Harmonic Balance Analysis of Lur'e Oscillator Network with Non-diffusive Weak Coupling

Bryan Lee and Tetsuya Iwasaki

Abstract—**The central pattern generator (CPG) is a group of interconnected neurons, existing in biological systems as a control center for oscillatory behaviors. We propose a new approach based on the multivariable harmonic balance to characterize the relationship between the oscillation profile (frequency, amplitude, phase) and interconnections within the CPG, modeled as weakly coupled oscillators. In particular, taking advantage of the weak coupling, we formulate a low-dimensional matrix whose eigenvalue/eigenvector capture the perturbation in the oscillation profile due to the coupling. Then we develop an algorithm to estimate the perturbed oscillation profile of a given CPG, and suggest an optimization to synthesize the interconnections to produce a given oscillation profile.**

Index Terms—**Network analysis and control, Cooperative control, Neural networks, Coupled oscillators**

I. Introduction

Located in the central nervous system of animals are neural oscillator circuits, called the central pattern generators (CPGs), which drive rhythmic behaviors of the body. The ability of CPGs to cooperate with external constraints and adapt to changing environment make them an attractive foundation for control design in many engineering applications. For example, CPG-inspired controllers have been designed for numerous robotic systems [1], [2] with such useful properties as gait adaptation [3], online trajectory generation [4], and resonance exploitation [5].

Whether modeling a biological CPG in nature or designing an artificial CPG for engineering applications, the main challenge remains to find the relationship between the neuronal connections and the resulting oscillation profile. Specifically, for a given CPG, it is of interest to find conditions under which a stable limit cycle exists and to predict the frequency, amplitude, and phase. For this purpose, the coupled-oscillator architecture of CPGs has facilitated the analysis.

In the literature, analysis and synthesis problems have been solved for coupled oscillators with diffusive coupling, based on the contracting/convergent systems [6]–[8] and Floquet theory [9]. However, diffusive coupling is not suitable for modeling of biological CPGs – for example, the synaptic interactions between segmental oscillators of leech CPG for swimming are active during steady swimming [10]. For oscillator networks with non-diffusive weak coupling, the phase reduction methods [11]–[13] simplify the synchronization analysis but ignore the amplitude variation and remove the oscillatory dynamics. The multivariable harmonic balance (MHB) provides a flexible framework for both analysis and synthesis of CPGs with non-diffusive coupling [14]. The method is not always accurate due to harmonic approximations but has been found effective in predicting oscillation profiles [15]–[17].

In this paper, we consider a network of m oscillators with non-diffusive weak coupling, and present a new characterization of the relationship between the interconnections and the oscillation profile through the MHB analysis. Specifically, the oscillator network is described as a Lur'e system, i.e., a feedback connection of linear dynamics and static nonlinearities. Exploiting the weak coupling, we condense the multi-dimensional oscillator dynamics and their interactions into scalar parameters to obtain an $m \times m$ matrix. The matrix captures the essential dynamics of the network such that its diagonal entries contain the intra-oscillator perturbations of the amplitudes and phases due to coupling and its eigenvector encodes the inter-oscillator phases. Reversing the analysis, we formulate an optimization to synthesize interconnections that produce a desired oscillation profile.

The benefits of our approach in comparison with existing MHB methods are that it (a) reduces computational cost of the analysis over [14], [16] through dimensional reduction achieved by exploiting the weakness of coupling, (b) covers a class of CPGs wider than or different from those in [14]–[16] as we allow possibly non-diffusive coupling with arbitrary dynamics for every neuronal connections, (c) provides more design flexibility over [14], [15] by allowing small variations of the oscillation profile due to coupling as design freedom, and (d) gives a rigorous proof for the MHB condition with a stability property, which was missing in [17].

We use the following notation. We denote by $\mathrm{Re}(x)$, $\mathrm{Im}(x)$, and $\angle x$ the real part, imaginary part, and phase angle of $x \in \mathbb{C}$, respectively. A scalar function $f : \mathbb{R} \mapsto \mathbb{R}$ acts on a vector $x \in \mathbb{R}^n$ elementwise to generate vector $f(x) \in \mathbb{R}^n$. Expressions $\mathrm{col}(\cdot)$ and $\mathrm{diag}(\cdot)$ denote the matrices obtained by stacking their arguments in a column and diagonal, respectively. For a function $F(x)$ of a scalar variable x, its derivative is denoted by $\dot{F}(x)$. For time signals $u(t)$ and $y(t)$, notation $y = f(s)u$ means $y(t) := \mathcal{L}^{-1}[f(s)\mathcal{L}[u(t)]]$ where $\mathcal{L}$ is the Laplace transform operator.

II. Problem Formulation

A. General Objective

We consider coupled m oscillators described by

$$q_i = M_o(s)\psi(q_i) + \sigma \sum_{j=1}^{m} \Delta_{ij}(s)\psi(q_j), \quad i \in \mathbb{I}_m, \qquad (1)$$

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where $q_i(t) \in \mathbb{R}^n$ are the membrane potentials and $\mathbb{I}_m := \{1, 2, \ldots, m\}$. Here, $\psi : \mathbb{R} \mapsto \mathbb{R}$ is an odd sigmoid function representing the threshold and saturation effects of the neuronal dynamics. Each oscillator is formed by a local group of n neurons, which we call a "segment," and the neurons are "intrasegmentally" connected through dynamics $M_o(s)$. In addition, "intersegmental" connections are made between oscillators through $\sigma\Delta_{ij}(s)$, and are assumed weak compared to the intrasegmental connections (small $\sigma > 0$). The segmental oscillators are assumed to be nominally identical with the same dynamics $M_o(s)$, but the terms $\sigma\Delta_{ii}(s)$ capture possible small variations. Note that system (1) is a Lur'e system described by

$$q = \mathcal{M}(s)\psi(q), \tag{2}$$

where $q := \mathrm{col}(q_1, \ldots, q_m) \in \mathbb{R}^{mn}$ and

$$\mathcal{M}(s) := \mathcal{M}_o(s) + \sigma\Delta(s), \quad \mathcal{M}_o(s) := I \otimes M_o(s), \tag{3}$$

with $\otimes$ denoting the Kronecker product. This class of systems cannot be captured by existing models in [14]–[16] which require identical dynamics for all neurons.

