

1 New Characterizations of Core Imputations of 2 Matching and b -Matching Games 3 (Extended Abstract)

4 **Vijay V. Vazirani** ✉

5 University of California, Irvine, Irvine, CA, USA

6 — Abstract —

7 We give new characterizations of core imputations for the following games:

- 8 1. The assignment game.
- 9 2. Concurrent games, i.e., general graph matching games having non-empty core.
- 10 3. The unconstrained bipartite b -matching game (edges can be matched multiple times).
- 11 4. The constrained bipartite b -matching game (edges can be matched at most once).

12 The classic paper of Shapley and Shubik [11] showed that core imputations of the assignment
13 game are precisely optimal solutions to the dual of the LP-relaxation of the game. Building on
14 this, Deng et al. [5] gave a general framework which yields analogous characterizations for several
15 fundamental combinatorial games. Interestingly enough, their framework does not apply to the
16 last two games stated above. In turn, we show that some of the core imputations of these games
17 correspond to optimal dual solutions and others do not. This leads to the tantalizing question of
18 understanding the origins of the latter.

19 We also present new characterizations of the profits accrued by agents and teams in core
20 imputations of the first two games. Our characterization for the first game is stronger than that for
21 the second; the underlying reason is that the characterization of vertices of the Birkhoff polytope is
22 stronger than that of the Balinski polytope.

23 **2012 ACM Subject Classification** Theory of computation → Algorithmic game theory

24 **Keywords and phrases** LP-duality theory, cooperative game theory, core of a game, assignment
25 game, general graph matching game, bipartite b -matching game

26 **Digital Object Identifier** 10.4230/LIPIcs...13

27 **Funding** *Vijay V. Vazirani*: Supported in part by NSF grants CCF-1815901 and CCF-2230414.



© Vijay V. Vazirani;

licensed under Creative Commons License CC-BY 4.0

Leibniz International Proceedings in Informatics

LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

28 **1** Introduction

29 The matching game forms one of the cornerstones of cooperative game theory and the *core*
 30 is a quintessential solution concept in this theory; the latter captures all possible ways of
 31 distributing the total worth of a game among individual agents in such a way that the grand
 32 coalition remains intact, i.e., a sub-coalition will not be able to generate more profits by itself
 33 and therefore has no incentive to secede from the grand coalition. The matching game can
 34 also be viewed as a matching market in which utilities of the agents are stated in monetary
 35 terms and side payments are allowed, i.e., it is a *transferable utility (TU) market*. For an
 36 extensive coverage of these notions, see the book by Moulin [9]. Due to space restrictions,
 37 we have not presented proofs in this version of the paper; for that, see the full version [12].

38 The classic paper of Shapley and Shubik [11] showed that the set of core imputations of
 39 the assignment game as the set of optimal solutions to the dual of the LP-relaxation of the
 40 maximum weight matching problem in the underlying graph. Among their other insights
 41 was a characterization of the two “antipodal” points — imputations which maximally favor
 42 one side of the bipartition¹ — in the core of this game. This in-depth understanding makes
 43 the assignment game a paradigmatic setting for studying the core; in turn, insights gained
 44 provide valuable guidance on profit-sharing in real-life situations.

45 Deng et al. [5] distilled the ideas underlying the Shapley-Shubik Theorem to obtain a
 46 general framework (see Section 4.1.1) which helps characterize the core of several games that
 47 are based on fundamental combinatorial optimization problems, including maximum flow
 48 in unit capacity networks both directed and undirected, maximum number of edge-disjoint
 49 s - t paths, maximum number of vertex-disjoint s - t paths, maximum number of disjoint
 50 arborescences rooted at a vertex r , and concurrent games (defined below).

51 In this paper, we study the core of the assignment game and some of its generalizations,
 52 including two versions of the bipartite b -matching game (Section 4); in the first version
 53 (Section 4.2) edges can be matched multiple number of times and in the second, edges can be
 54 matched at most once (Section 5). The intriguing aspect of the latter two games is that they
 55 don’t fall in framework of Deng et al.; see Section 4.1.1 for the reason. In turn, we show that
 56 some of the core imputations of these games correspond to optimal dual solutions and some
 57 not. This leads to a tantalizing question: is there a “mathematical structure” that produces
 58 the latter?

59 For the assignment game (Section 3), we start with the realization is that despite the
 60 in-depth work of Shapely and Shubik, and the passage of half a century, there are still basic
 61 questions about the core which have remained unexplored:

- 62 1. Do core imputations spread the profit more-or-less evenly or do they restrict them to
 63 certain well-chosen agents? If the latter, what characterizes these “chosen” agents?
- 64 2. By definition, under any core imputation, the sum of profits of two agents i and j is at
 65 least the profit they make by being matched, say w_{ij} . What characterizes pairs (i, j) for
 66 which this sum strictly exceed w_{ij} ?
- 67 3. How do core imputations behave in the presence of degeneracy?

68 An assignment game is said to be *degenerate* if the optimal assignment is not unique.
 69 Although Shapley and Shubik had mentioned this phenomenon, they brushed it away,
 70 claiming that “in the most common case” the optimal assignment will be unique, and if
 71 not, their suggestion was to perturb the edge weights to make the optimal assignment

¹ Much like the top and bottom elements in a lattice of stable matchings.

