



The Inverse Spectral Map for Dimers

T. George¹ · A. B. Goncharov² · R. Kenyon²

Received: 2 August 2022 / Accepted: 22 August 2023 / Published online: 12 September 2023

This is a U.S. Government work and not under copyright protection in the US; foreign copyright protection may apply 2023

Abstract

In 2015, Vladimir Fock proved that the spectral transform, associating to an element of a dimer cluster integrable system its spectral data, is birational by constructing an inverse map using theta functions on Jacobians of spectral curves. We provide an alternate construction of the inverse map that involves only rational functions in the spectral data.

Keywords Dimers · Cluster algebras · Integrable systems · Toric varieties

1 Introduction

The planar dimer model is a classical statistical mechanics model, involving the study of the set of *dimer covers* (perfect matchings) of a planar, edge-weighted graph. In the 1960s, Kasteleyn [15, 16] and Temperley and Fisher [22] showed how to compute the (weighted) number of dimer covers of planar graphs using the determinant of a signed adjacency matrix now known as the *Kasteleyn matrix*.

In mathematics the dimer model was popularized with the papers [8, 9] on the “Aztec diamond” and later with results on the local statistics [19], conformal invariance [17], and limit shapes [6], connections with algebraic geometry [20, 21], cluster varieties and integrability [13], and string theory [14].

Communicated by F W Nijhoff.

✉ T. George
georgete@umich.edu

A. B. Goncharov
alexander.goncharov@yale.edu

R. Kenyon
richard.kenyon@yale.edu

¹ Department of Mathematics, University of Michigan, 2844 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, USA

² Yale University, 12 Hillhouse, New Haven, CT 06511, USA

While the dimer model can be considered from a purely combinatorial point of view, it also has a rich integrable structure, first described in [13]. The integrable structure on dimers on graphs on the torus was found to generalize many well-known integrable systems, see for example [10] and [1]. What is especially important is that the related integrable system is of cluster nature, and this allows one to immediately quantize it, getting a quantum integrable system.

From the point of view of classical mechanics, associated to the dimer model on a bipartite graph on a torus (or equivalently a periodic bipartite planar graph) is a Poisson variety with a Hamiltonian integrable system. Underlying this system is an algebraic curve $\mathcal{C} = \{P(z, w) = 0\}$ (called the *spectral curve*) and a divisor on this curve—essentially a set of g distinct points $\{(p_1, q_1), \dots, (p_g, q_g)\}$ on $\mathcal{C}$. This is the *spectral data* associated to the model. It was shown in [20] that the map from the weighted graph to the spectral data was bijective, from the space of “face weights” (see below) to the moduli space of genus- g curves and effective degree- g divisors on the open spectral curve $\mathcal{C}^\circ$. Subsequently Fock [11] constructed the inverse spectral map (from the spectral data to the face weights), describing it in terms of theta functions over the spectral curve. The special case of genus 0 was described earlier in [18, 20] and an explicit construction in the case of genus 1 was more recently given in [2]. Positivity of Fock’s inverse map was studied in [3].

In the current paper, we show that the inverse map can be given an explicit *rational* expression in terms of the divisor points $(p_i, q_i) \in \mathcal{C}^\circ$ and the points of $\mathcal{C}$ at toric infinity. An exact statement is given in Theorem 3.10 below.

While Fock’s construction is very natural and interacts nicely with positivity, it involves theta functions. Our construction gives the inverse map as ratios of certain determinants in the spectral data and can be explicitly computed using computer algebra. We briefly describe our construction now. The spectral data is defined via a matrix $K = K(z, w)$ called the Kasteleyn matrix, whose rows are indexed by white vertices, columns by black vertices, and whose entries are Laurent polynomials in z and w . Let us consider the adjugate matrix of K :

$$Q = Q(z, w) = K^{-1} \det K.$$

The matrix Q is important when studying the probabilistic aspects of the dimer model (on the lift of the graph on the torus to the plane): the edge occupation variables form a determinantal process whose kernel is given by the Fourier coefficients of Q/P , as discussed in [21]. In the present work, we have a different use for Q : finding (a column of) the matrix Q from the spectral data allows us to reconstruct the face weights and thereby invert the spectral transform.

The points $(p_i, q_i) \in \mathcal{C}$ are defined to be the points where a column of Q , corresponding to a fixed white vertex $\mathbf{w}$, vanishes. We show that entries in the $\mathbf{w}$ -column of Q , which are Laurent polynomials, can be reconstructed from the spectral data by solving a linear system of equations. Some of the linear equations are easy to describe: for any black vertex $\mathbf{b}$, we have $Q_{\mathbf{bw}}(p_i, q_i) = 0$ for $i = 1, \dots, g$, which are g linear equations in the coefficients of the Laurent polynomial $Q_{\mathbf{bw}}$. However, these equations are usually not sufficient to determine the coefficients of $Q_{\mathbf{bw}}$. We find additional equations from the vanishing of $Q_{\mathbf{bw}}$ at certain points at infinity of the spectral curve

$\mathcal{C}$, and show that these equations determine Q_{bw} uniquely, up to a non-zero constant. We then give a procedure to reconstruct the weights from the $\mathbf{w}$ -column of Q .

A key construction in our approach is the extension of the Kasteleyn matrix K to a map of vector bundles on a toric stack, for which we make crucial use of the classification of line bundles on toric stacks and the computation of their cohomology developed in [5]. Toric stacks already appear implicitly in the context of the spectral transform in [20] and explicitly [23].

The article is organized as follows. In Sect. 2 we review the dimer cluster integrable system and the spectral transform. In Sect. 3, we state Theorem 3.1, which is our main result, and describe the reconstruction procedure. We work out two detailed examples in Sect. 4. Sections 5, 6 and 7 contain proofs of our results. In Appendix A, we review results from toric geometry. In Appendix B, we provide explicit combinatorial descriptions for some of our constructions. These are useful for computations.

2 Background

For further information about the material in this section see [13].

2.1 Dimer Models

Let Γ be a bipartite graph on the torus $\mathbb{T} \cong S^1 \times S^1$ such that the connected components of the complement of Γ —the faces—are contractible. We denote by $B(\Gamma)$ and $W(\Gamma)$ the black and white vertices of Γ , by $V(\Gamma)$ the vertices, and by $E(\Gamma)$ the edges of Γ . When the graph is clear from context, we will usually abbreviate these to B , W , V and E .

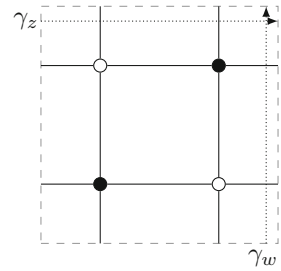
A *dimer model on the torus* is a pair $(\Gamma, [wt])$, where Γ is a bipartite graph on the torus as above and $[wt] \in H^1(\Gamma, \mathbb{C}^\times)$ (Here and throughout the paper, $\mathbb{C}^\times$ denotes the group of nonzero complex numbers under multiplication). For a loop L and a cohomology class $[wt]$, we denote by $[wt](L)$ the pairing between the cohomology and the homology. We orient edges from their black vertex to their white vertex. The cohomology class $[wt]$ can be represented by a cocycle wt which, using this orientation, can be identified with a $\mathbb{C}^\times$ -valued function on the edges of Γ called an *edge weight*.

The edge weight is well-defined modulo multiplication by coboundaries, which are functions on edges $e = \text{bw}$ given by $f(w)f(b)^{-1}$ for functions $f : V(\Gamma) \rightarrow \mathbb{C}^\times$. Note that the weight of a loop is not the product of its edge weights, but the “alternating product” of its edge weights: edges oriented against the orientation of the loop are multiplied with exponent -1 .

A *dimer cover* or *perfect matching* $\mathbf{m}$ of Γ is a subset of $E(\Gamma)$ such that each vertex of Γ is incident to exactly one edge in $\mathbf{m}$. Let $\mathcal{M}$ denote the set of dimer covers of Γ . If we fix a reference dimer cover $\mathbf{m}_0$, we get a function

$$\begin{aligned} \pi_{\mathbf{m}_0} : \mathcal{M} &\rightarrow H_1(\mathbb{T}, \mathbb{Z}) \\ \mathbf{m} &\mapsto [\mathbf{m} - \mathbf{m}_0]. \end{aligned}$$

Fig. 1 The fundamental rectangle R , along with the cycles γ_z, γ_w



Here $m - m_0$ is the 1-chain which assigns 1 to (oriented) edges of m and -1 to (oriented) edges of m_0 , so $m - m_0$ is a union of oriented cycles and doubled edges, whose homology class is $[m - m_0]$.

The *Newton polygon* of Γ is the polygon

$$N(\Gamma) := \text{Convex-hull}(\pi_{m_0}(\mathcal{M})) \subset H_1(\mathbb{T}, \mathbb{R})$$

defined modulo translation by $H_1(\mathbb{T}, \mathbb{Z})$. Changing the reference dimer cover from m_0 to m'_0 results in a translation of the polygon by $[m_0 - m'_0]$, so the Newton polygon does not depend on the choice.

We assume that Γ is such that $N(\Gamma)$ has interior. This is a nondegeneracy condition on Γ . (When N has empty interior, the graph Γ is equivalent under certain elementary transformations to a graph whose lift to $\mathbb{R}^2$ is disconnected, that is, has noncontractible faces; such a graph breaks into essentially one-dimensional components, and there is no integrable system.)

2.2 Zig-Zag Paths and the Newton Polygon

A *zig-zag path* in Γ is a closed path that turns maximally right at each black vertex and maximally left at each white vertex. The *medial graph* of Γ is the graph $\Gamma^\times$ that has a vertex v_e at the mid-point of each edge e of Γ and an edge between v_e and $v_{e'}$ whenever e and e' occur consecutively around a face of Γ . Note that by construction, each vertex of $\Gamma^\times$ has degree 4. A zig-zag path in Γ corresponds to a cycle in $\Gamma^\times$ that goes straight through each degree four vertex, i.e., at every vertex, the outgoing edge of the cycle is the one that is opposite the incoming one (see Fig. 2). Hereafter, when we say zig-zag path, we mean the corresponding cycle in the medial graph.

Let $\tilde{\Gamma}$ be the biperiodic graph on the plane given by the lift of Γ to the universal cover of $\mathbb{T}$. The bipartite graph Γ is said to be *minimal* if the lift of any zig-zag path does not self-intersect, and lifts of any two zig-zag paths do not have “parallel bigons”, where by *parallel bigon* we mean two consecutive intersections where both paths are oriented in the same direction from one to the next. For a minimal bipartite graph Γ on the torus, the Newton polygon has an alternative description in terms of the zig-zag paths of Γ . Namely, since Γ is embedded in $\mathbb{T}$, each zig-zag path α has a non-zero homology class $[\alpha] \in H_1(\mathbb{T}, \mathbb{Z})$. The polygon $N(\Gamma)$ is the unique convex integral polygon defined modulo translation in $H_1(\mathbb{T}, \mathbb{Z})$ whose integral primitive edge

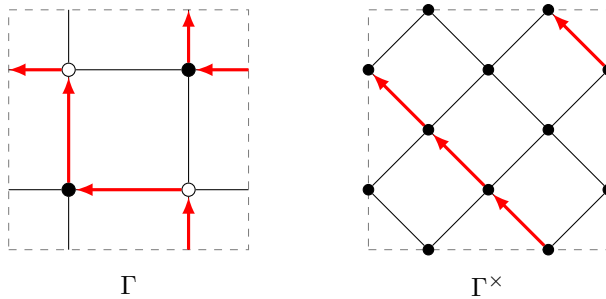
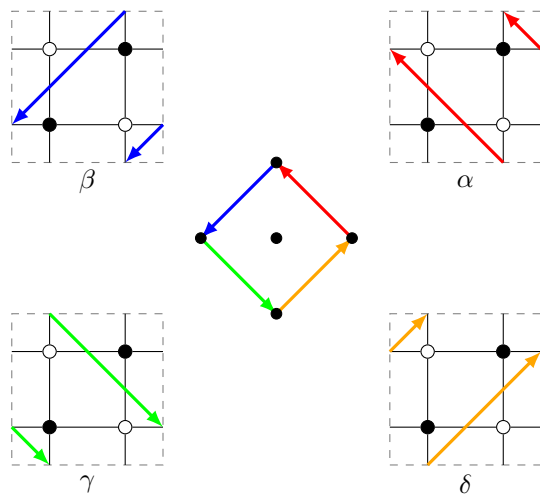


Fig. 2 A zig-zag path in a graph Γ and the corresponding cycle in the medial graph $\Gamma^\times$

Fig. 3 Zig-zag paths and Newton polygon for the bipartite graph in Fig. 1



vectors in counterclockwise order around N are given by the vectors $[\alpha]$ for all zig-zag paths α .

Example 2.1 Consider the fundamental domain for the square lattice shown in Fig. 1, and let γ_z, γ_w be cycles generating $H_1(\mathbb{T}, \mathbb{Z})$ as shown there. We will write homology classes in $H_1(\mathbb{T}, \mathbb{Z})$ in the basis (γ_z, γ_w) . There are four zig-zag paths labeled α, β, γ and δ with homology classes $(-1, 1), (-1, -1), (1, -1)$ and $(1, 1)$ respectively (Fig. 3), and therefore the Newton polygon is

$$\text{Convex-hull}\{(1, 0), (0, 1), (-1, 0), (0, -1)\}.$$

2.3 The Cluster Variety Assigned to a Newton Polygon

For a convex integral polygon $N \subset H_1(\mathbb{T}, \mathbb{R})$ defined modulo translation, consider the family of minimal bipartite graphs Γ with Newton polygon $N(\Gamma) = N$. Any two graphs Γ_1, Γ_2 in the family are related by certain *elementary transformations*; see Fig. 4. An elementary transformation $\Gamma_1 \rightarrow \Gamma_2$ gives rise to a birational map

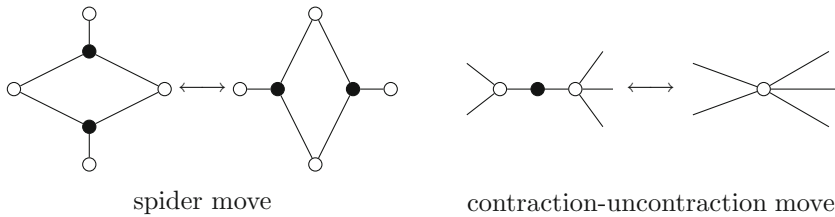


Fig. 4 The elementary transformations

$H^1(\Gamma_1, \mathbb{C}^\times) \dashrightarrow H^1(\Gamma_2, \mathbb{C}^\times)$. Gluing the tori $H^1(\Gamma, \mathbb{C}^\times)$ by these maps, we obtain a space $\mathcal{X}_N$, called the *dimer cluster Poisson variety*. It carries a canonical Poisson structure. The Poisson center is generated by the loop weights of the zig-zag paths. The space $\mathcal{X}_N$ is the phase space of the cluster integrable system. See details in [13].

2.4 Some Notation

Let Σ denote the normal fan of N (see Sect. A.2 and Figs. 7 and 10) so that the set of rays $\Sigma(1) = \{\rho\}$ of Σ is in bijection with the set of edges of N . We denote the edge of N whose inward normal is directed along the ray ρ by E_ρ , and the primitive vector along ρ by u_ρ .

Let $M := H_1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2$ and $M^\vee := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathbb{Z}^2$ be dual lattices and let $\langle *, * \rangle : M \times M^\vee \rightarrow \mathbb{Z}$ denote the duality pairing. Let us consider the algebraic torus with lattice of characters M :

$$\mathbb{T} := \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^2.$$

Let $M_{\mathbb{R}}$ (resp. $M_{\mathbb{R}}^\vee$) denote $M \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $M^\vee \otimes_{\mathbb{Z}} \mathbb{R}$), so that $N \subset M_{\mathbb{R}}$ and $\Sigma \subset M_{\mathbb{R}}^\vee$.

An elementary transformation $\Gamma_1 \rightarrow \Gamma_2$ induces a canonical bijection between zig-zag paths in Γ_1 and zig-zag paths in Γ_2 . Therefore, the set of zig-zag paths is canonically associated with N . We denote the set of zig-zag paths by Z , and for an edge E_ρ of N , we denote by Z_ρ the set of zig-zag paths α such that the primitive vector $[\alpha]$ is contained in E_ρ .

2.5 The Kasteleyn Matrix

Let R be a fundamental rectangle for $\mathbb{T}$, so that $\mathbb{T}$ is obtained by gluing together opposite sides of R . Let γ_z, γ_w be the oriented sides of R generating $H_1(\mathbb{T}, \mathbb{Z})$, as shown in Fig. 1. Let z (resp. w) denote the character χ^{γ_w} (resp. χ^{γ_z}), so the coordinate ring of $\mathbb{T}$ is $\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$.

Let $(*, *)_{\mathbb{T}}$ be the intersection pairing on $H_1(\mathbb{T}, \mathbb{Z})$. For $z, w \in \mathbb{C}^\times$ we multiply edge weights on edges crossing γ_z by $z^{\pm 1}$ and those crossing γ_w by $w^{\pm 1}$, with the sign determined by the orientation. Precisely, we multiply by

$$\phi(e) := z^{(e, \gamma_w)_{\mathbb{T}}} w^{(e, -\gamma_z)_{\mathbb{T}}}, \quad (1)$$

Here $(e, *)_{\mathbb{T}} := (l_e, *)_{\mathbb{T}}$ is the intersection index with the oriented loop l_e obtained by concatenating $e = bw$ with an oriented path contained in R from w to b . Let $H_1(\mathbb{T}, \mathbb{Z})^{\vee} := \text{Hom}_{\mathbb{Z}}(H_1(\mathbb{T}, \mathbb{Z}), \mathbb{Z})$ be the dual lattice of $H_1(\mathbb{T}, \mathbb{Z})$. There is an isomorphism $T := H_1(\mathbb{T}, \mathbb{Z})^{\vee} \otimes \mathbb{C}^{\times} \cong (\mathbb{C}^{\times})^2$, defined as follows. For each edge e of Γ , we associate a character, that is a group homomorphism $T \rightarrow \mathbb{C}^{\times}$:

$$\varphi(e) = z^{(e, \gamma_z)} w^{(e, \gamma_w)}, \quad (2)$$

where (e, γ_z) is the intersection index of the edge e and γ_z , and similarly (e, γ_w) . Explicitly we fix an embedding of Γ in the fundamental rectangle. Isotoping edges if necessary, we may assume that each edge of Γ intersects γ_z and γ_w only finitely many times. For an edge e , let $I_1, \dots, I_n$ be the intersection points of e with γ_z . We define $(e, \gamma_z) := \sum_{i=1}^n (e, \gamma_z)_{I_i}$, where $(e, \gamma_z)_{I_i} \in \{-1, 0, 1\}$ is the local intersection index, where we orient e from its black vertex to its white vertex.

A *Kasteleyn sign* is a cohomology class $[\epsilon] \in H^1(\Gamma, \mathbb{C}^{\times})$ such that for any loop L in Γ , $[\epsilon]([L])$ is -1 (resp., 1) if the number of edges in L is $0 \bmod 4$ (resp., $2 \bmod 4$). Given edge weights wt and ϵ representing $[wt]$ and $[\epsilon]$ respectively, one defines the *Kasteleyn matrix* $K = K(z, w)$, whose columns and rows are parameterized by $b \in B$ and $w \in W$ respectively:

$$K_{w,b} = \sum_{e \in E \text{ incident to } bw} wt(e) \epsilon(e) \phi(e). \quad (3)$$

It describes a map of free $\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$ -modules, called the *Kasteleyn operator*:

$$K : \mathbb{C}[z^{\pm 1}, w^{\pm 1}]^B \rightarrow \mathbb{C}[z^{\pm 1}, w^{\pm 1}]^W, \quad (4)$$

$$\delta_b \mapsto \sum_{w \in W} K_{w,b} \delta_w. \quad (5)$$

Theorem 2.2 (*Kasteleyn 1963, [16]*) *Fix a dimer cover m_0 , and let $\phi(m_0) = \prod_{e \in m_0} \phi(e)$. Then,*

$$\frac{1}{wt(m_0) \epsilon(m_0) \phi(m_0)} \det K = \sum_{m \in \mathcal{M}} \text{sign}([m - m_0]) [wt]([m - m_0]) \chi^{[m - m_0]},$$

where $\text{sign}([m - m_0]) \in \{\pm 1\}$ is a sign that depends only on the homology class $[m - m_0]$ and $[\epsilon]$.

The *characteristic polynomial* is the Laurent polynomial

$$P(z, w) := \frac{1}{wt(m_0) \epsilon(m_0) \phi(m_0)} \det K.$$

Its vanishing locus $\mathcal{C}^{\circ} := \{P(z, w) = 0\} \subset (\mathbb{C}^{\times})^2$ is called the (*open part of the*) *spectral curve*. Theorem 2.2 implies that N is the Newton polygon of $P(z, w)$. Although

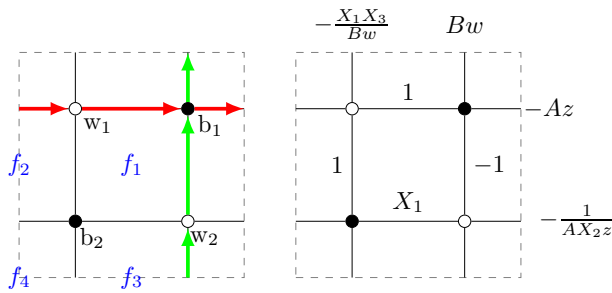


Fig. 5 Shown on the left is a labeling of vertices and faces of Γ , and two cycles a (red) and b (green) in Γ that generate $H_1(\mathbb{T}, \mathbb{Z})$. Shown on the right is a cocycle representing $[wt]$, along with ϵ and ϕ . The signs are due to ϵ , the z , w due to ϕ , and other weights are wt

the definition of the Kasteleyn matrix uses cocycles representing the cohomology classes wt and ϵ , the spectral curve does not depend on these choices.