The objective is to characterize the relationship between the neural connections $\mathcal{M}(s)$ and the profile of oscillations (frequency, amplitude, phase) of the weakly coupled segmental oscillators. The characterization should be simple and explicit to allow for computational analysis and synthesis.

B. Precise Problem Statement

To achieve the objective, we use the MHB method [14] and make a precise mathematical statement of the problem so that its solution provides a characterization of the oscillation profile. Assume a periodic solution $q(t)$ to (2) and consider the sinusoidal approximation $q(t) \approx \mathrm{Im}(\hat{q}e^{j\omega t})$ where $\omega \in \mathbb{R}$ is the frequency and $\hat{q} \in \mathbb{C}^{mn}$ is the phasor capturing the amplitude $\alpha := |\hat{q}|$ and phase $\angle\hat{q}$, where only the relative phases matter since (2) is autonomous. The static nonlinearity ψ, acting on $q(t)$ in (2), is approximated by

$$\psi(q) \approx \mathcal{K}(\alpha)q, \quad \mathcal{K}(\alpha) := \mathrm{diag}(\kappa(\alpha)), \tag{4}$$

where κ is the describing function of ψ defined by

$$\kappa(a) := \frac{1}{a\pi}\int_{-\pi}^{\pi} \psi(a\sin\theta)\sin\theta d\theta, \tag{5}$$

so that $\kappa(a)x$ is the first harmonic term of the Fourier series of $\psi(x)$ for $x := a\sin\omega t$. The dynamics of (2) in the neighborhood of the periodic solution $q(t)$ is thus approximated by the quasi-linear system

$$q = \mathcal{M}(s)\mathcal{K}(\alpha)q, \tag{6}$$

which has a solution $q = \mathrm{Im}(\hat{q}e^{j\omega t})$ if and only if

$$\hat{q} = \mathcal{M}(j\omega)\mathcal{K}(\alpha)\hat{q}, \quad \alpha := |\hat{q}|, \tag{7}$$

is satisfied. In this case, (6) has characteristic roots $s = \pm j\omega$.

Definition 1: *The quasi-linear system (6) is said to be* m-stable *if all the characteristic roots are in the open left half plane except for a simple pair on the imaginary axis. A pair $(\omega, \hat{q}) \in \mathbb{R} \times \mathbb{C}^{mn}$ satisfying the MHB equation (7) is said to be an* m-stable solution *if (6) with $\alpha := |\hat{q}|$ is m-stable.*

An m-stable solution $(\omega, \hat{q})$ of the MHB equation (7) predicts existence of a stable limit cycle for (2), and the oscillation profile for $q(t)$ is estimated as $q \approx \mathrm{Im}(\hat{q}e^{j\omega t})$. This claim is based on the harmonic approximation and does not have a theoretical guarantee, but has been supported by a number of numerical experiments [14]–[17]. We will rigorously characterize m-stable solutions, and demonstrate usefulness of the characterization by example systems through numerical simulations. To this end, let us introduce:

Assumption 1: *Consider an isolated segmental oscillator $r = M_o(s)\psi(r)$ where $r(t) \in \mathbb{R}^n$ represents one of the variables $q_i(t)$ in (1) with $\sigma = 0$. The MHB equation for this system admits an m-stable solution $(\omega_o, \hat{r}) \in \mathbb{R} \times \mathbb{C}^n$:*

$$\hat{r} = M_o(j\omega_o)K_o\hat{r}, \quad K_o := \mathcal{K}(a_o), \quad a_o := |\hat{r}| \in \mathbb{R}^n, \tag{8}$$

and the associated quasi-linear system $r = M_o(s)K_or$ is m-stable. Moreover, $\pm j\omega_o$ are not poles of $\Delta_{ij}(s)$ in (1).

Given a segmental oscillator satisfying Assumption 1, we will seek an m-stable solution $(\omega, \hat{q})$ to the MHB equation (7) for the coupled oscillators in (1). Note that, when uncoupled ($\sigma = 0$), the MHB equation (7) admits a solution $(\omega, \hat{q})$ with $\omega = \omega_o$ and $\hat{q}_i = \hat{r}e^{j\phi_i}$ where $\phi_i \in \mathbb{R}$ for $i \in \mathbb{I}_m$ are arbitrary. For a small $\sigma > 0$, we assume that the solution is slightly perturbed in the following form [15]:

$$\omega = \omega_o + \sigma\tilde{\omega} + O(\sigma^2), \quad \hat{q}_i = (I + \sigma P_i)\hat{r}e^{j\phi_i} + O(\sigma^2), \tag{9}$$

where $i \in \mathbb{I}_m$, $P_i \in \mathbb{C}^{n\times n}$ is a diagonal matrix, $\phi_i \in \mathbb{R}$ is the intersegmental phase, and $\tilde{\omega} \in \mathbb{R}$ is the frequency perturbation. The problem we address is the following:

Problem 1: *Consider the coupled oscillators in (1), which can be written as (2). Suppose Assumption 1 holds. Find a necessary and sufficient condition on $(\tilde{\omega}, P_i, \phi_i)$ for $i \in \mathbb{I}_m$, such that $(\omega, \hat{q})$ of the form (9) is an m-stable solution of (7) when $\sigma > 0$ is sufficiently small.*

A solution to this problem is given in the next section, and its applications to analysis and synthesis will be discussed in the sections that follow.

