leqslant -s(t)\tilde{\boldsymbol{f}}_i^T(t)H_{ij_*(t)}^{\mathcal{R}}(t)\tilde{\boldsymbol{f}}_i(t), \qquad (7)\end{aligned}$$

where $s(t) = 1$ if $\tilde{\boldsymbol{f}}_i^T H_{ij_*(t)}^{\mathcal{R}}(t)\tilde{\boldsymbol{f}}_i > 0$, otherwise $s(t) = -1$, and $j_*(t) \triangleq \arg\min_{j \in \mathcal{R}_i} d_{ij}(t)$. Then the augmented system (3) and (5) is globally asymptotically stable.

Proof: Consider the Lyapunov function $\hat{V}^{\mathcal{H}}(\boldsymbol{q}_i, \tilde{\boldsymbol{f}})$. Given (6), inequality (7) implies $\tilde{\boldsymbol{f}}_i^T(t)(\dot{\tilde{\boldsymbol{f}}}_i(t) + B^TQ^T\boldsymbol{q}_i(t)) < 0$ if $\tilde{\boldsymbol{f}}_i^T(t) \neq 0$ and $\tilde{\boldsymbol{f}}_i^T(t)(\dot{\tilde{\boldsymbol{f}}}_i(t) + B^TQ^T\boldsymbol{q}_i(t)) = 0$ if $\tilde{\boldsymbol{f}}_i^T(t) = 0$. Combining this with Lemma III.1, we have $\dot{\hat{V}}^{\mathcal{H}}(\mathbf{0}_4, \mathbf{0}_2) = 0$ and $\dot{\hat{V}}^{\mathcal{H}}(\boldsymbol{q}_i(t), \tilde{\boldsymbol{f}}_i(t)) < 0$ for all $\begin{bmatrix}\boldsymbol{q}_i(t) \\ \tilde{\boldsymbol{f}}_i(t)\end{bmatrix} \neq \mathbf{0}_6$. Then based on Theorem 4.2 in [21], each augmented state $(\boldsymbol{q}_i(t), \tilde{\boldsymbol{f}}_i(t))$, $i \in \mathcal{H}$, is globally asymptotically stable. ■

Robot stabilization. To drive the robots to the exit as much as possible, we consider the Lyapunov function $V^{\mathcal{R}}(\boldsymbol{z}_j) \triangleq \frac{1}{2}\boldsymbol{z}_j^T\boldsymbol{z}_j$. By Theorem 4.2 in [21], the stabilization of the robots is ensured if the Lie derivative $\dot{V}^{\mathcal{R}}(\mathbf{0}_2) = 0$ and for all $\boldsymbol{z}_j(t) \neq \mathbf{0}_2$, we have

$$\dot{V}^{\mathcal{R}}(\boldsymbol{z}_j(t)) = \boldsymbol{z}_j^T(t)\boldsymbol{u}_j(t) < 0. \qquad (8)$$

B. Human-aware optimal control.

In this section, we formally formulate the optimal control problem. Notice that the stabilization of the humans can be ensured if the control inputs always satisfy (7). Therefore, (7) is imposed as the hard constraints of the optimal control problem. Meanwhile, to drive the robots to the exit, (8) is imposed as soft constraints. Furthermore, control efforts should be minimized when the above objectives are satisfied. Formally, for each human i, the optimal control problem is formulated as follows.

$$\begin{aligned}&\min_{\boldsymbol{u}_j,\ j \in \mathcal{R}_i} \frac{1}{2}\sum_{j \in \mathcal{R}_i} \boldsymbol{u}_j^T(t)\boldsymbol{u}_j(t) + \boldsymbol{z}_j^T(t)\boldsymbol{u}_j(t) \qquad (9) \\ &\text{s.t.} \sum_{j \in \mathcal{R}_i}\Big(\tilde{\boldsymbol{f}}_i^T(t)H_{i,j}^{\mathcal{R}}(t)\boldsymbol{u}_j(t)\Big) + \tilde{\boldsymbol{f}}_i^T(t)\boldsymbol{h}_i(t) \\ &\qquad\qquad \leqslant -s(t)\tilde{\boldsymbol{f}}_i^T(t)H_{ij_*(t)}^{\mathcal{R}}(t)\tilde{\boldsymbol{f}}_i(t).\end{aligned}$$

Lemma III.3 below shows that problem (9) is always feasible.

