CYPMLI: WISL-MINIMIZED UNIMODULAR SEQUENCE DESIGN VIA POWER METHOD-LIKE ITERATIONS

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ABSTRACT

To facilitate target localization, active radar signals or sequences are designed to have low auto-correlation. This goal is typically achieved by the minimization of the auto-correlation integrated sidelobe level (ISL) metric, or the weighted more general version of ISL, known as the WISL metric. In this work, we propose an efficient approach to WISL minimization for unimodular sequence design that takes advantage of the low-cost and easily implementable *power method-like iterations*. Several numerical results are presented to illustrate the effectiveness of the proposed method.

Index Terms— Auto-correlation, power method-like iterations, radar signals, unimodular sequences, weighted integrated sidelobe level.

1. INTRODUCTION

Radar signal optimization has attracted a significant deal of interest in the last two decades owing to the emergence of stronger computational hardware as well as compatible adaptive illumination devices [1–3]. Particularly, the practitioners are more interested in unimodular (also known as constant-modulus) signals due to their minimal peak-to-average-ratio (PAR), which facilitates a uniform temporal and spatial power allocation, as well as the fact that the signal generation can be done using low-cost amplifiers, without being exposed to the perils of gain non-linearity [4–6].

Identifying and examining sequences with desirable aperiodic correlation properties is typically a more arduous task compared to sequences with good periodic correlation. This is because constructing sequences with exact impulsive aperiodic autocorrelation, unlike periodic correlations, is not feasible, as discussed in [7]. Therefore, the statistical signal processing literature has dedicated significant efforts to the search for low autocorrelation sequences [1].

A well-studied approach to the design of sequences with low (or good) correlation properties is through the minimization of the weighted integrated sidelobe level (WISL) [2, 4, 6]. The minimization of the WISL produces an optimization problem with a quartic objective in terms of the radar sequence which is deemed to be a difficult problem. Moreover, it deals with a considerable number of local optima, while many such local optima are in fact known to have a good quality for deployment [5, 8, 9]. In [4], the minimization of WISL has been accomplished by the weighted Cyclic Algorithm New (WeCAN). Majorization-minimization-based algorithms have been investigated to minimize the WISL or the weighted peak sidelobe level (WPSL) in [6]. To minimize the combination of the peak sidelobe level (PSL) and WISL, the iterative Coordinate-Descent (CD) framework was proposed in [8, 10].

In this paper, we will design sequences with good correlation in terms of the WISL metric by deploying one of the most efficient solvers for unimodular quadratic programming (UQP), known as the *power method-like iterations* (PMLI). Inspired by the power method, the PMLI algorithm takes advantage of simple matrix vector multiplications, leading to a low computational cost algorithm [5, 11–13]. In Section 2, we transform the quartic WISL objective to one that is quadratic. Section 3 presents a brief introduction to PMLI. The *Cy*clic *PMLI* (CyPMLI) algorithm is then proposed based on the cyclic application of PMLI to efficiently tackle the WISL minimization problem. The numerical results are presented in Section 4 to validate the performance of the proposed method to design a WISL-minimized unimodular sequence. Finally, Section 5 concludes the paper.

Notation: Throughout this paper, we use boldface lowercase, boldface uppercase, and calligraphic letters for vectors, matrices, and sets, respectively. $\mathbb{C}$ and $\mathbb{R}$ represent the set of complex and real numbers, respectively. $(\cdot)^\top$ and $(\cdot)^{\mathrm{H}}$ denote the vector/matrix transpose, and the Hermitian transpose, respectively. $\mathrm{Tr}(.)$ denotes the trace of the matrix argument. $\mathbf{I}_N$ is the identity matrix of size N. Finally, $\mathbf{0}_{N_1\times N_2}$ is the all zero matrix of size $N_1 \times N_2$.

