



Weak Solutions to the Stationary Cahn–Hilliard/Navier–Stokes Equations for Compressible Fluids

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Abstract

We are concerned with the Cahn–Hilliard/Navier–Stokes equations for the stationary compressible flows in a three-dimensional bounded domain. The governing equations consist of the stationary Navier–Stokes equations describing the compressible fluid flows and the stationary Cahn–Hilliard-type diffuse equation for the mass concentration difference. We prove the existence of weak solutions when the adiabatic exponent γ satisfies $\gamma > \frac{4}{3}$. The proof is based on the weighted total energy estimates and the new techniques developed to overcome the difficulties from the capillary stress.

Keywords Stationary equations · Weak solutions · Navier–Stokes · Cahn–Hilliard · Mixture of fluids · Diffuse interface

Mathematics Subject Classification 35Q35 · 76N10 · 35Q30 · 34K21 · 76T10

1 Introduction

The Cahn–Hilliard/Navier–Stokes system is one of the important diffuse interface models (cf. Anderson et al. 1998; Cahn and Hilliard 1958; Lowengrub and Truskinovsky 1998) describing the evolution of mixing fluids. The mixture is assumed to be macroscopically immiscible, with a partial mixing in a small interfacial region where

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the sharp interface is regularized by the Cahn–Hilliard-type diffusion in terms of the mass concentration difference. Roughly speaking, the Cahn–Hilliard equation is used for modeling the loss of mixture homogeneity and the formation of pure phase regions, while the Navier–Stokes equations describe the hydrodynamics of the mixture that is influenced by the order parameter, due to the surface tension and its variations, through an extra capillarity force term.

In this paper, we are interested in the following stationary Cahn–Hilliard/Navier–Stokes system for the mixture of compressible fluid flows in a three-dimensional bounded domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) = \operatorname{div}(\mathbb{S}_{ns} + \mathbb{S}_c - P\mathbb{I}) + \rho g, \\ \operatorname{div}(\rho u c) = \Delta \mu, \\ \rho \mu = \rho \frac{\partial f(\rho, c)}{\partial c} - \Delta c, \end{cases} \quad (1.1)$$

where ρ denotes the total density, u the mean velocity field, c the mass concentration difference of the two components, μ the chemical potential, and g the external force; the tensor

$$\mathbb{S}_{ns} = \lambda_1 \left(\nabla u + (\nabla u)^\top \right) + \lambda_2 \operatorname{div} u \mathbb{I}, \quad (1.2)$$

is the Navier–Stokes stress tensor, where $\mathbb{I}$ is the 3×3 identity matrix, and λ_1, λ_2 are constants such that

$$\lambda_1 > 0, \quad 2\lambda_1 + 3\lambda_2 \geq 0; \quad (1.3)$$

the tensor

$$\mathbb{S}_c = -\nabla c \otimes \nabla c + \frac{1}{2} |\nabla c|^2 \mathbb{I}, \quad (1.4)$$

is the capillary stress tensor; and

$$P = \rho^2 \frac{\partial f(\rho, c)}{\partial \rho}, \quad (1.5)$$

is the pressure with the free energy density (cf. Abels and Feireisl 2008; Lowengrub and Truskinovsky 1998)

$$f(\rho, c) = \rho^{\gamma-1} + H_1(c) \ln \rho + H_2(c), \quad (1.6)$$

where $\gamma > 1$ is the adiabatic exponent, and H_i ($i = 1, 2$) are two given functions. The corresponding evolutionary diffuse interface model was derived in (Abels and Feireisl 2008, Section 2.2) where the existence of weak solutions was obtained for $\gamma > \frac{3}{2}$. We refer the readers to Anderson et al. (1998), Cahn and Hilliard (1958), Lowengrub

and Truskinovsky (1998), Abels and Feireisl (2008), Liang and Wang (2020) for more discussions on the physics and models of mixing fluids with diffuse interfaces.

We briefly review some related results in the literature. For the stationary Navier–Stokes equations of compressible flows, the existence of weak solutions was studied in Lions (1998) with $\gamma > \frac{5}{3}$, Novotný–Strašcraba Novo and Novotný (2002) with $\gamma > \frac{3}{2}$, Frehse–Steinhauer–Weigant Frehse et al. (2012) with $\gamma > \frac{4}{3}$, Plotnikov–Weigant Plotnikov and Weigant (2015) with $\gamma > 1$, as well as in Jiang–Zhou Jiang and Zhou (2011) and Bresch–Burtea Bresch and Burtea (2021) for periodic domains. For the stationary Cahn–Hilliard/Navier–Stokes equations of incompressible flows, the existence of weak solutions was obtained in Biswas–Dharmatti–Mahendranath–Mohan Biswas et al. (2021), Ko–Pustejovska–Suli Ko et al. (2018), and Ko–Suli Ko and Suli (2019). For the compressible Cahn–Hilliard/Navier–Stokes equations, Liang–Wang in Liang and Wang (2020) proved the existence of weak solutions in case of the adiabatic exponent $\gamma > 2$. See (Lions 1998; Novo and Novotný 2002; Feireisl 2004; Frehse et al. 2012; Plotnikov and Weigant 2015; Jiang and Zhou 2011; Mucha et al. 2018; Mucha and Pokorný 2006; Biswas et al. 2021; Ko et al. 2018; Ko and Suli 2019; Liang and Wang 2020; Bresch and Burtea 2021; Biswas et al. 2021; Chen et al. 2020) and their references for more results.

In this paper, we shall continue our study on the existence of weak solutions and improve our previous result obtained in Liang and Wang (2020) for $\gamma > 2$ to the case of $\gamma > \frac{4}{3}$ for the stationary equations (1.1) subject to the following boundary conditions:

$$u = 0, \quad \frac{\partial c}{\partial n} = 0, \quad \frac{\partial \mu}{\partial n} = 0, \quad \text{on } \partial\Omega, \quad (1.7)$$

and the additional conditions:

$$\int \rho(x) dx = m_1 > 0, \quad \int \rho(x)c(x) dx = m_2, \quad (1.8)$$

with two given constants m_1 and m_2 , where n is the normal vector of $\partial\Omega$.

Before stating our main results, we introduce some notation that will be used throughout this paper. For two given matrices $\mathbb{A} = (a_{ij})_{3 \times 3}$ and $\mathbb{B} = (b_{ij})_{3 \times 3}$, we denote their scalar product by $\mathbb{A} : \mathbb{B} = \sum_{i,j=1}^3 a_{ij}b_{ij}$. For two vectors $a, b \in \mathbb{R}^3$, denote $a \otimes b = (a_i b_j)_{3 \times 3}$. We use $\int f = \int_{\Omega} f(x) dx$ for simplicity. For any $p \in [1, \infty]$ and integer $k \geq 0$, $W^{k,p}(\Omega)$ is the standard Sobolev space (cf. Adams 1975), and

$$\begin{aligned} W_0^{k,p} &= \{f \in W^{k,p} : f|_{\partial\Omega} = 0\}, \quad W_n^{k,p} = \left\{f \in W^{k,p} : \frac{\partial f}{\partial n}|_{\partial\Omega} = 0\right\}, \\ L^p &= W^{0,p}, \quad H^k = W^{k,2}, \quad H_0^k = W_0^{k,2}, \quad H_n^k = W_n^{k,2}, \\ \overline{L^p} &= \{f \in L^p : (f)_{\Omega} = 0\}, \end{aligned}$$

where $(f)_{\Omega} = \frac{1}{|\Omega|} \int f$ is the average of f over Ω .

As in Liang and Wang (2020), we define the weak solution as follows.

Definition 1.1 The vector of functions (ρ, u, μ, c) is called a weak solution to the problem (1.1)–(1.8), if

$$\rho \in L^{\gamma+\theta}(\Omega), \quad \rho \geq 0 \text{ a.e. in } \Omega, \quad u \in H_0^1(\Omega), \quad \mu \in H_n^1(\Omega), \quad c \in W_n^{2,p}(\Omega),$$

for some $p > \frac{6}{5}$ and $\theta > 0$, and the following properties hold true:

- (i) The system (1.1) is satisfied in the sense of distributions in Ω , and (1.8) holds for the given constants $m_1 > 0$ and $m_2 \in \mathbb{R}$.
- (ii) If (ρ, u) is prolonged by zero outside Ω , then both the equation (1.1₁) and

$$\operatorname{div}(b(\rho)u) + (b'(\rho)\rho - b(\rho)) \operatorname{div} u = 0$$

are satisfied in the sense of distributions in $\mathbb{R}^3$, where $b \in C^1([0, \infty))$ with $b'(z) = 0$ if z is large enough.

