Semi-random Impossibilities of Condorcet Criterion

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Abstract

The *Condorcet criterion* (CC) is a classical and well-accepted criterion for voting. Unfortunately, it is incompatible with many other desiderata including *participation* (PAR), *half-way monotonicity* (HM), *Maskin monotonicity* (MM), and *strategy-proofness* (SP). Such incompatibilities are often known as impossibility theorems, and are proved by worst-case analysis. Previous work has investigated the likelihood for these impossibilities to occur under certain models, which are often criticized of being unrealistic.

We strengthen previous work by proving the first set of *semi-random impossibilities* for voting rules to satisfy CC and the more general, group versions of the four desiderata: for any sufficiently large number of voters n, any size of the group $1 \leq B \leq \sqrt{n}$, any voting rule r, and under a large class of *semi-random* models that include Impartial Culture, the likelihood for r to satisfy CC and PAR, CC and HM, CC and MM, or CC and SP is $1 - \Omega(\frac{B}{\sqrt{n}})$. This matches existing lower bounds for CC&PAR ($B = 1$) and CC&SP and CC&HM ($B \leq \sqrt{n}$), showing that many commonly-studied voting rules are already asymptotically optimal in such cases.

1 Introduction

The *Condorcet criterion* of voting (Condorcet 1785) is a classical desideratum that has *"nearly universal acceptance"* (Saari 1995, p. 46). It requires a voting rule to choose the *Condorcet winner*—the alternative who beats other alternatives in head-to-head competitions—whenever it exists.

Unfortunately, it is well-known that the Condorcet criterion (CC for short) is incompatible with many other desiderata (a.k.a. *axioms*) when the number of alternatives m is at least 3. Such incompatibilities are often called *impossibility theorems*. For example, no voting rule satisfies

- CC and *participation* (PAR for short, which requires that no voter has incentive to abstain from voting), when $m \geq 4$ (Moulin 1988);
- CC and *half-way monotonicity* (HM for short, which requires that no voter has incentive to reverse his/her vote (Sanver and Zwicker 2009));
- CC and *Maskin monotonicity* (MM for short, which requires that any voter raising the position of the winner

relative to other alternatives does not change the winner (Maskin 1999)), as a special case of the Muller-Satterthwaite theorem (1977); or

- CC and *strategy-proofness* (SP for short, which requires that no agent has incentive to lie), as a special case of the Gibbard-Satterthwaite theorem (1973; 1975).

The four combinations of axioms are therefore denoted by CC&PAR, CC&HM, CC&MM, and CC&SP, respectively. Proofs for these impossibility theorems are based on worst-case analysis, by identifying a single instance of violation. Therefore, they do not preclude the possibility that such violations are rare in practice. Indeed, if so, "*then one need not be unduly worried*" (Pattanaik 1978).

Studying how rare such impossibilities are in practice has been a popular and active field of research (Gehrlein and Lepelley 2011; Diss and Merlin 2021). Recently, the topic was investigated using *smoothed analysis* (Spielman and Teng 2009; Baumeister, Hogrebe, and Rothe 2020; Xia 2020), which can be viewed as a worst average-case analysis under *semi-random* models (Feige 2021), following the frequentists' principle: the likelihood of violation of axioms is estimated under an adversarially chosen (i.e., worst-case) distribution for the votes from a given set of distributions. For example, the likelihood for CC or PAR to be violated is $\Theta(\frac{1}{\sqrt{n}})$ for many voting rules for n voters, under a large class of semi-random models (Xia 2021b).

While this is good news, as violations vanish at a $\Theta(\frac{1}{\sqrt{n}})$ rate, they are not rare enough when the cost of violation is high. For example, if a violation of CC or PAR leads to a revote, whose social cost is $\Theta(n)$, then the expected social cost is $\Theta(\sqrt{n})$, which is non-negligible. As another example, if everyone complaints on social media about the violation and gets -1 utility every time when seeing a complaint, then the social cost can be as high as $\Theta(n^2)$, meaning that the expected social cost is $\Omega(n^{1.5})$, or in other words, $\Omega(\sqrt{n})$ *per person*. In such situations, voting rules with rarer violations are desirable.

