Automated Market Making and Loss-Versus-Rebalancing*

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Abstract

We consider the market microstructure of constant function market makers (CFMMs) from the perspective of passive liquidity providers (LPs). In a Black-Scholes setting, we compare the CFMM's performance to that of a *rebalancing strategy*, which replicates the CFMM's trades at market prices. The CFMM systematically underperforms the rebalancing strategy, because it executes all trades at worse-than-market prices. The performance gap between the two strategies, "loss-versus-rebalancing" (LVR, pronounced "lever"), depends on the volatility of the underlying asset and the marginal liquidity of the CFMM bonding function. Our model's expressions for CFMM losses match actual losses from the Uniswap v2 WETH-USDC pair. LVR provides tradeable insight in both the *ex ante* and *ex post* assessment of CFMM LP investment decisions, and can also inform the design of CFMM protocols.

1. Introduction

In recent years, automated market makers (AMMs) and, more specifically, constant function market makers (CFMMs) such as Uniswap [Adams et al., 2020, 2021], have emerged as the dominant mechanism for decentralized exchange on blockchains. Compared to electronic limit order books (LOBs), which are the dominant market structure for traditional, centralized exchange-based electronic markets, CFMMs offer some advantages. First of all, they are efficient computationally. They have minimal storage needs, and matching computations can be done quickly, typically via

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constant-time closed-form algebraic computations. In an LOB, on the other hand, matching engine calculations may involve complex data structures and computations that scale with the number of orders. Thus, CFMMs are uniquely suited to the severely computation- and storage-constrained environment of the blockchain. Second, LOBs are not well-suited to a "long-tail" of illiquid assets. This is because they require the participation of active market makers. In contrast, CFMMs mainly rely on passive liquidity providers (LPs).

The goal of this paper is to understand the returns to providing liquidity in an CFMM, in a manner which is inspired by the Black and Scholes [1973] model of option pricing. The Black-Scholes model builds on the insight that options can be replicated by dynamically trading the underlying stock. From this insight, the model can be used to analyze option returns both qualitatively and quantitatively. Qualitatively, the Black-Scholes model shows how option returns are related to the underlying stock's return, volatility, and other parameters. Quantitatively, the model is realistic enough that it can be used to price options, by plugging in values for model parameters.

Analogous to the Black-Scholes approach, we aim to construct a model of CFMM LP returns, which both delivers qualitative insights about the factors that affect LP profitability, and is quantitatively realistic enough to bring to data. We begin with the idea of replicating the CFMM's trades by dynamically trading the underlying asset at market prices; we call this trading scheme the *rebalancing strategy*. The CFMM LP position systematically underperforms relative to the rebalancing strategy; we call the performance gap *loss-versus-rebalancing*, (or LVR, pronounced "lever"). The source of underperformance is *price slippage*: due to the passive nature of CFMM liquidity provision, whenever risky asset prices move, CFMMs trade at worse-than-market prices. We derive a simple expression for LVR, which depends only on two parameters: the volatility of the underlying asset, and the *marginal liquidity* of the CFMM bonding function. We then use our model to empirically analyze the Uniswap v2 WETH-USDC pair. Our model quantitatively performs well in matching LP returns. Our results have implications for measuring the returns to providing liquidity for CFMMs, as well as for redesigning CFMM to limit LVR and thus decrease the effective trading fees charged to CFMM traders.

We model trading between a risky asset and a numéraire. The two assets can be traded on a CFMM and a centralized exchange (CEX). We assume the CEX is infinitely deep, so the risky asset can be traded on the CEX with no price impact. As in the Black-Scholes model, we assume the risky asset's price follows a geometric Brownian motion with possibly stochastic volatility. The CFMM is described by an invariant curve $f(x, y) = L$; the CFMM is willing to make any trade such that it stays on this level curve. There are two kinds of traders in the model. Noise traders trade with the CFMM, contributing fees to CFMM LPs. Arbitrageurs trade with the CFMM and the CEX to maximize profits. We assume arbitrageurs pay no fees, implying that arbitrageurs ensure that the CFMM's price is always equal to the CEX price.

We define the *rebalancing strategy* as a trading strategy which holds whatever amount of the risky asset the CFMM holds at any point in time, but adjusts its positions in the risky asset by trading at CEX prices, rather than CFMM prices. Shorting the rebalancing strategy effectively *delta-hedges* the CFMM LP position. We show that, ignoring fees, CFMM LPs always do worse than the rebalancing strategy. We define loss-versus-rebalancing, or LVR, as the gap between the rebalancing strategy's performance, and the CFMM LP's performance. The intuition for this underperformance is related to the phenomenon of "sniping" in high-frequency trading settings. In the model of Budish et al. [2015], a market maker quotes prices to trade a risky asset. Whenever public information arrives causing the fair price of the risky asset to move, there is a "speed race" between the quoting market maker to cancel her order, and other traders to "snipe" the market maker's stale quotes.

CFMMs can be thought of as quoting market makers who never proactively update their price quotes; they only ever change prices in response to trades. Thus, whenever CEX prices move, CFMM quotes become "stale", giving arbitrageurs opportunities to profit by "sniping" the CFMM, until the point where CFMM prices are equal to CEX prices. CFMMs thus lose money from *price slippage*: every trade which the CFMM makes is executed at slightly worse prices than the rebalancing strategy, which buys and sells at CEX prices. LVR consists of the aggregate losses incurred from such price slippage.

Instantaneous LVR depends on only two parameters: the instantaneous variance of asset prices, and the marginal liquidity available — the slope of the CFMM's demand function for the risky asset — at the current price level in the pool. That is, CFMM losses from price slippage are greater when prices move more, and when the CFMM trades more aggressively in response to price movements. Asset price volatility is straightforwards to measure, and marginal liquidity can be calculated based on the formula for a CFMM's level sets, implying that our model can be used to measure LVR for any asset pair and CFMM invariant empirically.

The Black-Scholes model also implies that options can be delta-hedged by trading the underlying stock; a delta-hedged call option is a pure bet on whether the volatility implied by option prices is greater than realized volatility. Analogously, the concept of LVR can be used the basis of a trading strategy involving delta-hedging LP positions. A portfolio which holds a long position in the CFMM LP, and a short position in the rebalancing strategy, is always hedged to first-order at any point against directional movements in the risky asset's prices. At any point in time, the position is thus a bet on whether accrued trading fees are large enough to compensate for LVR losses due to price slippage; the strategy profits if fees are large relative to the product of price volatility and marginal liquidity, and loses money otherwise.

We use our model to empirically analyze the Uniswap v2 ETH-USDC trading pair. Unhedged LPing on ETH-USDC is very risky; however, this is mostly due to the fact that LPs are exposed to market risk in ETH prices. We show that *hedged LPing* — taking a long position in the CFMM LP, and a short position in the rebalancing strategy — is substantially less risky, with a Sharpe ratio of up to 18.2, depending on the rebalancing frequency. Moreover, our model-predicted LVR is able to match empirical hedged LP returns fairly well.

Next, we discuss connections between CFMM LP positions and the three classical ways that volatility can be traded: static (European) options, dynamic trading strategies, and variance swaps

[Carr and Madan, 2001]. We model the AMM reserves at the static payoff of a pool value function. This relates to Clark [2020], Fukasawa et al. [2022], and Deng and Zong [2022], who show that AMM LP payoffs, over any finite time horizon, can be replicated by shorting a bundle of European options. These option positions, in turn, are equivalent to dynamic trading strategies which sells (buys) the asset when prices increase (decrease). In our setting, the rebalancing strategy plays the role of delta-hedging. Finally, an delta-hedged LP position can be thought of as a generalized variance swap, whose payoff over any given time period is equal to realized variance weighted by the marginal liquidity of the AMMs.

A common benchmark used by practitioners to measure CFMM LP losses is "impermanent loss". Impermanent loss compares the performance of a CFMM LP position to a portfolio which simply holds the LP's initial bundle of assets; this differs from LVR, which compares LP performance to the rebalancing strategy. On the one hand, we show that the risk-neutral expectation of LVR and impermanent loss — and in fact with any other benchmark strategy which trades at market prices — is the same. On the other hand, LVR is the unique choice of benchmark which eliminates differences attributable to market risk. Mathematically, loss of an AMM LP position relative to any other benchmark can be thought of as LVR, plus a noise term due to difference between the market risk exposures of the benchmark and the AMM LP.

Our results have implications for AMM design. LVR can be used by CFMM protocol designers for guidance to set fees. This is because in a competitive market for liquidity provision, there should be no excess profits for LPs, and hence fees should balance with LVR. For example, since LVR scales with variance, one might imagine fee mechanisms that also scale with variance. Or, alternatively, protocols could be constructed that compare LVR versus fee income in a backward looking window, increasing fees if they are below LVR, and decreasing fees if they are above LVR. More speculatively, our results suggest a potential approach to redesign CFMMs to reduce or eliminate LVR: a CFMM which has access to a reliable and high-frequency price oracle could in principle quote prices arbitrarily close to market prices for the risky asset, thus eliminating losses from price slippage, and achieving payoffs arbitrarily close to that of the rebalancing strategy. Relatedly, CFMMs could sell special rights to arbitrage LPs to special wallets, "capturing" expected LVR and redistributing the profits to AMM LPs.

Related literature. Automated market makers have their origin in the classic literature on prediction markets and market scoring rules; see Pennock and Sami [2007] for a survey of this area. Constant function market makers, which are characterized by a invariant or bonding function, build on the utility-based market making framework of Chen and Pennock [2007]. In that framework, utility indifference conditions define a bonding function for binary payoff Arrow-Debreu securities. More recent interest in CFMMs has been prompted by an entirely new application: its functioning as a decentralized exchange mechanism, first proposed by Buterin [2016] and Lu and Köppelmann [2017]. The latter authors first suggested a constant product market maker, this was first analyzed by Angeris et al. [2019]. Angeris and Chitra [2020] and Angeris et al. [2021a,b] apply tools from convex analysis (e.g., the pool reserve value function) to study the more general case of constant function market makers, we employ some of those tools here. Angeris et al. [2021b] also analyze arbitrage profits, but do not relate them to the rebalancing strategy or express them in closed-form. A separate line of work seeks to design specific CFMMs with good properties by identifying good bonding functions [Port and Tiruviluamala, 2022, Wu and McTighe, 2022, Forgy and Lau, 2021, Krishnamachari et al., 2021].

Our paper relates to a sizable recent literature on automated market makers. Lehar and Parlour [2021] compare liquidity provision in limit order books and AMMs. In their model, as in ours, liquidity providers make profits from liquidity traders, and lose when risky asset prices move and arbitrageurs "snipe" stale CFMM quotes. Lehar and Parlour [2021] show theoretically and empirically that equilibrium pool size is smaller when asset volatility is higher, and characterize a number of other stylized facts of Uniswap liquidity pools. Capponi and Jia [2021] also show that CFMM LPs suffer losses when risky asset prices move, analyzing both the "rebalancing" arbitrage we study in this paper, as well as "reversal" arbitrage from exploiting noise traders. Capponi and Jia [2021] calculates the optimal convexity of the CFMM invariant, for trading off losses from arbitrage and increased price impact from investors.

A number of papers theoretically and empirically analyze DEX fees, and their effects on trade volume and price efficiency [Lehar et al., 2022, Hasbrouck et al., 2022]. Barbon and Ranaldo [2021], Foley et al. [2023], Lehar and Parlour [2021], and Han et al. [2021] empirically compare price impact, price efficiency, and net trading fees on centralized and decentralized exchanges. Other papers analyzing CFMMs include Brolley and Zoican [2023], Aoyagi [2020], Aoyagi and Ito [2021], Park [2021], and Fang [2022]. Arbitrage profits are a form of miner extractable value (MEV). Qin et al. [2022] empirically quantifies this and other types of CFMM related MEV, including "sandwich" attacks. Sandwich attacks are also considered by Zhou et al. [2021] empirically.

Black-Scholes-style options pricing models, like the ones developed in this paper, have been applied to weighted geometric mean market makers over a finite time horizon by Evans [2020], who also observes that constant product pool values are a super-martingale because of negative convexity. Clark [2020] replicates the payoff of a constant product market over a finite time horizon in terms of a static portfolio of European put and call options. Tassy and White [2020] compute the growth rate of a constant product market maker with fees. Lambert [2022] considers a number of related issues. Boueri [2021] considers the profitability of geometric mean market makers under geometric Brownian motion dynamics, his (re-)definition of "impermanent loss" in that setting is equivalent to LVR. Contemporaneous with the present work, the "convexity cost" component of the "predictable loss" of Cartea et al. [2022] is also equivalent to LVR.

We make a number of contributions relative to the literature. Qualitatively, our model highlights that the losses of CFMM LPs should be thought of as arising from *price slippage*, due to systematically trading at worse-than-market prices, in a manner similar to the "quote-sniping" losses of market makers in high-frequency trading models [Budish et al., 2015, Biais et al., 2015, Baldauf and Mollner, 2020, Aquilina et al., 2022]. This insight has implications for redesigning CFMMs to eliminate these losses. Besides this qualitative contribution, an important feature of our model is that it is quantitatively realistic, using continuous time and price spaces instead of the finite-time, finite-state models used in much of the prior literature. Our model is thus more amenable to fitting to data in practice; we show that our model matches empirical losses of CFMM LPs well. Moreover, our model highlights a simple trading strategy — delta-hedging the LP position by shorting the rebalancing strategy — which eliminates the market risk exposure of the CFMM, isolating the tradeoff between trading fees and sniping losses. Empirically, the delta-hedged LP strategy has much lower risk than the pure LP strategy.

Outline. The paper proceeds as follows. Section 2 describes our model. Section 3 contains our main results. Section 4 shows expressions for loss-versus-rebalancing for a number of CFMM invariants used in practice. Section 5 contains our empirical analysis. Section 6 discusses the relationship of CFMM LPs to options and other ways to trade volatility. Section 7 discusses the relationship between LVR and other benchmarks, such as impermanent loss. We discuss practical implications of our results for AMM design in Section 8, and conclude in Section 9. Proofs and supplementary results are contained in the appendix.

2. Model

In what follows, we describe the frictionless, continuous-time Black-Scholes setting of our model.

Assets. Fix a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$ where $\mathbb{Q}$ is a risk-neutral or equivalent martingale measure, satisfying the usual assumptions. Suppose there are two assets,[1] a risky asset x and a numéraire asset y. Without loss of generality, assume that the risk-free rate is zero. There is an infinitely deep centralized exchange, where the risky asset can be traded with zero fees. The price on the centralized exchange is observable, and evolves exogenously according to a geometric Brownian motion that is a continuous $\mathbb{Q}$-martingale, i.e.,

$$\frac{dP_t}{P_t} = \sigma_t \, dB_t^{\mathbb{Q}}, \quad \forall \, t \geq 0,$$

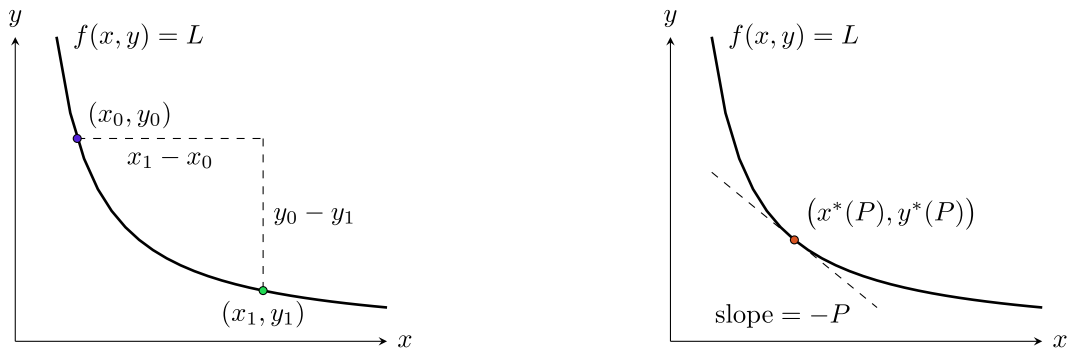
with a stochastic volatility process[2] given by $\sigma_t > 0$, and where $B_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion.