72 unique. However, this is far from satisfactory, since perturbing the weights destroys crucial
 73 information contained in the original instance and the outcome becomes a function of the
 74 vagaries of the randomness imposed on the instance.

75 The following broad idea helps answer all three questions. A well-known theorem in
 76 matching theory says that the LP-relaxation of the optimal assignment problem always has
 77 an integral optimal solution [8]. Therefore, the worth of the assignment game is given by the
 78 optimal objective function value of this LP. Next, the Shapley-Shubik Theorem says that the
 79 set of core imputations of this game are precisely the optimal solutions to the dual of this
 80 LP. These two facts naturally raise the question of viewing core imputations through the
 81 lens of complementarity; in turn, it leads to a resolution of all three questions.

82 The following setting, taken from [6] and [2], vividly captures the issues underlying
 83 profit-sharing in an assignment game. Suppose a coed tennis club has sets U and V of
 84 women and men players, respectively, who can participate in an upcoming mixed doubles
 85 tournament. Assume $|U| = m$ and $|V| = n$, where m, n are arbitrary. Let $G = (U, V, E)$ be a
 86 bipartite graph whose vertices are the women and men players and an edge (i, j) represents
 87 the fact that agents $i \in U$ and $j \in V$ are eligible to participate as a mixed doubles team
 88 in the tournament. Let w be an edge-weight function for G , where $w_{ij} > 0$ represents the
 89 expected earnings if i and j do participate as a team in the tournament. The total worth of
 90 the game is the weight of a maximum weight matching in G .

91 Assume that the club picks such a matching for the tournament. The question is how to
 92 distribute the total profit among the agents — strong players, weak players and unmatched
 93 players — so that no subset of players feel they will be better off seceding and forming their
 94 own tennis club. We will use this setting to discuss the issues involved in the questions raised
 95 above.

96 Under core imputations, the profit allocated to an agent is a function of the value he/she
 97 brings to the various sub-coalitions he/she belongs to, i.e., it is consistent with his/her
 98 negotiating power. Indeed, it is well known that core imputations provide profound insights
 99 into the negotiating power of individuals and sub-coalitions, see [9]. The first question
 100 provides further insights into this issue. Our answer to this question is that the core rewards
 101 only *essential* agents, namely those who are matched by *every* maximum weight matching,
 102 see Theorem 10.

103 Our answer to the second question is quite counter-intuitive: we show that a pair of
 104 players (i, j) get overpaid by core allocations if and only if they are so incompetent, as a
 105 team, that they don't participate in any maximum weight matching! Since i and j are
 106 incompetent as a team, w_{ij} is small. On the other hand, a least one of i and j does team up
 107 with other agents in maximum weight matchings — if not, (i, j) would have been matched in
 108 a maximum weight matching. Therefore, the sum of the profits of i and j exceeds w_{ij} in at
 109 least one core imputation; this is shown in Theorem 15.

110 Our insight into degeneracy is that it treats teams and agents in totally different ways,
 111 see Section 3.4. Section 2 discusses past approaches to degeneracy.

112 Whereas the core of the assignment game is always non-empty, that of the general graph
 113 matching game can be empty. Deng et al. [5] showed that the core of this game is non-empty
 114 if and only if the weights of maximum weight integral and fractional matchings concur. For
 115 this reason, we have named such games as *concurrent games*. As stated above, their core
 116 imputations are precisely the set of optimal solutions to the dual LP.

117 In the full paper [12], we study the three questions, raised above, for concurrent games as
 118 well.

119 **2 Related Works**

120 An imputation in the core has to ensure that *each* of the exponentially many sub-coalitions
 121 is “happy” — clearly, that is a lot of constraints. As a result, the core is non-empty only for
 122 a handful of games, some of which are mentioned in the Introduction. A different kind of
 123 game, in which preferences are cardinal, is based on the stable matching problem defined by
 124 Gale and Shapley [7]. The only coalitions that matter in this game are ones formed by one
 125 agent from each side of the bipartition. A stable matching ensures that no such coalition has
 126 the incentive to secede and the set of such matchings constitute the core of this game.

127 Over the years, researchers have approached the phenomenon of degeneracy in the
 128 assignment game from directions that are different from ours. Nunez and Rafels [10], studied
 129 relationships between degeneracy and the dimension of the core. They defined an agent
 130 to be *active* if her profit is not constant across the various imputations in the core, and
 131 non-active otherwise. Clearly, this notion has much to do with the dimension of the core,
 132 e.g., it is easy to see that if all agents are non-active, the core must be zero-dimensional.
 133 They prove that if all agents are active, then the core is full dimensional if and only if the
 134 game is non-degenerate. Furthermore, if there are exactly two optimal matchings, then the
 135 core can have any dimension between 1 and $m - 1$, where m is the smaller of $|U|$ and $|V|$;
 136 clearly, m is an upper bound on the dimension.

137 In another work, Chambers and Echenique [3] study the following question: Given the
 138 entire set of optimal matchings of a game on $m = |U|$, $n = |V|$ agents, is there an $m \times n$
 139 surplus matrix which has this set of optimal matchings. They give necessary and sufficient
 140 conditions for the existence of such a matrix.

