Learning for Edge-Weighted Online Bipartite Matching with Robustness Guarantees

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Abstract

Many problems, such as online ad display, can be formulated as online bipartite matching. The crucial challenge lies in the nature of sequentially-revealed online item information, based on which we make irreversible matching decisions at each step. While numerous expert online algorithms have been proposed with bounded worst-case competitive ratios, they may not offer satisfactory performance in average cases. On the other hand, reinforcement learning (RL) has been applied to improve the average performance, but it lacks robustness and can perform arbitrarily poorly. In this paper, we propose a novel RL-based approach to edge-weighted online bipartite matching with robustness guarantees (LOMAR), achieving both good average-case and worst-case performance. The key novelty of LOMAR is a new online switching operation which, based on a judicious condition to hedge against future uncertainties, decides whether to follow the expert's decision or the RL decision for each online item. We prove that for any $\rho \in [0, 1]$, LOMAR is ρ-competitive against any given expert online algorithm. To improve the average performance, we train the RL policy by explicitly considering the online switching operation. Finally, we run empirical experiments to demonstrate the advantages of LOMAR compared to existing baselines.

1. Introduction

Online bipartite matching is a classic online problem of practical importance (Mehta, 2013; Kim & Moon, 2020; Fahrbach et al., 2020; Antoniadis et al., 2020b; Huang & Shu, 2021; Gupta & Roughgarden, 2020). In a nutshell, online bipartite matching assigns online items to offline items in two separate sets: when an online item arrives, we need to match it to an offline item given applicable constraints (e.g., capacity constraint), with the goal of maximizing the total rewards collected (Mehta, 2013). For example, numerous applications, including scheduling tasks to servers, displaying advertisements to online users, recommending articles/movies/products, among many others, can all be modeled as online bipartite matching or its variants.

The practical importance, along with substantial algorithmic challenges, of online bipartite matching has received extensive attention in the last few decades (Karp et al., 1990; Fahrbach et al., 2020). Concretely, many algorithms have been proposed and studied for various settings of online bipartite matching, ranging from simple yet effective greedy algorithms to sophisticated ranking-based algorithms (Karp et al., 1990; Kim & Moon, 2020; Fahrbach et al., 2020; Aggarwal et al., 2011; Devanur et al., 2013). These expert algorithms typically have robustness guarantees in terms of the competitive ratio — the ratio of the total reward obtained by an online algorithm to the reward of another baseline algorithm (commonly the optimal offline algorithm) — even under adversarial settings given arbitrarily bad problem inputs (Karp et al., 1990; Huang & Shu, 2021). In some settings, even the optimal competitive ratio for adversarial inputs has been derived (readers are referred to (Mehta, 2013) for an excellent tutorial). The abundance of competitive online algorithms has clearly demonstrated the importance of performance robustness in terms of the competitive ratio, especially in safety-sensitive applications such as matching mission-critical items or under contractual obligations (Fahrbach et al., 2020). Nonetheless, as commonly known in the literature, the necessity of conservativeness to address the worst-case adversarial input means that the average performance is typically not optimal (see, e.g., (Christianson et al., 2022; Zeynali et al., 2021) for discussions in other general online problems).

More recently, online optimizers based on reinforcement learning (RL) (Chen et al., 2022; Georgiev & Lió, 2020; Wang et al., 2019; Alomrani et al., 2022; Du et al., 2019; Zuzic et al., 2020) have been proposed in the context of online bipartite matching as well as other online problems. Specifically, by exploiting statistical information of problem inputs, RL models are trained offline and then applied online

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to produce decisions given unseen problem inputs. These RL-based optimizers can often achieve high average rewards in many typical cases. Nonetheless, they may not have any performance robustness guarantees in terms of the competitive ratio. In fact, a crucial pain point is that the worst-case performance of many RL-based optimizers can be arbitrarily bad, due to, e.g., testing distribution shifts, inevitable model generalization errors, finite samples, and/or even adversarial inputs. Consequently, the lack of robustness guarantees has become a key roadblock for wide deployment of RL-based optimizers in real-world applications.

In this paper, we focus on an important and novel objective — achieving both good average performance and guaranteed worst-case robustness — for *edge-weighted* online bipartite matching (Fahrbach et al., 2020; Kim & Moon, 2020). More specifically, our algorithm, called `LOMAR` (Learning-based approach to edge-weighted Online bipartite MAtching with Robustness guarantees), integrates an expert algorithm with RL. The key novelty of `LOMAR` lies in a carefully-designed online *switching* step that dynamically switches between the RL decision and the expert decision online, as well as a switching-aware training algorithm. For both no-free-disposal and free-disposal settings, we design novel switching conditions as to when the RL decisions can be safely followed while still guaranteeing robustness of being ρ-competitive against *any* given expert online algorithms for any $\rho \in [0, 1]$. To improve the average performance of `LOMAR`, we train the RL policy in `LOMAR` by explicitly taking into account the introduced switching operation. Importantly, to avoid the "no supervision" trap during the initial RL policy training, we propose to approximate the switching operation probabilistically. Finally, we offer empirical experiments to demonstrate that `LOMAR` can improve the average cost (compared to existing expert algorithms) as well as lower the competitive ratio (compared to pure RL-based optimizers).

2. Related Works

Online bipartite matching has been traditionally approached by expert algorithms (Mehta, 2013; Karande et al., 2011; Huang et al., 2019; Devanur et al., 2013). A simple but widely-used algorithm is the (deterministic) greedy algorithm (Mehta, 2013), achieving reasonably-good competitive ratios and empirical performance (Alomrani et al., 2022). Randomized algorithms have also been proposed to improve the competitive ratio (Ting & Xiang, 2014; Aggarwal et al., 2011). In addition, competitive algorithms based on the primal-dual framework have also been proposed (Mehta, 2013; Buchbinder et al., 2009). More recently, multi-phase information and predictions have been leveraged to exploit stochasticity within each problem instance and improve the algorithm performance (Kesselheim et al., 2013). For example, (Korula & Pál, 2009) designs a secretary matching algorithm based on a threshold obtained using the information of phase one, and exploits the threshold for matching in phase two. Note that stochastic settings considered by expert algorithms (Mehta, 2013; Karande et al., 2011) mean that the arrival orders and/or rewards of different online items within each problem instance are stochastic. By contrast, as shown in (2), we focus on an unknown distribution of problem instances whereas the inputs within each instance can still be arbitrary.

Another line of algorithms utilize RL to improve the average performance (Wang et al., 2019; Georgiev & Lió, 2020; Chen et al., 2022; Alomrani et al., 2022). Even though heuristic methods (such as using adversarial training samples (Zuzic et al., 2020; Du et al., 2022)) are used to empirically improve the robustness, they do not provide any theoretically-proved robustness guarantees.

ML-augmented algorithms have been recently considered for various problems (Rutten et al., 2023; Jin & Ma, 2022; Christianson et al., 2022; Chłędowski et al., 2021; Lykouris & Vassilvitskii, 2021). By viewing the ML prediction as blackbox advice, these algorithms strive to provide good competitive ratios when the ML predictions are nearly perfect, and also bounded competitive ratios when ML predictions are bad. But, they still focus on the worst case without addressing the average performance or how the ML model is trained. By contrast, the RL model in `LOMAR` is trained by taking into account the switching operation and performs inference based on the actual state (rather than its own independently-maintained state as a blackbox). Assuming a given downstream algorithm, (Wang et al., 2021; Liu & Grigas, 2021; Wilder et al., 2019; Elmachtoub & Grigas, 2017; Du et al., 2021; Anand et al., 2021) focus on learning the ML model to better serve the end goal in completely different (sometimes, offline optimization) problems.

`LOMAR` is relevant to conservative bandits/RL (Wu et al., 2016; Kazerouni et al., 2017; Yang et al., 2022; Garcelon et al., 2020). With unknown reward functions (as well as transition models if applicable), conservative bandits/RL leverages an existing policy to safeguard the exploration process. But, they only consider the cumulative reward without addressing future uncertainties when deciding exploration vs. rolling back to an existing policy. Thus, as shown in Section 4, this cannot guarantee robustness in our problem. Also, constrained policy optimization (Yang et al., 2020; Kumar et al., 2020; Schulman et al., 2015; Achiam et al., 2017; Thomas et al., 2021; Berkenkamp et al., 2017) focuses on average (cost) constraints in the long run, whereas `LOMAR` achieves stronger robustness (relative to an expert algorithm) for any episode.

3. Problem Formulation

We focus on *edge-weighted* online bipartite matching, which includes un-weighted and vertex-weighted matching as special cases (Fahrbach et al., 2020; Kim & Moon, 2020). In the following, we also drop "edge-weighted" if applicable when referring to our problem.

3.1. Model

The agent matches items (a.k.a. vertices) between two sets $\mathcal{U}$ and $\mathcal{V}$ to gain as high total rewards as possible. Suppose that $\mathcal{U}$ is fixed and contains *offline* items $u \in \mathcal{U}$, and that the *online* items $v \in \mathcal{V}$ arrive sequentially: in each time slot, an online item $v \in \mathcal{V}$ arrives and the weight/reward information $\{w_{uv} \mid w_{u,\min} \leq w_{uv} \leq w_{u,\max}, u \in \mathcal{U}\}$ is revealed, where w_{uv} represents the reward when the online item v is matched to each offline $u \in \mathcal{U}$. We denote one problem instance by $\mathcal{G} = \{\mathcal{U}, \mathcal{V}, \mathcal{W}\}$, where $\mathcal{W} = \{w_{uv} \mid u \in \mathcal{U}, v \in \mathcal{V}\}$. We denote $x_{uv} \in \{0, 1\}$ as the matching decision indicating whether u is matched to v. Also, any offline item $u \in \mathcal{U}$ can be matched up to c_u times, where c_u is essentially the capacity for offline item u known to the agent in advance.

The goal is to maximize the total collected reward $\sum_{v\in\mathcal{V},u\in\mathcal{U}} x_{uv}w_{uv}$. With a slight abuse of notations, we denote $x_v \in \mathcal{U}$ as the index of item in $\mathcal{U}$ that is matched to item $v \in \mathcal{V}$. The set of online items matched to $u \in \mathcal{U}$ is denoted as $\mathcal{V}_u = \{v \in \mathcal{V} \mid x_{uv} = 1\}$.

The edge-weighted online bipartite matching problem has been mostly studied under two different settings: no free disposal and with free disposal (Mehta, 2013). In the no-free-disposal case, each offline item $u \in \mathcal{U}$ can only be matched strictly up to c_u times; in the free-disposal case, each offline item $u \in \mathcal{U}$ can be matched more than c_u times, but only the top c_u rewards are counted when more than c_u online items are matched to u. Compared to the free-disposal case, the no-free-disposal case is significantly more challenging with the optimal competitive ratio being 0 in the strong adversarial setting unless additional assumptions are made (e.g., $w_{u,\min} > 0$ for each $u \in \mathcal{U}$ (Kim & Moon, 2020) and/or random-order of online arrivals) (Fahrbach et al., 2020; Mehta, 2013). The free-disposal setting is not only analytically more tractable, but also is practically motivated by the display ad application where the advertisers (i.e., offline items $u \in \mathcal{U}$) will not be unhappy if they receive more impressions (i.e., online items $v \in \mathcal{V}$) than their budgets c_u, even though only the top c_u items count.