CFMM pool. The state of a CFMM pool is characterized by the reserves $(x, y) \in \mathbb{R}_+^2$, which describe the current holdings of the pool in terms of the risky asset and the numéraire, respectively. Define the feasible set of reserves $\mathcal{C}$ according to

$$\mathcal{C} \triangleq \{(x, y) \in \mathbb{R}_+^2 \ : \ f(x, y) = L\},$$

[1] This assumption is without loss of generality, we describe the multi-dimensional case where there are $n \geq 2$ assets, none of which need be the numéraire, in Appendix B.3.

[2] Volatility will be an important input in the analysis that follows. A natural question is how to calibrate volatility as a model parameter. As in the general application of Black-Scholes style models, for *ex ante* analysis of a possible future LP position, an implied volality is the appropriate input. On the other hand, for *ex post* LP return performance analysis as in Section 5, a historical or realized volatility is appropriate.



(a) Transitions between any two points on the bonding curve $f(x, y) = L$ are permitted, if an agent contributes the difference into the pool.

(b) Arbitrageurs ensures that, when the price is P, pool reserves shift to the point on the bonding curve where the slope is equal to $-P$.

Figure 1: Illustration of a CFMM.

where $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ is referred to as the *bonding function* or *invariant*, and $L \in \mathbb{R}$ is a constant. In other words, the feasible set is a level set of the bonding function. The pool is defined by a smart contract which allows an agent to transition the pool reserves from the current state $(x_0, y_0) \in \mathcal{C}$ to any other point $(x_1, y_1) \in \mathcal{C}$ in the feasible set, so long as the agent contributes the difference $(x_1 - x_0, y_1 - y_0)$ into the pool; see Figure 1a.

Example 1. *The constant product market maker is defined by the invariant $xy = L$.*

To simplify our analysis, we will also assume that, aside from trading with arriving liquidity demanding agents, the pool is static otherwise. In particular, we assume that the liquidity providers do not add ("mint") or remove ("burn") reserves over the time scale of our analysis. In other words, LPs are *passive*. Further, we ignore the details of the underlying blockchain on which the pool operates. In particular, we assume away any blockchain transaction fees such as "gas" fees, and also ignore the discrete-time nature of block updates.

Besides liquidity providers, there are two kinds of agents in the model: arbitrageurs and noise traders.

Arbitrageurs. There is a population of arbitrageurs, able to frictionlessly trade at the external market price, continuously monitoring the CFMM pool. When an arbitrageur interacts with the pool, we assume they maximize their immediate profit by exploiting any deviation from the external market price. In other words, they transfer the pool to a point in the feasible set $\mathcal{C}$ that allows them to extract maximum value assuming that they unwind their trade at the external market price P. Geometrically, the presence of arbitrageurs implies that, if the price of the risky asset is P, pool reserves will move to the point on the curve $f(x, y) = L$ where the slope of the bonding curve is equal to $-P$, as indicated in Figure 1b.

Noise traders. There is also a population of noise traders. Noise traders trade only in the CFMM pool, and trade for totally idiosyncratic reasons. There are many reasons in practice why certain market participants might prefer trading on CFMMs to CEXes: for example, certain market participants may not be able or willing to satisfy the know-your-customer requirements imposed by CEXes, or may not be willing to bear the credit risk associated with custodial centralized exchanges. Noise traders' trades have an initial impact on CFMM pool prices, but these effects are immediately offset by arbitrageurs, who immediately move the CFMM back to the CEX price P. Thus, from the LP's perspective, noise traders simply contribute a flow of fees. Denote by FEE_T the cumulative fees paid by noise traders up to time T. For simplicity, we assume fees are paid in units of the numéraire; this simplifies the analysis, since fees do not affect the level curve that the CFMM trades on.[3] In practice, fees are sometimes (e.g., in Uniswap v2 but not in Uniswap v3) reinvested into the pool reserves; another way to think about this assumption is to assume LPs immediately withdraw any accrued fees from the CFMM.

When discussing arbitrageurs, we will distinguish between two conceptually different forms of arbitrage activity. The first, which we call *rebalancing arbitrage*, is arbitrage of a pool when mispricing arises due to movements of the CEX price. The second, which we call *reversion arbitrage*, is arbitrage following the arrival of noise traders who move DEX prices away from P — this type of arbitrage is sometimes called "back-running". Our model allows us to quantify the magnitude of profits of rebalancing arbitrageurs, but not reversion arbitrageurs.

2.1. Discussion of Assumptions

We assume the CEX is infinitely deep, so trades have no price impact; this is analogous to the assumption in the Black-Scholes model that trades of the underlying stock have no price impact. In practice, for liquid trading pairs such as USDC-ETH, this assumption is likely to hold approximately in practice: a large majority of trade volume in the USD-ETH pair, around 90%, occurs on centralized exchanges relative to decentralized exchanges, suggesting that market depth on CEXes is likely higher than on DEXes. For less liquid tokens, which are not traded on centralized exchanges, this assumption may be less realistic. In this case, our model may still be a useful conceptual benchmark, analogous to the use of option pricing models to value options with illiquid underlying assets, such as employee stock options in privately held companies.

A number of other papers, such as Hasbrouck et al. [2022], Barbon and Ranaldo [2021], and Foley et al. [2023], analyze microfounded models of strategic liquidity provision on AMMs. We do not take a stance on the behavior of liquidity providers in this paper; instead, we simply take as given the level set which the CFMM LP is on at any given point in time. We will show that LVR only depends on price volatility, and the marginal liquidity of the CFMM level set, both of which are observable objects. Thus, given price volatility, any model of liquidity providers' strategic behavior which leads the CFMM LP to reach a given level set implies the same level of LVR. The cost of not modelling strategic LP behavior is that our framework cannot make sharp predictions about how the level of CFMM liquidity provision responds to changes in market design. However, the benefit is that our quantification of CFMM LP losses is robust to different underlying models of strategic LP behavior.

[3] The same assumption is made by Lehar and Parlour [2021].

We assume away many frictions to trading which are present in practice: we assume arbitrageurs pay no trading fees on CEXes or DEXes, we ignore gas fees, and we ignore the discrete, block-based nature of trading on blockchain CFMMs. As in Black-Scholes, these approximations allow us to derive particularly simple expressions for model outcomes. Accounting for fees will imply that the profits arbitrageurs make will tend to be lower than our expressions. In particular, in practice, arbitrageurs tend to engage in "gas races", bidding high gas fees so that block miners have an incentive to include their arbitrage trades in the blockchain first. These gas races will tend to redirect some of the profits from CEX-DEX arbitrage towards block miners.[4] We will show in our empirical analysis of Section 5 that our model appears to match the actual delta-hedged returns of LP positions from the data fairly closely, suggesting that the omission of fees from our model has a quantitatively small effect on estimated LP returns in the examples we analyze. Moreover, we note that the analysis of arbitrage profits in the presence of fees is the subject of follow-on work [Milionis et al., 2023], and we defer a more careful discussion of the impact of fees to that work. We also assume noise trader fees are paid in the numéraire, and we assume away minting and burning of LP shares for simplicity. However, we relax both these assumptions in the empirical application in Section 5.

3. Loss-Versus-Rebalancing

We proceed to analyze losses of CFMM LPs in the context of our model.

The pool value function $V(P)$. Figure 1b shows that the composition of the CFMM's reserve pool depends only on the risky asset's price: at any time t, if the risky asset's price is P_t, arbitrageurs will move the pool's reserves to the point on the $f(x, y)$ curve where the slope is P. The mark-to-market value of the pool's reserves at any point in time, $P_t x_t + y_t$, is thus also fully determined by the current price P_t. A convenient way to analyze the monetary value of pool reserves at any given point in time is to define the the *pool value function* $V : \mathbb{R}_+ \to \mathbb{R}_+$, as the solution to the optimization problem:

$$V(P) \triangleq \begin{array}{ll} \underset{(x,y)\in\mathbb{R}^2_+}{\text{minimize}} & Px + y \\ \text{subject to} & f(x,y) = L. \end{array} \tag{1}$$

The intuition behind (1) is that, at any point in time, arbitrageurs can access any point on the invariant curve $f(x, y) = L$; arbitrageur profits are maximized by minimizing the value of the pool's reserves. The minimizing choice of x, y is the tangency point illustrated in Figure 1b; $V(P)$ measures the monetary value of reserves, $Px + y$, at this minimal point. If we denote by V_t the value of pool reserves at time t, the presence of arbitrageurs implies that at any point, V_t is equal to $V(P_t)$. Geometrically, the pool value function implicitly defines a reparameterization of the pool state from primal coordinates (reserves) to dual coordinates (prices).

[4] CEX-DEX arbitrage is one form of "miner extractable value", or MEV, a set of circumstances in which miners' ability to determine the ordering of transactions allows them to extract monetary value; Daian et al. [2020] discusses MEV in detail.

We assume that the pool value function satisfies:

Assumption 1. *1. An optimal solution $(x^*(P), y^*(P))$ to the pool value optimization (1) exists for every $P \geq 0$.*

2. The pool value function $V(\cdot)$ is everywhere twice continuously differentiable.

3. For all $t \geq 0$,

$$\mathsf{E}^{\mathbb{Q}}\left[\int_0^t x^*(P_s)^2 \sigma_s^2 P_s^2 \, ds\right] < \infty.$$

Parts 1–2 are easily verified for many CFMMs, see Section 4 for examples. Part 3 is a square-integrability condition that will be used in Section 3. Parts 1–2 are a sufficient condition for the following:

Lemma 1. *For all prices $P \geq 0$, the pool value function satifies:*

1. $V(P) \geq 0$.

2. $V'(P) = x^*(P) \geq 0$.

3. $V''(P) = x^{*\prime}(P) \leq 0$.

Proof. The first part follows from the fact that $\mathcal{C} \subset \mathbb{R}_+^2$ and $P \geq 0$. The second part is the envelope theorem or Danskin's theorem [Bertsekas, 1971]. The third part follows from the concavity of $V(\cdot)$, as a pointwise minimum of a collection of affine functions. ■

Part 2 of Lemma 1 establishes that the slope of the pool value function is equal to the reserves in the risky asset. Part 3 establishes that the pool value function is concave. Note that this concavity does not depend on the nature of the feasible set $\mathcal{C}$ or the bonding function $f(\cdot)$. This part also establishes that the second derivative of the pool value function is the marginal liquidity available at the price level.

Note also that the optimization problem in (1) is isomorphic to the *expenditure minimization problem* from classical demand theory: under price P, a consumer minimizes total expenditures $Px + y$, subject to staying on the indifference curve $f(x, y) = L$. The solution to this problem is to choose the point where the level curve of $f(x, y)$ is tangent to the budget set. The envelope theorem thus gives that the first derivative of the expenditure function is the Hicksian demand function, which is isomorphic to $x^*(P)$, and the second derivative of the expenditure function is the slope of Hicksian demand.

The pool value function allows us to write the profit and loss of an CFMM, from time 0 to time t, as:

$$\text{LP } \mathsf{P\&L}_t = V_t - V_0 + \mathsf{FEE}_t. \tag{2}$$

In words, LP $\mathsf{P\&L}_t$ is the monetary value of the pool reserves at time t, minus the value at time 0, plus the cumulative fees collected until time t.

Rebalancing strategy R_t**.** The key idea of our paper is to decompose the change in pool value $V_t - V_0$ into the sum of the returns on a particular trading strategy, which we call the *rebalancing strategy*, and a residual term. Informally, the rebalancing strategy aims to hold exactly the same amount of the risky asset as the CFMM at any point in time. Whenever prices move in a way which causes the CFMM to buy or sell the underlying strategy, the rebalancing strategy makes exactly the same buys and sells; however, the rebalancing strategy executes these trades at CEX prices, rather than CFMM prices. An alternative way to think of the rebalancing strategy is that it aims to replicate the exposure of the CFMM to the risky asset at any point in time. Thus, taking a long position in the CFMM LP, and a short position in the rebalancing strategy, *delta hedges* the LP position, neutralizing first-order exposure to shifts in the risky asset's price.

We first define general trading strategies. A trading strategy is a process (x_t, y_t) defining holdings in the risky asset and numéraire at each time t. For a trading strategy to be *admissible*, we require that it be adapted, predictable, and satisfy

$$\mathsf{E}^{\mathbb{Q}}\left[\int_0^t x_s^2 \sigma_s^2 P_s^2 \, ds\right] < \infty, \quad \forall\, t \geq 0. \tag{3}$$

We further restrict admissible trading strategies to be *self-financing*, i.e., to satisfy

$$\underbrace{x_t P_t + y_t - (x_0 P_0 + y_0)}_{\mathsf{P\&L}_t} = \int_0^t x_s \, dP_s, \quad \forall\, t \geq 0. \tag{4}$$

Equation (4) states that the change in the profit of the strategy in a small period of time is equal to the holdings of the risky asset, x_s, times the change in price, dP_s. The total P&L of the strategy is just the integral of these instantaneous changes. Intuitively, a self-financing strategy executes all rebalancing trades at market prices; hence, trades do not affect the profit of the strategy, and no money needs to be injected into the trading strategy. In the special case where P_t is a martingale, so the expected profit from holding the risky asset is zero, any self-financing strategy makes zero profits in expectation, since any strategy which dynamically trades a asset with zero expected returns also has zero expected returns.[5]

The self-financing condition implies that, if we specify y_0, the initial amount of the numéraire, and $\{x_t\}_{t\geq 0}$ the amount of the risky asset that is held in any possible future history, the future path of the numéraire $\{y_t\}$ is implicitly determined through (4). The P&L of the resulting self-financing strategy can be directly expressed in terms of $\{x_t\}$, via the right side of (4). Intuitively, the RHS of (4) is the integrated form of the envelope formula: if the trading strategy holds a position x_t in the risky asset, the change in profits in an instant dt is $x_t dP_t$, the amount of the risky asset held times the change in price. Note that, since P_t is a $\mathbb{Q}$-martingale, the P&L process given by (4) is also $\mathbb{Q}$-martingale, and by (3) it is square-integrable.

[5]In the general case, the risky asset may have positive expected returns due to risk premia; self-financing strategies may thus have positive expected profits, proportional to how much portfolio weight they put on the risky asset. However, the positive expected returns of the strategy derive only from the risk premia on the underlying asset: if the strategy is delta-hedged, it makes zero expected profits.

