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From Three Dimensional Manifolds to Modular Tensor Categories

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Abstract: Using M-theory in physics, Cho et al. (JHEP 2020:115 (2020)) recently outlined a program that connects two parallel subjects of three dimensional manifolds, namely, geometric topology and quantum topology. They suggest that classical topological invariants such as Chern-Simons invariants of $SL(2, \mathbb{C})$-flat connections and $SL(2, \mathbb{C})$-adjoint Reidemeister torsions of a three manifold can be packaged together to produce a $(2 + 1)$-topological quantum field theory, which is essentially equivalent to a modular tensor category. It is further conjectured that every modular tensor category can be obtained from a three manifold and a semi-simple Lie group. In this paper, we study this program mathematically, and provide strong support for the feasibility of such a program. The program produces an algorithm to generate the potential modular T-matrix and the quantum dimensions of a candidate modular data. The modular S-matrix follows from essentially a trial-and-error procedure. We find premodular tensor categories that realize candidate modular data constructed from Seifert fibered spaces and torus bundles over the circle that reveal many subtleties in the program. We make a number of improvements to the program based on our examples. Our main result is a mathematical construction of the modular data of a premodular category from each Seifert fibered space with three singular fibers and a family of torus bundles over the circle with Thurston SOL geometry. The modular data of premodular categories from Seifert fibered spaces can be realized using Temperley-Lieb-Jones categories and the ones from torus bundles over the circle are related to metaplectic categories. We conjecture that a resulting premodular category is modular if and only if the three manifold is a $\mathbb{Z}_2$-homology sphere, and condensation of bosons in the resulting properly premodular categories leads to either modular or super-modular tensor categories.

Contents

1. Introduction

There are two parallel universes in three dimensional topology for the last several decades that do not intersect much: the classical Thurston world and the quantum Jones world. One famous conjecture that hints a deep connection of the two worlds is the volume conjecture. Recently M-theory in physics suggests another surprising different connection: classical topological invariants such as Chern-Simons invariants of $SL(2, \mathbb{C})$-flat connections and $SL(2, \mathbb{C})$-adjoint Reidemeister torsions of a three manifold X can be packaged together to produce a $(2 + 1)$-topological quantum field theory (TQFT) [4], which is essentially equivalent to a modular tensor category [19]. It is further conjectured in [4] that every modular tensor category can be obtained from a three manifold and a semi-simple Lie group. In this paper, we study this program mathematically, and provide strong support for such a program. The program as outlined in [4] produces an algorithm to generate the potential modular T-matrix and the quantum dimensions of a candidate modular data. The modular S-matrix follows from essentially a trial-and-error procedure. We find premodular tensor categories that realize candidate modular data from Seifert fibered spaces and torus bundles over the circle that reveal many subtleties in the program. Our main result is a mathematical construction of the modular data of a premodular category from each Seifert fibered space with three singular fibers and some torus bundles over the circle with Thurston SOL geometry. The modular data of the premodular categories from Seifert fibered spaces can be realized using Temperley-Lieb-Jones categories and the ones from torus bundles over the circle are related to metaplectic categories [9,20]. A more general study of the torus bundle case is in [6].

We conjecture that the resulting premodular category is modular if and only if the three manifold is a $\mathbb{Z}_2$-homology sphere, and condensation of bosons in the resulting properly premodular categories leads to either modular or super-modular tensor categories.

The program from three manifolds to modular tensor categories is a far-reaching progeny of the mysterious six-dimensional super-symmetric conformal field theories (SCFTs) spawned by M-theory. Our strong support for the program indirectly provides evidence for these 6d SCFTs. The dimension reduction or compactification of these 6d SCFTs to 3d depends on a three manifold X, and in general the resulting theory $T(X)$ is a super-conformal field theory. When X is non-hyperbolic, it is argued in [4] that $T(X)$ flows to a TQFT in the infrared limit and super-symmetry is decoupled, thus we obtain a (2+1)-TQFT labeled by X, hence a MTC $\mathcal{B}_X$. The program outlined in [4] centers on an algorithm to produce the quantum dimensions and topological twists of a MTC, and a trial-and-error algorithm for the modular S-matrix. The assumption on the three manifolds X in [4] includes that X is non-hyperbolic and the $SL(2, \mathbb{C})$ representation variety of the fundamental group $\pi_1(X)$ consists of finitely many conjugacy classes that all could be conjugated into either $SU(2)$ or $SL(2, \mathbb{R})$ subgroups of $SL(2, \mathbb{C})$. Our examples show that all but the non-hyperbolic assumption can be dropped. One subtlety is that we need to use indecomposable reducible representations in our torus bundle over the circle examples. We do not know whether or not MTCs could be constructed from hyperbolic three manifolds as the program as we formulated in this paper is more flexible. The main difficulty for more examples lies in the explicit calculation of Chern-Simons (CS) invariant and adjoint Reidemeister torsion of flat connections.

An $SL(2, \mathbb{C})$-representation of $\pi_1(X)$ is the same as a flat connection of the trivial $SL(2, \mathbb{C})$-bundle. There are two well-known invariants for a flat connection: the Chern-Simons (CS) invariant and the adjoint Reidemeister torsion. Each flat connection that satisfies certain conditions would give rise to an anyon type and the Reidemeister torsion is essentially the quantum dimension and the CS invariant is the conformal weight of the anyon.

For each Seifert fibered space with three singular fibers, we define a potential modular data inspired by the many examples in [4]. All those modular data can be realized by premodular categories obtained as a $\mathbb{Z}_2$-graded product of Temperley-Lieb-Jones categories. We expect that our results can be easily generalized to any number of the singular fibers if the adjoint Reidemeister torsions of the $SL(2, \mathbb{C})$ flat connections can be calculated because the CS invariants in this case are known. It is not clear if there are new MTCs among our examples. Going beyond Seifert fibered spaces, we analyze some torus bundles over the circle and identify the resulting premodular categories as the integral subcategories of $SO(N)_2$ for odd N. An important observation for the connection to Temperley-Lieb-Jones categories for Seifert fibered spaces is a relation between the slope of a singular fiber and the order of the Kauffman variable A in Temperley-Lieb-Jones theories [20]. Essentially the slope of a singular fiber determines a root of unity A, which allows us to realize all the candidate modular data from Seifert fibered spaces with three singular fibers.

The content of the paper is as follows. In Sect. 2, we outline our version of the program taking into account the many subtleties that we encountered in our examples. We also recall the definition of CS invariant and adjoint Reidemeister torsion, and collect some known results of CS invariants of Seifert fibered spaces. In Sect. 3, we study the Seifert fibered spaces and carry out the necessary calculations of CS and torsion invariants for our examples, and do the same for torus bundles over the circle in Sect. 4. Finally, in Sect. 5, we discuss some future directions and open questions.

2. A Program from 3-Manifolds to Modular Categories

The proposed program in [4] from three manifolds to MTCs came from physics, and the paper provides an algorithm to produce the potential modular T-matrix and all quantum dimensions of a candidate modular data from irreducible representations of the fundamental groups of three manifolds to $SL(2, \mathbb{C})$. Our results in Sects. 3 and 4 that realize candidate modular data from Seifert fibered spaces and torus bundles over the circle reveal many subtleties in the program as outlined in [4]. In this section, we follow the overall program of [4] and make a number of improvements to reformulate mathematically the construction of candidate modular data from three manifolds taking into account these new subtleties.

2.1. Representation and character variety. Suppose X is an orientable connected closed 3-manifold and G is a semi-simple Lie group. The set of representations of the fundamental group $\pi_1(X)$[1] to G consists of all group homomorphisms from $\pi_1(X)$ to G, denoted by $\text{Hom}(\pi_1(X), G)$, up to conjugation. The representation variety $\mathcal{R}(X, G)$ of $\pi_1(X)$ to G is simply $\text{Hom}(\pi_1(X), G)//G$—equivalence classes of representations up to conjugation.

In this paper, we will mainly consider the case $G = SL(2, \mathbb{C})$ and its higher dimensional irreducible representations Sym^j of dimension $j + 1$. Given such a representation $\rho : \pi_1(X) \to SL(n, \mathbb{C})$, its character is the function on $\pi_1(X)$ given by $\chi_\rho(x) = \text{Tr}(\rho(x))$ for $x \in \pi_1(X)$. The character variety $\chi(X, SL(n, \mathbb{C}))$ of X consists of all such character functions. We will also denote the representation variety $\mathcal{R}(X, SL(2, \mathbb{C}))$ and character variety $\chi(X, SL(2, \mathbb{C}))$ simply as $\mathcal{R}(X)$ and $\chi(X)$. In this paper, the topology of the spaces of the representation and character varieties is not important.

There are three obvious nontrivial automorphisms of $SL(2, \mathbb{C})$ by sending an element $g \in SL(2, \mathbb{C})$ to its complex conjuagte g^*, its transpose followed by inverse $(g^t)^{-1}$, and the composition $(g^\dagger)^{-1}$ of the previous two operations. For each representation of $\pi_1(X)$ to $SL(2, \mathbb{C})$, post-composing with one of the three automorphisms of $SL(2, \mathbb{C})$ gives rise to another representation, hence representations in $\mathcal{R}(X)$ come in group of four in general. Another obvious way to change a representation ρ in $\mathcal{R}(X, G)$ is to tensor ρ with a representation of $\pi_1(X)$ to the center $Z(G)$ of G. Representations of $\pi_1(X)$ to the center $Z(G)$ are in one-one correspondence with cohomology classes in the cohomology group $H^1(X, Z(G))$.

2.2. Non-hyperbolic three manifolds. The proposed program in [4] and in this section is to produce modular tensor categories (MTCs) from closed three manifolds and show that each MTC can be obtained from at least one three manifold. It is known that different three manifolds can lead to the same MTC. As suggested in [4], we will focus on non-hyperbolic three manifolds. There are seven non-hyperbolic geometries and six can be realized by Seifert fibered three manifolds with the exception SOL [17]. The geometry $\mathbb{S}^2 \times \mathbb{R}$ is not useful for our purpose as we need representations from the fundamental group to $SL(2, \mathbb{C})$ with non-Abelian images. We will mainly consider Seifert fibered spaces in this paper, but in Sect. 4, we will also study torus bundles over the circle with SOL geometry and more subtleties arise.
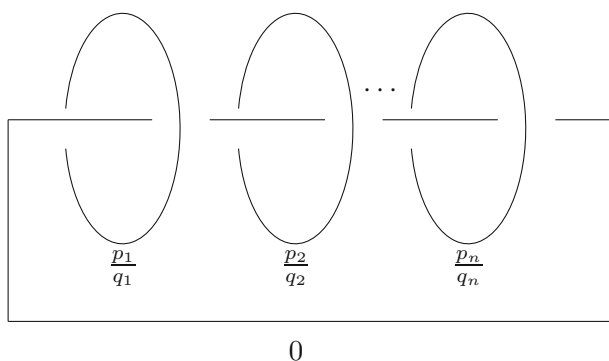
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[1] We omit the irrelevant base point.

Fig. 1. Surgery link of Seifert fibered space with base $\mathbb{S}^2$

Seifert fibered three manifolds X are those that can be foliated into disjoint union of circles and are completely enumerated [16]. In this paper, all our three manifolds are orientable, and we will denote the Seifert fibered spaces (SFSs) by the notation $X = \{b; (o, g); (p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n)\}$ as explained below. The quotient space of a SFS X, called the base orbifold B, by sending each circle, called a fiber, to a point is a topological surface. The symbol (o, g) means that the base topological surface B is an orientable closed surface of genus g.

Each fiber has a product neighbourhood $D^2 \times S^1$ in the SFS X except n singular fibers labeled by (p_i, q_i), $i = 1, \cdots, n$. The neighborhood of the i-th singular fiber is obtained from $D^2 \times [0, 1]$ by identifying the point $(x, 0)$, $x \in D^2$ with the point $(r_{a_i, p_i}(x), 1)$, where r_{a_i, p_i} is the rotation of the disk D^2 by the angle $2\pi a_i / p_i$, where $a_i \in \mathbb{Z}$ satisfies $a_i q_i = 1 \mod p_i$. The pair of coprime integers (p_i, q_i) are the corresponding surgery coefficient. The fundamental group of X fits into a short exact sequence $1 \to \pi_1(F) \to \pi_1(X) \to \pi_1^{orb}(B) \to 1$, where $\pi_1(F) \cong \mathbb{Z}$ for a regular fiber $F \cong \mathbb{S}^1$ and $\pi_1^{orb}(B)$ is the orbifold fundamental group of B (not the same as the fundamental group $\pi_1(B)$ of the topological surface B in general). The integer b in the notation is the obstruction class, which is also the order of the generator of $\pi_1(F)$ in $\pi_1^{orb}(B)$. Since we consider SFSs as three manifolds up to homeomorphism rather than as fibered spaces, we may always set b to 0.

The fundamental group of $X = \{b; (o, g); (p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n)\}$ has a presentation

$$\pi_1(X) = \langle a_j, b_j, x_i, h, \ j = 1, \cdots, g, i = 1, \cdots, n \ |$$
$$[a_j, h] = [b_j, h] = [x_i, h] = x_i^{p_i} h^{q_i} = 1, \ x_1 \cdots x_n [a_1, b_1] \cdots [a_g, b_g] = h^b \rangle. \quad (1)$$

In particular, the fundamental group of $X = \{0; (o, 0); (p_1, q_1), (p_2, q_2), (p_3, q_3)\}$ with base $\mathbb{S}^2$ and three singular fibers, denoted simply as $\{b; (p_1, q_1), (p_2, q_2), (p_3, q_3)\}$ sometimes, is

$$\pi_1(X) = \langle x_1, x_2, x_3, h | x_i^{p_i} h^{q_i} = 1, x_i h = h x_i, x_1 x_2 x_3 = h^b \rangle.$$

The orientable SFS $\{0; (o, 0); (p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n)\}$ with base $\mathbb{S}^2$ and n singular fibers has a surgery diagram shown in Fig. 1.

2.3. Chern-Simons invariant. Given an orientable connected closed three manifold X, a morphism ρ of its fundamental group $\pi_1(X)$ to a semi-simple Lie group G can be identified as the holonomy representation of a flat connection A_ρ on the trivial principal G-bundle over X. Therefore, in the following we will use the terms a representation ρ and a flat connection A interchangeably via such an identification.

Let X be a closed 3-manifold and $\rho : \pi_1(X) \longrightarrow \mathrm{SL}(2, \mathbb{C})$ be a holonomy representation. Denote by A_ρ the corresponding Lie algebra $\mathfrak{sl}(2, \mathbb{C})$-valued 1-form on X. The Chern-Simons (CS) invariant of ρ is defined as

$$\mathrm{CS}(\rho) = \frac{1}{8\pi^2} \int_X \mathrm{Tr}(dA_\rho \wedge A_\rho + \frac{2}{3} A_\rho \wedge A_\rho \wedge A_\rho) \quad \mathrm{mod}\ 1, \qquad (2)$$

where the integral with its coefficient in the front is well-defined up to integers.

The CS invariant $\mathrm{CS}(\rho)$ depends only on the character $\chi(\rho)$ of ρ [12], hence it descends from the representation variety $\mathcal{R}(X)$ to the character variety $\chi(X)$.

Auckly computed the CS invariant of SFSs for SU(2) representations in [1]. The CS invariant of SFSs for $\mathrm{SL}(2, \mathbb{C})$ representations may be known to experts. However, to make the paper self-contained, we provide a proof to compute that using method from [12].

Proposition 2.1. *Let* $X = \{0; (o, g); (p_1, q_1), (p_2, q_2), \cdots, (p_k, q_k)\}$ *be an SFS with the presentation of* $\pi_1(X)$ *given in Eq.* (1) *with* $b = 0$. *Choose integers* s_j *and* r_j *such that* $p_j s_j - q_j r_j = 1$. *Suppose* $\rho : \pi_1(X) \to \mathrm{SL}(2, \mathbb{C})$ *is non-Abelian such that* $Tr(\rho(x_j)) = 2\cos\frac{2\pi n_j}{p_j}$, *then*

$$CS(\rho) = \begin{cases} \displaystyle\sum_{j=1}^{k} \frac{r_j n_j^2}{p_j} \quad \mathrm{mod}\ 1, \quad \rho(h) = I \\[4mm] \displaystyle\sum_{j=1}^{k} (\frac{r_j n_j^2}{p_j} - \frac{q_j s_j}{4}) \quad \mathrm{mod}\ 1, \quad \rho(h) = -I \end{cases}$$

Remark 2.2. The formula for the CS invariant in Proposition 2.1 differs from that in [1] with a negative sign. We believe this discrepancy is due to conventions.

Before proving the proposition, we recall some facts in [12]

Let T be a torus and consider $\chi(T)$, the character variety of T to $\mathrm{SL}(2, \mathbb{C})$. It is direct to see that $\chi(T)$ can be identified with $\mathrm{Hom}(\pi_1(T), \mathbb{C}^*)/\sim$, where $f \sim g$ if $f = g^{\pm 1}$ where g^{-1} means point-wise inverse of g. We now describe a 'coordinate-version' of $\chi(T)$.

Let H be a group with the presentation,

$$H = \langle x, y, b \mid [x, y] = bxbx = byby = b^2 = 1 \rangle,$$

and define an action of H on $\mathbb{C}^2$ by

$$x(\alpha, \beta) = (\alpha + 1, \beta), \quad y(\alpha, \beta) = (\alpha, \beta + 1), \quad b(\alpha, \beta) = (-\alpha, -\beta).$$

Denote the image of $(\alpha, \beta) \in \mathbb{C}^2$ in the quotient space $\mathbb{C}^2/H$ by $[\alpha, \beta]$. Let $\vec{v} = (v_1, v_2)$ be any $\mathbb{Z}$-basis of $H_1(T)$, and define the map,

$$f_{\vec{v}} \colon \mathbb{C}^2/H \to \chi(T),$$

such that $f_{\vec{v}}[\alpha, \beta] \in \chi(T)$ sends

$$v_1 \mapsto e^{2\pi i \alpha}, \quad v_2 \mapsto e^{2\pi i \beta}.$$

It can be checked that $f_{\vec{v}}$ is a homeomorphism. A representation of $\pi_1(T)$ that induces the character $f_{\vec{v}}[\alpha, \beta]$ is given by,

$$v_1 \mapsto \begin{pmatrix} e^{2\pi i \alpha} & 0 \\ 0 & e^{-2\pi i \alpha} \end{pmatrix}, \quad v_2 \mapsto \begin{pmatrix} e^{2\pi i \beta} & 0 \\ 0 & e^{-2\pi i \beta} \end{pmatrix}.$$

Furthermore, the homeomorphism $f_{\vec{v}}$ is natural in the following sense. Let $\vec{w}$ be another basis such that $\vec{w} = \vec{v}A$ for some $A \in \mathrm{GL}(2, \mathbb{Z})$ (viewing $\vec{w}$ and $\vec{v}$ as row vectors), and define the map $\Phi_{\vec{v}, \vec{w}} \colon \mathbb{C}^2 \to \mathbb{C}^2$ by right multiplying (row) vectors of $\mathbb{C}^2$ by A on the right. Then $\Phi_{\vec{v}, \vec{w}}$ induces a homeomorphism, still denoted by $\Phi_{\vec{v}, \vec{w}}$, from $\mathbb{C}^2/H$ to $\mathbb{C}^2/H$, and the following diagram commutes,

$$
\begin{array}{ccc}
\mathbb{C}^2/H & \xrightarrow{\;\Phi_{\vec{v}, \vec{w}}\;} & \mathbb{C}^2/H \\
{\scriptstyle f_{\vec{v}}}\downarrow & \swarrow {\scriptstyle f_{\vec{w}}} & \\
\chi(T) & &
\end{array}
$$

Hence, we think of each $\mathbb{C}^2/H$ with a choice of basis $\vec{v}$ as a coordinate realization of $\chi(T)$. In fact, $\chi(T)$ is isomorphic to the direct limit[2] of $\{(\mathbb{C}^2/H)_{\vec{v}}, \ \Phi_{\vec{v}, \vec{w}}\}$,

$$\chi(T) \simeq \varinjlim (\mathbb{C}^2/H)_{\vec{v}},$$

where $(\mathbb{C}^2/H)_{\vec{v}}$ is a copy of $\mathbb{C}^2/H$ indexed by $\vec{v}$.