But can any voting rule do better, and if so, by how much? The answer lies in the lower bound on the likelihood of violations (under all rules), or equivalently, the upper bound on the likelihood of satisfying the axioms. In this paper, we address this question for the four combinations of axioms involving CC mentioned earlier, by proving *semi-*

random impossibilities (Xia 2020) under a large class of semi-random models that are more general and realistic than the commonly-used i.i.d. uniform distribution, known as the *Impartial Culture (IC)*. Therefore, the research question of this paper can be phrased as:

What are the semi-random impossibilities of CC?

More precisely, we consider the more general, group versions of $X \in \{\text{CC\&PAR}, \text{CC\&HM}, \text{CC\&MM}, \text{CC\&SP}\}$. For any $B \geq 1$, any collection of votes P (called a *profile*), and any voting rule r, we let $X(r, P, B) = 1$ if r satisfies CC at P and no group of at most B voters in P can collaboratively violate X; otherwise $X(r, P, B) = 0$. Then, given a set of distributions Π over the votes and n agents, the *semi-random version of X* (Xia 2020, 2021b) is defined as:

$$\widetilde{X}_\Pi^{\min}(r, n, B) \triangleq \inf_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}} (X(r, P, B) = 1) \quad (1)$$

That is, $\widetilde{X}_\Pi^{\min}(r, n, B)$ is the worst-case (lower bound) on the probability for X to be 1 under the profile P generated from a vector $\vec{\pi}$ of n distributions in Π^n, one for each agent. Notice that while agents' votes are independently generated, their underlying distributions are adversarially chosen and can be different. A high $\widetilde{X}_\Pi^{\min}$ value is desirable, because it implies that the expected satisfaction of X is high even under the worst distribution $\vec{\pi} \in \Pi^n$.

1.1 Our Contributions

The main results of this paper are four semi-random impossibility theorems: for $X = $ CC&PAR (Theorem 1), CC&HM (Theorem 2), CC&MM (Theorem 3), or CC&SP (Theorem 4), any $m \geq 4$ ($m \geq 3$ for CC&MM and CC&SP), any sufficiently large n, any $1 \leq B \leq \sqrt{n}$, any voting rule r, and any Π satisfying certain conditions (Assumption 1),

$$\widetilde{X}_\Pi^{\min}(r, n, B) = 1 - \Omega\left(\frac{B}{\sqrt{n}}\right)$$

In other words, no voting rule can guarantee that X is violated with probability smaller than $\Omega(\frac{B}{\sqrt{n}})$. The results also imply that every additional member in the group (up to $\sqrt{n}$) roughly increase the likelihood of violation by $\Theta(\frac{1}{\sqrt{n}})$. Specifically, when $B = \Omega(\sqrt{n})$, the likelihood of violation does not vanish even in large elections ($n \rightarrow \infty$).

Our results match the lower bound for CC&PAR when $B = 1$ (Xia 2021b) and for CC&SP and CC&HM[1] for every $B \leq \sqrt{n}$ (Xia 2022), which are achieved by many voting rules that satisfies CC, such as Copeland, maximin, ranked pairs, and Schulze—in contrast, for CC&PAR, positional scoring rules and STV are much worse, as their satisfactions are $1 - \Theta(1)$ (Xia 2021b).

Good or bad news? On the positive side, it is the first time, to the best of our knowledge, that the optimal likelihood of avoiding impossibility theorems that involve CC is known. It is surprising to us that many existing rules are already optimal. On the negative side, the tightness suggests that there is

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[1]This is because an upper bound for CC&SP is also an upper bound of CC&HM. We thank Dominik Peters for pointing this out to us.

little room for improvement, which can be a critical concern when the cost of violation is high. After all, we believe that these semi-random impossibility theorems are useful and informative in theory, as they reveal limitations of the optimal rules, as well as in practice, for the decision maker to choose the voting rule and decide the policies when a violation of axioms occurs.

Generality and limitations. The generality of the semi-random impossibilities proved in this paper largely depends on the restrictiveness of Assumption 1. We defer the formal technical definition and discussions to Section 2, and feel that the assumption is mild in practice, because it is satisfied by many single-agent preference models, including IC, the single-agent Mallows and single-agent Plackett-Luce with bounded parameters (Xia 2020). As a result, the $1 - \Omega(\frac{B}{\sqrt{n}})$ upper bound naturally holds under IC (Corollary 1).