`LOMAR` can handle both no-free-disposal and free-disposal settings. For better presentation of our key novelty and page limits, we focus on the no-free-disposal setting in the body of the paper, *while deferring the free-disposal setting to Appendix B*.

Specifically, the **offline** problem with no free disposal can be expressed as:

$$\max \sum_{x_{uv}\in\{0,1\},u\in\mathcal{U},v\in\mathcal{V}} x_{uv}w_{uv},$$
$$\text{s.t.,} \sum_{v\in\mathcal{V}} x_{uv} \leq c_u, \text{ and} \sum_{u\in\mathcal{U}} x_{uv} \leq 1, \forall u \in \mathcal{U}, v \in \mathcal{V} \tag{1}$$

where the constraints specify the offline item capacity limit and each online item $v \in \mathcal{V}$ can only be matched up to one offline item $u \in \mathcal{U}$. Given an online algorithm α, we use $f_u^\alpha(\mathcal{G})$ to denote the total reward collected for offline item $u \in \mathcal{U}$, and $R^\alpha(\mathcal{G}) = \sum_{u\in\mathcal{U}} f_u^\alpha(\mathcal{G})$ to denote the total collected reward. We will also drop the superscript α for notational convenience wherever applicable.

3.2. Objective

Solving the problem in (1) is very challenging in the online case, where the agent has to make irreversible decisions without knowing the future online item arrivals. Next, we define a generalized competitiveness as a metric of robustness and then present our optimization objective.

Definition 1 (Competitiveness)**.** An online bipartite matching algorithm α is said to be ρ-competitive with $\rho \geq 0$ against the algorithm π if for any problem instance $\mathcal{G}$, its total collected reward $R^\alpha(\mathcal{G})$ satisfies $R^\alpha(\mathcal{G}) \geq \rho R^\pi(\mathcal{G}) - B$, where $B \geq 0$ is a constant independent of the problem input, and R^π is the total reward of the algorithm π.

Competitiveness against a given online algorithm π (a.k.a., expert) is common in the literature on algorithm designs (Christianson et al., 2022): the greater $\rho \geq 0$, the better robustness of the online algorithm, although the average rewards can be worse. The constant $B \geq 0$ relaxes the strict competitive ratio by allowing an additive *regret* (Antoniadis et al., 2020a). When $B = 0$, the competitive ratio becomes the strict one. In practice, the expert algorithm π can be viewed as an existing solution currently in use, while the new RL-based algorithm is being pursued subject to a constraint that the collected reward must be at least ρ times of the expert. Additionally, if the expert itself has a competitive ratio of $\lambda \leq 1$ against the offline oracle algorithm (OPT), then it will naturally translate into `LOMAR` being $\rho\lambda$-competitive against OPT.

On top of worst-case robustness, we are also interested in the average award. Specifically, we focus on a setting where the problem instance $\mathcal{G} = \{\mathcal{U}, \mathcal{V}, \mathcal{W}\}$ follows an *unknown* distribution, whereas both the rewards $\mathcal{W}$ and online arrival order within each instance $\mathcal{G}$ can be adversarial.

Nonetheless, the average reward and worst-case robustness are different, and optimizing one metric alone does not necessarily optimize the other one (which is a direct byproduct of Yao's principle (Yao, 1977). In fact, there is a tradeoff between the average performance and worst-case robustness in general online problems (Christianson et al., 2022). The reason is that an online algorithm that maximizes the average reward prioritizes typical problem instances, while conservativeness is needed by a robust algorithm to mitigate the worst-case uncertainties and outliers.

In LOMAR, we aim to maximize the average reward subject to worst-case robustness guarantees as formalized below:

$$\max \mathbb{E}_{\mathcal{G}}\left[R^{\alpha}(\mathcal{G})\right] \tag{2a}$$

$$\text{s.t.}\quad R^{\alpha}(\mathcal{G}) \geq \rho R^{\pi}(\mathcal{G}) - B, \quad \forall \mathcal{G}, \tag{2b}$$

where the expectation $\mathbb{E}_{\mathcal{G}}\left[R^{\alpha}(\mathcal{G})\right]$ is over the randomness $\mathcal{G} = \{\mathcal{U}, \mathcal{V}, \mathcal{W}\}$. The worst-case robustness constraint for each problem instance is significantly more challenging than an average reward constraint. Our problem in (2) is novel in that it generalizes the recent RL-based online algorithms (Alomrani et al., 2022) by guaranteeing worst-case robustness; it leverages robustness-aware RL training (Section 5) to improve the average reward and hence also differs from the prior ML-augmented algorithms that still predominantly focus on the worst-case performance (Christianson et al., 2022; Wei & Zhang, 2020).

Some manually-designed algorithms focus on a *stochastic* setting where the arrival order is random and/or the rewards $\{w_{uv} \mid w_{u,\min} \leq w_{uv} \leq w_{u,\max}, u \in \mathcal{U}\}$ of each online item is independently and identically distributed (i.i.d.) within each problem instance $\mathcal{G}$ (Mehta, 2013). By contrast, our settings are significantly different — we only assume an unknown distribution for the entire problem instance $\mathcal{G} = \{\mathcal{U}, \mathcal{V}, \mathcal{W}\}$ while both the rewards $\mathcal{W}$ and online arrival order within each instance $\mathcal{G}$ can be arbitrary.

4. Design of Online Switching for Robustness

Assuming that the RL policy is already trained (to be addressed in Section 5), we now present the inference of LOMAR, which includes novel online *switching* to dynamically follow the RL decision or the expert decision, for robustness guarantees against the expert.

4.1. Online Switching

While switching is common in (ML-augmented) online algorithms, "*how to switch*" is highly non-trivial and a key merit for algorithm designs (Antoniadis et al., 2020a; Christianson et al., 2022; Rutten et al., 2023). To guarantee robustness (i.e., ρ-competitive against a given expert for any $\rho \in [0, 1]$), we propose a novel online algorithm (Algorithm 1). In the algorithm, we independently run an expert online algorithm π — the cumulative reward and item matching decisions are all maintained virtually for the expert, but not used as the actual decisions. Based on the performance of expert online algorithm, we design a robust constraint which serves as the condition for online switching.

Algorithm 1 Inference of Robust Learning-based Online Bipartite Matching (LOMAR)

```
Input: Competitiveness constraint $\rho \in [0, 1]$ and $B \geq 0$
1:  for $v = 1$ to $|\mathcal{V}|$ do
2:    Run the expert $\pi$ and get expert's decision $x^{\pi}_{v}$.
3:    If $x^{\pi}_{v} \neq$ skip: $\mathcal{V}^{\pi}_{x^{\pi}_{v},v} = \mathcal{V}^{\pi}_{x^{\pi}_{v},v-1} \bigcup \{v\}$,
      $R^{\pi}_{v} = R^{\pi}_{v-1} + w_{x^{\pi}_{v},v}$.
      //Update the virtual decision set and reward of the expert
4:    $s_u = w_{uv} - h_{\theta}(I_u, w_{uv}), \forall u \in \mathcal{U}$
      //Run RL model to get score $s_u$ with history information $I_u$
5:    $\tilde{x}_v = \arg\max_{u \in \mathcal{U}_a \bigcup \{\text{skip}\}} \left\{\{s_u\}_{u \in \mathcal{U}_a}, s_{\text{skip}}\right\}$, with
      $s_{\text{skip}} = 0$ and $\mathcal{U}_a = \{u \in \mathcal{U} \mid |\mathcal{V}_{u,v-1}| < c_u\}$.
      //Get RL decision $\tilde{x}_v$
6:    if Robust constraint in (3) is satisfied then
7:      Select $x_v = \tilde{x}_v$. //Follow RL
8:    else if $x^{\pi}_{v}$ is available (i.e., $|\mathcal{V}_{x^{\pi}_{v},v-1}| < c_{x^{\pi}_{v}}$) then
9:      Select $x_v = x^{\pi}_{v}$. //Follow the expert
10:   else
11:     Select $x_v =$ skip.
12:   end if
13:   If $x_v \neq$ skip, $\mathcal{V}_{x_v,v} = \mathcal{V}_{x_v,v-1} \bigcup \{v\}$,
      $R_v = R_{v-1} + w_{x_v,v}$.
      //Update the true decision set and reward
14: end for
```

Concretely, we define the set of items that is actually matched to offline item $u \in \mathcal{U}$ before the start of $(v+1)-$th step as $\mathcal{V}_{u,v}$, and the set of items that is virtually matched to offline item $u \in \mathcal{U}$ by expert before the start of $(v+1)-$th step as $\mathcal{V}^{\pi}_{u,v}$. Initially, we have $\mathcal{V}_{u,0} = \emptyset$, and $\mathcal{V}^{\pi}_{u,0} = \emptyset$. We also denote $\mathcal{U}_a$ as the set of available offline item and initialize it as $\mathcal{U}$. When an online item v arrives at each step, Algorithm 1 first runs the expert algorithm π, gets the expert decision x^{π}_{v} and update the virtual decision set and reward if the expert decision is not skipping this step. Then the RL policy gives the scores s_u of each offline item $u \in \mathcal{U}$. By assigning the score of skipping as 0 and comparing the scores, the algorithm obtain the RL action advice $\tilde{x}_v$. Then the algorithm perform online switching to guarantee robustness.

The most crucial step for safeguarding RL decisions is our online switching step: Lines 6–12 in Algorithm 1. The key idea for this step is to switch between the expert decision x^{π}_{v} and the RL decision $\tilde{x}_v$ in order to ensure that the actual online decision x_v meets the ρ-competitive requirement (against the expert π). Specifically, we follow the RL decision $\tilde{x}_v$ only if it can safely hedge against any future uncertainties (i.e., the expert's future reward increase); otherwise, we need to roll back to the expert's decision x^{π}_{v} to stay on track for robustness.

Nonetheless, naive switching conditions, e.g., only ensuring that the actual cumulative reward is at least ρ times of the expert's cumulative reward at each step (Wu et al., 2016; Yang et al., 2022), can fail to meet the competitive ratio requirement in the end. The reason is that, even though the competitive ratio requirement is met (i.e., $R_v \geq \rho R_v^{\pi} - B$) at the current step v, the expert can possibly obtain much higher rewards from future online items $v+1, v+2, \cdots$, if it has additional offline item capacity that the actual algorithm LOMAR does not have. Thus, we must carefully design the switching conditions to hedge against future risks.