- (iii) The following energy inequality is valid:

$$\int \left(\lambda_1 |\nabla u|^2 + (\lambda_1 + \lambda_2) (\operatorname{div} u)^2 + |\nabla \mu|^2 \right) dx \leq \int \rho g \cdot u.$$

We now state our main result.

Theorem 1.1 Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary. Assume that

$$\gamma > \frac{4}{3}, \tag{1.9}$$

and

$$g \in L^\infty(\Omega), \quad |H_i(c)| + |H'_i(c)| \leq \overline{H} \quad \forall c \in \mathbb{R}, \quad i = 1, 2, \tag{1.10}$$

for some constant $\overline{H} < \infty$. Then, for any given constants $m_1 > 0$ and m_2 , the problem (1.1)–(1.8) admits a weak solution (ρ, u, μ, c) in the sense of Definition 1.1.

The main contribution of this paper is to develop new ideas to improve the existence result of Liang and Wang (2020) from the adiabatic exponent $\gamma > 2$ in Liang and Wang (2020) to a wider range $\gamma > \frac{4}{3}$. Our approach is mainly motivated by the papers (Jiang and Zhou 2011; Plotnikov and Weigant 2015) where the authors studied the existence of weak solutions to the stationary Navier–Stokes equations of compressible fluids. In order to prove Theorem 1.1, we start with the approximate solution sequence $(\rho_\delta, u_\delta, \mu_\delta, c_\delta)$ stated in Proposition 2.1 in Sect. 2 and use the weighted total energy as in Jiang and Zhou (2011); Plotnikov and Weigant (2015) together with new techniques to handle the capillary stress to establish the uniform in δ bound on $(\rho_\delta, u_\delta, \mu_\delta, c_\delta)$ in (3.1). Then, we shall be able to take the limit as $\delta \rightarrow 0$ and complete the proof of Theorem 1.1 by means of the weak convergence arguments in Liang and Wang (2020). More precisely, our proof includes the following key ingredients and new ideas:

- (1) In light of Jiang and Zhou (2011), Plotnikov and Weigant (2015), for any given $x^* \in \overline{\Omega}$ we estimate the weighted total energy

$$\int_{\Omega} \frac{(\delta\rho^4 + P + \rho|u|^2)(x)}{|x - x^*|^\alpha} dx,$$

instead of

$$\int_{\Omega} \frac{(\delta\rho^4 + P)(x)}{|x - x^*|^\alpha} dx,$$

where the advantage is that the involved kinetic energy

$$\int_{\Omega} \frac{\rho|u|^2(x)}{|x - x^*|^\alpha} dx$$

helps us relax the restriction on γ .

- (2) In order to analyze the weighted total energy, we need to overcome the new difficulties caused by the capillary stress $\mathbb{S}_c$ in (1.4), besides the Navier–Stokes stress tensor $\mathbb{S}_{ns}$. In particular, we are required to control $\|\rho\mu\|_{L^{\frac{3}{2}}}^2$ appearing in (3.10) and (3.11). For this purpose, we make the following estimate

$$\begin{aligned} & \int_{\Omega \cap B_{r_0}(x^*)} \frac{(\delta\rho^4 + P + \rho|u|^2)(x)}{|x - x^*|^{\alpha^2}} dx \\ & \leq Cr_0^{\alpha(1-\alpha)} \int_{\Omega \cap B_{r_0}(x^*)} \frac{(\delta\rho^4 + P + \rho|u|^2)(x)}{|x - x^*|^\alpha} dx, \end{aligned}$$

where $r_0 > 0$ is small and $\alpha \in (0, 1)$. By virtue of the Finite Coverage Theorem, $\overline{\Omega}$ can be covered by a finite number of balls of radius r_0 centered at $x_1^*, \dots, x_K^*$, then

$$\begin{aligned} & \sup_{x^* \in \overline{\Omega}} \int_{\Omega} \frac{(\delta\rho^4 + P + \rho|u|^2)(x)}{|x - x^*|^{\alpha^2}} dx \\ & \leq \max_{1 \leq k \leq K} \int_{\Omega \cap B_{r_0}(x_k^*)} \frac{(\delta\rho^4 + P + \rho|u|^2)(x)}{|x - x^*|^{\alpha^2}} dx \\ & \leq Cr_0^{\alpha(1-\alpha)} \|\rho\mu\|_{L^{\frac{3}{2}}}^2 + \dots \end{aligned}$$

Next, we assume the following *a priori* bound

$$\mathbf{M} = \max\{1, \|\rho\|_{L^2}\} < \infty \quad (1.11)$$

that is uniform in $\delta > 0$. If we select $r_0 = r_0(\alpha, \mathbf{M})$ small enough such that

$$r_0^{\alpha(1-\alpha)} \|\rho\mu\|_{L^{\frac{3}{2}}}^2 \leq Cr_0^{\alpha(1-\alpha)} \mathbf{M}^{\frac{8}{3}} (\|\nabla\mu\|_{L^2}^2 + 1) \leq C(\|\nabla\mu\|_{L^2}^2 + 1),$$

we are able to derive the following estimate

$$\int_{\Omega} \frac{(\delta\rho^4 + P + \rho|u|^2)(x)}{|x - x^*|^{\alpha^2}} dx \leq C + C\|\nabla\mu\|_{L^2}^2 + \dots$$

- (3) With the above two key steps, we can show that there is a constant C that does not rely on $\mathbf{M}$, such that

$$\|\rho^\gamma\|_{L^s} \leq C + C\|\rho\mu\|_{L^{\frac{3}{2}}}^2 \leq C + C\|\rho\|_{L^2}^{\frac{4}{3}} \leq C + \frac{1}{2}\|\rho^\gamma\|_{L^s},$$

as long as $\gamma s > 2$. This yields $\|\rho^\gamma\|_{L^s} \leq 2C$, and then, we have the estimate $\|\rho\|_{L^2} \leq C_0$ for some positive constant C_0 independent of $\mathbf{M}$. By choosing the *a priori* bound $\mathbf{M} = 2C_0$, one can close the *a priori* assumption (1.11) and prove the existence of weak solutions in Theorem 1.1.

The rest of the paper is organized as follows. In Sect. 2, we present the approximate solutions constructed in Liang and Wang (2020) and provide some preliminary lemmas. In Sect. 3, we prove Theorem 1.1

2 Approximate Solutions and Preliminaries

We start with the following approximate solutions constructed in Liang and Wang (2020).

Proposition 2.1 (Theorem 4.1, Liang and Wang 2020) *Under the assumptions of Theorem 1.1, for any fixed parameter $\delta > 0$ and any given constants $m_1 > 0$ and m_2 , the system*

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) + \nabla \left(\delta\rho^4 + \rho^2 \frac{\partial f}{\partial \rho} \right) = \operatorname{div}(\mathbb{S}_{ns} + \mathbb{S}_c) + \rho g, \\ \operatorname{div}(\rho u c) = \Delta\mu, \\ \rho\mu = \rho \frac{\partial f}{\partial c} - \Delta c, \end{cases} \quad (2.1)$$

with the boundary conditions (1.7), admits a weak solution $(\rho_\delta, u_\delta, \mu_\delta, c_\delta)$ in the sense of distributions such that

$$\|\rho_\delta\|_{L^1} = m_1, \quad \int \rho_\delta c_\delta = m_2, \quad (2.2)$$

$$\rho_\delta \in L^5(\Omega), \quad \rho_\delta \geq 0 \text{ a.e. in } \Omega, \quad u_\delta \in H_0^1(\Omega), \quad (\mu_\delta, c_\delta) \in H_n^1(\Omega) \times H_n^1(\Omega), \quad (2.3)$$

and

$$\int \left(\lambda_1 |\nabla u_\delta|^2 + (\lambda_1 + \lambda_2) (\operatorname{div} u_\delta)^2 + |\nabla \mu_\delta|^2 \right) \leq \int \rho_\delta g \cdot u_\delta. \quad (2.4)$$

Lemma 2.1 *Let $(\rho_\delta, u_\delta, \mu_\delta, c_\delta)$ be the solution in Proposition 2.1. Then, we have*

$$\|\mu_\delta\|_{L^p} \leq C \left(1 + \|\nabla \mu_\delta\|_{L^2} \right) (1 + \|\rho_\delta\|_{L^{\frac{6}{5}}}), \quad p \in [1, 6], \quad (2.5)$$

where the constant C is independent of δ .