Lemma III.3. If $|a_i^{\mathcal{R}}| > 0$, then problem (9) is feasible for all $t \geqslant 0$. ■

The proof of the lemma is given in Section **??**.

Therefore, problem (9) can be solved by the Lagrangian dual method. The Lagrangian of (9) is given by

$$\begin{aligned}&L\Big(\{\boldsymbol{u}_j(t)\}_{j \in \mathcal{R}_i}, \lambda(t)\Big) \triangleq \sum_{j \in \mathcal{R}_i}\Big(\frac{1}{2}\boldsymbol{u}_j^T(t)\boldsymbol{u}_j(t) \\ &+ \boldsymbol{z}_j^T(t)\boldsymbol{u}_j(t)\Big) + \lambda_i(t)\Big(\tilde{\boldsymbol{f}}_i^T(t)\big(\sum_{j \in \mathcal{R}_i} H_{i,j}^{\mathcal{R}}(t)\boldsymbol{u}_j(t)\big) \\ &+ \tilde{\boldsymbol{f}}_i^T\boldsymbol{h}_i(t) + s(t)\tilde{\boldsymbol{f}}_i^T(t)H_{ij_*(t)}^{\mathcal{R}}(t)\tilde{\boldsymbol{f}}_i(t)\Big) \\ &\text{s.t. } \lambda_i(t) \geqslant 0. \qquad (10)\end{aligned}$$

By first minimizing (10) with respect to $\boldsymbol{u}_j(t)$ through setting the first derivative to zero and then maximizing with respect to $\lambda_i(t)$, we have the solution

$$\boldsymbol{u}_j(t) = -\big(\boldsymbol{z}_j(t) + \lambda_i(t)(H_{i,j}^{\mathcal{R}})^T(t)\tilde{\boldsymbol{f}}_i(t)\big), \qquad (11)$$

$$\lambda_i(t) = \big(s(t)\tilde{\boldsymbol{f}}_i^T(t)H_{ij_*(t)}^{\mathcal{R}}(t)\tilde{\boldsymbol{f}}_i(t) - \tilde{\boldsymbol{f}}_i^T(t)\sum_{j \in \mathcal{R}_i} H_{i,j}^{\mathcal{R}}(t)\boldsymbol{z}_j(t)$$

$$+\,\tilde{\boldsymbol{f}}_i^T(t)\boldsymbol{h}_i(t)\Big)\Big(\tilde{\boldsymbol{f}}_i^T(t)\big(\sum_{j\in\mathcal{R}_i} H_{i,j}^{\mathcal{R}}(t)(H_{i,j}^{\mathcal{R}}(t))^T\big)\tilde{\boldsymbol{f}}_i(t)\Big)^{-1}.$$

Next we analyze the performance of the controller derived in (11). Without loss of generality, let $\mathcal{R}_i = \{1,\cdots,n_i\}$. Denote $\boldsymbol{z}_i(t) \triangleq [\boldsymbol{z}_1^T(t),\cdots,\boldsymbol{z}_{n_i}^T(t)]^T$, $\boldsymbol{u}_i(t) \triangleq [\boldsymbol{u}_1^T(t),\cdots,\boldsymbol{u}_{n_i}^T(t)]^T$. Let $\beta \in [0,1)$. Then the following proposition provides a sufficient condition for the humans and the robots to asymptotically converge to the exit.