Next, we introduce a $\mathbb{C}^*$ bundle over $\chi(T)$. Define an action of H on $\mathbb{C}^2 \times \mathbb{C}^*$ lifting that on $\mathbb{C}^2$ by

$$
\begin{aligned}
x(\alpha, \beta; z) &= (\alpha + 1, \beta; z e^{2\pi i \beta}), \\
y(\alpha, \beta; z) &= (\alpha, \beta + 1; z e^{-2\pi i \alpha}), \\
b(\alpha, \beta; z) &= (-\alpha, -\beta; z).
\end{aligned}
$$

The canonical projection $\mathbb{C}^2 \times \mathbb{C}^* \to \mathbb{C}^2$ induces a projection

$$p \colon \mathbb{C}^2 \times \mathbb{C}^*/H \to \mathbb{C}^2/H,$$

which makes $\mathbb{C}^2 \times \mathbb{C}^*/H$ a $\mathbb{C}^*$ bundle over $\mathbb{C}^2/H$. Given two bases $\vec{v}, \ \vec{w}$ of $H_1(T)$ with $\vec{w} = \vec{v}A$, $\Phi_{\vec{v}, \vec{w}}$ can be covered by a bundle isomorphism. Explicitly, define $\tilde{\Phi}_{\vec{v}, \vec{w}} \colon \mathbb{C}^2 \times \mathbb{C}^*/H \to \mathbb{C}^2 \times \mathbb{C}^*/H$ which maps $[\alpha, \beta; z]$ to $[(\alpha, \beta)A; z^{\det(A)}]$. Then the following diagram commutes,

$$
\begin{array}{ccc}
(\mathbb{C}^2 \times \mathbb{C}^*/H)_{\vec{v}} & \xrightarrow{\;\tilde{\Phi}_{\vec{v}, \vec{w}}\;} & (\mathbb{C}^2 \times \mathbb{C}^*/H)_{\vec{w}} \\
{\scriptstyle p}\downarrow & & \downarrow {\scriptstyle p} \\
(\mathbb{C}^2/H)_{\vec{v}} & \xrightarrow{\;\Phi_{\vec{v}, \vec{w}}\;} & (\mathbb{C}^2/H)_{\vec{w}}
\end{array}
\tag{3}
$$

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[2] Here all maps involved are isomorphisms, so the notion of direct limit and inverse limit do not make a difference.

Let $\tilde{E}(T)$ be the direct limit of $\{(\mathbb{C}^2 \times \mathbb{C}^*/H)_{\vec{v}}, \ \tilde{\Phi}_{\vec{v},\vec{w}}\}$. Then Eq. 3 induces a map $p \colon \tilde{E}(T) \to \chi(T)$ which makes $\tilde{E}(T)$ a $\mathbb{C}^*$ bundle over $\chi(T)$, and the diagram below commutes,

$$
\begin{array}{ccc}
\tilde{E}(T) & \longleftarrow & (\mathbb{C}^2 \times \mathbb{C}^*/H)_{\vec{v}} \\
\downarrow{p} & & \downarrow{p} \\
\chi(T) & \underset{f_{\vec{v}}}{\longleftarrow} & (\mathbb{C}^2/H)_{\vec{v}}
\end{array}
$$

We often represent an element of $\tilde{E}(T)$ by a 'coordinate' $[\alpha, \beta; z]_{\vec{v}}$ with respect to a basis $\vec{v}$. Changing the basis to $\vec{w} = \vec{v}A$ induces the equality

$$
[\alpha, \beta; z]_{\vec{v}} = [(\alpha, \beta)A; z^{\det(A)}]_{\vec{w}},
$$

and when the bases involved are clear from the context, we will omit them.

We also need an 'orientation-version' of $\tilde{E}(T)$. Now assume T is *oriented*, and define $E(T)$ to be the direct limit of $\{(\mathbb{C}^2 \times \mathbb{C}^*/H)_{\vec{v}}, \ \tilde{\Phi}_{\vec{v},\vec{w}}\}$ where the limit is taken only over positive bases $\vec{v}$ of $H_1(T)$, namely, those $\vec{v}$ such that $v_1 \wedge v_2$ matches the orientation of T. Apparently, $E(T)$ and $E(-T)$ are both bundles over $\chi(T)$, and are both isomorphic to $\tilde{E}(T)$. However, it will be of conceptual convenience for latter calculations to distinguish $E(T)$ from $E(-T)$.

There is a fiber-wise pairing $\langle \ , \ \rangle$ defined on $E(T) \times E(-T)$ as follows. Given $e \in E(T)$, $e' \in E(-T)$ such that $p(e) = p(e')$, choose an arbitrary positive basis $\vec{v} = (v_1, v_2)$ of $H_1(T)$ and hence $\vec{v}' := (-v_1, v_2)$ is a positive basis of $H_1(-T)$, and write $e = [\alpha, \beta; z]_{\vec{v}}, e' = [-\alpha, \beta; z']_{\vec{v}'}$ (or $e' = [\alpha, -\beta; z']_{-\vec{v}'}$). Then $\langle e, e' \rangle := zz'$. It can be checked that the pairing is well defined.

Lastly, the above notions can be generalized to multiple tori in a natural way. Let $S = \sqcup_{i=1}^{k} T_i$ be a disjoint union of k oriented tori. Then $\chi(S) = \chi(T_1) \times \cdots \times \chi(T_k)$. The group H^k acts on $(\mathbb{C}^2)^k$ component-wise and the quotient is a 'coordinate-version' of $\chi(S)$. The action of H^k can also be lifted to $(\mathbb{C}^2)^k \times \mathbb{C}^*$ where the i-th component H_i in H^k acts on the i-th copy of $\mathbb{C}^2$ in $(\mathbb{C}^2)^k$ times $\mathbb{C}^*$, and $E(T)$ is the quotient of $(\mathbb{C}^2)^k \times \mathbb{C}^*$ by this action. For $n \leq k$, similar to the pairing above, there is a generalized 'pairing':

$$
E(T_1 \sqcup \cdots \sqcup T_k) \times E(-T_1 \sqcup \cdots \sqcup -T_m) \to E(T_{m+1} \sqcup \cdots \sqcup T_k).
$$

With the above notations, we recall several theorems in [12]. Let X be an oriented compact 3-manifold with toral boundaries $\partial X = \sqcup_{i=1}^{k} T_i$ and $\rho \colon \pi_1(X) \to \mathrm{SL}(2, \mathbb{C})$ be a holonomy representation. It is well-known that $\mathrm{CS}(\rho)$ in Eq (2) is not well defined since X has boundary. Let

$$
c_X(\rho) = e^{2\pi i\, \mathrm{CS}(\rho)}.
$$

Theorem 2.3. (Theorem 3.2 of [12]) *The Chern-Simons invariant defines a lifting* $c_X \colon \chi(X) \longrightarrow E(\partial X)$ *of the restriction map* r *from the character variety of* X *to the character variety of* ∂X,

$$
\begin{array}{ccc}
 & & E(\partial X) \\
 & {\scriptstyle c_X}\nearrow & \downarrow{p} \\
\chi(X) & \xrightarrow{\ r\ } & \chi(\partial X)
\end{array}
$$

Moreover, if $Y = X_1 \cup X_2$ is a closed oriented 3-manifold such that X_1 and X_2 are glued along toral boundaries $\partial X_1 = -\partial X_2$, then for $\chi \in \chi(Y)$, we have

$$e^{2\pi i \, CS(\chi)} = \langle c_{X_1}(\chi_1), c_{X_2}(\chi_2) \rangle,$$

where χ_i denotes the restriction of χ on X_i.

The following theorem is also due to [12] which the authors proved for the case of SU(2) representations (Theorem 2.7), but an almost identical proof also works for $SL(2, \mathbb{C})$ representations.

Theorem 2.4. *Let X be an oriented 3-manifold with toral boundaries $\partial X = \sqcup_{i=1}^{k} T_i$ and $\rho(t): \pi_1(X) \to SL(2, \mathbb{C})$ be a path of representations. Let $(\alpha_i(t), \beta_i(t))$ be a lift of $\chi \circ \rho(t)|_{T_i}$ to $\mathbb{C}^2$ with respect to some basis of $H_1(T_i)$. Suppose*

$$c_X(\rho(t)) = [\alpha_1(t), \beta_1(t), \cdots, \alpha_k(t), \beta_k(t); z(t)]$$

Then

$$z(1)z(0)^{-1} = \exp\left(2\pi i \sum_{j=1}^{k} \int_0^1 (\alpha_j \frac{d\beta_j}{dt} - \beta_j \frac{d\alpha_j}{dt}) \right)$$

In particular, if $\rho(1)$ is the trivial representation, then

$$c_X(\rho(0)) = \left[\alpha_1(0), \beta_1(0), \cdots, \alpha_k(0), \beta_k(0); \exp\left(-2\pi i \sum_{j=1}^{k} \int_0^1 (\alpha_j \frac{d\beta_j}{dt} - \beta_j \frac{d\alpha_j}{dt})\right) \right]$$

The following two facts are proved for SU(2) representations in [12] (Theorems 4.1 and 4.2, respectively). Similar methods combined with Theorems 2.3 and 2.4 above show that they also hold for $SL(2, \mathbb{C})$ representations.

Fact 1 Let X be an oriented 3-manifold with toral boundaries $\partial X = \sqcup_{i=1}^{n} T_i$. Assume $H_1(X)$ is torsion free. Choose a positive basis (μ_i, λ_i) for $H_1(T_i)$. Let $\{x_j \mid j = 1, \cdots, m\}$ be a basis of $H_1(X)$ and $\mu_i = \sum a_{ij} x_j$, $\lambda_i = \sum b_{ij} x_j$. Suppose that $\rho : \pi_1(X) \to SL(2, \mathbb{C})$ is an Abelian representation and $\mathrm{Tr}(\rho(x_j)) = e^{2\pi i \gamma_j} + e^{-2\pi i \gamma_j}$ for some $\gamma_j \in \mathbb{C}$. Then

$$c_X(\rho) = \left[\sum a_{1j}\gamma_j, \sum b_{1j}\gamma_j, \cdots, \sum a_{nj}\gamma_j, \sum b_{nj}\gamma_j; 1 \right]$$

Fact 2 Let F be a genus g oriented surface with k punctures. The fundamental group of F has the presentation,

$$\pi_1(F) = \langle a_1, b_1, \cdots, a_g, b_g, x_1, \cdots, x_k \mid [a_1, b_1] \cdots [a_g, b_g] x_1 \cdots x_k = 1 \rangle,$$

where x_j corresponds to the oriented boundary (induced from F) of the j-th puncture. Let $Y = F \times S^1$ be endowed with the product orientation and let $\tilde{h} = * \times S^1$ be the central element of $\pi_1(Y)$ corresponding to the oriented S^1 component. Then $\partial Y = \sqcup_{j=1}^{k} T_j$ with T_j the torus corresponding to the j-th puncture and $(x_j, \tilde{h})$ is a positive basis for $H_1(T_j)$. Suppose $\rho : \pi_1(Y) \longrightarrow SL(2, \mathbb{C})$ is a non-Abelian representation, which

implies $\mathrm{Tr}(\rho(\tilde{h})) = 2\cos 2\pi\beta$ for some $\beta \in \{0, \frac{1}{2}\}$. Suppose $\mathrm{Tr}(\rho(x_j)) = e^{2\pi i\alpha_j} + e^{-2\pi i\alpha_j}$ for some $\alpha_j \in \mathbb{C}$. Then

$$c_Y(\rho) = \left[\alpha_1, \beta, \cdots, \alpha_n, \beta; \exp\left(-2\pi i\beta \sum_{j=1}^{k} \alpha_j\right)\right].$$

Note that $c_Y(\rho)$ does not change under the replacement of some α_j by $-\alpha_j$.

The rest of the subsection is devoted to the proof of Proposition 2.1.

Proof. Let $Y = F \times S^1$ be as in **Fact 2** above with the chosen generators x_j and $\tilde{h}$. Set $h = \tilde{h}^{-1}$. Then X is obtained from Y by gluing k solid tori where the j-th solid torus A_j is glued along T_j by sending the meridian to $x_j^{p_j} h^{q_j}$. The generators x_j and h match those as presented in Eq. (1). Choose a meridian-longitude pair (μ_j, λ_j) for A_j such that (μ_j, λ_j) is a positive basis of $H_1(\partial A_j)$. The gluing of A_j to Y provides the transition of basis,

$$(\mu_j, \lambda_j) = (x_j, h)\begin{pmatrix} p_j & r_j \\ q_j & s_j \end{pmatrix}.$$

Since ρ is non-Abelian, $\rho(h)$ is $\pm I$. By assumption,

$$\mathrm{Tr}(\rho(x_j)) = \exp(\frac{2\pi in_j}{p_j}) + \exp(-\frac{2\pi in_j}{p_j}), \qquad \mathrm{Tr}(\rho(h)) = 2\cos(2\pi m), \ m = 0, \frac{1}{2}.$$

Therefore,

$$c_Y(\rho) = [\frac{n_1}{p_1}, -m, \cdots, \frac{n_k}{p_k}, -m; \exp(2\pi i\, m \sum_{j=1}^{k} \frac{n_j}{p_j})]_{(x_1, -h; \cdots; x_k, -h)}$$

$$c_{A_j}(\rho) = [0, \frac{r_j n_j}{p_j} + s_j m; 1]_{(\mu_j, \lambda_j)}$$

$$= [-q_j(\frac{r_j n_j}{p_j} + s_j m), r_j n_j + s_j p_j m; 1]_{(x_j, h)}$$

$$= [\frac{n_j}{p_j} - s_j\alpha_j, m + r_j\alpha_j; 1], \ (\text{setting } \alpha_j = n_j + q_j m)$$

$$= \left[\frac{n_j}{p_j} - s_j\alpha_j, m; \exp\left(2\pi i(r_j\alpha_j)(\frac{n_j}{p_j} - s_j\alpha_j)\right)\right]$$

$$= \left[\frac{n_j}{p_j}, m; \exp\left(2\pi i(r_j\alpha_j)(\frac{n_j}{p_j} - s_j\alpha_j) + 2\pi i(s_j\alpha_j)m\right)\right]$$

Note that the relation $x_j^{p_j} h^{q_j} = 1$ implies that α_j must be an integer. Applying the pairing on $c_Y(\rho)$ and each $c_{A_j}(\rho)$ one by one, we obtain,

$$\mathrm{CS}(\rho) = \sum_{j=1}^{k} (r_j\alpha_j \frac{n_j}{p_j} + s_j\alpha_j m + m\frac{n_j}{p_j})$$

$$= \sum_{j=1}^{k} \left(\frac{r_j n_j^2}{p_j} + s_j m(n_j + \alpha_j) \right)$$

$$= \sum_{j=1}^{k} \left(\frac{r_j n_j^2}{p_j} - s_j q_j m^2 \right).$$

<div align="right">□</div>

2.4. Adjoint reidemeister torsion. The Reidemeister torsion (R-torsion) $\tau(X)$ of a cellulation K_X of a manifold X uses the action of the fundamental group $\pi_1(X)$ on the universal cover $\widetilde{K_X}$ to measure the complexity of the cellulation of X. It is a topological invariant of X from determinants of matrices obtained from the incidences of the cells of $\widetilde{K_X}$. The R-torsion makes essential use of the bases in the chain complex of the universal cover, while the homology and homotopy groups do not see the geometric information encoded in the based chain complex. For our purpose, we need the non-Abelian generalization of R-torsion twisted by a representation $\rho : \pi_1(X) \to G$ for some semi-simple Lie group G, in particular the adjoint Reidemeister torsion for the adjoint representation of $SL(2, \mathbb{C})$. We recall some basics here, for more details, please refer to [14] and [18].

Let

$$C_* = (0 \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \longrightarrow 0)$$

be a chain complex of finite dimensional vector spaces over the field $\mathbb{C}$. Choose a basis c_i of C_i and a basis h_i of the i-th homology group $H_i(C_*)$. The torsion of C_* with respect to these choices of bases is defined as follows. For each i, let b_i be a set of vectors in C_i such that $\partial_i(b_i)$ is a basis of $\mathrm{Im}(\partial_i)$ and let $\tilde{h}_i$ denote a lift of h_i in $\mathrm{Ker}(\partial_i)$. Then the set of vectors $\bar{b}_i := \partial_{i+1}(b_{i+1}) \sqcup \tilde{h}_i \sqcup b_i$ is a basis of C_i. Let D_i be the transition matrix from c_i to $\bar{b}_i$. To be specific, each column of D_i corresponds to a vector in $\bar{b}_i$ being expressed as a linear combination of vectors in c_i. Define the torsion

$$\tau(C_*, c_*, h_*) := \left| \prod_{i=0}^{n} \det(D_i)^{(-1)^{i+1}} \right|$$

Remark 2.5. A few remarks are in order.

- The torsion, as it is denoted, does not depend on the choice of b_i and the lifting of h_i.
- Here we define the torsion as the norm of the usual torsion, thus we do not need to deal with sign ambiguities.

Let X be a finite CW-complex and (V, ρ) be a homomorphism $\rho : \pi_1(X) \longrightarrow SL(V)$. The vector space V turns into a left $\mathbb{Z}[\pi_1(X)]$-module. The universal cover $\tilde{X}$ has a natural CW structure from X, and its chain complex $C_*(\tilde{X})$ is a free *left* $\mathbb{Z}[\pi_1(X)]$-module via the action of $\pi_1(X)$ as covering transformations. View $C_*(\tilde{X})$ as a *right* $\mathbb{Z}[\pi_1(X)]$-module by $\sigma.g := g^{-1}.\sigma$ for $\sigma \in C_*(\tilde{X})$ and $g \in \pi_1(X)$. We define the twisted chain complex $C_*(X; \rho) := C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} V$. Let $\{e_\alpha^i\}_\alpha$ be the set of i-cells of X ordered in an arbitrary way. Choose a lifting $\tilde{e}_\alpha^i$ of e_α^i in $\tilde{X}$. It follows that $C_i(\tilde{X})$ is generated by $\{\tilde{e}_\alpha^i\}_\alpha$ as a free $\mathbb{Z}[\pi_1(X)]$-module (left or right). Choose a basis of $\{v_\gamma\}_\gamma$ of V. Then $c_i(\rho) := \{\tilde{e}_\alpha^i \otimes v_\gamma\}$ is a $\mathbb{C}$-basis of $C_i(X; \rho)$.

Definition 2.6. Let $\rho : \pi_1(X) \longrightarrow \mathrm{SL}(V)$ be a representation.