2. WISL METRIC: A QUARTIC TO QUADRATIC TRANSFORMATION

Consider the set of complex unimodular sequences defined as

$$\Omega^N = \left\{\mathbf{s}\in\mathbb{C}^N | s(l) = e^{\mathrm{j}\omega_l}, \omega_l \in [0, 2\pi),\ 0\le l\le N-1\right\}. \tag{1}$$

The autocorrelation function of the signal $\mathbf{s}$ is defined as:

$$\begin{aligned} r_k^{\mathcal{P}} &= \sum_{l=0}^{N-1} s(l)s^*(l+k)_{\text{mod } N} \\ r_k^{\mathcal{AP}} &= \sum_{l=0}^{N-k-1} s(l)s^*(l+k) = r_{-k}^{\mathcal{AP}\star} \end{aligned} \tag{2}$$

where $k\in\{0,\cdots,N-1\}$, in the periodic and aperiodic cases respectively, with the signal power calculated at $k=0$. The above functions can be written in a compact form as:

$$r_k^{\mathcal{P}} = \mathbf{s}^{\mathrm{H}}\mathbf{J}_k^{\mathcal{P}}\mathbf{s}, \quad r_k^{\mathcal{AP}} = \mathbf{s}^{\mathrm{H}}\mathbf{J}_k^{\mathcal{AP}}\mathbf{s}, \tag{3}$$

where $\mathbf{J}_k^{\mathcal{P}}$ and $\mathbf{J}_k^{\mathcal{AP}}$ are the periodic and aperiodic shift matrix operators that shift the vector argument by k indices. For the sake of brevity, in this paper we are going to focus on the aperiodic case that

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is deemed to be more challenging and denote the associated shift matrix as $\mathbf{J}_k$. Here $\mathbf{J}_k = \mathbf{J}_{-k}^{\top}$ is given by [14],

$$\mathbf{J}_k\left(i,j\right) = \begin{cases} 1 & i-j+k=0, \\ 0 & i-j+k\neq 0. \end{cases} \tag{4}$$

It is straightforward to verify that the derivations for the periodic case would be similar.

The weighted integrated sidelobe level (WISL) of the autocorrelation can be defined as [4]:

$$f(\mathbf{s}) = \sum_{k=1}^{N-1} \omega_k |r_k|^2, \tag{5}$$

with weights $\omega_k \geq 0$ that are predetermined based on the emphasis given to minimizing different autocorrelation lags. It is easy to see that the WISL formulation subsumes the ISL metric as a special case by simply considering unit $\{\omega_k\}$ [6].

The problem of interest is to design sequences $\mathbf{s}$ that minimize the WISL metric:

$$\underset{\mathbf{s}\in\Omega^N}{\text{minimize}} \quad f\left(\mathbf{s}\right). \tag{6}$$

Note that $f\left(\mathbf{s}\right)$ can be written as

$$\begin{aligned} f(\mathbf{s}) &= \sum_{k=1}^{N-1} \omega_k |\mathbf{s}^{\mathrm{H}}\mathbf{J}_k\mathbf{s}|^2, \\ &= \sum_{k=1}^{N-1} \omega_k \mathbf{s}^{\mathrm{H}}\mathbf{J}_k\mathbf{s}\mathbf{s}^{\mathrm{H}}\mathbf{J}_k\mathbf{s}, \end{aligned} \tag{7}$$

which is quartic with respect to $\mathbf{s}$. Therefore, optimizing the radar sequence $\mathbf{s}$ appears to be difficult. To find a quadratic alternative to $f(\mathbf{s})$, at first, we define $g\left(\mathbf{s}_1,\mathbf{s}_2\right)$ as follows:

$$g\left(\mathbf{s}_1,\mathbf{s}_2\right) = \sum_{k=1}^{N-1} \omega_k \mathbf{s}_1^{\mathrm{H}}\mathbf{J}_k\mathbf{s}_2\mathbf{s}_2^{\mathrm{H}}\mathbf{J}_k^{\mathrm{H}}\mathbf{s}_1, \tag{8}$$

where $f(\mathbf{s}) = g(\mathbf{s},\mathbf{s})$. Moreover, let

$$\mathbf{G}(\mathbf{s}) = \sum_{k=1}^{N-1} \omega_k \mathbf{J}_k\mathbf{s}\mathbf{s}^{\mathrm{H}}\mathbf{J}_k^{\mathrm{H}}. \tag{9}$$