Proposition III.4. Suppose

$$\tilde{\boldsymbol{f}}_i^T(t)H_i^{\mathcal{R}}(t)\boldsymbol{z}_i(t) \geqslant -\beta\|(H_i^{\mathcal{R}}(t))^T\tilde{\boldsymbol{f}}_i(t)\|_2\|\boldsymbol{z}_i(t)\|_2. \quad (12)$$

Suppose $|a_i^{\mathcal{R}}| > 0$ and there exists $\bar{d} > 0$ and some $k_1, k_2 \geqslant 0$ such that $d_{ij}(t) \leqslant \bar{d}$, $\|\dot{K}(t)\|_2 \leqslant k_1$ and $\|K(t)\|_2 \leqslant k_2$ for all $t \geqslant 0$. Then controller (11) renders that $\boldsymbol{z}_i(t)$ and $\boldsymbol{q}_i(t)$ are global uniformally asymptotically stable, for all $i \in \mathcal{H}$. ■

Proposition III.4 shows that in order to ensure global uniform asymptotic stability, the design of the desired force $\boldsymbol{f}_i^*(t)$, or $K(t)$, needs to satisfy three requirements: $\|\dot{K}(t)\|_2$ and $\|K(t)\|_2$ are bounded, and (12) is satisfied over time. The further design of $K(t)$ satisfying these requirements is part of the ongoing work.

IV. SIMULATION

In this section, we conduct Monte Carlo simulations to demonstrate the effectiveness of our controller and verify the theoretical results.

Parameter selection. In the simulation, we use parameters $a_i^{\mathcal{R}} = 1$, $r_i^{\mathcal{R}} = 1$, $b_i^{\mathcal{R}} = 1$, $a_i^{\mathcal{H}} = 0.01$, $r_i^{\mathcal{H}} = 1$, $b_i^{\mathcal{H}} = 1$. For each human i, we assign 3 robots into the group $\mathcal{R}_i$. The initial locations of the humans are uniformly sampled over $[-1,1]\times[-1,1]$ and the initial speeds of the humans are uniformly sampled over $[-0.1,0.1]\times[-0.1,0.1]$. We let $K(t) = 4[I_2, I_2]$ for all $t \geqslant 0$. For each human i with initial location $\boldsymbol{x}_i(0)$, to satisfy condition (12), we apply heuristics such that for each robot $j \in \mathcal{R}_i$, the initial location is generated as $\boldsymbol{z}_j(0) = 2\boldsymbol{x}_i(0)+\boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon} \sim \mathcal{N}(0, 0.25I_2)$.

Results. We conduct simulations of groups of $n_h = 1, 10, 20, 40$ humans respectively. Notice that social force always exists among the humans. Therefore, to facilitate the convergence of the other humans, each human i and the corresponding robots in $\mathcal{R}_i$ are immediately retrieved if $\|\boldsymbol{x}_i\| < 0.01$, mimicking the successful evacuation of human i. Figure 1 shows the trajectories of the robots and the humans, the convergence of the humans in terms of $\|\boldsymbol{x}_i(t)\|_2$ and the convergence of the robots in terms of $\|\boldsymbol{z}_i(t)\|_2$. The solid squares and the solid circles indicate the initial locations of the robots and the humans, respectively. The unit of the x-axis is second. Notice that all the humans and the robots reach sufficiently close to the origin within 8 seconds.

V. CONCLUSION

This paper studies the problem of multi-robot-assisted human evacuation. The problem is formulated stabilization of a nonlinear system. A novel optimization problem is derived to synthesize a controller with closed form. Preliminary sufficient conditions are established for global asymptotic stability. Simulation is conducted to verify the results.