1. We call ρ acyclic if $C_*(X; \rho)$ is acyclic. Assume ρ is acyclic. The torsion of X twisted by ρ is defined to be,

$$\tau(X; \rho) := \tau\bigg(C_*(X; \rho), \, c_*(\rho)\bigg).$$

2. Let $\mathrm{Adj} : \mathrm{SL}(V) \to \mathrm{SL}(\mathfrak{sl}(V))$ be the adjoint representation of $\mathrm{SL}(V)$ on its Lie algebra $\mathfrak{sl}(V)$. We call ρ *adjoint acyclic* if $\mathrm{Adj} \circ \rho$ is acyclic. Assume ρ is adjoint acyclic. Define the *adjoint Reidemeister torsion* of ρ to be,

$$\mathrm{Tor}(X; \rho) := \tau(X; \mathrm{Adj} \circ \rho).$$

Remark 2.7. In this paper, we will only deal with the adjoint Reidemeister torsion ρ. For that matter, we simply call it the torsion of ρ. When no confusion arises, we abbreviate $\mathrm{Tor}(X; \rho)$ as $\mathrm{Tor}(\rho)$.

The following tool will be useful in computing torsions.

Multiplicativity Lemma Let $0 \longrightarrow C'_* \longrightarrow C_* \longrightarrow C''_* \longrightarrow 0$ be an exact sequence of chain complexes. Assume that C_*, C'_*, C''_* are based by c_*, c'_*, c''_*, respectively, and their homology groups based by h_*, h'_*, h''_*, respectively. Associated to the short exact sequence is the long exact sequence H_* in homology

$$\cdots \longrightarrow H_j(C'_*) \longrightarrow H_j(C_*) \longrightarrow H_j(C''_*) \longrightarrow H_{j-1}(C'_*) \longrightarrow \cdots$$

with the reference bases. For each i, identify c'_i with its image in C_i and arbitrarily choose a preimage $\tilde{c}''_i$ of c''_i in C_i. If the transition matrix between the bases c_i and $c'_i \sqcup \tilde{c}''_i$ has determinant ± 1, we call c_*, c'_*, c''_* compatible. In this case, we have

$$\tau(C_*, c_*, h_*) \; = \; \tau(C'_*, c'_*, h'_*) \, \tau(C''_*, c''_*, h''_*) \, \tau(H_*, \{h_* \sqcup h'_* \sqcup h''_*\}).$$

2.5. Modular data from three manifolds. The modular data of an MTC or a pre-modular category consist of the modular S- and T- matrices. Given a three manifold X with certain conditions, [4] contains an algorithm for choosing the T-matrix and the first row of the S-matrix, i.e. all quantum dimensions. The next step for the full S-matrix is a trial-and-error algorithm based on finding the right loop operators for each simple object. When all the loop operators are given, then the modular data can be computed. There are no general algorithms to define loop operators, but in the cases of SFSs and SOL manifolds, we find the relevant loop operators completely.

2.5.1. From adjoint-acyclic non-Abelian characters to simple object types Each pre-modular category has a label set—the isomorphism classes of the simple objects, and a label is an isomorphism class of simple objects, so we will refer to a label also as a simple object type. In physics, an anyon model is a unitary MTC and a label is called an anyon type or a topological charge.

A candidate label from a three manifold X and $\mathrm{SL}(2, \mathbb{C})$ is morally an irreducible representation of the fundamental group $\pi_1(X)$ to $\mathrm{SL}(2, \mathbb{C})$. But the precise definition is more subtle and based on our examples later, we make the following definition. In particular, we discover that reducible but indecomposable representations cannot be discarded and play important roles in the construction of premodular categories from torus bundles over the circle. Our definition is specific for representations to $\mathrm{SL}(2, \mathbb{C})$ and we expect an appropriate generalization is needed for other Lie groups such as $\mathrm{SL}(n, \mathbb{C}), n \geq 3$.

Definition 2.8. Let $\chi \in \chi(X)$ be an SL(2, $\mathbb{C}$)-character of a three manifold X.

- χ is non-Abelian if at least one representation $\rho : \pi_1(X) \to$ SL(2, $\mathbb{C}$) with character χ is non-Abelian, i.e. ρ has non-Abelian image in SL(2, $\mathbb{C}$). The set of all non-Abelian characters of X is denoted by $\chi^{\mathrm{nab}}(X)$.
- A non-Abelian character χ is *adjoint-acyclic* if each non-Abelian representation $\rho : \pi_1(X) \to$ SL(2, $\mathbb{C}$) with character χ is adjoint-acyclic, namely, the chain complex associated with the universal cover $\tilde{X}$ twisted by Adj $\circ \rho$ is acyclic (see Definition 2.6), and furthermore, the adjoint Reidemeister torsion of all such non-Abelian representations with character χ are the same.
- A candidate label is an *adjoint-acyclic non-Abelian character*.
- A candidate label set $L(X)$ from a three manifolds X is a finite set of adjoint-acyclic non-Abelian characters in $\chi(X)$ with a pre-chosen character such that the difference of the CS invariant of each character $L(X)$ with that of the pre-chosen character is a rational number. The pre-chosen character is the candidate tensor unit.

Note that by definition, the adjoint Reidemeister torsion is well-defined for adjoint-acyclic non-Abelian characters. The CS invariant only depends on characters, and is hence also well-defined for such characters.

In this paper, our candidate label set is in general maximal in the sense it consists of all the adjoint-acyclic non-Abelian characters of the given three manifold. It is also true that the CS invariants of all the candidate labels including the candidate tensor unit are all rational in our examples. We are not aware of any example of a candidate label set for which not all CS invariants are rational numbers.

2.5.2. Vacuum choices, loop operators, and modular data Each simple object x of a premodular category $\mathcal{B}$ has a quantum dimension d_x and a topological twist θ_x. The set $Td(B) := \cup_{i \in L(\mathcal{B})}\{d_{x_i}, \theta_{x_i}\}$ will be called the twist-dimension set of $\mathcal{B}$, where $L(\mathcal{B})$ is the label set of $\mathcal{B}$ and $\{x_i, i \in L(\mathcal{B})\}$ form a complete representative set of simple objects of $\mathcal{B}$. A candidate label set of a three manifold X will lead to a candidate twist-dimension set in the following.

The choice of a tensor unit or vacuum from a collection of adjoint-acyclic non-Abelian characters is not unique in general and it is known that different choices could produce different premodular categories. Once a vacuum is chosen, then the adjoint Reidemeister torsion of each character is scaled to the absolute value of normalized quantum dimension and the difference of the CS invariant of the character with that of the vacuum is the conformal weight of the simple object up to a sign[3].

Given a three manifold X and a Lie group G, a central representation of $\pi_1(X)$ is a homomorphism from $\pi_1(X)$ to the center $Z(G)$ of G. For $G =$ SL(2, $\mathbb{C}$), a central representation of $\pi_1(X)$ is simply a homomorphism from $\pi_1(X)$ to $\mathbb{Z}_2$. The group of central representations can be identified with $H^1(X, \mathbb{Z}_2)$. A central representation $\sigma \in H^1(X, \mathbb{Z}_2)$ of $\pi_1(X)$ naturally acts on $\mathcal{R}(X)$ by tensoring $\rho \in \mathcal{R}(X)$, i.e. by sending ρ to $\rho \otimes \sigma$. Moreover, this action induces an action of central representations on the character variety $\chi(X)$.

Definition 2.9. 1. Given a candidate label set $L(X)$ from a three manifold X, a central representation σ is *bosonic* with respect to $L(X)$ if the action of σ keeps $L(X)$ invariant and preserves the CS invariant of every candidate label. If the action of σ

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[3] The sign and hence the negative sign in front of CS invariant below is not important and the choice is made to be the same as in [4].

changes the CS invariants of all candidate labels in $L(X)$ by either 0 or $\frac{1}{2}$, then χ is called *fermionic* if it is not bosonic.

2. Two candidate labels are centrally related if they are in the same orbit under the action of $H^1(X, \mathbb{Z}_2)$ and they have the same CS and torsion invariant.

Given a candidate label set $L(X)$ of X that is invariant under the action of $H^1(X, \mathbb{Z}_2)$, the candidate symmetric center $s(X)$ consists of all characters in $L(X)$ that are centrally related to the candidate tensor unit. Let $G_0(X)$ be the maximal subgroup of $H^1(X, \mathbb{Z}_2)$ such that $G_0(X)$ maps the candidate tensor unit onto $s(X)$. The action of $G_0(X)$ separates $L(X)$ into orbits $\{O_0, \cdots, O_m\}$, where each subset O_i of $L(X)$ consists of candidate labels that are centrally related to each other, and O_0 is the subset for the candidate vacuum.

We often represent a candidate label (a character) by arbitrarily choosing a representative (a representation of $\pi_1(M)$) for it.

Definition 2.10. A candidate label set $L(X) = \{\rho_\alpha\}$ of a three manifold X with ρ_0 the candidate vacuum is *admissible* if the following two equations hold with the notations as above:

$$\sum_{\rho_\alpha \in L(X)} \frac{1}{2\text{Tor}(\rho_\alpha)} = 1, \tag{4}$$

$$\left| \sum_\alpha \frac{\exp(-2\pi i \, \text{CS}(\rho_\alpha))}{2\text{Tor}(\rho_\alpha)} \right| = \frac{1}{s_L} \frac{\sqrt{|s(X)|}}{\sqrt{2\text{Tor}(\rho_0)}}, \tag{5}$$

where $s_L = 1$ if all central representations in $G_o(X)$ are bosonic and $s_L = \sqrt{2}$ if there is a fermionic one.

The conditions above are derived from the conjecture that the Mueger center of the potential premodular category is a collection of Abelian anyons parameterized by the subset O_0. In the condensed category, each subset O_i will be identified into a single composite object which has the same quantum dimension as that of any simple object in O_i and which splits into a number of simple objects of the same quantum dimension. The resulting condensed category is either modular or super-modular depending on if there is a fermion in the candidate Mueger center. In a particular case when X is a $\mathbb{Z}_2$ homology sphere, that is, $H^1(X, \mathbb{Z}_2) = 0$, Eq (5) reduces to,

$$\left| \sum_\alpha \frac{\exp(-2\pi i \, \text{CS}(\rho_\alpha))}{2\text{Tor}(\rho_\alpha)} \right| = \frac{1}{\sqrt{2\text{Tor}(\rho_0)}}. \tag{6}$$

Given an admissible candidate label set $L(X)$ with the chosen candidate tensor unit ρ_0, then the candidate twist-dimension set is constructed as follows:

$$\theta_\alpha = e^{-2\pi i (\text{CS}(\rho_\alpha) - \text{CS}(\rho_0))}, \tag{7}$$

$$D^2 = 2\text{Tor}(\rho_0) \tag{8}$$

$$d_\alpha^2 = \frac{D^2}{2\text{Tor}(\rho_\alpha)}, \tag{9}$$

where D^2 is the total dimension squared of the candidate premodular category.

Next, we discuss the construction of the S-matrix.

Definition 2.11. Given a three manifold X, a primitive loop operator of X is a pair (a, R), where a is a conjugacy class of the fundamental group $\pi_1(X)$ of X and R a finite dimensional irreducible representation of $SL(2, \mathbb{C})$.

Given an $SL(2, \mathbb{C})$-representation ρ of $\pi_1(X)$ and a primitive loop operator (a, R), then the weight of the loop operator (a, R) with respect to ρ is $W_\rho(a, R) := \text{Tr}_R(\rho(a))$. Denote by Sym^j the unique $(j + 1)$-dimensional irreducible representation of $SL(2, \mathbb{C})$. Then $W_\rho(a, \text{Sym}^j)$ can be computed from the Chebyshev polynomial $\Delta_j(t)$ defined recursively by,

$$\Delta_{j+2}(t) = t\Delta_{j+1}(t) - \Delta_j(t), \quad \Delta_0(t) = 1, \Delta_1(t) = t. \tag{10}$$

Explicitly,

$$W_\rho(a, \text{Sym}^j) = \Delta_j(t), \quad t = W_\rho(a, \text{Sym}^1) = \text{Tr}(\rho(a)). \tag{11}$$

From the above two equations, it follows that $W_\rho(a, \text{Sym}^j)$ only depends on the character χ induced by ρ. It is direct to check that,

$$\Delta_j(2\cos\theta) = \sin((j + 1)\theta)/\sin\theta, \quad \Delta_j(-t) = (-1)^j\Delta_n(t). \tag{12}$$

A fundamental assumption in constructing the S-matrix is that each candidate label ρ_α should correspond to a finite collection of primitive loop operators:

$$\rho_\alpha \mapsto \{(a_\alpha^\kappa, R_\alpha^\kappa)\}_\kappa. \tag{13}$$

By choosing a sign $\epsilon = \pm 1$, we define the W-symbols

$$W_\beta(\alpha) := \prod_\kappa W_{\epsilon\rho_\beta}(a_\alpha^\kappa, R_\alpha^\kappa) = \prod_\kappa \text{Tr}_{R_\alpha^\kappa}(\epsilon\rho_\beta(a_\alpha^\kappa)), \quad \rho_\alpha, \rho_\beta \in L(X). \tag{14}$$

The W-symbols and the un-normalized S-matrix $\tilde{S} = D\,S$ are related by,

$$W_\beta(\alpha) = \frac{\tilde{S}_{\alpha\beta}}{\tilde{S}_{0\beta}} \quad \text{or} \quad \tilde{S}_{\alpha\beta} = W_\beta(\alpha)W_0(\beta), \tag{15}$$

where 0 denotes the tensor unit ρ_0. In particular, the quantum dimension

$$d_\alpha = W_0(\alpha). \tag{16}$$

Remark 2.12. Currently it involves a guess-and-trial process to find the correspondence between candidate labels and loop operators. One on hand, we know the absolute value of $W_0(\alpha)$ from Eqs. (9) and (16). On the other hand, $W_0(\alpha)$ can also be computed by Eq. (14). Note that, to obtain $W_0(\alpha)$, only the loop operators corresponding to ρ_α are required. Hence, we can choose a set of loop operators for ρ_α so that the two ways of computing $W_0(\alpha)$ match (in absolute value). See the next remark for further validation of choices of loop operators. We leave it as a future direction to define the rigorous correspondence.

Remark 2.13. We expect that the resulting modular data corresponds to a MTC if and only if $H^1(X, \mathbb{Z}_2) = 0$. Note that, this is a purely topological condition, independent of the choice of loop operators. Hence, if $H^1(X, \mathbb{Z}_2) = 0$, we can also validate a choice of the loop operators by checking whether the resulting S and T matrices define a representation to $SL(2, \mathbb{Z})$.

Before closing this section, we summarize the construction of modular data and show various choices during the procedure. Given a closed 3-manifold X, first we choose a candidate label set $L(X)$ which is a finite set of adjoint-acyclic non-Abelian characters. A special character in $L(X)$ is prechosen as the tensor unit. The candidate label set is required to be admissible (see Definition 2.10). Then the topological twists (T-matrix) and quantum dimension squared of simple objects are given in Eqs. (7)–(9). Next, we associate to each character in $L(X)$ a finite collection of primitive loop operators (see Eq. (13)) from which and a choice of $\epsilon = \pm 1$ the W-symbols are defined. The S-matrix and the W-symbols are related to each other by Eq. (15).

3. Modular Tensor Categories from Seifert Fibered Spaces

In this section, we consider SFSs with three singular fibers and construct modular data associated with premodular categories. Throughout the section, set $M = \{0; (o, 0); (p_1, q_1), (p_2, q_2), (p_3, q_3)\}$, where each pair (p_k, q_k) are co-prime. So the underlying 2-manifold of the orbit surface Σ has genus 0 and both M and Σ are orientable.

3.1. Character varieties of seifert fibered spaces. For $M = \{0; (o, 0); (p_1, q_1), (p_2, q_2), (p_3, q_3)\}$, its fundamental group has the following presentation,

$$\pi_1(M) = \langle x_1, x_2, x_3, h \mid x_k^{p_k} h^{q_k} = 1, \ x_k h = h x_k, \ x_1 x_2 x_3 = 1, k = 1, 2, 3 \rangle$$

We look for all non-Abelian characters of $\pi_1(M)$ to $G = \mathrm{SL}(2, \mathbb{C})$.

Let $\rho : \pi_1(M) \to G$ be a non-Abelian representation. Since h is in the center of $\pi_1(M)$ and ρ is non-Abelian, $\rho(h)$ must be $\pm I$. It follows that each $\rho(x_k)$ has finite order, and is diagonalizable in particular. Moreover, any $\rho(x_k)$ does not commute with another $\rho(x_j)$. This implies neither $\rho(x_k)$ can be $\pm I$. Up to conjugation, we assume $\rho(x_k)$ take the following form (writing $\rho(x_k)$ simply as x_k),

$$x_1 = \begin{pmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{-i\alpha_1} \end{pmatrix}, x_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} e^{i\alpha_2} & 0 \\ 0 & e^{-i\alpha_2} \end{pmatrix}, x_3 \sim \begin{pmatrix} e^{i\alpha_3} & 0 \\ 0 & e^{-i\alpha_3} \end{pmatrix} \quad (17)$$

where $0 < \alpha_k < \pi$, $ad - bc = 1$, and b and c are not simultaneously zero. We have the following linear equations for a and d.

$$\mathrm{Tr}(x_2) = e^{i\alpha_2} + e^{-i\alpha_2} = a + d \tag{18}$$

$$\mathrm{Tr}(x_3) = e^{i\alpha_3} + e^{-i\alpha_3} = a e^{i\alpha_1} + d e^{-i\alpha_1} \tag{19}$$

Hence, given the α_k's, or equivalently $\mathrm{Tr}(x_k)$, a and d are uniquely determined, and $a = \bar{d}$. Moreover, when $|a| \neq 1$ implying $bc \neq 0$, this also determines ρ up to conjugacy. When $|a| = 1$ implying $bc = 0$, there are precisely two conjugacy classes with

$$x_2 = \begin{pmatrix} a & 1 \\ 0 & \bar{a} \end{pmatrix} \text{ or } x_2 = \begin{pmatrix} a & 0 \\ 1 & \bar{a} \end{pmatrix} \tag{20}$$

It can be checked that these two representations are complex conjugate to each other up to conjugacy, and that their characters take real values. They give rise to the same character. There are two types of non-Abelian representations. One type is irreducible satisfying $b, c \neq 0$. Characters of representations of this type one-to-one correspond to

conjugacy classes of representations [7]. The other type is reducible with exactly one of b, c zero. Each character of this type corresponds to two conjugacy classes.

To summarize, the triple $(\alpha_1, \alpha_2, \alpha_3)$ and $\mathrm{Tr}(h)$ uniquely determine the character. Next, we find all possible such triples.

If $h = I$, each $e^{i\alpha_k}$ is a p_k-th root of 1. If $h = -I$, then $e^{i\alpha_k}$ is a p_k-th root of 1 if q_k is even, and a p_k-th root of -1 if q_k is odd. We claim all triples satisfying the above conditions can be realized by some representations. Indeed, given such a triple $(\alpha_1, \alpha_2, \alpha_3)$, we define $\rho(x_1)$ and $\rho(x_2)$ as in Eq. (17) and let $\rho(x_3) := (\rho(x_1)\rho(x_2))^{-1}$. Eqs. (18), (19) determine a and d, and we arbitrarily choose b and c such that $ad - bc = 1$. Again, Eqs. (18), (19) guarantee that $\rho(x_k)$ so defined has eigenvalues $e^{\pm i\alpha_k}$, and therefore they satisfy all the relations in the presentation of $\pi_1(M)$.

Set $\alpha_k = \frac{2\pi n_k}{p_k}$ and $\rho(h) = e^{2\pi i\lambda}I$, $\lambda = 0, \frac{1}{2}$. If $\lambda = 0$ or if $\lambda = \frac{1}{2}$ and q_k is even, then n_k is an integer strictly between 0 and $\frac{p_k}{2}$. If $\lambda = \frac{1}{2}$ and q_k is odd, then n_k is a proper half integer strictly between 0 and $\frac{p_k}{2}$. The quadruple (n_1, n_2, n_3, λ) completely characterizes a character.