Proof Thanks to (1.6), (1.10), the boundary conditions (1.7), one has, from (2.1₄),

$$\int \rho_\delta \mu_\delta = \int \left(\rho_\delta \frac{\partial f}{\partial c_\delta} + \Delta c_\delta \right) = \int \rho_\delta \frac{\partial f}{\partial c_\delta} \leq C (\|\rho_\delta \ln \rho_\delta\|_{L^1} + 1). \quad (2.6)$$

Using (2.6) together with (2.2) and the embedding inequality guarantees that

$$\begin{aligned} \int \mu_\delta &= \frac{|\Omega|}{m_1} \int \rho_\delta (\mu_\delta)_\Omega \\ &= \frac{|\Omega|}{m_1} \int \rho_\delta \mu_\delta - \frac{|\Omega|}{m_1} \int \rho (\mu_\delta - (\mu_\delta)_\Omega) \\ &\leq C (\|\rho_\delta \ln \rho_\delta\|_{L^1} + 1) + C \|\rho_\delta\|_{L^{\frac{6}{5}}} \|\nabla \mu_\delta\|_{L^2}, \end{aligned}$$

which implies

$$\begin{aligned} \|\mu_\delta\|_{L^1} &\leq C \|\nabla \mu_\delta\|_{L^2} + C (\|\rho_\delta \ln \rho_\delta\|_{L^1} + 1) + C \|\rho_\delta\|_{L^{\frac{6}{5}}} \|\nabla \mu_\delta\|_{L^2} \\ &\leq C (1 + \|\nabla \mu_\delta\|_{L^2}) (1 + \|\rho_\delta\|_{L^{\frac{6}{5}}}). \end{aligned} \quad (2.7)$$

From (2.7) and the interpolation inequality, we obtain (2.5). The proof of Lemma 2.1 is completed. $\square$

The next lemma gives an embedding from H^1 to L^2 in a three-dimensional bounded domain, via the Green representation formula.

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary and $f \in L^2(\Omega)$ satisfy*

$$f \geq 0 \quad \text{and} \quad \int_\Omega \frac{f(x)}{|x - x^*|} dx \leq \mathbb{E}, \quad \forall x^* \in \Omega,$$

for some constant $\mathbb{E} > 0$. Then, there is a constant C which depends only on Ω , such that

(i) If $u \in H_0^1(\Omega)$, then

$$\int_{\Omega} |u|^2 f dx \leq C \mathbb{E} \|u\|_{H_0^1(\Omega)}^2. \quad (2.8)$$

(ii) If $\mu \in H_n^1(\Omega)$ and $(f)_{\Omega} = 0$, then

$$\int_{\Omega} \mu^2 f dx \leq C \mathbb{E} \|\nabla \mu\|_{L^2(\Omega)}^2. \quad (2.9)$$

Proof The proof of the case (i) can be found in (Plotnikov and Weigant 2015, Lemma 4). Here, we prove the case (ii). Let H be a solution to the Neumann boundary value problem:

$$\Delta H = f \in \overline{L^2} \text{ in } \Omega, \quad \text{with } \frac{\partial H}{\partial n} = 0 \text{ on } \partial\Omega. \quad (2.10)$$

Recalling the Green representation formula $H(x^*) = \int_{\Omega} G(x^*, x) f(x) dx$, we have

$$\|H\|_{L^\infty} \leq C \sup_{x^* \in \Omega} \int_{\Omega} \frac{f(x)}{|x - x^*|} dx \leq C \mathbb{E}. \quad (2.11)$$

Thanks to (2.10), using integration by parts yields

$$\int \mu^2 f = \int \mu^2 \Delta H = -2 \int \mu \nabla \mu \cdot \nabla H \leq 2 \|\nabla \mu\|_{L^2} \left(\int \mu^2 |\nabla H|^2 \right)^{\frac{1}{2}}. \quad (2.12)$$

From (2.12), we then derive the following estimate:

$$\begin{aligned} \int \mu^2 |\nabla H|^2 &= - \int \mu^2 H \Delta H - 2 \int \mu \nabla \mu H \nabla H \\ &\leq \|H\|_{L^\infty} \int |\mu|^2 f + 2 \|H\|_{L^\infty} \|\nabla \mu\|_{L^2} \left(\int |\mu|^2 |\nabla H|^2 \right)^{\frac{1}{2}} \\ &\leq 4 \|H\|_{L^\infty} \|\nabla \mu\|_{L^2} \left(\int |\mu|^2 |\nabla H|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

which implies

$$\left(\int |\mu|^2 |\nabla H|^2 \right)^{\frac{1}{2}} \leq 4 \|H\|_{L^\infty} \|\nabla \mu\|_{L^2}.$$

Substituting the above inequality into (2.12) gives that

$$\|\mu^2 f\|_{L^1} \leq 8 \|H\|_{L^\infty} \|\nabla \mu\|_{L^2}^2.$$

Then, (2.9) follows from (2.11). The proof of Lemma 2.2 is completed. $\square$

Finally, we present the properties of the Bogovskii operator whose proof is available in Galdi (1994); Novotný and Straškraba (2004).

Lemma 2.3 (Bogovskii) *Let Ω be a bounded Lipschitz domain. There is a linear operator $\mathcal{B} = (\mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3) : \overline{L^p} \rightarrow W_0^{1,p}$ for $p \in (1, \infty)$, such that, for $f \in \overline{L^p}$,*

(i)

$$\operatorname{div} \mathcal{B}(f) = f \text{ a.e. in } \Omega,$$

(ii)

$$\|\nabla \mathcal{B}(f)\|_{L^p} \leq C(p, \Omega) \|f\|_{L^p}.$$

3 Proof of Theorem 1.1

For the approximate solution $(\rho_\delta, u_\delta, \mu_\delta, c_\delta)$ given in Proposition 2.1, if we can show that there is a constant C uniform in δ such that

$$\|\delta \rho_\delta^4 + \rho^\gamma\|_{L^s} + \|u_\delta\|_{H_0^1} + \|\mu_\delta\|_{H_n^1} + \|c_\delta\|_{W_n^{2, \frac{3}{2}}} \leq C, \quad \gamma s > 2, \quad (3.1)$$

then from (3.1) we are able to control the possible oscillation of density and the nonlinearity in the free energy density (1.6), and hence, we can take the limit as $\delta \rightarrow 0$ to prove that the approximate solution $(\rho_\delta, u_\delta, \mu_\delta, c_\delta)$ converges weakly to some limit function which satisfies (1.1)–(1.8) in the sense of Definition 1.1. This convergence proof relies heavily on the compactness arguments in Feireisl (2004), Liang and Wang (2020), Lions (1998), Novotný and Straškraba (2004), and the details can be found in Liang and Wang (2020). Therefore, it suffices to prove the following proposition in order to complete the proof of Theorem 1.1.

Proposition 3.1 *Let the assumptions in Theorem 1.1 hold true. Assume that*

$$\frac{2}{\gamma} < s \leq \frac{3}{2} \quad \text{and} \quad \frac{4}{3} < \gamma \leq 2. \quad (3.2)$$

Then, the solutions $(\rho_\delta, u_\delta, \mu_\delta, c_\delta)$ stated in Proposition 2.1 satisfy (3.1).

Remark 3.1 In case when $\gamma > 2$, the existence of weak solutions to the problem (1.1)–(1.8) has been established in Liang and Wang (2020).

For the sake of simplicity of notation, in the proof of Proposition 3.1 we will drop the subscript in $(\rho_\delta, u_\delta, \mu_\delta, c_\delta)$ and denote it by (ρ, u, μ, c) .

Lemma 3.1 *Under the assumptions of Proposition 3.1, we have*

$$\left\| \delta \rho^4 + \rho^2 \frac{\partial f}{\partial \rho} \right\|_{L^s} \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho |u|^2\|_{L^s} + \|\rho \mu\|_{L^{\frac{6s}{3+2s}}}^2 \right), \quad (3.3)$$

where s is defined in (3.2). Here and below, the capital letter $C > 0$ denotes a generic constant which may rely on $m_1, m_2, \gamma, \bar{H}, \lambda_1, \lambda_2, |\Omega|, \|g\|_{L^\infty}$ but is independent of δ .