Example 2.3 Let a and b be the two cycles in Γ shown on the left of Fig. 5 whose projections to $\mathbb{T}$ generate $H_1(\mathbb{T}, \mathbb{Z})$. Let $[wt] \in H^1(\Gamma, \mathbb{C}^\times)$ and let $A := [wt]([a])$, $B := [wt]([b])$. For $i = 1, 2, 3$, let X_i denote the $[wt]([\partial f_i])$, where ∂f_i denotes the boundary of the face f_i (the weight of the fourth face is determined by the fact that the product of all face weights is 1). Then (X_1, X_2, X_3, A, B) generate the coordinate ring of $H^1(\Gamma, \mathbb{C}^\times)$. A cocycle representing $[wt]$ is shown on the right of Fig. 5, along with ϵ and ϕ . The Kasteleyn matrix and the spectral curve are:

$$K = \begin{pmatrix} b_1 & b_2 \\ 1 - Az & 1 - \frac{X_1 X_3}{Bw} \\ -1 + Bw & X_1 - \frac{1}{AX_2 z} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

$$P(z, w) = \left(1 + X_1 + \frac{1}{X_2} + X_1 X_3 \right) - Bw - \frac{X_1 X_3}{Bw} - \frac{1}{AX_2 z} - AX_1 z. \quad (6)$$

2.6 The Toric Surface Assigned to a Newton Polygon

In this section, we collect some notation regarding toric varieties, and refer the reader to the Appendices A.1 and A.2 for more details. A convex integral polygon $N \subset \mathbb{M}_{\mathbb{R}}$ determines a compactification X_N of the complex torus $\mathbb{T}$ called a *toric surface*, and a divisor D_N supported on the boundary $X_N - \mathbb{T}$, so that Laurent polynomials with Newton polygon N extend naturally to sections of the coherent sheaf $\mathcal{O}_{X_N}(D_N)$ (for background on the coherent sheaf associated to a divisor, see for example [7, Chapter 4]).

Denote by $|D_N|$ the projective space of non-zero global sections of the coherent sheaf $\mathcal{O}_{X_N}(D_N)$, considered modulo a multiplicative constant. Assigning to a section its vanishing locus, we see elements of $|D_N|$ as curves in X_N whose restrictions to $\mathbb{T}$ are defined by Laurent polynomials with Newton polygon contained in N .

The genus g of the generic curve in $|D_N|$ is equal to the number of interior lattice points in N . Recall that the edges $\{E_\rho\}$ of N are in bijection with the rays $\{\rho\}$ of Σ . Each edge E_ρ of N determines a projective line D_ρ which we call a *line at infinity* of X_N , and

$$X_N - T = \bigcup_{\rho \in \Sigma(1)} D_\rho.$$

The divisor D_N is given by

$$D_N = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho, \quad (7)$$

where $a_\rho \in \mathbb{Z}$ are such that

$$N = \bigcap_{\rho \in \Sigma(1)} \{m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -a_\rho\}. \quad (8)$$

The lines D_ρ intersect according to the combinatorics of N : precisely, for $\rho_1, \rho_2 \in \Sigma(1)$, the intersection $D_{\rho_1} \cap D_{\rho_2}$ is empty if $E_{\rho_1} \cap E_{\rho_2}$ is empty and a point if $E_{\rho_1} \cap E_{\rho_2}$ is a vertex of N . The intersection index of a generic curve in $|D_N|$ with the line D_ρ is equal to the number $|E_\rho|$ of primitive integral vectors in the edge E_ρ . The points of intersection are called *points at infinity*. Let $\mathcal{C} \in |D_N|$ denote the compactification of the open spectral curve $\mathcal{C}^\circ$, i.e., $\mathcal{C}$ is the closure of $\mathcal{C}^\circ$ in X_N . $\mathcal{C}$ is called the *spectral curve*.

2.7 Casimirs

Let α be a zig-zag path $\alpha = b_1 \rightarrow w_1 \rightarrow b_2 \rightarrow \cdots \rightarrow w_d \rightarrow b_1$ in Z_ρ . We define the *Casimir* C_α by

$$C_\alpha := (-1)^d [\epsilon]([\alpha])[wt]([\alpha]).$$

The Casimirs determine points at infinity of $\mathcal{C}$ as follows: since $[\alpha]$ is primitive and $\langle u_\rho, [\alpha] \rangle = 0$, we can extend it to a basis (x_1, x_2) of M with $[\alpha] = x_1$ and $\langle x_2, u_\rho \rangle = 1$. The affine open variety in X_N corresponding to the cone ρ is

$$U_\rho = \text{Spec } \mathbb{C}[x_1^{\pm 1}, x_2] \cong \mathbb{C}^\times \times \mathbb{C},$$

and $D_\rho \cap U_\rho$ is defined by $x_2 = 0$, and so the character $x_1^{-1} = \chi^{-[\alpha]}$ is a coordinate on the dense open torus $\mathbb{C}^\times = D_\rho \cap U_\rho$ in D_ρ . Therefore, the equation

$$\chi^{-[\alpha]}(v_\rho(\alpha)) = C_\alpha, \quad (9)$$

defines a point $v_\rho(\alpha)$ in D_ρ . In other words, the point is defined as the unique point on the line at infinity such that the monomial $z^i w^j$, where $-[\alpha] = (i, j)$, evaluates to C_α . We will prove later (see (42)) that these are precisely the points at infinity of $\mathcal{C}$.

Example 2.4 Consider the fundamental domain of the square lattice, whose zig-zag paths were listed in Example 2.1 and Fig. 3. The Casimirs are

$$C_\alpha = -\frac{B}{AX_1}, \quad C_\beta = -\frac{1}{ABX_2}, \quad C_\gamma = -\frac{AX_1X_2X_3}{B}, \quad C_\delta = -\frac{AB}{X_3}. \quad (10)$$

Let us denote the normal ray in Σ of a zig-zag path ω by $\rho(\omega)$, so $u_{\rho(\alpha)} = (-1, -1)$ etc. We choose $x_2 = \chi^{(0,-1)}$ so that $\langle (0, -1), u_{\rho(\alpha)} \rangle = 1$. Then we have $U_{\rho(\alpha)} = \text{Spec } \mathbb{C}[x_1 = z^{-1}w, x_2 = w^{-1}]$ and $D_{\rho(\alpha)} \subset U_{\rho(\alpha)}$ is given by $x_2 = 0$. In this case, $D_N = D_{\rho(\alpha)} + D_{\rho(\beta)} + D_{\rho(\gamma)} + D_{\rho(\delta)}$ and $P(z, w)$ is a global section of $\mathcal{O}_{X_N}(D_N)$. We trivialize $\mathcal{O}_{X_N}(D_N)$ over $U_{\rho(\alpha)}$ as follows:

$$\begin{aligned} \mathcal{O}_{X_N}(D_N)|_{U_{\rho(\alpha)}} &= \{t \in \mathbb{C}[z^{\pm 1}, w^{\pm 1}] : \text{div } t|_{U_{\rho(\alpha)}} + D_{\rho(\alpha)} \geq 0\} \cong \mathcal{O}_{U_{\rho(\alpha)}} \\ &t \mapsto tx_2 \end{aligned}$$

Then making the change of variables $z = \frac{1}{x_1x_2}$ and $w = \frac{1}{x_2}$, and multiplying by x_2 , the portion of the spectral curve $\mathcal{C}$ in U_ρ is cut out by

$$\left(1 + X_1 + \frac{1}{X_2} + X_1X_3\right)x_2 - B - \frac{X_1X_3}{B}x_2^2 - \frac{x_1x_2^2}{AX_2} - \frac{AX_1}{x_1},$$

so that $\mathcal{C} \cap D_{\rho(\alpha)}$ is given by

$$-B - \frac{AX_1}{x_1} = 0.$$

Therefore, $v(\alpha)$ is given by $\frac{z}{w} = \frac{1}{x_1} = C_\alpha$, which agrees with (9). The table below lists the points at infinity for each of the zig-zag paths.

Zig-zag path	Homology class	Basis x_1, x_2	Point at infinity
α	$(-1, 1)$	$(-1, 1), (0, -1)$	$x_1 = \frac{1}{C_\alpha}, x_2 = 0$
β	$(-1, -1)$	$(-1, -1), (0, -1)$	$x_1 = \frac{1}{C_\beta}, x_2 = 0$
γ	$(1, -1)$	$(1, -1), (0, 1)$	$x_1 = \frac{1}{C_\gamma}, x_2 = 0$
δ	$(1, 1)$	$(1, 1), (0, 1)$	$x_1 = \frac{1}{C_\delta}, x_2 = 0$

(11)

2.8 The Spectral Transform

Our next goal is to define the spectral transform, which plays the key role in this paper. We present two equivalent definitions of the spectral transform. The first is the original definition of Kenyon and Okounkov [20], and it is the one which we use in computations. However, it depends on the choice of the distinguished white vertex $\mathbf{w}$. The second is more invariant, and does not require choosing a distinguished white vertex $\mathbf{w}$.

Recall that for each edge E_ρ of N , we have $\#Z_\rho = \#\mathcal{C} \cap D_\rho$, but there is no canonical bijection between these sets. We define a *parameterization of the points at infinity by zig-zag paths* to be a choice of bijections $\nu = \{\nu_\rho\}_{\rho \in \Sigma(1)}$, where

$$\nu_\rho : Z_\rho \xrightarrow{\sim} \mathcal{C} \cap D_\rho. \quad (12)$$

For a curve $\mathcal{C} \in |D_N|$, we denote by $\text{Div}_\infty(\mathcal{C})$ the abelian group of divisors on $\mathcal{C}$ supported at the *infinity*, that is at $\mathcal{C} \cap D_N$.

Compactifications of the Kasteleyn operator will play a important role in this paper. The main ingredient in the construction of these compactifications is a combinatorial object called the *discrete Abel map* introduced by Fock [11] that encodes intersections with zig-zag paths. Let Γ be a minimal bipartite graph in $\mathbb{T}$ with Newton polygon N and spectral curve $\mathcal{C}$. The discrete Abel map

$$\mathbf{d} : B \cup W \cup F \rightarrow \text{Div}_\infty(\mathcal{C})$$

assigns to each vertex and face of Γ a divisor at infinity. It is defined uniquely up to a constant by the requirement that for a path γ from x to y , contained in the fundamental domain R , where x and y are either vertices or faces of Γ , we have

$$\mathbf{d}(y) - \mathbf{d}(x) = \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho} (\alpha, \gamma)_R \nu_\rho(\alpha).$$

Here $(\alpha, \gamma)_R$ is the intersection index in R , i.e., the signed number of intersections of α with γ . Since we require γ to be contained in R , this is well-defined, independent of the choice of path γ . Locally, the rule is as follows:

1. If $\mathbf{b}$ is a black vertex incident to a face f , and $\mathbf{b}$ and f are separated by $\alpha \in Z_\rho$, then $\mathbf{d}(\mathbf{b}) = \mathbf{d}(f) + \nu_\rho(\alpha)$.
2. If $\mathbf{w}$ is a white vertex incident to a face f , and $\mathbf{w}$ and f are separated by $\alpha \in Z_\rho$, then $\mathbf{d}(\mathbf{w}) = \mathbf{d}(f) - \nu_\rho(\alpha)$.

We normalize $\mathbf{d}$, setting the value of $\mathbf{d}$ at certain face f_0 of Γ to be 0. Then for any black vertex $\mathbf{b}$, face f , and white vertex $\mathbf{w}$ of $\tilde{\Gamma}$ we have:

$$\deg \mathbf{d}(\mathbf{b}) = 1, \quad \deg \mathbf{d}(f) = 0, \quad \deg \mathbf{d}(\mathbf{w}) = -1. \quad (13)$$

Remark 2.5 Only differences of the form $\mathbf{d}(y) - \mathbf{d}(x)$ will appear in our constructions later, so the choice of normalization does not play a role.

Example 2.6 Let us compute the discrete Abel map $\mathbf{d}$ for the square lattice in Fig. 5. We normalize $\mathbf{d}(f_1) = 0$. Then we have

$$\mathbf{d}(b_1) = v_{\rho(\gamma)}(\gamma), \quad \mathbf{d}(b_2) = v_{\rho(\alpha)}(\alpha), \quad \mathbf{d}(w_1) = -v_{\rho(\beta)}(\beta), \quad \mathbf{d}(w_2) = -v_{\rho(\delta)}(\delta),$$

where v is shown in table (11).

Definition 1. A *divisor spectral data* related to a Newton polygon N is a triple $(\mathcal{C}, S, v)$ where $\mathcal{C} \in |D_N|$ is a genus g curve on the toric surface X_N , S is a degree g effective divisor in $\mathcal{C}^\circ$, and $v = \{v_\rho\}$ are parameterizations of the divisors $D_\rho \cap \mathcal{C}$, see (12). Denote by $\mathcal{S}_N$ the moduli space parameterizing the divisor spectral data on N . Let us fix a *distinguished white vertex* $\mathbf{w}$ of Γ . Then there is a rational map (here and in the sequel, $\dashrightarrow$ means a rational map), called the *spectral transform*, defined by Kenyon and Okounkov [20],

$$\kappa_{\Gamma, \mathbf{w}} : \mathcal{X}_N \dashrightarrow \mathcal{S}_N \quad (14)$$

defined on the dense open subset $H^1(\Gamma, \mathbb{C}^\times)$ of $\mathcal{X}_N$ by $[wt] \mapsto (\mathcal{C}, S, v)$ as follows:

1. $\mathcal{C}$ is the spectral curve.
2. For generic $[wt]$, $\mathcal{C}$ is a smooth curve and coker K is the pushforward of a line bundle on $\mathcal{C}^\circ$. Let $s_{\mathbf{w}}$ be the section of coker K given by the $\mathbf{w}$ -entry of the cokernel map. S is defined to be the divisor of this section. In Corollary 6.3, we show that S has degree g . Then S is the set of g points in $\mathcal{C}^\circ$ where the $\mathbf{w}$ -column of the adjugate matrix $Q = Q(z, w) = K^{-1} \det K$ vanishes.
3. The parameterization of points at infinity by zig-zag paths v is defined as follows: $v_\rho(\alpha)$ is the point in $\mathcal{C} \cap D_\rho$ satisfying $\chi^{-[\alpha]} = C_\alpha$ (see Sect. 2.7). We call $v_\rho(\alpha)$ the *point at infinity associated to α* .

Definition 2. A *line bundle spectral data* related to a Newton polygon N is a triple $(\mathcal{C}, \mathcal{L}, v)$ where $\mathcal{C} \in |D_N|$ is a genus g curve on the toric surface X_N , $\mathcal{L}$ is a degree $g - 1$ line bundle on $\mathcal{C}$, and v is a parameterization of points at infinity by zig-zag paths. Denote by $\mathcal{S}'_N$ the moduli space parameterizing the line bundle spectral data on N .

The spectral transform is a rational map

$$\kappa_{\Gamma, \mathbf{d}} : \mathcal{X}_N \dashrightarrow \mathcal{S}'_N$$

defined on the dense open subset $H^1(\Gamma, \mathbb{C}^\times)$ of $\mathcal{X}_N$ by $[wt] \mapsto (\mathcal{C}, \mathcal{L}, v)$, where:

1. $\mathcal{C}$ is the spectral curve.
2. Let $K|_{\mathcal{C}^\circ}$ denote the restriction of the Kasteleyn matrix to $\mathcal{C}^\circ$. The discrete Abel map $\mathbf{d}$ determines an extension $\overline{K}$ of $K|_{\mathcal{C}^\circ}$ to a morphism of locally free sheaves on $\mathcal{C}$; see Sect. 6. The coherent sheaf $\mathcal{L}$ is defined as the cokernel of $\overline{K}$. When $\mathcal{C}$ is a

smooth curve, which happens for generic $[wt]$, $\mathcal{L}$ is a line bundle. The convention $\deg \mathbf{d}(\mathbf{w}) = -1$ implies that $\deg \mathcal{L} = g - 1$; see Proposition 6.4.

3. The parameterizations of the divisors $D_N \cap \mathcal{C}$ are defined by associating to a zig-zag path α the point at infinity $v_\rho(\alpha)$.

Since ρ is determined by α , we will use the simpler notation $v(\alpha) := v_\rho(\alpha)$ hereafter.

The two types of spectral data are equivalent. Given a degree g effective divisor S , we have (Proposition 6.4)

$$\mathcal{L} \cong \mathcal{O}_{\mathcal{C}}(S + \mathbf{d}(\mathbf{w})). \quad (15)$$

On the other hand, given a line bundle $\mathcal{L}$ and a white vertex $\mathbf{w}$, we can recover S as follows. Consider the Abel-Jacobi map

$$\begin{aligned} A^g : \text{Sym}^g \mathcal{C} &\rightarrow \text{Jac}(\mathcal{C}), \\ E &\mapsto \mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(E + \mathbf{d}(\mathbf{w})). \end{aligned}$$

Then A^g is birational by the Abel-Jacobi theorem [4, Corollary 4.6]. We obtain $S = (A^g)^{-1}(\mathcal{O}_{\mathcal{C}})$.

Example 2.7 We compute the spectral transform for our running example of the square lattice. Let us take the distinguished white vertex to be $\mathbf{w} = \mathbf{w}_1$.

$$Q = \begin{pmatrix} \overset{\mathbf{w}_1}{X_1 - \frac{1}{AX_2z}} & \overset{\mathbf{w}_2}{-1 + \frac{X_1X_3}{Bw}} \\ 1 - Bw & 1 - Az \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \quad (16)$$

Solving $Q_{b_1\mathbf{w}}(p, q) = Q_{b_2\mathbf{w}}(p, q) = 0$, we get

$$p = \frac{1}{AX_1X_2}, \quad q = \frac{1}{B}. \quad (17)$$

Therefore, the spectral transform is:

$$\begin{aligned} \kappa_{\Gamma, \mathbf{w}} : H^1(\Gamma, \mathbb{C}^\times) &\dashrightarrow \mathcal{S}_N \\ (X_1, X_2, X_3, A, B) &\mapsto (\mathcal{C}, (p, q), v), \end{aligned}$$

where $\mathcal{C} = \{P(z, w) = 0\}$ with $P(z, w)$ as is in (6), $S = (p, q)$ is a single point (the genus $g = 1$ since N in Fig. 3 has one interior lattice point) and v is as shown in table (11).

3 The Main Theorem

Below we introduce functions $V_{\mathbf{bw}}$ on the moduli space $\mathcal{S}_N$ of spectral data, relying on results in the remaining Sects. 5, 6, 7. They are defined for any pair $(\mathbf{bw}) \in B \times W$

of black and white vertices, and defined as the solution to a system of linear equations $\mathbb{V}_{\mathbf{bw}}$.

The main result of the paper is the following.

Theorem 3.1 *For the distinguished white vertex $\mathbf{w}$, the pull-back of the function $\mathbb{V}_{\mathbf{bw}}$ under the spectral map coincides, up to a multiplicative constant, with the $\mathbf{bw}$ matrix element $Q_{\mathbf{bw}}$ of the adjugate matrix $Q := K^{-1} \det K$ of the Kasteleyn matrix K . That is,*

$$Q_{\mathbf{bw}} = c \cdot \kappa_{\Gamma, \mathbf{w}}^*(\mathbb{V}_{\mathbf{bw}}), \quad (18)$$

where c depends on $\mathbf{b}$ (and $\mathbf{w}$).

As an application of this result, we get an explicit description of the inverse to the spectral map (14); see Sect. 3.2.

The next few sections discuss the structure of the system of linear equations $\mathbb{V}_{\mathbf{bw}}$. Detailed examples are given in Sect. 4.

3.1 The Matrix $\mathbb{V}_{\mathbf{bw}}$

The system of linear equations $\mathbb{V}_{\mathbf{bw}}$ is in the variables $(a_m)_{m \in N_{\mathbf{bw}} \cap \mathbf{M}}$ where $N_{\mathbf{bw}} \subset \mathbf{M}_{\mathbb{R}}$ is a convex polygon, introduced in Sect. 3.1.1.2, and called the *small Newton polygon*. There is one system for every pair $(\mathbf{bw}) \in B \times W$. The system $\mathbb{V}_{\mathbf{bw}}$ is of the form $(\text{matrix})(a_m) = 0$; we also denote this matrix by $\mathbb{V}_{\mathbf{bw}}$. Therefore, the columns of the matrix $\mathbb{V}_{\mathbf{bw}}$ are indexed by the lattice points $N_{\mathbf{bw}} \cap \mathbf{M}$. By Corollary 5.3, the polygon $N_{\mathbf{bw}}$ is the Newton polygon of the Laurent polynomial $Q_{\mathbf{bw}}$.