We then define the rebalancing strategy to be the self-financing trading that starts initially holding $(x^*(P_0), y^*(P_0))$ (the same position as the CFMM), and continuously and frictionlessly rebalances to maintain a position in the risky asset given by $x_t \triangleq x^*(P_t)$. Let R_t denote the monetary value of the rebalancing strategy at time t; that is, if the rebalancing strategy holds x_t, y_t at time t, R_t is $P_t x_t + y_t$. Applying the self-financing condition (4) the rebalancing portfolio has value:

$$R_t = V_0 + \int_0^t x^*(P_s)\, dP_s, \quad \forall\, t \geq 0. \tag{5}$$

Because of Assumption 1 Part 3, the rebalancing strategy is admissible and R_t is a square-integrable $\mathbb{Q}$-martingale. In particular, being a self-financing strategy, the rebalancing strategy breaks even in expectation under the risk-neutral measure $\mathbb{Q}$; it only makes expected returns to the extent that the underlying risky asset has nonzero risk premia.

As a matter of accounting, we then express the change in pool value from time 0 to time t as the sum of the rebalancing strategy's profits, and a residual term which we will define as *loss-versus-rebalancing*:

$$V_t - V_0 = V_0 + \mathsf{LVR}_t$$

$$\mathsf{LVR}_t \triangleq R_t - V_t \tag{6}$$

LVR_t can also be thought of as the losses from a delta-hedged LP position, ignoring fees. In other words, a strategy which takes a long position in the CFMM LP position, and a short position in the rebalancing strategy, pays $V_t - R_t$ at time t, disregarding any fees collected. The core contribution of our paper is the characterization of LVR_t in the following theorem.

Theorem 1. *Loss-versus-rebalancing takes the form:*

$$\mathsf{LVR}_t = \int_0^t \ell(\sigma_s, P_s)\, ds, \quad \forall\, t \geq 0, \tag{7}$$

where we define, for $P \geq 0$, the instantaneous LVR *by:*

$$\ell(\sigma, P) \triangleq \frac{\sigma^2 P^2}{2} \left| x^{*\prime}(P) \right| \geq 0. \tag{8}$$

$\ell(\sigma, P)$ is always positive, so LVR *is a non-negative, non-decreasing, and predictable process. Moreover, the cumulative profits of rebalancing arbitrageurs up to time t is equal to* LVR_t.

A rigorous proof that LVR is equal to (12), and that rebalancing arbitrage profits are equal to LVR, is contained in Appendix A. Here, we present an intuitive derivation, based on Figure 2. The core intuition is that the CFMM systematically loses money relative to the rebalancing strategy due to *price slippage*: every trade made by the CFMM is made at slightly worse prices than the rebalancing strategy.

Suppose the market price changes from P_t to $P_t + dP_t$. Arbitrageurs thus trade with the CFMM, moving from point A to point B on the CFMM invariant curve. Let dx_t denote the amount of the risky asset sold, indicated by the green horizontal line. When the price moves from P_t to $P_t + dP_t$,

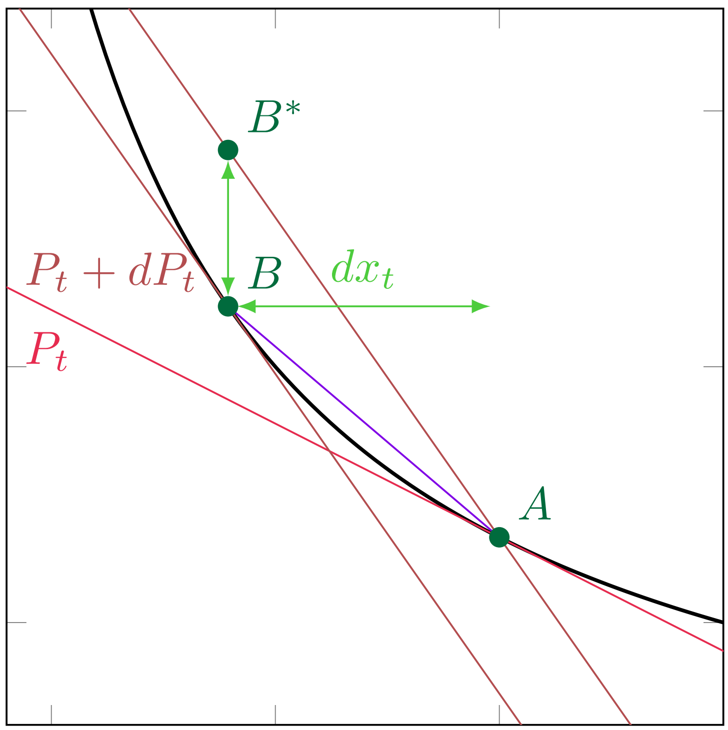


Figure 2: LVR and a stylized depiction of CFMM LP price slippage. Suppose prices begin at P_t, the slope of the red line; the CFMM reserves then begin at point A. If prices increase to $P_t + dP_t$, the slope of the brown line, the CFMM trades to point B. The rebalancing strategy trades instead at the price $P_t + dP_t$, to point B^*. LVR is the vertical gap between B and B^*.

the rebalancing strategy sells exactly the same amount, dx_t, of the risky asset as the CFMM does. However, the rebalancing strategy trades at the CEX price, $P_t + dP_t$; it trades along the the brown line of slope $P_t + dP_t$ passing through A, reaching point B^*, which is higher than point B. Thus, after prices move, the LP position and the rebalancing strategy hold the same amount of the risky asset, but the rebalancing strategy holds more cash. The gap is equal to the height of the line connecting B and B^*.

To calculate the height of the $B - B^*$ line, note that the rebalancing strategy trades at the slope of the brown line, which is $P_t + dP_t$. The CFMM trades at the slope of the purple line — that is, the secant line connecting points A and B. Since the tangency lines at points A and B have slopes P_t and $P_t + dP_t$ respectively, the secant line has slope $P_t + \frac{dP_t}{2}$. Thus, the $B - B^*$ line has height:

$$dx_t \left((P_t + dP_t) - \left(P_t + \frac{dP_t}{2} \right) \right) = \frac{dx_t \, dP_t}{2}. \tag{9}$$

This is thus the loss of the CFMM, relative to the rebalancing strategy. This is also exactly the profit extracted by arbitrageurs when prices move: arbitrageurs purchase quantity dx from the CFMM at price $p + \frac{dp}{2}$, and selling to the CEX at price $P_t + dP_t$, thus earning a profit of $\frac{dx_t dP_t}{2}$.

Next, we write the amount traded dx_t as a function of dP_t:

$$dx_t = \left| \frac{dx^*(P)}{dP} \right| dP_t = \left| x^{*\prime}(P_t) \right| \, dP_t, \tag{10}$$

where $x(P_t)$ is the demand function of the CFMM. $|x^{*\prime}(P_t)|$ can be thought of as the marginal liquidity provided by the CFMM: how much of the risky asset it trades when prices move a small amount.

Expression (9) then becomes:

$$\frac{dx_t \, dP_t}{2} = \left| x^{*\prime}(P_t) \right| \frac{(dP_t)^2}{2}. \tag{11}$$

Now, for a geometric Brownian motion, in a small amount of time dt, the quadratic variation $(dP_t)^2$ is equal to $\sigma_t^2 P_t^2$, that is, the instantaneous variance σ_t^2 multiplied by the square of the price. Hence, plugging in to (11), in any instant of time dt, the CFMM LP position loses:

$$\left| x^{*\prime}(P_t) \right| \frac{\sigma_t^2 P_t^2}{2}. \tag{12}$$

This is (8) of Theorem 1.

Figure 2 thus illustrates that LVR — that is, the losses CFMMs incur relative to the rebalancing strategy — arises entirely from *price slippage*. The fact that CFMMs rebalance, selling when prices rise and buying when prices fall, is not in itself the source of losses. The rebalancing strategy makes exactly the same trades of the risky asset as the CFMM LP position, but does not lose money because it executes all trades at CEX prices. For that matter, *any* trading strategy which executes all trades at CEX prices exactly breaks even under the risk-neutral measure. LVR arises from the fact that CFMMs execute all trades at worse-than-market prices.

Price slippage is intrinsic to the design of CFMMs. CFMMs are fully passive liquidity providers, making markets for risky assets without access to external price feeds from centralized exchanges. CFMMs rely on arbitrage to "inform" them about current market prices: whenever CEX prices move, the CFMM's quotes become "stale", offering to trade some quantity of the risky asset at better-than-market prices. Arbitrageurs trade the CFMM against the CEX until the CFMM's price is equal to the CEX price, and these profitable trades are exhausted. In other words, the slippage built into CFMM design is what gives arbitrageurs the incentive to align CFMM prices with CEX prices.

Our work is thus related to models of "quote sniping" in high-frequency trading. Budish et al. [2015] analyzes a model in which, whenever prices move, bid-ask quotes become "stale", creating a speed race between the quoting market maker to update her quotes, and arbitrageurs to "snipe" the stale quote. In these models, purely public information creates "informed-trader" risk, because arbitrageurs occasionally win speed races and are able to act on public information before the quoting market maker can. In relation to these models, CFMMs can be thought of like quoting market makers that, by design, never update prices proactively. An CFMM only ever increases its quoted price when it receives orders to buy the risky asset; in other words, CFMMs' price quotes only ever move when they are sniped. Thus, any movement in CEX prices causes CFMM quotes to become stale, creating a speed race to snipe the CFMM. CFMMs always lose these races, and loss-versus-rebalancing consists of the cumulative losses LPs suffer from getting "sniped" to trade at worse-than-market prices.

Comparative statics. Theorem 1 states that the magnitude of slippage losses only depends on two parameters of the model: instantaneous volatility, σ_t; and the marginal liquidity of the CFMM bonding function $|x^{*\prime}(P_t)|$. Intuitively, CFMMs lose money from trading at worse-than-market prices; they lose more when volatility σ_t is high and prices move more, and they lose more when marginal liquidity, $|x^{*\prime}(P_t)|$, is high, and the CFMM trades larger quantities when prices move. In Appendix A.2, we derive an expression for $|x^{*\prime}(P)|$ in terms of the CFMM bonding function $f(\cdot)$, and show that $|x^{*\prime}(P)|$ is related to the curvature of the level sets of $f(\cdot)$: CFMMs with "flatter" bonding curves have higher marginal liquidity.[6]

The idea that the losses of CFMM LP positions are related to volatility and curvature are not new to the finance literature [Aoyagi, 2020, Aoyagi and Ito, 2021, Lehar and Parlour, 2021, Capponi and Jia, 2021, O'Neill, 2022]. Our contribution is to build a model which, like the Black-Scholes model for option prices, both delivers qualitative comparative statics, and is also quantitatively realistic enough to be used to measure CFMM losses in practice.

Volatility versus informed trading. Our results are reminiscent of classic results in market microstructure, which state that market makers charge fees to make up for losses from adverse se-

[6] An interesting implication of these results for the design of CFMM invariants is that, in our model, under our assumptions, the only feature of CFMMs which matters for losses is the curvature of the CFMM bonding function, which determines $x^{*\prime}(P)$. Any two CFMM invariants for an asset pair which have the same local convexity at any given price have the same LVR.

lection. However, the literature on AMMs have analyzed two slightly different narratives for the source of the adverse selection CFMM LPs face. The first, reminiscent of microstructure models such as Glosten and Milgrom [1985], is that LPs lose money when traders with knowledge of *future* market prices trade with the CFMM, creating "wrong-way" risk. This channel is often called "informed trading". The second, which we focus on in this paper, is that LPs lose money when traders with knowledge of *current* market prices on the CEX snipe the CFMM LP. We will refer to this channel as "volatility" or "slippage".

In our model, the losses of CFMM LPs arise entirely from the latter "slippage" effect. Slippage is straightforwards to quantify, because it depends on easily measurable objects: the volatility of CEX price movements, and the CFMM's marginal utility. A number of recent papers in the finance literature have pointed out qualitatively that CFMM losses are linked to the volatility of the underlying asset; our contribution relative to these papers is to is to show that this relationship can be quantified, under exactly the canonical Black-Scholes model of risky asset prices.[7] Informed trading is more difficult to quantify in practice, since it requires estimating the extent to which order flow tends to be informative about future price movements. In the context of our model, since we assumed the CEX is infinitely deep, informed traders would only trade on the CEX, since they have lower price impact. Thus, in our model, informed trading does not directly contribute to adverse selection losses to CFMM LPs.

Decomposing LP P&L. Next, we plug the LVR expression into (2), to decompose the $\mathsf{P\&L}$ of the LP position into three components,

$$\text{LP } \mathsf{P\&L}_t = \underbrace{\mathsf{FEE}_t}_{\text{accumulated fees}} + \underbrace{V_t - V_0}_{\substack{\text{change in pool}\\ \text{reserve value}}} = \underbrace{\int_0^t x^*(P_s)\, dP_s}_{\text{market risk}} + \underbrace{\mathsf{FEE}_t - \mathsf{LVR}_t}_{\text{fees minus } \mathsf{LVR}} \tag{13}$$

The right side of expression (13) decomposes the profit of an CFMM LP position, from time 0 to time t, into three components. The first is "market risk", which from (5) is exactly the $\mathsf{P\&L}$ of the rebalancing strategy. The CFMM is long the risky asset; hence, at any given point in time, it accrues gains and losses when the risky asset's price fluctuates. However, market risk contributes nothing to the CFMM's profits under the risk-neutral measure; equivalently, the market risk component of the CFMM's profits can be costlessly hedged, simply by shorting however much of the underlying asset the CFMM holds at any point in time, as the rebalancing strategy does. Besides market risk, CFMM LP positions attain positive returns from the strictly increasing process FEE_t, and negative returns from the strictly decreasing process $-\mathsf{LVR}_t$.

The Black-Scholes framework for classic options indicates that the directional risk exposure of options can be hedged, simply by taking positions in the underlying asset. Option market makers use this principle in practice, delta-hedging the directional risk of their options portfolios, and profiting from collecting bid-ask spreads and betting on differences between realized volatility

[7] Outside of the finance literature, some earlier papers in the crypto literature have derived related quantification results, such as Angeris et al. [2020]. Our contribution relative to this literature is to construct a clean "rebalancing strategy" benchmark, and to link these results to ideas about adverse selection in market making.

and the volatility implied by option prices. Analogous to this, the decomposition in (13) also corresponds to a tradable strategy: one can *delta-hedge* CFMM LP positions, simply by combining a long position in the CFMM LP with a short position in the rebalancing strategy. The time-t payoff of the long-LP, short-rebalancing-strategy position is:

$$\text{delta-hedged LP } \mathsf{P\&L}_t = \text{LP } \mathsf{P\&L}_t - R_t = \underbrace{\mathsf{FEE}_t - \mathsf{LVR}_t}_{\text{fees minus } \mathsf{LVR}}. \tag{14}$$

Intuitively, the delta-hedged LP position is always short as much ETH in the rebalancing strategy as it is long in the LP position, and is thus insulated against directional movements in ETH prices. This is thus a pure bet on whether fees are large enough to offset losses from slippage.[8] Using the expression for LVR from Theorem 1, the delta-hedged CFMM LP position is thus affected by three components: the magnitude of trading fees, the volatility of the underlying asset, and the marginal liquidity of the CFMM LP curve.