4.2. Robustness Constraint

In the no-free-disposal case, an offline item $u \in \mathcal{U}$ cannot receive any additional online items if it has been matched for c_u times up to its capacity. By assigning more online items to $u \in \mathcal{U}$ than the expert algorithm at step v, LOMAR can possibly receive a higher cumulative reward than the expert's cumulative reward. But, such advantages are just *temporary*, because the expert may receive an even higher reward in the future by filling up the unused capacity of item u. Thus, to hedge against the future uncertainties, LOMAR chooses the RL decisions only when the following condition is satified:

$$\begin{aligned} R_{v-1} + w_{\tilde{x}_v, v} \geq & \rho\Big(R_v^{\pi} \sum_{u \in \mathcal{U}} \big(|\mathcal{V}_{u,v-1}| - |V_{u,v}^{\pi}| \\ & + \mathbb{I}_{u=\tilde{x}_v}\big)^+ \cdot w_{u,\max}\Big) - B, \end{aligned} \tag{3}$$

where $\mathbb{I}_{u=\tilde{x}_v} = 1$ if and only if $u = \tilde{x}_v$ and 0 otherwise, $(\cdot)^+ = \max(\cdot, 0)$, $\rho \in [0, 1]$ and $B \geq 0$ are the hyperparameters indicating the desired robustness with respect to the expert algorithm π.

The interpretation of (3) is as follows. The left-hand side is the total reward of LOMAR after assigning the online item v based on the RL decision (i.e. $\tilde{x}_t$). The right-hand side is the expert's cumulative cost R_v^{π}, plus the term $\sum_{u \in \mathcal{U}} \big(|\mathcal{V}_{u,v-1}| - |V_{u,v}^{\pi}| + \mathbb{I}_{u=\tilde{x}_v}\big)^+ \cdot w_{u,\max}$ which indicates the maximum reward that can be possibly received by the expert in the future. This reservation term is crucial, especially when the expert has more unused capacity than LOMAR. Specifically, $|\mathcal{V}_{u,v-1}|$ is the number of online items (after assigning $v-1$ items) already assigned to the offline item $u \in \mathcal{U}$, and hence $\big(|\mathcal{V}_{u,v-1}| - |V_{u,v}^{\pi}| + \mathbb{I}_{u=\tilde{x}_v}\big)^+$ represents the number of more online items that LOMAR has assigned to u than the expert if LOMAR follows the RL decision at step v. If LOMAR assigns fewer items than the expert for an offline item $u \in \mathcal{U}$, there is no need for any hedging because LOMAR is guaranteed to receive more rewards by filling up the item u up to the expert's assignment level.

The term $w_{u,\max}$ in (3) is the set as the maximum possible reward for each decision. Even when $w_{u,\max}$ is unknown in advance, LOMAR still applies by simply setting $w_{u,\max} = \infty$. In this case, LOMAR will be less "greedy" than the expert and never use more resources than the expert at any step.

While we have focused on the no-free-disposal setting to highlight the key idea of our switching condition (i.e., not following the RL decisions too aggressively by hedging against future reward uncertainties), the free-disposal setting requires a very different switching condition, which we defer to Appendix B due to the page limit.

4.3. Robustness Analysis

We now formally show the competitive ratio of LOMAR. The proof is available in the appendix.

Theorem 4.1. *For any $0 \leq \rho \leq 1$ and $B \geq 0$ and any expert algorithm π, LOMAR achieves a competitive ratio of ρ against the algorithm π, i.e., $R \geq \rho R^{\pi} - B$ for any problem input.*

The hyperparameters $0 \leq \rho \leq 1$ and $B \geq 0$ govern the level of robustness we would like to achieve, at the potential expense of average reward performance. For example, by setting $\rho = 1$ and $B = 0$, we achieve the same robustness as the expert but leave little to no freedom for RL decisions. On the other hand, by setting a small $\rho > 0$ and/or large B, we provide higher flexibility to RL decisions for better average performance, while potentially decreasing the robustness. In fact, such tradeoff is necessary in the broad context of ML-augmented online algorithms (Rutten et al., 2022; Christianson et al., 2022). Additionally, in case of multiple experts, we can first combine these experts into a single expert and then apply LOMAR as if it works with a single combined expert.

While the competitive ratio of all online algorithms against the optimal offline algorithm is zero in the no-free-disposal and general adversarial setting, there exist provably competitive online expert algorithms under some technical assumptions and other settings (Mehta, 2013). For example, the simple greedy algorithm achieves $\left(1 + \max_{u \in \mathcal{U}} \frac{w_{u,\max}}{w_{u,\min}}\right)^{-1}$ under bounded weights assumptions for the adversarial no-free-disposal setting (Kim & Moon, 2020), and $\frac{1}{2}$ for the free-disposal setting (Fahrbach et al., 2020), and there also exist $1/e$-competitive algorithms against the optimal offline algorithm for the random-order setting (Mehta, 2013). Thus, an immediate result follows.

Corollary 4.1.1. *For any $0 \leq \rho \leq 1$ and $B \geq 0$, by using Algorithm 1 and an expert online algorithm π that is λ-competitive against the optimal offline algorithm OPT, then under the same assumptions for π to be λ-competitive, LOMAR is $\rho\lambda$-competitive against OPT.*

Corollary 4.1.1 provides a general result that applies to any λ-competitive expert algorithm π under its respective required assumptions. For example, if the expert π assumes an adversarial or random-order setting, then Corollary 4.1.1 holds under the same adversarial or random-order setting.

5. RL Policy Training with Online Switching

The prior ML-augmented online algorithms typically assume a standalone RL model that is pre-trained without considering what the online algorithm will perform (Christianson et al., 2022). Thus, while the standalone RL model may perform well on its own, its performance can be poor when directly used in `LOMAR` due to the added online switching step. In other words, there will be a training-testing mismatch. To rectify the mismatch, we propose a novel approach to train the RL model in `LOMAR` by explicitly considering the switching operation.

RL architecture. For online bipartite matching, there exist various network architectures, e.g., fully-connected networks and scalable invariant network for general graph sizes. The recent study (Alomrani et al., 2022) has shown using extensive empirical experiments that the invariant network architecture, where each offline-online item pair runs a separate neural network with shared weights among all the item pairs, is empirically advantageous, due to its scalability to large graphs and good average performance.

We denote the RL model as $h_\theta(I_u, w_{uv})$ where θ is the network parameter. By feeding the item weight w_{uv} and applicable history information I_u for each offline-online item pair (u, v), we can use the RL model to output a *threshold* for possible item assignment, following threshold-based algorithms (Alomrani et al., 2022; Huang et al., 2019; Mehta, 2013). The history information I_u includes, but is not limited to, the average value and variance of weights assigned to u, average in-degree of u, and maximum weight for the already matched items. More details about the information can be found in the appendix. Then, with the RL output, we obtain a score $s_u = w_{uv} - h_\theta(I_u, w_{uv})$ for each possible assignment.

Policy training. Training the RL model by considering switching in Algorithm 1 is non-trivial. Most critically, the initial RL decisions can perform arbitrarily badly upon policy initialization, which means that the initial RL decisions are almost always overridden by the expert's decisions for robustness. Due to following the expert's decisions, the RL agent almost always receive a good reward, which actually has nothing to do with the RL's own decisions and hence provides little to no supervision to improve the RL policy. Consequently, this creates a *gridlock* for RL policy training. While using an offline pre-trained standalone RL model without considering online switching (e.g., (Alomrani et al., 2022)) as an initial policy may partially address this gridlock, this is certainly inefficient as we have to spend resources for training another RL model, let alone the likelihood of being trapped into the standalone RL model's suboptimal policies (e.g. local minimums).

Algorithm 2 Policy Training with Online Switching

```
Input: Competitiveness constraint $\rho \in [0, 1]$ and $B \geq 0$,
    initial model weight $\theta$ of RL model.
for $i = 1$ to n do
    for $v = 1$ to $|\mathcal{V}|$ do
        Calculate the actual item selection probability
            $p_\theta(x_v|I_u)$ from Eqn. (4).
        Sample from $p_\theta(x_v|I_u)$ to get the item selection
            $x_v$, then collect the reward $w_{x_v,v}$ for item $v$.
        Update the capacity of the offline item $x_v$ after the
            assignment $\mathcal{V}_{x_v,v}$.
    end for
    Collect node matching results for the problem instance and add them into a trajectory set.
end for
Estimate policy gradient $\nabla_\theta \hat{R}_\theta$ based on Eqn. (5).
Update the RL model weight $\theta$ with $\theta = \theta + \alpha \nabla_\theta \hat{R}_\theta$;
```

To address these issues, during training, we introduce a softmax probability to approximate the otherwise non-differentiable switching operation. Specifically, the switching probability depends on the cumulative reward difference R_{diff} in the switching condition, which is

$$R_{diff} = R_{v-1} + w_{\tilde{x}_v,v} + B - \rho \cdot \Big(R_v^\pi + \sum_{u \in \mathcal{U}} \big(|\mathcal{V}_{u,v-1}| - |V^\pi_{u,v}| + \mathbb{I}_{u=\tilde{x}_v} \big)^+ \cdot w_{u,\max} \Big)$$

Then, the probability of following RL is $p_{os} = \frac{e^{R_{diff}/t}}{1+e^{R_{diff}/t}}$, where t is the softmax function's temperature. Importantly, this softmax probability is differentiable and hence allows backpropagation to train the RL model weight θ to maximize the expected total reward while being aware of the switching operation for robustness. Next, with *differentiable* switching, we train the RL model by policy gradient (Williams, 1992) to optimize the policy parameter θ. Denote $\tau = \{x_1, \cdots, x_v\}$ as an action trajectory sample and $p_\theta(x_v|I_u)$ as the probability of matching offline item u to online item v:

$$p_\theta(x_v|I_u) = (1 - p_{os}) \cdot \tilde{p}_\theta(x_v|I_u) + p_{os} \cdot p_\theta^\pi(x_v|I_u), \quad (4)$$

where $\tilde{p}_\theta(x_v|I_u)$ is the RL's item selection probability obtained with the RL's output score s_u, and $p_\theta^\pi(x_v|I_u)$ is the item selection probability for expert π. If the expert's item selection is not available (i.e., Line 11 in Algorithm 1), then x_v^π will be replaced with skip when calculating (4).