The equations in $\mathbb{V}_{\mathbf{bw}}$, i.e., the rows of the matrix $\mathbb{V}_{\mathbf{bw}}$ are defined in Sect. 3.1.2. There are two types:

1. There is a row for each of the points $(p_1, q_1), \dots, (p_g, q_g)$ of the divisor S on the spectral curve. The entry of the row in column $m \in N_{\mathbf{bw}} \cap \mathbf{M}$ is $\chi^m(p_i, q_i)$.
2. The remaining rows correspond to certain zig-zag paths α . The entries in the row corresponding to α are certain monomials in C_α .

Let us proceed to the precise definition of the matrix $\mathbb{V}_{\mathbf{bw}}$.

3.1.1 Columns of the Matrix $\mathbb{V}_{\mathbf{bw}}$

We now describe the small Newton polygons, whose lattice points correspond to columns of $\mathbb{V}_{\mathbf{bw}}$.

3.1.1.1. Rational Abel Map $\mathbf{D}$.

Recall the set $\{D_\rho\}$ of lines at infinity of the toric surface X_N . Consider the $\mathbb{Q}$ -vector space $\text{Div}_{\mathbf{T}}^{\mathbb{Q}}(X_N)$ of $\mathbb{Q}$ -divisors at infinity, defined as the $\mathbb{Q}$ -vector space with a basis given by the divisors D_ρ :

$$\text{Div}_{\mathbf{T}}^{\mathbb{Q}}(X_N) := \bigoplus_{\rho \in \Sigma(1)} \mathbb{Q} D_\rho.$$

We define a *rational Abel map*

$$\mathbf{D} : V \rightarrow \operatorname{Div}_T^{\mathbb{Q}}(X_N)$$

which assigns to each vertex v of the graph Γ a $\mathbb{Q}$ -divisor at infinity $\mathbf{D}(v)$ as follows:

1. Normalize $\mathbf{D}(w) = 0$. As in the case of $\mathbf{d}$, the choice of normalization plays no role, and we can replace 0 with any $\mathbb{Q}$ -divisor.
2. For any path γ contained in R from v_1 to v_2 ,

$$\mathbf{D}(v_2) - \mathbf{D}(v_1) = \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho} \frac{(\alpha, \gamma)_R}{|E_\rho|} D_\rho,$$

where $(\cdot, \cdot)_R$ is the intersection index in R , i.e., the signed number of intersections of α with γ .

The following lemma follows from definitions.

Lemma 3.2 *Let α, β be the zig-zag paths through $e = bw$, with $\alpha \in Z_\sigma, \beta \in Z_\rho$. Then, we have*

$$\mathbf{D}(w) - \mathbf{D}(b) = -\frac{1}{|E_\sigma|} D_\sigma - \frac{1}{|E_\rho|} D_\rho - \operatorname{div} \phi(e) \quad (19)$$

where $\phi(e)$ is the character defined in (1) and $\operatorname{div} \phi(e)$ denotes its (Weil) divisor as in (49).

3.1.1.2 Small Newton Polygons

Recall the divisor D_N at infinity of X_N , see (7). Given an edge $e = bw$, we define a $\mathbb{Q}$ -divisor at infinity

$$Y_{bw} := D_N - \mathbf{D}(w) + \mathbf{D}(b) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho: b \in \alpha} \frac{1}{|E_\rho|} D_\rho. \quad (20)$$

Here the double sum is over all zig-zag paths α passing through b . We define $b_\rho \in \mathbb{Q}$ as the multiplicities of the projective lines at infinity D_ρ in the divisor Y_{bw} :

$$Y_{bw} = \sum_{\rho \in \Sigma(1)} b_\rho D_\rho. \quad (21)$$

Definition 3.3 The *small Newton polygon* N_{bw} is the polygon defined by the formula

$$N_{bw} = \bigcap_{\rho \in \Sigma(1)} \{m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -b_\rho\}. \quad (22)$$



Fig. 6 The two small polygons in Example 3.4. The big black dot denotes the origin, while the other black dots are integral points

There is a canonical bijection between divisors D in $\text{Div}_{\mathbb{T}}^{\mathbb{Q}}(X_N)$ and convex polygons P with rational intercepts (see Proposition A.2 for its importance in toric geometry):

$$D = \sum_{\rho \in \Sigma(1)} a_{\rho} D_{\rho} \leftrightarrow P = \bigcap_{\rho \in \Sigma(1)} \{m \in M_{\mathbb{R}} : \langle m, u_{\rho} \rangle \geq -a_{\rho}\}, \quad a_{\rho} \in \mathbb{Q}. \quad (23)$$

Therefore, N_{bw} is the polygon associated to the divisor Y_{bw} in (23).

The polygon N_{bw} may not be integral. We will consider only integral points in it. The convex hull of the integral points in N_{bw} contains the Newton polygon of Q_{bw} (Corollary 5.3).

Example 3.4 We compute the small polygons for the square lattice in Fig. 5. Recall that we chose $\mathbf{w} = \mathbf{w}_1$. Since there is only one zig-zag path in each homology direction, the rational Abel map $\mathbf{D}$ is obtained from $\mathbf{d}$ by replacing the point at infinity with the corresponding line at infinity, so from Example 2.6, we have

$$\mathbf{D}(\mathbf{b}_1) = D_{\rho(\gamma)}, \quad \mathbf{D}(\mathbf{b}_2) = D_{\rho(\alpha)}, \quad \mathbf{D}(\mathbf{w}_1) = -D_{\rho(\beta)}, \quad \mathbf{D}(\mathbf{w}_2) = -D_{\rho(\delta)}.$$

We have $D_N = D_{\rho(\alpha)} + D_{\rho(\beta)} + D_{\rho(\gamma)} + D_{\rho(\delta)}$, using which we compute

$$\begin{aligned} Y_{\mathbf{b}_1 \mathbf{w}_1} &= (D_{\rho(\alpha)} + D_{\rho(\beta)} + D_{\rho(\gamma)} + D_{\rho(\delta)}) - (-D_{\rho(\beta)}) + D_{\rho(\gamma)} - (D_{\rho(\alpha)} + D_{\rho(\beta)} \\ &\quad + D_{\rho(\gamma)} + D_{\rho(\delta)}) = D_{\rho(\beta)} + D_{\rho(\gamma)}, \\ Y_{\mathbf{b}_2 \mathbf{w}_1} &= (D_{\rho(\alpha)} + D_{\rho(\beta)} + D_{\rho(\gamma)} + D_{\rho(\delta)}) - (-D_{\rho(\beta)}) + D_{\rho(\alpha)} - (D_{\rho(\alpha)} + D_{\rho(\beta)} \\ &\quad + D_{\rho(\gamma)} + D_{\rho(\delta)}) = D_{\rho(\alpha)} + D_{\rho(\beta)}. \end{aligned}$$

Therefore,

$$\begin{aligned} N_{\mathbf{b}_1 \mathbf{w}_1} &= \{-i - j \geq 0\} \cap \{i - j \geq -1\} \cap \{i + j \geq -1\} \cap \{-i + j \geq 0\}, \\ N_{\mathbf{b}_2 \mathbf{w}_1} &= \{-i - j \geq -1\} \cap \{i - j \geq -1\} \cap \{i + j \geq 0\} \cap \{-i + j \geq 0\}, \end{aligned}$$

see Fig. 6. Note that the convex hulls of the lattice points are the Newton polygons of $Q_{\mathbf{b}_1 \mathbf{w}_1}$ and $Q_{\mathbf{b}_2 \mathbf{w}_1}$ in (16).

3.1.2 Rows of the Matrix $\mathbb{V}_{\text{bw}}$

Recall that the variables in $\mathbb{V}_{\text{bw}}$ are $(a_m)_{m \in N_{\text{bw}} \cap M}$. We identify a Laurent polynomial $F = \sum_{m \in M} b_m \chi^m$ with its vector of coefficients $(b_m)_{m \in M}$. The equations in $\mathbb{V}_{\text{bw}}$ are of two types:

1. For each $1 \leq i \leq g$, we have the linear equations

$$\sum_{m \in N_{\text{bw}} \cap M} a_m \chi^m(p_i, q_i) = 0, \quad (24)$$

so the entry of the corresponding row of $\mathbb{V}_{\text{bw}}$ in column m is $\chi^m(p_i, q_i)$.

2. Recall the notation $\lfloor x \rfloor$ for the largest integer n such that $n \leq x$. Given a $\mathbb{Q}$ -divisor $D = \sum_{\rho \in \Sigma(1)} b_\rho D_\rho$, we define a divisor with integral coefficients

$$\lfloor D \rfloor := \sum_{\rho \in \Sigma(1)} \lfloor b_\rho \rfloor D_\rho.$$

Recall the divisor Y_{bw} in (20). For a divisor D at infinity, let $D|_{\mathcal{C}}$ denote the divisor corresponding to the intersection of D with $\mathcal{C}$. Precisely, if $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$, then $D|_{\mathcal{C}} := \sum_{\rho \in \Sigma(1)} a_\rho \sum_{\alpha \in Z_\rho} v(\alpha)$. We have a linear equation for every zig-zag path α such that $v(\alpha)$ appears in

$$-D_N|_{\mathcal{C}} + \mathbf{d}(\mathbf{w}) - \mathbf{d}(\mathbf{b}) + \sum_{\alpha \in Z} v(\alpha) + \lfloor Y_{\text{bw}} \rfloor|_{\mathcal{C}}. \quad (25)$$

Suppose $\alpha \in Z_\rho$ is a zig-zag path that contributes an equation. We extend $[\alpha]$ to a basis (x_1, x_2) of M , where $x_1 := [\alpha]$ and $\langle x_2, u_\rho \rangle = 1$, so that for any $m \in M$, we can write

$$\chi^m = x_1^{b_m} x_2^{c_m}, \quad b_m, c_m \in \mathbb{Z}.$$

Let N_{bw}^ρ be the set of lattice points in N_{bw} closest to the edge E_ρ of N i.e., the set of points in N_{bw} that minimize the functional $\langle *, u_\rho \rangle$. Then the equation associated with α is

$$\sum_{m \in N_{\text{bw}}^\rho \cap M} a_m C_\alpha^{-b_m} = 0. \quad (26)$$

So the entry in column $m \in N_{\text{bw}}^\rho \cap M$ is the monomial $C_\alpha^{-b_m}$, and the entries in the other columns are 0. Choosing a different basis vector x_2 leads to the same equation multiplied by a monomial in C_α .

Remark 3.5 When all the sides of the Newton polygon are primitive, we call the Newton polygon *simple*. In this case, we have $\lfloor Y_{\text{bw}} \rfloor = Y_{\text{bw}}$ and $\mathbf{d}(\mathbf{w}) - \mathbf{d}(\mathbf{b}) = (\mathbf{D}(\mathbf{w}) - \mathbf{D}(\mathbf{b}))|_{\mathcal{C}}$. Then Formula (25) simplifies considerably to

$$\sum_{\alpha \in Z: \mathbf{b} \notin \alpha} v(\alpha). \quad (27)$$

So for a simple Newton polygon the Casimir rows of the matrix $\mathbb{V}_{\mathbf{bw}}$, i.e., the rows providing equations (26), are parameterized by the zig-zag paths α which do not contain the vertex $\mathbf{b}$.

3.1.3 The Functions $V_{\mathbf{bw}}$

The number of rows of $\mathbb{V}_{\mathbf{bw}}$ is at least as large as the number of columns minus one, but not necessarily equal. However, Proposition 7.3 shows that there is a unique solution to $\mathbb{V}_{\mathbf{bw}}$ up to a multiplicative constant. Therefore,

$$V_{\mathbf{bw}} := \sum_{m \in N_{\mathbf{bw}} \cap M} a_m \chi^m, \quad (28)$$

is uniquely defined up to a multiplicative constant (where $(a_m)_{m \in N_{\mathbf{bw}} \cap M}$ is a solution to $\mathbb{V}_{\mathbf{bw}}$). Only ratios of the values of these functions that are independent of the multiplicative constant appear in the inverse map, see Sect. 3.2.

Remark 3.6 When the equations in $\mathbb{V}_{\mathbf{bw}}$ are linearly independent (so there is exactly one less equation than the number of variables), we can prepend to $\mathbb{V}_{\mathbf{bw}}$ the equation $\sum_{m \in N_{\mathbf{bw}} \cap M} a_m \chi^m$ to get a square matrix, which we denote by $\mathbb{V}_{\mathbf{bw}}^\chi$. Then the function $V_{\mathbf{bw}}$ is the determinant:

$$V_{\mathbf{bw}} = \det \mathbb{V}_{\mathbf{bw}}^\chi.$$

Indeed, given an $(n-1) \times n$ matrix (a_{ij}) , the system of linear equations $\sum_{j=1}^n a_{ij} x_j = 0$ has a solution given by the signed maximal minors A_j of the matrix A :

$$x_j = (-1)^j A_j.$$

Here A_j is the determinant of the matrix obtained by deleting the j -th column of A . Therefore, the determinant of the augmented matrix $\mathbb{V}_{\mathbf{bw}}^\chi$ recovers the expression $V_{\mathbf{bw}}$ in (28).

Example 3.7 We compute the linear system of equations $\mathbb{V}_{\mathbf{bw}}$ for the square lattice in Fig. 5 with $\mathbf{w} = \mathbf{w}_1$. Since both black vertices are contained in every zig-zag path, the formula (27) is 0, so there are no equations of type 2 in $\mathbb{V}_{\mathbf{bw}}$ for $\mathbf{b} \in B$. Therefore,

$$\mathbb{V}_{\mathbf{b}_1 \mathbf{w}} = (1 \ p^{-1}), \quad \mathbb{V}_{\mathbf{b}_2 \mathbf{w}} = (1 \ q).$$

By Remark 3.6, we get

$$V_{\mathbf{b}_1 \mathbf{w}} = \begin{vmatrix} 1 & z^{-1} \\ 1 & p^{-1} \end{vmatrix}, \quad V_{\mathbf{b}_2 \mathbf{w}} = \begin{vmatrix} 1 & w \\ 1 & q \end{vmatrix}. \quad (29)$$

Using (17), we have

$$\kappa_{\Gamma, \mathbf{w}}^*(V_{\mathbf{b}_1 \mathbf{w}}) = AX_1 X_2 - \frac{1}{z} = AX_2 Q_{\mathbf{b}_1 \mathbf{w}},$$

$$\kappa_{\Gamma, \mathbf{w}}^*(V_{b_2 \mathbf{w}}) = \frac{1}{B} - w = \frac{1}{B} Q_{b_2 \mathbf{w}},$$

verifying the conclusion of Theorem 3.1.

3.2 Reconstructing Weights via Functions $V_{b\mathbf{w}}$.

Take a white vertex w and a zig-zag path α containing w . The pair (w, α) determines a wedge $W := b \xrightarrow{e} w \xrightarrow{e'} b'$, where w is a white vertex incident to the vertices b, b' such that bwb' is a part of α . Recall $\phi(e)$ from (1), and the Kasteleyn sign $\epsilon(e)$. We assign to this wedge the ratio

$$r_W := -\frac{\epsilon(e')\phi(e')V_{b'\mathbf{w}}}{\epsilon(e)\phi(e)V_{b\mathbf{w}}}(\nu(\alpha)). \quad (30)$$

Note that we use the distinguished white vertex $\mathbf{w}$ in the expression rather than w . The expression is in fact independent of w , as we will see in the proof of Theorem 3.10 below.

Remark 3.8 The ratio on the right is a rational function on the curve. We evaluate the ratio at the point at infinity of the spectral curve $\nu(\alpha)$ corresponding to the zig-zag path α , see (12). To do this, we first extend $[\alpha]$ to a basis (x_1, x_2) of M with $[\alpha] = x_1$ and $\langle x_2, u_\rho \rangle = 1$, as explained in Sect. 2.7. Then $\nu(\alpha)$ is given by $\frac{1}{x_1} = C_\alpha, x_2 = 0$. The numerator and denominator in (30) vanish to the same order in x_2 by Corollary 6.2 below, so after factoring out and canceling the highest power of x_2 in the numerator and denominator, we can evaluate at $x_1 = \frac{1}{C_\alpha}, x_2 = 0$ to get a well-defined number.

Let $L = b_1 \rightarrow w_1 \rightarrow b_2 \rightarrow \dots \rightarrow b_\ell = b_1$ be an oriented loop on Γ . It is a concatenation of wedges $W_i := b_{i-1}w_ib_i, i = 1, \dots, \ell$ (with i taking values cyclic modulo ℓ) provided by the white vertices. Denote by α_i the zig-zag path assigned to the wedge W_i . We define a cohomology class $[\omega]$ by

$$[\omega]([L]) := \prod_{i=1}^{\ell} r_{W_i}. \quad (31)$$

Lemma 3.9 *The product (31) does not depend on the ambiguities of the multiplicative constants in the involved functions $V_{b\mathbf{w}}$.*

Proof For each black vertex b_i in L , $V_{b_i\mathbf{w}}$ appears twice in (31), once each in the numerator and denominator, and so the multiplicative constants cancel out. $\square$

Theorem 3.10 *The cohomology classes $[wt]$ and $\kappa_{\Gamma, \mathbf{w}}^*[\omega]$ are equal.*

Proof Let $b \xrightarrow{e} w \xrightarrow{e'} b'$ be a wedge with zig-zag path $\alpha \in Z_\rho$. The restriction of the characteristic polynomial $P(z, w)|_{D_\rho}$ is the partition function of those dimers whose homology class in N lies on E_ρ . From the explicit construction of external dimers

in [13] (that is, dimers whose homology classes are in ∂N), we have that each dimer with homology class in E_ρ uses exactly one of the edges e or e' . Since $Q_{\text{bw}}(z, w)$ is the partition function of dimers with the vertices bw removed, we have

$$P|_{D_\rho} = wt(e)\epsilon(e)\phi(e)Q_{\text{bw}}|_{D_\rho} + wt(e')\epsilon(e')\phi(e')Q_{\text{b'w}}|_{D_\rho}.$$

Since $v(\alpha)$ is on the spectral curve, $P(v(\alpha)) = 0$, from which we get

$$\frac{wt(e)}{wt(e')} = -\frac{\epsilon(e')\phi(e')Q_{\text{b'w}}}{\epsilon(e)\phi(e)Q_{\text{bw}}}(v(\alpha)). \quad (32)$$

We have $\text{corank}(K) = 1$ at smooth points of $\mathcal{C}$. Note that $KQ|_{\mathcal{C}} = 0$. Therefore, for generic wt , since $\mathcal{C}$ is smooth, Q is a rank 1 matrix given by

$$Q = \ker K^* \otimes \text{coker } K.$$

This implies that

$$\frac{Q_{\text{bw}}}{Q_{\text{b'w}}}(v(\alpha)) = \frac{Q_{\text{bw}}}{Q_{\text{b'w}}}(v(\alpha)).$$

□

Example 3.11 Consider the cycle a in Fig. 5 given by the red horizontal path. We write it as the concatenation of the two wedges W_1 and W_2 represented by (w_1, δ) and (w_1, γ) respectively. From Table (11), we know that in the basis $x_1 = zw, x_2 = w$, the point $v(\delta)$ is given by $x_1 = \frac{1}{C_\delta}, x_2 = 0$. Using (29), and making the substitution $z = \frac{x_1}{x_2}, w = x_2$, we get

$$\begin{aligned} r_{W_1} &= -\frac{-1 \cdot w^{-1} \cdot V_{\text{b}_2\text{w}}}{-1 \cdot z \cdot V_{\text{b}_1\text{w}}}(v(\delta)) \\ &= -\frac{1}{zw} \frac{q - w}{p^{-1} - z^{-1}}(v(\delta)) \\ &= \frac{-(q - x_2)}{x_1 p^{-1} - x_2} \left(\frac{1}{C_\delta}, 0 \right) \\ &= -pqC_\delta. \end{aligned}$$

Similarly, from table (11) we know that in the basis $x_1 = \frac{z}{w}, x_2 = w$, the point $v(\gamma)$ is given by $x_1 = \frac{1}{C_\gamma}, x_2 = 0$. Using (29), and making the substitution $z = x_1 x_2, w = x_2$, we get

$$\begin{aligned} r_{W_2} &= -\frac{1 \cdot 1 \cdot V_{\text{b}_1\text{w}}}{-1 \cdot w^{-1} \cdot V_{\text{b}_2\text{w}}}(v(\gamma)) \\ &= w \frac{p^{-1} - z^{-1}}{q - w}(v(\gamma)) \end{aligned}$$

$$\begin{aligned} &= x_2 \frac{p^{-1} - \frac{1}{x_1 x_2}}{q - x_2} \left(\frac{1}{C_\gamma}, 0 \right) \\ &= -\frac{C_\gamma}{q}. \end{aligned}$$

Therefore, $[\omega]([a]) = pC_\gamma C_\delta$, and using (10) and (17), we have

$$\begin{aligned} \kappa_{\Gamma, \mathbf{w}}^*[\omega]([a]) &= \left(\frac{1}{AX_1 X_2} \right) \cdot \left(-\frac{AX_1 X_2 X_3}{B} \right) \cdot \left(-\frac{AB}{X_3} \right) \\ &= A. \end{aligned}$$

4 Examples

In this section, we work out two detailed examples.

4.1 Primitive Genus 2 Example

Consider the hexagonal graph Γ with Newton polygon N and normal fan Σ as shown in Fig. 7. We label the vertices of Γ as in Fig. 8. We label the zig-zag paths by α, β, γ , and denote the ray of Σ dual to $\tau \in \{\alpha, \beta, \gamma\}$ by σ_τ .