III. Main Result

Let us first provide a perturbation analysis of the MHB equation (7) with (3) when the coupling strength $\sigma > 0$ is arbitrarily small. The result is proven in Appendix A.

Lemma 1: *Consider the weakly coupled oscillators described by (2) with (3). Suppose Assumption 1 holds, and let $(P_i, \phi_i, \tilde{\omega}) \in \mathbb{C}^{n\times n} \times \mathbb{R} \times \mathbb{R}$ be given for $i \in \mathbb{I}_m$, where P_i are diagonal. Then $(\omega, \hat{q})$ of the form (9) satisfies the MHB equation (7) with $\alpha := |\hat{q}|$ up to $O(\sigma)$ if and only if* [1]

$$\Big(j\tilde{\omega}\dot{\mathcal{M}}_o\mathcal{K}_o + \mathcal{M}_oS + \Delta\mathcal{K}_o + (\mathcal{M}_o\mathcal{K}_o - I)P\Big)(I \otimes \hat{r})e^{j\phi} = 0, \tag{10}$$

$$\begin{aligned} &P := \mathrm{diag}(P_1, \ldots, P_m), && \mathcal{K}_o := I \otimes K_o, \\ &S := \mathrm{diag}(S_1, \ldots, S_m), && S_i := \dot{K}_o\mathrm{Re}(P_i)\mathfrak{A}, \\ &\mathfrak{A} := \mathrm{diag}(a_o), && \dot{K}_o := \mathrm{diag}(\dot{\kappa}(a_o)). \end{aligned} \tag{11}$$

For computational verification of (10), $\kappa(x)$ and $\dot{\kappa}(x)$ can be calculated via numerical integration and derivative using (5).

[1] Δ denotes $\Delta(j\omega_o)$. This notation applies to all transfer functions.

For $\mathcal{M}_o(s) = CXB$ with $X := (sI - A)^{-1}$, its derivative is given by $\dot{\mathcal{M}}_o(s) = C\dot{X}B = -CX^2B$, which can be verified by taking the derivative of $(sI - A)X = I$.

Next we give a condition for the MHB solution in Lemma 1 to be m-stable when $\sigma > 0$ is sufficiently small. The proof (Appendix B) is based on an eigenvalue perturbation result, applied to system (6) in the state space.

Lemma 2: *Consider the weakly coupled oscillators described by (2) with (3). Suppose Assumption 1 holds, and let $(P_i, \phi_i, \tilde{\omega}) \in \mathbb{C}^{n\times n} \times \mathbb{R} \times \mathbb{R}$ be given for $i \in \mathbb{I}_m$, where P_i are diagonal. Suppose $(\omega, \hat{q})$ of the form (9) satisfies the MHB equation (7) with $\alpha := |\hat{q}|$. Then the associated quasi-linear system (6) is m-stable for sufficiently small $\sigma > 0$ if and only if matrix Λ_p defined by*

$$\begin{aligned}&\Lambda_p := \beta(I \otimes \hat{\ell}^*)\Big(\mathcal{M}_o S + \Delta\mathcal{K}_o\Big)(I \otimes \hat{r}),\\ &\beta := -1/(\hat{\ell}^* \dot{M}_o K_o \hat{r})\end{aligned} \tag{12}$$

has all the eigenvalues in the open left half plane except for a pair on the imaginary axis, where S and $\mathcal{K}_o$ are defined in (11), and $\hat{\ell}^*$ is the left eigenvector of $M_o K_o$ associated with eigenvalue 1, normalized such that $\hat{\ell}^* \hat{r} = 1$.

Combining the above two lemmas, we have a characterization of m-stable solutions to the MHB equation (7). Since the stability condition is given in terms of the matrix Λ_p in (12), we break down the MHB equation (10) into two equations: one gives an eigenvalue condition on Λ_p, and the other captures the remaining constraints.

Proposition 1: *Consider the weakly coupled oscillators described by (2) with (3). Suppose Assumption 1 holds, and let $(P_i, \phi_i, \tilde{\omega}) \in \mathbb{C}^{n\times n} \times \mathbb{R} \times \mathbb{R}$ be given for $i \in \mathbb{I}_m$, where P_i are diagonal. Then $(\omega, \hat{q})$ of the form (9) satisfies (7) with $\alpha := |\hat{q}|$ up to $O(\sigma)$ and the associated quasi-linear system (6) is m-stable for sufficiently small $\sigma > 0$ if and only if*

$$\begin{aligned}&\mathrm{eig}\big(\beta(U + \Lambda)\big)\backslash\{j\tilde{\omega}\} \subset \mathbb{C}^-,\\ &\beta(U + \Lambda)e^{j\phi} = j\tilde{\omega}e^{j\phi},\\ &(W + \Omega)e^{j\phi} = 0,\end{aligned} \tag{13}$$