4. Examples

In this section, we calculate LVR for a number of specific CFMM examples.

Example 2 (Weighted Geometric Mean Market Maker / Balancer)**.** *Consider the bonding function* $f(x,y) \triangleq x^\theta y^{1-\theta}$*, for* $\theta \in (0,1)$*. Solving the pool value optimization* (1) *allows us to obtain the closed-form optimal solutions*

$$x^*(P) = L\left(\frac{\theta}{1-\theta}\frac{1}{P}\right)^{1-\theta}, \quad y^*(P) = L\left(\frac{1-\theta}{\theta}P\right)^{\theta}.$$

Then,

$$V(P) = \frac{L}{\theta^\theta(1-\theta)^{1-\theta}}P^\theta, \quad V''(P) = -L\theta^{1-\theta}(1-\theta)^\theta\frac{1}{P^{2-\theta}},$$

and

$$\ell(\sigma, P) = \frac{\sigma^2}{2}\theta(1-\theta)V(P).$$

The weighted geometric mean market maker generalizes the constant product market maker. For these market makers, the instantaneous LVR normalized per dollar of pool reserves is constant, i.e.,

$$\frac{\ell(\sigma,P)}{V(P)} = \frac{\sigma^2}{2}\theta(1-\theta). \tag{15}$$

In fact, with a minor caveat, weighted geometric market makers are the *only* CFMMs for which this is true. We discuss this in Appendix B.2. Finally, observe that LVR is maximized when $\theta = 1/2$,

[8] Note, in addition, that delta-hedging an CFMM is in fact simpler than delta-hedging an option. Options cannot be delta-hedged in a model-free way: in the Black-Scholes framework, for example, the delta of an option depends on volatility. In contrast, CFMMs can be delta-hedged in a fully model-free way, since the rebalancing strategy simply shorts as much ETH as the LP position holds at any point in time.

and goes to zero as $\theta \to \{0,1\}$.[9]

Example 3 (Constant Product Market Maker / Uniswap v2)**.** *Taking $\theta = 1/2$ in Example 2, we have that*

$$V(P) = 2L\sqrt{P}, \quad \ell(\sigma, P) = \frac{L\sigma^2}{4}\sqrt{P}, \quad \frac{\ell(\sigma, P)}{V(P)} = \frac{\sigma^2}{8}. \tag{16}$$

This example shows that the constant product market maker admits particularly simple expressions for LVR: $\ell(\sigma, P)/V(P)$, the loss per unit time as a fraction of mark-to-market pool value, is simply 1/8 times the instantaneous variance. This formula is straightforward to apply empirically: for example, if the ETH-USDC volatility is $\sigma = 5\%$ (daily), this formula implies that the ETH-USD LP pool loses approximately $\sigma^2/8 = 3.125$ (bp) in pool value to LVR daily.

Example 4 (Uniswap v3 Range Order)**.** *For prices in the liquidity range $[P_a, P_b]$, consider the bonding function of Adams et al. [2021],*

$$f(x, y) \triangleq \left(x + L/\sqrt{P_b}\right)^{1/2} \left(y + L\sqrt{P_a}\right)^{1/2}.$$

Solving the pool value optimization (1),

$$x^*(P) = L\left(\frac{1}{\sqrt{P}} - \frac{1}{\sqrt{P_b}}\right), \quad y^*(P) = L\left(\sqrt{P} - \sqrt{P_a}\right).$$

Then, for $P \in (P_a, P_b)$,

$$V(P) = L\left(2\sqrt{P} - P/\sqrt{P_b} - \sqrt{P_a}\right), \quad V''(P) = -\frac{L}{2P^{3/2}},$$

so that

$$\ell(\sigma, P) = \frac{L\sigma^2}{4}\sqrt{P}.$$

Observe that the instantaneous LVR is the same in Example 3. However, the pool value $V(P)$ is lower. Indeed $V(P) \to 0$ if $P_a \uparrow P$ and $P_b \downarrow P$, so

$$\lim_{\substack{P_a \to P \\ P_b \to P}} \frac{\ell(\sigma, P)}{V(P)} = +\infty,$$

i.e., the instantaneous LVR per dollar of pool reserves can be arbitrarily high in this case, if the liquidity range is sufficiently narrow. This is consistent with the idea that range orders "concentrate" liquidity.

Example 5 (Linear Market Maker / Limit Order)**.** *For $K > 0$, consider the linear bonding function*

[9] See also Proposition 1 of Evans [2020], evaluating a weighted geometric mean market maker over a finite horizon using risk-neutral pricing.

$f(x,y) \triangleq Kx + y$. Solving the pool value optimization (1),

$$x^*(P) = \begin{cases} L/K & \text{if } P < K, \\ 0 & \text{if } P \geq K, \end{cases} \quad y^*(P) = \begin{cases} 0 & \text{if } P < K, \\ L & \text{if } P \geq K. \end{cases}$$

Hence, this pool can be viewed as similar to a resting limit order[10] that is, depending on the relative value of the price P_t versus limit price K, either an order to buy (if $P_t \geq K$) or an order to sell (if $P_t < K$) up to L/K units of the risky asset at price K. In this case,

$$V(P) = L \min \{P/K, 1\}.$$

Observe that $V(\cdot)$ does not satisfy the smoothness requirement of Assumption 1 Part 2: the first derivative is discontinuous at the limit price $P = K$. Thus, the characterization of Theorem 1 does not apply.[11]

5. Empirical Analysis

Next, we bring our model to data to evaluate whether LVR matches the returns of LP positions in practice. Repeating (14), we have

$$\underbrace{\text{LP } \mathsf{P\&L}_t - \int_0^t x^*(P_s)\, dP_s}_{\text{delta-hedged LP } \mathsf{P\&L}} = \underbrace{\mathsf{FEE}_t - \mathsf{LVR}_t}_{\text{fees minus } \mathsf{LVR}}. \tag{17}$$

The left side of (17) can be thought of as the P&L from a delta-hedged LP position: the P&L of the LP position, minus that of the rebalancing strategy. This quantity can be estimated empirically under very weak assumptions. The profits of the rebalancing strategy are simply the returns on a portfolio which holds just as much of the risky asset as the LP position holds at any point in time, adjusting holdings always at market prices. The P&L of an LP position over any period of time can be calculated simply as the mark-to-market value of pool reserves, at CEX prices at the start and end of the time period, accounting for mints, burns, swaps, and trading fees.[12]

[10] While the linear market maker is *statically* identical to a resting limit order, observe that they are *dynamically* different. In particular, once the price level K is crossed, in a traditional LOB, the limit order is filled and removed from the order book. With a linear market maker, the order remains in the pool at the same price and quantity, but with opposite direction.

[11] Note that the pool value function remains concave and the pool value process is a super-martingale. Hence, from the Doob-Meyer decomposition, a non-negative monotonic running cost process exists. However, this process is not described by (7)–(8). Instead, it can be constructed using the concept of "local time" and the Itô-Tanaka-Meyer formula, but we will not pursue such a generalization here [see, e.g., Carr and Jarrow, 1990].

[12] Note that delta-hedging an LP position does not incur any flow gas costs, since simply holding an LP position in a CFMM, without doing any minting or burning, does not require spending any gas. Thus, compared to executing this trading strategy in practice over a fixed time period, the only fees that the left side of (17) does not account for are the transaction fees from executing the rebalancing strategy on a CEX; any financing costs for maintaining a short position on a CEX; and two one-time gas costs, for minting an LP position at the start of the period and then burning it at the end of the period.

The right side of (17) can be thought of as our model's prediction for the delta-hedged $\mathsf{P\&L}$, i.e., left side of (17). The first term on the right side corresponds to trading fees, which are observable. The second term is LVR, which we can measure as a function of realized volatility using expressions (7) and (8) of Theorem 1. In this way, the degree to which the right side of (17) is close to the left side measures the effectiveness of LVR in quantifying LP returns.

We bring the model to data using the WETH-USDC trading pair[13] on Uniswap v2 for the period from August 1, 2021 to July 31, 2022. Details of the data sources we use, and how we measure various quantities, are described in Appendix C. Essentially, to measure the left side of (17), we measure the $\mathsf{P\&L}$ of an LP position simply as the mark-to-market value of pool reserves, periodically valuing "mints" and "burns" — that is, tokens withdrawn or deposited from the LP position — at market prices. We measure the profits of the rebalancing strategy simply by rebalancing to match the CFMM LP holdings at a number of different discrete time frequencies. For example, suppose we rebalance each minute, and suppose we observe that the CFMM LP position holds 10,000 ETH at 12:01am on January 1st, 2022. The rebalancing strategy then holds 10,000 ETH at 12:01am, so the $\mathsf{P\&L}$ of the rebalancing strategy from 12:01am to 12:02am is simply $10,000\,(P_{12:02am} - P_{12:01am})$, the amount of ETH held times the change in ETH prices over the next minute. In general, if the rebalancing strategy holds x_t^{RB} of the risky asset at time t until time $t + \Delta t$, then $\Delta\text{RB } \mathsf{P\&L}_t$, the rebalancing strategy's net profit from period t to $t + \Delta t$, is:

$$\Delta\text{RB } \mathsf{P\&L}_t = x_t^{RB}\,(P_{t+\Delta t} - P_t)\,. \tag{18}$$

Expression (18) is the discrete-time analog of the envelope formula expression, (5), for the returns on any strategy which trades at market prices. Note that $\Delta\text{RB } \mathsf{P\&L}_t$ is not directly affected by rebalancing trades – changes in x_t^{RB} over time – because these rebalancing trades are made at fair market prices on the CEX, and we assumed CEX trades have no price impact. We calculate total profits of the rebalancing strategy over any time period by summing the increments (18) over time. As we will show below, our results are relatively insensitive to the rebalancing horizon chosen.

To measure the right side of (17), we observe FEE_t, fees paid into the LP pool over any given time period. For LVR_t, since Uniswap v2 is a constant-product CFMM, percentage LVR has the particularly simple expression in (16) of Example 3,

$$\mathsf{LVR}_t = \int_0^t \frac{\sigma_s^2}{8} \times V(P_s)\,ds. \tag{19}$$

We measure LVR in each period simply by plugging in realized volatility and pool value to a version of equation (19) that is discretized over time.

Note that, empirically, we measure the total fees paid by all kinds of traders. This differs slightly from our model, where we assume arbitrage traders pay no fees. Practically, since fees are simply an increasing process which potentially compensates for LVR, whether fees arise from noise trade or

[13] "WETH", or "wrapped ETH" is a variation of ETH that is compliant with the ERC-20 token standard. For our purposes, we will view ETH and WETH as equivalent.

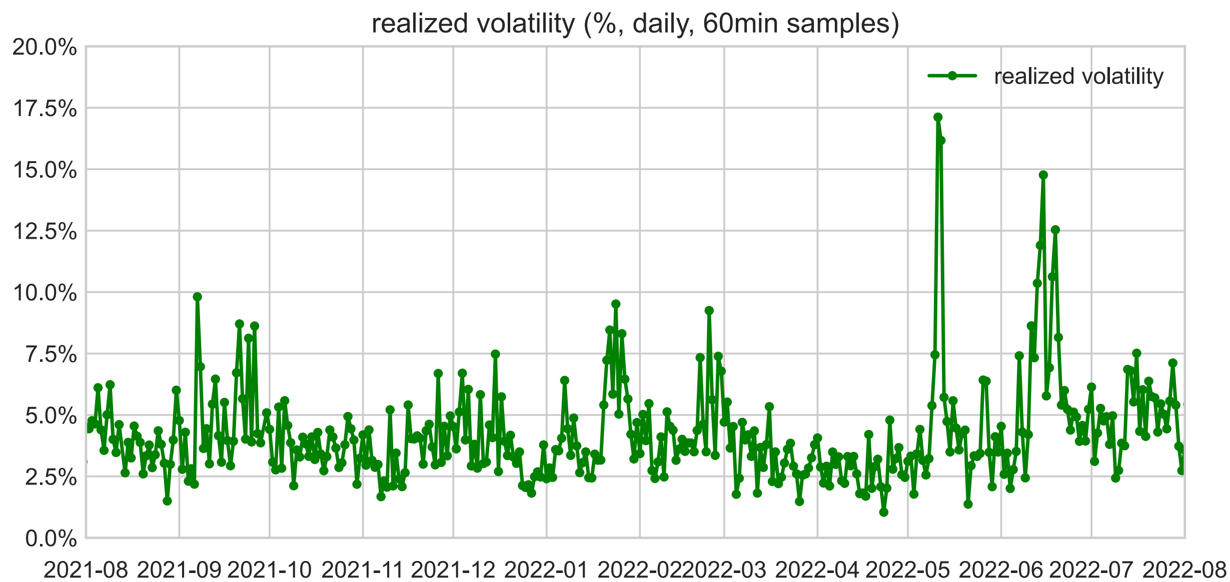


Figure 3: Daily realized volatility for the ETH-USDC pair, computed from Binance minutely closing prices, sampled at 60 minute intervals.

arbitrage trade does not substantially impact the returns on LP positions. If we assumed arbitrage traders paid trading fees in the model, this would decrease the amount of arbitrage: instead of prices on the CFMM moving immediately to match CEX prices at all times, prices would have to move more than fees in order for arbitrage trade to have nonnegative payoffs. The analysis of arbitrage profits in the presence of fees is the subject of follow-on work [Milionis et al., 2023].

5.1. Empirical Results

The daily realized volatility estimates are illustrated in Figure 3. Here, it is clear that, not only is the volatility of this asset pair high, but the volatility in turn is also highly volatile, varying by a factor of five over the observation interval.

In Figure 4, we see the daily average aggregate value for this pool over the time period. The average pool value was \$209 million, and the pool value ranged between \$90 million and \$310 million.

Next, we show why it is important to account for the profits of the rebalancing strategy in analyzing LP returns. The `pool_pnl` series in Figure 5a shows the raw aggregate LP $\mathsf{P\&L}_t$ (i.e., without delta-hedging by subtracting the rebalancing strategy). The pool $\mathsf{P\&L}$ fluctuates wildly, and ultimately loses money. In particular, as shown in Table 1, the pool has an overall annualized return of -6.2%, and a Sharpe ratio of -0.1.