APPENDIX

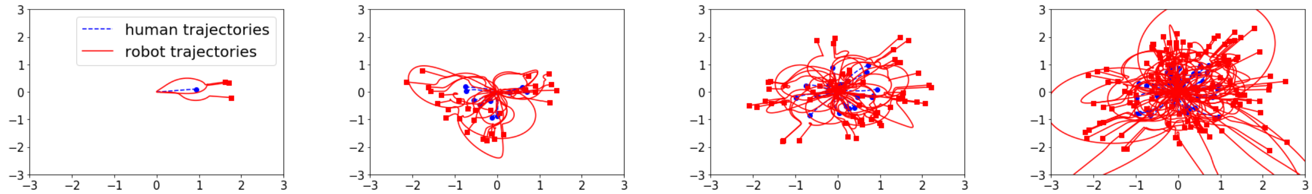
A. Derivation of $\dot{\tilde{\boldsymbol{f}}}_i$

Recall that $a_{ij}^{\mathcal{R}}(t) = a_i^{\mathcal{R}}\exp[\frac{r_i-d_{ij}(t)}{b_i^{\mathcal{R}}}]/d_{ij}(t)$. Then we have

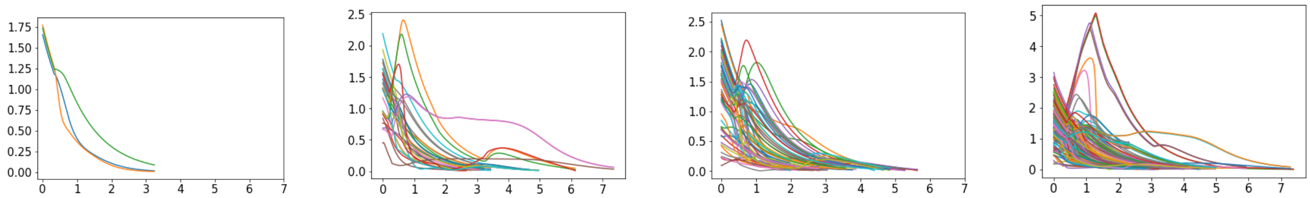
$$\dot{a}_{ij}^{\mathcal{R}}(t) = -\frac{a_i^{\mathcal{R}}}{b_i^{\mathcal{R}}}\exp[\frac{r_i-d_{ij}(t)}{b_i^{\mathcal{R}}}]\frac{\dot{d}_{ij}(t)}{d_{ij}(t)} - a_i^{\mathcal{R}}\exp[\frac{r_i-d_{ij}(t)}{b_i^{\mathcal{R}}}]\frac{\dot{d}_{ij}(t)}{d_{ij}^2(t)},$$

where $\dot{d}_{ij}(t) = \frac{(\boldsymbol{x}_i(t)-\boldsymbol{z}_j(t))^T(\dot{\boldsymbol{x}}_j(t)-\dot{\boldsymbol{z}}_j(t))}{d_{ij}(t)}$. The the derivative of $\sum_{j\in\mathcal{R}}\dot{\boldsymbol{f}}_{ij}^{soc}(t)$ is given by