where $\mathbb{C}^-$ is the open left half plane,

$$\begin{aligned}&U = \mathrm{diag}(u_1, \ldots, u_m), \quad \Lambda_{ij} = \hat{\ell}^* \Delta_{ij} K_o \hat{r},\\ &W = \mathrm{diag}(w_1, \ldots, w_m), \quad \Omega_{ij} = N^* \Delta_{ij} K_o \hat{r},\\ &u_i := \hat{\ell}^* M_o S_i \hat{r}, \quad S_i := \dot{K}_o \mathrm{Re}(P_i)\mathrm{diag}(a_o),\\ &w_i := N^*(j\tilde{\omega}\dot{M}_o K_o + M_o S_i + (M_o K_o - I)P_i)\hat{r},\end{aligned}$$

and $N \in \mathbb{C}^{n\times(n-1)}$ is the orthogonal complement of $\hat{r}$.

Proof. Multiplying $\beta(I \otimes \hat{\ell}^*)$ and $(I \otimes N^*)$ from the left, it can be verified that (10) is equivalent to the equalities in (13). The eigenvalue condition in (13) follows from Lemma 2 once we verify that Λ_p in (12) is equal to $\beta(U + \Lambda)$. ∎

To gain insights, let us consider a simple case where the neuronal dynamics within each segment are nominally identical and represented by a scalar transfer function $f_o(s)$. In this case, $M_o(s)$ is given by the product of $f_o(s)$ and a constant matrix $\bar{M}_o$ capturing the connection topology, strength, and inhibitory/excitatory property. We also focus on the intersegmental oscillation properties and choose to ignore the small intrasegmental variations due to the weak coupling. Specifically, we assume $P_i = p_i I$ in (9) so that $p_i, \phi_i \in \mathbb{R}$ capture the intersegmental amplitude and phase, while the oscillation profile within each segment remains to be captured by the nominal phasor $\hat{r}$. With these two simplifications, Proposition 1 reduces to the following.

Corollary 1: *Consider Proposition 1. Suppose*

$$P_i = p_i I, \quad M_o(s) = f_o(s)\bar{M}_o,$$

where $p_i \in \mathbb{R}$ and $\bar{M}_o \in \mathbb{R}^{n\times n}$ are constant parameters and $f_o(s)$ is a scalar transfer function. Then conditions

$$\begin{aligned}&\mathrm{eig}(\Lambda_p)\backslash\{j\tilde{\omega}\} \subset \mathbb{C}^-,\\ &\Lambda_p e^{j\phi} = j\tilde{\omega}e^{j\phi}, \quad \Lambda_p := \gamma\mathcal{P} + \beta\Lambda,\\ &\Omega_p e^{j\phi} = 0, \qquad\quad \Omega_p := \mathcal{P} \otimes v + \Omega,\end{aligned} \tag{14}$$

are equivalent to (13), where

$$\begin{aligned}&\mathcal{P} := \mathrm{diag}(p_1, \ldots, p_m), \quad \gamma := \beta\hat{\ell}^* M_o \dot{K}_o \mathrm{diag}(a_o)\hat{r},\\ &\beta = -f_o/\dot{f}_o, \quad v := N^* M_o \dot{K}_o \mathrm{diag}(a_o)\hat{r},\end{aligned}$$

Proof. The result follows by noting that

$$\begin{aligned}&\dot{M}_o K_o \hat{r} = \dot{f}_o \bar{M}_o K_o \hat{r} = (\dot{f}_o/f_o)\hat{r},\\ &\beta = -1/(\hat{\ell}^* \dot{M}_o K_o \hat{r}) = -f_o/\dot{f}_o,\\ &w_i = N^* M_o \dot{K}_o \mathrm{diag}(a_o)\hat{r}p_i \quad \Rightarrow \quad W = \mathcal{P} \otimes v.\end{aligned}$$ ∎

Without coupling ($\sigma = 0$), the quasi-linear system (6) has eigenvalue $j\omega_o$ with multiplicity m and the rest are in the open left half plane due to Assumption 1. With weak coupling (small $\sigma > 0$), the segmental oscillations with frequency near ω_o would be coordinated with orbital stability if the $m-1$ of the eigenvalues at $j\omega_o$ move to the left. Because the eigenvalues of Λ_p are the derivatives of the eigenvalues $j\omega_o$ with respect to σ, the orbital stability is expected when Λ_p satisfies (14). The condition in (14) gives a simple characterization of m-stable solutions and its practical use will be discussed in the next two sections.

IV. Analysis

This section will address the analysis problem: Given a CPG model (1), estimate the oscillation profile of a stable periodic solution, if any. Specifically, we assume that each segmental oscillator $r = M_o(s)\psi(r)$ in isolation has a stable limit cycle on which $r(t) \approx \mathrm{Im}[\hat{r}e^{j\omega_o t}]$. When weakly coupled as in (1) with a small $\sigma > 0$, the segmental oscillators may coordinate with specific intersegmental phases, while exhibiting small variations in the amplitude and frequency:

$$q_i(t) \approx \mathrm{Im}[(1 + \sigma p_i)\hat{r}e^{j((\omega_o+\sigma\tilde{\omega})t+\phi_i)}].$$

Corollary 1 states that the oscillation profile can be estimated by solving (14) for $(\tilde{\omega}, \phi, \mathcal{P})$.