We can take $X_i = [wt]([\partial f_i])$, $i = 1, \dots, 4$, and $A = [wt]([a])$, $B = [wt]([b])$ as coordinates on $H^1(\Gamma, \mathbb{C})$ (see Fig. 8).

The Casimirs are

$$C_\alpha = -\frac{B^2 X_1 X_2 X_4}{A}, \quad C_\beta = -\frac{X_3}{AB^3 X_1^2 X_4}, \quad C_\gamma = \frac{A^2 B X_1}{X_2 X_3}. \quad (33)$$

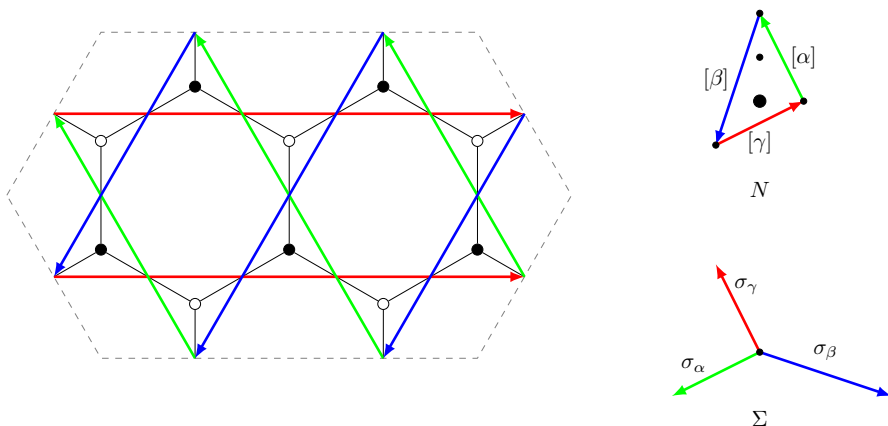


Fig. 7 A hexagonal graph, its Newton polygon N and normal fan Σ , with zig-zag paths and rays labeled

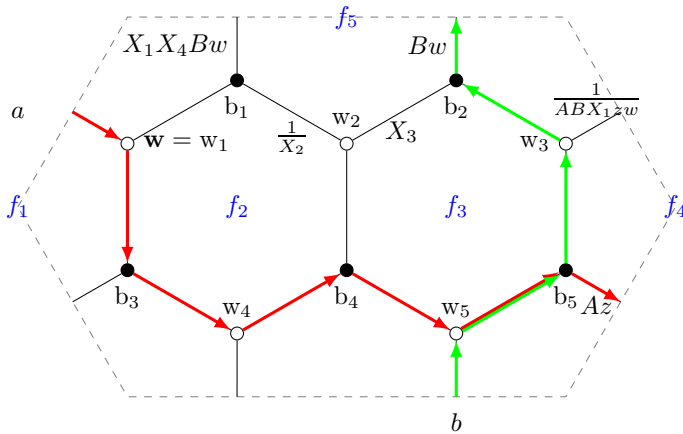


Fig. 8 Labeling of the vertices and faces of Γ , a cocycle representing $[wt]$ and ϕ , where $X_i = [wt](\partial f_i)$, $A = [wt](\partial a)$, $B = [wt](\partial b)$, and a and b are the red and green cycles respectively. The Kasteleyn sign ϵ is 1 for all edges. If no weight or ϕ is indicated for an edge, it means that it is 1

The Kasteleyn matrix is

$$K = \begin{pmatrix} & b_1 & b_2 & b_3 & b_4 & b_5 \\ & 1 & 0 & 1 & 0 & Az \\ & \frac{1}{X_2} & X_3 & 0 & 1 & 0 \\ & 0 & 1 & \frac{1}{ABX_1zw} & 0 & 1 \\ X_1X_4Bw & 0 & 0 & 1 & 1 & 0 \\ 0 & Bw & 0 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{matrix}$$

Let $P(z, w) = \det K$ and $\mathcal{C} = \{P(z, w) = 0\}$. The spectral transform is $\kappa_{\Gamma, \mathbf{w}} = (\mathcal{C}, S, \nu) \in \mathcal{S}_N$, where since the interior of N contains two lattice points, the divisor $S = (p_1, q_1) + (p_2, q_2)$ is a sum of two points, where

$$\begin{aligned} p_1 &= -\frac{\sqrt{(-BX_1X_2X_3X_4 - BX_1X_2X_4 - B)^2 - 4B^2X_1X_2X_4} + BX_1X_2X_3X_4 - BX_1X_2X_4 + B}{2ABX_1}, \\ q_1 &= \frac{-\sqrt{(-BX_1X_2X_3X_4 - BX_1X_2X_4 - B)^2 - 4B^2X_1X_2X_4} + BX_1X_2X_3X_4 + BX_1X_2X_4 + B}{2B^2X_1X_2X_4}, \\ p_2 &= -\frac{\sqrt{(-BX_1X_2X_3X_4 - BX_1X_2X_4 - B)^2 - 4B^2X_1X_2X_4} + BX_1X_2X_3X_4 - BX_1X_2X_4 + B}{2ABX_1}, \\ q_2 &= \frac{\sqrt{(-BX_1X_2X_3X_4 - BX_1X_2X_4 - B)^2 - 4B^2X_1X_2X_4} + BX_1X_2X_3X_4 + BX_1X_2X_4 + B}{2B^2X_1X_2X_4}. \end{aligned} \quad (34)$$

The points at infinity are given by the following table:

Zig-zag path	Homology class	Basis x_1, x_2	Point at infinity
α	$(-1, 2)$	$(-1, 2), (0, -1)$	$x_1 = \frac{1}{C_\alpha}, x_2 = 0$
β	$(-1, -3)$	$(-1, -3), (0, -1)$	$x_1 = \frac{1}{C_\beta}, x_2 = 0$
γ	$(2, 1)$	$(2, 1), (-1, 0)$	$x_1 = \frac{1}{C_\gamma}, x_2 = 0$

(35)

The discrete Abel map $\mathbf{D}$ is given by

$$\begin{aligned} \mathbf{D}(\mathbf{w}) &= 0, & \mathbf{D}(\mathbf{b}_1) &= D_\beta + D_\gamma, & \mathbf{D}(\mathbf{b}_2) &= -D_\alpha + 2D_\beta + D_\gamma, \\ \mathbf{D}(\mathbf{b}_3) &= D_\alpha + D_\beta, & \mathbf{D}(\mathbf{b}_4) &= 2D_\beta, & \mathbf{D}(\mathbf{b}_5) &= -D_\alpha + 3D_\beta, \end{aligned}$$

and $D_N = 2D_\alpha + 2D_\beta + D_\gamma$. Since $\mathbf{D}(\mathbf{w}) = 0$ and every black vertex $\mathbf{b}$ is contained in every zig-zag path, we have

$$\begin{aligned} Y_{\mathbf{bw}} &= 2D_\alpha + 2D_\beta + D_\gamma + \mathbf{D}(\mathbf{b}) - D_\alpha - D_\beta - D_\gamma \\ &= \mathbf{D}(\mathbf{b}) + D_\alpha + D_\beta. \end{aligned}$$

Using this, we compute

$$\begin{aligned} Y_{\mathbf{b}_1\mathbf{w}} &= D_\alpha + 2D_\beta + D_\gamma, & Y_{\mathbf{b}_2\mathbf{w}} &= 3D_\beta + D_\gamma, & Y_{\mathbf{b}_3\mathbf{w}} &= 2D_\alpha + 2D_\beta, \\ Y_{\mathbf{b}_4\mathbf{w}} &= D_\alpha + 3D_\beta, & Y_{\mathbf{b}_5\mathbf{w}} &= 4D_\beta. \end{aligned}$$

The small polygons are shown in Fig. 9. Since the Newton polygon N is primitive, we are in the setting of Remark 3.5. Every zig-zag path contains every black vertex, so the expression (27) is 0. Therefore, there are no equations of type 2 in the linear system $\mathbb{V}_{\mathbf{bw}}$ for any black vertex $\mathbf{b}$. Since $g = 2$, we have two equations of type 1 for every black vertex $\mathbf{b}$. Moreover, we note that each of the small polygons in Fig. 9 contains exactly three lattice points, so by Remark 3.6, we get

$$\begin{aligned} \mathbf{V}_{\mathbf{b}_1\mathbf{w}} &= \begin{vmatrix} 1 & w & z^{-1}w^{-1} \\ 1 & q_1 & p_1^{-1}q_1^{-1} \\ 1 & q_2 & p_2^{-1}q_2^{-1} \end{vmatrix}, & \mathbf{V}_{\mathbf{b}_2\mathbf{w}} &= \begin{vmatrix} 1 & z^{-1} & z^{-1}w^{-1} \\ 1 & p_1^{-1} & p_1^{-1}q_1^{-1} \\ 1 & p_2^{-1} & p_2^{-1}q_2^{-1} \end{vmatrix}, & \mathbf{V}_{\mathbf{b}_3\mathbf{w}} &= \begin{vmatrix} 1 & w & w^2 \\ 1 & q_1 & q_1^2 \\ 1 & q_2 & q_2^2 \end{vmatrix}, \\ \mathbf{V}_{\mathbf{b}_4\mathbf{w}} &= \begin{vmatrix} 1 & w & z^{-1} \\ 1 & q_1 & p_1^{-1} \\ 1 & q_2 & p_2^{-1} \end{vmatrix}, & \mathbf{V}_{\mathbf{b}_5\mathbf{w}} &= \begin{vmatrix} 1 & z^{-1}w & z^{-1} \\ 1 & p_1^{-1}q_1 & p_1^{-1} \\ 1 & p_2^{-1}q_2 & p_2^{-1} \end{vmatrix}. \end{aligned}$$

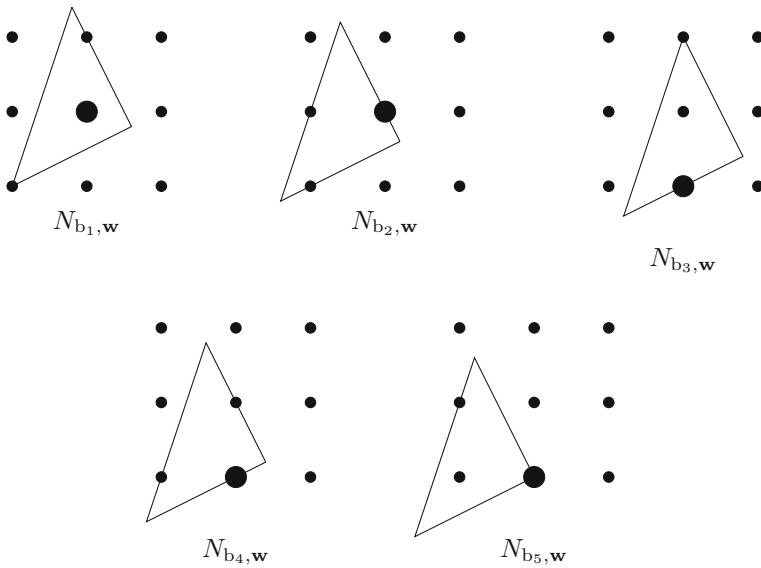


Fig. 9 The small polygons for the hexagonal graph

The boundary of the face f_2 is the concatenation of the three wedges W_1 , W_2 and W_3 represented by (w_2, α) , (w, β) and (w_4, γ) respectively. We compute

$$r_{W_1} = -\frac{V_{b_1 w_2}}{V_{b_4 w_2}}(\nu(\alpha)) = -\frac{\begin{vmatrix} 1 & w & z^{-1}w^{-1} \\ 1 & q_1 & p_1^{-1}q_1^{-1} \\ 1 & q_2 & p_2^{-1}q_2^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & w & z^{-1} \\ 1 & q_1 & p_1^{-1} \\ 1 & q_2 & p_2^{-1} \end{vmatrix}}(\nu(\alpha)).$$

To evaluate at $\nu(\alpha)$, as explained in Remark 3.8, we extend $[\alpha] = (-1, 2)$ to the basis (x_1, x_2) of M , where $x_1 = [\alpha] = (-1, 2)$ and $x_2 = (0, -1)$. Then $\nu(\alpha)$ is given by $x_1 = \frac{1}{C_\alpha}$, $x_2 = 0$. Expressing z, w in the basis (x_1, x_2) as $z = \frac{1}{x_1 x_2^2}$, $w = \frac{1}{x_2}$, we get

$$\begin{aligned} r_{W_1} &= -\frac{\begin{vmatrix} 1 & \frac{1}{x_2} & x_1 x_2^3 \\ 1 & q_1 & p_1^{-1}q_1^{-1} \\ 1 & q_2 & p_2^{-1}q_2^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{1}{x_2} & x_1 x_2^2 \\ 1 & q_1 & p_1^{-1} \\ 1 & q_2 & p_2^{-1} \end{vmatrix}}\left(\frac{1}{C_\alpha}, 0\right) = -\frac{\begin{vmatrix} x_2 & 1 & x_1 x_2^4 \\ 1 & q_1 & p_1^{-1}q_1^{-1} \\ 1 & q_2 & p_2^{-1}q_2^{-1} \end{vmatrix}}{\begin{vmatrix} x_2 & 1 & x_1 x_2^3 \\ 1 & q_1 & p_1^{-1} \\ 1 & q_2 & p_2^{-1} \end{vmatrix}}\left(\frac{1}{C_\alpha}, 0\right) = -\frac{\begin{vmatrix} 0 & 1 & 0 \\ 1 & q_1 & p_1^{-1}q_1^{-1} \\ 1 & q_2 & p_2^{-1}q_2^{-1} \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 0 \\ 1 & q_1 & p_1^{-1} \\ 1 & q_2 & p_2^{-1} \end{vmatrix}} \\ &= -\frac{p_1 q_1 - p_2 q_2}{q_1 q_2 (p_1 - p_2)}, \end{aligned}$$

where we factored out x_2 from the numerator and denominator and then evaluated at $(x_1, x_2) = (\frac{1}{C_\alpha}, 0)$.

For W_2 , letting $(x_1, x_2) = ((-1, -3), (0, -1))$ we have $z = \frac{x_2^3}{x_1}$, $w = \frac{1}{x_2}$, and $v(\beta)$ is given by $x_1 = \frac{1}{C_\beta}$, $x_2 = 0$. Therefore, we get

$$\begin{aligned} r_{W_2} &= -\frac{V_{b_3w}}{V_{b_1w}}(v(\beta)) = -\frac{\begin{vmatrix} 1 & w & w^2 \\ 1 & q_1 & q_1^2 \\ 1 & q_2 & q_2^2 \end{vmatrix}}{\begin{vmatrix} 1 & w & z^{-1}w^{-1} \\ 1 & q_1 & p_1^{-1}q_1^{-1} \\ 1 & q_2 & p_2^{-1}q_2^{-1} \end{vmatrix}}(v(\beta)) = -\frac{\begin{vmatrix} 1 & \frac{1}{x_2} & \frac{1}{x_2^2} \\ 1 & q_1 & q_1^2 \\ 1 & q_2 & q_2^2 \end{vmatrix}}{\begin{vmatrix} 1 & \frac{1}{x_2} & \frac{x_1}{x_2^2} \\ 1 & q_1 & p_1^{-1}q_1^{-1} \\ 1 & q_2 & p_2^{-1}q_2^{-1} \end{vmatrix}}\left(\frac{1}{C_\beta}, 0\right) \\ &= -\frac{\begin{vmatrix} 0 & 0 & 1 \\ 1 & q_1 & q_1^2 \\ 1 & q_2 & q_2^2 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & \frac{1}{C_\beta} \\ 1 & q_1 & p_1^{-1}q_1^{-1} \\ 1 & q_2 & p_2^{-1}q_2^{-1} \end{vmatrix}} = -C_\beta. \end{aligned}$$

Finally, for W_3 , letting $(x_1, x_2) = ((2, 1), (-1, 0))$ we have $z = \frac{1}{x_2}$, $w = x_1x_2^2$, and $v(\gamma)$ is given by $x_1 = \frac{1}{C_\gamma}$, $x_2 = 0$. Therefore, we get

$$\begin{aligned} r_{W_3} &= -\frac{V_{b_4w}}{V_{b_3w}}(v(\gamma)) = -\frac{\begin{vmatrix} 1 & w & z^{-1} \\ 1 & q_1 & p_1^{-1} \\ 1 & q_2 & p_2^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & w & w^2 \\ 1 & q_1 & q_1^2 \\ 1 & q_2 & q_2^2 \end{vmatrix}}(v(\gamma)) = -\frac{\begin{vmatrix} 1 & x_1x_2^2 & x_2 \\ 1 & q_1 & p_1^{-1} \\ 1 & q_2 & p_2^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & x_1x_2^2 & x_1^2x_2^4 \\ 1 & q_1 & q_1^2 \\ 1 & q_2 & q_2^2 \end{vmatrix}}\left(\frac{1}{C_\gamma}, 0\right) \\ &= -\frac{\begin{vmatrix} 1 & 0 & 0 \\ 1 & q_1 & p_1^{-1} \\ 1 & q_2 & p_2^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & q_1 & q_1^2 \\ 1 & q_2 & q_2^2 \end{vmatrix}} = \frac{p_1q_1 - p_2q_2}{p_1p_2q_1q_2(q_1 - q_2)}. \end{aligned}$$

Putting everything together, we get

$$X_2 = - \frac{\begin{vmatrix} 0 & 1 & 0 \\ 1 & q_1 & p_1^{-1}q_1^{-1} \\ 1 & q_2 & p_2^{-1}q_2^{-1} \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 0 \\ 1 & q_1 & p_1^{-1} \\ 1 & q_2 & p_2^{-1} \end{vmatrix}} \frac{\begin{vmatrix} 0 & 0 & 1 \\ 1 & q_1 & q_1^2 \\ 1 & q_2 & q_2^2 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & \frac{1}{C_\beta} \\ 1 & q_1 & p_1^{-1}q_1^{-1} \\ 1 & q_2 & p_2^{-1}q_2^{-1} \end{vmatrix}} \frac{\begin{vmatrix} 1 & 0 & 0 \\ 1 & q_1 & p_1^{-1} \\ 1 & q_2 & p_2^{-1} \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & q_1 & q_1^2 \\ 1 & q_2 & q_2^2 \end{vmatrix}} = \frac{C_\beta(p_1q_1 - p_2q_2)^2}{p_1p_2q_1^2q_2^2(p_1 - p_2)(q_1 - q_2)},$$

with similar formulas for X_1, X_3, X_4, A, B . It may be easily verified that these invert the spectral transform by plugging in the formulas (33) and (34) into the right-hand side and simplifying using computer algebra.

4.2 Non-primitive Example

Consider the square-octagon graph Γ with Newton polygon N and normal fan Σ as shown in Fig. 10. We label the vertices of Γ as in Fig. 11. We label the rays of Σ by $\sigma_\alpha, \sigma_\beta, \sigma_\gamma, \sigma_\delta$ and the two zig-zag paths dual to ray σ_τ by $\{\tau_1, \tau_2\}$, for $\tau \in \{\alpha, \beta, \gamma, \delta\}$.