For an integer $p > 0$, denote by $[0 \cdots p]$ the set of integers $\{0, 1, \cdots, p\}$, and by $[0 \cdots p]^e$ (resp. $[0 \cdots p]^o$) the subset of even (resp. odd) integers in $[0 \cdots p]$. The non-Abelian character variety of M is given as follows,

$$\chi^{\mathrm{nab}}(M) = \left\{ \left(\frac{j_1 + 1}{2}, \frac{j_2 + 1}{2}, \frac{j_3 + 1}{2}, \frac{1}{2} \right) \mid j_k \in [0 \cdots p_k - 2]^{\epsilon_k} \right\}$$
$$\sqcup \left\{ \left(\frac{j_1 + 1}{2}, \frac{j_2 + 1}{2}, \frac{j_3 + 1}{2}, 0 \right) \mid j_k \in [0 \cdots p_k - 2]^o \right\}, \tag{21}$$

where $\epsilon_k = $ 'e' if q_k is odd, and $\epsilon_k = $ 'o' otherwise. For $(n_1, n_2, n_3, \lambda) \in \chi^{\mathrm{nab}}(M)$, a corresponding representation ρ has $e^{\pm \frac{2\pi i n_k}{p_k}}$ as the eigenvalue of $\rho(x_k)$ and $\rho(h) = e^{2\pi i\lambda}I$.

The size of $\chi^{\mathrm{nab}}(M)$ is

$$|\chi^{\mathrm{nab}}(M)| = \lfloor \frac{p_1}{2} \rfloor \lfloor \frac{p_2}{2} \rfloor \lfloor \frac{p_3}{2} \rfloor + \lfloor \frac{p_1 - 1}{2} \rfloor \lfloor \frac{p_2 - 1}{2} \rfloor \lfloor \frac{p_3 - 1}{2} \rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

For instance, if all the q_k's are odd, then $\chi^{\mathrm{nab}}(M)$ can also be written as,

$$\chi^{\mathrm{nab}} = \left\{ \left(\frac{j_1 + 1}{2}, \frac{j_2 + 1}{2}, \frac{j_3 + 1}{2}, \frac{(j_1 + 1) \mod 2}{2} \right) \right.$$
$$\left. \mid j_k \in [0 \cdots p_k - 2], \ j_1 = j_2 = j_3 \mod 2 \right\}$$

3.2. Torsion of seifert fibered spaces.

Freed computed torsions of Brieskorn homology spheres for the adjoint representations of irreducible $SU(2)$ representations in [8]. Kitano computed torsions of SFSs for irreducible $SL(2, \mathbb{C})$ representations in [13]. However, we need to compute torsions of SFSs for the adjoint representations of nonAbelian $SL(2, \mathbb{C})$ representations containing both irreducible and reducible ones. This may be known to experts, but we did not find a reference for explicitly doing so. To make the paper self-contained, we provide a detailed derivation of these torsions, generalizing the work of [8] and [13].

Let X be the SFS $\{0; (o, 0); (p_1, q_1), (p_2, q_2), (p_3, q_3)\}$. Decompose X as $\cup_{i=0}^{3} A_i \cup B$ along $\cup_{i=0}^{3} T_i$ where $B = (S^2 - 4pts) \times S^1$, and A_0, $A_i (i = 1, 2, 3)$ are solid tori attached to B by index 1, $\frac{p_i}{q_i}$ along T_0, T_i, respectively. Let $\rho : \pi_1(X) \longrightarrow \mathrm{SL}(2, \mathbb{C})$ be a non-Abelian representation, $V = \mathfrak{sl}(2, \mathbb{C})$ be the adjoint representation of ρ with the basis

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

From Sect. 3.1, ρ is parametrized by (n_1, n_2, n_3, h) where $0 < n_i < \frac{p_i}{2}$, $n_i \in \frac{1}{2}\mathbb{Z}$, $h = 0, \frac{1}{2}$. Assume that $r_i, s_i \in \mathbb{Z}$, such that $p_i s_i - r_i q_i = 1$.

Proposition 3.1. *When ρ is nonAbelian, $C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} V$ is acyclic and*

$$Tor(X; \rho) = \frac{p_1 p_2 p_3}{\prod_{i=1}^{3} 4 \sin^2 \frac{2\pi r_i n_i}{p_i}}$$

Proof. Denote $C_* \otimes_{\mathbb{Z}[\pi_1(X)]} V$ by $C_{*,\rho}$, twisted homology by H_*, and the matrix of element in π_1 under ρ by the same letter.

Given CW structure on X, we have the following exact chain sequence

$$0 \longrightarrow \bigoplus_{i=0}^{3} C_{*,\rho}(T_i) \longrightarrow \bigoplus_{i=0}^{3} C_{*,\rho}(A_i) \oplus C_{*,\rho}(B) \longrightarrow C_{*,\rho}(X) \longrightarrow 0$$

and long exact sequence

$$0 \longrightarrow \bigoplus_{i=0}^{3} H_3(T_i) \longrightarrow \bigoplus_{i=0}^{3} H_3(A_i) \oplus H_3(B) \longrightarrow H_3(X) \longrightarrow \cdots$$

$$\longrightarrow \bigoplus_{i=0}^{3} H_0(T_i) \longrightarrow \bigoplus_{i=0}^{3} H_0(A_i) \oplus H_0(B) \longrightarrow H_0(X) \longrightarrow 0$$

Since the Reidemeister torsion is invariant under simple homotopy, we can just consider the simple homotopy types of the above spaces. A_i is simple homotopy equivalent to a 1-complex, and T_i and B are each simple homotopy equivalent to a 2-complex. Thus we have

$$H_3(A_i) = H_3(B) = H_3(T_i) = 0, \quad H_2(A_i) = 0.$$

Construct their cell structure as follows.

$$C_0(B) = < v_B >, C_0(T_i) = < v_{T_i} >, C_0(A_i) = < v_{A_i} >$$
$$C_1(B) = < x_1, x_2, x_3, h >, C_1(T_i) = < m_i, l_i >, C_1(A_i) = < b_i >$$
$$C_2(B) = < u_{1,B}, u_{2,B}, u_{3,B} >, C_2(T_i) = < u_{T_i} >$$

where v_* are base points of connected spaces, x_i generate $\pi_1(S^2 - 4pts)$, $h = * \times S^1 \in \pi_1(S^2 - 4pts \times S^1)$, m_i, l_i are meridians and longitudes of T_i respectively, b_i are longitudes of boundary of A_i, $u_{i,B}$ are squares with boundary $x_i h x_i^{-1} h^{-1}$, u_{T_i} are squares with boundary $m_i l_i m_i^{-1} l_i^{-1}$. $T_i (i = 1, 2, 3)$ are attached to $x_i \times h$ by identity map and

boundary of A_i by $\begin{pmatrix} s_i & -q_i \\ -r_i & p_i \end{pmatrix}$. T_0 is attached to $x_1 x_2 x_3 \times h$ and boundary of A_0 by identity map. x_1, x_2, x_3, h generate $\pi_1(X)$ as follows.

$$\pi_1(X) = < x_1, x_2, x_3, h | x^{P_i} h^{q_i} = 1, x_i h = h x_i, x_1 x_2 x_3 = 1 >$$

For matrix under ρ, we have

$$x_i \sim \begin{pmatrix} \zeta_i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_i^{-1} \end{pmatrix}, h = I$$

where ζ_i is a p_i-th root of unity. $m_i = x_i$, $b_i = x_i^{r_i}$, $l_i = h$. Here we use 1-cell with ends points attached as element in π_1.

The work of [8] can be generalized to irreducible representations of $SL(2, \mathbb{C})$. Thus we focus on reducible and nonAbelian representations. According to (20), taking upper triangular ones for example, they have the following form.

$$x_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}, x_2 = \begin{pmatrix} a_2 & 1 \\ 0 & a_2^{-1} \end{pmatrix}, x_3 = \begin{pmatrix} a_1^{-1} a_2^{-1} & -a_1 \\ 0 & a_1 a_2 \end{pmatrix}$$

where $a_1, a_2, a_3 = a_1^{-1} a_2^{-1}$ are roots of 1 or -1.
For adjoint representation, we have

$$x_1 = \begin{pmatrix} a_1^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_1^2 \end{pmatrix}, x_2 = \begin{pmatrix} a_2^{-2} & 2a_2^{-1} & -1 \\ 0 & 1 & -a_2 \\ 0 & 0 & a_2^2 \end{pmatrix}$$

$$x_3 = \begin{pmatrix} a_1^2 a_2^2 & -2a_2 & -a_1^{-2} \\ 0 & 1 & a_1^{-2} a_2^{-1} \\ 0 & 0 & a_1^{-2} a_2^{-2} \end{pmatrix} \tag{22}$$

Let $w_i^{\pm}$ be the eigenvectors of x_i for eigenvalue $\zeta_i = a_i^{-2} = e^{\frac{4\pi i n_i}{p_i}}$, $\zeta_i^{-1} = e^{-\frac{4\pi i n_i}{p_i}}$ respectively and w_i^0 be the eigenvector of x_i for eigenvalue 1. Then $w_i^{\pm}$ are the eigenvectors of x_i^r for $\zeta_i^{r_i}$ and w_i^0 be the eigenvector of $x_i^{r_i}$ for 1. By scaling, assume that $|[w_i^{\pm} w_i^0]| = 1$ in V. According to (22) , $w_1^{\pm}, w_2^-$ is a basis of V. Similarly, for lower triangular ones in (20), $w_1^{\pm}, w_2^+$ is a basis of V.

For $T_i (i = 1, 2, 3)$, we have

$$0 \longrightarrow C_{2,\rho}(T_i) \xrightarrow{\partial_2} C_{1,\rho}(T_i) \xrightarrow{\partial_1} C_{0,\rho}(T_i) \longrightarrow 0$$

where

$$\partial_2 = \begin{pmatrix} O \\ x_i - I \end{pmatrix}, \partial_1 = (x_i - I \ O)$$

We have

$$H_2(T_i) = <\tilde{u}_{T_i} \otimes w_i^0>$$
$$H_1(T_i) = <\tilde{m}_i \otimes w_i^0, \tilde{l}_i \otimes w_i^0>$$
$$H_0(T_i) = <\tilde{v}_{T_i} \otimes w_i^0>$$

Choose preference basis h_* for $H_*(T_i)$ as above and similarly with others. Without confusion, we omit h_* in the expression as c_*.

$$
\begin{aligned}
\tau(C_{*,\rho}(T_i)) &= \left| \frac{[\tilde{l}_i \otimes (x_i - I)w_i^{\pm}, \tilde{m}_i \otimes w_i^0, \tilde{l}_i \otimes w_i^0, \tilde{m}_i \otimes w_i^{\pm}]}{[\tilde{u}_{T_i} \otimes w_i^0, \tilde{u}_{T_i} \otimes w_i^{\pm}][\tilde{v}_{T_i} \otimes w_i^0, \tilde{v}_{T_i} \otimes (x_i - I)w_i^{\pm}]} \right| \\
&= \left| \frac{[\tilde{l}_i \otimes (\zeta_i^{\pm 1} - 1)w_i^{\pm}, \tilde{m}_i \otimes w_i^0, \tilde{l}_i \otimes w_i^0, \tilde{m}_i \otimes w_i^{\pm}]}{[\tilde{u}_{T_i} \otimes w_i^0, \tilde{u}_{T_i} \otimes w_i^{\pm}][\tilde{v}_{T_i} \otimes w_i^0, \tilde{v}_{T_i} \otimes (\zeta_i^{\pm 1} - 1)w_i^{\pm}]} \right| \\
&= 1
\end{aligned}
\tag{23}
$$

For T_0, we have $\partial_2 = 0$, $\partial_1 = 0$.

$$H_2(T_0) = <\tilde{u}_{T_0} \otimes e_i> \quad (i = 1, 2, 3)$$
$$H_1(T_0) = <\tilde{m}_0 \otimes e_i, \tilde{l}_0 \otimes e_i>$$
$$H_0(T_0) = <\tilde{v}_{T_0} \otimes e_i>$$

$$\tau(C_{*\rho}(T_0)) = 1 \tag{24}$$

For $A_i (i = 1, 2, 3)$, we have

$$0 \longrightarrow C_{1,\rho}(A_i) \longrightarrow C_{0,\rho}(A_i) \longrightarrow 0$$

where $\partial_1 = x_i^{r_i} - I$.
We have

$$H_1(A_i) = <\tilde{b}_i \otimes w_i^0>$$
$$H_0(A_i) = <\tilde{v}_{A_i} \otimes w_i^0>$$

$$
\begin{aligned}
\tau(C_{*,\rho}(A_i)) &= \left| \frac{[\tilde{b}_i \otimes w_i^0, \tilde{b}_i \otimes w_i^{\pm}]}{[\tilde{v}_{A_i} \otimes (x_i^{r_i} - I)w_i^{\pm}, \tilde{v}_{A_i} \otimes w_i^0]} \right| \\
&= \left| \frac{[\tilde{b}_i \otimes w_i^0, \tilde{b}_i \otimes w_i^{\pm}]}{[\tilde{v}_{A_i} \otimes (\zeta_i^{\pm r_i} - 1)w_i^{\pm}, \tilde{v}_{A_i} \otimes w_i^0]} \right| \\
&= \frac{1}{|\zeta_i^{r_i} - 1||\zeta_i^{-r_i} - 1|}
\end{aligned}
\tag{25}
$$

For A_0, we have $\partial_1 = 0$.

$$H_1(A_0) = < \tilde{b}_0 \otimes e_i > \quad (i = 1, 2, 3)$$
$$H_0(A_0) = < \tilde{v}_{A_0} \otimes e_i >$$

$$\tau(C_{*\rho}(A_0)) = 1 \tag{26}$$

For B, we have

$$0 \longrightarrow C_{2,\rho}(B) \xrightarrow{\partial_2} C_{1,\rho}(B) \xrightarrow{\partial_1} C_{0,\rho}(B) \longrightarrow 0$$

where

$$\partial_2 = \begin{pmatrix} O & O & O \\ O & O & O \\ O & O & O \\ x_1 - I & x_2 - I & x_3 - I \end{pmatrix}, \partial_1 = \begin{pmatrix} x_1 - I & x_2 - I & x_3 - I & O \end{pmatrix}$$

We have

$$H_2(B) = < \tilde{u}_{i,B} \otimes w_i^0, (\tilde{u}_{1,B} + \tilde{u}_{2,B}x_1 + \tilde{u}_{3,B}x_2x_1) \otimes e_i > \quad (i = 1, 2, 3)$$
$$H_1(B) = < \tilde{x}_i \otimes w_i^0, (\tilde{x}_1 + \tilde{x}_2x_1 + \tilde{x}_3x_2x_1) \otimes e_i >$$

Since the rank of matrix ∂_1 is 3, we have $H_0(B) = 0$. Also, $H_3(B) = 0$ since B is simple homotopy equivalent to a 2-complex.

$\tau(C_{*,\rho}(B))$
$$= |[\tilde{u}_{i,B} \otimes w_i^0, \tilde{u} \otimes e_i, \tilde{u}_{1,B} \otimes w_1^\pm, \tilde{u}_{2,B} \otimes w_2^-]^{-1}$$
$$[\tilde{v}_B \otimes (x_1 - I)w_1^\pm, , \tilde{v}_B \otimes (x_2 - I)w_2^-]^{-1}$$
$$[\tilde{x}_i \otimes w_i^0, \tilde{x} \otimes e_i, \tilde{h} \otimes (x_1 - I)w_1^\pm, \tilde{h} \otimes (x_2 - I)w_2^-, \tilde{x}_1 \otimes w_1^\pm, \tilde{x}_2 \otimes w_2^-]|$$
$$= |[\tilde{u}_{i,B} \otimes w_i^0, \tilde{u} \otimes e_i, \tilde{u}_{1,B} \otimes w_1^\pm, , \tilde{u}_{2,B} \otimes w_2^-]^{-1}$$
$$[\tilde{v}_B \otimes (\zeta_1^{\pm 1} - 1)w_1^\pm, \tilde{v}_B \otimes (\zeta_2^{-1} - 1)w_2^-]^{-1}$$
$$[\tilde{x}_i \otimes w_i^0, \tilde{x} \otimes e_i, \tilde{h} \otimes (\zeta_1^{\pm 1} - 1)w_1^\pm, \tilde{h} \otimes (\zeta_2^{-1} - 1)w_2^-, \tilde{x}_1 \otimes w_1^\pm, \tilde{x}_2 \otimes w_2^-]|$$
$$= |[\tilde{u}_{i,B} \otimes w_i^0, \tilde{u} \otimes e_i, \tilde{u}_{1,B} \otimes w_1^\pm, , \tilde{u}_{2,B} \otimes w_2^-]^{-1}[\tilde{v}_B \otimes w_1^\pm, \tilde{v}_B \otimes w_2^-]^{-1}$$
$$[\tilde{x}_i \otimes w_i^0, \tilde{x} \otimes e_i, \tilde{h} \otimes w_1^\pm, \tilde{h} \otimes w_2^-, \tilde{x}_1 \otimes w_1^\pm, \tilde{x}_2 \otimes w_2^-]|$$
$$= 1 \tag{27}$$

where $\tilde{x} = \tilde{x}_1 + \tilde{x}_2x_1 + \tilde{x}_3x_2x_1$, $\tilde{u} = \tilde{u}_{1,B} + \tilde{u}_{2,B}x_1 + \tilde{u}_{3,B}x_2x_1$.

In the long exact sequence for twisted homology group, we have isomorphisms

$$0 \longrightarrow \bigoplus_{i=1}^{3} H_*(T_i) \longrightarrow \bigoplus_{i=1}^{3} H_*(A_i) \oplus H_*(B) \longrightarrow 0$$

Then $C_{*,\rho}(X)$ is acyclic as follows.

We have

$$0 \longrightarrow \bigoplus_{i=0}^{3} H_0(T_i) \longrightarrow \bigoplus_{i=0}^{3} H_0(A_i) \longrightarrow 0$$

where $\partial(\tilde{v}_{T_i} \otimes w_i^0) = \tilde{v}_{A_i} \otimes w_i^0$, $\partial(\tilde{v}_{T_0} \otimes e_i) = \tilde{v}_{A_0} \otimes e_i$, $det(\partial) = 1$.

$$0 \longrightarrow \bigoplus_{i=0}^{3} H_1(T_i) \longrightarrow \bigoplus_{i=0}^{3} H_1(A_i) \oplus H_1(B) \longrightarrow 0$$

where $\partial(\tilde{m}_i \otimes w_i^0) = (\tilde{x}_i - \tilde{b}_i Q_i) \otimes w_i^0$, $\partial(\tilde{l}_i \otimes w_i^0) = \tilde{b}_i P_i \otimes w_i^0$, $\partial(\tilde{m}_0 \otimes e_i) = (\tilde{x}_1 + \tilde{x}_2 x_1 + \tilde{x}_3 x_1 x_2) \otimes e_i$, $\partial(\tilde{l}_0 \otimes e_i) = \tilde{b}_0 \otimes e_i$, $Q_i = \sum_{j=1}^{q_i} x^{-jr_i}$, $P_j = \sum_{j=0}^{p_i-1} x^{jr_i}$, $det(\partial) = p_1 p_2 p_3$.

$$0 \longrightarrow \bigoplus_{i=0}^{3} H_2(T_i) \longrightarrow H_2(B) \longrightarrow 0$$

where $\partial(\tilde{u}_{T_i} \otimes w_i^0) = \tilde{u}_{i,B} \otimes w_i^0$, $\partial(\tilde{u}_0 \otimes e_i) = (\tilde{u}_{1,B} + \tilde{u}_{2,B} x_1 + \tilde{u}_{3,B} x_2 x_1) \otimes e_i$, $det(\partial) = 1$.