In (14), $j\tilde{\omega}$ is the "maximal" eigenvalue of Λ_p with the largest real part. The corresponding eigenvector $e^{j\phi}$ gives the estimated intersegmental phases ϕ_i, provided the eigenvector has the same magnitude for all its entries. Thus, the amplitude parameter $\mathcal{P}$ should be selected such that the maximal eigenvalue λ is purely imaginary, and the associated eigenvector v satisfies the uniform magnitude property. We propose a heuristic algorithm to numerically search for such $\mathcal{P}$ based on a fixed point iteration. Define the mapping $\bar{\mathcal{P}} = \mu(\mathcal{P})$ as follows. For a given $\mathcal{P}$, let (λ, v) be the maximal eigenvalue-eigenvector pair of Λ_p, let $(j\tilde{\omega}, e^{j\phi})$ be the projection of (λ, v) onto the set $(j\mathbb{R}, e^{j\mathbb{R}^m})$, and let $\bar{\mathcal{P}}$ be the solution $\mathcal{P}$ to the second equation in (14). Then a solution $(\tilde{\omega}, \phi, \mathcal{P})$ to the first two constraints in (14) satisfies $\mathcal{P} = \mathrm{Re}[\mu(\mathcal{P})]$. Thus the fixed point iteration $\mathcal{P}_{k+1} = \mathrm{Re}[\mu(\mathcal{P}_k)]$, or its relaxation, may provide a solution at convergence.

Algorithm 1

1) Initialize $k = 0$ and $p_k = 0 \in \mathbb{R}^m$, and choose positive scalars $\varepsilon_1 < 1$ and $\varepsilon_2 \ll 1$.
2) Let $\mathcal{P}_k := \mathrm{diag}(p_k)$ and compute the maximal eigenvalue $\lambda \in \mathbb{C}$ and the associated eigenvector $v \in \mathbb{C}^m$ of $\gamma\mathcal{P}_k + \beta\Lambda$. Set $\tilde{\omega} := \mathrm{Im}(\lambda)$ and $\phi := \angle v$.
3) Update p_k by
$$\begin{aligned} &\bar{p}_k := \Phi^*(j\tilde{\omega}I - \beta\Lambda)e^{j\phi}/\gamma, \quad \Phi := \mathrm{diag}(e^{j\phi}), \\ &p_{k+1} = p_k + \varepsilon_1 \mathrm{Re}(\bar{p}_k - p_k), \end{aligned}$$
4) If $\|p_{k+1} - p_k\| < \varepsilon_2$, then set $\mathcal{P} := \mathcal{P}_k$ and stop. Otherwise increment k and go to step 2.

This algorithm is heuristic, its convergence is not guaranteed, and the third constraint in (14) is ignored. However, our numerical experience suggests that it works well for the purpose of obtaining a rough estimate for the intersegmental properties. Since the analysis involves eigenvalue computation for $m \times m$ matrix Λ_p, the computational cost is reduced in comparison with [14], which ignores the weak coupling structure and involves $nm \times nm$ matrix $\mathcal{M}(j\omega_o)$.

Example 1. We consider a model of the leech CPG for swimming described as a chain of 17 segmental oscillators. The model is identical to the one in [18] except that intersegmental time-delay $e^{-k\tau_d s}$ is replaced by its approximation

$$f_k(s) := \frac{\alpha_k}{1 + \tau_k s}, \qquad \begin{aligned} \alpha_k &:= 1/\cos(k\omega_o\tau_d), \\ \tau_k &:= \tan(k\omega_o\tau_d)/\omega_o, \end{aligned}$$

where $k\tau_d$ is the communication delay over k segments, ω_o is the nominal frequency, and (α_k, τ_k) are chosen such that $f_k(j\omega_o) = e^{-j\omega_o k\tau_d}$. We will estimate the oscillation profile using Corollary 1. The existing methods [14]–[17] cannot be applied since the model does not belong to the classes of CPGs they considered.

Each segmental oscillator has three neurons, and the phasor $\hat{r} \in \mathbb{C}^3$ and the nominal frequency ω_o can be computed using Proposition 2 in [14]. The amplitude is uniform and

$$a_o = |\hat{r}| = 13.45 \times \mathrm{col}(1, 1, 1), \quad \omega_o = 14.43.$$

The actual frequency from simulation of the segmental oscillator is $\omega_o^{\mathrm{sim}} = 15.26$. Using Algorithm 1 with $\varepsilon_1 = 0.1$ and $\varepsilon_2 = 10^{-9}$, we have computed $(\tilde{\omega}, \mathcal{P}, \phi)$ to estimate the oscillation profile. In Fig. 1, history of the first entry of $q_i(t) \in \mathbb{R}^3$ for $i \in \mathbb{I}_{17}$, and amplitude and phase of $\hat{q}_i$ from simulation (blue) and estimation (red) are shown for three neurons in each segment. The time course shows convergence to the stable limit cycle within a few cycles, and the peaks of the 17 curves in the steady state shift from left to right (linear decrease of the phase plots) with peak values largest in the middle segments (top blue curve in the amplitude plot). While the phase estimate closely agrees with the simulated phase, the amplitude estimate (single red curve), which is assumed uniform within a segment, provides a reasonable approximation to the average amplitudes within each segment (average of blue curves).

The frequency perturbation $\tilde{\omega} = -9.24$ is negative, resulting in the estimated frequency $\omega = \omega_o + \sigma\tilde{\omega} = 13.88$ with $\sigma = 0.06$, which roughly agrees with $\omega^{\mathrm{sim}} = 14.53$ from simulation. The quasi-linear system has eigenvalues $0.04 \pm 13.86j$ and the rest with the maximum real part -0.51, approximately satisfying the m-stability (and hence correctly suggesting orbital stability of the limit cycle) although the third condition in (14) is ignored in the analysis.