141 **3 The Core of the Assignment Game**

142 In this section, we provide answers to the three questions, for assignment games, which were
 143 raised in the Introduction.

144 **3.1 Definitions and Preliminary Facts**

145 The *assignment game*, $G = (U, V, E)$, $w : E \rightarrow \mathcal{R}_+$, has been defined in the Introduction.
 146 We start by giving definitions needed to state the Shapley-Shubik Theorem.

147 **► Definition 1.** *The set of all players, $U \cup V$, is called the grand coalition. A subset of the*
 148 *players, $(S_u \cup S_v)$, with $S_u \subseteq U$ and $S_v \subseteq V$, is called a coalition or a sub-coalition.*

149 **► Definition 2.** *The worth of a coalition $(S_u \cup S_v)$ is defined to be the maximum profit that*
 150 *can be generated by teams within $(S_u \cup S_v)$ and is denoted by $p(S_u \cup S_v)$. Formally, $p(S_u \cup S_v)$*
 151 *is the weight of a maximum weight matching in the graph G restricted to vertices in $(S_u \cup S_v)$*
 152 *only. $p(U \cup V)$ is called the worth of the game. The characteristic function of the game is*
 153 *defined to be $p : 2^{U \cup V} \rightarrow \mathcal{R}_+$.*

154 **► Definition 3.** *An imputation² gives a way of dividing the worth of the game, $p(U \cup V)$,*
 155 *among the agents. It consists of two functions $u : U \rightarrow \mathcal{R}_+$ and $v : V \rightarrow \mathcal{R}_+$ such that*
 156 $\sum_{i \in U} u(i) + \sum_{j \in V} v(j) = p(U \cup V)$.

² Some authors prefer to call this a pre-imputation, while using the term imputation when individual rationality is also satisfied.

157 ► **Definition 4.** An imputation (u, v) is said to be in the core of the assignment game if for
 158 any coalition $(S_u \cup S_v)$, the total worth allocated to agents in the coalition is at least as large
 159 as the worth that they can generate by themselves, i.e., $\sum_{i \in S_u} u(i) + \sum_{j \in S_v} v(j) \geq p(S)$.

160 We next describe the characterization of the core of the assignment game given by Shapley
 161 and Shubik [11]³.

162 As stated in Definition 2, the worth of the game, $G = (U, V, E)$, $w : E \rightarrow \mathcal{R}_+$, is the
 163 weight of a maximum weight matching in G . Linear program (1) gives the LP-relaxation of
 164 the problem of finding such a matching. In this program, variable x_{ij} indicates the extent to
 165 which edge (i, j) is picked in the solution. Matching theory tells us that this LP always has
 166 an integral optimal solution [8]; the latter is a maximum weight matching in G .

$$\begin{aligned}
 & \max && \sum_{(i,j) \in E} w_{ij} x_{ij} \\
 & \text{s.t.} && \sum_{(i,j) \in E} x_{ij} \leq 1 \quad \forall i \in U, \\
 & && \sum_{(i,j) \in E} x_{ij} \leq 1 \quad \forall j \in V, \\
 & && x_{ij} \geq 0 \quad \forall (i,j) \in E
 \end{aligned} \tag{1}$$

168 Taking u_i and v_j to be the dual variables for the first and second constraints of (1), we
 169 obtain the dual LP:

$$\begin{aligned}
 & \min && \sum_{i \in U} u_i + \sum_{j \in V} v_j \\
 & \text{s.t.} && u_i + v_j \geq w_{ij} \quad \forall (i,j) \in E, \\
 & && u_i \geq 0 \quad \forall i \in U, \\
 & && v_j \geq 0 \quad \forall j \in V
 \end{aligned} \tag{2}$$

171 ► **Theorem 5.** (Shapley and Shubik [11]) The imputation (u, v) is in the core of the assignment
 172 game if and only if it is an optimal solution to the dual LP, (2).

173 By Theorem 5, the core of the assignment game is a convex polyhedron. Shapley and
 174 Shubik shed further light on the structure of the core by showing that it has two special
 175 imputations which are furthest apart and so can be thought of as antipodal imputations.
 176 In the tennis club setup, one of these imputations maximizes the earnings of women players
 177 and the other maximizes the earnings of men players.

178 Finally, we state a fundamental fact about LP (1); its corollary will be used in a crucial
 179 way in Theorems 10 and 15.

180 ► **Theorem 6.** (Birkhoff [1]) The vertices of the polytope defined by the constraints of LP
 181 (1) are 0/1 vectors, i.e., they are matchings in G .

³ Shapley and Shubik had described this game in the context of the housing market in which agents are of two types, buyers and sellers. They had shown that each imputation in the core of this game gives rise to unique prices for all the houses. In this paper we will present the assignment game in a variant of the tennis setting given in the Introduction; this will obviate the need to define “prices”, hence leading to simplicity.