According to Multiplicativity lemma, Eqs. 23, 24, 25, 26, 27 and the calculations about homology above, we have

$$\text{Tor}(C_{*,\rho}(X)) = \frac{p_1 p_2 p_3}{\prod_{i=1}^{3} 4 \sin^2 \frac{2\pi r_i n_i}{p_i}}$$

$\square$

3.3. Modular data from Seifert fibered spaces. We will show that the modular data constructed from 3-component SFSs are related to the Temperley-Lieb-Jones categories at root of unit. So let us collect some basic facts about those. For references, see for instance [20].

Let A be a complex number such that $A^4 \neq 1$. For an integer n, define the quantum integer $[n]_A = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}$. So $[0]_A = 0$, $[1]_A = 1$, $[2]_A = A^2 + A^{-2}$. For each A, usually called the Kauffman variable, such that A^4 is a primitive r-th root of unity for some integer $r \geq 2$, there is an associated premodular category, called the Temperley-Lieb-Jones category and denoted by $\text{TLJ}(A)$. The category has the label set (simple objects) $[0 \cdots p - 2]$ where the label 0 is the unit object. For $i, j \in [0 \cdots p - 2]$, the quantum dimension is

$$d_j(A) = (-1)^j [j+1]_A = (-1)^j \frac{A^{2j+2} - A^{-2j-2}}{A^2 - A^{-2}},$$

the twist is

$$\theta_j(A) = (-A)^{j(j+2)},$$

and the (un-normalized) S-matrix is

$$\tilde{S}_{ij}(A) = (-1)^{i+j} [(i+1)(j+1)]_A.$$

The total dimension can be computed directly,

$$D(A) = \frac{\sqrt{2r}}{|A^2 - A^{-2}|}.$$

Denote by $\mathrm{TLJ}(A)_0$ (resp. $\mathrm{TLJ}(A)_0$) the subcategory linearly spanned by even (resp. odd) labels. We call $\mathrm{TLJ}(A)_0$ and $\mathrm{TLJ}(A)_1$ the even and odd subcategory of $\mathrm{TLJ}(A)$, respectively. The even and odd subcategory has the same dimension, both equal to $\frac{D(A)}{\sqrt{2}}$.

It is well known that if A is a primitive $4r$-th root of unity, then $\mathrm{TLJ}(A)$ is non-degenerate. If r is odd and A is a primitive $2r$-th root of unity, then $\mathrm{TLJ}(A)$ is degenerate, but the even subcategory $\mathrm{TLJ}(A)_0$ is non-degenerate.

Now we consider the construction of modular data. As before, set $M = \{0; (o, 0);$ $(p_1, q_1), (p_2, q_2), (p_3, q_3)\}$. Here each pair (p_k, q_k) are co-prime. Choose integers s_k and r_k such that $p_k s_k - q_k r_k = 1$. If q_k is odd, set $c_k = p_k q_k s_k - r_k$. Otherwise, set $c_k = p_k q_k s_k - r_k(p_k - 1)^2$. Let $A_k = -\exp(\frac{2\pi i}{4p_k} c_k)$. Note that while c_k depends on the choice of s_k and r_k, A_k does not. Moreover, A_k is a primitive $4p_k$-th root of unity if q_k is odd, a primitive $2p_k$-th root of unity if $q_k = 0 \mod 4$, and a primitive p_k-th root of unity if $q_k = 2 \mod 4$. In the latter two cases, p_k clearly must be odd. Hence, in all cases, A_k^4 is a primitive p_k-th root of unity.

If some q_k's are even, we re-arrange the elements of $\chi^{\mathrm{nab}}(M)$ as follows. For (p, q) co-prime, $j \in [0 \cdots p - 2]$, let

$$n_{p,q}(j) = \begin{cases} \frac{p-1-j}{2}, & q \text{ even and } j \text{ even} \\ \frac{j+1}{2}, & \text{otherwise} \end{cases}$$

Then from Eq. (21), $\chi^{\mathrm{nab}}(M)$ can also be written as

$$\left\{ (n_{p_1,q_1}(j_1), n_{p_2,q_2}(j_2), n_{p_3,q_3}(j_3), \tfrac{1}{2}) \mid j_k \in [0 \cdots p_k - 2]^e, k = 1, 2, 3 \right\}$$
$$\sqcup \left\{ (n_{p_1,q_1}(j_1), n_{p_2,q_2}(j_2), n_{p_3,q_3}(j_3), 0) \mid j_k \in [0 \cdots p_k - 2]^o, k = 1, 2, 3 \right\}$$
$$\tag{28}$$

Thus, the elements of $\chi^{\mathrm{nab}}(M)$ are indexed by $\vec{j} \in \prod_{k=1}^{3}[0 \cdots p_k - 2]^e \sqcup \prod_{k=1}^{3} [0 \cdots p_k - 2]^o$. Given such a $\vec{j} = (j_1, j_2, j_3)$, denote a corresponding representation by $\rho_{\vec{j}}$. (The choice of a representative is irrelevant.)

Proposition 3.1 shows that all non-Abelian characters of M are adjoint acyclic and Proposition 2.1 shows that the CS invariants of non-Abelian characters are all rational. We choose the candidate label set $L(M)$ to be $\chi^{\mathrm{nab}}(M)$.

We propose the correspondence between $L(M)$ and loop operators by the following map,

$$\rho_{\vec{j}} \mapsto \left\{ (x_k^{c_k}, \mathrm{Sym}^{j_k}) \mid k = 1, 2, 3 \right\}. \tag{29}$$

Moreover, we designate $\rho_{\vec{0}} = \rho_{(0,0,0)}$ as the unit object, which of course corresponds to the loop operator

$$\mathbf{1} = \rho_{\vec{0}} \mapsto \left\{ (x_k^{c_k}, \mathrm{Sym}^0) \mid k = 1, 2, 3 \right\}. \tag{30}$$

The following two lemmas are direct consequences of Propositions 2.1 and 3.1, respectively.

Lemma 3.2. *Let M, c_k, A_k be given as above. For each $\vec{j} = (j_1, j_2, j_3) \in \prod_{k=1}^{3}$*
$[0 \cdots p_k - 2]^e \sqcup \prod_{k=1}^{3}[0 \cdots p_k - 2]^o$ with $\rho_{\vec{j}}$ a corresponding representation, then

$$CS(\rho_{\vec{j}}) = \sum_{k=1}^{3} \frac{-c_k}{4p_k}(j_k + 1)^2. \tag{31}$$

As a consequence,

$$e^{-2\pi i CS(\rho_{\vec{j}})} = \prod_{k=1}^{3}(-A_k)^{(j_k+1)^2} = (-A_1 A_2 A_3) \prod_{k=1}^{3} \theta_{j_k}(A_k). \tag{32}$$

Proof. Note that for $\vec{j} = (j_1, j_2, j_3), \mathrm{Tr}(\rho_{(j_1, j_2, j_3)}(x_i)) = 2\cos\frac{2\pi n_{p_i, q_i}(j_i)}{p_i}$. The formula
above then follows from Proposition 2.1.   □

Lemma 3.3. *Let M, c_k, A_k be given as above and let $D = D(A_1)D(A_2)D(A_3)/2$.*
For each $\vec{j} = (j_1, j_2, j_3) \in \prod_{k=1}^{3}[0 \cdots p_k - 2]^e \sqcup \prod_{k=1}^{3}[0 \cdots p_k - 2]^o$ with $\rho_{\vec{j}}$ a
corresponding representation, then

$$Tor(\rho_{\vec{j}}) = \prod_{k=1}^{3} \frac{p_k}{4\sin^2(\frac{\pi r_k(j_k+1)}{p_k})}, \tag{33}$$

and hence,

$$\left(2Tor(\rho_{\vec{j}})\right)^{-\frac{1}{2}} = 2\prod_{k=1}^{3}\left|\frac{d_{j_k}(A_k)}{D(A_k)}\right| = \frac{|\prod_{k=1}^{3} d_{j_k}(A_k)|}{D}. \tag{34}$$

The main result of the section is the following theorem.

Theorem 3.4. *Let $M = \{0; (p_1, q_1), (p_2, q_2), (p_3, q_3)\}$ and $\{A_k\}_{k=1,2,3}$ be given as*
above. With the operators and tensor unit given in Eqs. (29) and (30), respectively, the
modular data constructed from M matches that of the following pre-modular category,

$$\mathcal{B} := \left(\boxtimes_{k=1}^{3} TLJ(A_k)_0 \right) \bigoplus \left(\boxtimes_{k=1}^{3} TLJ(A_k)_1 \right)$$

Proof. Since A_k^4 is a primitive p_k-th root of unity, the label set for $\mathcal{B}$ is clearly $L :=$
$\prod_{k=1}^{3}[0 \cdots p_k - 2]^e \sqcup \prod_{k=1}^{3}[0 \cdots p_k - 2]^o$, the same index set for $L(M)$. The modular
data of $\mathcal{B}$ can be easily expressed in terms of that of the individual TLJ(A_k). For $\vec{i}, \vec{j} \in L$,

$$d_{\vec{j}} = \prod_{k=1}^{3} d_{j_k}(A_k), \quad \theta_{\vec{j}} = \prod_{k=1}^{3} \theta_{j_k}(A_k), \quad \tilde{S}_{\vec{i}\,\vec{j}} = \prod_{k=1}^{3} \tilde{S}_{i_k j_k}(A_k).$$

Also, the total dimension of $\mathcal{B}$ is $D = D(A_1)D(A_2)D(A_3)/2$.

Lemma 3.2 shows that, up to a global phase, the Chern-Simons invariant gives the
twist,

$$e^{-2\pi i CS(\rho_{\vec{j}})} = \theta_{\vec{j}},$$

and Lemma 3.3 shows that the torsion matches the absolute value of the normalized quantum dimension,

$$\left(2\mathrm{Tor}(\rho_{\vec{j}})\right)^{-\frac{1}{2}} = \frac{d_{\vec{j}}}{D}.$$

Lastly, We check the S-matrix computed from loop operators. Given $\vec{i} = (i_1, i_2, i_3)$, $\vec{j} = (j_1, j_2, j_3) \in L$, we have (choosing $\epsilon = -1$)

$$W_{\vec{i}}(\vec{j}) = \prod_{k=1}^{3} \mathrm{Tr}_{\mathrm{Sym}^{j_k}}(-\rho_{\vec{i}}(x_k^{c_k})).$$

Note that,

$$\mathrm{Tr}\left(\rho_{\vec{i}}(x_k^{c_k})\right) = 2\cos\frac{2n_{p_k,q_k}(i_k)\pi c_k}{p_k} = 2\cos\frac{(i_k+1)\pi c_k}{p_k},$$

where the second equality holds irrelevant of the parity of q_k. Combining the previous two equations, we get

$$W_{\vec{i}}(\vec{j}) = \prod_{k=1}^{3} \Delta_{j_k}(-2\cos\frac{(i_k+1)\pi c_k}{p_k}) = \prod_{k=1}^{3}(-1)^{j_k}\frac{\sin\frac{(i_k+1)(j_k+1)\pi c_k}{p_k}}{\sin\frac{(i_k+1)\pi c_k}{p_k}},$$

where $\Delta_{j_k}(\cdot)$ is the Chebyshev polynomial (see Eq. (12)). Therefore, the $(\vec{j}, \vec{i})$-entry of the potential un-normalized S matrix is,

$$W_{\vec{i}}(\vec{j})W_{\vec{0}}(\vec{i}) = \prod_{k=1}^{3}(-1)^{i_k+j_k}\frac{\sin\frac{(i_k+1)(j_k+1)\pi c_k}{p_k}}{\sin\frac{\pi c_k}{p_k}}$$

$$= \prod_{k=1}^{3}\tilde{S}(A_k)_{j_k i_k},$$

which is precisely $\tilde{S}_{\vec{j}\vec{i}}$ of $\mathcal{B}$. $\square$

The premodular category produced in the previous theorem may not be modular in general, and it depends crucially on the topology of the three manifold. For a three-component SFS M, it is a $\mathbb{Z}_2$ homology sphere, i.e., $H^1(M, \mathbb{Z}_2) = 0$, if and only if

$$p_1 p_2 p_3\left(\frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3}\right) \in 2\mathbb{Z} + 1$$

Lemma 3.5. *Assume that r is odd. Suppose that*

$$T(p, j, l, *) = \sum_{m \in [p]^*}\left(e^{(j+l)mr\frac{\pi}{p}i} - e^{(j-l)mr\frac{\pi}{p}i} - e^{(-j+l)mr\frac{\pi}{p}i} + e^{(-j-l)mr\frac{\pi}{p}i}\right)$$

where $* = 1, 0$, and $[p]^*$ denotes the set of odd integers from 1 to $p-1$ if $*$ is 1 and the set of even integers in the same range otherwise.

When p is odd, $j \neq l$, $j + l$ is odd,

$$T(p, j, l, *) = \begin{cases} 0 & j + l \neq p \\ (-1)^* p & j + l = p \end{cases}$$

When p is odd, $j \neq l$, $j + l$ is even,

$$T(p, j, l, *) = 0$$

When p is odd, $j = l$,

$$T(p, j, l, *) = -p$$

When p is even, $j \neq l$, $j + l$ is odd,

$$T(p, j, l, *) = 0$$

When p is even, $j \neq l$, $j + l$ is even,

$$T(p, j, l, *) = \begin{cases} 0 & j + l \neq p \\ (-1)^* p & j + l = p \end{cases}$$

When p is even, $j = l$,

$$T(p, j, l, 0) = \begin{cases} -p & j + l \neq p \\ 0 & j + l = p \end{cases}$$

$$T(p, j, l, 1) = \begin{cases} -p & j + l \neq p \\ -2p & j + l = p \end{cases}$$

Proof. We prove the lemma by direct computation.
When p is odd, $j \neq l$, $j + l$ is odd,

$$
T(p, j, l, 1) = \sum_{m=1, m \text{ odd}}^{p-2} \left(e^{(j+l)mr\frac{\pi}{p}i} - e^{(j-l)mr\frac{\pi}{p}i} + e^{(j-l)(p-m)r\frac{\pi}{p}i} - e^{(j+l)(p-m)r\frac{\pi}{p}i} \right)
$$

$$
= \sum_{m=1, m \text{ odd}}^{p-2} \left(e^{(j+l)mr\frac{\pi}{p}i} - e^{(j-l)mr\frac{\pi}{p}i} \right) + \sum_{m=2, \text{ even}}^{p-1} \left(e^{(j-l)mr\frac{\pi}{p}i} - e^{(j+l)mr\frac{\pi}{p}i} \right)
$$

$$
= -\sum_{m=1}^{p-1} \left(-e^{(j+l)r\frac{\pi}{p}i} \right)^m + \sum_{m=1}^{p-1} \left(-e^{(j-l)r\frac{\pi}{p}i} \right)^m
$$

$$
= \begin{cases} 0 & j + l \neq p \\ -p & j + l = p \end{cases}
$$

$$
= -T(p, j, l, 0)
$$

Similarly, we get other cases. $\square$

Proposition 3.6. *Given a three-component SFS M, the premodular category $\mathcal{B}_M$ produced in Theorem 3.4 is modular if and only if M is a $\mathbb{Z}_2$ homology sphere.*

Proof. Since the structure from Sect. 3 respects the change of parametrization of Seifert fiber space, it suffices to verify the following 5 cases for $(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$.

$$(\frac{odd}{odd}, \frac{odd}{odd}, \frac{odd}{odd}), (\frac{odd}{odd}, \frac{odd}{odd}, \frac{even}{odd}), (\frac{odd}{odd}, \frac{even}{odd}, \frac{even}{odd}),$$
$$(\frac{even}{odd}, \frac{even}{odd}, \frac{even}{odd}), (\frac{odd}{odd}, \frac{odd}{odd}, \frac{odd}{even})$$

The first two cases correspond to $\mathbb{Z}_2$-homology sphere. In the following, we will explicitly calculate S^2, which directly implies the proposition.

When q_1, q_2, q_3 are odd, $j_1 = j_2 = j_3 \mod 2$, $l_1 = l_2 = l_3 \mod 2$. Up to a scalar,

$$S_{(j_1, j_2, j_3),(l_1,l_2,l_3)} = (-1)^{j_1+l_1} \prod_{k=1}^{3} \sin j_k l_k r_k \frac{\pi}{p_k}$$

$(S^2)_{(j_1, j_2, j_3),(l_1,l_2,l_3)}$

$$= \sum_{(m_1,m_2,m_3)} (-1)^{j_1+m_1+m_1+l_1} \prod_{k=1}^{3} \sin j_k m_k r_k \frac{\pi}{p_k} \sin m_k l_k r_k \frac{\pi}{p_k}$$

$$= (-1)^{j_1+l_1} \sum_{(m_1,m_2,m_3)} \prod_{k=1}^{3} -\frac{1}{4} (e^{(j_k+l_k)m_k r_k \frac{\pi}{p_k} i} - e^{(j_k-l_k)m_k r_k \frac{\pi}{p_k} i} - e^{(-j_k+l_k)m_k r_k \frac{\pi}{p_k} i}$$
$$+ e^{(-j_k-l_k)m_k r_k \frac{\pi}{p_k} i})$$

$$= (-1)^{j_1+l_1} (\sum_{(m_1,m_2,m_3), m_i \text{ odd}} + \sum_{(m_1,m_2,m_3), m_i \text{ even}})...$$

$$= (-1)^{j_1+l_1} (\prod_{k=1}^{3} T(p_k, j_k, l_k, 1) + \prod_{k=1}^{3} T(p_k, j_k, l_k, 0))$$

When p_1, p_2, p_3 are odd,

$$(S^2)_{(j_1, j_2, j_3),(l_1,l_2,l_3)} = \begin{cases} 0 & (j_1, j_2, j_3) \neq (l_1, l_2, l_3) \\ \frac{p_1 p_2 p_3}{32} & (j_1, j_2, j_3) = (l_1, l_2, l_3) \end{cases}$$

When p_1, p_2 are odd, p_3 is even,

$$(S^2)_{(j_1, j_2, j_3),(l_1,l_2,l_3)} = \begin{cases} 0 & (j_1, j_2, j_3) \neq (l_1, l_2, l_3) \\ \frac{p_1 p_2 p_3}{32} & (j_1, j_2, j_3) = (l_1, l_2, l_3) \end{cases}$$

Thus $S^2 = cI$ for the above two cases.
When p_1 is odd, p_2, p_3 are even,

$$(S^2)_{(1,1,1),(l_1,l_2,l_3)} = \begin{cases} \frac{p_1 p_2 p_3}{32} & (l_1, l_2, l_3) = (1, 1, 1), (1, p_2 - 1, p_3 - 1) \\ 0 & \text{otherwise} \end{cases}$$

$$(S^2)_{(1,p_2-1,p_3-1),(l_1,l_2,l_3)} = \begin{cases} \dfrac{p_1p_2p_3}{32} & (l_1,l_2,l_3) = (1,1,1), (1,p_2-1,p_3-1) \\ 0 & \text{otherwise} \end{cases}$$

$$(S^2)_{(1,1,1)} = (S^2)_{(1,p_2-1,p_3-1)}$$

When p_1, p_2, p_3 are even,

$$(S^2)_{(1,1,1),(l_1,l_2,l_3)} = \begin{cases} \dfrac{p_1p_2p_3}{32} & (l_1,l_2,l_3) = (1,1,1), (1,p_2-1,p_3-1) \\ 0 & \text{otherwise} \end{cases}$$

$$(S^2)_{(1,p_2-1,p_3-1),(l_1,l_2,l_3)} = \begin{cases} \dfrac{p_1p_2p_3}{32} & (l_1,l_2,l_3) = (1,1,1), (1,p_2-1,p_3-1) \\ 0 & \text{otherwise} \end{cases}$$

$$(S^2)_{(1,1,1)} = (S^2)_{(1,p_2-1,p_3-1)}$$

S^2 is degenerate for above two cases.