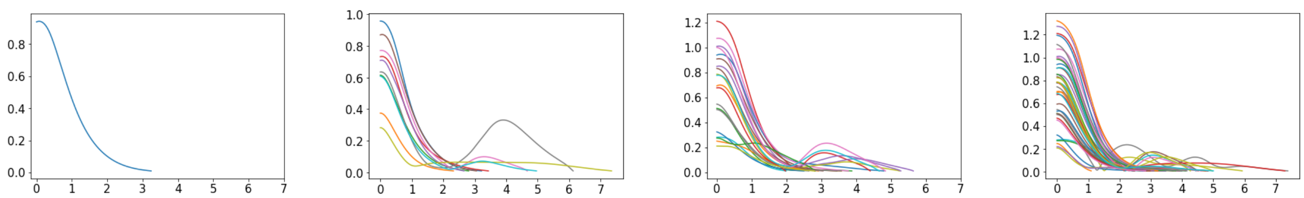
$$\begin{aligned}
&\sum_{j\in\mathcal{R}_i}\dot{\boldsymbol{f}}_{ij}^{soc}(t)\\
&=\sum_{j\in\mathcal{R}_i}\Big(\dot{a}_{ij}^{\mathcal{R}}(t)(\boldsymbol{x}_i(t)-\boldsymbol{z}_j(t)) + a_{ij}^{\mathcal{R}}(t)(\dot{\boldsymbol{x}}_i-\dot{\boldsymbol{z}}_j)\Big)\\
&=\sum_{j\in\mathcal{R}_i}\Big(\dot{a}_{ij}^{\mathcal{R}}(t)(\boldsymbol{x}_i(t)-\boldsymbol{z}_j(t)) + a_{ij}^{\mathcal{R}}(t)(\boldsymbol{v}_i(t)-\boldsymbol{u}_j(t))\Big)\\
&=\sum_{j\in\mathcal{R}_i}\Big(\big(-\frac{a_i^{\mathcal{R}}}{b_i^{\mathcal{R}}}\exp[\frac{r_i-d_{ij}(t))}{b_i^{\mathcal{R}}}]\frac{\dot{d}_{ij}(t)}{d_{ij}(t)}\\
&\quad - a_i^{\mathcal{R}}\exp[\frac{r_i-d_{ij}(t)}{b_i^{\mathcal{R}}}]\frac{\dot{d}_{ij}(t)}{d_{ij}^2(t)}\big)(\boldsymbol{x}_i(t)-\boldsymbol{z}_j(t))\\
&\quad + a_{ij}^{\mathcal{R}}(t)(\boldsymbol{v}_i(t)-\boldsymbol{u}_j(t))\Big)\\
&=\sum_{j\in\mathcal{R}_i}\Big(-\big((\frac{a_i^{\mathcal{R}}}{b_i^{\mathcal{R}}}+\frac{a_i^{\mathcal{R}}}{d_{ij}(t)})\exp[\frac{r_i-d_{ij}(t)}{b_i^{\mathcal{R}}}]\big)\\
&\quad\cdot\frac{(\boldsymbol{x}_i(t)-\boldsymbol{z}_j(t))^T(\dot{\boldsymbol{x}}_i(t)-\dot{\boldsymbol{z}}_j(t))(\boldsymbol{x}_i(t)-\boldsymbol{z}_j(t))}{d_{ij}^2(t)}\\
&\quad + a_{ij}^{\mathcal{R}}(t)(\boldsymbol{v}_i(t)-\boldsymbol{u}_j(t))\Big)\\
&=\sum_{j\in\mathcal{R}_i}\Big(-\big((\frac{1}{b_i^{\mathcal{R}}}+\frac{1}{d_{ij}(t)})(\hat{\boldsymbol{a}}_{ij}^{\mathcal{R}}(t))^T\big)\\
&\quad\cdot(\dot{\boldsymbol{x}}_i(t)-\dot{\boldsymbol{z}}_j(t))(\boldsymbol{x}_i(t)-\boldsymbol{z}_j(t)) + a_{ij}^{\mathcal{R}}(t)(\boldsymbol{v}_i(t)-\boldsymbol{u}_j(t))\Big)\\
&=\sum_{j\in\mathcal{R}_i}\Big(-(\frac{1}{b_i^{\mathcal{R}}}+\frac{1}{d_{ij}(t)})(\hat{\boldsymbol{a}}_{ij}^{\mathcal{R}}(t))^T\dot{\boldsymbol{x}}_i(t)(\boldsymbol{x}_i(t)-\boldsymbol{z}_j(t))\\
&\quad + (\frac{1}{b_i^{\mathcal{R}}}+\frac{1}{d_{ij}(t)})(\hat{\boldsymbol{a}}_{ij}^{\mathcal{R}}(t))^T\boldsymbol{u}_j(t)(\boldsymbol{x}_i(t)-\boldsymbol{z}_j(t))\big)\Big)\\
&\quad + a_{ij}^{\mathcal{R}}(t)(\boldsymbol{v}_i(t)-\boldsymbol{u}_j(t))\\
&=\sum_{j\in\mathcal{R}_i}\Big(-(\frac{1}{b_i^{\mathcal{R}}}+\frac{1}{d_{ij}(t)})\hat{\boldsymbol{a}}_{ij}^{\mathcal{R}}(t)(\boldsymbol{x}_i(t)-\boldsymbol{z}_j(t))^T\boldsymbol{v}_i(t)\\
&\quad + (\frac{1}{b_i^{\mathcal{R}}}+\frac{1}{d_{ij}(t)})\hat{\boldsymbol{a}}_{ij}^{\mathcal{R}}(t)(\boldsymbol{x}_i(t)-\boldsymbol{z}_j(t))^T\boldsymbol{u}_j(t)\Big)\\
&\quad + a_{ij}^{\mathcal{R}}(t)(\boldsymbol{v}_i(t)-\boldsymbol{u}_j(t))\\
&=\sum_{j\in\mathcal{R}_i}\Big(-H_{ij}^{\mathcal{R}}(t)\boldsymbol{v}_i(t) + H_{ij}^{\mathcal{R}}(t)\boldsymbol{u}_j(t)\Big).
\end{aligned}$$