During the training process, our goal is to maximize the expected total reward $R_\theta = \mathbb{E}_{\tau \sim p_\theta}[w_{x_v,v}]$. Thus, at each

	DRL-OS		LOMAR ($\rho = 0.4$)		LOMAR ($\rho = 0.6$)		LOMAR ($\rho = 0.8$)		Greedy	
Test	AVG	CR	AVG	CR	AVG	CR	AVG	CR	AVG	CR
$\rho = 0.4$	12.315	0.800	**12.364**	**0.819**	12.288	0.804	12.284	0.804	11.000	0.723
$\rho = 0.6$	11.919	0.787	11.982	0.807	**11.990**	**0.807**	11.989	0.800	11.000	0.723
$\rho = 0.8$	11.524	**0.773**	11.538	0.766	11.543	0.762	**11.561**	0.765	11.000	0.723

Table 1. Comparison under different ρ. In the top, LOMAR ($\rho = x$) means LOMAR is trained with the value of $\rho = x$. The average reward and competitive ratio are represented by AVG and CR, respectively — the higher, the better. The highest value in each testing setup is highlighted in bold. The AVG and CR for DRL are **12.909** and **0.544** respectively. The average reward for OPT is **13.209**.

training step, given an RL policy with parameter θ, we sample n action trajectories $\{\tau_i = \{x_{1,i}, \cdots, x_{v,i}\}, i \in [n]\}$ and record the corresponding rewards. We can get the approximated average reward as $\hat{R}_\theta = \frac{1}{n}\sum_{i=1}^{n} w^i_{x_{i,v},v}$, and calculate the gradient as

$$\nabla_\theta \hat{R}_\theta = \sum_{i=1}^{n}\left(\sum_{v\in\mathcal{V}} \nabla_\theta \log p_\theta(x_{v,i}|I_{u,i})\right)\left(\sum_{v\in\mathcal{V}} w^i_{x_{v,i},v}\right) \tag{5}$$

Then, we update the parameter θ by $\theta = \theta + \alpha\nabla_\theta \hat{R}_\theta$, where α is the step size. This process repeats until convergence and/or the maximum number of iterations is reached.

At the beginning of the policy training, we can set a high temperature t to encourage the RL model to explore more aggressively, instead of sticking to the expert's decisions. As the RL model performance continuously improves, we can reduce the temperature in order to make the RL agent more aware of the downstream switching operation. The training process is performed offline as in the existing RL-based optimizers (Alomrani et al., 2022; Du et al., 2022) and described in Algorithm 2 for one iteration.

6. Experiment

6.1. Setup

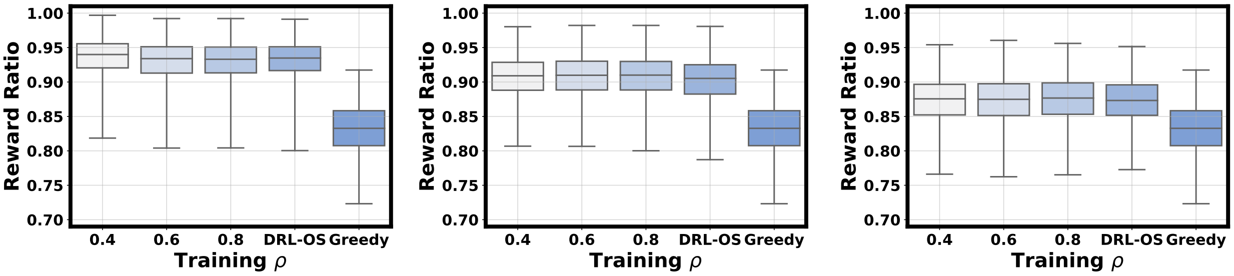
We conduct experiments based on the movie recommendation application. Specifically, when an user (i.e., online item v) arrives, we recommend a movie (i.e., offline item u) to this user and receive a reward based on the user-movie preference information. We choose the MovieLens dataset (Harper & Konstan, 2015), which provides a total of 3952 movies, 6040 users and 100209 ratings. We preprocess the dataset to sample movies and users randomly from the dataset to generate subgraphs, following the same steps as used by (Dickerson et al., 2019) and (Alomrani et al., 2022). In testing dataset, we empirically evaluate each algorithm using average reward (**AVG**) and competitive ratio (**CR**, against OPT), which represents the average performance and worst case performance, respectively. Thus, the value of CR is the empirically worst reward ratio in the testing dataset. For fair comparison, all the experimental settings like capacity c_u follow those used in (Alomrani et al., 2022). More details about the setup and training are in Appendix A.

Baseline Algorithms. We consider the following baselines. All the RL policies are trained offline with the same architecture and applicable hyperparameters. **OPT:** The offline optimal oracle has the complete information about the bipartite graph. We use the Gurobi optimizer to find the optimal offline solution. **Greedy:** At each step, Greedy selects the available offline item with highest weight. **DRL:** It uses the same architecture as in LOMAR, but does not consider online switching for training or inference. That is, the RL model is both trained and tested with $\rho = 0$. More specifically, our RL architecture has 3 fully connected layers, each with 100 hidden nodes. **DRL-OS (DRL-OnlineSwitching):** We apply online switching to the same RL policy used by DRL during inference. That is, the RL model is trained with $\rho = 0$, but tested with a different $\rho > 0$. This is essentially an existing ML-augmented algorithm that uses the standard practice (i.e., pre-train a standalone RL policy) (Christianson et al., 2022).

Our baselines include all those considered in (Alomrani et al., 2022). In the no-free-disposal setting, the best competitive ratio is 0 in general adversarial cases (Mehta, 2013). Here, we use Greedy as the expert, because the recent study (Alomrani et al., 2022) has shown that Greedy performs better than other alternatives and is a strong baseline.

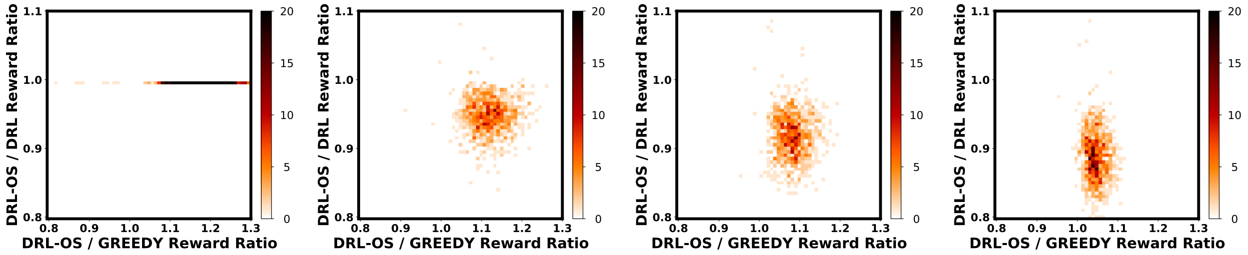
6.2. Results

Reward comparison. We compare LOMAR with baseline algorithms in Table 1. First, we see that DRL has the highest average reward, but its empirical competitive ratio is the lowest. The expert algorithm Greedy is fairly robust, but has a lower average award than RL-based policies. Second, DRL-OS can improve the competitive ratio compared to DRL. But, its RL policy is trained alone without being aware of the online switching. Thus, by making the RL policy aware of online switching, LOMAR can improve the average reward compared to DRL-OS. Specifically, by training LOMAR using the same ρ as testing it, we can obtain both the highest average cost and the highest competitive ratio. One exception is the minor decrease of competitive ratio when $\rho = 0.8$ for testing. This is likely due to the dataset and a few hard instances can affect the empirical competitive ratio, which also explains why the empirical competitive ratio is not necessarily monotonically increasing in the $\rho \in [0, 1]$. Nonetheless, unlike DRL that may



(a) Testing with $\rho = 0.4$ (b) Testing with $\rho = 0.6$ (c) Testing with $\rho = 0.8$

Figure 1. Boxplot for reward ratio with different ρ within testing dataset. Greedy and DRL-OS are also shown here for comparison. The best average performance in each figure is achieved by choosing the same ρ during training and testing.



(a) $\rho = 0.0$ (i.e., DRL) (b) $\rho = 0.4$ (c) $\rho = 0.6$ (d) $\rho = 0.8$

Figure 2. Histogram of bi-competitive reward ratios of DRL-OS (against Greedy and DRL) under different ρ.

only work well empirically without guarantees, LOMAR offers provable robustness while exploiting the power of RL to improve the average performance. The boxplots in Fig. 1 visualizes the reward ratio distribution of LOMAR, further validating the importance of switching-aware training.

Impact of ρ. To show the impact of ρ, we calculate the bi-competitive reward ratios. Specifically, for each problem instance, the bi-competitive ratio compares the actual reward against those of Greedy and RL model, respectively. To highlight the effect of online switching, we focus on DRL-OS (i.e., training the RL with $\rho = 0$) whose training process of RL model is not affected by ρ, because the RL model trained with $\rho > 0$ in LOMAR does not necessarily perform well on its own and the reward ratio of LOMAR to its RL model is not meaningful. The histogram of the bi-competitive ratios are visualized in Fig. 2. When $\rho = 0$, the ratio of DRL-OS/ DRL is always 1 unsurprisingly, since DRL-OS are essentially the same as DRL in this case (i.e., both trained and tested with $\rho = 0$). With a large ρ (e.g. 0.8) for testing, the reward ratios of DRL-OS/Greedy for most samples are around 1, which means the robustness is achieved, as proven by our theoretical analysis. But on the other hand, DRL-OS has limited flexibility and can less exploit the good average performance of DRL. Thus, the hyperparameter $\rho \in [0, 1]$ governs the tradeoff between average performance and robustness relative to the expert and, like other hyperparameters, can be tuned to maximize the average performance subject to the robustness requirement.

We also consider a crowdsourcing application, as provided by the gMission dataset (Chen et al., 2014). Additional results for gMission are deferred to Appendix A.

7. Conclusion

In this paper, we propose LOMAR for edge-weighted online bipartite matching. LOMAR includes a novel online switching operation to decide whether to follow the expert's decision or the RL decision for each online item arrival. We prove that for any $\rho \in [0, 1]$, LOMAR is ρ-competitive against any expert online algorithms, which directly translates a bounded competitive ratio against OPT if the expert algorithm itself has one. We also train the RL policy by explicitly considering the online switching operation so as to improve the average performance. Finally, we run empirical experiments to validate LOMAR.

There are also interesting problems that remain open, such as how to incorporate multiple RL models or experts and what the performance bound of LOMAR is compared to pure RL in terms of the average reward.

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Appendix

In the appendix, we show the experimental setup and additional results (Appendix A), algorithm details for the free-disposal setting (Appendix B), and finally the proof of Theorem 4.1 (Appendix C).

A. Experimental Settings and Additional Results

Our implementation of all the considered algorithms, including LOMAR, is based on the source codes provided by (Alomrani et al., 2022), which includes codes for training the RL model, data pre-proposing and performance evaluation. We conduct experiments on two real-world datasets: MovieLens (Harper & Konstan, 2015) and gMission (Chen et al., 2014).

A.1. MovieLens

A.1.1. Setup and Training

We first sample u_0 movies from the original MovieLens dataset (Harper & Konstan, 2015). We then sample v_0 users and make sure each user can get at least one movie; otherwise, we remove the users that have no matched movies, and resample new users. After getting the topology graph, we use Gurobi to find the optimal matching decision. In our experiment, we set $u_0 = 10$ and $v_0 = 60$ to generate the training and testing datasets. The number of graph instances in the training and testing datasets are 20000 and 1000, respectively. For the sake of reproducibility and fair comparision, our settings follows the same setup of (Alomrani et al., 2022). In particular, the general movie recommendation problem belongs to online submodular optimization, but it can actually be equivalently mapped to edge-weighted online bipartite matching with no free disposal under the setting considered in (Alomrani et al., 2022). So by default, the capacity c_u for each offline node is set as 1 and $w_{u,\max} = 5$. While LOMAR can use any RL architecture, we follow the design of *inv-ff-hist* proposed by (Alomrani et al., 2022), which empirically demonstrates the best performance among all the considered architectures.