182 ► **Corollary 7.** *Any fractional matching in a bipartite graph is a convex combination of*
 183 *integral matchings.*

184 3.2 The first question: Allocations made to agents by core imputations

185 ► **Definition 8.** *A generic player in $U \cup V$ will be denoted by q . We will say that q is:*

- 186 1. *essential if q is matched in every maximum weight matching in G .*
- 187 2. *viable if there is a maximum weight matching M such that q is matched in M and another,*
 188 *M' such that q is not matched in M' .*
- 189 3. *subpar if for every maximum weight matching M in G , q is not matched in M .*

190 ► **Definition 9.** *Let y be an imputation in the core. We will say that q gets paid in y if*
 191 *$y_q > 0$ and does not get paid otherwise. Furthermore, q is paid sometimes if there is at least*
 192 *one imputation in the core under which q gets paid, and it is never paid if it is not paid*
 193 *under every imputation.*

194 ► **Theorem 10.** *For every player $q \in (U \cup V)$:*

195 *q is paid sometimes $\iff q$ is essential*

196 Theorem 10 is equivalent to the following. For every player $q \in (U \cup V)$:

197 *q is never paid $\iff q$ is not essential*

198 Thus core imputations pay only essential players and each of them is paid in some core
 199 imputation. Since we have assumed that the weight of each edge is positive, so is the worth
 200 of the game, and all of it goes to essential players. Hence we get:

201 ► **Corollary 11.** *In the assignment game, the set of essential players is non-empty and in*
 202 *every core imputation, the entire worth of the game is distributed among essential players;*
 203 *moreover, each of them is paid in some core imputation.*

204 ► **Remark 12.** Theorem 5 and Corollary 11 are of much consequence.

- 205 1. Corollary 11 reveals the following surprising fact: the set of players who are allocated
 206 profits in a core imputation is *independent* of the set of teams that play.
- 207 2. The identification of these players, and the exact manner in which the total profit is
 208 divided among them, follows the negotiating process. In turn, this process identifies
 209 agents who play in *all* possible maximum weight matchings.
- 210 3. Perhaps the most remarkable aspect of Theorem 5 is that each possible outcome of this
 211 very real process is captured by an inanimate object, namely an optimal solution to the
 212 dual, LP (2).

213 By Corollary 11, core imputations reward only essential players. This raises the following
 214 question: Can't a non-essential player, say q , team up with another player, say p , and secede,
 215 by promising p almost all of the resulting profit? The answer is "No", because the dual (2)
 216 has the constraint $y_q + y_p \geq w_{qp}$. Therefore, if $y_q = 0$, $y_p \geq w_{qp}$, i.e., p will not gain by
 217 seceding together with q .

218 3.3 The second question: Allocations made to teams by core 219 imputations

220 ► **Definition 13.** *By a mixed doubles team we mean an edge in G ; a generic one will be*
 221 *denoted as $e = (u, v)$. We will say that e is:*

- 222 1. essential if e is matched in every maximum weight matching in G .
 223 2. viable if there is a maximum weight matching M such that $e \in M$, and another, M' such
 224 that $e \notin M'$.
 225 3. subpar if for every maximum weight matching M in G , $e \notin M$.

226 ► **Definition 14.** Let y be an imputation in the core of the game. We will say that e is fairly
 227 paid in y if $y_u + y_v = w_e$ and it is overpaid if $y_u + y_v > w_e$ ⁴. Finally, we will say that e is
 228 always paid fairly if it is fairly paid in every imputation in the core.

229 ► **Theorem 15.** For every team $e \in E$:

230 e is always paid fairly $\iff e$ is viable or essential

231 Negating both sides of the implication proved in Theorem 15, we get the following
 232 implication. For every team $e \in E$:

233 e is subpar $\iff e$ is sometimes overpaid

234 Clearly, this statement is equivalent to the statement proved Theorem 15 and hence
 235 contains no new information. However, it provides a new viewpoint. These two equivalent
 236 statements yield the following assertion, which at first sight seems incongruous with what we
 237 desire from the notion of the core and the just manner in which it allocates profits:

238 *Whereas viable and essential teams are always paid fairly, subpar teams are sometimes*
 239 *overpaid.*

240 How can the core favor subpar teams over viable and essential teams? An explanation
 241 is provided in the Introduction, namely a subpar team (i, j) gets overpaid because i and j
 242 create worth by playing in competent teams with other players.

243 Finally, we observe that contrary to Corollary 11, which says that the set of essential
 244 players is non-empty, the set of essential teams may be empty.

245 3.4 The third question: Degeneracy

246 Next we use Theorems 10 and 15 to get insights into degeneracy. Clearly, if an assignment
 247 game is non-degenerate, then every team and every player is either always matched or always
 248 unmatched in the set of maximum weight matchings in G , i.e., there are no viable teams
 249 or players. Since viable teams and players arise due to degeneracy, in order to understand
 250 the phenomenon of degeneracy, we need to understand how viable teams and players behave
 251 with respect to core imputations; this is done in the next corollary.

252 ► **Corollary 16.** In the presence of degeneracy, imputations in the core of an assignment
 253 game treat:

- 254 ■ viable players in the same way as subpar players, namely they are never paid.
- 255 ■ viable teams in the same way as essential teams, namely they are always fairly paid.

256 4 The Core of Bipartite b -Matching Games

257 In this section, we will define two versions of the *bipartite b -matching game* and we will study
 258 their core imputations; both versions generalize the assignment game.

⁴ Observe that by the first constraint of the dual LP (2), these are the only possibilities.