It is also interesting to observe that if either $\mathbf{s}_1$ or $\mathbf{s}_2$ are fixed, minimizing $g\left(\mathbf{s}_1,\mathbf{s}_2\right)$ with respect to the other variable can be done via a UQP formulation [5, 11]:

$$\underset{\mathbf{s}_j\in\Omega^N}{\text{minimize}} \quad \mathbf{s}_j^{\mathrm{H}}\mathbf{G}(\mathbf{s}_i)\mathbf{s}_j, \quad i\neq j\in\{1,2\}. \tag{10}$$

Let λ_m be the maximum eigenvalue of $\mathbf{G}(\mathbf{s}_i)$, where $\lambda_m\mathbf{I}\succeq\mathbf{G}(\mathbf{s}_i)$. Thus, $\tilde{\mathbf{G}}(\mathbf{s}_i) = \lambda_m\mathbf{I} - \mathbf{G}(\mathbf{s}_i)$ is positive definite. As a result, (10) can be reformulated as

$$\underset{\mathbf{s}_j\in\Omega^N}{\text{maximize}} \quad \mathbf{s}_j^{\mathrm{H}}\tilde{\mathbf{G}}(\mathbf{s}_i)\mathbf{s}_j, \quad i\neq j\in\{1,2\}. \tag{11}$$

Note that a diagonal loading with $\lambda_m\mathbf{I}$ has no effect on the solution of (10) due to the fact that $\mathbf{s}_j^{\mathrm{H}}\tilde{\mathbf{G}}(\mathbf{s}_i)\mathbf{s}_j = \lambda_m N - \mathbf{s}_j^{\mathrm{H}}\mathbf{G}(\mathbf{s}_i)\mathbf{s}_j$.

In order to guarantee that a minimization of $g\left(\mathbf{s}_1,\mathbf{s}_2\right)$ leads to minimize $f(\mathbf{s})$, a connection must be made to show that $\mathbf{s}_1$ and $\mathbf{s}_2$ obtained from (11) are convergent. We may consider:

$$\begin{aligned} &\underset{\mathbf{s}_j\in\Omega^N}{\text{maximize}} \quad \mathbf{s}_j^{\mathrm{H}}\tilde{\mathbf{G}}(\mathbf{s}_i)\mathbf{s}_j, \quad i\neq j\in\{1,2\} \\ &\text{subject to} \quad \mathbf{s}_i = \mathbf{s}_j. \end{aligned} \tag{12}$$

By adding the norm-2 error between $\mathbf{s}_1$ and $\mathbf{s}_2$ as a *regularizer* with the Lagrangian multiplier to (11), we have:

$$\underset{\mathbf{s}_j\in\Omega^N}{\text{minimize}} \quad \mathbf{s}_j^{\mathrm{H}}\mathbf{G}(\mathbf{s}_i)\mathbf{s}_j + \eta\left\|\mathbf{s}_i-\mathbf{s}_j\right\|_2^2, \quad i\neq j\in\{1,2\}, \tag{13}$$

where η is the Lagrangian multiplier. The regularizer $\left\|\mathbf{s}_i-\mathbf{s}_j\right\|_2^2$ is a quadratic function as well. Consequently, the objective function of (13) is recast as

$$\begin{aligned} &\mathbf{s}_j^{\mathrm{H}}\mathbf{G}(\mathbf{s}_i)\mathbf{s}_j + \eta\left\|\mathbf{s}_i-\mathbf{s}_j\right\|_2^2, \\ &= \mathbf{s}_j^{\mathrm{H}}\mathbf{G}(\mathbf{s}_i)\mathbf{s}_j - 2\eta\,\mathrm{Re}\left(\mathbf{s}_j^{\mathrm{H}}\mathbf{s}_i\right) + 2\eta N, \\ &= \begin{pmatrix}\mathbf{s}_j\\ 1\end{pmatrix}^{\mathrm{H}} \underbrace{\begin{pmatrix}\mathbf{G}(\mathbf{s}_i) & -\eta\mathbf{s}_i \\ -\eta\mathbf{s}_i^{\mathrm{H}} & 2\eta N\end{pmatrix}}_{\mathcal{G}(\mathbf{s}_i)} \begin{pmatrix}\mathbf{s}_j\\ 1\end{pmatrix}. \end{aligned} \tag{14}$$