Proof For any $h \in L^{\frac{s}{s-1}}$, we test Eq. (2.1₂) against $\mathcal{B}(h - (h)_\Omega)$ and deduce that

$$\begin{aligned} & \int \left(\delta \rho^4 + \rho^2 \frac{\partial f}{\partial \rho} \right) h \\ &= (h)_\Omega \int \left(\delta \rho^4 + \rho^2 \frac{\partial f}{\partial \rho} \right) - \int \rho g \cdot \mathcal{B}(h - (h)_\Omega) + \int \mathbb{S}_{ns} : \nabla \mathcal{B}(h - (h)_\Omega) \\ & \quad - \int \rho u \otimes u : \nabla \mathcal{B}(h - (h)_\Omega) + \int \mathbb{S}_c : \nabla \mathcal{B}(h - (h)_\Omega) \\ &\leq C \|h\|_{L^{\frac{s}{s-1}}} \left(1 + \|\delta \rho^4 + \rho^2 \frac{\partial f}{\partial \rho}\|_{L^1} + \|\nabla u\|_{L^2} + \|\rho |u|^2\|_{L^s} + \|\nabla c\|_{L^{2s}}^2 \right), \end{aligned} \quad (3.4)$$

where we have used (2.2), Lemma 2.3, the fact $\frac{s}{s-1} > 2$, and the following inequality:

$$\begin{aligned} & \|h\|_{L^2} + \|\mathcal{B}(h - (h)_\Omega)\|_{L^\infty} + \|\nabla \mathcal{B}(h - (h)_\Omega)\|_{L^2} + \|\nabla \mathcal{B}(h - (h)_\Omega)\|_{L^{\frac{s}{s-1}}} \\ &\leq C \|h\|_{L^{\frac{s}{s-1}}}. \end{aligned}$$

Now, we choose

$$h = \left(\frac{|\delta \rho^4 + \rho^2 \frac{\partial f}{\partial \rho}|}{\|\delta \rho^4 + \rho^2 \frac{\partial f}{\partial \rho}\|_{L^s}} \right)^{s-1} \in L^{\frac{s}{s-1}}$$

and derive from (3.4) that

$$\begin{aligned} & \left\| \delta \rho^4 + \rho^2 \frac{\partial f}{\partial \rho} \right\|_{L^s} \\ &\leq C \left(1 + \left\| \delta \rho^4 + \rho^2 \frac{\partial f}{\partial \rho} \right\|_{L^1} + \|\nabla u\|_{L^2} + \|\rho |u|^2\|_{L^s} + \|\nabla c\|_{L^{2s}}^2 \right). \end{aligned} \quad (3.5)$$

Next, by (1.6), (1.10), and the interpolation theorem, it holds that

$$\begin{aligned} \left\| \rho \frac{\partial f}{\partial c} \right\|_{L^{\frac{6s}{3+2s}}}^2 &\leq C + C \|\rho \ln \rho\|_{L^{\frac{6s}{3+2s}}}^2 \\ &\leq C + C \|\rho^\gamma\|_{L^s}^{\frac{(4s-3)}{3(\gamma s-1)} + \eta} \\ &\leq C + C \left\| \delta \rho^4 + \rho^2 \frac{\partial f}{\partial \rho} \right\|_{L^s}^{\frac{(4s-3)}{3(\gamma s-1)} + \eta}. \end{aligned} \quad (3.6)$$

Since $\gamma > \frac{4}{3}$, if $\eta > 0$ is small, one has

$$\frac{(4s-3)}{3(\gamma s-1)} + \eta < 1.$$

Utilizing (1.7) and (3.6), we obtain

$$\begin{aligned} \|\nabla c\|_{L^{2s}}^2 &\leq C \|\nabla^2 c\|_{L^{\frac{6s}{3+2s}}}^2 \leq C \|\Delta c\|_{L^{\frac{6s}{3+2s}}}^2 \\ &\leq C \left\| \rho \frac{\partial f}{\partial c} \right\|_{L^{\frac{6s}{3+2s}}}^2 + C \|\rho \mu\|_{L^{\frac{6s}{3+2s}}}^2 \\ &\leq C + \frac{1}{2} \left\| \delta \rho^4 + \rho^2 \frac{\partial f}{\partial \rho} \right\|_{L^s}^2 + C \|\rho \mu\|_{L^{\frac{6s}{3+2s}}}^2. \end{aligned} \quad (3.7)$$

Substituting (3.7) into (3.5), we conclude (3.3). The proof of Lemma 3.1 is completed. $\square$

Remark 3.2 Due to the boundary condition $\frac{\partial \mu}{\partial n} = 0$ and the coupling of the chemical potential μ with the density ρ , the restriction $\gamma > \frac{4}{3}$ seems critical in our proof especially when closing *a priori* estimates on the pressure function. See also Lemmas 3.4, 3.5.

Next, we shall deduce some weighted estimates on the pressure and kinetic energy together, i.e., the weighted total energy motivated by Jiang and Zhou (2011), Plotnikov and Weigant (2015).

As in Frehse et al. (2012), we introduce

$$\xi(x) = \frac{\phi(x) \nabla \phi(x)}{\left(\phi(x) + |x - x^*|^{\frac{2}{2-\alpha}} \right)^\alpha} \quad \text{with } x, x^* \in \overline{\Omega}, \alpha \in (0, 1), \quad (3.8)$$

where the function $\phi(x) \in C^2(\overline{\Omega})$ can be regarded as the distance function when $x \in \Omega$ is close to the boundary, smoothly extended to the whole domain Ω . In particular,

$$\begin{cases} \phi(x) > 0 \text{ in } \Omega \text{ and } \phi(x) = 0 \text{ on } \partial\Omega, \\ |\phi(x)| \geq k_1 \text{ if } x \in \Omega \text{ and } \text{dist}(x, \partial\Omega) \geq k_2, \\ \nabla \phi = \frac{x - \tilde{x}}{\phi(x)} = \frac{x - \tilde{x}}{|x - \tilde{x}|} \text{ if } x \in \Omega \text{ and } \text{dist}(x, \partial\Omega) = |x - \tilde{x}| \leq k_2, \end{cases} \quad (3.9)$$

where the constants $k_i > 0$, $i = 1, 2$, are given. See, for example, (Ziemer 1989, Exercise 1.15) for details.

Lemma 3.2 Let (ρ, u, μ, c) be the solutions stated in Proposition 2.1. Then, for $\alpha \in (0, 1)$, the following properties hold:

(i) In case of $x^* \in \partial\Omega$, we have

$$\begin{aligned} & \int_{B_{k_2}(x^*) \cap \Omega} \frac{(\delta\rho^4 + \rho^\gamma + \rho|u|^2)(x)}{|x - x^*|^\alpha} dx \\ & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho|u|^2\|_{L^{\frac{3}{2}}} + \|\rho\mu\|_{L^{\frac{3}{2}}}^2 \right), \end{aligned} \quad (3.10)$$

where k_2 is taken from (3.9), and C is independent of x^* .

(ii) In case of $x^* \in \Omega$, we have

$$\begin{aligned} & \int_{B_r(x^*)} \frac{(\delta\rho^4 + \rho^\gamma + \rho|u|^2)(x)}{|x - x^*|^\alpha} dx \\ & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho|u|^2\|_{L^{\frac{3}{2}}} + \|\rho\mu\|_{L^{\frac{3}{2}}}^2 \right), \end{aligned} \quad (3.11)$$

where $r = \frac{1}{3}\text{dist}(x^*, \partial\Omega) > 0$, and C is independent of r or x^* .

Proof In order to prove Lemma 3.2, we borrow some ideas developed in Frehse et al. (2012), Mucha et al. (2018), Plotnikov and Weigant (2015) and modify the proof in Liang and Wang (2020).

Write the function $f(\rho, c)$ in (1.6) as

$$f(\rho, c) = \rho^\gamma + (H_1(c) + \overline{H}) \ln \rho + H_2(c) - \overline{H} \ln \rho = \tilde{f}(\rho, c) - \overline{H} \ln \rho,$$

where

$$\tilde{f}(\rho, c) = \rho^\gamma + (H_1(c) + \overline{H}) \ln \rho + H_2(c).$$

Then, we have

$$\rho^2 \frac{\partial f(\rho, c)}{\partial \rho} = \rho^2 \frac{\partial \tilde{f}(\rho, c)}{\partial \rho} - \rho \overline{H}, \quad (3.12)$$

and

$$\rho^2 \frac{\partial \tilde{f}(\rho, c)}{\partial \rho} = (\gamma - 1)\rho^\gamma + \rho (H_1(c) + \overline{H}) \geq (\gamma - 1)\rho^\gamma \geq 0, \quad (3.13)$$

due to (1.10) and (2.3).