259 **4.1 Definitions and Preliminary Facts**

260 As in the assignment game, let $G = (U, V, E)$, $w : E \rightarrow \mathcal{R}_+$ be the underlying bipartite graph
 261 and edge-weight function. Let function $b : U \cup V \rightarrow \mathbf{Z}_+$ give a bound on the number of times
 262 a vertex can be matched. Under the *unconstrained bipartite b -matching game*, each edge
 263 can be matched multiple number of times and under the *constrained bipartite b -matching*
 264 *game*, each edge can be matched at most once. Observe that even in the first version, limits
 265 imposed by b on vertices will impose limits on edges — thus edge (i, j) can be matched at
 266 most $\min\{b_i, b_j\}$ times.

267 The *worth* of a coalition $(S_u \cup S_v)$, with $S_u \subseteq U, S_v \subseteq V$, is the weight of a maximum
 268 weight b -matching in the graph G restricted to vertices in $(S_u \cup S_v)$ only; we will denote this
 269 by $p(S_u \cup S_v)$. Whether an edge can be matched at most once or more than once depends
 270 on the version of the problem we are dealing with. $p(U \cup V)$ is called the *worth of the game*.
 271 The *characteristic function* of the game is defined to be $p : 2^{U \cup V} \rightarrow \mathcal{R}_+$. Definitions 3 and 4,
 272 defining an imputation and the core, carry over unchanged from the assignment game.

273 The tennis setting, given in the Introduction, provides a vivid description of these two
 274 variants of the b -matching game as well. Let K denote the maximum b -value of a vertex and
 275 assume that the tennis club needs to enter mixed doubles teams into K tennis tournaments.
 276 In the first variant, a team can play in multiple tournaments and in the second version, a
 277 team can play in at most one tournament. In both cases, a player i can play in at most b_i
 278 tournaments. The goal of the tennis club is to maximize its profit over all the tournaments
 279 and hence picks a maximum weight b -matching in G . An imputation in the core gives a way
 280 of distributing the profit in such a way that no sub-coalition has an incentive to secede.

281 Linear program (3) gives the LP-relaxation of the problem of finding a maximum weight
 282 b -matching for the unconstrained version. In this program, variable x_{ij} indicates the extent
 283 to which edge (i, j) is picked in the solution; observe that there is no upper bound on the
 284 variables x_{ij} since an edge can be matched any number of times.

$$\begin{aligned}
 & \max \quad \sum_{(i,j) \in E} w_{ij} x_{ij} \\
 & \text{s.t.} \quad \sum_{(i,j) \in E} x_{ij} \leq b_i \quad \forall i \in U, \\
 & \quad \quad \sum_{(i,j) \in E} x_{ij} \leq b_j \quad \forall j \in V, \\
 & \quad \quad x_{ij} \geq 0 \quad \forall (i, j) \in E
 \end{aligned} \tag{3}$$

286 Taking u_i and v_j to be the dual variables for the first and second constraints of (3), we
 287 obtain the dual LP:

$$\begin{aligned}
 & \min \quad \sum_{i \in U} b_i u_i + \sum_{j \in V} b_j v_j \\
 & \text{s.t.} \quad u_i + v_j \geq w_{ij} \quad \forall (i, j) \in E, \\
 & \quad \quad u_i \geq 0 \quad \forall i \in U, \\
 & \quad \quad v_j \geq 0 \quad \forall j \in V
 \end{aligned} \tag{4}$$

289 Linear program (5) gives the LP-relaxation of the problem of finding a maximum weight

290 b -matching for the constrained version. Observe that in this program, variables x_{ij} are upper
 291 bounded by 1, since an edge can be matched at most once.

$$\begin{aligned}
 & \max \quad \sum_{(i,j) \in E} w_{ij} x_{ij} \\
 & \text{s.t.} \quad \sum_{(i,j) \in E} x_{ij} \leq b_i \quad \forall i \in U, \\
 & \quad \sum_{(i,j) \in E} x_{ij} \leq b_j \quad \forall j \in V, \\
 & \quad x_{ij} \leq 1 \quad \forall (i,j) \in E, \\
 & \quad x_{ij} \geq 0 \quad \forall (i,j) \in E
 \end{aligned} \tag{5}$$

293 ► **Remark 17.** In both in LPs (3) and (5), the matrices of coefficients of the constraints are
 294 totally unimodular [8], and therefore both LPs always have integral optimal solutions.

295 Taking u_i , v_j and z_{ij} to be the dual variables for the first, second and third constraints
 296 of (5), we obtain the dual LP:

$$\begin{aligned}
 & \min \quad \sum_{i \in U} b_i u_i + \sum_{j \in V} b_j v_j + \sum_{(i,j) \in E} z_{ij} \\
 & \text{s.t.} \quad u_i + v_j + z_{ij} \geq w_{ij} \quad \forall (i,j) \in E, \\
 & \quad u_i \geq 0 \quad \forall i \in U, \\
 & \quad v_j \geq 0 \quad \forall j \in V, \\
 & \quad z_{ij} \geq 0 \quad \forall (i,j) \in E
 \end{aligned} \tag{6}$$

298 4.1.1 The Framework of Deng et al. [5]