However, these returns are largely driven by market risk: at any point in time, the pool maintains half of its value in ETH, and ETH prices varied significantly over this interval. The `hedged_pnl` series in Figure 5 illustrate hedged $\mathsf{P\&L}$, that is, LP $\mathsf{P\&L}_t$ minus the profits of the

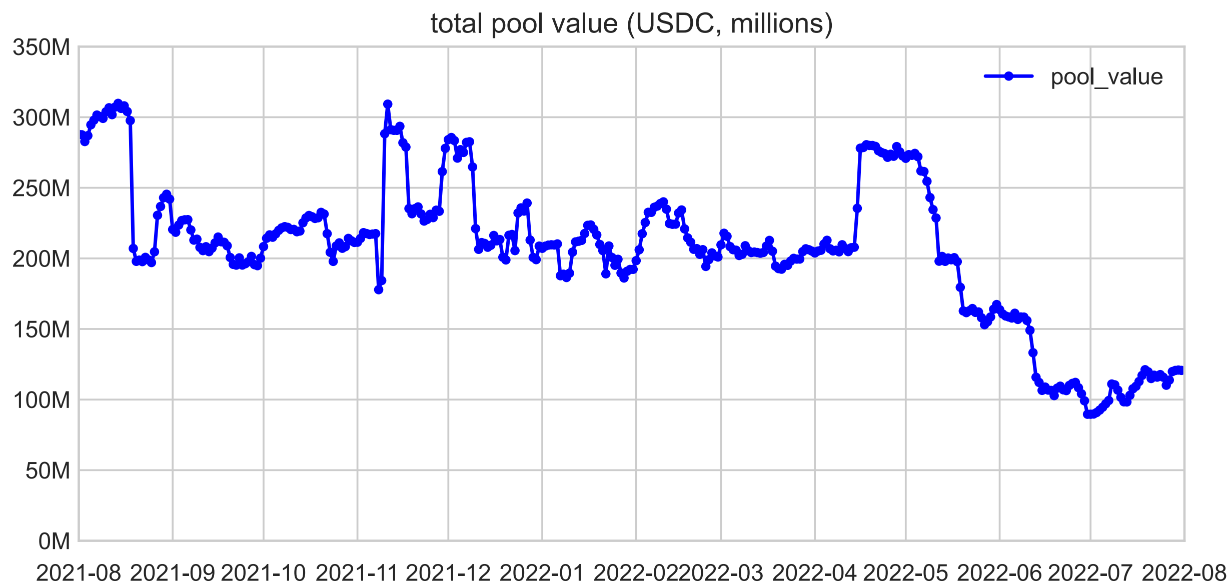


Figure 4: The daily average pool value of the Uniswap v2 WETH-USDC pair.

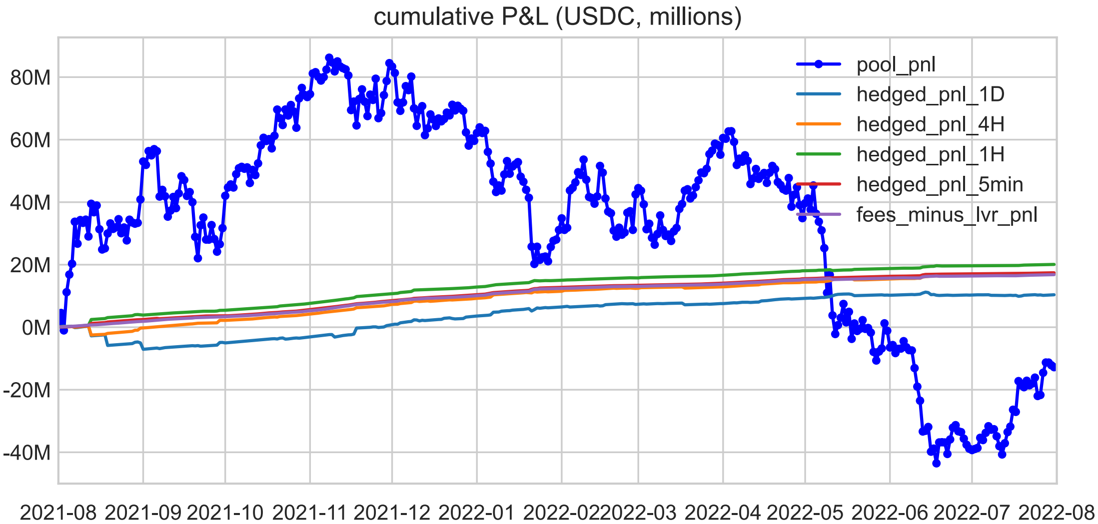
	return (%, annual)	Sharpe ratio (annual)
`pool_pnl`	-6.2	-0.1
`hedged_pnl_1D`	5.0	1.8
`hedged_pnl_4H`	8.2	5.5
`hedged_pnl_1H`	9.7	10.8
`hedged_pnl_5min`	8.4	18.2
`fees_minus_lvr_pnl`	8.2	17.0

Table 1: Overall return statistics for the cumulative P&L series of Figure 5.

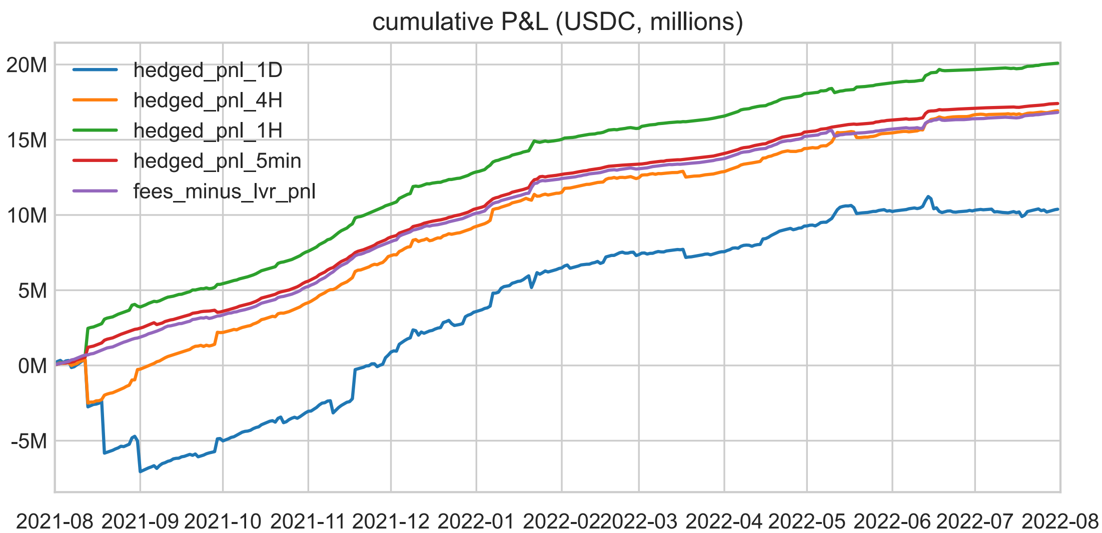
rebalancing strategy, which is the left side of (17). This hedges directional exposure to ETH prices to first-order, and is just a bet on whether fees are greater than LVR. Visually, these lines are substantially less volatile than the raw LP $\mathsf{P\&L}_t$. This is quantified in Table 1, where we see that a delta-hedged LP position can achieve very high Sharpe ratios, and that, in general, Sharpe ratios increase with more frequent rebalancing. Moreover, the delta-hedged LP position actually makes positive returns.[14]

Accuracy of the model. In Figure 6, we analyze the accuracy of our model, that is, the difference between the left and right sides of (17), for various choices of frequency of rebalancing. This analyzes how well our model is able to predict delta-hedged LP returns in the data. Fees minus LVR in our model tracks the pattern of hedged LP P&L: differences between the two seem to be stationary over the observed interval. Moreover, as the frequency of rebalancing increases, the differences are smaller in magnitude. This is consistent with LVR as being a continuous rebalancing

[14]Note that these returns assume no trading or financing costs for the rebalancing strategy.



(a) Including the aggregate pool P&L series.



(b) Excluding the aggregate pool P&L series.

Figure 5: Cumulative pool P&L, delta-hedged P&L, and predicted P&L from our expressions for LVR, for the Uniswap v2 WETH-USDC trading pair. In the first panel, the `pool_pnl` series shows the raw P&L of the LP position. In both panels, the various `hedged_pnl` series show delta-hedged LP P&L, that is, the P&L from a long position in the pool, and a short position in the rebalancing strategy, rebalanced at various frequencies (daily, every four hours, every hour, every five minutes). The `fees_minus_lvr_pnl` series shows fees minus LVR, this is the delta-hedged P&L predicted by our theory. The data source for prices is Binance, and LP P&L is calculated using data on Uniswap trades, mints, and burns from the Ethereum blockchain. Details of how we calculate these quantities are in Appendix C.

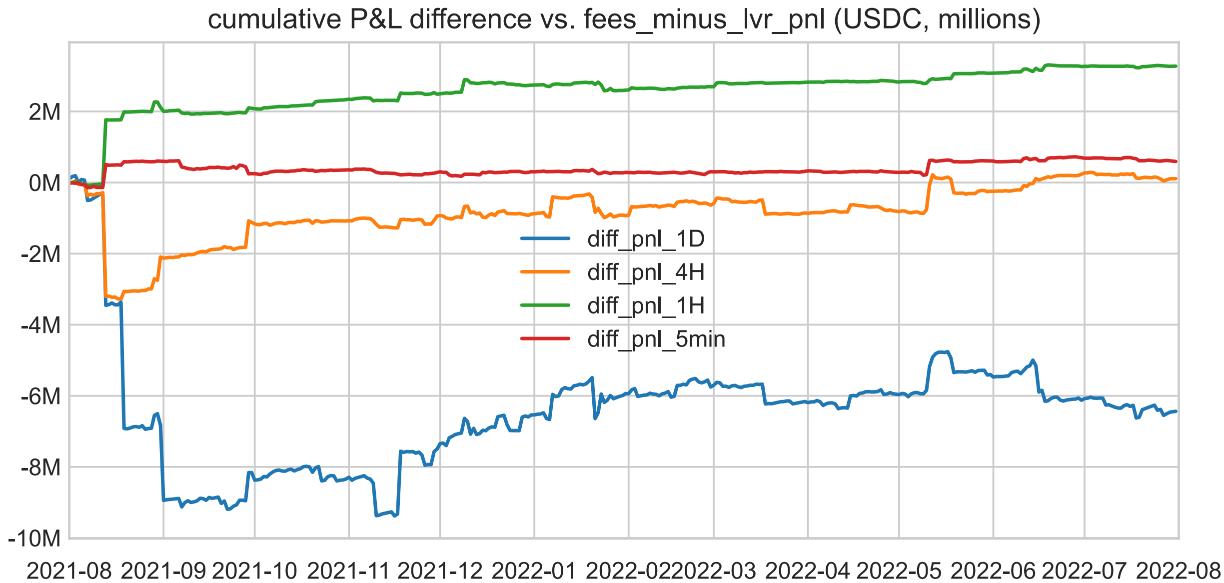


Figure 6: Cumulative differences between the delta-hedged P&L (various `hedged_pnl` series of Figure 5) and predicted P&L from our model (`fees_minus_lvr_pnl` series of Figure 5).

approximation.

6. Option Pricing

We have shown that CFMM LPs behave like a bet on volatility, in the sense that LVR is large when volatility is high. In this section, we briefly discuss the relationship of CFMM LPs to three classical and inter-related ways that volatility can be traded [Carr and Madan, 2001]: static (European) options positions, dynamic trading strategies, and variance swaps. In Appendix B.1, we also demonstrate the equivalences between CFMM LP positions and these concepts in a simple two-step binomial tree model.

6.1. Static Options Positions

Our results are related to Clark [2020], Fukasawa et al. [2022], and Deng and Zong [2022], who show that, over any finite time horizon, an AMM LP position's payoff, ignoring fees, can be replicated by shorting a bundle of European options. Technically, this follows from the facts that the CFMM's asset position and value are both path-independent: if the price is P_T at time T, the CFMM holds $x^*(P_T)$ of the risky asset and has pool value $V_T = V(P_T)$ corresponding to the final "payoff function" $V(\cdot)$, regardless of the path that prices took to reach P_T. More intuitively, at any time T, the CFMM simply offers a menu of quantities of the asset to buy or sell at any given price, identically to a portfolio of European options. Ignoring fees, the CFMM exactly breaks even if prices do not move, $P_T = P_0$, and loses money otherwise; hence, the CFMM LP position is essentially equivalent to *giving away* a bundle of European options. This intuition is consistent with the fact that the $V(\cdot)$ is a *concave* function (cf. Lemma 1).

Expected LVR until period T can be thought of as the value of the European options given away. This analogy gives another intuition for the comparative statics of expected LVR. European options are worth more when volatility is higher, so LVR is increasing in the volatility of the underlying asset. When the marginal liquidity of the AMM bonding curve is greater, the replicating portfolio of European options is larger: AMMs that trade more aggressively essentially give away more European options, also increasing LVR.

As previous papers have discussed, the European option replication result also implies that, over any finite time horizon, the exposure of the AMM LP position to underlying prices can be *totally* hedged, by taking a long position in the replicating bundle of European options. This trade — a long position in the AMM LP, plus a short position in the replicating bundle of European options — is essentially a *trading fee swap*, betting on whether accrued trading fees from time 0 to T are greater than European option premia of the replicating portfolio at time 0. The trader enters an LP position, and pays a premium for buying the replicating bundle of European options upfront. The AMM LP position then loses no money from price movements; the total position profits if the accrued trading fees until time T are greater than the European option premia paid upfront, and loses otherwise.

6.2. Dynamic Trading Strategies

Classic options theory implies that static option positions are equivalent to dynamically trading the underlying asset in a certain way. The static option position is a combination of short straddles and strangles, selling out-of-money calls and puts. This position is equivalent to a dynamic trading strategy which sells the asset when prices increase, and buys when prices increase. This is exactly what the AMM LP position does: observe that, from Lemma 1 Part 3, $x^*(\cdot)$ is non-increasing. If prices decrease slightly from P_0 to $P_t < P_0$, the rebalancing strategy responds by buying the risky asset. The rebalancing strategy thus makes a profit, relative to simply holding the initial position $x^*(P_0)$, if prices increase back to P_0, and makes a loss if prices decrease further from P_t. This argument holds symmetrically for price decreases, implying that the rebalancing strategy makes losses if prices diverge from P_0, and profits when prices make small movements away from P_0 and back. In the special case where the risky asset's price is a random walk, the rebalancing strategy thus breaks even on on average. In contrast, when prices move away from P_0 and back, the CFMM reverts to the initial value $V(P_0)$, exactly breaking even: there is no profit from price convergence, to offset the losses the CFMM makes when prices diverge from P_0.

6.3. Variance Swaps

Finally, as discussed by Fukasawa et al. [2022], variance can be traded directly by trading swaps on realized variance. The VIX is such a contract, operating on a fixed finite time horizon. Applying

Lemma 1 Part 3, the instantaneous LVR of (8) can be re-written as

$$\ell(\sigma, P) = \tfrac{1}{2} \times (\sigma P)^2 \times |x^{*\prime}(P)|.$$

Here, the first component, $(\sigma P)^2$, is the instantaneous variance or quadratic variation of the price, i.e., for small Δt, $\mathrm{Var}[P_{t+\Delta t}|P_t = P] \approx (\sigma P)^2 \, \Delta t$. Recalling that $x^*(P)$ is the total quantity of risky asset held by the pool if the price is P, the second component, $|x^{*\prime}(P)|$ corresponds to the *marginal* liquidity available from the pool at price level P. Now, integrating over time, we have that

$$\mathsf{LVR}_t = \tfrac{1}{2} \int_0^t (\sigma_s P_s)^2 \times |x^{*\prime}(P_s)| \, ds = \tfrac{1}{2} \int_0^t |x^{*\prime}(P_s)| \, d[P]_s, \quad \forall \; t \geq 0.$$

This expression is the payoff of the floating leg of a continuously sampled generalized variance swap [Carr and Lee, 2009, see, e.g.,], specifically a price variance swap that is weighted by marginal liquidity.