The input to our considered RL model is the edge weights w_{uv} revealed by the online items plus some historical information, which includes: Mean and variances of each offline node's weights; Average degree of each offline nodes; Normalized step size; Percentage of offline nodes connected to the current node; Statistical information of these already matched nodes' weights (maximum, minimum, mean and variance); Ratio of matched offline node; Ratio of skips up to now; Normalized reward with respect to the offline node number. For more details of the historical information, readers are referred to Table 1 in (Alomrani et al., 2022).

For applicable algorithms (i.e., DRL, DRL-OS, and LOMAR), we train the RL model for 300 epochs in the training dataset with a batch size of 100. In LOMAR, the parameter $B = 0$ is used to follow the strict definition of competitive ratio. We test the algorithms on the testing dataset to obtain the average reward and the worst-case competitive ratio empirically. By setting $\rho = 0$ for training, LOMAR is equivalent to the vanilla inv-ff-hist RL model (i.e., DRL) used in (Alomrani et al., 2022). Using the same problem setup, we can reproduce the same results shown in (Alomrani et al., 2022), which reaffirms the correctness of our data generation and training process.

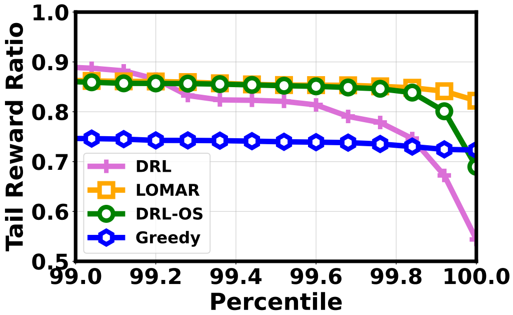


Figure 3. Tail reward ratio comparison. In this experiment, we set $\rho = 0.4$ for DRL-OS and LOMAR.

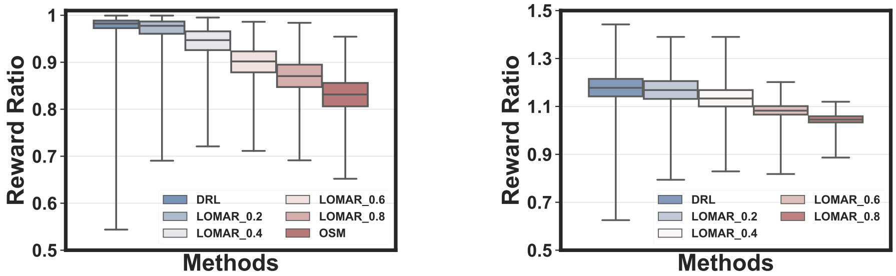
Additionally, training the RL model in LOMAR usually takes less than 8 hours on a shared research cluster with one NVIDIA K80 GPU, which is almost the same as the training the model for DRL in a standalone manner (i.e., setting $\rho = 0$ without considering online switching).

A.1.2. Additional Results

In Table 1, we have empirically demonstrated that LOMAR achieves the best tradeoff between the average reward and competitive ratio. In Fig 3, we further demonstrate that LOMAR not only achieves a better worst-case competitive ratio (at 100.0%). The tail reward ratio of LOMAR is also good compared to the baseline algorithms. Specifically, we show the percentile of reward ratios (compared to the optimal offline algorithm) — the 100% means the worst-case empirical reward ratio (i.e., competitive ratio). We see that DRL has a bad high-percentile reward ratio and lacks performance robustness, although its lower-percentile cost ratio is better. This is consistent with the good average performance of LOMAR. Because of online switching, both DRL-OS and LOMAR achieve better robustness, and LOMAR is even better due to its awareness of the online switching operation during its training process. The expert Greedy has a fairly stable competitive ratio, showing its good robustness. But, it can be outperformed by other algorithms when we look at lower-percentile reward ratio.

A.1.3. Results for another expert algorithm

Optimally competitive expert algorithms have been developed under the assumptions of random oder and/or i.i.d. rewards of different online items. In particular, by considering the random order setting, OSM (online secretary matching) has the optimal competitive ratio of $1/e$ (Kesselheim et al., 2013). Note that the competitive ratio for OSM is average over the random order of online items, while the rewards can be adversarially chosen. We show the empirical results in Fig. 4. As OSM skips the first $|\mathcal{V}|/e$ online items, it actually does not perform (in terms of the empirical worst-case cost ratio) as well as the default expert Greedy in our experiments despite its guaranteed competitive ratio against OPT. That said, we still observe the same trend as using Greedy for the expert: by tuning $\rho \in [0, 1]$, LOMAR achieves a good average performance while guaranteeing the competitiveness against the expert OSM (and against OPT as OSM itself is optimally competitive against OPT).



(a) Reward ratio against OPT

(b) Reward ratio against OSM

Figure 4. Reward ratio distribution (OSM as the expert)

Fig. 4 shows the empirical results in our testing dataset, which does not strictly satisfy the random order assumption required by OSM. Next, to satisfy the random order assumption, we select a typical problem instance and randomly vary the arrival orders of online items. We show the cost ratio averaged over the random arrival order in Table 2. Specifically, we calculate each cost ratio by 100 different random orders, and repeat this process 100 times. We show the mean and stand deviation of the average cost ratios (each averaged over 100 different random orders). We see that LOMAR improves the average cost ratio compared to OSM under the random order assumption. While DRL has a better average cost for this particular instance, it does not provide any guaranteed worst-case robustness as LOMAR.

	DRL	LOMAR $\rho = 0.2$	LOMAR $\rho = 0.4$	LOMAR $\rho = 0.6$	LOMAR $\rho = 0.8$	OSM
Mean	0.9794	0.9688	0.9431	0.9095	0.8799	0.8459
Std	0.0074	0.0082	0.0078	0.0086	0.0084	0.0084

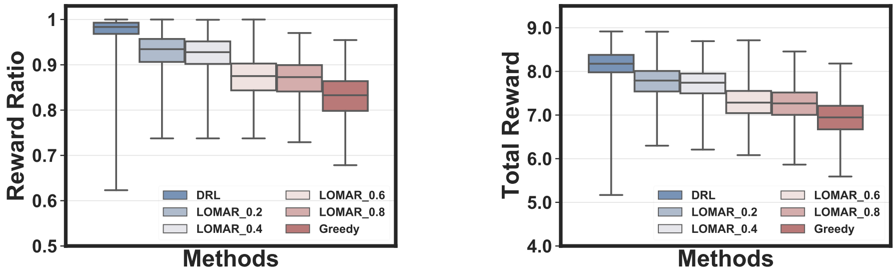
Table 2. Reward ratio (averaged over the random arrival order) for a typical graph instance

A.1.4. Results for the free-disposal setting

For the free-disposal setting, we use the same parameter (e.g. RL architecture, learning rates) and datasets as in the no-free-disposal case. By modifying the implementation of the public codes released by (Alomrani et al., 2022) that focus on no-free-disposal matching, we consider a 5 × 60 graph and allow each offline node to be matched with multiple online items, while only the maximum reward for each offline node is considered. In Table 3 and Fig. 5, we use Greedy as the expert and evaluate LOMAR with different ρ parameters. The empirical results show a similar trend as our experiments under the no-free-disposal setting: with a smaller ρ, the average performance of LOMAR is closer to DRL.

	DRL	LOMAR $\rho = 0.2$	LOMAR $\rho = 0.4$	LOMAR $\rho = 0.6$	LOMAR $\rho = 0.8$	Greedy
AVG	8.172	7.764	7.712	7.298	7.256	6.932
CR	0.623	0.738	0.738	0.738	0.729	0.678

Table 3. Average reward and competitive ratio comparison between different algorithms. The average reward of OPT is 8.359.



(a) Reward ratio against OPT (b) Total reward

Figure 5. Reward ratio and total reward distributions for the free-disposal setting (Greedy as the expert).

A.2. gMission

The gMission dataset (Chen et al., 2014) considers a crowdsourcing application, where the goal is to assign the tasks (online items) to workers (offline items). The edge weight between a certain online task and each worker can be calculated by the product of the task reward and the worker's success probability, which is determined by the physical location of workers and the type of tasks. Our goal is to maximize the total reward given the capacity of each worker, which perfectly fits into our formulation in Eqn. (1).

We use the same data processing and RL architecture design as introduced in Section A.1.1. We train LOMAR with different ρ in the gMission dataset by setting $u_0 = 10$, $v_0 = 60$, $w_{u,\max} = 1$. Again, we use Greedy as the expert, which is an empirically strong baseline algorithm as shown in (Alomrani et al., 2022). Our results are all consistent with those presented in (Alomrani et al., 2022).

A.2.1. Testing on 10×60

In our the first result, we generate a testing dataset with $u_0 = 10$ and $v_0 = 60$, which is the same setting as our training dataset. In other words, the training and testing datasets have similar distributions. Specifically, Greedy's average reward and competitive ratio are 4.508 and 0.432, while these two values for DRL are 5.819 and 0.604, respectively. Thus, DRL performs outperforms Greedy in both average performance and the worst-case performance.

	DRL-OS		LOMAR $\rho = 0.4$		LOMAR $\rho = 0.6$		LOMAR $\rho = 0.8$		LOMAR $\rho = 0.9$	
ρ in Testing	AVG	CR	AVG	CR	AVG	CR	AVG	CR	AVG	CR
0.4	5.553	**0.599**	**5.573**	0.598	5.553	0.598	5.523	0.598	5.535	0.598
0.6	5.389	0.591	**5.429**	**0.619**	5.420	0.619	5.403	0.623	5.402	0.623
0.8	5.102	0.543	**5.115**	**0.543**	5.111	0.523	5.110	0.521	5.107	0.521
0.9	4.836	0.495	4.836	0.495	4.839	0.495	4.839	0.540	**4.839**	**0.540**

Table 4. Performance comparison in gMission 10×60 for different ρ. LOMAR with $\rho = y$ means LOMAR is trained with $\rho = y$.

Next, we show the results for LOMAR and DRL-OS under different $\rho \in [0, 1]$ in Table 4. In general, by setting a larger ρ for inference, both LOMAR and DRL-OS are closer to the expert algorithm Greedy, because there is less freedom for the RL decisions. As a result, when ρ increases during inference, the average rewards of both DRL-OS and LOMAR decrease, although they have guaranteed robustness whereas DRL does not. Moreover, by training the RL model with explicit awareness of online switching, LOMAR can have a higher average cost than DRL-OS, which reconfirms the benefits of training the RL model by considering its downstream operation. Interestingly, by setting ρ identical for both training and testing, the average reward may not always be the highest for LOMAR. This is partially because of the empirical testing dataset. Another reason is that, in this test, DRL alone already performs the best (both on average and in the worst case).