Thus, the final UQP formulation for (6) is given by

$$\underset{\mathbf{s}_j\in\Omega^N}{\text{maximize}} \quad \begin{pmatrix}\mathbf{s}_j\\ 1\end{pmatrix}^{\mathrm{H}} \underbrace{\begin{pmatrix}\lambda_M\mathbf{I}-\mathbf{G}(\mathbf{s}_i) & \eta\mathbf{s}_i \\ \eta\mathbf{s}_i^{\mathrm{H}} & \lambda_M-2\eta N\end{pmatrix}}_{\hat{\mathbf{G}}(\mathbf{s}_i)} \begin{pmatrix}\mathbf{s}_j\\ 1\end{pmatrix}, \tag{15}$$

where λ_M is the maximum eigenvalue of $\mathcal{G}(\mathbf{s}_i)$, and $\hat{\mathbf{G}}(\mathbf{s}_i) = \lambda_M\mathbf{I} - \mathcal{G}(\mathbf{s}_i)$.

3. CYPMLI ALGORITHM

Due to the NP-hard nature of UQP, it has a highly multi-modal optimization objective. Finding the local optima of UQP is not only useful to tackle the problem itself, but also to improve the UQP approximate solutions obtained by semi-definite relaxation (SDR) or other optimization algorithms. PMLI is a computationally efficient procedure which resembles the well-studied *power iteration* for computing the dominant eigenvalue/vector pairs of matrices [5]. In particular, it was demonstrated in [5, 11, 12] that UQP solutions can be efficiently approximated by deploying the PMLI.

Assume $\hat{\mathbf{G}}$ is positive definite and $\left\{\bar{\mathbf{s}}_j^{(t+1)}\right\}_{t=0}^{\infty}$ be a sequence of unimodular codes where $\bar{\mathbf{s}}_j^{(t+1)}$ is the minimizer of the following criterion:

$$\underset{\bar{\mathbf{s}}_j^{(t+1)}\in\Omega^N}{\text{minimize}} \quad \left\|\bar{\mathbf{s}}_j^{(t+1)} - \hat{\mathbf{G}}(\bar{\mathbf{s}}_i^{(t)})\bar{\mathbf{s}}_j^{(t)}\right\|_2, \tag{16}$$

where $\bar{\mathbf{s}}_j = \begin{pmatrix}\mathbf{s}_j^{\top} & 1\end{pmatrix}^{\top}$. Note that

$$\left\|\bar{\mathbf{s}}_j^{(t+1)} - \hat{\mathbf{G}}(\bar{\mathbf{s}}_i^{(t)})\bar{\mathbf{s}}_j^{(t)}\right\|_2^2 = \text{const} - 2\,\mathrm{Re}\left(\bar{\mathbf{s}}_j^{(t+1)\mathrm{H}}\hat{\mathbf{G}}(\bar{\mathbf{s}}_i^{(t)})\bar{\mathbf{s}}_j^{(t)}\right). \tag{17}$$

If $\mathbf{s}_j^{(t+1)} \neq \mathbf{s}_j^{(t)}$ and $\hat{\mathbf{G}}\succ 0$,

$$\left(\bar{\mathbf{s}}_j^{(t+1)} - \bar{\mathbf{s}}_j^{(t)}\right)^{\mathrm{H}}\hat{\mathbf{G}}(\bar{\mathbf{s}}_i^{(t)})\left(\bar{\mathbf{s}}_j^{(t+1)} - \bar{\mathbf{s}}_j^{(t)}\right) > 0, \tag{18}$$