7. Other Benchmarks and "Impermanent Loss"

In this section, we consider the possibility of alternative benchmarks aside from the rebalancing strategy. We first define a broad class of benchmark strategies: the only restrictions we impose on these strategies are that they begin holding the same position in the risky asset as the CFMM, and that they adjust holdings at CEX prices. Specifically, we define a benchmark as a self-financing trading strategy, described by a position $\bar{x}_t$ in the risky asset. We assume that initial holdings match the pool, i.e., $(\bar{x}_0, \bar{y}_0) \triangleq \big(x^*(P_0), y^*(P_0)\big)$. We assume that $\bar{x}_t$ satisfies the square-integrability condition (3), so that the resulting trading strategy is admissible. Denote the value of that strategy by $\bar{R}_t$, so that

$$\bar{R}_t = V_0 + \int_0^t \bar{x}_s \, dP_s, \quad \forall \; t \geq 0.$$

For any such benchmark, we can thus define the *loss-versus-benchmark* according to $\mathsf{LVB}_t \triangleq \bar{R}_t - V_t$.

One benchmark of particular interest is a strategy that simply holds the initial position, i.e., $x_t^{\mathsf{HODL}} \triangleq x^*(P_0)$, with value

$$R_t^{\mathsf{HODL}} = V_0 + \int_0^t x^*(P_0) \, dP_s = V_0 + x^*(P_0) \, (P_t - P_0) \, , \quad \forall \; t \geq 0.$$

Loss versus the HODL benchmark is often discussed among practitioners as "impermanent loss" or "divergence loss" [e.g., Engel and Herlihy, 2021]. Motivated by the aforementioned analysis, in our view this is more accurately described as "loss-versus-holding": $\mathsf{LVH}_T \triangleq R_t^{\mathsf{HODL}} - V_t$. The following result characterizes the loss process LVB_t as a function of the underlying benchmark strategy $\bar{x}_t$.

Corollary 1. *For all $t \geq 0$,*

$$\mathsf{LVB}_t = \mathsf{LVR}_t + \underbrace{\int_0^t \left[\bar{x}_s - x^*(P_s)\right] \, dP_s}_{\triangleq \Delta(\bar{x})_t} . \tag{20}$$

The loss process has quadratic variation

$$[\mathsf{LVB}]_t = [\Delta(\bar{x})]_t = \int_0^t \left[\bar{x}_s - x^*(P_s)\right]^2 \, \sigma_s^2 P_s^2 \, ds \geq [\mathsf{LVR}]_t = 0. \tag{21}$$

Therefore, among all benchmark strategies, the rebalancing strategy uniquely defines a loss process with minimal (zero) quadratic variation.

Proof. The first part is an immediate corollary of Theorem 1 and (5). The second part follows from the Itô isometry. ■

There are two ways to interpret Corollary 1. On the one hand, in (20), the expected value of $\Delta(\bar{x})_t$ is always 0 under the risk-neutral measure. Thus, the risk-neutral expectation of LVB is the same for *any* choice of benchmark, including LVR and the HODL benchmark. This is because CFMM LP losses arise from trading at off-market prices: *any* benchmark which trades at market prices, in expectation, does equally well under the risk-neutral measure, and thus the gap between any market benchmark and LVR is equal in expectation. In this sense, the expected losses of CFMM LPs appear invariant to the particular choice of market-based benchmark.

On the other hand, LVR is the *unique* choice of benchmark which eliminates differences in performance between the CFMM and the benchmark strategy due to market risk, and isolating losses due to price slippage. All benchmarks outperform the CFMM LP position by the same amount in expectation; however, on any given price path P_t, any given benchmark may over- or under-perform to the CFMM LP position, because the benchmark may adopt different holding strategies for the risky asset from the CFMM. As an example, we showed in Section 5 that the CFMM LP position underperforms a benchmark which sells all ETH and holds $\bar{x}_t = 0$ throughout, because of the fact that the CFMM LP holds a larger ETH position and ETH prices dropped over the time horizon we analyze, implying the misleading conclusion that the CFMM LP position underperformed a market-based benchmark.

The LVR benchmark is useful because the rebalancing strategy exactly matches the risky asset holdings of the CFMM, removing differences in market risk exposure and isolates losses due to slippage. Theorem 1 showed that LVR is a strictly increasing process: it is *always* positive, regardless of the path prices take. Expression (21) thus shows that the rebalancing strategy is the unique choice of benchmark which minimizes the quadratic variation of the loss process: that is, any other choice of benchmark can be thought of as LVR, plus a noise term which has mean 0 under risk-neutral measure, caused by differences in market risk exposures. Thus, in our view, benchmarks other than the rebalancing strategy confound two concepts: LVR, which captures losses of the CFMM

LP position due to trading at off-market prices, and $\Delta(\bar{x})$, which captures differences between the risky asset holdings of the CFMM and the benchmark.

A common argument in the industry discourse for the benchmark of "impermanent loss" states that, as long as prices revert to their initial values, AMM holdings will also revert to their initial state, resulting in zero losses. We analyze this argument in Appendix B.1. In a slight departure from our baseline model, we assume prices evolve according to a two-step binomial tree. In the first step of the tree, the risky asset's price either increases or decreases; in the second step, prices can either revert to their initial level, or diverge further. The basic trading strategy of an CFMM is to sell the risky asset when its price increases, and buy when its price decreases. Trading in this manner is a bet on mean reversion: when implemented trading at CEX prices, as the rebalancing strategy does, the strategy profits if prices mean-revert, and loses if prices diverge further, thus breaking even on average. The CFMM executes the same trades as the rebalancing strategy, but attains worse prices on each trade. Thus, the CFMM exactly breaks even if prices mean-revert, and loses more money than the rebalancing strategy if prices diverge, thus losing money on average. This example illustrates that, for an CFMM to perform well, it is not sufficient to break even when prices revert; a trading strategy which sells into price rises and buys into price decreases must actually make strictly positive profits when prices revert, in order to compensate for the losses it makes when prices diverge. An CFMM's performance should thus be benchmarked to the rebalancing strategy — making the same trades at CEX prices — rather than the intuitive but misleading benchmark that the CFMM should break even upon price reversion.

8. Discussion and Implications

Besides their positive value for understanding and quantifying the losses from AMM LPing, our results also suggest ways that these losses could be reduced or eliminated. We have shown that LVR changes based on market conditions: LVR is greater when volatility is high. This suggests the utilization of dynamic trading fee rules: trading fees could be adjusted based on volatility or variance. These adjustments could in principle be based on the recent past (e.g., historical realized volatility or realized LVR), or on future predictions (e.g., options-implied volatility).

We showed that CFMM LPs lose money from price slippage: when CEX prices move, CFMM quotes become "stale" and are vulnerable to sniping by rebalancing arbitrageurs. It is in principle possible to eliminate this slippage: if an AMM had access to a high-frequency oracle for the CEX price P_t, the AMM could in principle quote prices arbitrarily close to P_t, up to the desired asset position $x^*(P_t)$. Quoting prices this way would reduce arbitrageurs' profits, allowing the AMM to achieve a payoff arbitrarily close to that of the rebalancing strategy. This design has a number of risks — it relies heavily on the accuracy of the oracle for P_t, and leaves open the potential for oracle manipulation — but in principle an oracle-based AMM could substantially reduce or eliminate LVR.

A related AMM design, proposed by Jump Crypto, revolves around selling the right to arbitrage the pool to certain special wallets, and redistributing profits to AMM LPs. Suppose a particular crypto wallet address, which we call the “authorized participant” or AP, had the unique right to trade with the AMM paying zero fees. The AP would then have a large advantage in arbitrage trading, since the AP could profitably trade against arbitrarily small price movements, whereas prices would have to move at least as much as the AMM’s percentage trading fees for non-AP wallets to profit from arbitrage trade. The AP would thus be able to capture essentially the entirety of fees from arbitrage trade.

An AMM protocol aiming to reduce LVR could thus run an AP wallet itself, doing CEX-DEX arbitrage, and redistributing arbitrage profits to LPs. Alternatively, AMM protocols could simply run periodic auctions — over longer periods, such as weeks or months — in which wallets can bid for the right to be authorized participants for some period of time. In principle, potential arbitrageurs should bid the ex-ante expectation of arbitrage profits, which includes, LVR; the protocol can then redistribute these profits to LPs.[15] Both these methods capture either LVR of its expectation, and redistribute the profits to pool LPs. We believe that these are interesting directions for future AMM design research.

9. Conclusion

In this paper, we constructed a model of AMM LP profits. We defined the losses suffered by LPs as “loss-versus-rebalancing”, or LVR; this is the gap between the profits of an AMM LP position, and the returns from a trading strategy which perfectly mimics the AMM’s position in the risky asset, but performs all trades at market prices. LVR arises from the fact that AMMs always trade at off-market prices, leaving money to arbitrageurs trading the AMM against a CEX. LVR is greater when prices are more volatile, and when the AMM’s “marginal liquidity” is greater, that is, it trades more aggressively in response to price movements. A delta-hedged AMM LP position — a position which is long the LP position, and short the rebalancing strategy — is profitable if the AMM collects more in fees than it loses in LVR. The model is quantitatively realistic enough to be brought to data; we show that our expressions for LVR predict AMM LP losses fairly accurately in practice. Our results have implications for how to redesign AMMs to reduce or eliminate LVR, which could lower the effective trading fees paid by market participants relying on AMMs for token pair liquidity.

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[15]Note that one subtlety about these designs is that, in addition to rebalancing arbitrage profits, APs would be able to capture reversion arbitrage profits; hence, LVR provides a lower bound for how much revenue wallets could capture from having preferential access to arbitrage the pool.

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A. Proofs

A.1. Proof of Theorem 1

First, we show that we show that LVR_t is equal to expression (7). The smoothness condition of Assumption 1 Part 2 allows us to apply Itô's lemma to $V(\cdot)$ to obtain

$$\begin{aligned} dV_t &= V'(P_t)\, dP_t + \tfrac{1}{2} V''(P_t)\, (dP_t)^2 \\ &= V'(P_t)\, dP_t + \tfrac{1}{2} V''(P_t)\, \sigma^2 P_t^2\, dt \\ &= x^*(P_t)\, dP_t + \tfrac{1}{2} V''(P_t)\, \sigma^2 P_t^2\, dt, \end{aligned} \tag{22}$$

where the last step follows from Lemma 1 Part 2. Comparing with (5), we obtain (7). Finally, the fact that $\ell(\sigma, P) \geq 0$ follows from Lemma 1 Part 3.

Next, we show that the cumulative profits of rebalancing arbitrageurs is equal to LVR_t. We start with a discrete approximation to the arbitrage profit, indexed by $N \geq 1$. Suppose arbitrageurs arrive sequentially, so that the ith arbitrageur arrives at time τ_i, for $1 \leq i \leq N$. For convenience, set $\tau_0 \triangleq 0$ and $\tau_{N+1} \triangleq T$. For each $1 \leq i \leq N$, at time τ_i, the ith arbitrageur observes the price P_{τ_i}, rebalances the pool from $\big(x^*(P_{\tau_{i-1}}), y^*(P_{\tau_{i-1}})\big)$ to $\big(x^*(P_{\tau_i}), y^*(P_{\tau_i})\big)$. In other words, the arbitrageur purchases $x^*(P_{\tau_{i-1}}) - x^*(P_{\tau_i})$ units of the risky asset from the CFMM at average price :

$$P_i^{\mathsf{CFMM}} \triangleq -\frac{y^*(P_{\tau_i}) - y^*(P_{\tau_{i-1}})}{x^*(P_{\tau_i}) - x^*(P_{\tau_{i-1}})}.$$

The arbitrageur can then sell these units on the external market at price P_{τ_i} and earn profits (in the numéraire) from the difference in price according to

$$\left(P_{\tau_i} - P_i^{\mathsf{CFMM}}\right)\left[x^*(P_{\tau_{i-1}}) - x^*(P_{\tau_i})\right] = P_{\tau_i}\left[x^*(P_{\tau_{i-1}}) - x^*(P_{\tau_i})\right] + \left[y^*(P_{\tau_{i-1}}) - y^*(P_{\tau_i})\right].$$

Denote by $\mathsf{ARB}_T^{(N)}$ the aggregate arbitrage profits. Summing over $1 \leq i \leq N$, telescoping the sum, and applying summation-by-parts yields

$$\begin{aligned}
\mathsf{ARB}_T^{(N)} &\triangleq \sum_{i=1}^{N}\left\{P_{\tau_i}\left[x^*(P_{\tau_{i-1}}) - x^*(P_{\tau_i})\right] + \left[y^*(P_{\tau_{i-1}}) - y^*(P_{\tau_i})\right]\right\} \\
&= \sum_{i=1}^{N} P_{\tau_i}\left[x^*(P_{\tau_{i-1}}) - x^*(P_{\tau_i})\right] + y^*(P_0) - y^*(P_{\tau_N}) \\
&= P_0 x^*(P_0) + y^*(P_0) + \sum_{i=0}^{N} x^*(P_{\tau_i})\left[P_{\tau_{i+1}} - P_{\tau_i}\right] - P_T x^*(P_{\tau_N}) - y^*(P_{\tau_N}).
\end{aligned}$$

Observe that the sum in the final expression is the discrete approximation of an Itô integral. Assume that the time partition mesh over $[0, T]$ shrinks to zero as $N \to \infty$. Taking the limit as $N \to \infty$ and passing to continuous time, the sum converges to an Itô integral, which is well-defined under Assumption 1 Part 3. Further, $\tau_N \to T$, so that $P_{\tau_N} \to P_T$, and $x^*(P_{\tau_N}) \to x^*(P_T)$, $y^*(P_{\tau_N}) \to y^*(P_T)$. Thus, it holds that

$$\mathsf{ARB}_T \triangleq \lim_{N\to\infty} \mathsf{ARB}_T^{(N)} = V(P_0) + \int_0^T x^*(P_t)\, dP_t - V(P_T).$$

Hence, the cumulative profits of rebalancing arbitrageurs from time 0 to time T are equal to LVR, defined in (6).

A.2. LVR, Marginal Liquidity, and Bonding Function Curvature

The following proposition expresses the marginal liquidity of a CFMM, $x'(P)$, in terms of derivatives of the CFMM bonding function f.

Proposition 1. *If the CFMM bonding function $f(x, y)$ is twice continuously differentiable, the marginal liquidity at price P is:*

$$\frac{dx}{dP} = \frac{\frac{\partial f}{\partial y}}{\left(\frac{\partial^2 f}{\partial x^2} + P^2 \frac{\partial^2 f}{\partial y^2} - 2P \frac{\partial^2 f}{\partial x \partial y}\right)} \tag{23}$$

Qualitatively, Proposition 1 implies that marginal liquidity, $x'(P)$, is related to the curvature of the CFMM invariant curves. The denominator of (33) is equal to P^2 times the negative of the determinant of the bordered Hessian of f. f is strictly quasiconcave — that is, the upper level sets of f are convex — if and only if this determinant is positive; moreover, the magnitude of the determinant is related to the curvature of the level curves of f [Simon et al., 1994, p. 542]. Thus, CFMM invariants with "flatter", more linear level curves will have greater marginal liquidity $\frac{dx}{dP}$, and also greater LVR.