Hence, by setting a smaller ρ for inference, LOMAR works better empirically though it is trained under a different ρ. Nonetheless, this does not void the benefits of guaranteed robustness in LOMAR. The empirically better performance of DRL lacks guarantees, which we show as follows.

A.2.2. Testing on 100×100

In our second test, we consider an opposite case compared to the first one. We generate a testing dataset with $u_0 = 100$ and $v_0 = 100$, which is different from the training dataset setting. As a result, the training and testing datasets have very different distributions, making DRL perform very badly. Specifically, Greedy's average reward and competitive ratio are 40.830 and 0.824, and these two values for DRL are 32.938 and 0.576, respectively. DRL has an even lower average reward than Greedy, showing its lack of performance robustness.

	DRL-OS		LOMAR $\rho = 0.4$		LOMAR $\rho = 0.6$		LOMAR $\rho = 0.8$		LOMAR $\rho = 0.9$	
ρ in Testing	AVG	CR	AVG	CR	AVG	CR	AVG	CR	AVG	CR
0.4	33.580	0.604	37.030	0.707	38.199	0.750	38.324	0.750	**38.538**	**0.766**
0.6	34.973	0.680	37.490	0.731	38.518	0.762	38.505	0.756	**38.727**	**0.767**
0.8	37.939	0.758	38.866	0.775	39.502	0.782	39.385	**0.794**	**39.552**	0.781
0.9	39.772	0.794	40.057	0.803	**40.377**	0.806	40.239	**0.812**	40.332	0.798

Table 5. Performance comparison on gMission 100×100 for different ρ. LOMAR with $\rho = y$ means LOMAR is trained with $\rho = y$.

We show the results for LOMAR and DRL-OS under different $\rho \in [0, 1]$ in Table 5. In general, by setting a larger ρ for inference, both LOMAR and DRL-OS are closer to the expert algorithm Greedy. As Greedy works empirically much better than DRL in terms of the average performance and the worst-case performance, both LOMAR and DRL-OS have better performances when we increase ρ to let Greedy safeguard the RL decisions more aggressively. Moreover, by training the RL model with explicit awareness of online switching, LOMAR can have a higher average cost than DRL-OS, which further demonstrates the benefits of training the RL model by considering its downstream operation. Also, interestingly, by setting ρ identical for both training and testing, the average reward may not be the highest for LOMAR, partially because of the empirical testing dataset. Another reason is that, in this test, DRL alone already performs very badly (both on average and in the worst case) due to the significant training-testing distributional discrepancy. Hence, by setting a higher ρ, LOMAR works better empirically though it is tested under a different ρ. An exception is when testing LOMAR with $\rho = 0.9$: setting $\rho = 0.6/0.8$ for training makes LOMAR perform slightly better in terms of the average performance and worst-case performance, respectively. But, setting $\rho = 0.9$ for training still brings benefits to LOMAR compared to DRL-OS that does not consider the downstream online switching operation.

To sum up, our experimental results under different settings demonstrate: LOMAR's empirical improvement in terms of the average reward compared to DRL-OS; the improved competitive ratio of LOMAR and DRL-OS compared to DRL, especially when the training-testing distributions differ significantly; and the improved average reward of LOMAR compared to Greedy when RL is good. Therefore, LOMAR can exploit the power of RL while provably guaranteeing the performance robustness.

B. Free Disposal

The offline version of bipartite matching with free disposal can be expressed as:

$$\begin{aligned} &\textbf{With Free Disposal:} \quad \max_{x_{uv} \in \{0,1\}, u \in \mathcal{U}} \sum \left(\max_{\mathcal{S} \subseteq \mathcal{V}_u, |\mathcal{S}| \le c_u} \sum_{v \in \mathcal{S}} w_{uv} \right) \\ &\text{s.t.} \quad \mathcal{V}_u = \{v \in \mathcal{V} \,|\, x_{uv} = 1\} \;\; \forall u \in \mathcal{U}, \;\; \sum_{u \in \mathcal{U}} x_{uv} \le 1, \;\; \forall v \in \mathcal{V}, \end{aligned} \tag{6}$$

where $\mathcal{V}_u = \{v \in \mathcal{V} \,|\, x_{uv} = 1\}$ is the set of online items matched to $u \in \mathcal{U}$ and the objective $\max_{\mathcal{S} \in \mathcal{V}_u, |\mathcal{S}| \le c_u} \sum_{v \in \mathcal{S}} x_{uv} w_{uv}$ indicates that only up to top c_u rewards are counted for $u \in \mathcal{U}$.

In the free-disposal setting, it is more challenging to design the switching conditions to guarantee the robustness. The reason is the additional flexibility allowed for matching decisions — each offline item $u \in \mathcal{U}$ is allowed to be matched more than c_u times although only up to top c_u rewards actually count (Mehta, 2013; Fahrbach et al., 2020). For example, even though LOMAR and the expert assign the same number of online items to an offline item $u \in \mathcal{U}$ and LOMAR is better than the expert

Algorithm 3 Inference of LOMAR (Free Disposal)

```
Initialization: The actual set of items matched to $u \in \mathcal{U}$ is $\mathcal{V}_{u,v}$ after sequentially-arriving item $v$'s assignment with $\mathcal{V}_{u,0} = \emptyset$, the actual remaining capacity is $b_u = c_u$ for $u \in \mathcal{U}$, and the actual cumulative reward is $R_0 = \sum_{u\in\mathcal{U}} f_u(\mathcal{V}_{u,0}) = 0$. The same notations apply to the expert algorithm $\pi$ by adding the superscript $\pi$. Competitive ratio requirement $\rho \in [0,1]$ and slackness $B \geq 0$ with respect to the expert algorithm $\pi$.
for $v = 1$ to $|\mathcal{V}|$ do
    Run the algorithm $\pi$ and match the item $v$ to $u \in \mathcal{U}$ based on the expert's decision $u = x^\pi_v$.
    Update the expert's decision set and reward for offline item $u = x^\pi_v$:
    $\mathcal{V}^\pi_{x^\pi_v,v} = \mathcal{V}^\pi_{x^\pi_v,v-1} \bigcup \{v\}$ and $f_{x^\pi_v} = f_{x^\pi_v}(\mathcal{V}^\pi_{x^\pi_v,v})$.
    Update the expert's cumulative reward $R^\pi_v = \sum_{u\in\mathcal{U}} f_u$
    for $u$ in $\mathcal{U}$ do
        Collect the available history information $I_u$ about item $u$
        Run the RL model to get score: $s_u = w_{uv} - h_\theta(I_u, w_{uv})$ where $\theta$ is the network weight
    end for
    Calculate the probability of choosing each item $u$: $\{\{\tilde{s}_u\}_{u\in\mathcal{U}}\} = \mathrm{softmax}\{\{s_u\}_{u\in\mathcal{U}}\}$.
    Obtain RL decision: $\tilde{x}_v = \arg\max_{u\in\mathcal{U}\bigcup\{\text{skip}\}} \{\{\tilde{s}_u\}_{u\in\mathcal{U}}\}$.
    Find $\Delta f_{\tilde{x}_v}$ in Eqn. (9) and $G\left(\tilde{x}_v, \{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v}\}u \in \mathcal{U}\right)$ in Eqn. (15)
    if $R_{v-1} + \Delta f_{\tilde{x}_v} \geq \rho\left(R^\pi_v + G\left(\tilde{x}_v, \{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v}\}_{u\in\mathcal{U}}\right)\right) - B$ then
        Select $x_v = \tilde{x}_v$.   //Follow the ML action
    else
        Select $x_v = x^\pi_v$.   //Follow the expert
    end if
    Update assignment and reward: $\mathcal{V}_{x_v,v} = \mathcal{V}_{x_v,v-1} \bigcup \{v\}$ and $R_v = \sum_{u\in\mathcal{U}} f_u(\mathcal{V}_{u,v})$
end for
```

at a certain step, future high-reward online items can still be assigned to $u \in \mathcal{U}$, increasing the expert's total reward or even equalizing the rewards of LOMAR and the expert (i.e., high-reward future online items become the top c_u items for $u \in \mathcal{U}$ for both LOMAR and the expert). Thus, the temporarily "higher" rewards received by LOMAR must be hedged against such future uncertainties. Before designing our switching condition for the free-disposal setting, we first define the set containing the top c_u online items for $u \in \mathcal{U}$ after assignment of v:

$$\mathcal{E}_{u,v}(\mathcal{V}_{u,v}) = \arg\max_{\mathcal{E}\subseteq\mathcal{V}_{u,v}, |\mathcal{E}|=c_u} \sum_{v\in\mathcal{E}} w_{uv}, \tag{7}$$

where $\mathcal{V}_{u,v}$ is the set of all online items matched to $u \in \mathcal{U}$ so far after assignment of item $v \in \mathcal{V}$. When there are fewer than c_u items in $\mathcal{V}_{uv}$, we will simply add null items with reward 0 to $\mathcal{E}_{u,v}$ such that $|\mathcal{E}_{u,v}| = c_u$. We also sort the online items denoted as $e_{u,i}$, for $i = 1, \cdots, c_u$, contained in $\mathcal{E}_{u,v}$ according to their weights in an increasing order such that $w_{u,e_{u,1}} \leq \cdots \leq w_{u,e_{u,c_u}}$. Similarly, we define the same top-c_u item set for the expert algorithm π by adding the superscript π.

Next, we define the following value which indicates the maximum possible additional reward for the expert algorithm π if LOMAR simply switches to the expert and follows it for all the future steps $v+1, v+2, \cdots$:

$$G\left(\tilde{x}_v, \{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v}\}_{u\in\mathcal{U}}\right) = \sum_{u\in\mathcal{U}} \left(\max_{i=1,\cdots,c_u} \sum_{j=1}^{i} (w_{u,e_{u,j}} - w_{u,e^\pi_{u,j}}) \right)^+, \tag{8}$$

where $e^\pi_{u,j} \in \mathcal{E}^\pi_u(\mathcal{V}^\pi_{u,v})$, and $e_{u,j} \in \mathcal{E}_u(\tilde{\mathcal{V}}_{u,v})$ in which $\tilde{\mathcal{V}}_{u,v} = \mathcal{V}_{u,v-1}$ if $\tilde{x}_v \neq u$ and $\tilde{\mathcal{V}}_{u,v} = \mathcal{V}_{u,v-1} \bigcup \{v\}$ if $\tilde{x}_v = u$.