which implies

$$\begin{aligned} \bar{\mathbf{s}}_j^{(t+1)\mathrm{H}}\hat{\mathbf{G}}\bar{\mathbf{s}}_j^{(t+1)} &> 2\,\mathrm{Re}\left(\bar{\mathbf{s}}_j^{(t+1)\mathrm{H}}\hat{\mathbf{G}}\bar{\mathbf{s}}_j^{(t)}\right) - \bar{\mathbf{s}}_j^{(t)\mathrm{H}}\hat{\mathbf{G}}\bar{\mathbf{s}}_j^{(t)} \\ &> \bar{\mathbf{s}}_j^{(t)\mathrm{H}}\hat{\mathbf{G}}\bar{\mathbf{s}}_j^{(t)}, \end{aligned} \tag{19}$$

as $\mathrm{Re}\left(\bar{\mathbf{s}}_j^{(t+1)\mathrm{H}}\hat{\mathbf{G}}(\bar{\mathbf{s}}_i^{(t)})\bar{\mathbf{s}}_j^{(t)}\right) > \bar{\mathbf{s}}_j^{(t)\mathrm{H}}\hat{\mathbf{G}}(\bar{\mathbf{s}}_i^{(t)})\bar{\mathbf{s}}_j^{(t)}$ [5]. Therefore, the UQP objective is increasing through the PMLI. The desired vector

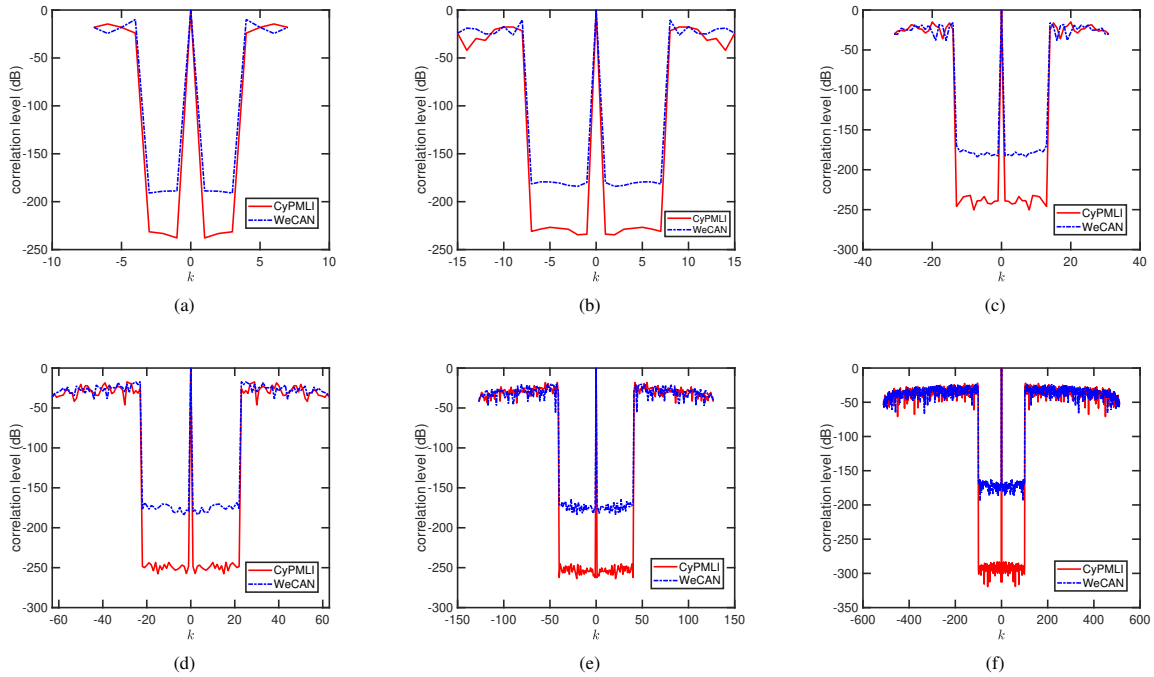

Fig. 1. Correlation levels obtained by CyPMLI and WeCAN for: a) $N = 8$, b) $N = 16$, c) $N = 32$, d) $N = 64$, e) $N = 128$, f) $N = 512$.