Proposition 1 is also useful because, for any CFMM invariant, expression (23) can be used to calculate $\frac{dx}{dP}$, and thus compute LVR in practice.

A.2.1. Proof of Proposition 1

The Lagrangian of the pool expenditure minimization problem, (1), is:

$$\Lambda = Px + y + \lambda \left[f\left(x, y\right) - L \right]$$

The optimal solution is characterized by the FOCs:

$$\frac{\partial \Lambda}{\partial x} :\ P + \lambda \frac{\partial f}{\partial x} = 0 \tag{24}$$

$$\frac{\partial \Lambda}{\partial y} :\ 1 + \lambda \frac{\partial f}{\partial y} = 0 \tag{25}$$

$$\frac{\partial \Lambda}{\partial \lambda} :\ f\left(x, y\right) - L = 0 \tag{26}$$

Now, we will take $\frac{dx}{dP}$ by applying the implicit function theorem to this system of first-order conditions. The derivatives of the FOCs are:

$$\frac{\partial}{\partial P}\frac{\partial \Lambda}{\partial x} :\ 1$$
$$\frac{\partial}{\partial P}\frac{\partial \Lambda}{\partial y} :\ 0$$
$$\frac{\partial}{\partial P}\frac{\partial \Lambda}{\partial \lambda} :\ 0$$

$$\frac{\partial}{\partial x}\frac{\partial \Lambda}{\partial x} :\ \lambda \frac{\partial^2 f}{\partial x^2}$$
$$\frac{\partial}{\partial x}\frac{\partial \Lambda}{\partial y} :\ \lambda \frac{\partial^2 f}{\partial x \partial y}$$
$$\frac{\partial}{\partial x}\frac{\partial \Lambda}{\partial \lambda} :\ \frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial y}\frac{\partial \Lambda}{\partial x} :\ \lambda \frac{\partial^2 f}{\partial x \partial y}$$
$$\frac{\partial}{\partial y}\frac{\partial \Lambda}{\partial y} :\ \lambda \frac{\partial^2 f}{\partial y^2}$$
$$\frac{\partial}{\partial y}\frac{\partial \Lambda}{\partial \lambda} :\ \frac{\partial f}{\partial y}$$

$$\frac{\partial}{\partial\lambda}\frac{\partial\Lambda}{\partial x}: \; -\frac{\partial f}{\partial x}$$

$$\frac{\partial}{\partial\lambda}\frac{\partial\Lambda}{\partial y}: \; -\frac{\partial f}{\partial y}$$

$$\frac{\partial}{\partial\lambda}\frac{\partial\Lambda}{\partial\lambda}: \; 0$$

Hence, we wish to solve the following system of equations for $\frac{dx}{dP}$:

$$\frac{\partial}{\partial P}\frac{\partial\Lambda}{\partial x} + \frac{\partial}{\partial x}\frac{\partial\Lambda}{\partial x}\frac{dx}{dP} + \frac{\partial}{\partial y}\frac{\partial\Lambda}{\partial x}\frac{dy}{dP} + \frac{\partial}{\partial\lambda}\frac{\partial\Lambda}{\partial x}\frac{d\lambda}{dP} = 0$$

$$\frac{\partial}{\partial P}\frac{\partial\Lambda}{\partial y} + \frac{\partial}{\partial x}\frac{\partial\Lambda}{\partial y}\frac{dx}{dP} + \frac{\partial}{\partial y}\frac{\partial\Lambda}{\partial y}\frac{dy}{dP} + \frac{\partial}{\partial\lambda}\frac{\partial\Lambda}{\partial y}\frac{d\lambda}{dP} = 0$$

$$\frac{\partial}{\partial P}\frac{\partial\Lambda}{\partial\lambda} + \frac{\partial}{\partial x}\frac{\partial\Lambda}{\partial\lambda}\frac{dx}{dP} + \frac{\partial}{\partial y}\frac{\partial\Lambda}{\partial\lambda}\frac{dy}{dP} + \frac{\partial}{\partial\lambda}\frac{\partial\Lambda}{\partial\lambda}\frac{d\lambda}{dP} = 0$$

Substituting, we have:

$$1 + \lambda\frac{\partial^2 f}{\partial x^2}\frac{dx}{dP} + \lambda\frac{\partial^2 f}{\partial x\partial y}\frac{dy}{dP} - \frac{\partial f}{\partial x}\frac{d\lambda}{dP} = 0 \tag{27}$$

$$0 + \lambda\frac{\partial^2 f}{\partial x\partial y}\frac{dx}{dP} + \lambda\frac{\partial^2 f}{\partial y^2}\frac{dy}{dP} - \frac{\partial f}{\partial y}\frac{d\lambda}{dP} = 0 \tag{28}$$

$$\frac{\partial f}{\partial x}\frac{dx}{dP} + \frac{\partial f}{\partial y}\frac{dy}{dP} = 0 \tag{29}$$

Now, applying (24) and (25), we have:

$$\frac{\partial f}{\partial x} = P\frac{\partial f}{\partial y} \tag{30}$$

Thus, we can simplify (29) to:

$$\frac{dy}{dP} = -P\frac{dx}{dP}$$

Substituting into (27) and (28), we have:

$$1 + \lambda\frac{\partial^2 f}{\partial x^2}\frac{dx}{dP} + \lambda\frac{\partial^2 f}{\partial x\partial y}\left(-P\frac{dx}{dP}\right) - \frac{\partial f}{\partial x}\frac{d\lambda}{dP} = 0$$

$$0 + \lambda\frac{\partial^2 f}{\partial x\partial y}\frac{dx}{dP} + \lambda\frac{\partial^2 f}{\partial y^2}\left(-P\frac{dx}{dP}\right) - \frac{\partial f}{\partial y}\frac{d\lambda}{dP} = 0$$

Rearranging,

$$\left(\frac{\partial^2 f}{\partial x^2} - P\frac{\partial^2 f}{\partial x\partial y}\right)\lambda\frac{dx}{dP} = \frac{\partial f}{\partial x}\frac{d\lambda}{dP} - 1 \tag{31}$$

$$\left(\frac{\partial^2 f}{\partial x\partial y} - P\frac{\partial^2 f}{\partial y^2}\right)\lambda\frac{dx}{dP} = \frac{\partial f}{\partial y}\frac{d\lambda}{dP} \tag{32}$$

Now, multiply (32) by P, applying (30), and subtract (31), to get:

$$\left(\frac{\partial^2 f}{\partial x^2} + P^2 \frac{\partial^2 f}{\partial y^2} - 2P \frac{\partial^2 f}{\partial x \partial y}\right) \lambda \frac{dx}{dp} = -1$$

Now,

$$\frac{dx}{dP} = \frac{-1}{\lambda \left(\frac{\partial^2 f}{\partial x^2} + P^2 \frac{\partial^2 f}{\partial y^2} - 2P \frac{\partial^2 f}{\partial x \partial y}\right)}$$

Now, to solve for λ, we simply use (25). Hence, we have:

$$\frac{dx}{dP} = \frac{\frac{\partial f}{\partial y}}{\left(\frac{\partial^2 f}{\partial x^2} + P^2 \frac{\partial^2 f}{\partial y^2} - 2P \frac{\partial^2 f}{\partial x \partial y}\right)} \tag{33}$$

This is (23), and thus we have proven Proposition 1.

B. Other Results

B.1. LVR and Impermanent Loss in a Two-Step Binomial Tree

In this appendix, we consider the performance of the CFMM and the rebalancing strategy, as well as the simple buy-and-hold benchmark, on a two-step binomial tree. This discrete-time model is a departure from our baseline model, but usefully illustrates the intuitions behind how the rebalancing strategy behaves relative to the buy-and-hold strategy, why the CFMM strategy under-performs the rebalancing strategy, and why the benchmark behind the "impermanent loss" concept — that the CFMM should not lose money if prices revert to their original state — is inappropriate.

The binomial tree is a two-period discrete-time model, where prices can either go up or down in each time period. The tree is depicted in Panel A of Figure 7. The price begins at $P_0 = 1$. In the first step of the tree, the price can then increase to $P_1^U = 1.4$, or decrease to $P_1^D = 0.6$. In the second step, from P_1^U, the price can increase to $P_2^{UU} = 1.8$, or decrease to the original level $P_1^{UD} = 1$. From P_1^D, the price can increase to the original level $P_2^{DU} = 1$, or decrease further to $P_2^{DD} = 0.4$. Note that the tree is set up so that, assuming 0 interest rates, the risk-neutral probabilities of up and down movements are all 0.5 — in other words, prices are martingales if the probabilities of up and down movements are equal, so no strategy trading at market prices should be able to make or lose money in expectation.

We then calculate the asset positions and profits of three strategies in each tree state: the buy-and-hold strategy, which simply holds the initial endowment $(x_0, y_0) = 1, 1$ forever; the CFMM; and the rebalancing strategy. To calculate these quantities, for the CFMM, we simply compute what (x_t, y_t) is as a function of the tree price and the initial state (x_0, y_0). For the rebalancing strategy, in each tree state, if the CFMM trades from (x_t, y_t) to (x_{t+1}, y_{t+1}), we assume the rebalancing strategy trades to x_{t+1} at the price P_{t+1}; hence, the rebalancing strategy's cash position is:

$$y_{t+1} = y_t - P_t\left(x_{t+1} - x_t\right)$$

The results are shown in Figure 7. Panel B considers the profits of the buy-and-hold strategy, which simply holds the initial endowment $(x_0, y_0) = 1, 1$ forever. This strategy is exposed to market risk, making mark-to-market profits if prices rise and losing if prices fall.

Panel C shows the performance of the rebalancing strategy. Since this strategy trades at market prices, it does not make or lose money in expectation under the risk-neutral measure. In period 1, the rebalancing strategy has exactly the same profits as the buy-and-hold strategy in each state. However, the rebalancing strategy sells the risky asset when prices increase to $P_1^U = 1.4$. This is essentially a bet that prices will decrease; indeed, if prices decrease to $P_2^{UD} = 1$, the rebalancing strategy makes 2.062 in profits, which is slightly higher than the buy-and-hold's profit of 2.000. Conversely, if prices increase further to $P_2^{UU} = 1.8$, the rebalancing strategy makes 2.738, slightly lower than the buy-and-hold strategy's return of 2.800. On average, the rebalancing strategy makes 2.400 conditional on reaching P_1^U, like the buy-and-hold strategy, or any other strategy trading at market prices.

Analogously, if prices first decrease to $P_1^D = 0.6$, the rebalancing strategy buys the risky asset; it profits relative to buy-and-hold when prices increase to $P_2^{DU} = 1$, and loses if prices decrease further to $P_2^{DD} = 0.2$. This example shows that the rebalancing strategy is essentially a bet on price convergence. The rebalancing strategy sells into price increases and buys into price decreases; this pays off if prices mean-revert, and loses money if prices diverge further. However, the rebalancing strategy does not make or lose money on average.

The performance of the CFMM is shown in Panel D. The CFMM makes the same trades as the rebalancing strategy; thus, x_t is the same in each tree state in panels C and D. However, the CFMM always trades at worse-than-market prices; as a result, y_t is higher state-by-state on the rebalancing strategy compared to the CFMM. A natural definition of loss-versus-rebalancing in discrete time would be the gap between the CFMM's y_t-position, and the rebalancing strategy's y_t-position, in each state.

Note that the CFMM strategy has the elegant feature that, if prices revert to 1 — as in states P_2^{UD} and P_2^{DU} — the CFMM always holds exactly the same position as the buy-and-hold strategy. This is the basis of the colloquially popular idea that CFMM losses are "impermanent". However, the CFMM loses money in expectation: the average profit of the CFMM, across states, is lower than the buy-and-hold strategy in both periods 1 and 2. Essentially, the CFMM bets on convergence, like the rebalancing strategy, but does so in a very inefficient way. The CFMM exactly breaks even if prices revert to their initial state, and loses money — even relative to the rebalancing strategy — if prices diverge. The CFMM strategy thus cannot break even without fees, because there is no state of the world where the CFMM strategy makes positive amounts. Using the options analogy, an CFMM LP position is a dynamic trading strategy which performs like giving away a straddle without collecting premia: the position loses money whenever prices move, and exactly breaks even when prices end where they started.

This example also shows that the gap between the simple CFMM strategy and the buy-and-hold strategy can be decomposed into two distinct components: the performance of the rebalancing strategy relative to buy-and-hold, and the performance of the CFMM relative to the rebalancing strategy. The gap between the rebalancing strategy and buy-and-hold is a bet on convergence: the rebalancing strategy makes money if prices converge, and loses if prices diverge. The gap between the CFMM and the rebalancing strategy, which is the discrete-time version of LVR, is a systematic loss from slippage which accrues whenever prices move.

Relationships to option strategies. The binomial tree example also helps illustrate the analogy between CFMM LP payoffs and European options The time-2 difference between the payoffs of the rebalancing strategy and the buy-and-hold strategy is positive when prices mean-revert, and negative when prices diverge. Hence, payoffs are similar to those of a short European straddle or strangle position, which involves selling calls and puts which expire after two periods. The positive payoff when prices revert can be thought of as the option premia collected from selling the straddle, and the negative payoffs when prices diverge can be thought of as the payouts to the option buyer, which are made if either the call or the put sold expire in-the-money. The CFMM LP position has a similar pattern of payoffs, but makes 0 profits if prices end where they started. An CFMM LP position, ignoring fees, can thus be thought of like giving away a straddle position, without collecting any upfront option premia. Viewed this way, the equivalence between the rebalancing strategy and the static European strangle on the binomial tree reflects the classic idea that static option positions can be replicated by dynamically trading the underlying asset; in this case, European straddles and strangles are replicated by a strategy which sells the risky asset when prices increase and buys when prices decrease.