Step 1: Proof of (3.10): From (3.8) and (3.9), we see that $\xi \in L^\infty \cap W_0^{1,p}$ with $p \in [2, \frac{3}{\alpha})$. Furthermore, by (3.9) and the fact $\frac{2}{2-\alpha} > 1$, one has

$$\phi(x) < \phi(x) + |x - x^*|^{\frac{2}{2-\alpha}} \leq C|x - x^*|. \quad (3.14)$$

With (3.9) and (3.14), one deduces that, for $\text{dist}(x, \partial\Omega) \leq k_2$,

$$\begin{aligned} C + \frac{C}{|x - x^*|^\alpha} \geq \text{div}\xi(x) &\geq -C + \frac{(1-\alpha)}{2} \frac{|\nabla\phi(x)|^2}{\left(\phi(x) + |x - x^*|^{\frac{2}{2-\alpha}}\right)^\alpha} \\ &\geq -C + \frac{C}{|x - x^*|^\alpha}. \end{aligned} \quad (3.15)$$

Thanks to (3.12), we multiply (2.1₂) by ξ to obtain

$$\begin{aligned} &\int \left(\delta\rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} \right) \text{div}\xi + \int \rho u \otimes u : \nabla \xi \\ &= - \int \rho g \cdot \xi + \int (\mathbb{S}_{ns} + \mathbb{S}_c) : \nabla \xi + \overline{H} \int \rho \text{div}\xi. \end{aligned} \quad (3.16)$$

By (1.10), (2.2), (2.3), (3.15), and the fact $\xi \in L^\infty \cap W_0^{1,3}$, we estimate the right-hand side of (3.16) as

$$\begin{aligned} \left| - \int \rho g \cdot \xi + \int (\mathbb{S}_{ns} + \mathbb{S}_c) : \nabla \xi \right| &\leq C(\alpha) \left(1 + \|\nabla u\|_{L^2} + \|\nabla c\|_{L^3}^2 \right) \\ &\leq C(\alpha) \left(1 + \|\nabla u\|_{L^2} + \|\Delta c\|_{L^{\frac{3}{2}}}^2 \right), \end{aligned} \quad (3.17)$$

and

$$\left| \overline{H} \int \rho \text{div}\xi \right| \leq C \left(1 + \int \frac{\rho(x)}{|x - x^*|^\alpha} dx \right). \quad (3.18)$$

For the left-hand side of (3.16), it holds from (3.13) and (3.15) that

$$\begin{aligned} \int \left(\delta\rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} \right) \text{div}\xi &\geq -C \int \left(\delta\rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} \right) \\ &\quad + C \int_{\Omega \cap B_{k_2}(x^*)} \frac{\left(\delta\rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} \right)}{|x - x^*|^\alpha}. \end{aligned} \quad (3.19)$$

By (3.9), one has

$$\partial_j \partial_i \phi = \frac{\partial_i (x - \tilde{x})^j}{\phi} - \frac{\partial_j \phi \partial_i \phi}{\phi}.$$

Then,

$$\int \frac{\phi \rho u \otimes u \partial_j \partial_i \phi}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^\alpha} = \int \frac{\rho |u|^2}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^\alpha} - \int \frac{\rho |u \cdot \nabla \phi|^2}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^\alpha}.$$

Thus, we have the following computation and estimate:

$$\begin{aligned}
 & \int \rho u \otimes u : \nabla \xi \\
 &= \int \frac{\rho |u|^2}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^\alpha} - \alpha \int \frac{\phi \rho (u \cdot \nabla \phi)^2}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^{\alpha+1}} \\
 & \quad - \alpha \int \frac{\phi \rho (u \cdot \nabla |x - x^*|^{\frac{2}{2-\alpha}})(u \cdot \nabla \phi)}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^{\alpha+1}} \\
 & \geq (1 - \alpha) \int \frac{\rho |u|^2}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^\alpha} - \alpha \int \frac{\phi \rho (u \cdot \nabla |x - x^*|^{\frac{2}{2-\alpha}})(u \cdot \nabla \phi)}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^{\alpha+1}} \\
 & \geq \frac{(1 - \alpha)}{2} \int \frac{\rho |u|^2}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^\alpha} - C \int \frac{\phi^2 \rho |u|^2 |x - x^*|^{\frac{2\alpha}{2-\alpha}}}{\left(\phi + |x - x^*|^{\frac{2}{2-\alpha}}\right)^{\alpha+2}} \\
 & \geq C \int_{\Omega \cap B_{k_2}(x^*)} \frac{\rho |u|^2}{|x - x^*|^\alpha} - C \|\rho |u|^2\|_{L^1},
 \end{aligned} \tag{3.20}$$

where we have used (3.14) and the Cauchy inequality. Therefore, taking (3.17)–(3.20) into account, using (3.7), (3.3), and $\frac{6s}{3+2s} < \frac{3}{2}$, we deduce from (3.16) that

$$\begin{aligned}
 & \int_{\Omega \cap B_{k_2}(x^*)} \frac{\delta \rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} + \rho |u|^2}{|x - x^*|^\alpha} \\
 & \leq C \left(\|\delta \rho^4 + \rho^2 \frac{\partial f}{\partial \rho}\|_{L^1} + \|\nabla u\|_{L^2} + \|\rho |u|^2\|_{L^1} + \|\Delta c\|_{L^{\frac{3}{2}}}^2 \right) \\
 & \quad + C \int \frac{\rho(x)}{|x - x^*|^\alpha} dx \\
 & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho |u|^2\|_{L^{\frac{3}{2}}} + \|\rho \mu\|_{L^{\frac{3}{2}}}^2 \right) + C \int \frac{\rho(x)}{|x - x^*|^\alpha} dx.
 \end{aligned} \tag{3.21}$$

Finally, thanks to (2.3) and (3.13), one has

$$\begin{aligned}
 C \int \frac{\rho(x)}{|x - x^*|^\alpha} dx &= C \left(\int_{\Omega \setminus B_{k_2}(x^*)} + \int_{\Omega \cap B_{k_2}(x^*)} \right) \frac{\rho(x)}{|x - x^*|^\alpha} dx \\
 &\leq C + C \int_{\Omega \cap B_{k_2}(x^*)} \frac{\rho(x)}{|x - x^*|^\alpha} dx \\
 &\leq C + \frac{1}{2} \int_{\Omega \cap B_{k_2}(x^*)} \frac{\rho^2 \frac{\partial \tilde{f}}{\partial \rho}}{|x - x^*|^\alpha}.
 \end{aligned} \tag{3.22}$$

Substituting (3.22) back into (3.21), we obtain (3.10).

Step 2: Proof of (3.11): Let $\text{dist}(x^*, \partial\Omega) = 3r > 0$, and χ be the smooth cutoff function satisfying

$$\chi(x) = 1 \text{ if } x \in B_r(x^*), \quad \chi(x) = 0 \text{ if } x \notin B_{2r}(x^*), \quad |\nabla \chi(x)| \leq 2r^{-1}. \quad (3.23)$$

If we multiply (2.1₂) by $\frac{x-x^*}{|x-x^*|^\alpha} \chi^2$, we get

$$\begin{aligned} & \int \left(\delta \rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} \right) \frac{3-\alpha}{|x-x^*|^\alpha} \chi^2 + \int \rho u \otimes u : \nabla \left(\frac{x-x^*}{|x-x^*|^\alpha} \chi^2 \right) \\ &= - \int \rho g \cdot \frac{x-x^*}{|x-x^*|^\alpha} \chi^2 + \int (\mathbb{S}_{ns} + \mathbb{S}_c) : \nabla \left(\frac{x-x^*}{|x-x^*|^\alpha} \chi^2 \right) \\ & \quad - 2 \int \left(\delta \rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} \right) \chi \frac{\nabla \chi \cdot (x-x^*)}{|x-x^*|^\alpha} + \overline{H} \int \rho \frac{3-\alpha}{|x-x^*|^\alpha} \chi^2. \end{aligned} \quad (3.24)$$