We can take $X_i := [wt](\partial f_i)$, $i = 1, \dots, 7$, and $A := [wt](a)$, $B := [wt](b)$ as coordinates on $H^1(\Gamma, \mathbb{C})$ (see Fig. 11). The Casimirs are

$$\begin{aligned} C_{\alpha_1} &= X_1 X_3 X_7 B, & C_{\alpha_2} &= \frac{B X_2 X_3 X_4 X_6 X_7}{X_1 X_5}, & C_{\beta_1} &= \frac{X_2}{A X_1 X_5}, & C_{\beta_2} &= \frac{1}{A X_7}, \\ C_{\gamma_1} &= \frac{X_5}{B X_1 X_3}, & C_{\gamma_2} &= \frac{X_6}{B}, & C_{\delta_1} &= \frac{A X_1}{X_2 X_6}, & C_{\delta_2} &= \frac{A X_1 X_5}{X_2 X_3 X_4 X_6 X_7}. \end{aligned}$$

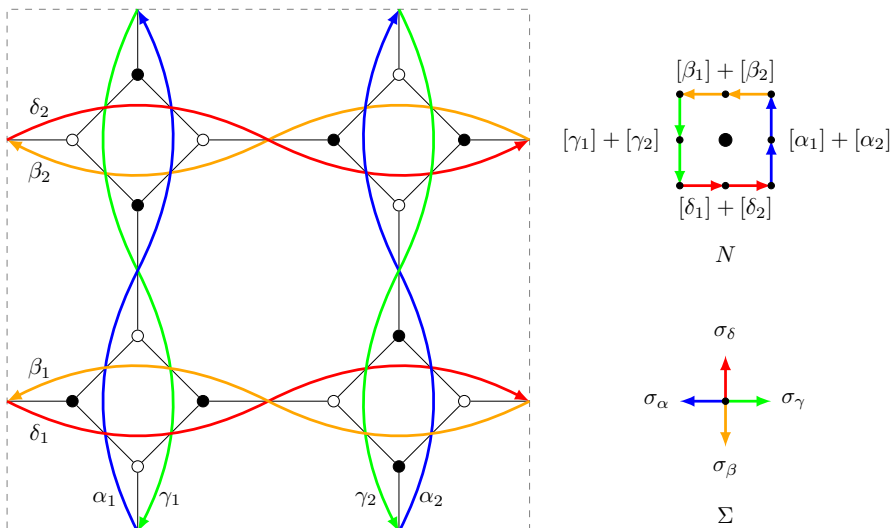


Fig. 10 A square-octagon graph, its Newton polygon N and normal fan Σ , with zig-zag paths and rays labeled

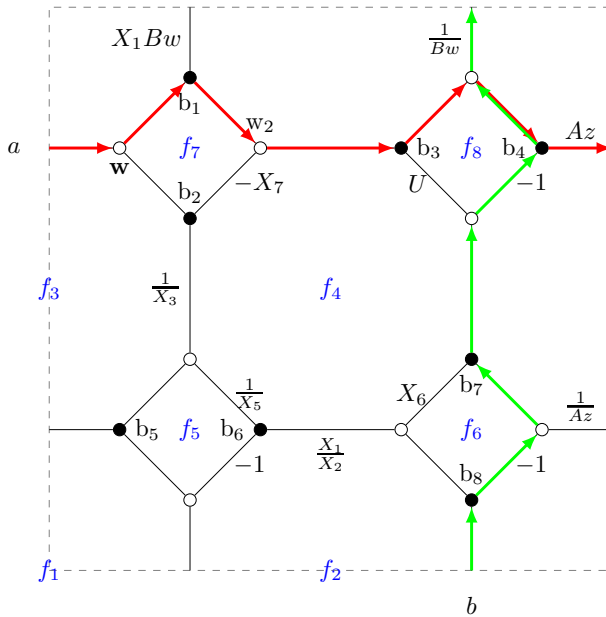


Fig. 11 Labeling of the vertices and faces of Γ , and a cocycle and Kasteleyn sign, where $X_i = [wt](\partial f_i)$, $A = [wt](a)$, $B = [wt](b)$ and $U = \frac{X_1 X_5}{X_2 X_3 X_4 X_6 X_7}$. The edges with no weight indicated have weight 1

Since the Newton polygon N has only one interior lattice point, the divisor $S = (p, q)$ consists of a single point. The Kasteleyn matrix is

$$K = \begin{pmatrix} 1 & 1 & 0 & Az & 0 & 0 & 0 & 0 \\ 1 & -X_7 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & \frac{1}{Bw} \\ 0 & 0 & \frac{X_1 X_5}{X_2 X_3 X_4 X_6 X_7} & -1 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{X_3} & 0 & 0 & 1 & \frac{1}{X_5} & 0 & 0 \\ BwX_1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{X_1}{X_2} & X_6 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{Az} & 0 & 1 & -1 \end{pmatrix}.$$

Let $P(z, w) = \det K$ and $\mathcal{C} = \overline{\{P(z, w) = 0\}}$. The spectral transform is $\kappa_{\Gamma, w} = (\mathcal{C}, S, \nu) \in \mathcal{S}_N$, where

$$p = -\frac{X_2 X_4 X_6 (X_3 X_5 X_6 X_7 (X_1^2 (X_4 + 1) + X_2 X_4) + X_1 X_2 X_3^2 X_4 X_6^2 X_7^2 + X_1 X_5^2)}{A(X_1 X_5 + X_2 X_3 X_4 X_6 X_7)} \\ \times \frac{1}{(X_3 X_4 X_6 X_7 (X_1^2 X_5 + X_2 (X_5 + 1)(X_6 + 1)) + X_1 X_5 (X_6 (X_4 + X_5 + 1) + X_5 + 1))}, \\ q = \frac{X_5 (-X_3 X_4 X_6 X_7 (X_1^2 + X_2 X_6 + X_2) - X_1 X_5 (X_6 + 1))}{BX_1 X_3 X_7 (X_1 X_5 (X_4 X_6 + X_6 + 1) + X_2 X_3 X_4 X_6 (X_6 + 1) X_7)}.$$

The table below lists the points at infinity for each of the zig-zag paths:

Zig-zag path	Homology class	Basis x_1, x_2	Point at infinity
α_1	(0,1)	(0,1),(-1,0)	$x_1 = \frac{1}{C_{\alpha_1}}, x_2 = 0$
α_2			$x_1 = \frac{1}{C_{\alpha_2}}, x_2 = 0$
β_1	(-1,0)	(-1,0),(0,-1)	$x_1 = \frac{1}{C_{\beta_1}}, x_2 = 0$
β_2			$x_1 = \frac{1}{C_{\beta_2}}, x_2 = 0$
γ_1	(0,-1)	(0,-1),(1,0)	$x_1 = \frac{1}{C_{\gamma_1}}, x_2 = 0$
γ_2			$x_1 = \frac{1}{C_{\gamma_2}}, x_2 = 0$
δ_1	(1,0)	(1,0),(0,1)	$x_1 = \frac{1}{C_{\delta_1}}, x_2 = 0$
δ_2			$x_1 = \frac{1}{C_{\delta_2}}, x_2 = 0$

(36)

The discrete Abel map $\mathbf{D}$ is given by $\mathbf{D}(\mathbf{w}) = 0$ and

$$\begin{aligned}
 \mathbf{D}(\mathbf{b}_1) &= \frac{1}{2}D_\gamma + \frac{1}{2}D_\delta, & \mathbf{D}(\mathbf{b}_2) &= \frac{1}{2}D_\beta + \frac{1}{2}D_\gamma, \\
 \mathbf{D}(\mathbf{b}_3) &= \frac{1}{2}(-D_\alpha + D_\beta + D_\gamma + D_\delta), & \mathbf{D}(\mathbf{b}_4) &= -D_\alpha + \frac{1}{2}D_\beta + D_\gamma + \frac{1}{2}D_\delta \\
 \mathbf{D}(\mathbf{b}_5) &= D_\beta, & \mathbf{D}(\mathbf{b}_6) &= -\frac{1}{2}D_\alpha + D_\beta + \frac{1}{2}D_\gamma \\
 \mathbf{D}(\mathbf{b}_7) &= -\frac{1}{2}D_\alpha + \frac{1}{2}D_\beta + D_\gamma, & \mathbf{D}(\mathbf{b}_8) &= -\frac{1}{2}D_\alpha + D_\beta + D_\gamma - \frac{1}{2}D_\delta.
 \end{aligned}$$

We have $D_N = D_\alpha + D_\beta + D_\gamma + D_\delta$, using which we compute

$$\begin{aligned}
 Y_{\mathbf{b}_1\mathbf{w}} &= \frac{1}{2}D_\alpha + D_\beta + D_\gamma + D_\delta, & Y_{\mathbf{b}_2\mathbf{w}} &= \frac{1}{2}D_\alpha + D_\beta + D_\gamma + D_\delta, \\
 Y_{\mathbf{b}_3\mathbf{w}} &= D_\beta + \frac{3}{2}D_\gamma + D_\delta, & Y_{\mathbf{b}_4\mathbf{w}} &= D_\beta + \frac{3}{2}D_\gamma + D_\delta, \\
 Y_{\mathbf{b}_5\mathbf{w}} &= \frac{1}{2}D_\alpha + \frac{3}{2}D_\beta + D_\gamma + \frac{1}{2}D_\delta, & Y_{\mathbf{b}_6\mathbf{w}} &= \frac{1}{2}D_\alpha + \frac{3}{2}D_\beta + D_\gamma + \frac{1}{2}D_\delta, \\
 Y_{\mathbf{b}_7\mathbf{w}} &= \frac{3}{2}D_\beta + \frac{3}{2}D_\gamma + \frac{1}{2}D_\delta, & Y_{\mathbf{b}_8\mathbf{w}} &= \frac{3}{2}D_\beta + \frac{3}{2}D_\gamma + \frac{1}{2}D_\delta.
 \end{aligned}$$

The corresponding small polygons are shown in Fig. 12. Therefore, we have

$$\mathbf{V}_{\mathbf{b}_1\mathbf{w}} = a_{(-1,-1)}z^{-1}w^{-1} + a_{(0,-1)}w^{-1} + a_{(1,-1)}zw^{-1} + a_{(-1,0)}z^{-1} + a_{(0,0)} + a_{(1,0)}z,$$

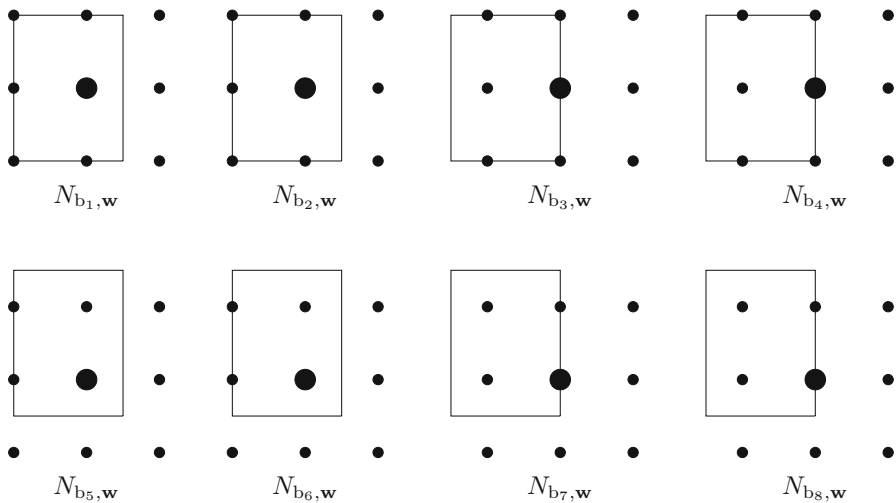


Fig. 12 The small polygons for the square-octagon graph

where the a_m satisfy the system of equations $\mathbb{V}_{b_1, w}$ that we now determine. We have the equation of type 1:

$$a_{(-1,-1)}p^{-1}q^{-1} + a_{(0,-1)}q^{-1} + a_{(1,-1)}pq^{-1} + a_{(-1,0)}p^{-1} + a_{(0,0)} + a_{(1,0)}p = 0.$$

To find the zig-zag paths that contribute equations of type 2, we compute (25). We have

$$\begin{aligned} -D_N|_C &= v(\alpha_1) + v(\alpha_2) + v(\beta_1) + v(\beta_2) + v(\gamma_1) + v(\gamma_2) + v(\delta_1) + v(\delta_2), \\ \mathbf{d}(\mathbf{w}) - \mathbf{d}(\mathbf{b}_1) &= -v(\gamma_1) - v(\delta_2), \\ [Y_{b_1, w}]|_C &= v(\beta_1) + v(\beta_2) + v(\gamma_1) + v(\gamma_2) + v(\delta_1) + v(\delta_2), \end{aligned}$$

using which we get that (25) is equal to $v(\beta_1) + v(\beta_2) + v(\gamma_2) + v(\delta_1)$, so we have four equations of type 2, one for each of the zig-zag paths $\beta_1, \beta_2, \gamma_2, \delta_1$.

Therefore, we have 5 equations and 6 variables, so we are in the setting of Remark 3.6 where $V_{b_1, w} = \det V_{b_1, w}^X$. Computing the equations of type 2, we get

$$V_{b_1, w} = \begin{vmatrix} z^{-1}w^{-1} & w^{-1} & zw^{-1} & z^{-1} & 1 & z \\ p^{-1}q^{-1} & q^{-1} & pq^{-1} & p^{-1} & 1 & p \\ 1 & C_{\beta_1} & 0 & 0 & 0 & 0 \\ 1 & C_{\beta_2} & 0 & 0 & 0 & 0 \\ C_{\gamma_2} & 0 & 1 & 0 & C_{\gamma_2}^{-1} & 0 \\ 0 & 0 & 0 & 0 & C_{\delta_1} & 1 \end{vmatrix}.$$

In like fashion, for V_{b_2w} , we have an equation of type 1 and four equations of type 2 for the zig-zag paths $\beta_1, \gamma_2, \delta_1, \delta_2$. We compute

$$V_{b_2w} = \begin{vmatrix} z^{-1}w & w & z^{-1} & 1 & z^{-1}w^{-1} & w^{-1} \\ p^{-1}q & q & p^{-1} & 1 & p^{-1}q^{-1} & q^{-1} \\ 1 & C_{\beta_1} & 0 & 0 & 0 & 0 \\ C_{\gamma_2} & 0 & 1 & 0 & C_{\gamma_2}^{-1} & 0 \\ 0 & 0 & 0 & 0 & C_{\delta_1} & 1 \\ 0 & 0 & 0 & 0 & C_{\delta_2} & 1 \end{vmatrix}.$$

We write the boundary of the face f_7 as the concatenation of the two wedges W_1 and W_2 represented by (w, γ_1) and (b_2, α_1) respectively. We have

$$r_{W_1} = \frac{V_{b_2w}}{V_{b_1w}}(v(\gamma_1)) = \frac{\begin{vmatrix} C_{\gamma_1} & 0 & 1 & 0 & C_{\gamma_1}^{-1} & 0 \\ p^{-1}q & q & p^{-1} & 1 & p^{-1}q^{-1} & q^{-1} \\ 1 & C_{\beta_1} & 0 & 0 & 0 & 0 \\ C_{\gamma_2} & 0 & 1 & 0 & C_{\gamma_2}^{-1} & 0 \\ 0 & 0 & 0 & 0 & C_{\delta_1} & 1 \\ 0 & 0 & 0 & 0 & C_{\delta_2} & 1 \end{vmatrix}}{\begin{vmatrix} C_{\gamma_1} & 0 & 1 & 0 & C_{\gamma_1}^{-1} & 0 \\ p^{-1}q^{-1} & q^{-1} & pq^{-1} & p^{-1} & 1 & p \\ 1 & C_{\beta_1} & 0 & 0 & 0 & 0 \\ 1 & C_{\beta_2} & 0 & 0 & 0 & 0 \\ C_{\gamma_2} & 0 & 1 & 0 & C_{\gamma_2}^{-1} & 0 \\ 0 & 0 & 0 & 0 & C_{\delta_1} & 1 \end{vmatrix}},$$

where to evaluate at $v(\gamma_1)$, we use the basis x_1, x_2 from table (36). Similarly, we compute

$$r_{W_2} = -\frac{V_{b_1w}}{V_{b_2w}}(v(\alpha_1)) = -\frac{\begin{vmatrix} 0 & C_{\alpha_1}^{-1} & 0 & 1 & 0 & C_{\alpha_1} \\ p^{-1}q^{-1} & q^{-1} & pq^{-1} & p^{-1} & 1 & p \\ 1 & C_{\beta_1} & 0 & 0 & 0 & 0 \\ 1 & C_{\beta_2} & 0 & 0 & 0 & 0 \\ C_{\gamma_2} & 0 & 1 & 0 & C_{\gamma_2}^{-1} & 0 \\ 0 & 0 & 0 & 0 & C_{\delta_1} & 1 \end{vmatrix}}{\begin{vmatrix} 0 & C_{\alpha_1}^{-1} & 0 & 1 & 0 & C_{\alpha_1} \\ p^{-1}q & q & p^{-1} & 1 & p^{-1}q^{-1} & q^{-1} \\ 1 & C_{\beta_1} & 0 & 0 & 0 & 0 \\ C_{\gamma_2} & 0 & 1 & 0 & C_{\gamma_2}^{-1} & 0 \\ 0 & 0 & 0 & 0 & C_{\delta_1} & 1 \\ 0 & 0 & 0 & 0 & C_{\delta_2} & 1 \end{vmatrix}}.$$

It can be verified using computer algebra that $X_7 = r_{W_1}r_{W_2}$.

5 The Small Polygons

In the remaining sections, we prove the results stated in Sect. 3. In order to invert the spectral transform, we want to first reconstruct the $Q_{\mathbf{bw}}$, the entries of the $\mathbf{w}$ -column of the adjugate matrix, from the spectral data. To do this, we need to first find the Newton polygon of the $Q_{\mathbf{bw}}$, which we call the small polygons and denote by $N_{\mathbf{bw}}$. Explicitly, $N_{\mathbf{bw}}$ is the convex hull of homology classes of dimer covers of $\Gamma - \{\mathbf{b}, \mathbf{w}\}$. However, it appears difficult to describe $N_{\mathbf{bw}}$ in a direct combinatorial way. Instead, we will re-express the problem in terms of toric geometry. The key to doing this is an extension of the Kasteleyn matrix, which is a map of trivial sheaves on T , to a map of locally free sheaves on a compactification of T . We are led to consider a stacky toric surface $\mathcal{X}_N$ instead of the toric surface X_N , because such an extension does not exist on X_N unless the polygon has only primitive sides.

The basics of stacky toric surfaces are recalled in detail in Appendix A.3. For the convenience of the reader we reproduce some notation.

Let Σ be the normal fan of N . There is a *stacky fan* $\Sigma = (\Sigma, \beta)$ where

$$\begin{aligned}\beta : \mathbb{Z}^{\Sigma(1)} &\rightarrow \mathbf{M}^\vee, \\ \delta_\rho &\mapsto |E_\rho|u_\rho,\end{aligned}$$

where u_ρ is the primitive normal to E_ρ . We identify the set of rays $\Sigma(1)$ of the fan Σ with the components D_ρ of the divisor at infinity

$$\rho \leftrightarrow \tau_\rho = \mathbb{R}_{\geq 0}u_\rho.$$

We assign to Σ a smooth *toric Deligne-Mumford stack* $\mathcal{X}_N$, which contains the torus T as a dense open subset.

We consider the stack rather than the toric surface since we construct an extension of the Kasteleyn operator to a compactification of the torus T in Lemma 5.1. There is no such extension on the toric surface when the Newton polygon is not simple, but there is one on the stack.

5.1 Extension of the Kasteleyn Operator

Define for each black vertex $\mathbf{b}$ the line bundle

$$\mathcal{E}_{\mathbf{b}} := \mathcal{O}_{\mathcal{X}_N} \left(\mathbf{D}(\mathbf{b}) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in \mathbb{Z}_\rho : \mathbf{b} \in \alpha} \frac{1}{|E_\rho|} D_\rho \right),$$

and for each white vertex $\mathbf{w}$, the line bundle

$$\mathcal{F}_{\mathbf{w}} := \mathcal{O}_{\mathcal{X}_N}(\mathbf{D}(\mathbf{w})).$$

Let

$$\mathcal{E} := \bigoplus_{b \in B} \mathcal{E}_b, \quad \mathcal{F} := \bigoplus_{w \in W} \mathcal{F}_w.$$

They are locally free sheaves of the same rank $\#B = \#W$ on $\mathcal{X}_N$.

Proposition 5.1 *The Kasteleyn operator K extends to a map of locally free sheaves on $\mathcal{X}_N$:*

$$\tilde{K} : \mathcal{E} \rightarrow \mathcal{F}. \quad (37)$$

Proof By definition,

$$K_{wb} = \sum_{e \in E(\Gamma) \text{ incident to } bw} wt(e) \epsilon(e) \phi(e).$$

We need to show that for any edge e with vertices bw , the character $\phi(e)$ is a global section of

$$\mathcal{H}om_{\mathcal{X}_N}(\mathcal{E}_b, \mathcal{F}_w) \cong \mathcal{O}_{\mathcal{X}_N}(\mathbf{D}(w) - \mathbf{D}(b) + \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho : b \in \alpha} \frac{1}{|E_\rho|} D_\rho).$$

Let $m \in M$ be such that $\phi(e) = \chi^m$ and let $D := \mathbf{D}(w) - \mathbf{D}(b) + \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho : b \in \alpha} \frac{1}{|E_\rho|} D_\rho$. By Proposition A.2, χ^m is a global section of the line bundle $\mathcal{O}_{\mathcal{X}_N}(D)$ if and only if $m \in P_D \cap M$. Using (53), this is equivalent to showing that for every edge $e = bw$, we have

$$\operatorname{div} \phi(e) + \mathbf{D}(w) - \mathbf{D}(b) + \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho : b \in \alpha} \frac{1}{|E_\rho|} D_\rho \geq 0,$$

where $\operatorname{div} \phi(e) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$ as in (49).

Let α, β be the zig-zag paths through e , with $\alpha \in Z_\sigma, \beta \in Z_\rho$. Then by Lemma 3.2, we have

$$\mathbf{D}(w) - \mathbf{D}(b) = -\frac{1}{|E_\sigma|} D_\sigma - \frac{1}{|E_\rho|} D_\rho - \operatorname{div} \phi(e).$$

This implies

$$\operatorname{div} \phi(e) + \mathbf{D}(w) - \mathbf{D}(b) + \sum_{\tau \in \Sigma(1)} \sum_{\gamma \in Z_\tau : b \in \gamma} \frac{1}{|E_\tau|} D_\tau = \sum_{\tau \in \Sigma(1)} \sum_{\substack{\gamma \in Z_\tau : b \in \gamma \\ \gamma \neq \alpha, \beta}} \frac{1}{|E_\tau|} D_\tau \geq 0. \quad (38)$$

□

The small polygon N_{bw} is by definition the Newton polygon of Q_{bw} . By Proposition A.2, this is equivalent to saying that Q_{bw} is a global section of a line bundle $\mathcal{O}_{\mathcal{X}_N}(Y_{\text{bw}})$, where Y_{bw} is the divisor associated to N_{bw} by the correspondence (53). Now that we have shown that K is a global section of $\mathcal{H}\mathbb{I}_{\mathcal{X}_N}(\mathcal{E}, \mathcal{F})$, we can take exterior powers to find which line bundle $\mathcal{O}_{\mathcal{X}_N}(Y_{\text{bw}})$ the minor Q_{bw} of K is a global section of.