$\bar{\mathbf{s}}_j^{(t+1)}$ of (15) and (16) is readily evaluated by PMLI in each iteration as

$$\bar{\mathbf{s}}_j^{(t+1)} = e^{\mathrm{j}\arg\left(\hat{\mathbf{G}}(\bar{\mathbf{s}}_i^{(t)})\bar{\mathbf{s}}_j^{(t)}\right)}, \tag{20}$$

or equivalently,

$$\mathbf{s}_j^{(t+1)} = e^{\mathrm{j}\arg\left(\left(\lambda_M\mathbf{I}-\mathbf{G}(\mathbf{s}_i^{(t)})\right)\mathbf{s}_j^{(t)}+\eta\mathbf{s}_i^{(t)}\right)}. \tag{21}$$

It is worth pointing out that the difference of recursions for (20) and (21) compared to that of (11), i.e., $\mathbf{s}_j^{(t+1)} = e^{\mathrm{j}\arg\left(\left(\lambda_m\mathbf{I}-\mathbf{G}(\mathbf{s}_i^{(t)})\right)\mathbf{s}_j^{(t)}\right)}$ is merely in including more *momentum* or the effect of η at each iteration. Such power method-like iterations are already shown to be convergent in terms of the signals, implying that $\mathbf{s}_1$ and $\mathbf{s}_2$ will be converging to each other as well. We call our algorithm, *C*yclic *PMLI* (CyPMLI) which is summarized in Algorithm 1.

To guarantee the convergence of PMLI and a monotonically increasing objective function, $\hat{\mathbf{G}}(\mathbf{s}_i^{(t)})$ must be positive definite which is achieved by choosing proper λ_M. The following theorem is proposed to select λ_M for our algorithm:

Theorem 1. *To guarantee the positive definiteness of* $\hat{\mathbf{G}}(\bar{\mathbf{s}}^{(t)})$ *in each iteration, its maximum eigenvalue must meet the following relation:*

$$\lambda_M \geq \max\left\{\lambda_{max}, \frac{\lambda_{max}+2N\eta+\sqrt{(\lambda_{max}-2N\eta)^2+4N\eta^2}}{2}\right\}, \tag{22}$$
$$= \frac{\lambda_{max}+2N\eta+\sqrt{(\lambda_{max}-2N\eta)^2+4N\eta^2}}{2},$$

where η *is the Lagrangian multiplier,* N *is the sequence length, and* λ_{max} *is the maximum eigenvalue of* $\mathbf{G}$ *defined in (9).*

Proof. According to the Schur complement, $\hat{\mathbf{G}}(\mathbf{s}_i)$ is positive definite if and only if [15, 16]:

- $\lambda_M\mathbf{I} - \mathbf{G}(\mathbf{s}_i)$ is positive definite.
- $(\lambda_M - 2\eta N) - \eta^2\mathbf{s}_i^{\mathrm{H}}\left(\lambda_M\mathbf{I} - \mathbf{G}(\mathbf{s}_i)\right)^{-1}\mathbf{s}_i > 0.$

The first condition implies that $\lambda_M \geq \lambda_{\max}$ where $\lambda_{\max}$ is the maximum eigenvalue of $\mathbf{G}(\mathbf{s}_i)$. To achieve a boundary of λ_M from the second condition, one can utilize the below relation:

$$\begin{aligned}&(\lambda_M - 2\eta N) - \eta^2\mathbf{s}_i^{\mathrm{H}}\left(\lambda_M\mathbf{I} - \mathbf{G}(\mathbf{s}_i)\right)^{-1}\mathbf{s}_i \geq \\ &(\lambda_M - 2\eta N) - \eta^2\mathbf{s}_i^{\mathrm{H}}\left(\lambda_M - \lambda_{\max}\right)^{-1}\mathbf{I}\mathbf{s}_i.\end{aligned} \tag{23}$$