From the following computation,

$$\begin{aligned} & \partial_i \left(\frac{x^j - (x^*)^j}{|x-x^*|^\alpha} \chi^2 \right) \\ &= \frac{\partial_i (x^j - (x^*)^j)}{|x-x^*|^\alpha} \chi^2 - \alpha \frac{(x^j - (x^*)^j)(x^i - (x^*)^i)}{|x-x^*|^{\alpha+2}} \chi^2 + 2\chi \frac{x^j - (x^*)^j}{|x-x^*|^\alpha} \partial_i \chi, \end{aligned}$$

one sees that the second term on the left-hand side of (3.24) satisfies

$$\begin{aligned} & \int \rho u \otimes u : \nabla \left(\frac{x-x^*}{|x-x^*|^\alpha} \chi^2 \right) \\ & \geq (1-\alpha) \int \frac{\rho |u|^2}{|x-x^*|^\alpha} \chi^2 + 2 \int \frac{\chi \rho (u \cdot \nabla \chi)(u \cdot (x-x^*))}{|x-x^*|^\alpha} \\ & \geq \frac{1-\alpha}{2} \int \frac{\rho |u|^2}{|x-x^*|^\alpha} \chi^2 - C \int_{B_{2r}(x^*) \setminus B_r(x^*)} \frac{\rho |u|^2}{|x-x^*|^\alpha}, \end{aligned} \quad (3.25)$$

where the constant C is independent of r , and for the last inequality, we have used $|\nabla \chi| |x-x^*| \leq 4$ for any $x \in B_{2r}(x^*) \setminus B_r(x^*)$. Owing to (3.23), (3.7), and the fact

$$\nabla \left(\frac{x-x^*}{|x-x^*|^\alpha} \chi^2 \right) \in L^3,$$

we have the following estimates:

$$\begin{aligned} & \left| - \int \rho g \cdot \frac{x-x^*}{|x-x^*|^\alpha} \chi^2 + \int (\mathbb{S}_{ns} + \mathbb{S}_c) : \nabla \left(\frac{x-x^*}{|x-x^*|^\alpha} \chi^2 \right) \right| \\ & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\Delta c\|_{L^{\frac{3}{2}}}^2 \right), \end{aligned} \quad (3.26)$$

and

$$\begin{aligned} & \left| -2 \int \left(\delta \rho^4 + \rho^2 \frac{\partial f}{\partial \rho} \right) \chi \frac{\nabla \chi \cdot (x - x^*)}{|x - x^*|^\alpha} + \overline{H} \int \rho \frac{3 - \alpha}{|x - x^*|^\alpha} \chi^2 \right| \\ & \leq C \int_{B_{2r}(x^*) \setminus B_r(x^*)} \frac{\left(\delta \rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} \right)}{|x - x^*|^\alpha} + C \int_{B_{2r}(x^*)} \frac{\rho(x)}{|x - x^*|^\alpha} dx, \end{aligned} \quad (3.27)$$

where C is independent of r .

With the above three estimates (3.25)–(3.27) in hand, we deduce from (3.24) that

$$\begin{aligned} & \int_{B_r(x^*)} \frac{\left(\delta \rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} + \rho |u|^2 \right) (x)}{|x - x^*|^\alpha} dx \\ & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\Delta c\|_{L^{\frac{3}{2}}}^2 \right) \\ & \quad + C \int_{B_{2r}(x^*) \setminus B_r(x^*)} \frac{\left(\delta \rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} + \rho |u|^2 \right) (x)}{|x - x^*|^\alpha} dx \\ & \quad + C \int_{B_{2r}(x^*)} \frac{\rho(x)}{|x - x^*|^\alpha} dx. \end{aligned} \quad (3.28)$$

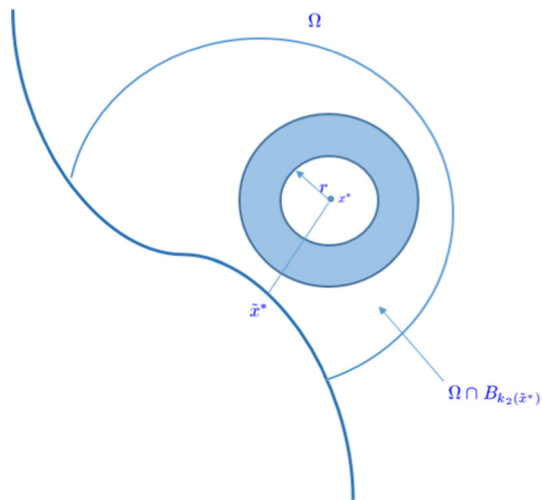
By (3.21)–(3.22) and the following estimate

$$\begin{aligned} C \int_{B_{2r}(x^*)} \frac{\rho(x)}{|x - x^*|^\alpha} dx &= C \left(\int_{B_r(x^*)} + \int_{B_{2r}(x^*) \setminus B_r(x^*)} \right) \frac{\rho(x)}{|x - x^*|^\alpha} dx \\ &\leq C + \frac{1}{2} \int \frac{\rho^2 \frac{\partial \tilde{f}}{\partial \rho}(x)}{|x - x^*|^\alpha} + C \int_{B_{2r}(x^*) \setminus B_r(x^*)} \frac{\rho^2 \frac{\partial \tilde{f}}{\partial \rho}(x)}{|x - x^*|^\alpha} dx, \end{aligned}$$

we obtain from (3.28) that

$$\begin{aligned} & \int_{B_r(x^*)} \frac{\left(\delta \rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} + \rho |u|^2 \right) (x)}{|x - x^*|^\alpha} dx \\ & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho |u|^2\|_{L^{\frac{3}{2}}} + \|\rho \mu\|_{L^{\frac{3}{2}}}^2 \right) \\ & \quad + C \int_{B_{2r}(x^*) \setminus B_r(x^*)} \frac{\left(\delta \rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} + \rho |u|^2 \right) (x)}{|x - x^*|^\alpha} dx. \end{aligned} \quad (3.29)$$

It remains to deal with the last term in (3.29). To this end, we use the ideas developed in Liang and Wang (2020) and divide the proof into two cases: (1) $x^* \in \Omega$ is far away from the boundary; (2) $x^* \in \Omega$ is close to the boundary.

Fig. 1 Near boundary points

- (1) For the case of $\text{dist}(x^*, \partial\Omega) = 3r \geq \frac{k_2}{2} > 0$ with k_2 being taken from (3.9), it is clear that

$$\int_{B_{2r}(x^*) \setminus B_r(x^*)} \frac{(\delta\rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} + \rho|u|^2)(x)}{|x - x^*|^\alpha} dx \leq C(k_2) \left\| \delta\rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} + \rho|u|^2 \right\|_{L^1}. \quad (3.30)$$

With (3.30), as well as (3.12)–(3.13), (3.22), Lemma 3.1, we deduce from (3.29) that

$$\begin{aligned} & \int_{B_r(x^*)} \frac{(\delta\rho^4 + \rho^\gamma + \rho|u|^2)(x)}{|x - x^*|^\alpha} dx \\ & \leq C \int_{B_r(x^*)} \frac{(\delta\rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} + \rho|u|^2)(x)}{|x - x^*|^\alpha} dx \\ & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho|u|^2\|_{L^{\frac{3}{2}}} + \|\rho\mu\|_{L^{\frac{3}{2}}}^2 + \left\| \delta\rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} + \rho|u|^2 \right\|_{L^1} \right) \\ & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho|u|^2\|_{L^{\frac{3}{2}}} + \|\rho\mu\|_{L^{\frac{3}{2}}}^2 \right). \end{aligned} \quad (3.31)$$

- (2) For the case of $x^* \in \Omega$ close to the boundary, that is, $\text{dist}(x^*, \partial\Omega) = 3r < \frac{k_2}{2}$, let $|x^* - \tilde{x}^*| = \text{dist}(x^*, \partial\Omega)$ with $\tilde{x}^* \in \partial\Omega$. Then, one deduces (see Fig. 1) that

$$4|x - x^*| \geq |x - \tilde{x}^*|, \quad \forall x \notin B_r(x^*). \quad (3.32)$$

Making use of (3.32) and (3.21), (3.22), we have the following estimate

$$\begin{aligned}
 & C \int_{B_{2r}(x^*) \setminus B_r(x^*)} \frac{\left(\delta \rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} + \rho |u|^2 \right) (x)}{|x - x^*|^\alpha} dx \\
 & \leq C \int_{\Omega \cap B_{k_2}(\tilde{x}^*)} \frac{\left(\delta \rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} + \rho |u|^2 \right) (x)}{|x - \tilde{x}^*|^\alpha} dx \\
 & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho |u|^2\|_{L^{\frac{3}{2}}} + \|\rho \mu\|_{L^{\frac{3}{2}}}^2 \right).
 \end{aligned} \tag{3.33}$$

This inequality (3.33) and (3.12), (3.13) ensure that (3.29) leads to

$$\begin{aligned}
 & \int_{B_r(x^*)} \frac{\left(\delta \rho^4 + \rho^\gamma + \rho |u|^2 \right) (x)}{|x - x^*|^\alpha} dx \\
 & \leq C \int_{B_r(x^*)} \frac{\left(\delta \rho^4 + \rho^2 \frac{\partial \tilde{f}}{\partial \rho} + \rho |u|^2 \right) (x)}{|x - x^*|^\alpha} dx \\
 & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho |u|^2\|_{L^{\frac{3}{2}}} + \|\rho \mu\|_{L^{\frac{3}{2}}}^2 \right).
 \end{aligned} \tag{3.34}$$

Therefore, the desired estimate (3.11) follows immediately from (3.31) and (3.34). The proof of Lemma 3.2 is completed. $\square$

The next lemma provides a refined estimate on the weighted energy obtained in Lemma 3.2.