299 In this section, we present the framework of Deng et al. [5], which was mentioned in the
 300 Introduction, and point out why it does not apply to the two versions of the b -matching game.
 301 Let $T = \{1, \dots, n\}$ be the set of n agents of the game. Let $w \in \mathbb{R}_+^m$ be an m -dimensional
 302 non-negative real vector specifying the weights of certain objects; in the assignment game,
 303 the objects are edges of the underlying graph. Let A be an $n \times m$ matrix with 0/1 entries
 304 whose i^{th} row corresponds to agent $i \in T$. Let x be an m -dimensional vector of variables
 305 and $\mathbb{1}$ be the n -dimensional vector of all 1s. Assume that the worth of the game is given by
 306 the objective function value of following integer program.

$$\begin{aligned}
 & \max \quad w \cdot x \\
 & \text{s.t.} \quad Ax \leq \mathbb{1}, \\
 & \quad x \in \{0, 1\}
 \end{aligned} \tag{7}$$

308 Moreover, for a sub-coalition, $T' \subseteq T$ assume that its worth is given by the integer
 309 program obtained by replacing A by A' in (7), where A' picks the set of rows corresponding
 310 to agents in T' . The LP-relaxation of (7) is:

$$\begin{aligned}
 & \max \quad w \cdot x \\
 & \text{s.t.} \quad Ax \leq \mathbb{1}, \\
 & \quad x \geq 0
 \end{aligned} \tag{8}$$

13:10 New Characterizations of Core Imputations

312 Deng et al. proved that if LP (8) always has an integral optimal solution, then the set of
313 core imputations of this game is exactly the set of optimal solutions to the dual of LP (8).

314 As stated in Remark 17, the matrices of coefficients of both LPs (3) and (5) are totally
315 unimodular and therefore these LPs always have integral optimal solutions. However, they
316 still don't fall in the above-stated framework because their right-hand-sides are b values of
317 the vertices and not \mathcal{K} .

318 4.2 The Core of the Unconstrained Bipartite b -Matching Game

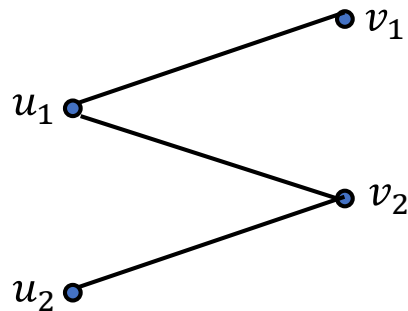
319 Let I denote an instance of this game and let $C(I)$ denote its set of core imputations. We
320 will show in Theorem 18 that corresponding to every optimal solution to the dual LP (4),
321 there is an imputation in $C(I)$. Let $D(I)$ denote the set of all such core imputations. Since
322 $D(I) \neq \emptyset$, we get Corollary 19 stating that the core of this game is non-empty. Next, we
323 will give an instance I such that $D(I) \subset C(I)$, i.e., unlike the assignment game, I has core
324 imputations that don't correspond to optimal solutions to the dual LP.

325 The correspondence between optimal solutions to the dual LP (4) and core imputations
326 in $D(I)$ is as follows. Given an optimal solution (u, v) , define the profit allocation to $i \in U$
327 to be $\alpha_i = b_i \cdot u_i$ and that to $j \in V$ to be $\beta_j = b_j \cdot v_j$.

328 ► **Theorem 18.** *The profit-sharing method (α, β) , which corresponds to an optimal solution*
329 *(u, v) to the dual LP (4), is an imputation in the core of the unconstrained bipartite b -matching*
330 *game.*

331 ► **Corollary 19.** *The core of the unconstrained bipartite b -matching game is always non-empty.*

332 ► **Remark 20.** Observe that the mapping given from optimal solutions to the dual LP (4) to
333 core imputations in $D(I)$ is a bijection.



■ **Figure 1** The graph for Example 21.

334 ► **Example 21.** For the bipartite b -matching game defined by the graph of Figure 1, let the b
335 values be 2, 1, 2, 1 for u_1, u_2, v_1, v_2 , and let the edge weights be 1, 3, 1 for $(u_1, v_1), (u_1, v_2), (u_2, v_2)$.

336 In this section, we will view the game defined in Example 21 as an unconstrained bipartite
337 b -matching game and will show that it has a set of core imputations which do not correspond
338 to optimal dual solutions, i.e., they lie in $C(I) - D(I)$. The optimal matching picks edges

339 $(u_1, v_1), (u_1, v_2)$ once each, for a total profit of 4. The unique optimal dual solution is 1, 0, 0, 2
 340 for u_1, u_2, v_1, v_2 , and the corresponding core imputation is 2, 0, 0, 2.

341 Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be the profits allocated to u_1, u_2, v_1, v_2 . The solutions of the system
 342 of linear inequalities (9), for non-negative values of the variables, capture all possible core
 343 imputations, i.e., the set $C(I)$.

$$\begin{aligned}
 & \alpha_1 + \beta_1 \geq 2 \\
 & \alpha_1 + \beta_2 \geq 3 \\
 & \alpha_1 + \beta_1 + \beta_2 \geq 4 \\
 344 & \alpha_2 + \beta_2 \geq 1 \\
 & \alpha_1 + \alpha_2 + \beta_2 \geq 3 \\
 & \alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 4
 \end{aligned} \tag{9}$$

345 On solving this system, we find that $\alpha_1, \alpha_2, \beta_1, \beta_2$ should be $1+a, 0, b, 1+c$, where a, b, c
 346 are non-negative and satisfy the system (10).

$$\begin{aligned}
 & a + b \geq 1 \\
 347 & a + c \geq 1 \\
 & a + b + c = 2
 \end{aligned} \tag{10}$$

348 A fourth constraint, $b \leq 1$ follows from the last two in this system. The solution
 349 $a = 1, b = 0, c = 1$ gives the core imputation corresponding to the unique optimal dual
 350 solution; the rest give the remaining core imputations, e.g., the imputation 3, 0, 0, 1.