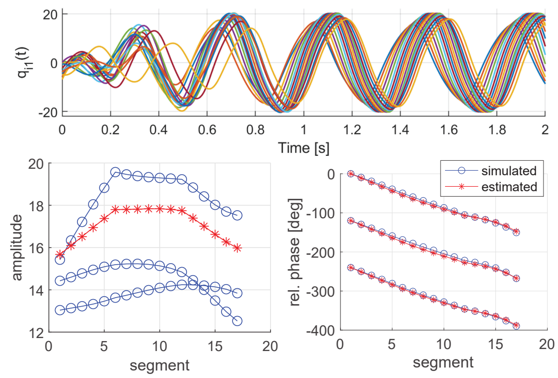

Fig. 1. MHB analysis of the leech CPG

V. Synthesis

This section will address the synthesis problem: Given a segmental oscillator $r = M_o(s)\psi(r)$ and desired intersegmental phase $\phi \in \mathbb{R}^m$, determine the intersegmental connections $\Delta(s)$ such that the CPG model (1) with small $\sigma > 0$ possesses a stable limit cycle on which $q_i(t)$ oscillates with phase ϕ_i for $i \in \mathbb{I}_m$. While this problem is for the design of a synthetic CPG, it can also be interpreted as the modeling of a biological CPG to identify the neuronal connections that reproduce the observed intersegmental phase ϕ.

In the general synthesis problem, the perturbations of the frequency $\tilde{\omega}$ and amplitude $\mathcal{P}$ may also be specified as desired, or alternatively, may be left as design variables to facilitate satisfaction of the phase specification. The latter approach may be preferred when the intersegmental connections are subject to dynamical/structural constraints, such as a communication delay or distributed network topology, which make the former approach infeasible. In this case, the phase specification may be satisfied at the expense of slight perturbations in the frequency and amplitudes from the nominal values of the segmental oscillator.

To formalize the synthesis, let the (k, ℓ) entry of $\Delta(s)$, denoted by $\delta_{k\ell}(s)$, specify the connection from the ℓ^{th} neuron to the k^{th} neuron as $\delta_{k\ell}(s) = g_{k\ell}(s)x_{k\ell}$ where $g_{k\ell}(s)$ captures the dynamics of the synaptic connection and $x_{k\ell}$ captures the excitatory/inhibitory type and strength of that connection. Then the connection parameter vector x is defined by stacking the scalar parameters $x_{k\ell}$ for possible connections. Since there may exist multiple solutions for $\Delta(s)$, let us consider the ℓ_1-norm minimization:

$$\min_x \|x\|_1 \text{ s.t. } \begin{cases} \Lambda_p e^{j\phi} = j\tilde{\omega}e^{j\phi}, \quad \Omega_p e^{j\phi} = 0, \\ W^*(\Lambda_p + \Lambda_p^*)W < \gamma I, \end{cases} \tag{15}$$

where $W \in \mathbb{C}^{(m-1)\times m}$ is the orthogonal complement of $e^{j\phi}$. The use of the ℓ_1-norm is motivated by the LASSO regularization, which attempts to minimize the number of nonzero weights in a linear regression model. For the synthesis of coupled oscillators, using the ℓ_1-norm seeks the essential connections. Noting that Λ_p and Ω_p depend linearly on x, the problem is convex and easily solved.

We may also allow small perturbations in the frequency and amplitudes by adding $(\tilde{\omega}, \mathcal{P})$ as additional optimization variables in (15) while keeping convexity. This provides design flexibility for the synthesis of a CPG. With the existing approaches [14], [15], the desired amplitudes and phases of all mn neurons have to be specified. Such tight specifications can lead to infeasible design especially in the presence of constraints on the network topology. In contrast, our approach allows us to specify only the intersegmental phases, leaving the small intersegmental amplitude variations as a design freedom and possibly making the design feasible.

The Lyapunov inequality in (15) is a sufficient condition for the eigenvalue condition in (14) when $\gamma \le 0$ since $\gamma/2$ is an upper bound on $\mathrm{Re}(\lambda_i)$ for $i \in \mathbb{I}_{m-1}$ where λ_i for $i \in \mathbb{I}_m$ are the eigenvalues of Λ_p with $\lambda_m = j\tilde{\omega}$. Since the eigenvalues of the quasi-linear system (6) near the imaginary axis are approximately equal to $j\omega_o + \sigma\lambda_i$, the magnitude of $\sigma\gamma/2$ estimates a lower bound on the rate of convergence to the target oscillation – the larger, the faster.

Example 2. We consider the design of coupled oscillators (1) for the specifications in Table I. The segmental oscillator is found using Proposition 4 of [14] as

$$\begin{array}{l} M_o(s) = f_o(s)\bar{M}_o, \\ f_o(s) := 1/(1+\tau_o s), \end{array} \qquad \bar{M}_o := \begin{bmatrix} 0.79 & 0.65 & 0 \\ 0 & 1.14 & 1.20 \\ -2.68 & 0 & 1.58 \end{bmatrix},$$

where the ℓ_1-norm of $\mathrm{vec}(\bar{\mathrm{M}}_\mathrm{o})$ is minimized. For the intersegmental connections, we assume $\Delta_{ij}(s) = f_k(s)\bar{\Delta}_{ij}$ with constant $\bar{\Delta}_{ij}$ that is allowed to be nonzero if $k := |i-j| = 1$, and $f_k(s)$ is defined as in Example 1 with $\omega_o = \pi$ and $\tau_d = 0.015$. The entries of nonzero $\bar{\Delta}_{ij}$ are stacked into vector $x \in \mathbb{R}^{72}$ and are optimized to achieve the oscillation profile in Table I. As in Example 1, the existing methods cannot be used for the synthesis unless $f_k(s) = f_o(s)$.