When q_1, q_2 are odd, q_3 is even, $j_1 = j_2 \mod 2$, $l_1 = l_2 \mod 2$, $j_3 = 0 \mod 2$, $l_3 = 0 \mod 2$.

$$(S^2)_{(j_1,j_2,j_3),(l_1,l_2,l_3)} = \prod_{k=1}^{2} T(p_k,j_k,l_k,1)T(p_3,j_3,l_3,0) + \prod_{k=1}^{3} T(p_k,j_k,l_k,0)$$

When p_1, p_2, p_3 are odd,

$$(S^2)_{(1,1,2),(l_1,l_2,l_3)} = \begin{cases} -\dfrac{p_1p_2p_3}{32} & (l_1,l_2,l_3) = (1,1,2), (p_1-1,p_2-1,2) \\ 0 & \text{otherwise} \end{cases}$$

$$(S^2)_{(p_1-1,p_2-1,2),(l_1,l_2,l_3)} = \begin{cases} -\dfrac{p_1p_2p_3}{32} & (l_1,l_2,l_3) = (1,1,2), (p_1-1,p_2-1,2) \\ 0 & \text{otherwise} \end{cases}$$

S^2 is degenerate.   □

It is worth noting even if every TLJ(A_k) appearing in the construction of $\mathcal{B}_M$ in Theorem 3.4 is not modular, $\mathcal{B}_M$ could still be modular. For instance, for the SFS $M_0 = (0; (o,0); (5,1),(3,2),(5,4))$, the corresponding Kauffman variables are $A_1 = -e^{\frac{i\pi}{10}}$, $A_2 = -e^{\frac{i\pi}{3}}$, $A_3 = -e^{\frac{2i\pi}{5}}$. It is direct to see that TLJ(A_1) is modular, but TLJ(A_2) and TLJ(A_3) are not. However, M_0 is a $\mathbb{Z}_2$ homology sphere, by Proposition 3.6, $\mathcal{B}_{M_0}$ is modular, a rank-8 MTC.

3.4. Examples: realization of $SU(2)_k$. Here we study a special class of SFSs with three components, namely, $M(r) := \{0; (o,0); (3,1),(3,1),(r,1)\}$. We show explicitly that different choice of characters as the unit object may lead to different theories. In fact, it will be proved that from $M(r)$ we can construct either the MTC $SU(2)_{r-2}$ or TLJ($e^{\frac{2\pi i}{4r}}$).

For each integer $r \geq 2$, there is a unitary MTC, usually denoted by $SU(2)_{r-2}$ [3], which is closely related to the Temperley-Lieb-Jones categories. Here $r-2$ is called the level of the MTC. It has the same label set as TLJ($e^{\frac{2\pi i}{4r}}$), but differs from it in modular

data by some signs. Explicitly, setting $A = e^{\frac{2\pi i}{4r}}$, the modular data for $SU(2)_{r-2}$ is given as follows,

$$\theta_j = A^{j(j+2)} = e^{\frac{2\pi i \, j(j+2)}{4r}},$$

$$\tilde{S}_{ij} = [(i+1)(j+1)]_A = \frac{\sin \frac{(i+1)(j+1)\pi}{r}}{\sin \frac{\pi}{r}}.$$

In particular, its quantum dimensions are all positive (since it is unitary),

$$d_j = [j+1]_A = \frac{\sin \frac{(j+1)\pi}{r}}{\sin \frac{\pi}{r}},$$

and the total dimension is

$$D = \sqrt{\frac{r}{2}} \frac{1}{\sin \frac{\pi}{r}}.$$

Note that $d_j = |d_j(A)|$ and $D = D(A)$, where $d_j(A)$ and $D(A)$ are the quantum dimension of j and total dimension of $TLJ(A)$, respectively.

We will use notations from Sects. 3.1 and 3.3. The non-Abelian characters of $M(r)$ is given by

$$\chi^{\text{nab}}(M(r)) = \left\{ \left(\frac{1}{2}, \frac{1}{2}, \frac{j+1}{2}, \frac{1}{2} \right) \mid (0, 0, j) \in \{0\} \times \{0\} \times [0 \cdots r - 2]^e \right\}$$

$$\sqcup \quad \left\{ \left(1, 1, \frac{j+1}{2}, 0 \right) \mid (1, 1, j) \in \{1\} \times \{1\} \times [0 \cdots r - 2]^o \right\}.$$

(35)

Thus, each $j \in [0 \cdots r - 2]$ corresponds to a non-Abelian character indexed by $(j \bmod 2, j \bmod 2, j)$. We denote the corresponding representation by ρ_j (instead of using the triple as the subscript). The eigenvalues of $\rho_j(x_3)$ are $e^{\pm \frac{(j+1)\pi i}{r}}$. The eigenvalues of $\rho_j(x_1)$ and those of $\rho_j(x_2)$ are both $e^{\pm \frac{a_j \pi i}{3}}$, where $a_j = 1$ if j even and $a_j = 2$ otherwise.

Also, it is direct to see that $c_1 = c_2 = c_3 = 1$, and $A_1 = A_2 = -e^{\frac{\pi i}{6}}$, $A_3 = -e^{\frac{2\pi i}{4r}}$.

In Sect. 3.3, we chose the candidate label set $L(M(r))$ to be $\chi^{\text{nab}}(M(r))$, and defined the following map from $\chi^{\text{nab}}(M(r))$ to loop operators,

$$\rho_j = \mapsto \left\{ (x_1, \text{Sym}^{j \bmod 2}), (x_2, \text{Sym}^{j \bmod 2}), (x_3, \text{Sym}^j) \right\}.$$

(36)

It can be checked directly that for $i, j \in [0 \cdots r - 2]$, $\text{Tr}(\rho_i(x_1)) = \text{Tr}(\rho_i(x_2)) = \pm 1$, and it follows that,

$$W_i(j) = \text{Tr}_{\text{Sym}^{j \bmod 2}}(-\rho_i(x_1)) \, \text{Tr}_{\text{Sym}^{j \bmod 2}}(-\rho_i(x_2)) \, \text{Tr}_{\text{Sym}^j}(-\rho_i(x_3))$$

$$= \text{Tr}_{\text{Sym}^j}(-\rho_i(x_3)).$$

Hence, we may as well choose a simplified map to loop operators,

$$\rho_j \mapsto \{(x_3, \text{Sym}^j)\}.$$

(37)

The unit object was chosen to be ρ_0 which corresponds to the loop operator (x_3, Sym^0). By Theorem 3.4, the modular data match that of the premodular category,

$$\mathcal{B}_{M(r)} = \left(\boxtimes_{k=1}^3 \mathrm{TLJ}(A_k)_0 \right) \bigoplus \left(\boxtimes_{k=1}^3 \mathrm{TLJ}(A_k)_1 \right). \tag{38}$$

Note that $\mathrm{TLJ}(A_1) = \mathrm{TLJ}(-e^{\frac{\pi i}{6}})$ has label set $\{0, 1\}$, the twists $\theta_0 = 1, \theta_1 = i$, and un-normalized S-matrix,

$$\tilde{S} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

This means that $\mathcal{B}_{M(r)}$ has the same twists for even labels and S-matrix as $\mathrm{TLJ}(A_3)$. The twists for odd labels differ by a minus sign between the two theories. Let $A(r) = -A_3 = e^{\frac{2\pi i}{4r}}$. Note that a change of the Kauffman variable from A to $-A$ does not change the S-matrix. It follows that $\mathcal{B}_{M(r)}$ and $\mathrm{TLJ}(A(r))$ has the same modular data. In fact, they are isomorphic.

Therefore, by using the loop operator correspondence in Eq. (37) and letting ρ_0 be the unit object, we recover the MTC $\mathrm{TLJ}(A(r))$.

Now we examine an alternative choice of the unit object. Since $M(r)$ is a $\mathbb{Z}_2$ homology sphere, a potential unit object ρ_{α_0} can be determined by the equation,

$$\left| \sum_{\rho \in \chi^{\mathrm{nab}}(M(r))} \frac{\exp(-2\pi i \mathrm{CS}(\rho))}{2\mathrm{Tor}(\rho)} \right| = (2\mathrm{Tor}(\rho_{\alpha_0}))^{-\frac{1}{2}}. \tag{39}$$

Such a ρ_{α_0} would have quantum dimension in absolute value equal to 1 in any MTC produced by $M(r)$. Since we already know that we can produce $\mathrm{TLJ}(A(r))$ from $M(r)$ and the only non-unit object in $\mathrm{TLJ}(A(r))$ whose quantum dimension is 1 in absolute value is ρ_{r-2}, we can choose ρ_{r-2} as the unit object in a new theory.

In this case, we reverse the previous order of the simple objects. Denote by $\tilde{\rho}_j := \rho_{r-2-j}, j \in [0 \cdots r-2]$. Set $\tilde{\rho}_0 = \rho_{r-2}$ as the unit object. The correspondence between characters and loop operators is now defined as,

$$\tilde{\rho}_j \mapsto (x_3, \mathrm{Sym}^j). \tag{40}$$

We claim that with above choice of unit object and loop operators, the modular data produced from $M(r)$ matches that of $\mathrm{SU}(2)_{r-2}$ where $\tilde{\rho}_j$ corresponds to j in the label set of $\mathrm{SU}(2)_{r-2}$. See Sect. 3.3 for a collection of facts about $\mathrm{SU}(2)_{r-2}$.

Firstly, by Lemma 3.2, up to an irrelevant phase factor,

$$\mathrm{CS}(\rho_j) = -\frac{j(j+2)}{4r} + \frac{1 - (-1)^j}{4} \quad \mathrm{mod}\ 1. \tag{41}$$

Then rewriting above equation in terms of $\tilde{\rho}_j$, we get, again up to an irrelevant factor,

$$\mathrm{CS}(\tilde{\rho}_j) = -\frac{j(j+2)}{4r} \quad \mathrm{mod}\ 1. \tag{42}$$

Thus,

$$e^{-2\pi i \mathrm{CS}(\tilde{\rho}_j)} = e^{\frac{2\pi i \, j(j+2)}{4r}} \tag{43}$$

is the twist θ_j of $\mathrm{SU}(2)_{r-2}$.

Next, we check the S-matrix.

$$W_0(j) = \mathrm{Tr}_{\mathrm{Sym}^j}(-\tilde{\rho}_0(x_3)) = \Delta_j(2\cos\frac{\pi}{r}) = \frac{\sin\frac{(j+1)\pi}{r}}{\sin\frac{\pi}{r}}, \tag{44}$$

and the (j, i)-entry of the potential S-matrix is,

$$W_i(j)W_0(i) = \mathrm{Tr}_{\mathrm{Sym}^j}(-\tilde{\rho}_i(x_3))\, W_0(i) \tag{45}$$

$$= \Delta_j(2\cos\frac{(i+1)\pi}{r})\Delta_i(2\cos\frac{\pi}{r}) \tag{46}$$

$$= \frac{\sin\frac{(i+1)(j+1)\pi}{r}}{\sin\frac{\pi}{r}}, \tag{47}$$

which is $\tilde{S}_{ji}$ of $\mathrm{SU}(2)_{r-2}$.

Lastly, by Lemma 3.3,

$$\left(2\mathrm{Tor}(\tilde{\rho}_j)\right)^{-\frac{1}{2}} = \left(2\mathrm{Tor}(\rho_{r-2-j})\right)^{-\frac{1}{2}} = \frac{|d_{r-2-j}(A_3)|}{D(A_3)}, \tag{48}$$

where we used the fact that in $\mathrm{TLJ}(A_1) = \mathrm{TLJ}(A_2)$, the two simple objects have quantum dimensions ± 1 and thus the dimension of the category is $D(A_1) = \sqrt{2}$. Also note that $A_3 = -e^{\frac{2\pi i}{4r}}$, then $|d_{r-2-j}(A_3)| = |d_j(A_3)|$ and $D(A_3)$ are equal to the quantum dimension d_j and the total dimension D, respectively, in $\mathrm{SU}(2)_{r-2}$. Hence, the torsion invariant computes the normalized quantum dimension,

$$\left(2\mathrm{Tor}(\tilde{\rho}_j)\right)^{-\frac{1}{2}} = \frac{d_j}{D}. \tag{49}$$

To summarize, for the SFS $M(r)$, two choices of the unit object together with appropriate definition of loop operators produce the MTCs $\mathrm{TLJ}(e^{\frac{2\pi i}{4r}})$ and $\mathrm{SU}(2)_{r-2}$, with the former non-unitary and the latter unitary.

3.5. Graded product of graded premodular categories.
In Sect. 3.3, we have seen that the premoduar category resulting from three-component SFSs is formed from three Temperley-Lieb-Jones categories, by taking the Deligne product of the even sectors, that of the odd sectors, and suming them up. Here we generalize the operation.

Definition 3.7. Let $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ and $\mathcal{D} = \oplus_{g\in G}\mathcal{D}_g$ be two G-graded premodular tensor categories for some finite group G (which must be Abelian). The graded product of $\mathcal{C}$ and $\mathcal{D}$ is again a G-graded premodular category $\mathcal{C}\boxtimes_{gr}\mathcal{D} = \oplus_{g\in G}(\mathcal{C}\boxtimes_{gr}\mathcal{D})_g$ such that $(\mathcal{C}\boxtimes_{gr}\mathcal{D})_g := \mathcal{C}_g\boxtimes\mathcal{D}_g$.

The monoidal and braiding structure on $\mathcal{C} \boxtimes_{gr} \mathcal{D}$ is defined in the obvious way which make it into a premodular category. Another way to see this is that $\mathcal{C} \boxtimes_{gr} \mathcal{D}$ is a full subcategory of the premodular category $\mathcal{C} \boxtimes \mathcal{D}$ and is closed under tensor product and braiding. The graded product operation $\boxtimes_{gr}$ is associative up to canonical equivalence.

For a Kauffman variable A, $\text{TLJ}(A)$ is a $\mathbb{Z}_2$-graded premodular category with $\text{TLJ}(A)_0$ spanned by even labels and $\text{TLJ}(A)_1$ odd labels. Hence, Theorem 3.4 states that, for a three-component SFS $M = \{0; (o, 0); (p_1, q_1), (p_2, q_2), (p_3, q_3)\}$ with $A_k, k = 1, 2, 3$ defined as in Sect. 3.3, the premodular category resulting from M is $\mathcal{B}_M = \text{TLJ}(A_1) \boxtimes_{gr} \text{TLJ}(A_2) \boxtimes_{gr} \text{TLJ}(A_3)$.

The graded product operation provides method to construct new premodular categories from old ones. A very interesting question is when the graded product of two pre-modular categories is modular. For instance, take $A_1 = -e^{\frac{i\pi}{6}}$, $A_2 = -e^{-\frac{i\pi}{5}}$. Here A_1 is a primitive 12-th root of unity and A_2 a primitive 5-th root of unity. Hence $\text{TLJ}(A_1)$ is modular of rank 2 and $\text{TLJ}(A_2)$ is none modular of rank 4. Their S-matrices are given by,

$$\tilde{S}(A_1) = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad \tilde{S}(A_2) = \begin{pmatrix} 1 & \varphi & \varphi & 1 \\ \varphi & -1 & -1 & \varphi \\ \varphi & -1 & -1 & \varphi \\ 1 & \varphi & \varphi & 1 \end{pmatrix}, \tag{50}$$

where $\varphi = \frac{1}{2}(1 - \sqrt{5})$. Then the S-matrix of $\text{TLJ}(A_1) \boxtimes_{gr} \text{TLJ}(A_2)$ with its simple objects ordered as $\{0 \boxtimes 0, 0 \boxtimes 2, 1 \boxtimes 1, 1 \boxtimes 3\}$ is,

$$\tilde{S} = \begin{pmatrix} 1 & \varphi & -\varphi & -1 \\ \varphi & -1 & 1 & -\varphi \\ -\varphi & 1 & 1 & -\varphi \\ -1 & -\varphi & -\varphi & -1 \end{pmatrix}, \tag{51}$$

which can be checked straightforwardly to be non-degenerate. Thus $\text{TLJ}(A_1) \boxtimes_{gr} \text{TLJ}(A_2)$ is modular.

We leave the question of when the graded product of two arbitrary graded (and more generally multiple) premodular categories is modular as a future direction. In the rest of this section, we focus on the case where the group is $\mathbb{Z}_2$ and study a special class of $\mathbb{Z}_2$-graded modular categories, namely $SU(2)_k$. For basic facts, see Sect. 3.4.

Let $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$ be a $\mathbb{Z}_2$-graded MTC. Denote by I the label set of $\mathcal{C}$ and partition $I = I_0 \sqcup I_1$ where I_α consists of objects of I that are in the $\mathcal{C}_\alpha$ sector. To avoid confusion, when there is more than one MTC present, we write $I(\mathcal{C})$, $\tilde{S}(\mathcal{C})$, etc.

Proposition 3.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathbb{Z}_2$-graded MTCs. Then $\mathcal{C} \boxtimes_{gr} \mathcal{D}$ is a proper (i.e., degenerate) premodular category if and only if there exist $i \in I(\mathcal{C})$, $j \in I(\mathcal{D})$, scalars $c_0(\mathcal{C})$, $c_1(\mathcal{C})$, $c_0(\mathcal{D})$, and $c_1(\mathcal{D})$, such that,*

1. i and j belong to sectors of the same parity;
2. the following equations concerning S-entries hold:

$$\tilde{S}(\mathcal{C})_{ik} = \begin{cases} c_0(\mathcal{C})d_k(\mathcal{C}) & k \in I_0(\mathcal{C}) \\ c_1(\mathcal{C})d_k(\mathcal{C}) & k \in I_1(\mathcal{C}) \end{cases} \qquad \tilde{S}(\mathcal{D})_{jk} = \begin{cases} c_0(\mathcal{D})d_k(\mathcal{D}) & k \in I_0(\mathcal{D}) \\ c_1(\mathcal{D})d_k(\mathcal{D}) & k \in I_1(\mathcal{D}) \end{cases}$$

3. $c_0(\mathcal{C})/c_1(\mathcal{C}) = c_1(\mathcal{D})/c_0(\mathcal{D}) \neq 1$.

Proof. The main idea is to show that the conditions presented in the statement of the proposition are equivalent to the property that in the S-matrix of $\mathcal{C} \boxtimes_{gr} \mathcal{D}$, the row corresponding to the object $i \boxtimes j$ is proportional to the first row (i.e., the row corresponding to the unit object). □

Remark 3.9. In the above proposition, the conditions $c_0(\mathcal{C})/c_1(\mathcal{C}) \neq 1$ and $c_1(\mathcal{D})/c_0(\mathcal{D}) \neq 1$ are used to eliminate the trivial case where i and j are both the unit object. When neither of i nor j is the unit object, those conditions automatically hold since otherwise the S-matrix of $\mathcal{C}$ or $\mathcal{D}$ would be degenerate. Also, note that if either $\mathcal{C}_0$ or $\mathcal{D}_0$ is non-degenerate, then i and j must be in the sector of odd parity.

For $m \geq 0$, $SU(2)_m$ is a $\mathbb{Z}_2$-graded MTC with $(SU(2)_m)_0$ spanned by even labels and $(SU(2)_m)_1$ by odd labels.