The interpretation is as follows. Suppose that LOMAR follows the RL decision for online item v. If it has a higher cumulative reward for the j-th item in the top-c_u item set $\mathcal{E}_{u,v}$ than the expert algorithm π, then the expert can still possibly offset the reward difference $w_{u,e_{u,j}} - w_{u,e^\pi_{u,j}}$ by receiving a high-reward future online item that replaces the j-th item for both LOMAR and the expert. Nonetheless, in the free-disposal model, the items in the top-c_u set $\mathcal{E}_{u,v}$ are removed sequentially — the lowest-reward item will be first removed from the sorted set $\mathcal{E}_{u,v}$, followed by the next lowest-reward item, and so on. Thus, in order for a high-reward item to replace the i-th item in the sorted set $\mathcal{E}_{u,v}$, the first $(i-1)$ items have to be removed first by other high-reward online items. As a result, if LOMAR has a lower reward for the j-th item (for $j \leq i$) in the top-c_u item set $\mathcal{E}_{u,v}$ than the expert algorithm π, then it will negatively impact the expert's additional reward gain in the future. Therefore, for item $u \in \mathcal{U}$ we only need to find the highest total reward difference, $\left(\max_{i=1,\cdots,c_u} \sum_{j=1}^{i}(w_{u,e_{u,j}} - w_{u,e^{\pi}_{u,j}})\right)^+$, that can be offset for the expert algorithm π by considering that i items are replaced by future high-reward online items for $i = 1, \cdots, c_u$. If $\max_{i=1,\cdots,c_u} \sum_{j=1}^{i}(w_{u,e_{u,j}} - w_{u,e^{\pi}_{u,j}})$ is negative (i.e., the expert algorithm cannot possibly gain higher rewards than LOMAR by receiving high-reward online items to replace its existing ones), then we use 0 as the hedging reward.

Finally, by summing up the hedging rewards for all the offline items $u \in \mathcal{U}$, we obtain the total hedging reward in Eqn. (8). Based on this hedging reward, we have the condition (Line 28 in Algorithm 1 for LOMAR to follow the RL decision: $R_{v-1} + \Delta f_{\tilde{x}_v} \geq \rho\left(R^{\pi}_v + G\left(\tilde{x}_v, \{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^{\pi}_{u,v}\}_{u\in\mathcal{U}}\right)\right) - B$, where $\Delta f_{\tilde{x}_v}$ defined below is the additional reward if $\tilde{x}_v$ is not skip, which would be obtained by following the RL decision:

$$\Delta f_{\tilde{x}_v} = f_{\tilde{x}_v}(\mathcal{V}_{\tilde{x}_v,v} \bigcup \{v\}) - f_{\tilde{x}_v}(\mathcal{V}_{\tilde{x}_v,v-1}), \tag{9}$$

in which $f_u = f_u(\mathcal{V}') = \max_{\mathcal{S}\in\mathcal{V}',|\mathcal{S}|\leq c_u} \sum_{v\in\mathcal{S}} w_{uv}$ is the reward function for an offline item $u \in \mathcal{U}$ in the free-disposal model. The condition means that if LOMAR can maintain the competitive ratio ρ against the expert algorithm π by being able to hedge against any future uncertainties even in the worst case, then it can safely follow the RL decision $\tilde{x}_v$ at step v.

Training with free disposal. The training process for the free-disposal setting is the same as that for the no-free-disposal setting, except for we need to modify reward difference R_{diff} based on the switching condition (i.e., Line 13 of Algorithm 3) for the free-disposal setting. The R_{diff} is obtained by subtracting right hand side from the left hand side of the switching condition, which is used to calculate $p_\theta(x_v \mid I_u)$ in Line 3 of Algorithm 2.

C. Proof of Theorem 4.1

The key idea of proving Theorem 4.1 is to show that there always exist feasible actions (either following the expert or skip) while being able to guarantee the robustness if we follow the switching condition. Next, we prove Theorem 4.1 for the no-free-disposal and free-disposal settings, respectively.

C.1. No Free Disposal

Denote $\mathcal{V}_{u,v}$ as the actual set of items matched to $u \in \mathcal{U}$ after making decision for v. Denote $\mathcal{V}^{\pi}_{u,v}$ as the expert's set of items matched to $u \in \mathcal{U}$. We first prove a technical lemma.

Lemma C.1. *Assuming that the robustness condition is met after making the decision for $v-1$, i.e. $R_{v-1} \geq \rho\left(R^{\pi}_{v-1} + \sum_{u\in\mathcal{U}}\left(|\mathcal{V}_{u,v-1}| - |V^{\pi}_{u,v-1}|\right)^+ \cdot w_{u,\max}\right) - B$. If at the step when v arrives and the expert's decision x^{π}_v is not available for matching, then $x_v = \text{skip}$ always satisfies $R_v \geq \rho\left(R^{\pi}_v + \sum_{u\in\mathcal{U}}\left(|\mathcal{V}_{u,v}| - |V^{\pi}_{u,v}|\right)^+ \cdot w_{u,\max}\right) - B$.*

Proof. If the item x^{π}_v is not available for matching, it must have been consumed before v arrives, which means $|\mathcal{V}_{x^{\pi}_v,v-1}| - |V^{\pi}_{x^{\pi}_v,v-1}| \geq 1$ (since otherwise the expert cannot choose x^{pi}_v either). Since $x_v = \text{skip}$, we have $R_v = R_{v-1}$ and $\mathcal{V}_{u,v} = \mathcal{V}_{u,v-1}, \quad \forall u \in \mathcal{U}$. Then, by the robustness assumption of the previous step, we have

$$\begin{aligned}
R_v = R_{v-1} \geq & \rho\left(R^{\pi}_{v-1} + \sum_{u\in\mathcal{U}}\left(|\mathcal{V}_{u,v-1}| - |V^{\pi}_{u,v-1}|\right)^+ \cdot w_{u,\max}\right) - B \\
\geq & \rho\left(R^{\pi}_{v-1} + w_{x^{\pi}_v,v} - w_{x^{\pi}_v,\max} + \sum_{u\in\mathcal{U}}\left(|\mathcal{V}_{u,v-1}| - |V^{\pi}_{u,v-1}|\right)^+ \cdot w_{u,\max}\right) - B \\
= & \rho\left(R^{\pi}_v + \sum_{u\in\mathcal{U}}\left(|\mathcal{V}_{u,v}| - |V^{\pi}_{u,v}|\right)^+ \cdot w_{u,\max}\right) - B
\end{aligned} \tag{10}$$

where the last equality holds because $R^{\pi}_v = R^{\pi}_{v-1} + w_{x^{\pi}_v,v}$, and $(|\mathcal{V}_{u,v}| - |\mathcal{V}^{\pi}_{u,v}|)^+ - (|\mathcal{V}_{u,v-1}| - |\mathcal{V}^{\pi}_{u,v-1}|)^+ = -1$ if $u = x^{\pi}_v$, and $(|\mathcal{V}_{u,v}| - |\mathcal{V}^{\pi}_{u,v}|)^+ - (|\mathcal{V}_{u,v-1}| - |\mathcal{V}^{\pi}_{u,v-1}|)^+ = 0$ otherwise. □

Next we prove by induction that the condition

$$R_v \geq \rho \left(R_v^{\pi} + \sum_{u \in \mathcal{U}} \left(|\mathcal{V}_{u,v}| - |V_{u,v}^{\pi}| \right)^+ \cdot w_{u,\max} \right) - B \tag{11}$$

holds for all steps by Algorithm 1.

At the first step, if $\tilde{x}_v$ is not the same as x_v^{π} and $w_{\tilde{x}_v,v} \geq \rho \left(w_{x_v^{\pi},v} + w_{u,\max} \right) - B$, we select the RL decision $x_v = \tilde{x}_v$, and the robustness condition (11) is satisfied. Otherwise, we select the expert action $x_v = x_v^{\pi}$ and the condition still holds since the reward is non-negative, $\rho \leq 1$ and $B \geq 0$.

Then, assuming that the robustness condition in (11) is satisfied after making the decision for $v - 1$, we need to prove it is also satisfied after making the decision for v. If the condition in (3) in Algorithm 1 is satisfied, then $x_v = \tilde{x}_v$ and so (11) holds naturally. Otherwise, if the expert action x_v^{π} is available for matching, then we select expert action $x_v = x_v^{\pi}$. Then, we have $w_{x_v^{\pi},v} \geq 0$ and $|\mathcal{V}_{u,v}| - |V_{u,v}^{\pi}| = |\mathcal{V}_{u,v-1}| - |V_{u,v-1}^{\pi}|, \quad \forall u \in \mathcal{U}$, hence the condition (11) still holds. Other than these two cases, we also have the option to "skip", i.e. $x_v = \mathrm{skip}$. By Lemma C.1, the condition (11) still holds. Therefore, we prove that the condition (11) holds for every step.

After the last step $v = |\mathcal{V}|$, we must have

$$R_v \geq \rho \left(R_v^{\pi} + \sum_{u \in \mathcal{U}} \left(|\mathcal{V}_{u,\hat{v}}| - |V_{u,\hat{v}}^{\pi}| \right)^+ \cdot w_{u,\max} \right) - B \geq \rho R_v^{\pi} - B \tag{12}$$

where R_v and R_v^{π} are the total rewards of `LOMAR` and the expert algorithm π after the last step $v = |\mathcal{V}|$, respectively. This completes the proof for the no-free-disposal case.

C.2. With Free Disposal

We now turn to the free-disposal setting which is more challenging than the no-free-disposal setting because of the possibility of using future high-reward items to replace existing low-reward ones.

We first denote $\Delta f_{x_v^{\pi}}$ as the actual additional reward obtained by following the expert's decision x_v^{π},

$$\Delta f_{x_v^{\pi}} = f_{x_v^{\pi}}(\mathcal{V}_{x_v^{\pi},v} \bigcup \{v\}) - f_{x_v^{\pi}}(\mathcal{V}_{x_v^{\pi},v-1}), \tag{13}$$

Additionally, we denote $\Delta f_{x_v^{\pi}}^{\pi}$ as the expert's additional reward of choosing x_v^{π}, where

$$\Delta f_{x_v^{\pi}}^{\pi} = f_{x_v^{\pi}}(\mathcal{V}_{x_v^{\pi},v}^{\pi} \bigcup \{v\}) - f_{x_v^{\pi}}(\mathcal{V}_{x_v^{\pi},v-1}^{\pi}). \tag{14}$$

For presentation convenience, we rewrite the hedging reward as $\tilde{G}\left(\{\mathcal{V}_{u,v}\}_{u \in \mathcal{U}}, \{\mathcal{V}_{u,v}^{\pi}\}_{u \in \mathcal{U}}\right)$ as

$$\tilde{G}\left(\{\mathcal{V}_{u,v}\}_{u \in \mathcal{U}}, \{\mathcal{V}_{u,v}^{\pi}\}_{u \in \mathcal{U}}\right) = \sum_{u \in \mathcal{U}} \left(\max_{i=1,\cdots,c_u} \sum_{j=1}^{i} (w_{u,e_{u,j}} - w_{u,e_{u,j}^{\pi}}) \right)^+, \tag{15}$$

where $e_{u,j}^{\pi} \in \mathcal{E}_u^{\pi}(\mathcal{V}_{u,v}^{\pi})$, $e_{u,j} \in \mathcal{E}_u(\mathcal{V}_{u,v})$, and $\mathcal{E}_u$ is defined in Eqn. (7).