We have found that the optimization (15) over $(x, \tilde{\omega})$ with uniform amplitudes ($\mathcal{P} = 0$) was infeasible for any $\gamma < 0$ due to the restriction of the nearest neighbor coupling ($k = 1$). Hence, we let $\mathcal{P}$ be an additional free variable in (15) to allow small amplitude variations. With variables $(x, \tilde{\omega}, \mathcal{P})$, the value of $\gamma < 0$ does not alter the essential result since it just scales the solution and the scaling freedom can be absorbed into σ. The design result with $\sigma\gamma = -0.1$ is shown in Fig. 2, where the optimized amplitudes $(1+\sigma p_i)a_o$ and specified phases $\angle\hat{r} + \phi_i$ for $i \in \mathbb{I}_5$ are compared with the simulated values. We see slight perturbations in the amplitudes from a_o as intended, making the optimization feasible. The intersegmental phases are closely matched with the specification. The optimized frequency $\omega_o + \sigma\tilde{\omega} = 3.07$ and simulated frequency $\omega = 2.98$ are both close to the nominal frequency $\omega_o = \pi$ of the segmental oscillator. Simulated time courses of $q_{i1}(t)$ indicated fast convergence from random initial conditions, confirming stability of the limit cycle. Finally, we note that the ℓ_1 optimization in (15) eliminated 37 out of the 72 connections, with 2 to 6 retained out of 9 connections between two adjacent oscillators.

TABLE I. Design specifications

threshold nonlinearity in neurons	$\psi(x) = \tanh(x)$,
# of neurons in a segmental oscillator, n	3
nominal frequency, ω_o [rad/s]	π
intrasegmental amplitude, a_o	1,2,3
intrasegmental phase, $\angle\hat{r}$ [deg]	0,60,120
intrasegmental time constant, τ_o [s]	0.2
# of segmental oscillators, m	5
intersegmental phase, ϕ [deg]	0,-35,-50,-90,-180

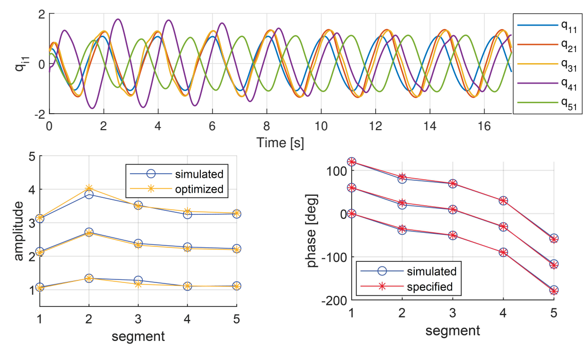

Fig. 2. MHB synthesis of CPG via optimization (15)

VI. Conclusion

Exploiting the weakness of the coupling, we have transformed the MHB condition into a simple eigenvalue-eigenvector equation. The resulting condition enabled the analysis of a given CPG to predict the oscillation profile, as well as the synthesis of intersegmental connections to achieve phase coordination with specified or unspecified intersegmental variation of amplitudes. Consideration of amplitude variation in weakly coupled oscillators will help to model biological observations realistically as well as design engineering systems with added design freedom. For example, the leech CPG for swimming may be modeled as coupled oscillators based on experimental data. Membrane potential data $q_i(t)$ of a segmental oscillator, partially known topology of the neuronal connections, and physiological parameters, such as synaptic time constants, are known from the literature. Solving the optimization (15), one can determine the synaptic strengths and identify missing connections to develop a high fidelity model, which could be validated by further experiments.

Appendix

A. Proof of Lemma 1

The MHB equation (7) can be written as

$$\hat{q}_i = M_o(j\omega)\mathcal{K}(\alpha_i)\hat{q}_i + \sigma \sum_{j=1}^{m} \Delta_{ij}(j\omega)\mathcal{K}(\alpha_j)\hat{q}_j, \tag{16}$$

where $\alpha_i := |\hat{q}_i|$ and $i \in \mathbb{I}_m$. From (9), we have

$$\begin{aligned} &\alpha_i = \big(I + \sigma\mathrm{Re}(P_i)\big)a_o + O(\sigma^2), \\ &K(\alpha_i) = K_o + \sigma\dot{K}_o\mathrm{Re}(P_i)\mathrm{diag}(a_o) + O(\sigma^2), \\ &M_o(j\omega) = M_o(j\omega_o) + j\tilde{\omega}\sigma\dot{M}_o(j\omega_o) + O(\sigma^2), \\ &\Delta_{ij}(j\omega) = \Delta_{ij}(j\omega_o) + O(\sigma), \end{aligned} \tag{17}$$

where $K_o := K(a_o)$ and we used the approximation

$$x, y \in \mathbb{C} \quad \Rightarrow \quad |(1+\sigma y)x| = \big(1+\sigma\mathrm{Re}(y)\big)|x| + O(\sigma^2).$$

Directly substituting these relationships and (9) into (16), neglecting $O(\sigma^2)$ terms, and dividing by σ yield

$$\Big(j\tilde{\omega}\dot{M}_o K_o + M_o S_i + (M_o K_o - I)P_i\Big)v_i + \sum_{j=1}^{m} \Delta_{ij} K_o v_j = 0,$$

for $i \in \mathbb{I}_m$, where $v_i := \hat{r}e^{j\phi_i}$. Here, we note that the $O(\sigma^0)$ terms vanish due to Assumption 1. Assembling the above equations for $i \in \mathbb{I}_m$, we obtain (10).