(a) Human-robot trajectories, $n_h = 1$ (b) Human-robot trajectories, $n_h = 10$ (c) Human-robot trajectories, $n_h = 20$ (d) Human-robot trajectories, $n_h = 40$



(e) Convergence of $\boldsymbol{z}_i(t)$, $n_h = 1$ (f) Convergence of $\boldsymbol{z}_i(t)$, $n_h = 10$ (g) Convergence of $\boldsymbol{z}_i(t)$, $n_h = 20$ (h) Convergence of $\boldsymbol{z}_i(t)$, $n_h = 40$



(i) Convergence of $\boldsymbol{x}_i(t)$, $n_h = 1$ (j) Convergence of $\boldsymbol{x}_i(t)$, $n_h = 10$ (k) Convergence of $\boldsymbol{x}_i(t)$, $n_h = 20$ (l) Convergence of $\boldsymbol{x}_i(t)$, $n_h = 40$

Fig. 1: Trajectories and the convergence of humans and robots for $n_h = 1, 10, 20, 40$.

Similarly, recall that $a_{ij}^{\mathcal{H}}(t) = \frac{a_i^{\mathcal{H}}}{d_{ij}(t)} \exp[\frac{r_i - d_{ij}(t)}{b_i^{\mathcal{H}}}]$, and its derivative is given by

$$\begin{aligned}\dot{a}_{ij}^{\mathcal{H}}(t) = &-\frac{a_i^{\mathcal{H}}}{b_i^{\mathcal{H}}} \exp[\frac{r_i - d_{ij}(t)}{b_i^{\mathcal{H}}}] \frac{\dot{d}_{ij}(t)}{d_{ij}(t)} \\ &- a_i^{\mathcal{H}} \exp[\frac{r_i - d_{ij}(t)}{b_i^{\mathcal{H}}}] \frac{\dot{d}_{ij}(t)}{d_{ij}^2(t)},\end{aligned}$$

The the derivative of $\sum_{j \in \mathcal{H}\setminus\{i\}} \dot{\boldsymbol{f}}_{ij}^{soc}(t)$ is given by