Lemma C.2. *Assuming that the robustness condition is met after making the decision for $v - 1$, i.e. $R_{v-1} \geq \rho\left(R_{v-1}^{\pi} + \tilde{G}\left(\{\mathcal{V}_{u,v-1}\}_{u \in \mathcal{U}}, \{\mathcal{V}_{u,v-1}^{\pi}\}_{u \in \mathcal{U}}\right)\right) - B$. At step v, we have $\Delta f_{x_v^{\pi}} - \Delta f_{x_v^{\pi}}^{\pi} \geq G\left(x_v^{\pi}, \{\mathcal{V}_{u,v-1}\}_{u \in \mathcal{U}}, \{\mathcal{V}_{u,v}^{\pi}\}_{u \in \mathcal{U}}\right) - \tilde{G}\left(\{\mathcal{V}_{u,v-1}\}_{u \in \mathcal{U}}, \{\mathcal{V}_{u,v-1}^{\pi}\}_{u \in \mathcal{U}}\right)$.*

Proof. We begin with "$G\left(x_v^{\pi}, \{\mathcal{V}_{u,v-1}\}_{u \in \mathcal{U}}, \{\mathcal{V}_{u,v}^{\pi}\}_{u \in \mathcal{U}}\right) - \tilde{G}\left(\{\mathcal{V}_{u,v-1}\}_{u \in \mathcal{U}}, \{\mathcal{V}_{u,v-1}^{\pi}\}_{u \in \mathcal{U}}\right)$" in Lemma C.2. By definition, it can be written as

$$\begin{aligned} & G\left(x_v^{\pi}, \{\mathcal{V}_{u,v-1}\}_{u \in \mathcal{U}}, \{\mathcal{V}_{u,v}^{\pi}\}_{u \in \mathcal{U}}\right) - \tilde{G}\left(\{\mathcal{V}_{u,v-1}\}_{u \in \mathcal{U}}, \{\mathcal{V}_{u,v-1}^{\pi}\}_{u \in \mathcal{U}}\right) \\ = & \left(\max_{i=1,\cdots,c_u} \sum_{j=1}^{i} (w_{u,\hat{e}_{u,j}} - w_{u,\hat{e}_{u,j}^{\pi}}) \right)^+ - \left(\max_{i=1,\cdots,c_u} \sum_{j=1}^{i} (w_{u,e_{u,j}} - w_{u,e_{u,j}^{\pi}}) \right)^+ \end{aligned} \tag{16}$$

where $u = x^\pi_v$, $\hat{e}^\pi_{u,j} \in \mathcal{E}^\pi_u(\mathcal{V}^\pi_{u,v-1}\bigcup\{v\})$, and $\hat{e}_{u,j} \in \mathcal{E}_u(\mathcal{V}_{u,v-1}\bigcup\{v\})$. Besides, $e^\pi_{u,j} \in \mathcal{E}^\pi_u(\mathcal{V}^\pi_{u,v-1})$, and $e_{u,j} \in \mathcal{E}_u(\mathcal{V}_{u,v-1})$.

To prove the lemma, we consider four possible cases for $w_{u,v}$ to cover all the cases.

Case 1: If the reward for v is small enough such that $w_{u,v} < w_{u,e_{u,1}}$ and $w_{u,v} < w_{u,e^\pi_{u,1}}$, then $v \notin \mathcal{E}_u(\mathcal{V}_{u,v-1}\bigcup\{v\})$ and $v \notin \mathcal{E}_u(\mathcal{V}^\pi_{u,v-1}\bigcup\{v\})$. Then we have $\Delta f_{x^\pi_v} = \Delta f^\pi_{x^\pi_v} = 0$, since both the expert and LOMAR cannot gain any reward from the online item v. From Eqn. (16), we can find that the right-hand side is also 0. Therefore, the conclusion in Lemma C.2 holds with the equality activated.

Case 2: If the reward for v is large enough such that $w_{u,v} > w_{u,e_{u,1}}$ and $w_{u,v} > w_{u,e^\pi_{u,1}}$, then $v \in \mathcal{E}_u(\mathcal{V}_{u,v-1}\bigcup\{v\})$ and $v \in \mathcal{E}_u(\mathcal{V}^\pi_{u,v-1}\bigcup\{v\})$. In other words, we will remove the smallest-reward item $e_{u,1} \notin \mathcal{E}_u(\mathcal{V}_{u,v-1}\bigcup\{v\})$ and $e^\pi_{u,1} \notin \mathcal{E}_u(\mathcal{V}^\pi_{u,v-1}\bigcup\{v\})$. Then

$$G\left(x^\pi_v, \{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v}\}_{u\in\mathcal{U}}\right) - \tilde{G}\left(\{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v-1}\}_{u\in\mathcal{U}}\right) \leq -w_{u,e_{u,1}} + w_{u,e^\pi_{u,1}}$$

The inequality holds because $(w_{u,e_{u,1}} - w_{u,e^\pi_{u,1}})^+ \geq w_{u,e_{u,1}} - w_{u,e^\pi_{u,1}}$. In this case, $\Delta f_{x^\pi_v} = w_{u,v} - w_{u,e_{u,1}}$ and $\Delta f^\pi_{x^\pi_v} = w_{u,v} - w_{u,e^\pi_{u,1}}$. Therefore, the conclusion in Lemma C.2 holds.

Case 3: If the reward for v satisfies $w_{u,v} \geq w_{u,e_{u,1}}$ and $w_{u,v} \leq w_{u,e^\pi_{u,1}}$, then $v \in \mathcal{E}_u(\mathcal{V}_{u,v-1}\bigcup\{v\})$ and $v \notin \mathcal{E}_u(\mathcal{V}^\pi_{u,v-1}\bigcup\{v\})$. In other words, even if $v \in \mathcal{E}_u(\mathcal{V}_{u,v-1}\bigcup\{v\})$ (i.e., the online item v produces additional rewards for LOMAR), the reward of v is still smaller than the smallest reward for the expert. Then, only the lowest reward of LOMAR will be kicked out. Then we have $G\left(x^\pi_v, \{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v}\}_{u\in\mathcal{U}}\right) - \tilde{G}\left(\{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v-1}\}_{u\in\mathcal{U}}\right) \leq w_{u,v} - w_{u,e_{u,1}}$, the equality activates if $G\left(x^\pi_v, \{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v}\}_{u\in\mathcal{U}}\right) \geq 0$. In this case, $\Delta f_{x^\pi_v} = w_{u,v} - w_{u,e_{u,1}}$ and $\Delta f^\pi_{x^\pi_v} = 0$. Therefore, the conclusion in Lemma C.2 still holds.

Case 4: If the reward for v satisfies $w_{u,v} \leq w_{u,e_{u,1}}$ and $w_{u,v} \geq w_{u,e^\pi_{u,1}}$, then in this case, only the current smallest-reward item is replaced with v for the expert, while the reward of LOMAR remains unchanged. Thus, we have

$$G\left(x^\pi_v, \{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v}\}_{u\in\mathcal{U}}\right) - \tilde{G}\left(\{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v-1}\}_{u\in\mathcal{U}}\right) = w_{u,e^\pi_{u,1}} - w_{u,v}.$$

In this case, $\Delta f_{x^\pi_v} = 0$ and $\Delta f^\pi_{x^\pi_v} = w_{u,v} - w_{u,e^\pi_{u,1}}$. Then the conclusion in Lemma C.2 still holds with the equality activated. □

We next prove by induction that the condition

$$R_v \geq \rho\left(R^\pi_v + \tilde{G}\left(\{\mathcal{V}_{u,v}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v}\}_{u\in\mathcal{U}}\right)\right) - B \tag{17}$$

holds for all steps by Algorithm 1.

At the first step, by using $x_v = x^\pi_v$, we have $R_v = R^\pi_v$ and $\tilde{G}\left(\{\mathcal{V}_{u,v}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v}\}_{u\in\mathcal{U}}\right) = 0$, and it is obvious that the condition in (17) is satisfied. Thus, there is at least one solution $x_v = x^\pi_v$ for our robustness condition in (17).

Starting from the second step, assume that after the step $v-1$, we already have

$$R_{v-1} \geq \rho\left(R^\pi_{v-1} + \tilde{G}\left(\{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v-1}\}_{u\in\mathcal{U}}\right)\right) - B \tag{18}$$

If the condition in Line 13 of Algorithm 3 is already satisfied, we can just use $x_v = \tilde{x}_v$, which directly satisfies (17). Otherwise, we need to follow the expert by setting $x_v = x^\pi_v$. Now we will prove $x_v = x^\pi_v$ satisfies the robustness condition at any step v.

From Lemma C.2, since $0 \leq \rho \leq 1$ and $\Delta f_{x^\pi_v} \geq 0$ we have

$$\Delta f_{x^\pi_v} \geq \rho\Delta f_{x^\pi_v} \geq \rho\left(\Delta f^\pi_{x^\pi_v} + G\left(x^\pi_v, \{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v}\}_{u\in\mathcal{U}}\right) - \tilde{G}\left(\{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}^\pi_{u,v-1}\}_{u\in\mathcal{U}}\right)\right).$$

Then, by substituting it back to Eqn. (18), we have

$$\begin{aligned}R_{v-1} + \Delta f_{x_v^\pi} \geq &\rho \left(\Delta f_{x_v^\pi}^\pi + G\left(x_v^\pi, \{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}_{u,v}^\pi\}_{u\in\mathcal{U}}\right) - \tilde{G}\left(\{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}_{u,v-1}^\pi\}_{u\in\mathcal{U}}\right) \right) \\ &+ \rho \left(R_{v-1}^\pi + \tilde{G}\left(\{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}_{u,v-1}^\pi\}_{u\in\mathcal{U}}\right) \right) - B \\ =&\rho \left(R_{v-1}^\pi + \Delta f_{x_v^\pi}^\pi + G\left(x_v^\pi, \{\mathcal{V}_{u,v-1}\}_{u\in\mathcal{U}}, \{\mathcal{V}_{u,v}^\pi\}_{u\in\mathcal{U}}\right) \right) - B \\ =&\rho \left(R_v^\pi + \tilde{G}\left(\{\mathcal{V}_{u,v}\}_{u\in\mathcal{U}}, \{\mathcal{V}_{u,v}^\pi\}_{u\in\mathcal{U}}\right) \right) - B.\end{aligned} \tag{19}$$

Therefore, after the last step v, LOMAR must satisfy

$$R_v \geq \rho \left(R_v^\pi + \tilde{G}\left(\{\mathcal{V}_{u,v}\}_{u\in\mathcal{U}}, \{\mathcal{V}_{u,v}^\pi\}_{u\in\mathcal{U}}\right) \right) - B \geq \rho R_v^\pi - B,$$

where R_v and R_v^π are the total rewards of LOMAR and the expert algorithm π after the last step $v = |\mathcal{V}|$, respectively. Thus, we complete the proof for the free-disposal setting.