B. Proof of Lemma 2

Let (A_o, B_o, C_o) be a minimal realization of $M_o(s)$. Then the system $r = M_o(s)K_o r$ can be written as

$$\dot{x}_o = \bar{A}_o x_o, \quad \bar{A}_o := A_o + B_o K_o C_o, \quad r = C_o x_o.$$

Due to Assumption 1, the system is m-stable with an eigenvalue at $j\omega_o$. Note that the MHB equation (8) and the definition of $\hat{\ell}^*$ imply

$$\begin{array}{ll} \hat{r} = C_o\hat{x}_o, & \hat{x}_o := (j\omega_o I - A_o)^{-1}B_o K_o \hat{r}, \\ \hat{\ell}^* = \hat{y}_o^* B_o K_o/\beta, & \hat{y}_o^* := \beta\hat{\ell}^* C_o (j\omega_o I - A_o)^{-1}, \end{array} \tag{18}$$

where β is chosen such that $\hat{y}_o^*\hat{x}_o = 1$, and we also have

$$(\bar{A}_o - j\omega_o I)\hat{x}_o = 0, \quad \hat{y}_o^*(\bar{A}_o - j\omega_o I) = 0. \tag{19}$$

Now, consider the quasi-linear system (6). Note that

$$\begin{array}{l} \mathcal{M}_o(s) = C(sI - A)^{-1}B, \\ A := I \otimes A_o, \quad B := I \otimes B_o, \quad C := I \otimes C_o, \end{array}$$

and let a minimal realization of $\Delta(s)$ be given by

$$\Delta(s) = H(sI - F)^{-1}G.$$

Then the quasi-linear system (6) is described by

$$\dot{\mathrm{x}} = \mathcal{A}\mathrm{x}, \quad \mathrm{x} := \mathrm{col}(x, \xi),$$
$$\mathcal{A} := \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} + \begin{bmatrix} B \\ G \end{bmatrix} \mathcal{K}(\alpha) \begin{bmatrix} C & \sigma H \end{bmatrix}.$$

From (17), we have

$$\alpha = \mathrm{col}(\alpha_1, \ldots, \alpha_m), \quad \mathcal{K}(\alpha) = \mathcal{K}_o + \sigma S + O(\sigma^2).$$

We then see that

$$\mathcal{A} = \begin{bmatrix} A + B\mathcal{K}_o C & 0 \\ G\mathcal{K}_o C & F \end{bmatrix} + \sigma \begin{bmatrix} BSC & B\mathcal{K}_o H \\ GSC & G\mathcal{K}_o H \end{bmatrix} + O(\sigma^2).$$

Note from (19) that

$$\begin{array}{l} (A + B\mathcal{K}_o C)X_o = j\omega_o X_o, \quad X_o := I \otimes \hat{x}_o, \\ Y_o^*(A + B\mathcal{K}_o C) = j\omega_o Y_o^*, \quad Y_o := I \otimes \hat{y}_o. \end{array}$$

Therefore,

$$\begin{bmatrix} A + B\mathcal{K}_o C & 0 \\ G\mathcal{K}_o C & F \end{bmatrix} \begin{bmatrix} X_o \\ \Xi \end{bmatrix} = j\omega_o \begin{bmatrix} X_o \\ \Xi \end{bmatrix},$$
$$\begin{bmatrix} Y_o \\ 0 \end{bmatrix}^* \begin{bmatrix} A + B\mathcal{K}_o C & 0 \\ G\mathcal{K}_o C & F \end{bmatrix} = j\omega_o \begin{bmatrix} Y_o \\ 0 \end{bmatrix}^*,$$
$$\Xi := (j\omega_o I - F)^{-1} G\mathcal{K}_o C X_o,$$

From Theorem 4.1 in [19], the eigenvalue $j\omega_o$ of $\mathcal{A}$ at $\sigma = 0$, with geometric multiplicity m, is perturbed to the open left half plane except for one on the imaginary axis with sufficiently small $\sigma > 0$ if all the eigenvalues of

$$\Lambda_p := \begin{bmatrix} Y_o \\ 0 \end{bmatrix}^* \begin{bmatrix} BSC & B\mathcal{K}_o H \\ GSC & G\mathcal{K}_o H \end{bmatrix} \begin{bmatrix} X_o \\ \Xi \end{bmatrix}$$

have negative real parts, except for one on the imaginary axis which is equal to $j\tilde{\omega}$ where $\tilde{\omega}$ gives the first order term in (9). Direct calculation gives

$$\Lambda_p = Y_o^* B(S + \mathcal{K}_o\Delta(j\omega_o)\mathcal{K}_o)CX_o.$$

Noting from (18) that

$$\begin{array}{l} I \otimes \hat{r} = CX_o, \quad I \otimes \hat{\ell}^* = Y_o^* B\mathcal{K}_o/\beta, \\ Y_o^* B = \beta(I \otimes \hat{\ell}^*)\mathcal{M}_o(j\omega_o), \end{array}$$

we can verify that Λ_p is given by (12).

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