$$\begin{aligned}&\sum_{j \in \mathcal{H}\setminus\{i\}} \dot{\boldsymbol{f}}_{ij}^{soc}(t) \\ &= \sum_{j \in \mathcal{H}\setminus\{i\}} \Big(\dot{a}_{ij}^{\mathcal{H}}(t)(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)) + a_{ij}^{\mathcal{H}}(t)(\dot{\boldsymbol{x}}_i(t) - \dot{\boldsymbol{x}}_j(t)) \Big) \\ &= \sum_{j \in \mathcal{H}\setminus\{i\}} \Big(\dot{a}_{ij}^{\mathcal{H}}(t)(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)) + a_{ij}^{\mathcal{H}}(t)(\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)) \Big) \\ &= \sum_{j \in \mathcal{H}\setminus\{i\}} \Big(\Big(-\frac{a_i^{\mathcal{H}}}{b_i^{\mathcal{H}}} \exp[\frac{r_i - d_{ij}(t)}{b_i^{\mathcal{H}}}] \frac{\dot{d}_{ij}(t)}{d_{ij}(t)} \\ &\quad - a_i^{\mathcal{H}} \exp[\frac{r_i - d_{ij}(t)}{b_i^{\mathcal{H}}}] \frac{\dot{d}_{ij}(t)}{d_{ij}^2(t)} \Big)(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)) \\ &\quad + a_{ij}^{\mathcal{H}}(t)(\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)) \Big) \\ &= \sum_{j \in \mathcal{H}\setminus\{i\}} \Big(-(\frac{a_i^{\mathcal{H}}}{b_i^{\mathcal{H}}} + \frac{a_i^{\mathcal{H}}}{d_{ij}(t)}) \exp[(r_i - d_{ij}(t))/b_i^{\mathcal{H}}] \\ &\quad \cdot \frac{(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t))^T (\dot{\boldsymbol{x}}_i(t) - \dot{\boldsymbol{x}}_j(t))(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t))}{d_{ij}^2(t)} \\ &\quad + a_{ij}^{\mathcal{H}}(t)(\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)) \Big) \\ &= \sum_{j \in \mathcal{H}\setminus\{i\}} \Big(-(\frac{1}{b_i^{\mathcal{H}}} + \frac{1}{d_{ij}(t)})(\hat{\boldsymbol{a}}_{ij}^{\mathcal{H}}(t))^T (\dot{\boldsymbol{x}}_i(t) - \dot{\boldsymbol{x}}_j(t)) \\ &\quad \cdot (\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)) + a_{ij}^{\mathcal{H}}(t)(\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)) \Big) \\ &= \sum_{j \in \mathcal{H}\setminus\{i\}} \Big(-(\frac{1}{b_i^{\mathcal{H}}} + \frac{1}{d_{ij}(t)})(\hat{\boldsymbol{a}}_{ij}^{\mathcal{H}}(t))^T (\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)) \\ &\quad \cdot (\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)) + a_{ij}^{\mathcal{H}}(t)(\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)) \Big) \\ &= \sum_{j \in \mathcal{H}\setminus\{i\}} \Big(-(\frac{1}{b_i^{\mathcal{H}}} + \frac{1}{d_{ij}(t)}) \hat{\boldsymbol{a}}_{ij}^{\mathcal{H}}(t)(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t))^T \\ &\quad \cdot (\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)) + a_{ij}^{\mathcal{H}}(t)(\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)) \Big) \\ &= -\sum_{j \in \mathcal{H}\setminus\{i\}} \Big(H_{ij}^{\mathcal{H}}(t)(\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t)) \Big).\end{aligned}$$

Notice that

$$\dot{\boldsymbol{f}}_i^*(t) = -K(t)\boldsymbol{q}_i(t) - \dot{K}(t)\boldsymbol{q}_i(t).$$

Then we have the derivative of $\tilde{\boldsymbol{f}}_i(t)$ by combining the above results.

References

[1] W. Grosshandler, N. Bryner, D. Madrzykowski, and K. Juntz, "Report of the technical investigation of the Station Nightclub fire," *Journal of Machine Learning Research*, vol. 12, no. 6, 2011.

[2] P. Robinette, P. A. Vela, and A. M. Howard, "Information propagation applied to robot-assisted evacuation," in *Proc. Int. Conf. Robotics and Automation (ICRA)*, 2012, pp. 856–861.

[3] A. R. Wagner, “Robot-guided evacuation as a paradigm for human-robot interaction research,” *Frontiers in Robotics and AI*, vol. 8, 2021.

[4] B. Tang, C. Jiang, H. He, and Y. Guo, “Human mobility modeling for robot-assisted evacuation in complex indoor environments,” *IEEE Trans. Human-Machine Systems*, vol. 46, no. 5, pp. 694–707, 2016.

[5] E. Boukas, I. Kostavelis, A. Gasteratos, and G. C. Sirakoulis, “Robot guided crowd evacuation,” *IEEE Trans. Automation Science and Engineering*, vol. 12, no. 2, pp. 739–751, 2014.