Taking the determinant of the map (37), we see that $\det \tilde{K}$ is a global section of the line bundle

$$\begin{aligned} \text{Hom}_{\mathcal{X}_N} \left(\bigwedge_{b \in B} \mathcal{E}_b, \bigwedge_{w \in W} \mathcal{F}_w \right) &\cong \mathcal{O}_{\mathcal{X}_N} \left(\sum_{w \in W} \mathbf{D}(w) - \sum_{b \in B} \left(\mathbf{D}(b) \right. \right. \\ &\quad \left. \left. - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho: b \in \alpha} \frac{1}{|E_\rho|} D_\rho \right) \right). \end{aligned} \quad (39)$$

Lemma 5.2 *Let D_N be the divisor associated to N by the correspondence (53) between divisors and polygons. Then one has*

$$\sum_{w \in W} \mathbf{D}(w) - \sum_{b \in B} \left(\mathbf{D}(b) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho: b \in \alpha} \frac{1}{|E_\rho|} D_\rho \right) = D_N. \quad (40)$$

Therefore,

$$\det \tilde{K} \in H^0(\mathcal{X}_N, \mathcal{O}_{\mathcal{X}_N}(D_N)).$$

Proof Let a_ρ be the coefficient of D_ρ in D_N . Let (i_1, i_2) be a vertex of P contained in E_ρ and let $\mathbf{m}$ be the associated extremal dimer cover. We pair up black and white vertices in the sum according to $\mathbf{m}$:

$$\sum_{e=\text{bw} \in \mathbf{m}} \left(\mathbf{D}(w) - \mathbf{D}(b) + \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho: b \in \alpha} \frac{1}{|E_\rho|} D_\rho \right).$$

Now we observe that if e is not contained in any zig-zag path in Z_ρ , then D_ρ does not appear in the summand, and if e is contained in a zig-zag path associated to E_ρ , then D_ρ appears twice but with opposite signs, modulo contributions from intersections of edges with γ_z, γ_w . Therefore, there is no net contribution to the coefficient of D_ρ except for the intersections of edges in $\mathbf{m}$ with γ_z, γ_w , which is the same as in

$$-\sum_{e \in \mathbf{m}} \text{div } \phi(e) = -\text{div } z^{i_1} w^{i_2},$$

which is a_ρ . Comparing with (39), we see that (40) implies the second statement. $\square$

Now we consider the codimension 1 exterior power, where we remove $\{\mathbf{b}, \mathbf{w}\}$. Let $\tilde{Q}$ be the adjugate matrix of $\tilde{K}$. Set

$$Y_{\mathbf{bw}} := D_N - \mathbf{D}(\mathbf{w}) + \mathbf{D}(\mathbf{b}) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho: \mathbf{b} \in \alpha} \frac{1}{|E_\rho|} D_\rho.$$

Corollary 5.3 $\tilde{Q}_{\mathbf{bw}} \in H^0(\mathcal{X}_N, \mathcal{O}_{\mathcal{X}_N}(Y_{\mathbf{bw}}))$.

We therefore arrive at the definition of the small polygon $N_{\mathbf{bw}}$ given in Definition 3.3 by the correspondence (53).

5.2 Points at Infinity

In this section, we prove that the points at infinity of $\mathcal{C}$ are as described in Sect. 2.7. We use the notations $U_\Sigma \subset \mathbb{C}^{\Sigma(1)}$ and the standard coordinates (z_ρ) on $\mathbb{C}^{\Sigma(1)}$ from Appendix A.3. The toric variety X_N is the quotient U_Σ/H , where H is the kernel of the map $(\mathbb{C}^\times)^{\Sigma(1)} \rightarrow \mathbf{T}$ sending $(z_\rho)_{\rho \in \Sigma(1)}$ to $(\prod_\rho z_\rho^{\langle(1,0), u_\rho\rangle}, \prod_\rho z_\rho^{\langle(0,1), u_\rho\rangle})$. There is a canonical map $\pi : U_\Sigma \rightarrow X_N$ given by sending (z_ρ) to $H \cdot (z_\rho^{|E_\rho|})$ which induces the coarse moduli space map $\pi : \mathcal{X}_N \rightarrow X_N$. The spectral curve $\mathcal{C}$ is cut out by the section $P = P(z, w)$ of $\mathcal{O}_{X_N}(D_N)$. The pullback π^*P defines a section of $\mathcal{O}_{\mathcal{X}_N}(D_N)$ which is a G -invariant section of $\mathcal{O}_{U_\Sigma}$, so it vanishes on a G -invariant subvariety $\mathcal{C}_{U_\Sigma}$. Each point at infinity of $\mathcal{C}$ corresponds to a G -invariant set of points at infinity of $\mathcal{C}_{U_\Sigma}$, so we will determine the points at infinity of $\mathcal{C}$ from the points at infinity of $\mathcal{C}_{U_\Sigma}$. By Lemma 5.2, $\pi^*P = \det \tilde{K}$ so the points at infinity of $\mathcal{C}_{U_\Sigma}$ are obtained by setting $z_\rho = \det \tilde{K} = 0$ for $\rho \in \Sigma(1)$.

From (38) and Proposition A.2, for $e = \mathbf{bw}$, we get that $\phi(e)$ corresponds to the G -invariant section of $\mathcal{O}_{U_\Sigma}$ given by

$$\phi(e) = \prod_{\tau \in \Sigma(1)} \prod_{\substack{\gamma \in Z_\tau: \mathbf{b} \in \gamma \\ \gamma \neq \alpha, \beta}} z_\tau. \quad (41)$$

The divisor D_ρ in X_N corresponds to $\{z_\rho = 0\} \subset U_\Sigma$. $\phi(e)$ vanishes on $\{z_\rho = 0\}$ precisely when there is a zig-zag path $\alpha \in Z_\rho$ such that $\mathbf{b}$ is contained in α but $\mathbf{w}$ is not contained in α . This implies that when restricted to $\{z_{\tilde{\rho}} = 0\}$, after reordering the black and white vertices, the extended Kasteleyn operator $\tilde{K}$ takes a block-upper-triangular form

$$\begin{pmatrix} \tilde{K}|_{\alpha_1} & & & * \\ & \tilde{K}|_{\alpha_2} & & * \\ & & \ddots & \vdots \\ & & & \tilde{K}|_{\alpha_n} & * \\ & & & & \tilde{K}|_{\Gamma - \{\alpha_1, \dots, \alpha_n\}} \end{pmatrix},$$

where $Z_\rho = \{\alpha_1, \dots, \alpha_n\}$, $\tilde{K}|_{\alpha_i}$ is the restriction of $\tilde{K}$ to the black and white vertices in α_i , and the $*$'s denote some possibly nonzero blocks whereas any block that has not been indicated is zero. In particular, the nonzero blocks $*$ are only in the last column (in these blocks, b is not in α but w is in α). Note that the zig-zag paths $\alpha_1, \dots, \alpha_n$ do not share any vertices because of minimality since otherwise we would have a parallel bigon, so the blocks $\tilde{K}|_{\alpha_1}, \dots, \tilde{K}|_{\alpha_n}$ do not overlap.

If $\alpha \in Z_\rho$ is $b_1 \rightarrow w_1 \rightarrow b_2 \rightarrow \dots \rightarrow w_d \rightarrow b_1$, the determinant of the block $\tilde{K}|_\alpha$ is

$$\begin{aligned} \det \tilde{K}|_\alpha &= \det \begin{pmatrix} \tilde{K}_{w_1 b_1} & & & & \tilde{K}_{w_d b_1} \\ \tilde{K}_{w_1 b_2} & \tilde{K}_{w_2 b_2} & & & \\ & \tilde{K}_{w_2 b_3} & & & \\ & & \ddots & \ddots & \\ & & & \tilde{K}_{w_{d-1} b_d} & \tilde{K}_{w_d b_d} \end{pmatrix} \\ &= \prod_{i=1}^d \tilde{K}_{w_i b_i} - (-1)^d \prod_{i=1}^d \tilde{K}_{w_{i-1} b_i} \\ &= - \prod_{i=1}^d (wt(b_i w_{i-1}) \epsilon(b_i w_{i-1}) \phi(b_i w_i)) \left(\prod_{i=1}^d \frac{\phi(b_i w_{i-1})}{\phi(b_i w_i)} - C_\alpha \right), \end{aligned}$$

where we have used the definition of the Kasteleyn matrix (see (1), (3)). Plugging in (41) and using the fact that α intersects a zig-zag path $\beta \in Z_\tau \setminus [\alpha], u_\tau$ times, we get

$$\prod_{i=1}^d \frac{\phi(b_i w_{i-1})}{\phi(b_i w_i)} = \prod_{\tau \in \Sigma(1)} z_\tau^{-|E_\tau| \langle [\alpha], u_\tau \rangle}.$$

Therefore, the points at infinity of $\mathcal{C}_{U_\Sigma}$ are given by setting $z_\rho = 0$ and $\prod_{\tau \in \Sigma(1)} z_\tau^{-|E_\tau| \langle [\alpha], u_\tau \rangle} = C_\alpha$. The point at infinity of $\mathcal{C}$ is the point obtained by applying π to any of these points. From the definition of π , we get that $\prod_{\tau \in \Sigma(1)} z_\tau^{-|E_\tau| \langle [\alpha], u_\tau \rangle} = \pi^* \chi^{-[\alpha]}$, so the point at infinity of $\mathcal{C}$ is given by

$$\chi^{-[\alpha]} = C_\alpha. \quad (42)$$

6 Behaviour of the Laurent Polynomial $Q_{bw}(z, w)$ at infinity

We proved in Corollary 5.3 that the Laurent polynomial $Q_{bw}(z, w)$ lies in the finite dimensional vector space $H^0(\mathcal{X}_N, \mathcal{O}_{\mathcal{X}_N}(Y_{bw}))$. We need some additional constraints on $Q_{bw}(z, w)$ to determine it. Corollary 6.3 provides g linear equations for the coefficients of $Q_{bw}(z, w)$ coming from the vanishing of $Q_{bw}(z, w)$ at the g points of the divisor S_w . We obtain additional equations from the behaviour of $Q_{bw}(z, w)$ at the points at infinity of the spectral curve, which we study in this section.

Recall that X_N is the toric surface associated to N compactifying T . The restriction of the Kasteleyn operator to the open spectral curve $\mathcal{C}^\circ$ is a map of trivial sheaves:

$$K|_{\mathcal{C}^\circ} : \bigoplus_{b \in B} \mathcal{O}_{\mathcal{C}^\circ} \longrightarrow \bigoplus_{w \in W} \mathcal{O}_{\mathcal{C}^\circ}.$$

Recall the correspondence between divisors D and invertible sheaves with rational sections $(\mathcal{L}, s)$: given a divisor D , the corresponding invertible sheaf $\mathcal{L} = \mathcal{O}_{\mathcal{C}}(D)$ is defined on an open U by

$$H^0(U, \mathcal{O}_{\mathcal{C}}(D)) := \{t \in K(\mathcal{C})^\times : (\operatorname{div} t + D)|_U \geq 0\} \cup \{0\},$$

with the obvious restriction maps, where $K(\mathcal{C})^\times$ denotes the nonzero rational functions on $\mathcal{C}$. The rational section s corresponds to the rational function 1. On the other hand, given $(\mathcal{L}, s)$, we obtain D as the divisor $\operatorname{div} s$. Moreover, there is a correspondence between rational functions t on $\mathcal{C}$ and rational sections $\bar{t}$ of $\mathcal{L}$ given by $t \mapsto \bar{t} := st$. In particular,

$$\operatorname{div} \bar{t} = \operatorname{div} t + \operatorname{div} s = \operatorname{div} t + D, \quad (43)$$

and so $\bar{t}$ is regular if and only if $\operatorname{div} t + D \geq 0$.

A similar proof to Proposition 5.1 shows that the Kasteleyn matrix K , which is a matrix of rational functions on $\mathcal{C}$, defines a regular map $\bar{K}$ of locally free sheaves on $\mathcal{C}$ extending $K|_{\mathcal{C}^\circ}$, providing an exact sequence

$$0 \rightarrow \mathcal{M} \rightarrow \bigoplus_{b \in B} \mathcal{O}_{\mathcal{C}}(\mathbf{d}(b) - \sum_{\alpha \in Z: b \in \alpha} v(\alpha)) \xrightarrow{\bar{K}} \bigoplus_{w \in W} \mathcal{O}_{\mathcal{C}}(\mathbf{d}(w)) \rightarrow \mathcal{L} \rightarrow 0, \quad (44)$$

where $\mathcal{M}$ and $\mathcal{L}$ are the kernel and cokernel of the map $\bar{K}$ respectively. When we say $\bar{K}$ is regular, we mean that each entry $\bar{K}_{wb}$ is a regular section of the corresponding $\mathcal{H}\mathcal{U}$ line bundle

$$\operatorname{Hom}_{\mathcal{O}_{\mathcal{C}}}(\mathcal{O}_{\mathcal{C}}(\mathbf{d}(b) - \sum_{\alpha \in Z: b \in \alpha} v(\alpha)), \mathcal{O}_{\mathcal{C}}(\mathbf{d}(w))).$$

For generic dimer weights, Σ is smooth, and $\mathcal{M}$ and $\mathcal{L}$ are line bundles (so $\bar{Q}$ has rank 1). Let $\bar{s}_b$ and $\bar{s}_w$ be sections of

$$\mathcal{M}^\vee \otimes \mathcal{O}_{\mathcal{C}}(\mathbf{d}(b) - \sum_{\alpha \in Z: b \in \alpha} v(\alpha)) \text{ and } \mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(\mathbf{d}(w))^\vee$$

respectively, given by the b -entry of the kernel map and w -entry of the cokernel map respectively. Since $\bar{Q}$ has rank 1, we have $\bar{Q}_{bw} = \bar{s}_b \bar{s}_w$. Denote by S_b and S_w the effective divisors on the open spectral curve $\mathcal{C}^\circ$ given by vanishing of the b -row and w -column of Q respectively, or equivalently, the vanishing of $\bar{s}_b$ and $\bar{s}_w$ respectively.

Lemma 6.1

$$\begin{aligned}\operatorname{div}_{\mathcal{C}} \bar{s}_b &= S_b + \sum_{\alpha \in Z: b \notin \alpha} v(\alpha), \\ \operatorname{div}_{\mathcal{C}} \bar{s}_w &= S_w,\end{aligned}$$

Proof By the definition, $(\operatorname{div}_{\mathcal{C}} \bar{s}_b)|_{\mathcal{C}^\circ} = S_b$ and $(\operatorname{div}_{\mathcal{C}} \bar{s}_w)|_{\mathcal{C}^\circ} = S_w$, so it only remains to find their orders of vanishing at infinity.

Let $U \subset \mathcal{C}$ be a neighbourhood of $v(\alpha)$ that does not contain any other point at infinity. Let u be a local parameter in U that vanishes to order 1 at $v(\alpha)$ and nowhere else. When restricted to U , each of the line bundles in the source and target of $\bar{K}$ in (44) is of the form $\mathcal{O}_U(kv(\alpha))$ for some $k \in \mathbb{Z}$. We trivialize $\mathcal{O}_U(kv(\alpha))$ as follows:

$$\begin{aligned}\mathcal{O}_U(kv(\alpha)) &\xrightarrow{\cong} \mathcal{O}_U \\ f &\mapsto u^k f.\end{aligned}$$

Let us order the black and white vertices so that the vertices on α come first. Then in U , we have

$$\bar{K} = \begin{pmatrix} K_1 & B \\ uA & K_2 \end{pmatrix} + O(u),$$

where K_1, K_2 are the restrictions of $\bar{K}$ to α and $\Gamma - \alpha$ respectively. Since $\operatorname{corank} \bar{K} = 1$ and since we know $\operatorname{corank} K_1 > 0$ from the computation of the determinant in Sect. 5.2, we have $\operatorname{corank} K_1 = 1$ and that K_2 is invertible. Let $v \in \ker K_1$. Then,

$$\ker \bar{K} = (v, -uK_2^{-1}Av) + O(u).$$

If any entry in v or $K_2^{-1}Av$ is 0, then it means that some $\bar{s}_b$ is identically 0. Let $\bar{Q}$ denote the adjugate matrix of $\bar{K}$. Since $\bar{Q}$ has rank 1, we have $\bar{Q}_{bw} = \bar{s}_b \bar{s}_w = 0$, so we will have $\bar{Q}_{bw} = 0$ for all $w \in W$. On the other hand, Q_{bw} is the signed partition function for dimer covers of $\Gamma \setminus \{b, w\}$, so if we choose w such that bw is an edge of Γ used in a dimer cover, then $Q_{bw} \neq 0$ for generic dimer weights, a contradiction. Therefore, the entries of $\ker \bar{K}$ are nonzero when $u \neq 0$, so $\bar{s}_b$ has a simple zero at $v(\alpha)$ for all $b \notin \alpha$ and has no zeroes or poles for $b \in \alpha$.

Similarly, let $v' \in \ker K_1^*$. We have

$$\ker \bar{K}^* = (v', -(K_2^*)^{-1}Bv') + O(u).$$

For generic dimer weights, none of the entries of $(K_2^*)^{-1}Bv'$ can vanish, so $\bar{s}_w$ has no zeroes or poles at $v(\alpha)$. $\square$

Corollary 6.2 $\operatorname{div}_{\mathcal{C}} Q_{bw} = S_b + S_w - D_N|_{\mathcal{C}} + \mathbf{d}(w) - \mathbf{d}(b) + \sum_{\alpha \in Z} v(\alpha).$

Proof Let $\overline{Q}$ denote the adjugate matrix of $\overline{K}$. Since $\overline{Q}$ has rank 1, we have $\overline{Q}_{bw} = \overline{s}_b \overline{s}_w$, so that

$$\operatorname{div}_{\mathcal{C}} \overline{Q}_{bw} = S_b + S_w + \sum_{\alpha \in Z: b \neq \alpha} v(\alpha).$$

A computation similar to Corollary 5.3 shows that

$$\overline{Q}_{bw} \in H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-D_N|_{\mathcal{C}} + \mathbf{d}(w) - \mathbf{d}(b) + \sum_{\alpha \in Z: b \in \alpha} v(\alpha))).$$

Q_{bw} is the rational function corresponding to the rational section $\overline{Q}_{bw}$. Therefore, using (43), we have

$$\begin{aligned} \operatorname{div}_{\mathcal{C}} Q_{bw} &= \operatorname{div}_{\mathcal{C}} \overline{Q}_{bw} - D_N|_{\mathcal{C}} + \mathbf{d}(w) - \mathbf{d}(b) + \sum_{\alpha \in Z: b \in \alpha} v(\alpha) \\ &= S_b + S_w - D_N|_{\mathcal{C}} + \mathbf{d}(w) - \mathbf{d}(b) + \sum_{\alpha \in Z} v(\alpha). \end{aligned}$$

□

Corollary 6.3 *We have for all $b \in B$, $w \in W$, $\deg S_b = \deg S_w = g$, where g is the genus of $\mathcal{C}$.*

Proof Let ω_* denote the canonical divisor of $*$. We have $\omega_{X_N} = -\sum_{\rho \in \Sigma(1)} D_{\rho}$ ([7, Theorem 8.2.3]). By the adjunction formula, we get $\omega_{\mathcal{C}} = D_N|_{\mathcal{C}} - \sum_{\alpha \in Z} v(\alpha)$. Since Q_{bw} is a rational function on $\mathcal{C}$, we have $\deg \operatorname{div}_{\mathcal{C}} Q_{bw} = 0$. Since $\deg(\mathbf{d}(w) - \mathbf{d}(b)) = -2$ and $\deg \omega_{\mathcal{C}} = 2g - 2$, we get $\deg(S_b + S_w) = 2g$. By symmetry under interchanging B and W , we get $\deg S_b = \deg S_w = g$. □

Recall that the number g is also equal to the number of interior lattice points in N for generic $\mathcal{C} \in |D_N|$.

Proposition 6.4 *The line bundle $\mathcal{L}$ is isomorphic to $\mathcal{O}_{\mathcal{C}}(S_w + \mathbf{d}(w))$ for any $w \in W$. It has degree $g - 1$.*

Proof By Lemma 6.1, $\overline{s}_w$ is a section of $\mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(\mathbf{d}(w))^{\vee}$ with divisor S_w . Therefore, we must have

$$\mathcal{L} \otimes \mathcal{O}_{\mathcal{C}}(\mathbf{d}(w))^{\vee} \cong \mathcal{O}_{\mathcal{C}}(S_w),$$

which implies $\mathcal{L} \cong \mathcal{O}_{\mathcal{C}}(S_w + \mathbf{d}(w))$. Since $\deg S_w = g$ and $\deg \mathbf{d}(w) = -1$, we get $\deg \mathcal{L} = g - 1$. □

7 Equations for the Laurent Polynomial Q_{bw}

Since Q_{bw} has Newton polygon N_{bw} , we have

$$Q_{\text{bw}} = \sum_{m \in N_{\text{bw}} \cap M} a_m \chi^m,$$

for some $a_m \in \mathbb{C}$. We know that Q_{bw} vanishes on S_{w} , which gives g linear equations among the $(a_m)_{m \in N_{\text{bw}} \cap M}$. However these g linear equations are not usually sufficient to determine Q_{bw} , so we need to find some additional equations. These additional equations will come from the vanishing of Q_{bw} at the points at infinity.