By choosing λ_M in a such way to guarantee the lower term in (23) to be the positive value, the second condition of the aforementioned Schur complement is met as well. Thus, the optimal value of λ_M may satisfy the following inequality which is obtained from (23) considering $\mathbf{s}^{\mathrm{H}}\mathbf{s} = N$:

$$\lambda_M^2 - (2N\eta + \lambda_{\max})\lambda_M + 2N\eta\lambda_{\max} - \eta^2 N \geq 0. \tag{24}$$

It is straightforward to verify that to satisfy both $\lambda_M \geq \lambda_{\max}$, and (24), the relation (22) must be met for λ_M. Also, it is easy to verify that the second term of the $\max(\cdot)$ in (22) is greater than $\lambda_{\max}$ as follows:

$$\begin{aligned}\lambda_{\max} + 2N\eta + \sqrt{(\lambda_{\max}-2N\eta)^2+4N\eta^2} \geq 2\lambda_{\max}, \\ \sqrt{(\lambda_{\max}-2N\eta)^2+4N\eta^2} \geq \lambda_{\max} - 2N\eta,\end{aligned} \tag{25}$$

where the last inequality is trivial and completes the proof. □

According to Theorem 1, λ_M should not only be larger the the maximum eigenvalue of $\mathbf{G}$, but should also satisfy the tighter inequality discussed in (22). Although the theorem proposes a condition on λ_M, it efficiently ensures there is enough momentum η to guarantee the solutions $\mathbf{s}_1$ and $\mathbf{s}_2$ are convergent.

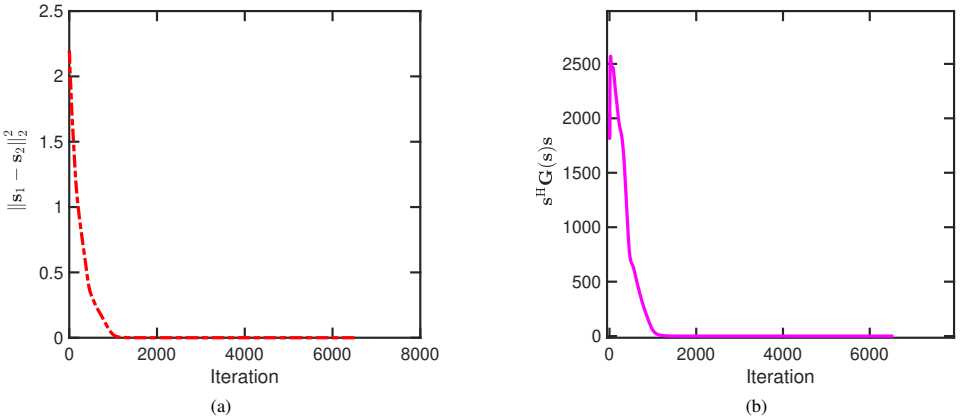

Fig. 2. Decreasing behaviour of (a) the regularizer $\|\mathbf{s}_1-\mathbf{s}_2\|_2^2$, and (b) the WISL during the iterations of the CyPMLI algorithm. As can be observed, by increasing the number of iterations, (a) $\mathbf{s}_1$ and $\mathbf{s}_2$ converges to each other and to the optimal sequence $\mathbf{s}^\star$. (b) demonstrates that the CyPMLI properly minimizes the WISL (7).

Algorithm 1 The CyPMLI algorithm for WISL minimization.

Input: Initialization values $\mathbf{s}_1^{(0)}$ and $\mathbf{s}_2^{(0)}$, weights of WISL $\{\omega_k\}$, Lagrangian multiplier η, total number of iterations $\mathcal{E}$.
Output: Optimized unimodular sequence $\mathbf{s}^\star$.