[6] C. Jiang, Z. Ni, Y. Guo, and H. He, “Learning human–robot interaction for robot-assisted pedestrian flow optimization,” *IEEE Trans. Systems, Man, and Cybernetics: Systems*, vol. 49, no. 4, pp. 797–813, 2017.

[7] A. Bahamid, A. M. Ibrahim, A. Ibrahim, I. Z. Zahurin, and A. N. Wahid, “Intelligent robot-assisted evacuation: A review,” in *Journal of Physics: Conference Series*, vol. 1706, no. 1, 2020, p. 012159.

[8] Z. Wan, C. Jiang, M. Fahad, Z. Ni, Y. Guo, and H. He, “Robot-assisted pedestrian regulation based on deep reinforcement learning,” *IEEE Trans. Cybernetics*, vol. 50, no. 4, pp. 1669–1682, 2018.

[9] L. F. Henderson, “On the fluid mechanics of human crowd motion,” *Transportation research*, vol. 8, no. 6, pp. 509–515, 1974.

[10] M. Okada and T. Ando, “Optimization of personal distribution for evacuation guidance based on vector field,” in *Proc. IEEE/RSJ Int. Conf. Intelligent Robots and Systems (IROS)*, 2011, pp. 3673–3678.

[11] Z. Liu, B. Wu, and H. Lin, “Coordinated robot-assisted human crowd evacuation,” in *Proc. IEEE Conf. Decision and Control (CDC)*, 2018, pp. 4481–4486.

[12] T. Zheng, Z. Yuan, M. Nayyar, A. Wagner, M. Zhu, and H. Lin, “Multi-robot-assisted human crowd evacuation using navigation velocity fields,” in *Proc. IEEE Conf. Decision and Control (CDC)*, 2022, To appear.

[13] L. Shan, L. Chang, S. Xu, C. Jiang, and Y. Guo, “Robot-assisted pedestrian flow control of a controlled pedestrian corridor,” *International Journal of Advanced Robotic Systems*, vol. 15, no. 6, p. 1729881418814694, 2018.

[14] M. Zhou, H. Dong, P. A. Ioannou, Y. Zhao, and F.-Y. Wang, “Guided crowd evacuation: approaches and challenges,” *IEEE/CAA Journal of Automatica Sinica*, vol. 6, no. 5, pp. 1081–1094, 2019.

[15] X. Wang, X. Zheng, and Y. Cheng, “Evacuation assistants: An extended model for determining effective locations and optimal numbers,” *Physica A: Statistical Mechanics and its Applications*, vol. 391, no. 6, pp. 2245–2260, 2012.

[16] Y. Yang, D. V. Dimarogonas, and X. Hu, “Optimal leader-follower control for crowd evacuation,” in *Proc. IEEE Conf. Decision and Control (CDC)*, 2013, pp. 2769–2774.

[17] M. Zhou, H. Dong, Y. Zhao, Y. Zhang, and P. A. Ioannou, “Optimal number and location planning of evacuation leader in subway stations,” *IFAC-PapersOnLine*, vol. 51, no. 9, pp. 410–415, 2018.

[18] Y. Ma, R. K. K. Yuen, and E. W. M. Lee, “Effective leadership for crowd evacuation,” *Physica A: Statistical Mechanics and its Applications*, vol. 450, pp. 333–341, 2016.

[19] D. Helbing, I. J. Farkas, P. Molnar, and T. Vicsek, “Simulation of pedestrian crowds in normal and evacuation situations,” *Pedestrian and evacuation dynamics*, vol. 21, no. 2, pp. 21–58, 2002.

[20] C.-T. Chen, *Linear System Theory and Design, 3rd Ed.* Oxford University Press, Inc., 1999.

[21] H. K. Khalil, “Nonlinear systems third edition,” *Patience Hall*, vol. 115, 2002.

[22] G. Strang, *Linear algebra and its applications*. Delhi, India: Cengage Learning, 2006.

[23] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge university press, 2004.