Theorem 3.10. *For $m, n \geq 0$, $SU(2)_m \boxtimes_{gr} SU(2)_n$ is an MTC if and only if the pair (m, n) have different parity. In particular, $SU(2)_m \boxtimes_{gr} SU(2)_m$ is always degenerate.*

Proof. In $SU(2)_m$, the un-normalized S-matrix is given by,

$$\tilde{S}_{ab} = \frac{\sin \frac{(a+1)(b+1)\pi}{m+2}}{\sin \frac{\pi}{m+2}}.$$

Hence, $\tilde{S}_{mb} = (-1)^b \tilde{S}_{0b} = (-1)^b d_b$. For (m, n) with the same parity, with the notation from the statement of Proposition 3.8, we choose $i = m, j = n$. Then the relevant constants are $c_0(SU(2)_m) = c_0(SU(2)_n) = 1$, $c_1(SU(2)_m) = c_1(SU(2)_n) = -1$ which satisfies the conditions stated in that proposition, and hence $SU(2)_m \boxtimes_{gr} SU(2)_n$ is degenerate. For the converse direction, it can be seen that the only non-unit simple object in $SU(2)_m$ for which $c_0(SU(2)_m)$ and $c_1(SU(2)_m)$ exist is the object m. Therefore, if (m, n) have different parity, the only pair of indexes for (i, j) is (m, n) which contradicts the first condition of Proposition 3.8. This implies that $SU(2)_m \boxtimes_{gr} SU(2)_n$ is non-degenerate. □

Example 3.11. By Theorem 3.10, $SU(2)_2 \boxtimes_{gr} SU(2)_3$ is an MTC of rank 6. Its un-normalized S-matrix and T-matrix are given by,

$$\tilde{S} = \begin{pmatrix} 1 & \frac{1}{2}\left(1 + \sqrt{5}\right) & 1 & \frac{1}{2}\left(1 + \sqrt{5}\right) & \frac{1+\sqrt{5}}{\sqrt{2}} & \sqrt{2} \\ \frac{1}{2}\left(1 + \sqrt{5}\right) & -1 & \frac{1}{2}\left(1 + \sqrt{5}\right) & -1 & -\sqrt{2} & \frac{1+\sqrt{5}}{\sqrt{2}} \\ 1 & \frac{1}{2}\left(1 + \sqrt{5}\right) & 1 & \frac{1}{2}\left(1 + \sqrt{5}\right) & -\frac{1+\sqrt{5}}{\sqrt{2}} & -\sqrt{2} \\ \frac{1}{2}\left(1 + \sqrt{5}\right) & -1 & \frac{1}{2}\left(1 + \sqrt{5}\right) & -1 & \sqrt{2} & -\frac{1+\sqrt{5}}{\sqrt{2}} \\ \frac{1+\sqrt{5}}{\sqrt{2}} & -\sqrt{2} & -\frac{1+\sqrt{5}}{\sqrt{2}} & \sqrt{2} & 0 & 0 \\ \sqrt{2} & \frac{1+\sqrt{5}}{\sqrt{2}} & -\sqrt{2} & -\frac{1+\sqrt{5}}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{4i\pi}{5}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{\frac{4i\pi}{5}} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{27i\pi}{40}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -ie^{\frac{3i\pi}{8}} \end{pmatrix}$$

Since $SU(2)_2 \boxtimes_{gr} SU(2)_3$ contains the even part of $SU(2)_3$ as a subcategory which is itself an MTC (Fibonacci), $SU(2)_2 \boxtimes_{gr} SU(2)_3$ must split. In fact, $SU(2)_2 \boxtimes_{gr} SU(2)_3 \simeq$ Fib $\boxtimes$ TLJ$(-ie^{\frac{\pi i}{8}})$.

4. Modular Tensor Categories from SOL Geometry

4.1. Character varieties of torus bundles over the circle. One of the non-hyperbolic geometries is SOL and some examples of closed manifolds are torus bundles over the circle with Anosov monodromy maps.

Let M be a torus bundle over S^1 with the monodromy map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ where $|a + d| > 2$. Its fundamental group has the presentation,

$$\pi_1(M) = \langle x, y, h \mid x^a y^c = h^{-1}xh, \ x^b y^d = h^{-1}yh, \ xyx^{-1}y^{-1} = 1 \rangle, \quad (52)$$

where x and y are the meridian and longitude, respectively, on the torus, and h corresponds to a loop around the S^1 component. We consider non-Abelian characters of M to $SL(2, \mathbb{C})$. Let $\rho : \pi_1(M) \to SL(2, \mathbb{C})$ be a non-Abelian representation.

First, we consider the case where $\rho(x)$ is diagonalizable. Up to conjugation, assume $\rho(x)$ is diagonal. Since y commutes with x, $\rho(y)$ is also diagonal, and moreover, $\rho(x)$ and $\rho(y)$ cannot be both contained in the center $\{\pm I\}$. (Otherwise, the image of ρ would be Abelian.) If $\rho(x) \neq \pm I$, it follows from the relation $x^a y^c = h^{-1}xh$ that $\rho(h)$, up to conjugation, simply permutes the two eigenvectors of $\rho(x)$. The same conclusion is obtained if $\rho(y) \neq \pm I$. Hence, we may assume ρ takes the following form (abbreviating $\rho(x)$ simply as x),

$$x = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad y = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (53)$$

where $\text{Im}(\alpha) \geq 0$ and either $\alpha \neq \pm 1$ or $\beta \neq \pm 1$. The presentation of $\pi_1(M)$ yields the following equations for ρ,

$$\alpha^{a+1}\beta^c = \alpha^b \beta^{d+1} = 1, \quad (54)$$

from which we deduce the relations,

$$\alpha^{a+d+2} = \beta^{a+d+2} = 1. \quad (55)$$

Let $N = |a+d+2|$. Hence α and β are both N-th root of unity. Set $\alpha = e^{\frac{2\pi i k}{N}}$, $\beta = e^{\frac{2\pi i l}{N}}$ such that $0 \leq k \leq \frac{N}{2}$, $0 \leq l < N$, and either $k \neq 0, \frac{N}{2}$ or $l \neq 0, \frac{N}{2}$. Then, Eq. (54) can be equivalently written as,

$$\begin{aligned} (a + 1)\,k + c\,l = 0 \quad \text{mod } N \\ b\,k + (d + 1)\,l = 0 \quad \text{mod } N \end{aligned} \quad (56)$$

The solutions to Eq. (56) depend on a number of conditions involving a, b, c, and d. When at least one of $a + 1$, c, b, $d + 1$ is co-prime to N, there is a compact form to organize all the solutions. For instance, when (c, N) are co-prime, the solutions are simply given by,

$$l = -\tilde{c}(a + 1)k \quad \text{mod } N, \quad k = 1, \cdots, \lfloor \frac{N - 1}{2} \rfloor, \quad (57)$$

where $\tilde{c}$ is the multiplicative inverse of c in $\mathbb{Z}_N$. The representations thus obtained are all irreducible.

Now we consider the case where $\rho(x)$ is not diagonalizable. Then neither is $\rho(y)$ diagonalizable. Up to conjugation, we may assume that $\rho(x)$ and $\rho(y)$ are both upper triangular, each have a single eigenvalue $+1$ or -1 lying on the diagonal, and $\rho(h)$ is diagonal. Thus, ρ takes the form,

$$x = (-1)^{\epsilon_x} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y = (-1)^{\epsilon_y} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, \tag{58}$$

where $\epsilon_x, \epsilon_y \in \{0, 1\}$ and $u \neq 0$. From the presentation of $\pi_1(M)$, we deduce the equations to be satisfied,

$$\begin{aligned} (a+1)\,\epsilon_x + c\,\epsilon_y &= 0 \quad \mathrm{mod}\ 2 \\ b\,\epsilon_x + (d+1)\,\epsilon_y &= 0 \quad \mathrm{mod}\ 2 \end{aligned} \tag{59}$$

$$c\,u^2 + (a-d)u - b = 0, \quad v^2 = \frac{1}{cu+a}. \tag{60}$$

Equation 60 is equivalent to,

$$(v + v^{-1})^2 = a + d + 2, \quad u = \frac{v^{-2} - a}{c}. \tag{61}$$

From Eq. (61), we see that for each fixed ϵ_x and ϵ_y, there are four inequivalent representations, but only two characters. We choose a representative for each character by setting,

$$u = \frac{d - a + \sqrt{(a+d)^2 - 4}}{2c}, \quad v^2 = \frac{1}{cu+a} = \frac{a + d - \sqrt{(a+d)^2 - 4}}{2}. \tag{62}$$

The solution set to Eq. (59) depends on the parity of the entries of the monodromy matrix. Let P be the quadruple that records the parity of the entries $(a, d; b, c)$ and we use 'e' to denote for 'even' and 'o' for 'odd'. For instance, $P = (e, e; o, e)$ means b is odd and the rest are even. The solutions contain the following possible values for ϵ_x and ϵ_y,

- $\epsilon_x = 0$, $\epsilon_y = 0$;
- $\epsilon_x = 1$, $\epsilon_y = 1$, only if $P = (e, e; o, o)$ or $P = (o, o; e, e)$;
- $\epsilon_x = 0$, $\epsilon_y = 1$, only if $P = (o, o; o, e)$ or $P = (o, o; e, e)$;
- $\epsilon_x = 1$, $\epsilon_y = 0$, only if $P = (o, o; e, o)$ or $P = (o, o; e, e)$.

Note that the last three cases above all imply that $N = |a+d+2|$ is even and all possible configurations of P that have N even are contained in one (or more) of the last three cases.

To summarize, the non-Abelian characters of M contain two types, the irreducible and the reducible ones. The irreducible characters take the form of Eq. (53) and are determined by Eq. (56). The reducible characters take the form of Eq. (58) and are determined by Eq. (62) and the possible values of ϵ_x and ϵ_y discussed above.

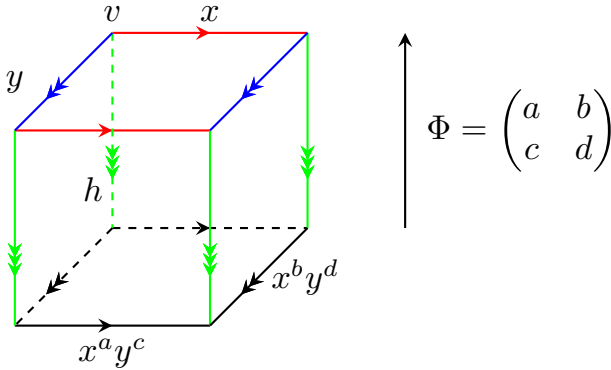$$\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Fig. 2. A cell structure for the torus bundle with monodromy matrix Φ For convenience but no other purposes, mark the vertical edges green, the horizontal on the top face red, and the 45°-slope edges on the top face blue. Edges of the same color and the same arrow are identified. The front and back faces are identified by the obvious map, and so are the left and right side faces. The bottom face is identified to the top via the monodromy map Φ. Hence, the single-arrow edge and the double-arrow edge at the bottom face are homotopic to $x^a y^c$ and $x^b y^d$, respectively

4.2. Torsion and Chern-Simons invariant of torus bundles. In this subsection, we compute the torsion and Chern-Simons invariant for the torus bundle over the circle M with the monodromy map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ where $|a + d| > 2$. Its fundamental group has a presentation given in Eq. (52).

Construct a cell structure for M as follows. See Fig. 2. The cell structure contains,

- a single 0-cell v;
- three 1-cells corresponding to the generators x, y, and h in the presentation of $\pi_1(M)$;
- three 2-cells corresponding to the three relations in the presentation of $\pi_1(M)$. Explicitly, denote them by s_1, s_2 and s_3 such that $\partial s_1 = yxy^{-1}x^{-1}$, $\partial s_2 = h^{-1} xh(x^a y^c)^{-1}$, and $\partial s_3 = h(x^b y^d)h^{-1}y^{-1}$. Graphically, s_1, s_2 and s_3 correspond to the top face, the back face, and the left face, respectively, in Fig. 2 with the induced orientation of the cube.
- a single 3-cell t. Think of a 3-cell as a cube. Then the attaching map is determined by the identification of faces described in Fig. 2.

Let V be a representation $\rho : \pi_1(M) \to GL(V)$, and let $\{v_j \mid j = 1, 2, \cdots\}$ be an arbitrary basis of V. We now construct the chain complex. For simplicity, assume that $a, b, c, d \geq 0$, $a \geq c$, $b \geq d$. Other cases can be dealt similarly. Fix an arbitrary preimage $\tilde{v}$ of v. For each other cell σ, fix a lifting $\tilde{\sigma}$ starting at the base point $\tilde{v}$. We have the following chain complex,

$$0 \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

where $C_i = C_i(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} V$. As a vector space, C_i has the following basis, $C_3 = \mathrm{span}\{\tilde{t} \otimes v_j \mid j = 1, 2, \cdots\}$, $C_2 = \mathrm{span}\{\tilde{s}_i \otimes v_j \mid i = 1, 2, 3, \ j = 1, 2, \cdots\}$, $C_1 = \mathrm{span}\{\tilde{\sigma} \otimes v_j \mid \sigma = x, y, h, \ j = 1, 2, \cdots\}$, $C_0 = \mathrm{span}\{\tilde{v} \otimes v_j \mid j = 1, 2, \cdots\}$. We present the boundary map ∂_i as a block matrix with each entry a $\dim(V) \times \dim(V)$ block. Also, denote $S : \mathbb{Z}[\pi_1(M)] \to \mathbb{Z}[\pi_1(M)]$ the antipode map that sends a group

element $g \in \pi_1(M)$ to its inverse g^{-1} and linearly extends to the whole ring. Lastly, for a matrix A with entries in $\mathbb{Z}[\pi_1(M)]$, $\rho \circ S(A)$ is meant applying $\rho \circ S$ to every entry of A. With the above conventions, the boundary map is given by,

$$\partial_3 = \rho \circ S \begin{pmatrix} 1 - hw(x,y) \\ 1 - y \\ 1 - x \end{pmatrix}$$

$$\partial_2 = \rho \circ S \begin{pmatrix} y - 1 & 1 - h\sum_{i=1}^{a-1} x^i & h\sum_{i=1}^{b-1} x^i \\ 1 - x & -hx^a \sum_{i=1}^{c-1} y^i & hx^b \sum_{i=1}^{d-1} y^i - 1 \\ 0 & x - 1 & 1 - y \end{pmatrix}$$

$$\partial_1 = \rho \circ S \left(x - 1 \ \ y - 1 \ \ h - 1 \right)$$

where w is a polynomial of x, y with the sum of its coefficients equal to 1.

For each of the non-Abelian characters of $\pi_1(M)$ to $\mathrm{SL}(2, \mathbb{C})$, we will compute its torsion below and show (implicitly) that the associated chain complex is always acyclic and the torsion does not depend on the representation chosen in the equivalence class of a character.

For an irreducible representation ρ given in Eq. (53) that satisfies Eq. (56), its adjoint representation has the form,

$$x = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}, \quad y = \begin{pmatrix} \beta^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta^{-2} \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Denote by I and O and 3×3 identity matrix and zero matrix, respectively, and let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define the block matrices,

$$K_1 = \begin{pmatrix} A \\ O \\ B \end{pmatrix}, \quad K_2 = \begin{pmatrix} O & A \\ I & O \\ O & B \end{pmatrix}, \quad K_3 = \left(I \right).$$

It can be checked directly that the columns (as vectors in C_{i-1}) of $\partial_i K_i$ is a basis of $\mathrm{Im}(\partial_i)$. Set $K_4 = K_0$ to be the empty matrix. Now for $i = 0, 1, 2, 3$, let

$$A_i = \left(\partial_{i+1} K_{i+1} \,' K_i \right),$$

then the columns of A_i give a basis for C_i. By direct calculations, we obtain the torsion,

$$\mathrm{Tor}(\rho) = \left| \frac{\det(A_1)\det(A_3)}{\det(A_0)\det(A_2)} \right| = \frac{|a + d + 2|}{4}.$$

Now we compute the torsion of the reducible representations ρ given in Eq. (58). The associated adjoint representation takes the form,

$$x = \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & -2u & -u^2 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} v^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{v^2} \end{pmatrix},$$

which are clearly independent on the sign terms ϵ_x and ϵ_y. Let,

$$
A = \begin{pmatrix} 0\,0\,0 \\ 1\,0\,0 \\ 0\,1\,0 \end{pmatrix}, \quad B = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,1 \\ 0\,0\,0 \end{pmatrix}, \quad C = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \\ 1\,0\,0 \end{pmatrix},
$$

$$
D = \begin{pmatrix} 0\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix}, \quad E = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \\ 0\,0\,1 \end{pmatrix}, \quad F = \begin{pmatrix} 1\,0\,0 \\ 0\,0\,0 \\ 0\,1\,0 \end{pmatrix}.
$$

Define the block matrices,

$$
K_1 = \begin{pmatrix} E \\ O \\ F \end{pmatrix}, \quad K_2 = \begin{pmatrix} A\;O \\ B\;C \\ O\;D \end{pmatrix}, \quad K_3 = \begin{pmatrix} I \end{pmatrix}.
$$

The matrices K_i have the same properties as outlined in the case of irreducible representations above, and in the same way define the matrices A_i. It can be computed that,

$$
\mathrm{Tor}(\rho) = \left| \frac{\det(A_1)\det(A_3)}{\det(A_0)\det(A_2)} \right| = |a + d + 2|.
$$

Some details for the derivation are as follows, where the condition $cu^2 + (a-d)u - b = 0$ is used to simplify expressions,

$$
\begin{aligned}
\mathrm{Tor}(\rho) &= \left| \frac{(2cu + a - d)(b - u + du)(a - b + 1 + (c - d - 1)u)}{u(1 - cu - a)^2(u - 1)} \right| \\
&= \left| \frac{(2cu + a - d)(b - u + du)(a - b + 1 + (c - d - 1)u)}{(cu^2 + (a - 1)u)(cu^2 + (a - 1 - c)u - a + 1)} \right| \\
&= \left| \frac{(2cu + a - d)(b - u + du)(a - b + 1 + (c - d - 1)u)}{((d - 1)u + b)((d - c - 1)u + b - a + 1)} \right| \\
&= \left| \frac{(d - c - 1)u + b - a + 1}{2(c - d - 1)cu^2 + (2c(a - b + 1) + (a - d)(c - d - 1))u + (a - d)(a - b + 1)} \right| \\
&= \left| \frac{(2c(a - b + 1) - (a - d)(c - d - 1))u + (a - d)(a - b + 1) + 2b(c - d - 1)}{(d - c - 1)u + b - a + 1} \right| \\
&= \left| \frac{(a + d + 2)((d - c - 1)u + b - a + 1)}{(d - c - 1)u + b - a + 1} \right| \\
&= |a + d + 2|.
\end{aligned}
$$

Now, we compute the CS invariant of M. Any irreducible representation of $\pi_1(M)$ to $SL(2, \mathbb{C})$ can be conjugated to one into $SU(2)$ (see Eq. 53), and Kirk and Klassen computed its CS invariant in [11]. Here we use methods in Sect. 2.3 to compute the CS invariant of both irreducible and reducible but indecomposable ones, the latter of which can not be conjugated to $SU(2)$.