B.2. Weighted Geometric Mean Market Makers

Weighted geometric mean market makers have the special property that the instantaneous LVR per dollar of pool value, i.e., $\ell(\sigma, P)/V(P)$, is a constant. The following theorem establishes that these are essentially the only CFMMs for which this is true:

Theorem 2. *Suppose a CFMM satisfies*

$$\frac{\ell(\sigma, P)}{V(P)} = c(\sigma), \quad \forall\ P \geq 0. \tag{34}$$

Then, we have

$$V(P) = L_1 P^{\theta(\sigma)} + L_2 P^{1-\theta(\sigma)}, \tag{35}$$

for free constants $L_1, L_2 \geq 0$, where

$$\theta(\sigma) \triangleq \frac{1 - \sqrt{1 - 8c(\sigma)/\sigma^2}}{2} \leq \frac{1}{2}.$$

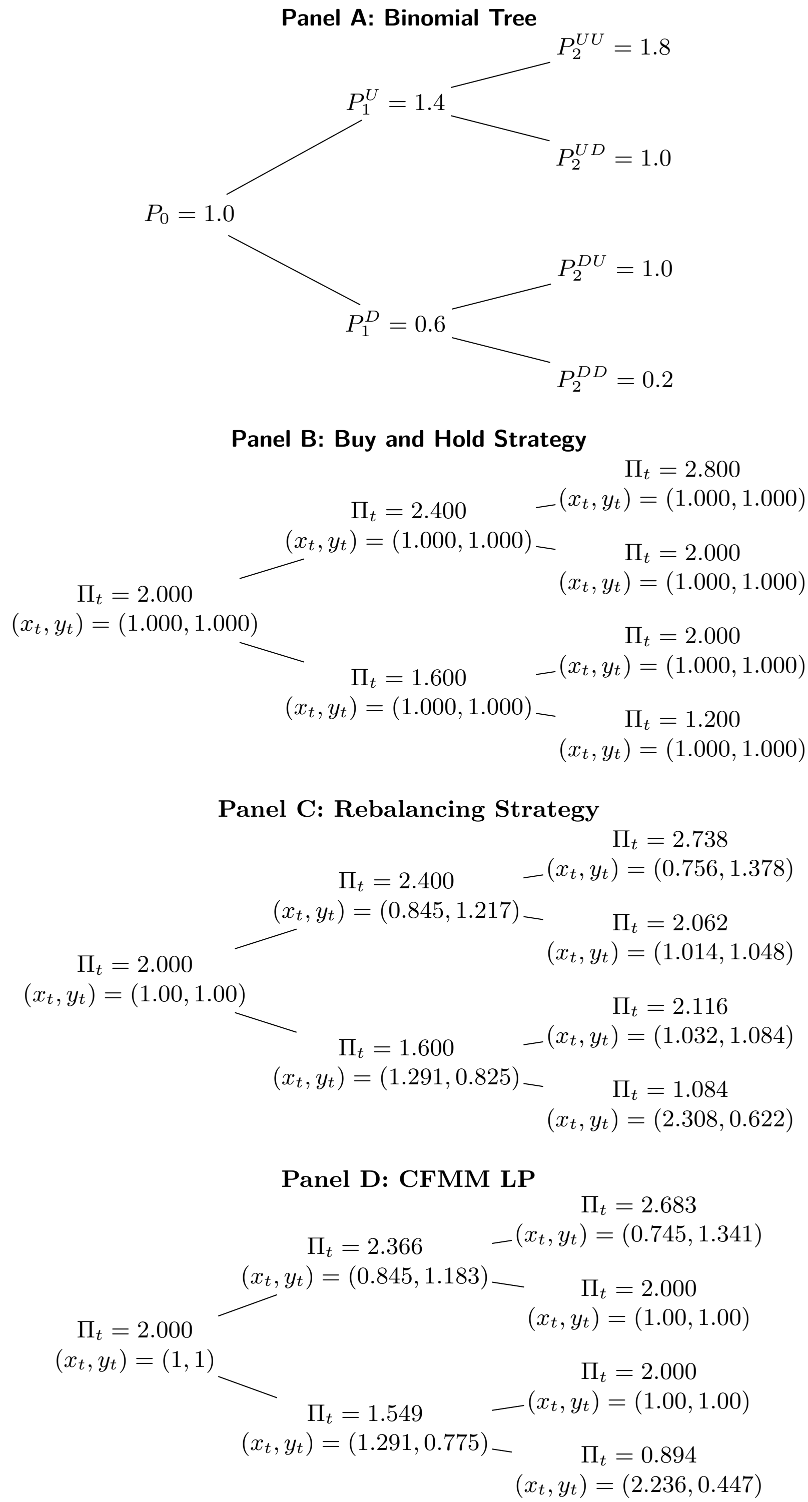

Figure 7: The performance of buy-and-hold, a constant-product CFMM, and the rebalancing strategy, on a two-step binomial tree. Panel A depicts the binomial tree. The performance of the buy-and-hold strategy is shown in Panel B; the rebalancing strategy is shown in panel C, and the constant product CFMM is shown in panel D. In each panel, x_t, y_t are the holdings of the strategy, and Π_t is the pool value, $y_t + P_t x_t$.

Comparing with Example 2, observe that (35) states that is the pool can only be the "sum" of θ and $1-\theta$ weighted geometric mean market makers. The two degrees of freedom are intuitive, since θ and $1-\theta$ are exchangeable in (15).

Proof of Theorem 2. We construct the following ODE from the (34) along with (8),

$$P^2 V''(P) + \bar{c} V(P) = 0,$$

with constant $\bar{c} \triangleq 2c/\sigma^2$. Make the substitution $P = e^z$, to arrive at the equivalent ODE,

$$V''(z) - V'(z) + \bar{c} V(z) = 0,$$

which when solved, along with the known limit condition from (1) that $V(z) \to 0$ as $z \to -\infty$, by the usual method of linear ODEs results in the generic solution,

$$V(P) = L_1 P^{\frac{1-\sqrt{1-4\bar{c}}}{2}} + L_2 P^{\frac{1+\sqrt{1-4\bar{c}}}{2}} = L_1 P^{\theta} + L_2 P^{1-\theta}.$$

Note that the above calculation is allowed because the quantity under the root is necessarily non-negative, as if it were not, then $V(P)$ would not be everywhere concave, which must be the case by Lemma 1. ■

B.3. Multi-Dimensional Generalization

In this section, we describe the multi-dimensional generalization of our results. Specifically, denote by vectors $x \in \mathbb{R}^n_+$ the reserves in $n \geq 2$ assets (none of which need be the numéraire), and $P_t \in \mathbb{R}^n_+$ a vector of prices (in terms of the numéraire). We assume that the price vector evolves according to geometric Brownian motion, i.e.,

$$dP_t = \operatorname{diag}(P_t) \Sigma_t^{1/2} \, dB_t^{\mathbb{Q}}, \quad \forall\, t \geq 0,$$

with covariance matrix of returns $\Sigma_t \in \mathbb{R}^{n \times n}$, $\Sigma_t \succeq 0$, and where $B_t^{\mathbb{Q}}$ is a standard $\mathbb{Q}$-Brownian motion on $\mathbb{R}^n$.

Given a bonding function $f \colon \mathbb{R}^n_+ \to \mathbb{R}$, define the pool value function $V \colon \mathbb{R}^n_+ \to \mathbb{R}_+$ according to

$$\begin{aligned} V(P) \triangleq \quad & \underset{x \in \mathbb{R}^n_+}{\text{minimize}} \quad P^\top x \\ & \text{subject to} \quad f(x) = L. \end{aligned}$$

Analogous to Assumption 1, we will assume that an optimal solution $x^*(P)$ exists for all $P \in \mathbb{R}^n_+$, that $V(\cdot)$ is twice continuously differentiable, and a suitable square-integrability condition on $x^*(\cdot)$.

Analogous to Lemma 1, we have

Lemma 2. *For all prices $P \in \mathbb{R}^n_+$, the pool value function satisfies:*

(i) $V(P) \geq 0$.

(ii) $\nabla V(P) = x^*(P) \geq 0$.

(iii) $\nabla^2 V(P) = \nabla x^*(P) \preceq 0$.

Define the rebalancing strategy by $x_t = x^*(P_t)$, with value

$$R_t = V_0 + \int_0^t x^*(P_s)^\top dP_s, \quad \forall\, t \geq 0.$$

Then, we have the following multi-dimensional analog of Theorem 1:

Theorem 3. *Loss-versus-rebalancing takes the form*

$$\mathsf{LVR}_t = \int_0^t \ell(\Sigma_s, P_s)\, ds, \quad \forall\, t \geq 0,$$

where we define, for $P \geq 0$, the instantaneous LVR

$$\ell(\Sigma, P) \triangleq -\tfrac{1}{2} \operatorname{tr}\left[\operatorname{diag}(P) \Sigma \operatorname{diag}(P)\, \nabla x^*(P)\right] \geq 0,$$

where we have applied Lemma 2. In the case where $\Sigma = \sigma^2 I$, i.e., i.i.d. assets, we have that

$$\ell(\Sigma, P) = -\frac{\sigma^2}{2} \operatorname{tr}\left[\operatorname{diag}(P)^2\, \nabla x^*(P)\right] = -\frac{\sigma^2}{2} \sum_{i=1}^n P_i^2 \frac{\partial}{\partial P_i} x^*(P) \geq 0.$$

In particular, LVR *is a non-negative, non-decreasing, and predictable process.*

Proof. Applying Itô's lemma to $V_t = V(P)$,

$$\begin{aligned} dV_t &= \nabla V(P_t)^\top\, dP_t + \tfrac{1}{2} (dP_t)^\top \nabla^2 V(P_t)\, dP_t \\ &= x^*(P_t)^\top\, dP_t + \tfrac{1}{2} \operatorname{tr}\left[\Sigma_t^{1/2} \operatorname{diag}(P) \nabla^2 V(P_t) \operatorname{diag}(P) \Sigma_t^{1/2}\right] dt \\ &= dR_t - \ell(\Sigma_t, P_t)\, dt. \end{aligned}$$

The rest of the result follows as in the proof of Theorem 1. ■

C. Data and Measurement

C.1. Data

Prices. We download minute-level USDC-ETH prices from the Binance API. We use close prices at the end of each minute for P_t.

Uniswap. We download data on the Uniswap v2 WETH-USDC pool from Dune Analytics, a data provider which aggregates data from the Ethereum blockchain into SQL databases. The queries we use to extract this data are included in Appendix C.2.

Mints and burns. In each minute, we observe the gross amounts of each asset in which are withdrawn through "burns", and deposited through "mints". Let $(x_t^{\text{mint}}, y_t^{\text{mint}})$ and $(x_t^{\text{burn}}, y_t^{\text{burn}})$

be the total amounts of each asset x and y which are minted and burned respectively, between time $t-1$ and time t. We will value minted and burned assets at the time-t closing price P_t. Thus, the monetary value of mints and burns respectively are:

$$\Pi_t^{\text{mint}} \triangleq y_t^{\text{mint}} + P_t x_t^{\text{mint}}, \quad \Pi_t^{\text{burn}} \triangleq y_t^{\text{burn}} + P_t x_t^{\text{burn}}.$$

Let (x_t, y_t) be the total asset holdings of the pool at time t. As in the model, define the pool value at time t by

$$V_t \triangleq y_t + P_t x_t.$$

We can calculate $\Delta\text{LP P\&L}_t$, the change in P&L of the pool, from period $t-1$ to t, as

$$\Delta\text{LP } \mathsf{P\&L}_t \triangleq V_t + \Pi_t^{\text{burn}} - \Pi_t^{\text{mint}} - V_{t-1}. \tag{36}$$

In words, this is the value of pool reserves at time t valued at price P_t, plus burned assets and minus minted assets valued at P_t, minus the value of pool reserves at time $t-1$ valued at P_{t-1}. Note that, in contrast to our simplifying assumption in the model that fees are paid in the numéraire, in practice in Uniswap v2 fees are paid directly into the pool reserves; hence, the pool P&L includes transaction fees paid into the pool.

Rebalancing strategy. We rebalance the pool at different time frequencies. For each rebalancing frequency, we compute the returns of a strategy which at any point in time holds as much ETH as the pool holds at the start of the period. For example, if the rebalancing frequency is daily, we set x_t^{RB} at any minute t equal to the LP pool reserves at the start of the day containing the minute t. We then calculate the returns on the rebalancing strategy using expression (18), that is:

$$\Delta\text{RB } \mathsf{P\&L}_t = x_t^{RB}\left(P_{t+1} - P_t\right). \tag{37}$$

Fees. In each minute, we compute the gross amount of each asset in the pair bought and sold. The Uniswap v2 pool has a fixed fee rate of 30bps on the contributed asset; we thus calculate fees in each asset by multiplying the gross amount contributed of each asset by 0.003. Call these fees x_t^{fee} and y_t^{fee} in period t. We value fees at the period t price; thus, the monetary value of fees in period t, which we will call $\Delta\mathsf{FEE}_t$, is:

$$\Delta\mathsf{FEE}_t \triangleq y_t^{fee} + P_t x_t^{fee}. \tag{38}$$

LVR. We compute a realized daily volatility using USDC-ETH prices from the Binance API sampled at 60 minute intervals. Let $\Delta\mathsf{LVR}_t$ be the increment of LVR in period t. As in Example 3, we then calculate $\Delta\mathsf{LVR}_t$ simply as

$$\Delta\mathsf{LVR}_t \triangleq \frac{\hat{\sigma}_t^2}{8} \times V_t \times \Delta t, \tag{39}$$

where $\hat{\sigma}_t$ denotes the realized daily volatilty estimate for the day containing period t, and $\Delta t = 1/(24 \times 60)$ corresponds to a one minute period. This is a discrete approximation of (19).

Adding everything up. We then calculate the cumulative returns, the left side of (17), as:

$$\mathsf{LP\ P\&L}_t - \int_0^t x^*(P_s)\,dP_s \triangleq \sum_{t=1}^{T} \left(\Delta\mathsf{P\&L}_t - \Delta\mathrm{RB}\ \mathsf{P\&L}_t\right),$$

using the definitions of $\Delta\mathrm{LP}\ \mathsf{P\&L}_t$ and $\Delta\mathrm{RB}\ \mathsf{P\&L}_t$ in (36) and (37) respectively. Note that different rebalancing frequencies give slightly different values of for $\Delta\mathrm{RB}\ \mathsf{P\&L}_t$. We calculate the right side of (17) as:

$$\mathsf{FEE}_t - \mathsf{LVR}_t \triangleq \sum_{t=1}^{T} \left(\Delta\mathsf{FEE}_t - \Delta\mathsf{LVR}_t\right),$$

using the definitions of $\Delta\mathsf{FEE}_t$ and $\Delta\mathsf{LVR}_t$ in (38) and (39) respectively.

C.2. Dune SQL Queries

This appendix contains the SQL queries we use on Dune to extract Uniswap v2 WETH-USDC data.

Mints.

```
SELECT to_char(evt_block_time, 'YYYY-MM-DD"T"HH24:MI:SSOF') AS ts, *
FROM uniswap_v2."Pair_evt_Mint"
WHERE contract_address = '\xb4e16d0168e52d35cacd2c6185b44281ec28c9dc'
ORDER BY evt_block_number, evt_index ASC
```

Burns.

```
SELECT to_char(evt_block_time, 'YYYY-MM-DD"T"HH24:MI:SSOF') AS ts, *
FROM uniswap_v2."Pair_evt_Burn"
WHERE contract_address = '\xb4e16d0168e52d35cacd2c6185b44281ec28c9dc'
ORDER BY evt_block_number, evt_index ASC
```

Trades.

```
SELECT
date_trunc('minute', evt_block_time) AS minute,
SUM("amount0In") as "amount0In",
SUM("amount1In") as "amount1In",
SUM("amount0Out") as "amount0Out",
SUM("amount1Out") as "amount1Out"
FROM uniswap_v2."Pair_evt_Swap"
WHERE contract_address =  '\xb4e16d0168e52d35cacd2c6185b44281ec28c9dc'
GROUP BY 1
ORDER BY 1 ASC
```

Pool reserves.

```
SELECT
minute,
latest_reserves[3] AS reserve0,
latest_reserves[4] AS reserve1
FROM
(SELECT date_trunc('minute', evt_block_time) AS minute,
        (SELECT MAX(ARRAY[evt_block_number, evt_index, reserve0, reserve1]))
             AS latest_reserves
         FROM uniswap_v2."Pair_evt_Sync"
         WHERE contract_address =  '\xb4e16d0168e52d35cacd2c6185b44281ec28c9dc'
         GROUP BY 1) AS day_reserves
ORDER BY 1 ASC
```