7.1 Additional Linear Equations for Q_{bw}

The fact that the Newton polygon of Q_{bw} is the small polygon N_{bw} imposes certain inequalities on the order of vanishing of Q_{bw} at points at infinity of $\mathcal{C}$. Corollary 6.2 imposes additional constraints that are linear equations in the coefficients of Q_{bw} . Inverting this linear system gives $(a_m)_{m \in N_{\text{bw}} \cap M}$ and therefore Q_{bw} .

We now give the precise statement. For a $\mathbb{Q}$ -divisor $D = \sum_{\rho \in \Sigma(1)} b_\rho D_\rho$, we define a $(\mathbb{Z})$ -divisor $\lfloor D \rfloor := \sum_{\rho \in \Sigma(1)} \lfloor b_\rho \rfloor D_\rho$, where $\lfloor x \rfloor$ is the largest integer n such that $n \leq x$. It is the pushforward of D by the canonical projection $\mathcal{X}_N \rightarrow X_N$.

Proposition 7.1 *The extra linear equations for $(a_m)_{m \in N_{\text{bw}} \cap M}$ from vanishing of Q_{bw} at points at infinity correspond to the points in*

$$-D_N|_{\mathcal{C}} + \mathbf{d}(\mathbf{w}) - \mathbf{d}(\mathbf{b}) + \sum_{\alpha \in Z} v(\alpha) + \lfloor Y_{\text{bw}} \rfloor|_{\mathcal{C}}. \quad (45)$$

Proof A generic Laurent polynomial F of the form $\sum_{m \in N_{\text{bw}} \cap M} a_m \chi^m$ has order of vanishing

$$\text{div}_{\mathcal{C}} F|_{\mathcal{C}} \geq -\lfloor Y_{\text{bw}} \rfloor|_{\mathcal{C}}$$

at the points at infinity of $\mathcal{C}$. From Corollary 6.2, we have that $\text{div}_{\mathcal{C}} Q_{\text{bw}} = S_{\text{b}} + S_{\text{w}} - D_N|_{\mathcal{C}} + \mathbf{d}(\mathbf{w}) - \mathbf{d}(\mathbf{b}) + \sum_{\alpha \in Z} v(\alpha)$. The discrepancy provides the extra equations. $\square$

Now we describe these extra linear equations explicitly. Suppose $\alpha \in Z_\rho$ is a zig-zag path that contributes a linear equation. We extend $[\alpha]$ to a basis $([\alpha] = x_1, x_2)$ of M such that $\langle x_2, u_\rho \rangle = 1$, so that for any $m \in M$, we can write

$$\chi^m = x_1^{b_m} x_2^{c_m}, \quad b_m, c_m \in \mathbb{Z}.$$

Let N_{bw}^ρ be the set of lattice points in N_{bw} closest to the edge E_ρ of N i.e., the set of points in N_{bw} that minimize the functional $\langle *, u_\rho \rangle$.

Proposition 7.2 Suppose $Q_{\mathbf{bw}} = \sum_{m \in N_{\mathbf{bw}} \cap \mathbf{M}} a_m \chi^m$ and suppose $\alpha \in Z_\rho$ is a zig-zag path that contributes a linear equation. Then, the linear equation given by α is:

$$\sum_{m \in N_{\mathbf{bw}}^\rho \cap \mathbf{M}} a_m C_\alpha^{-b_m} = 0.$$

Proof The affine open variety in X_N corresponding to the cone ρ is

$$U_\rho = \text{Spec } \mathbb{C}[x_1^{\pm 1}, x_2] \cong \mathbb{C}^\times \times \mathbb{C},$$

and $D_\rho \cap U_\rho$ is defined by $x_2 = 0$.

A generic curve $\mathcal{C}$ meets D_ρ transversely at $v(\alpha)$, and therefore we may take x_2 as a uniformizer of the local ring $\mathcal{O}_{\mathcal{C}, v(\alpha)}$ at $v(\alpha)$. For each $m \in N_{\mathbf{bw}}^\rho \cap \mathbf{M}$, we have

$$\chi^m = x_1^{b_m} x_2^p, \quad b_\gamma, p \in \mathbb{Z},$$

where p is the same for all of them, and is the coefficient of $v(\alpha)$ in $-[E_{\mathbf{bw}}]|_{\mathcal{C}}$. Then using $x_1^{-1} = C_\alpha$ at $v(\alpha)$, we have

$$Q_{\mathbf{bw}} = \left(\sum_{m \in N_{\mathbf{bw}}^\rho \cap \mathbf{M}} a_m C_\alpha^{-b_m} \right) x_2^p + O(x_2^{p+1}). \quad (46)$$

Since α contributes a linear equation, (46) must vanish at order x_2^p , so $\sum_{m \in N_{\mathbf{bw}}^\rho \cap \mathbf{M}} a_m C_\alpha^{-b_m} = 0$. □

7.2 The System of Linear Equations $\mathbb{V}_{\mathbf{bw}}$

Recall from Sect. 3 the system of linear equations $\mathbb{V}_{\mathbf{bw}}$. These are linear equations in the variables $(a_m)_{m \in N_{\mathbf{bw}} \cap \mathbf{M}}$. Recall also that the matrix $\mathbb{V}_{\mathbf{bw}}$ is defined such that these equations are given by

$$\mathbb{V}_{\mathbf{bw}}(a_m) = 0.$$

It is not necessarily a square matrix. However, we have:

Proposition 7.3 For generic spectral data, $Q_{\mathbf{bw}}$ is the unique solution of the linear system of equations $\mathbb{V}_{\mathbf{bw}}$ modulo scaling.

Remark 7.4 1. While the definition of $\mathbb{V}_{\mathbf{bw}}$ makes sense for all $w \in W$, Proposition 7.3 only holds when $w = \mathbf{w}$ since $(p_i, q_i)_{i=1}^g$ depends on $\mathbf{w}$.
2. For generic spectral data, the equations (24) are linearly independent, but the equations (26) may not be.

The rest of this section is devoted to the proof of Proposition 7.3. Consider following the exact sequence on X_N , obtained by tensoring the closed embedding exact sequence of $i : \mathcal{C} \hookrightarrow X_N$ by $\mathcal{O}_{X_N}(\lfloor Y_{\text{bw}} \rfloor)$.

$$0 \rightarrow \mathcal{O}_{X_N}(\lfloor Y_{\text{bw}} \rfloor - D_N) \rightarrow \mathcal{O}_{X_N}(\lfloor Y_{\text{bw}} \rfloor) \rightarrow i_* \mathcal{O}_{\mathcal{C}}(\lfloor Y_{\text{bw}} \rfloor|_{\mathcal{C}}) \rightarrow 0.$$

The following is a portion of the long exact sequence of cohomology.

$$0 \rightarrow H^0(X_N, \lfloor Y_{\text{bw}} \rfloor - D_N) \rightarrow H^0(X_N, \lfloor Y_{\text{bw}} \rfloor) \rightarrow H^0(\mathcal{C}, \lfloor Y_{\text{bw}} \rfloor|_{\mathcal{C}}). \quad (47)$$

We need the following technical lemma.

Lemma 7.5 *The restriction map $H^0(X_N, \lfloor Y_{\text{bw}} \rfloor) \rightarrow H^0(\mathcal{C}, \lfloor Y_{\text{bw}} \rfloor|_{\mathcal{C}})$ is injective.*

Proof If $\chi^m \in H^0(X_N, \lfloor Y_{\text{bw}} \rfloor - D_N)$, then $\text{div } \chi^m + \lfloor Y_{\text{bw}} \rfloor - D_N \geq 0$. This implies that

$$\begin{aligned} \text{div } \chi^m + Y_{\text{bw}} - D_N &= \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho} \langle m, u_\rho \rangle \frac{D_\rho}{|E_\rho|} - \mathbf{D}(\mathbf{w}) + \mathbf{D}(\mathbf{b}) \\ &\quad - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho : \mathbf{b} \in \alpha} \frac{1}{|E_\rho|} D_\rho \geq 0. \end{aligned} \quad (48)$$

Let γ be a cycle in $\mathbb{T}$ with homology class m . The total number of signed intersections of γ with all zig-zag paths is zero. This number is the sum of the coefficients of $\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho} \langle m, u_\rho \rangle \frac{D_\rho}{|E_\rho|}$. Let $\mathbf{w}'$ be any white vertex adjacent to $\mathbf{b}$. Then we have

$$-\mathbf{D}(\mathbf{w}) + \mathbf{D}(\mathbf{b}) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho : \mathbf{b} \in \alpha} \frac{1}{|E_\rho|} D_\rho = (\mathbf{D}(\mathbf{w}') - \mathbf{D}(\mathbf{w})) - \sum_{\rho \in \Sigma(1)} \sum_{\substack{\alpha \in Z_\rho : \mathbf{b} \in \alpha \\ \mathbf{bw}' \notin \alpha}} \frac{1}{|E_\rho|} D_\rho.$$

The sum of the coefficients of $\mathbf{D}(\mathbf{w}') - \mathbf{D}(\mathbf{w})$ is the signed number of intersections with zig-zag paths of any path in R from $\mathbf{w}$ to $\mathbf{w}'$, which is also 0. Since the coefficients of the last term $-\sum_{\rho \in \Sigma(1)} \sum_{\substack{\alpha \in Z_\rho : \mathbf{b} \in \alpha \\ \mathbf{bw}' \notin \alpha}} \frac{1}{|E_\rho|} D_\rho$ are strictly negative, the sum in (48)

cannot be non-negative. Therefore, $H^0(X_N, \lfloor Y_{\text{bw}} \rfloor - D_N) = 0$, which by (47) means that the map $H^0(X_N, \lfloor Y_{\text{bw}} \rfloor) \rightarrow H^0(\mathcal{C}, \lfloor Y_{\text{bw}} \rfloor|_{\mathcal{C}})$ is injective. $\square$

Proof of Proposition 7.3 1. Existence: By Theorem 7.3 of [13], the map $\kappa_{\Gamma, \mathbf{w}}$ is dominant. So a generic spectral data is in the image of $\kappa_{\Gamma, \mathbf{w}}$. For such a spectral data, Q_{bw} satisfies:

- (a) The system of equations (24) because, by definition of the spectral transform, Q_{bw} vanishes at the points of the divisor $S = \sum_{i=1}^g (p_i, q_i)$.
- (b) The equations (26) by Proposition 7.2.

2. Uniqueness: Suppose V_{bw} is a solution of $\mathbb{V}_{\text{bw}}$. Since V_{bw} has Newton polygon N_{bw} , we have $\text{div}_{\mathcal{C}} F|_{\mathcal{C}} \geq -\lfloor Y_{\text{bw}} \rfloor|_{\mathcal{C}}$ as in the proof of Proposition 7.1. The additional equations in Proposition 7.1 then imply that

$$\text{div}_{\mathcal{C}} V_{\text{bw}} \geq S + D,$$

where $D := -D_N|_{\mathcal{C}} + \mathbf{d}(\mathbf{w}) - \mathbf{d}(\mathbf{b}) + \sum_{\alpha \in Z} v(\alpha)$ satisfies $\deg D = -2g$. Therefore, $V_{\text{bw}}|_{\mathcal{C}}$ can be identified with a section of $\mathcal{O}_{\mathcal{C}}(-D)$ vanishing at the points of S . Let $\omega_{\mathcal{C}}$ denote the canonical divisor of $\mathcal{C}$ as in Sect. 6. By the Riemann-Roch theorem,

$$h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-D)) - h^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-D)) = \deg(-D) - g + 1 = g + 1.$$

By Serre duality, $h^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(-D)) = h^0(\mathcal{C}, \omega_{\mathcal{C}}(D))$, which equals 0 since $\omega_{\mathcal{C}}(D)$ has negative degree -2 . For generic S that avoids the base locus of $\mathcal{O}_{\mathcal{C}}(-D)$, the requirement that the section of $\mathcal{O}_{\mathcal{C}}(-D)$ corresponding to V_{bw} vanishes at each of the g points of S imposes g independent conditions, and therefore determines $V_{\text{bw}}|_{\mathcal{C}}$ uniquely up to multiplication by a nonzero complex number. By Lemma 7.5, V_{bw} is unique up to multiplication by a nonzero complex number. $\square$

Remark 7.6 It is easy to see using Riemann-Roch that the number of equations in $\mathbb{V}_{\text{bw}}$ is equal to $h^0(\mathcal{C}, \lfloor Y_{\text{bw}} \rfloor|_{\mathcal{C}}) - 1$. On the other hand, the number of variables is $h^0(X_N, \lfloor Y_{\text{bw}} \rfloor)$. However, the map in Lemma 7.5 is not necessarily an isomorphism (there may be sections on the curve that are not restrictions of sections on the surface), so we only have the inequality

$$\# \text{ equations in } \mathbb{V}_{\text{bw}} \geq \# \text{ variables} - 1.$$

Acknowledgements The work of A.G. was supported by the NSF grants DMS-1900743, DMS-2153059. Work of R. K. was supported by NSF grant DMS-1940932 and the Simons Foundation grant 327929.

Declarations

Competing Interests The authors have no conflicts of interest to declare that are relevant to the content of this article.

A Toric Geometry

In A.1 and A.2, we give a brief background on toric varieties; further details can be found in the books [12] and [7].

A.1 Toric Varieties

A *toric variety* X over $\mathbb{C}$ is an algebraic variety containing the complex algebraic torus $T \cong (\mathbb{C}^\times)^n$ as a Zariski open subset, such that the action of T on itself extends to an action of T on X .

Let M be a lattice, and let $M^\vee := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ denote the dual lattice. Let $T := M^\vee \otimes \mathbb{C}^\times = \text{Hom}(M, \mathbb{C}^\times)$ be the complex algebraic torus with the lattice of characters M . We denote by $\chi^m : T \rightarrow \mathbb{C}^\times$ the character associated to $m \in M$. Let $\langle *, * \rangle$ be the pairing between M and $M^\vee$. In our case $M = H_1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2$, so $M^\vee = H^1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2$ and $T = H^1(\mathbb{T}, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^2$. We have $\chi^{(i,j)}(z, w) = z^i w^j$.

A *fan* Σ is a collection of cones in the real vector space $M_{\mathbb{R}}^\vee := M^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, which is just the Lie algebra of the real torus $T(\mathbb{R})$, such that

1. Each face of a cone $\sigma \in \Sigma$ is also in Σ .
2. The intersection of two cones $\sigma_1, \sigma_2 \in \Sigma$ is a face of each of them.

Each cone $\sigma \in \Sigma$ gives rise to an affine toric variety

$$U_\sigma = \text{Spec } \mathbb{C}[S_\sigma],$$

where $S_\sigma = \sigma^\vee \cap M$ is a semigroup, $\sigma^\vee$ is the cone dual to σ , and $\mathbb{C}[S_\sigma]$ is its semigroup algebra:

$$\mathbb{C}[S_\sigma] = \left\{ \sum_{m \in S_\sigma} c_m \chi^m : c_m \in \mathbb{C}, c_m = 0 \text{ for all but finitely many } m \in S_\sigma \right\}.$$

If $\tau \subset \sigma$, then U_τ is an open subset of U_σ . Gluing the affine toric varieties $U_{\sigma_1}, U_{\sigma_2}$ along $U_{\sigma_1 \cap \sigma_2}$ for all cones $\sigma_1, \sigma_2 \in \Sigma$, we get the toric variety X_Σ associated to Σ .

In particular, if $\sigma = \{0\}$, then $S_\sigma = M$, so $\mathbb{C}[S_\sigma] = \mathbb{C}[M]$ and $U_\sigma = T$. So X_Σ contains T .

We define the action $T \times U_\sigma \rightarrow U_\sigma$ via the dual map of the algebras of functions:

$$\begin{aligned} \mathbb{C}[S_\sigma] &\longrightarrow \mathbb{C}[M] \otimes \mathbb{C}[S_\sigma], \\ \chi^m &\longmapsto \chi^m \otimes \chi^m. \end{aligned}$$

When $\sigma = \{0\}$, this is the action of T on itself. The action of T on U_σ is compatible with the gluing, and therefore gives an action of T on X_Σ .

We denote by $\Sigma(r)$ the set of r -dimensional cones of Σ . There is an inclusion-reversing bijection between T -orbit closures in X_Σ and cones in Σ . Under this bijection, each ray $\rho \in \Sigma(1)$ corresponds to a T -invariant divisor D_ρ . Let u_ρ be the primitive vector generating ρ . Then, the (Weil) divisor of the character χ^m is

$$\text{div } \chi^m = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho. \quad (49)$$

The following fundamental exact sequence computes the class group of Weil divisors of X_Σ :

$$\begin{aligned} 0 \rightarrow M \rightarrow \mathbb{Z}^{\Sigma(1)} \rightarrow \text{Cl}(X_\Sigma) \rightarrow 0, \\ m \mapsto (\langle m, u_\rho \rangle)_{\rho \in \Sigma(1)}. \end{aligned} \quad (50)$$

A.2 Polygons and Projective Toric Surfaces

Given a convex integral polygon N in the plane $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, we construct the *normal fan* Σ of N as follows:

1. $\Sigma(0) = \{0\}$.
2. For each edge E_ρ of N , let $u_\rho \in M^\vee$ be the primitive inward normal vector to E_ρ , providing an element of $\Sigma(1)$ given by the ray spanned by u_ρ .
3. For each vertex v of N , we get an element of $\Sigma(2)$ by taking the convex hull of the two rays in $\Sigma(1)$ associated to the two edges incident to v in N .

The normal fan Σ gives rise to a toric surface denoted below by X_N . The orbit-cone correspondence assigns to each edge E_ρ of N a divisor $D_\rho \cong \mathbb{P}^1$. These divisors intersect according to the combinatorics of N . Their union is the *divisor at infinity* $X_N - T$.

In fact the polygon N determines a pair (X_N, D_N) , where D_N is an ample divisor at infinity:

$$D_N := \sum_{\rho \in \Sigma(1)} a_\rho D_\rho,$$

where a_ρ is such that the edge E_ρ of N is contained in the line $\{m \in M \otimes \mathbb{R} : \langle m, u_\rho \rangle = -a_\rho\}$.

The linear system of hyperplane sections $|D_N|$ has the following properties:

1. $H^0(X_N, \mathcal{O}_{X_N}(D_N)) \cong \bigoplus_{m \in N \cap M} \mathbb{C} \cdot \chi^m$.
2. The genus of a generic curve C in $|D_N|$ is the number of interior lattice points of N .
3. Curves in $|D_N|$ intersect the divisor D_ρ with multiplicity $|E_\rho|$ (the number of primitive vectors in E_ρ).

A.3 Toric Stacks

We follow [5, Sect. 2]. Given a convex integral polygon $N \subset M_{\mathbb{R}}$, we define a *stacky fan* Σ as the following data:

1. The normal fan Σ of N , defined above.
2. For each ray $\rho \in \Sigma(1)$, the vector $|E_\rho|u_\rho$ generating the ray ρ .

We define a fan $\tilde{\Sigma} \subset \mathbb{R}^{\Sigma(1)}$ as follows: for $\sigma \in \Sigma$, we define $\tilde{\sigma} \in \tilde{\Sigma}$ by

$$\tilde{\sigma} = \text{cone}(e_\rho : \rho \in \sigma(1)) \subset \mathbb{R}^{\Sigma(1)},$$

where $\{e_\rho\}$ is the standard basis in ρ in $\mathbb{R}^{\Sigma(1)}$, and $\sigma(1)$ denotes the rays of Σ incident to σ . Then $\tilde{\Sigma}$ is the fan generated by the cones $\tilde{\sigma}$ and their faces.

Let U_Σ be the toric variety of the fan $\tilde{\Sigma}$. It is of the form $\mathbb{C}^{\Sigma(1)}$ – (closed codimension 2 subset).

Consider the following map, modifying the map (50) for polygons N with a non-primitive side:

$$\begin{aligned}\beta : M &\rightarrow \mathbb{Z}^{\Sigma(1)} \\ m &\mapsto (|E_\rho| \langle m, u_\rho \rangle)_\rho.\end{aligned}$$

Applying the functor $\text{Hom}_{\mathbb{Z}}(*, \mathbb{C}^\times)$, we get a surjective map $(\mathbb{C}^\times)^{\Sigma(1)} \rightarrow T$. Denote by G its kernel. So there is an exact sequence

$$1 \rightarrow G \rightarrow (\mathbb{C}^\times)^{\Sigma(1)} \rightarrow T \rightarrow 1. \quad (51)$$

So G is a subgroup of the torus $(\mathbb{C}^\times)^{\Sigma(1)}$ of the toric variety U_Σ . Therefore, G acts on U_Σ .

Explicitly, $\lambda = (\lambda_\rho) \in (\mathbb{C}^\times)^{\Sigma(1)}$ is in G if and only if

$$\prod_{\rho \in \Sigma(1)} \lambda_\rho^{|E_\rho| \langle m, u_\rho \rangle} = 1 \quad (52)$$

for all $m \in M$. Let $z = (z_\rho) \in \mathbb{C}^{\Sigma(1)}$ denote the standard coordinates on $\mathbb{C}^{\Sigma(1)}$. The action of G on U_Σ is $\lambda \cdot z = (\lambda_\rho z_\rho)$.