$\triangleright$ $\mathbf{s}_1^{(t)}$ and $\mathbf{s}_2^{(t)}$ are the solutions at the t-th iteration.
$\triangleright$ The objective function is $g\left(\bar{\mathbf{s}}_i,\bar{\mathbf{s}}_j\right)=\bar{\mathbf{s}}_j^{\mathrm{H}}\hat{\mathbf{G}}(\mathbf{s}_i)\bar{\mathbf{s}}_j$.
for $t=0:\mathcal{E}-1$ **do**
  $i\leftarrow 0$.
  $\bar{\mathbf{s}}_2^{(i)}\leftarrow\bar{\mathbf{s}}_2^{(t)}$.
  while until convergence **do**
    $\mathbf{s}_2^{(i+1)}\leftarrow e^{\,\mathrm{j}\arg\left(\begin{pmatrix}\mathbf{I}_N\\ \mathbf{0}_{1\times N}\end{pmatrix}^\top\hat{\mathbf{G}}(\mathbf{s}_1^{(t)})\bar{\mathbf{s}}_2^{(i)}\right)}$.
    $i\leftarrow i+1$.
  end while
  $\mathbf{s}_2^{(t+1)}\leftarrow\mathbf{s}_2^{(i)}$.
  $j\leftarrow 0$.
  $\bar{\mathbf{s}}_1^{(j)}\leftarrow\bar{\mathbf{s}}_1^{(t)}$.
  while until convergence **do**
    $\mathbf{s}_1^{(j+1)}\leftarrow e^{\,\mathrm{j}\arg\left(\begin{pmatrix}\mathbf{I}_N\\ \mathbf{0}_{1\times N}\end{pmatrix}^\top\hat{\mathbf{G}}(\mathbf{s}_2^{(t+1)})\bar{\mathbf{s}}_1^{(j)}\right)}$.
    $j\leftarrow j+1$.
  end while
end for
return $\mathbf{s}^\star\leftarrow\mathbf{s}_1^{(\mathcal{E})}$ or $\mathbf{s}_2^{(\mathcal{E})}$.

4. NUMERICAL INVESTIGATION FOR CYPMLI

To scrutinize our method, we consider the design of a unimodular sequence of length N, where $N\in\{8,16,32,64,128,512\}$. Suppose that we are interested in suppressing the correlation terms $\{r_1,\cdots,r_c\}$. Therefore, ω_k in Eq. (5) will have the following form:

$$\omega_k=\begin{cases}1, & k\in[1,c]\\ 0, & k\in[c+1,N]\end{cases}. \tag{26}$$

We compare our method with WeCAN. Both methods are initialized by randomly generated sequences. The correlation levels of the designed sequences are shown in Fig. 1 for different values of N which are averaged over 15 experiments. In this example, we afford both algorithms the same design time for fair comparison. As can be seen, it appears that the obtained unimodular sequences by CyPMLI have lower correlation sidelobes at the required lags comparing to that of WeCAN, with the correlation level defined in dB as $20\log_{10}\left|\frac{r_k}{r_0}\right|$. To compare the complexity cost of our approach with WeCAN, we designed a unimodular sequence of length $N=100$. In this case, we impose the stopping criterion such that the designed sequences by both algorithms have the same correlation levels. The CPU time for WeCAN is $23s$ while CyPMLI requires $0.3s$ to achieve such performance. Therefore, we may conclude the complexity of CyPMLI is less than that of WeCAN, presumably due to using a simple matrix-vector multiplications in CyPMLI.

To further investigate the performance of the CyPMLI algorithm, we present the decreasing behaviour of the regularizer $\|\mathbf{s}_1-\mathbf{s}_2\|_2^2$ and the WISL metric (7), in Fig. 2(a) and Fig. 2(b), respectively. As can be seen in Fig. 2(a), in the iterative process of the CyPMLI algorithm, the sequences $\mathbf{s}_1$ and $\mathbf{s}_2$ are getting closer and after near 1000 iterations, they virtually approach to each other. By using the bi-quadratic transformation, we turn the WISL (7) to (13). In Fig. 2(b), we show that the CyPMLI algorithm can simultaneously optimize both bi-quadratic (13) and quartic (7) WISL formulations.

5. SUMMARY

To design a unimodular sequence with good correlation properties, we proposed the CyPMLI algorithm. In this approach, at first we present the UQP formulation for the WISL minimization problem and then solve this UQP problem via a cyclic application of the PMLI. In the numerical results, we showcased the effectiveness of the proposed approach by comparing it to WeCAN which is a well-known approach for WISL minimization.

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