Let T_i $(i = A, B)$ be two copies of the torus, and I be the interval $[0, 1]$. Then M is obtained by gluing the two $T_i \times I$ such that $T_B \times \{0\}$ is glued to $T_A \times \{1\}$ via the identity map and $T_B \times \{1\}$ is glued to $T_A \times \{0\}$ via the map $\begin{pmatrix} a\;b \\ c\;d \end{pmatrix}$. Let (μ_i, λ_i) be a positive basis of $H_1(T_i)$ so that, under the embedding $T_i \times I \hookrightarrow M$, μ_i and λ_i are sent to x and y, respectively. For $\kappa = 0, 1$, denote by μ_i^κ the element of $H_1(T_i \times \{\kappa\})$ that

corresponds to μ_i in $H_1(T_i \times I)$, and by λ_i^κ in a similar way. Then (μ_i^1, λ_i^1) is a positive basis for $H_1(T_i \times \{1\})$ and $(-\mu_i^0, \lambda_i^0)$ is a positive basis for $H_1(T_i \times \{0\})$. These basis are identified as follows,

$$(\mu_B^0, \lambda_B^0) = (\mu_A^1, \lambda_A^1), \qquad (\mu_B^1, \lambda_B^1) = (\mu_A^0, \lambda_A^0) \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Set $N = |a + d + 2|$. For an irreducible representation ρ in Eq. (53) where $\alpha = e^{\frac{2\pi i k}{N}}$ and $\beta = e^{\frac{2\pi i l}{N}}$, we have

$$
\begin{aligned}
c_{T_i \times I}(\rho) &= [\frac{k}{N}, \frac{l}{N}, \frac{k}{N}, \frac{l}{N}; 1]_{(\mu_i^1, \lambda_i^1),(\mu_i^0, \lambda_i^0)} \\
&= [\frac{k}{N}, \frac{l}{N}, -\frac{k}{N}, \frac{l}{N}; 1]_{(\mu_i^1, \lambda_i^1),(-\mu_i^0, \lambda_i^0)}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
c_{T_A \times I}&(\rho) \\
&= [\frac{k}{N}, \frac{l}{N}, \frac{k}{N}, \frac{l}{N}; 1]_{(\mu_A^1, \lambda_A^1),(\mu_A^0, \lambda_A^0)} \\
&= [\frac{k}{N}, \frac{l}{N}, \frac{ak+cl}{N}, \frac{bk+dl}{N}; 1]_{(\mu_A^1, \lambda_A^1),(\mu_B^1, \lambda_B^1)} \\
&= [\frac{k}{N}, \frac{l}{N}, -\frac{k}{N}, \frac{bk+dl}{N}; \exp(2\pi i (-\nu)\frac{bk+dl}{N})], \quad (\nu := \frac{(a+1)k+cl}{N}) \\
&= [\frac{k}{N}, \frac{l}{N}, -\frac{k}{N}, -\frac{l}{N}; \exp(2\pi i (-\nu)\frac{bk+dl}{N} + 2\pi i (-\mu)\frac{k}{N})], \quad (\mu := \frac{bk+(d+1)l}{N}) \\
&= [\frac{k}{N}, \frac{l}{N}, -\frac{k}{N}, \frac{l}{N}; \exp(2\pi i f)]_{(\mu_A^1, \lambda_A^1),(-\mu_B^1, \lambda_B^1)}
\end{aligned}
$$

where,

$$f = \nu \frac{bk+dl}{N} + \mu \frac{k}{N} = \frac{k\mu - l\nu}{N} + \mu\nu.$$

Note that, by Eq. (56), μ and ν are both integers. Also,

$$
\begin{aligned}
c_{T_B \times I}(\rho) &= [\frac{k}{N}, \frac{l}{N}, -\frac{k}{N}, \frac{l}{N}; 1]_{(\mu_B^1, \lambda_B^1),(-\mu_B^0, \lambda_B^0)} \\
&= [\frac{k}{N}, \frac{l}{N}, -\frac{k}{N}, \frac{l}{N}; 1]_{(\mu_B^1, \lambda_B^1),(-\mu_A^1, \lambda_A^1)}
\end{aligned}
$$

By taking the pairing on $c_{T_A \times I}(\rho)$ and $c_{T_B \times I}(\rho)$, we obtain that,

$$\mathrm{CS}(\rho) = f = \frac{k\mu - l\nu}{N}. \tag{63}$$

For reducible representations $\rho_{\epsilon_x, \epsilon_y}$ in Eq. (58) depending on the values of ϵ_x and ϵ_y (see Sect. 4.1), the computation of the CS invariant proceeds in the exactly the same way as for irreducible representations by making the substitution,

$$\frac{k}{N} \to \frac{\epsilon_x}{2}, \quad \frac{l}{N} \to \frac{\epsilon_y}{2}.$$

Consequently, by setting

$$\nu = \frac{(a+1)\epsilon_x + c\epsilon_y}{2}, \quad \mu = \frac{b\epsilon_x + (d+1)\epsilon_y}{2},$$

we obtain that,

$$\begin{aligned} \mathrm{CS}(\rho_{\epsilon_x,\epsilon_y}) &= \frac{\epsilon_x\mu - \epsilon_y\nu}{2} = \frac{\epsilon_x\mu + \epsilon_y\nu}{2} \\ &= \frac{(a+d+2)\epsilon_x\epsilon_y + b\epsilon_x + c\epsilon_y}{4} \end{aligned} \tag{64}$$

It can be checked that $\mathrm{CS}(\rho_{\epsilon_x,\epsilon_y}) \in \frac{1}{2}\mathbb{Z}$.

4.3. Modular data from torus bundles over the circle. In this subsection, let M be a torus bundle over the circle with the monodromy map given by a matrix,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

We assume that $N := a + d + 2 > 4$ is odd and (c, N) are co-prime. It is direct to see that b and c are both odd, while a and d have different parity. Set $N = 2r + 1$. Denote by $\tilde{c} \in \mathbb{Z}_N$ the multiplicative inverse of c in $\mathbb{Z}_N$.

The non-Abelian character variety of M to $\mathrm{SL}(2,\mathbb{C})$ consists of the representations $\chi^{\mathrm{nab}}(M) = \{\rho_+, \rho_-, \rho_k, k = 1, \cdots, r\}$ which are defined as follows. For $\rho_\pm$,

$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad h \mapsto \begin{pmatrix} v_\pm & 0 \\ 0 & v_\pm^{-1} \end{pmatrix} \tag{65}$$

where

$$u = \frac{d - a + \sqrt{(a+d)^2 - 4}}{2c}, \quad v_\pm = \pm\frac{1}{\sqrt{cu + a}}. \tag{66}$$

For $\rho_k, k = 1, \cdots, r$,

$$x \mapsto \begin{pmatrix} e^{\frac{2\pi i k}{N}} & 0 \\ 0 & e^{-\frac{2\pi i k}{N}} \end{pmatrix}, \quad y \mapsto \begin{pmatrix} e^{\frac{-2\pi i \tilde{c}(a+1)k}{N}} & 0 \\ 0 & e^{\frac{2\pi i \tilde{c}(a+1)k}{N}} \end{pmatrix}, \quad h \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{67}$$

In Sect. 4.1, we computed the adjoint torsion and CS of representations of $\pi_1(M)$. In particular, it implies that all non-Abelian characters are adjoint-acyclic and their CS invariants are all rational numbers. As with the example of SFSs, we choose the candidate label set $L(M) = \chi^{\mathrm{nab}}(M)$. According to Sect. 4.1, the torsion of these representations are given by

$$\mathrm{Tor}(\rho_\pm) = N, \quad \mathrm{Tor}(\rho_k) = \frac{N}{4}. \tag{68}$$

The Chern-Simons invariant of $\rho_\pm$ is 0 by Eq. (64).

Lemma 4.1. *For $k = 1, \cdots, r$, the Chern-Simons invariant of ρ_k is given by,*

$$CS(\rho_k) = -\frac{\tilde{c}k^2}{N}. \tag{69}$$

Proof. This can be derived from Eq. (63). □

We will show below that the premodular categories obtained from the torus bundles are related to quantum group categories associated with $\mathfrak{so}_{2r+1}$.

For an odd integer $N = 2r + 1 > 0$, let $\mathfrak{so}_N$ (Type B) be the Lie algebra of $SO(N)$. Given $q = e^{\frac{m\pi i}{2N}}$ such that q^2 is a primitive $2N$-th root of unity (thus m is odd and (m, N) are co-prime), there is an associated premodular category $\mathcal{C}(\mathfrak{so}_N, q, 2N)$ of rank $r + 4$. See [15] and references therein. When $m = 1$, the corresponding category is always an MTC, and is denoted by $SO(N)_2$ in physics literature. The MTC has the label set,

$$\{\mathbf{1}, Z\} \sqcup \{Y_1, \cdots, Y_r\} \sqcup \{X_1, X_2\}. \tag{70}$$

We will mainly be interested in the (adjoint) monoidal subcategory $\mathcal{C}(\mathfrak{so}_N, q, 2N)_{ad}$ linearly spanned by the objects $\mathbf{1}, Z, Y_1, \cdots, Y_r$. So only modular data on this subcategory is given below.

The twists are,

$$\theta_{\mathbf{1}} = \theta_Z = 1, \quad \theta_{Y_k} = q^{2(Nk-k^2)}, \ k = 1, \cdots, r. \tag{71}$$

The un-normalized S-matrix is,

$$\tilde{S}_{\alpha\beta} = \begin{cases} 1 & \alpha \in \{\mathbf{1}, Z\}, \beta \in \{\mathbf{1}, Z\} \\ 2 & \alpha \in \{\mathbf{1}, Z\}, \beta \in \{Y_1, \cdots, Y_r\} \end{cases} \tag{72}$$

$$\tilde{S}_{kj} := \tilde{S}_{Y_k Y_j} = 2(q^{4kj} + q^{-4kj}) = 4\cos\frac{2\pi m\, kj}{N}. \tag{73}$$

In particular, there are only two values for quantum dimensions, $d_{\mathbf{1}} = d_Z = 1$ and $d_k := d_{Y_k} = 2$. The total dimension is $D = \sqrt{2N}$. Note that $\mathcal{C}(\mathfrak{so}_N, q, 2N)_{ad}$ is a proper premodular category of rank $r + 2$.

Remark 4.2. The label set as ordered in Eq. (70) correspond to the labels $\{0, 2\lambda_1, \lambda_1, \cdots, \lambda_{r-1}, 2\lambda_r, \lambda_r, \lambda_r + \lambda_1\}$ in [15]. Although the S-matrix in [15] is only given for the root $q = e^{\frac{\pi i}{2N}}$, the case for other roots can be easily deduced by either applying a Galois action to the original S-matrix or using the formula

$$\tilde{S}_{\lambda\mu} = \theta_\lambda^{-1}\theta_\mu^{-1}\sum_\nu N_{\lambda^*\mu}^\nu \theta_\nu d_\nu.$$

Now, for the torus bundle defined at the beginning of the subsection, recall that $N = a + d + 2$ is odd, and $\tilde{c}c = 1 \in \mathbb{Z}_N$. Let $m = -2\tilde{c} - N \in \mathbb{Z}$ which is well defined up to multiples of $2N$. For clarity, fix an arbitrary representative for m, and let $q = e^{\frac{m\pi i}{2N}}$. Note that m is odd and co-prime to $2N$. Hence q^2 is a primitive $2N$-th root of unity.

We propose the following correspondence between $\chi^{\mathrm{nab}}(M)$ and loop operators,

$$\begin{aligned} \rho_\pm &\mapsto (x, \mathrm{Sym}^0), \\ \rho_k &\mapsto (x^{mk}, \mathrm{Sym}^1). \end{aligned} \tag{74}$$

and designate ρ_+ as the unit object,

$$\rho_+ = \mathbf{1} \mapsto (x, \mathrm{Sym}^0). \tag{75}$$

Theorem 4.3. *Let M be the torus bundle over the circle with the monodromy matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $N = a + d + 2 > 4$ is odd and (c, N) are co-prime. With the choice of loop operators and unit object in Eqs.* (74) *and* (75)*, respectively, and q as above, the modular data constructed from M matches that of $\mathcal{C}(\mathfrak{so}_N, q, 2N)_{ad}$, the adjoint subcategory of $\mathcal{C}(\mathfrak{so}_N, q, 2N)$.*

Proof. For convenience, we also write $\rho_\pm$ and ρ_k simply as $\pm$ and k, respectively. The correspondence between $\chi^{nab}(M)$ and label set of $\mathcal{C}(\mathfrak{so}_N, q, 2N)_{ad}$ is,

$$\rho_+ \leftrightarrow \mathbf{1}, \quad \rho_- \leftrightarrow Z, \quad \rho_k \leftrightarrow Y_k, \ k = 1, \cdots, r.$$

We first check the twists. By Eq. (71),

$$\theta_{Y_k} = q^{2(Nk-k^2)} = e^{-\frac{2\pi i}{N} \frac{Nk-k^2}{2}(2\tilde{c}+N)} = e^{2\pi i \frac{\tilde{c}k^2}{N}}.$$

Note that in the last equality, we used the fact that $(Nk - k^2)/2$ is an integer. By Lemma 4.1, we immediately have

$$\theta_{Y_k} = e^{-2\pi i CS(\rho_k)}.$$

Of course, for $\rho_\pm$, a similar relation to the above holds trivially.

Next, we verify quantum dimension.

$$W_+(\pm) = 1, \quad W_+(k) = \text{Tr}_{\text{Sym}^1}(\rho_+(x^{mk})) = 2. \tag{76}$$

This means that the total dimension is $D = \sqrt{2N}$ (equal to the dimension of $\mathcal{C}(\mathfrak{so}_N, q, 2N)_{ad}$), and by Eq. (68), for each $\rho \in \chi^{nab}(M)$, the normalized quantum dimension matches the torsion,

$$\frac{W_+(\rho)}{D} = (2\text{Tor}(\rho))^{-\frac{1}{2}}.$$

Lastly, for the S-matrix computed from the W matrix,

$$\tilde{S}_{\alpha\beta} = 1, \quad \alpha, \beta \in \{+, -\}.$$
$$\tilde{S}_{\alpha k} = W_k(\alpha) W_+(k) = 2,$$
$$\tilde{S}_{k\alpha} = W_\alpha(k) W_+(\alpha) = 2, \quad \alpha \in \{+, -\}.$$
$$\tilde{S}_{kj} = W_j(k) W_+(j) = 2 \, \text{Tr}_{\text{Sym}^1}(\rho_j(x^{mk})) = 4 \cos \frac{2\pi m k j}{N}, \quad k, j = 1, \cdots, r.$$

This matches the S-matrix of $\mathcal{C}(\mathfrak{so}_N, q, 2N)_{ad}$ in Eqs. (72) and (73). ∎

Torus bundles are not $\mathbb{Z}_2$ homology sphere by Eq. (52), and the adjoint subcategory of $\mathcal{C}(\mathfrak{so}_N, q, 2N)$ is a properly premodular category by Eq. (72). Hence the above theorem verifies the conjecture on the non-degeneracy of the resulting premodular category for torus bundles considered in this paper.

Remark 4.4. In this subsection, we restricted ourselves to the case where $N = a+d+2 > 4$ is odd and (c, N) are co-prime. In other cases, it seems less straightforward to derive the character variety and the structure of the character variety depends on the parity of N (among other factors). This is expected, since we conjecture in the general case the corresponding premodular category is also related to the adjoint subcategory of some $\mathcal{C}(\mathfrak{so}_N, q, l)$ whose structure varies dramatically depending on the parity of N and the value of N modulo 4 in the case of even N. We leave this as a future direction.

5. Full Data of Modular Categories and Beyond

The structure theory of MTCs is naturally divided into two parts: one is the classification of modular data (MD), and the other is for a fixed modular data, the classification of modular isotopes (MIs)[4]. The missing steps in the program from three manifolds to MTCs are then an algorithm to define loop operators for an admissible candidate label set, hence a candidate MD, and the F-matrices for the fusion structures beyond MD.

Physics point to a framework that is a generalization of gauging finite group symmetries [2,5] to continuous non-Abelian Lie group symmetries such as SU(2). One hint from physics is the form of the primitive loop operators in this paper: a pair (a, R), where a is a conjugacy class of the fundamental group, some kind of flux, and R is an irreducible representation of SU(2), some charge of the SU(2) symmetry. The F-matrices are difficult to find, so we wonder if they depend on more than topology: some geometric information of the given three manifolds.

5.1. Towards the full data.

5.1.1. From non-Abelian characters to loop operators
The identification of a simple object type with a non-Abelian character is based on the relation between a simple object type and a loop operator in the solid torus. In a $(2 + 1)$-TQFT, the rank of an MTC is the same as the dimension of the vector space $V(T^2)$ associated to the torus T^2 from the TQFT. One basis $\{e_a\}$ of the vector space $V(T^2)$ consists of labeled core curves of a solid torus by a complete representative set of simple objects $\{a\}$. Then each basis element e_a can be obtained as the image of a loop operator O_a on e_0—the basis element associated to the vacuum, i.e. $|e_a> = O_a|e_0>$.

Suppose a non-Abelian character corresponds to a primitive loop operator (a, R) of the three manifold X. Then a can be represented by a knot K_a in X. The knot complement of K_a in X determines a vector in $V(T^2)$ from the reduction of 6d SCFT onto X, which should be related to e_a, hence a simple object type eventually.

5.1.2. From flatness equations to pentagons
One possible relation between pentagon equations and flatness is that the flatness of $SL(2, \mathbb{C})$-connections corresponding to the fundamental group representations can be translated into pentagon equations for the F-matrices. It is known that pentagon equations can be interpreted as flatness equations for bi-unitary connections on finite graphs (see e.g. [10]).

5.2. Towards gauging $SU(2)$ R-symmetry.
An R-symmetry of a super-symmetric theory is an outer automorphism of the super-Poincare group that fixes the Poincare group. It is pointed out in [4] that the R-symmetries in infrared could be different from those in ultra-violet. Hence we could have an SU(2) R-symmetry for the residual topological theory in infrared, which is probably often trivial. We believe that the MTCs obtained from three manifolds in this program are actually the results of gauging such SU(2) R-symmetries of the residual topological theory in infrared, which generalizes gauging of finite group symmetries [2,5].

---

[4] A terminology due to C. Delaney: distinct MTCs with the same MD are called modular isotopes of each other.

5.3. Towards quantum double of infinite discrete groups. An interesting class of MTCs comes from the representation categories of quantum doubles of finite groups. A naive generalization to infinite discrete groups does not work. The program in this paper can be regarded as a first step in this direction for the class of 3-manifold groups. The choice of the simple Lie group serves as an analogue of a level in quantum groups.

5.4. Climbing the dimension ladder. Two interesting classes of quantum algebras are vertex operator algebras (VOAs) and MTCs. The bulk-edge correspondence of topological phases of matter makes them into a unified theory of two and three dimensions. The program in this paper suggests an inversion of dimensions: MTCs and VOAs could also fit into a unified theory of three and four dimensional manifolds, where 4-manifolds with 3-manifold boundaries could give rise to VOAs that realize the boundary MTCs.

5.5. Open questions. There are many other interesting open questions in this program. One obvious one is to extend our results to more examples such as Seifert fibered spaces with more than three fibers and the remaining cases of our torus bundles over the circle examples. It is also not clear how to obtain MTCs which are not self-dual. As mentioned in Sect. 2, representations of $SL(2, \mathbb{C})$ come in group of four and a natural guess is that one of the four is the dual anyon type. If so, then which one? The dual representation is a candidate. Another general direction is what operations on MTCs that standard topological constructions of three manifolds such as connected sum and torus decomposition correspond to. Connect sum should correspond to Deligne product.

Our adjoint-acyclic condition for a representation ρ is closely related to $H^1(\pi, \text{Adj} \circ \rho) = 0$. Are they equivalent? It should be equivalent for irreducible representations, but for the indecomposable reducible ones, it is not clear.

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