Lemma 3.3 *Let the assumptions in Lemma 3.2 hold true. Assume that there is a constant $\mathbf{M}$ uniform in δ , such that*

$$\mathbf{M} = \max\{1, \|\rho\|_{L^2}\} < \infty. \tag{3.35}$$

Then,

$$\sup_{x^* \in \Omega} \int_{\Omega} \frac{\left(\delta \rho^4 + \rho^\gamma + \rho |u|^2 \right) (x)}{|x - x^*|^{\alpha^2}} dx \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho |u|^2\|_{L^{\frac{3}{2}}} + \|\nabla \mu\|_{L^2}^2 \right). \tag{3.36}$$

Proof If $x^* \in \partial\Omega$, it holds that, for any $r \in (0, k_2)$,

$$\begin{aligned} & \frac{1}{r^{\alpha(1-\alpha)}} \int_{B_r(x^*) \cap \Omega} \frac{(\delta\rho^4 + \rho^\gamma + \rho|u|^2)(x)}{|x - x^*|^{\alpha^2}} dx \\ & \leq \int_{B_r(x^*) \cap \Omega} \frac{(\delta\rho^4 + \rho^\gamma + \rho|u|^2)(x)}{|x - x^*|^\alpha} dx \\ & \leq \int_{B_{k_2}(x^*) \cap \Omega} \frac{(\delta\rho^4 + \rho^\gamma + \rho|u|^2)(x)}{|x - x^*|^\alpha} dx. \end{aligned} \quad (3.37)$$

Combining (3.37) with (3.10), we obtain for any $r \in (0, k_2)$,

$$\begin{aligned} & \int_{B_r(x^*) \cap \Omega} \frac{(\delta\rho^4 + \rho^\gamma + \rho|u|^2)(x)}{|x - x^*|^{\alpha^2}} dx \\ & \leq Cr^{\alpha(1-\alpha)} \left(1 + \|\nabla u\|_{L^2} + \|\rho|u|^2\|_{L^{\frac{3}{2}}} + \|\rho\mu\|_{L^{\frac{3}{2}}}^2 \right), \end{aligned} \quad (3.38)$$

where the constant C is independent of r or x^* . Using (2.2), (2.5), and the interpolation inequality, we have the following estimate:

$$\begin{aligned} \|\rho\mu\|_{L^{\frac{3}{2}}}^2 & \leq C\|\rho\|_{L^2}^2\|\mu\|_{L^6}^2 \\ & \leq C\|\rho\|_{L^2}^2 \left(1 + \|\rho\|_{L^{\frac{6}{5}}} \|\nabla\mu\|_{L^2} \right)^2 \\ & \leq C\mathbf{M}^{\frac{8}{3}} \left(1 + \|\nabla\mu\|_{L^2}^2 \right), \end{aligned} \quad (3.39)$$

where, and in what follows, the constant C is independent of $\mathbf{M}$. Choose r_0 small so that

$$r_0 \leq \min \left\{ \frac{k_2}{2}, \mathbf{M}^{\frac{-8}{3\alpha(1-\alpha)}} \right\}. \quad (3.40)$$

It follows from (3.38) that

$$\begin{aligned} & \int_{B_{r_0}(x^*) \cap \Omega} \frac{(\delta\rho^4 + \rho^\gamma + \rho|u|^2)(x)}{|x - x^*|^{\alpha^2}} dx \\ & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho|u|^2\|_{L^{\frac{3}{2}}} \right) + Cr_0^{\alpha(1-\alpha)} \|\rho\mu\|_{L^{\frac{3}{2}}}^2 \\ & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho|u|^2\|_{L^{\frac{3}{2}}} \right) + Cr_0^{\alpha(1-\alpha)} \mathbf{M}^{\frac{8}{3}} \left(1 + \|\nabla\mu\|_{L^2}^2 \right) \\ & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho|u|^2\|_{L^{\frac{3}{2}}} + \|\nabla\mu\|_{L^2} \right), \end{aligned} \quad (3.41)$$

where, for the last two inequalities, we have used (3.39) and (3.40).

If $x^* \in \Omega$, we use the similar arguments to obtain

$$\begin{aligned}
 & \int_{B_{r_0}(x^*)} \frac{(\delta\rho^4 + \rho^\gamma + \rho|u|^2)(x)}{|x - x^*|^{\alpha^2}} dx \\
 & \leq Cr_0^{\alpha(1-\alpha)} \int_{B_{r_0}(x^*)} \frac{(\delta\rho^4 + \rho^\gamma + \rho|u|^2)(x)}{|x - x^*|^\alpha} dx \\
 & \leq Cr_0^{\alpha(1-\alpha)} \left(1 + \|\nabla u\|_{L^2} + \|\rho|u|^2\|_{L^{\frac{3}{2}}} + \|\rho\mu\|_{L^{\frac{3}{2}}}^2 \right) \\
 & \leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho|u|^2\|_{L^{\frac{3}{2}}} + \|\nabla\mu\|_{L^2}^2 \right).
 \end{aligned} \tag{3.42}$$

As a result of (3.41) and (3.42), we conclude (3.36) by using the Finite Coverage Theorem, as the domain $\bar{\Omega}$ is bounded. The proof of Lemma 3.3 is completed. $\square$

The final two Lemmas 3.4 and 3.5 are devoted to proving the desired inequality (3.1) and the *a priori* bound (3.35).

Lemma 3.4 *Let the assumptions in Proposition 3.1 hold true. Then,*

$$\|u\|_{H_0^1} + \|\nabla\mu\|_{L^2} \leq C. \tag{3.43}$$

Proof Define

$$\mathbb{A} = \int \rho|u|^2|u|^{2(1-\theta)} \quad \text{with} \quad \theta = \frac{3\gamma - 4}{8\gamma}. \tag{3.44}$$

By (3.2), one has

$$\theta \in (0, \frac{1}{8}]. \tag{3.45}$$

Thanks to (2.2) and the Hölder inequality, it holds that

$$\|\rho u\|_{L^1} \leq \|\rho|u|^2|u|^{2(1-\theta)}\|_{L^1}^{\frac{1}{2(2-\theta)}} \|\rho\|_{L^1}^{\frac{3-2\theta}{2(2-\theta)}} \leq C\mathbb{A}^{\frac{1}{2(2-\theta)}} \tag{3.46}$$

and

$$\|\rho|u|^2\|_{L^{\frac{3}{2}}} \leq \|\rho|u|^2|u|^{2(1-\theta)}\|_{L^1}^{\frac{1}{2-\theta}} \|\rho\|_{L^1}^{\frac{(1-\theta)}{2-\theta}} \leq C\mathbb{A}^{\frac{1}{2-\theta}}. \tag{3.47}$$

By means of (1.3), (1.10), (2.4), (3.46), we get

$$\int (|\nabla u|^2 + |\nabla\mu|^2) \leq C\|\rho u\|_{L^1} \leq C\mathbb{A}^{\frac{1}{2(2-\theta)}}. \tag{3.48}$$

Let

$$\alpha^2 = 1 - \frac{\theta}{2}. \quad (3.49)$$

One calculates as the following,

$$\frac{\rho|u|^{2(1-\theta)}}{|x-x^*|} = \left(\frac{\rho|u|^2}{|x-x^*|^{\alpha^2}} \right)^{1-\theta} \left(\frac{\rho^\gamma}{|x-x^*|^{\alpha^2}} \right)^{\frac{\theta}{\gamma}} \left(\frac{1}{|x-x^*|^{\frac{\gamma}{2(\gamma-1)} + \alpha^2}} \right)^{\frac{(\gamma-1)\theta}{\gamma}} \quad (3.50)$$

where $\frac{\gamma}{2(\gamma-1)} + \alpha^2 < 3$ since $\gamma > \frac{4}{3}$. Hence, utilizing (3.36), (3.47), (3.48), we integrate (3.50) and obtain

$$\begin{aligned} \int \frac{\rho|u|^{2(1-\theta)}(x)}{|x-x^*|} dx &\leq \int \frac{\rho|u|^2(x)}{|x-x^*|^{\alpha^2}} dx + \int \frac{\rho^\gamma(x)}{|x-x^*|^{\alpha^2}} dx + C \\ &\leq C \int_{\Omega} \frac{(\delta\rho^4 + \rho^\gamma + \rho|u|^2)(x)}{|x-x^*|^{\alpha^2}} dx \\ &\leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho|u|^2\|_{L^{\frac{3}{2}}} + \|\nabla\mu\|_{L^2}^2 \right) \\ &\leq C \left(1 + \mathbb{A}^{\frac{1}{2-\theta}} \right). \end{aligned} \quad (3.51)$$