351 For an arbitrary instance I , one can clearly capture all possible core imputations via an
 352 exponential sized system of inequalities of the type \geq , one corresponding to each coalition
 353 $(S_u \cup S_v)$; its r.h.s. will be $p(S_u \cup S_v)$ and its l.h.s. will be the sum of all variables denoting
 354 profits accrued to vertices in this coalition. Note that all the variables of this system will be
 355 constrained to be non-negative and it will have one equality corresponding to the worth of
 356 the grand coalition; the latter is the last equality in system (9).

357 The following question arises: is there a smaller system which accomplishes this task?
 358 We observe that it suffices to include in the system only those coalitions whose induced
 359 subgraph is connected. This is so because if the induced subgraph for coalition $(S_u \cup S_v)$
 360 has two or more connected components, then the sum of the inequalities for the connected
 361 components yields the inequality for coalition $(S_u \cup S_v)$. In particular, if the underlying
 362 graph of instance I is sparse, this may lead to a much smaller system. Observe that the
 363 system (9), for Example 21, follows from this idea.

364 ► **Remark 22.** Since for the unconstrained bipartite b -matching game, the optimal dual
 365 solutions don't capture all core imputations, the characterizations established in Theorems
 366 10 and 15 for the assignment game, don't carry over. However, if one restricts to core
 367 imputations in the set $D(I)$ only, one can see that suitable modifications of these statements
 368 do hold.

369 **5 The Core of the Constrained Bipartite b -Matching Game**

370 Our results for this game are related to, though not identical with, those for the unconstrained
 371 version. In Theorem 23, we will show that corresponding to every optimal solution to the

13:12 New Characterizations of Core Imputations

372 dual LP (6), there is a set of core imputations. This theorem yields Corollary 24 stating that
 373 the core of this game is also non-empty. Finally, we will give an instance which has core
 374 imputations that don't correspond to optimal solutions to the dual LP.

The corresponding to an optimal solution to the dual LP (4), (u, v, z) , we define a set of imputations as follows. For each edge (i, j) define two new variables c_{ij} and d_{ij} ; both are constrained to be non-negative. Furthermore, consider all possible ways of splitting z_{ij} into c_{ij} and d_{ij} , i.e., $z_{ij} = c_{ij} + d_{ij}$. Observe that if $x_{ij} = 0$ then $z_{ij} = 0$ and therefore $c_{ij} = d_{ij} = 0$. Define the profit allocation to $i \in U$ to be

$$\alpha_i = b_i \cdot u_i + \sum_{(i,j) \in E} c_{ij}$$

and that to $j \in V$ to be

$$\beta_j = b_j \cdot v_j + \sum_{(i,j) \in E} d_{ij}.$$

375 Taken over all possible ways of splitting all z_{ij} s, this gives a set of imputations.

376 ► **Theorem 23.** *All profit-sharing methods (α, β) , which correspond to the optimal solution*
 377 *(u, v, z) to the dual LP (6), are imputations in the core of the constrained bipartite b -matching*
 378 *game.*

379 ► **Corollary 24.** *The core of the constrained bipartite b -matching game is always non-empty.*

380 In this section, we will view the game defined in Example 21 as a constrained bipartite
 381 b -matching game and will again show that it has a set of core imputations which do not
 382 correspond to optimal dual solutions. The optimal matching picks edges $(u_1, v_1), (u_1, v_2)$
 383 once each, for a total profit of 4. Unlike the unconstrained case, this time, the optimal dual
 384 is not unique. The optimal dual solutions are given by $1, 0, 0, 2 - a$, for vertices u_1, u_2, v_1, v_2 ,
 385 and $0, a, 0$ for edges $(u_1, v_1), (u_1, v_2), (u_2, v_2)$, where $a \in [0, 1]$. The corresponding core
 386 imputations are $3 - b, 0, 0, 1 + b$, for the four vertices u_1, u_2, v_1, v_2 , where $b \in [0, 1]$.

As in the unconstrained case, let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be the profits allocated to u_1, u_2, v_1, v_2 . This time, the system of linear inequalities whose solutions capture all possible core imputations is given by system (9) after replacing the first inequality by

$$\alpha_1 + \beta_1 \geq 1.$$

387 This is so because edge (u_1, v_1) can be matched twice under the the unconstrained bipartite
 388 b -matching game, but only once under the constrained version. As before, non-negativity is
 389 imposed on all these variables. On solving this system, we find that $\alpha_1, \alpha_2, \beta_1, \beta_2$ should be
 390 $1, 0, b, 1 + c$, where a, b, c are non-negative and satisfy the system (11).

$$\begin{aligned} a + b &\geq 1 \\ a + c &\geq 2 \\ a + b + c &= 3 \end{aligned} \tag{11}$$

392 Solutions of this system which do not correspond to dual solutions include $1, 0, 0, 3$ and
 393 $0, 0, 1, 3$. Observe that neither of these is a core imputation for the unconstrained bipartite
 394 b -matching game. The method given in Section 4.2, for finding a smaller system, holds for
 395 this case as well and so does Remark 22.