Definition A.1 The toric stack $\mathcal{X}_N$ is the smooth Deligne-Mumford stack $[U_\Sigma/G]$.

A.4 Example: A Stacky $\mathbb{P}^2$.

Consider the polygon N given by the convex-hull of $\{(0, 0), (2, 0), (0, 2)\}$. The rays of its normal fan Σ are generated by $u_1 = (1, 0)$, $u_2 = (0, 1)$, $u_3 = (-1, -1)$ with $|E_1| = |E_2| = |E_3| = 2$. The fan $\tilde{\Sigma} \subset \mathbb{R}^3$ is generated by the cones

$$\tilde{\sigma}_1 = \text{cone}(e_2, e_3), \quad \tilde{\sigma}_2 = \text{cone}(e_1, e_3), \quad \tilde{\sigma}_3 = \text{cone}(e_1, e_2),$$

and their faces, where $\{e_i\}$ is the standard basis of $\mathbb{R}^3$. These cones define affine varieties

$$\begin{aligned}U_1 &= \text{Spec } \mathbb{C}[X_1^{\pm 1}, X_2, X_3], & U_2 &= \text{Spec } \mathbb{C}[X_1, X_2^{\pm 1}, X_3], \\ U_3 &= \text{Spec } \mathbb{C}[X_1, X_2, X_3^{\pm 1}],\end{aligned}$$

respectively. The face $\tilde{\sigma}_{12} := \tilde{\sigma}_1 \cap \tilde{\sigma}_2 = \text{cone}(e_3)$ defines the affine variety $U_{12} = \text{Spec } \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}, X_3]$, identified with $U_1 \cap U_2$. Similarly, we define U_{23} and U_{13} .

Gluing U_i and U_j along the U_{ij} for all i, j , we see that the toric variety U_Σ of $\tilde{\Sigma}$ is $\mathbb{C}^3 - 0$. The map $M \rightarrow \mathbb{Z}^{\Sigma(1)}$ is

$$\begin{aligned}\mathbb{Z}^2 &\rightarrow \mathbb{Z}^3 \\ (1, 0) &\mapsto (2, 0, -2) \\ (0, 1) &\mapsto (0, 2, -2).\end{aligned}$$

The group G is the kernel of

$$\begin{aligned}(\mathbb{C}^\times)^3 &\rightarrow (\mathbb{C}^\times)^2 \\ (t_1, t_2, t_3) &\mapsto \left(\left(\frac{t_1}{t_3} \right)^2, \left(\frac{t_2}{t_3} \right)^2 \right).\end{aligned}$$

Thus, $G = \{(\pm\lambda, \pm\lambda, \lambda) : \lambda \in \mathbb{C}^\times\}$ and it acts on $\mathbb{C}^3 - 0$ by multiplication. The quotient $[\mathbb{C}^3 - 0/G]$ is a stacky $\mathbb{P}^2$.

A.5 Line Bundles and Divisors on Toric Stacks

A line bundle on the quotient stack $\mathcal{X}_N = [U_\Sigma/G]$ is the same thing as a G -equivariant line bundle on U_Σ . The Picard group of U_Σ is trivial, so line bundles on $\mathcal{X}_N$ correspond to the various G -linearizations of $\mathcal{O}_{U_\Sigma}$.

Proposition A.1 (Borisov and Hua, 2009 [5, Proposition 3.3]) *There is an isomorphism, describing the Picard group of $\mathcal{X}_N$ via divisors D_ρ :*

$$\begin{aligned}\mathbb{Z}^{\Sigma(1)}/\beta^*\mathbf{M} &\cong \text{Pic } \mathcal{X}_N, \\ (b_\rho)_\rho &\mapsto \mathcal{O}_{\mathcal{X}_N} \left(\sum_{\rho \in \Sigma(1)} \frac{b_\rho}{|E_\rho|} D_\rho \right).\end{aligned}$$

The line bundle $\mathcal{O}_{\mathcal{X}_N} \left(\sum_{\rho \in \Sigma(1)} \frac{b_\rho}{|E_\rho|} D_\rho \right)$ is the trivial line bundle $\mathcal{O}_{U_\Sigma} = U_\Sigma \times \mathbb{C}$ with the G -linearization

$$\begin{aligned}G \times (U_\Sigma \times \mathbb{C}) &\rightarrow U_\Sigma \times \mathbb{C} \\ \lambda \cdot (z, t) &\mapsto \left(\lambda \cdot z, t \prod_{\rho \in \Sigma(1)} \lambda_\rho^{b_\rho} \right).\end{aligned}$$

Let $D = \sum_{\rho \in \Sigma(1)} \frac{b_\rho}{|E_\rho|} D_\rho$ be a divisor at infinity on $\mathcal{X}_N$. We assign to D a polygon P_D in $\mathbf{M}_\mathbb{R}$ defined by the intersection of the half planes provided by the coefficients of D :

$$P_D := \bigcap_{\rho \in \Sigma(1)} \left\{ m \in \mathbf{M}_\mathbb{R} : \langle m, u_\rho \rangle \geq -\frac{b_\rho}{|E_\rho|} \right\}. \quad (53)$$

A global section of a line bundle on $\mathcal{X}_N$ is the same thing as a G -invariant global section of $\mathcal{O}_{U_\Sigma}$. As in the case of toric varieties, global sections of toric line bundles are identified with integral points in the associated polygons:

Proposition A.2 (Borisov and Hua, 2009 [5, Proposition 4.1]) *We have*

$$H^0(\mathcal{X}_N, \mathcal{O}_{\mathcal{X}_N}(D)) \cong \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m.$$

The G -invariant section of $\mathcal{O}_{U_\Sigma}$ corresponding to χ^m , $m \in P_D \cap M$, is $\prod_{\rho \in \Sigma(1)} z_\rho^{a_\rho}$, where $a_\rho = |E_\rho| \langle m, u_\rho \rangle + b_\rho$.

Proof We have $H^0(U_\Sigma, \mathcal{O}_{U_\Sigma}) = \mathbb{C}[z_\rho : \rho \in \Sigma(1)]$. The global section $\prod_{\rho \in \Sigma(1)} z_\rho^{a_\rho}$ is G -invariant if and only if

$$\prod_{\rho \in \Sigma(1)} \lambda_\rho^{b_\rho} \cdot \prod_{\rho \in \Sigma(1)} z_\rho^{a_\rho} = \prod_{\rho \in \Sigma(1)} (z_\rho \lambda_\rho)^{a_\rho} \text{ for all } \rho \in \Sigma(1),$$

which is equivalent to the equations $\prod_{\rho \in \Sigma(1)} \lambda_\rho^{b_\rho - a_\rho} = 1$ for all $\rho \in \Sigma(1)$. By exactness of (51), this is equivalent to the existence of $m \in M$ such that $a_\rho - b_\rho = |E_\rho| \langle m, u_\rho \rangle$ for all $\rho \in \Sigma(1)$. $\square$

B Combinatorial Rules for the Linear System of Equations $\mathbb{V}_{\text{bw}}$

In this appendix, we collect some combinatorial rules that facilitate the computation of the small polygons and equations in $\mathbb{V}_{\text{bw}}$.

B.1 Equivalent Description of the Small Polygons

Consider the lines

$$L_\rho := \{m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle = -b_\rho\} \quad (54)$$

that form the boundary of the small Newton polygon N_{bw} . We give an alternate description of these lines. Recall that $\tilde{\Gamma}$ be the biperiodic graph on the plane given by the lift of Γ to the universal cover of $\mathbb{T}$. The zig-zag paths in $\tilde{\Gamma}$ for a given ρ divide the plane into an infinite collection of strips $\mathcal{S}_\rho(d)$ parameterized by $d \in \frac{1}{|E_\rho|} \mathbb{Z}$ such that

$$\mathcal{S}_\rho(d) \cap V(\tilde{\Gamma}) = \{v \in V(\tilde{\Gamma}) : [D_\rho] \mathbf{D}(v) = d\},$$

where for a divisor D , $[D_\rho]D$ denotes the coefficient of D_ρ in D . We assign to each strip $\mathcal{S}_\rho(d)$ a line $L_\rho(d)$ in $M_{\mathbb{R}}$ parallel to E_ρ , using the following rule illustrated on Fig. 13:

1. The line associated to a strip $\mathcal{S}_\rho(d)$ contains the side E_ρ if and only if either

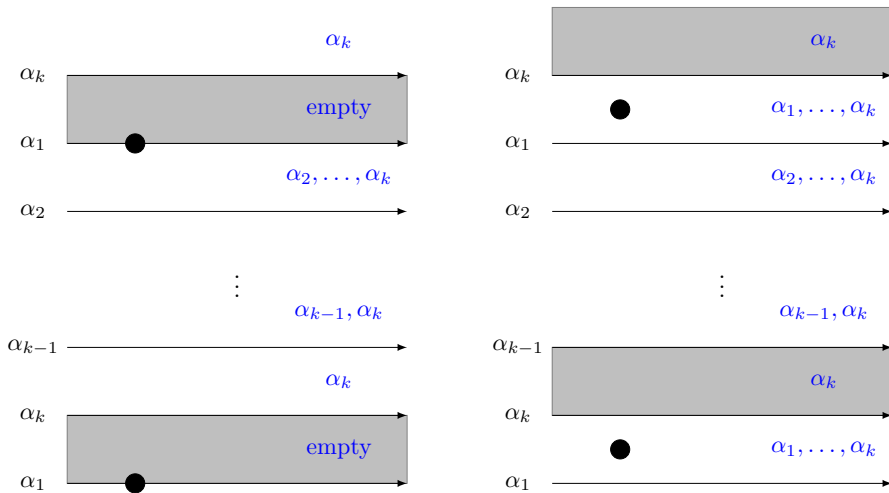


Fig. 13 The lifts of zig-zag paths $\alpha_1, \dots, \alpha_k$ in Z_ρ divide the plane into strips. The side L_ρ of the small polygon N_{bw} and the columns of the matrix $\mathbb{V}_{bw}$ are determined by the strips containing b and w . The black vertex b is the black dot. On the left panel, b is on a zigzag path, and on the right, it is between two zigzag paths. Written inside each strip in blue is the subset of Z_ρ that gives rise to equations in $\mathbb{V}_{bw}$ if w is contained in that strip. Exceptional strips are shaded

- i) The strip $\mathcal{S}_\rho(d)$ is on the right (when facing in the direction of the path) of a zig-zag path $\alpha_1 \in Z_\rho$, and α_1 contains b .
 - ii) The strip $\mathcal{S}_\rho(d)$ contains b , and b is not in a zigzag path in Z_ρ , or
2. Moving to the strip to the left shifts the line $1/|E_\rho|$ steps to the left.

We call the strip to the left of the one whose line contains E_ρ , and all strips obtained by its translations by $H_1(\mathbb{T}, \mathbb{Z})$, *exceptional strips*.

Proposition B.1 *If we associate lines to strips as above, the boundary of the small Newton polygon N_{bw} is given by the lines $\{L_\rho(d_\rho)\}$, where $d_\rho \in \frac{1}{|E_\rho|}\mathbb{Z}$ is determined by the condition $w \in \mathcal{S}_\rho(d_\rho)$, that is, it is the index of the strip containing w in the direction ρ .*

Proof In order for the line L_ρ in (54) to contain E_ρ , we must have $b_\rho = 0$, where b_ρ is the coefficient of D_ρ in (21). We have to consider two cases.

1. There is a zig-zag path $\alpha \in Z_\rho$ such that b is contained in α . We need $[D_\rho](\mathbf{D}(b)) = \frac{1}{|E_\rho|} + [D_\rho](\mathbf{D}(w))$, which means w is contained in the strip $\mathcal{S}$ to the right of the one containing b , with α separating the two strips.
2. No zig-zag path in Z_ρ contains b . In this case, we need the coefficients $[D_\rho](\mathbf{D}(b)) = [D_\rho](\mathbf{D}(w))$, which means w is in the strip $\mathcal{S}$ containing b .

If w_2 is a white vertex in the strip to the left of the strip containing a white w_1 , then $[D_\rho](\mathbf{D}(w_2)) = [D_\rho](\mathbf{D}(w_1)) + \frac{1}{|E_\rho|}$. So if we define $b_\rho(w_1)$ and $b_\rho(w_2)$ as in (21) with $w = w_1$ and $w = w_2$ respectively, then $b_\rho(w_2) = b_\rho(w_1) + \frac{1}{|E_\rho|}$. Note that the

line (54) which bounds N_{bw_2} is given by

$$L_\rho(w_2) := \{m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle = b_\rho(w_2)\}.$$

The similar line which bounds N_{bw_1} is

$$L_\rho(w_1) := \{m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle = b_\rho(w_1)\},$$

so the line $L_\rho(w_2)$ is obtained from the line $L_\rho(w_1)$ by shifting $1/|E_\rho|$ steps to the left. $\square$

B.2 The Equations in $\mathbb{V}_{bw}$

We describe the equations of type 2 in Sect. 3.1.2.

Let $\rho \in \Sigma(1)$ be a ray and let $Z_\rho = \{\alpha_1, \dots, \alpha_k\}$, where $\alpha_1, \dots, \alpha_k$ are labeled in cyclic order. Their lifts to the universal cover of the torus divides it into strips, see Fig. 13. We denote by $\mathcal{S}_i$ the strip immediately to the right of α_i .

Proposition B.2 *The set of extra linear equations is described as follows:*

1. *One of these zig-zag paths contains b . We can assume it is α_1 . Then the subset of Z_ρ that contributes an equation to $\mathbb{V}_{bw}$ is:*

$$\begin{aligned} &\text{empty if } w \in \mathcal{S}_k; \\ &\alpha_{i+1}, \dots, \alpha_k \text{ if } w \in \mathcal{S}_i \text{ for some } i \neq k. \end{aligned} \quad (55)$$

2. *The vertex b is not in any of zig-zag paths in Z_ρ . Then the subset of Z_ρ is*

$$\begin{aligned} &\alpha_1, \dots, \alpha_k \text{ if } w \in \mathcal{S}_k; \\ &\alpha_{i+1}, \dots, \alpha_k \text{ if } w \in \mathcal{S}_i, \text{ for some } i \neq k. \end{aligned} \quad (56)$$

Proof Plugging

$$Y_{bw} = D_N - \mathbf{D}(w) + \mathbf{D}(b) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho : b \in \alpha} \frac{1}{|E_\rho|} D_\rho.$$

into (45), we first observe that (45) does not change if we replace w by its any translate on the universal cover because $\mathbf{d}(w) - \mathbf{d}(b)$ changes by the same amount as $[E_{bw}]|_{\mathcal{C}}$ but with the opposite sign. Therefore we may assume that among all its possible translates in the universal cover, the strip $\mathcal{S}_i$ is the one that is immediately to the right of b . Then we have

$$(\mathbf{d}(w) - \mathbf{d}(b))|_{\mathcal{C} \cap D_\rho} = -v(\alpha_1) - \dots - v(\alpha_i)$$

and the coefficient of D_ρ in $(\mathbf{D}(w) - \mathbf{D}(b))$ is $-\frac{i}{k}$.

Now we distinguish two cases:

1. Suppose $\mathbf{b}$ is contained in α_1 , so that the coefficient of D_ρ in $\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho: \mathbf{b} \in \alpha} \frac{1}{|E_\rho|} D_\rho$ is $\frac{1}{k}$. Then we have

$$\left[-\mathbf{D}(\mathbf{w}) + \mathbf{D}(\mathbf{b}) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho: \mathbf{b} \in \alpha} \frac{1}{|E_\rho|} D_\rho \right] \Big|_{\mathcal{C} \cap D_\rho} = 0,$$

so that (45) is $v(\alpha_{i+1}) + \cdots + v(\alpha_k)$, which proves (55).

2. Suppose $\mathbf{b}$ is not contained in any of the α_i , so that the coefficient of D_ρ in $\sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho: \mathbf{b} \in \alpha} \frac{1}{|E_\rho|} D_\rho$ is 0. We have

$$\left[-\mathbf{D}(\mathbf{w}) + \mathbf{D}(\mathbf{b}) - \sum_{\rho \in \Sigma(1)} \sum_{\alpha \in Z_\rho: \mathbf{b} \in \alpha} \frac{1}{|E_\rho|} D_\rho \right] \Big|_{\mathcal{C} \cap D_\rho} = \begin{cases} 0 & \text{if } i \neq k, \\ \sum_{j=1}^k v(\alpha_j) & \text{if } i = k. \end{cases}$$

This gives

$$(45) = \begin{cases} v(\alpha_{i+1}) + \cdots + v(\alpha_k) & \text{if } i \neq k, \\ \sum_{j=1}^k v(\alpha_j) & \text{if } i = k. \end{cases}$$

We obtain (56). □

References

1. Affolter, N., George, T., Ramassamy, S.: Cross-ratio dynamics and the dimer cluster integrable system (2021). [arXiv:2108.12692](https://arxiv.org/abs/2108.12692)
2. Boutillier, C., Cimasoni, D., de Tilière, B.: Elliptic dimers on minimal graphs and genus 1 Harnack curves (2020). [arXiv:2007.14699](https://arxiv.org/abs/2007.14699)
3. Boutillier, C., Cimasoni, D., de Tilière, B.: Minimal bipartite dimers and higher genus Harnack curves (2021)
4. Beauville, A.: Theta functions, old and new. In: Open Problems and Surveys of Contemporary Mathematics. Surveys in Modern Mathematics, vol. 6, pp. 99–132. International Press, Somerville, MA (2013)
5. Borisov, L., Hua, Z.: On the conjecture of King for smooth toric Deligne-Mumford stacks. Adv. Math. **221**(1), 277–301 (2009). <https://doi.org/10.1016/j.aim.2008.11.017>
6. Cohn, H., Kenyon, R., Propp, J.: A variational principle for domino tilings. J. Am. Math. Soc. **14**(2), 297–346 (2001). <https://doi.org/10.1090/S0894-0347-00-00355-6>
7. Cox, D.A., Little, J.B., Schenck, H.K.: Toric Varieties. Graduate Studies in Mathematics, vol. 124. American Mathematical Society, Providence, RI (2011). <https://doi.org/10.1090/gsm/124>
8. Elkies, N., Kuperberg, G., Larsen, M., Propp, J.: Alternating-sign matrices and domino tilings. I. J. Algebraic Comb. **1**(2), 111–132 (1992). <https://doi.org/10.1023/A:1022420103267>
9. Elkies, N., Kuperberg, G., Larsen, M., Propp, J.: Alternating-sign matrices and domino tilings. II. J. Algebraic Comb. **1**(3), 219–234 (1992). <https://doi.org/10.1023/A:1022483817303>

10. Fock, V.V., Marshakov, A.: Loop groups, clusters, dimers and integrable systems. In: *Geometry and Quantization of Moduli Spaces*, pp. 1–66. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser/Springer, Cham (2016)
11. Fock, V.V.: Inverse spectral problem for GK integrable systems (2015). [arXiv:1503.00289](https://arxiv.org/abs/1503.00289)
12. Fulton, W.: *Introduction to Toric Varieties*. Annals of Mathematics Studies, vol. 131. Princeton University Press, Princeton, NJ. The William H. Roever Lectures in Geometry (1993). <https://doi.org/10.1515/9781400882526>
13. Goncharov, A.B., Kenyon, R.: Dimers and cluster integrable systems. *Ann. Sci. Éc. Norm. Supér.* (4) **46**(5), 747–748 (2013). <https://doi.org/10.24033/asens.2201>
14. Hanany, A., Kennaway, K.D.: Dimer models and toric diagrams (2005). [hep-th/0503149](https://arxiv.org/abs/hep-th/0503149)
15. Kasteleyn, P.W.: The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice. *Physica* **27**, 1209–1225 (1961)
16. Kasteleyn, P.W.: Dimer statistics and phase transitions. *J. Math. Phys.* **4**, 287–293 (1963). <https://doi.org/10.1063/1.1703953>
17. Kenyon, R.: Conformal invariance of domino tiling. *Ann. Probab.* **28**(2), 759–795 (2000). <https://doi.org/10.1214/aop/1019160260>
18. Kenyon, R.: The Laplacian and Dirac operators on critical planar graphs. *Invent. Math.* **150**(2), 409–439 (2002). <https://doi.org/10.1007/s00222-002-0249-4>
19. Kenyon, R.: Local statistics of lattice dimers. *Ann. Inst. H. Poincaré Probab. Stat.* **33**(5), 591–618 (1997). [https://doi.org/10.1016/S0246-0203\(97\)80106-9](https://doi.org/10.1016/S0246-0203(97)80106-9)
20. Kenyon, R., Okounkov, A.: Planar dimers and Harnack curves. *Duke Math. J.* **131**(3), 499–524 (2006). <https://doi.org/10.1215/S0012-7094-06-13134-4>
21. Kenyon, R., Okounkov, A., Sheffield, S.: Dimers and amoebae. *Ann. Math.* (2) **163**(3), 1019–1056 (2006). <https://doi.org/10.4007/annals.2006.163.1019>
22. Temperley, H.N.V., Fisher, M.E.: Dimer problem in statistical mechanics-an exact result. *Philos. Mag.* **8**(6), 1061–1063 (1961)
23. Treumann, D., Williams, H., Zaslow, E.: Kasteleyn operators from mirror symmetry. *Sel. Math. (N.S.)*, **25**, 4, Paper No. 60, 36 (2019). <https://doi.org/10.1007/s00029-019-0506-7>

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.