From (3.48), (3.51), and Part (i) in Lemma 2.2, one deduces

$$\begin{aligned} \mathbb{A} &\leq \|\nabla u\|_{L^2}^2 \sup_{x^* \in \Omega} \int \frac{\rho|u|^{2(1-\theta)}(x)}{|x-x^*|} dx \\ &\leq C \mathbb{A}^{\frac{1}{2(2-\theta)}} \left(1 + \mathbb{A}^{\frac{1}{2-\theta}} \right) \\ &\leq 1 + C \mathbb{A}^{\frac{3}{2(2-\theta)}}, \end{aligned}$$

which together with (3.45) yields

$$\mathbb{A} \leq C. \quad (3.52)$$

Combining (3.52) with (3.48), we get (3.43). The proof of Lemma 3.4 is completed. $\square$

Lemma 3.5 *Let the assumptions in Theorem 3.1 hold true. Then,*

$$\|\delta\rho^4 + \rho^\gamma\|_{L^s} + \|\mu\|_{L^6} + \|c\|_{W_n^{2,\frac{3}{2}}} \leq C. \quad (3.53)$$

Proof Owing to (3.44) and (3.49), one has

$$\frac{3\gamma - 4\alpha^2}{3\gamma - 4} \in (0, 3).$$

By (3.2), (3.51), (3.52), and the Hölder inequality, we have the following estimate,

$$\begin{aligned} \int \frac{\rho^{\frac{4}{3}}(x)}{|x-x^*|} dx &\leq \left(\int \frac{\rho^\gamma(x)}{|x-x^*|^{\alpha^2}} dx \right)^{\frac{4}{3\gamma}} \left(\int \frac{dx}{|x-x^*|^{\frac{3\gamma-4\alpha^2}{3\gamma}}} \right)^{\frac{3\gamma-4}{3\gamma}} \\ &\leq C \left(\int \frac{\rho^\gamma(x)}{|x-x^*|^{\alpha^2}} dx \right)^{\frac{4}{3\gamma}} \\ &\leq C. \end{aligned} \quad (3.54)$$

Hence, using (3.48), (3.52), (3.54), Part (ii) in Lemma 2.2, we find

$$\|(\rho^{\frac{4}{3}} - (\rho^{\frac{4}{3}})_\Omega) \mu^2\|_{L^1} \leq \|\nabla \mu\|_{L^2}^2 \left(1 + \sup_{x^*} \int \frac{\rho^{\frac{4}{3}}(x)}{|x-x^*|} dx \right) \leq C. \quad (3.55)$$

On the other hand, it follows from (2.2), (3.54), Lemma 2.1, Lemma 3.4, and the interpolation inequality that

$$\|(\rho^{\frac{4}{3}})_\Omega \mu^2\|_{L^1} \leq (\rho^{\frac{4}{3}})_\Omega \|\mu^2\|_{L^1} \leq \|\mu\|_{L^2}^2 \leq C \|\rho\|_{L^2}^{\frac{2}{3}}. \quad (3.56)$$

Therefore, utilizing (3.55), (3.56) and the fact $\frac{6s}{3+2s} \leq \frac{3}{2}$, we conclude

$$\begin{aligned} \|\rho \mu\|_{L^{\frac{6s}{3+2s}}}^2 &\leq C \|\rho \mu\|_{L^{\frac{3}{2}}}^2 \\ &\leq C \|\rho^{\frac{4}{3}} \mu^2\|_{L^1} \|\rho\|_{L^2}^{\frac{2}{3}} \\ &\leq C \left(\|(\rho^{\frac{4}{3}} - (\rho^{\frac{4}{3}})_\Omega) \mu^2\|_{L^1} + \|(\rho^{\frac{4}{3}})_\Omega \mu^2\|_{L^1} \right) \|\rho\|_{L^2}^{\frac{2}{3}} \\ &\leq C \|\rho\|_{L^2}^{\frac{4}{3}}. \end{aligned} \quad (3.57)$$

Substituting (3.57) into (3.3), using (3.47), (3.48), (3.52), we get

$$\begin{aligned} \|\delta \rho^4 + \rho^\gamma\|_{L^s} &\leq \left\| \delta \rho^4 + \rho^2 \frac{\partial f}{\partial \rho} \right\|_{L^s} \\ &\leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho |u|^2\|_{L^s} + \|\rho \mu\|_{L^{\frac{6s}{3+2s}}}^2 \right) \\ &\leq C \left(1 + \|\nabla u\|_{L^2} + \|\rho |u|^2\|_{L^{\frac{3}{2}}} + \|\rho \mu\|_{L^{\frac{3}{2}}}^2 \right) \\ &\leq C \left(1 + \|\rho\|_{L^2}^{\frac{4}{3}} \right). \end{aligned} \quad (3.58)$$

Thanks to (3.2), one has

$$C \|\rho\|_{L^2}^{\frac{4}{3}} \leq C \|\rho^\gamma\|_{L^{\frac{2}{\gamma}}}^{\frac{4}{3\gamma}} \leq C + \frac{1}{2} \|\rho^\gamma\|_{L^s} \leq C + \frac{1}{2} \|\delta\rho^4 + \rho^\gamma\|_{L^s}. \quad (3.59)$$

The combination of (3.58) with (3.59) gives rise to

$$\|\delta\rho^4 + \rho^\gamma\|_{L^s} \leq \overline{C}, \quad (\gamma s > 2), \quad (3.60)$$

where $\overline{C}$ depends only on $m_1, \gamma, \overline{H}, \lambda_1, \lambda_2, |\Omega|, \|g\|_{L^\infty}$. From (3.60), there is a constant C_0 independent of δ , such that

$$\|\rho\|_{L^2} \leq C_0,$$

and hence, we are allowed to select in (3.35)

$$\mathbf{M} = 2C_0 \quad (3.61)$$

and close the *a priori* assumption in (3.35).

It only remains to derive the bound of $\|c\|_{W_n^{2, \frac{3}{2}}}$. From (1.6), (3.57), (3.60), it follows that

$$\begin{aligned} \|\nabla^2 c\|_{L^{\frac{3}{2}}} &\leq C \|\Delta c\|_{L^{\frac{3}{2}}} \\ &\leq C \left\| \rho \frac{\partial f}{\partial c} \right\|_{L^{\frac{3}{2}}} + C \|\rho\mu\|_{L^{\frac{3}{2}}} \\ &\leq C. \end{aligned} \quad (3.62)$$

From (2.3) and (3.60), the same argument as (2.7) yields

$$\begin{aligned} \int c &= \frac{|\Omega|}{m_1} \int \rho(c)_\Omega = \frac{|\Omega|}{m_1} \int \rho c - \frac{|\Omega|}{m_1} \int \rho(c - (c)_\Omega) \\ &= \frac{|\Omega|m_2}{m_1} - \frac{|\Omega|}{m_1} \int \rho(c - (c)_\Omega) \\ &\leq C + C \|\rho\|_{L^{\frac{6}{5}}} \|\nabla c\|_{L^2} \\ &\leq C + C \|\nabla c\|_{L^2}, \end{aligned}$$

which implies

$$\|c\|_{L^1} \leq \|c - (c)_\Omega\|_{L^1} + \|(c)_\Omega\|_{L^1} \leq C + C \|\nabla c\|_{L^2}. \quad (3.63)$$

Then, (3.63) and (3.62) provide us the following estimate:

$$\|c\|_{W_n^{2, \frac{3}{2}}} \leq C. \quad (3.64)$$

In conclusion, the desired estimate (3.53) follows from (2.5), (3.48), (3.52), (3.60), and (3.64). The proof of Lemma 3.5 is completed. $\square$

Therefore, the proof of Proposition 3.1 and hence Theorem 1.1 is completed.

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