396 ▶ Remark 25. In the assignment game, core imputations were precisely optimal dual solutions.
 397 On the other hand, in both versions of the bipartite b -matching game, core imputations
 398 are obtained from optimal dual solutions via specific operations. As stated in Remark
 399 20, for the unconstrained version, there is a bijection between optimal dual solutions and
 400 core imputations in $D(I)$. In contrast, for the constrained version, the set of imputations
 401 corresponding to optimal dual solutions may not be disjoint.

402 Let us illustrate the last point of Remark 25 via Example 21. Consider the two optimal
 403 dual solutions obtained by setting $a = 0$ and $a = 1$, namely $1, 0, 0, 2$, for vertices u_1, u_2, v_1, v_2 ,
 404 and $0, 0, 0, 0$ for edges $(u_1, v_1), (u_1, v_2), (u_2, v_2)$; and $1, 0, 0, 1$, for vertices u_1, u_2, v_1, v_2 , and
 405 $0, 1, 0$ for edges $(u_1, v_1), (u_1, v_2), (u_2, v_2)$. Both these optimal duals yield the core imputation
 406 assigning profits of $2, 0, 0, 2$ for u_1, u_2, v_1, v_2 .

407 6 Discussion

408 Our most important open question is to shed light on the origins of core imputations, for
 409 the two bipartite b -matching games, which do not correspond to optimal dual solutions. Is
 410 there a “mathematical structure” that produces them? A related question is to determine
 411 the complexity of the following question for these two games: Given an imputation for a
 412 game, decide if it belongs to the core. We believe this question should be co-NP-complete.
 413 On the other hand, the following question is clearly in P: Given an imputation for a game I ,
 414 decide if it lies in $D(I)$.

415 As stated in Section 3.1, for the assignment game, Shapley and Shubik were able to
 416 characterize “antipodal” points in the core. An analogous understanding of the core of the
 417 general graph matching games having non-empty core will be desirable.

418 For the assignment game, Demange, Gale and Sotomayor [4] give an auction-based
 419 procedure to obtain a core imputation; it turns out to be optimal for the side that proposes,
 420 as was the case for the deferred acceptance algorithm of Gale and Shapley [7] for stable
 421 matching. Is there an analogous procedure for obtaining an imputation in the core of the
 422 general graph matching games having non-empty core?

423 7 Acknowledgements

424 I wish to thank Hervé Moulin for asking the interesting question of extending results obtained
 425 for the assignment game to general graph matching games having a non-empty core. I also
 426 wish to thank Federico Echenique, Hervé Moulin and Thorben Trobst for several valuable
 427 discussions.

428 ——— References ———

- 429 1 Garrett Birkhoff. Three observations on linear algebra. *Univ. Nac. Tacuman, Rev. Ser. A*,
 430 5:147–151, 1946.
- 431 2 Péter Biró, Walter Kern, and Daniël Paulusma. Computing solutions for matching games.
 432 *International journal of game theory*, 41(1):75–90, 2012.
- 433 3 Christopher P Chambers and Federico Echenique. The core matchings of markets with
 434 transfers. *American Economic Journal: Microeconomics*, 7(1):144–64, 2015.
- 435 4 Gabrielle Demange, David Gale, and Marilda Sotomayor. Multi-item auctions. *Journal of*
 436 *political economy*, 94(4):863–872, 1986.
- 437 5 Xiaotie Deng, Toshihide Ibaraki, and Hiroshi Nagamochi. Algorithmic aspects of the core of
 438 combinatorial optimization games. *Mathematics of Operations Research*, 24(3):751–766, 1999.

13:14 New Characterizations of Core Imputations

- 439 **6** Kimmo Eriksson and Johan Karlander. Stable outcomes of the roommate game with transfer-
440 able utility. *International Journal of Game Theory*, 29(4):555–569, 2001.
- 441 **7** David Gale and Lloyd S Shapley. College admissions and the stability of marriage. *The*
442 *American Mathematical Monthly*, 69(1):9–15, 1962.
- 443 **8** L. Lovász and M.D. Plummer. *Matching Theory*. North-Holland, Amsterdam–New York, 1986.
- 444 **9** Hervé Moulin. *Cooperative microeconomics: a game-theoretic introduction*, volume 313.
445 Princeton University Press, 2014.
- 446 **10** Marina Núñez and Carles Rafels. On the dimension of the core of the assignment game. *Games*
447 *and Economic Behavior*, 64(1):290–302, 2008.
- 448 **11** Lloyd S Shapley and Martin Shubik. The assignment game I: The core. *International Journal*
449 *of Game Theory*, 1(1):111–130, 1971.
- 450 **12** Vijay V Vazirani. New characterizations of core imputations of matching and b -matching
451 games. *arXiv preprint arXiv:2202.00619*, 2022.