The major limitations are, first, the constant in $\Omega(\frac{B}{\sqrt{n}})$ may be exponentially large in m, though it does not depend on n, B, or r. Second, the semi-random model in this paper assumes that the votes are statistically independent (but not necessarily identically distributed). These are common limitations/assumptions in preference modeling, see, e.g., (Thurstone 1927; Berry, Levinsohn, and Pakes 1995; Train 2009). Addressing them may require breakthroughs in probability theory and are important and challenging directions.

Proof overview. The high-level idea is surprisingly simple: for each X studied in this paper, in step 1, we leverage existing proof of the (worst-case) impossibility theorem to identify sufficiently many profiles where X is violated. Then, in step 2, we prove that there exists $\vec{\pi} \in \Pi^n$ under which with $\Omega(\frac{B}{\sqrt{n}})$ probability, a profile falls in the set identified in step 1.

Nevertheless, the actual calculations are technical challenging due to the generality of r. In step 1, we introduce a *rotated template* by scaling up an existing proof diagram (e.g., (Peters 2019, Chapter 1)) to identify profiles where X is violated, and prove that there are sufficiently many such violation profiles by upper-bounding the number of times each of them is identified by the rotated template. Then in step 2, we use an averaging argument over all $n!$ permutations of a carefully chosen $\vec{\pi}$ to convert the problem to the likelihood about the histogram of profiles, which is then tackled by applying the point-wise concentration bound (Xia 2021a, Lemma 1).

The idea and techniques have the potential to leverage other (worst-case) impossibility theorems to their semi-random versions. See Section 4 for more discussions.

1.2 Related Work and Discussions

Condorcet criterion (CC) is satisfied by many commonly-studied voting rules. Prominent exceptions are positional scoring rules (Fishburn 1974b) and multi-round-score-based elimination rules, such as STV. Much previous work aimed at theoretically characterizing the *Condorcet efficiency*, which is the probability for the Condorcet winner to win conditioned on its existence (Fishburn 1974c,a; Paris 1975; Gehrlein and Fishburn 1978; Newenhizen 1992).

Participation (PAR) was introduced to study voting rules that avoid the *no-show paradox* (Fishburn and Brams 1983). Moulin (1988) proved that when $m \geq 4$ and $n \geq 25$, no voting rule satisfies CC and PAR simultaneously. The bound on n was characterized to be 12 by simplified, SAT-solver-based proofs (Brandt, Geist, and Peters 2017; Peters 2019). The likelihood of PAR satisfaction by popular voting rules under IC was investigated in a series of work as summarized by Gehrlein and Lepelley (2011, Chapter 4.2.2), and also more recently by Brandt, Hofbauer, and Strobel (2021).

Half-way monotonicity (HM) was introduced to study voting rules that avoid the *preference reversal paradox*, and was proved to be incompatible with CC (Sanver and Zwicker 2009). Peters (2017) used SAT solvers to characterized the number of voters under which the impossibility holds.

Maskin monotonicity (MM) was introduced to characterize Nash implementability (Maskin 1999). The Muller-Satterthwaite theorem (Muller and Satterthwaite 1977) establishes the equivalence between MM and SP in the worst-case sense: a voting rule satisfies MM if and only if it satisfies SP.

Strategy-proofness (SP) cannot be satisfied by any non-dictatorial and unanimous voting rules when $m \geq 3$, due to the Gibbard-Satterthwaite theorem (Gibbard 1973; Satterthwaite 1975). SP is stronger than HM, because the latter uses a special form of manipulation (by reversing the truthful vote). At a high-level, PAR can be viewed as a weak form of SP that prevents manipulation by abstention, though PAR is not weaker than SP by definition, because PAR reasons about elections of different sizes. A quantitative Gibbard-Satterthwaite theorem (under IC) was proved for $m = 3$ by Friedgut et al. (2011), and was subsequently developed in (Dobzinski and Procaccia 2008; Xia and Conitzer 2008; Isaksson, Kindler, and Mossel 2010), and the case for general m was resolved by Mossel and Racz (2015).

Semi-random CC&PAR, CC&HM, CC&MM, and CC&SP. We are not aware of any semi-random impossibility theorem about the satisfaction of CC&PAR, CC&HM, or CC&MM, even under IC. For SP, the quantitative Gibbard-Satterthwaite theorem by Mossel and Racz (2015) establishes an $1 - \Omega(\frac{1}{n^{67}})$ upper bound under IC for any voting rule that is sufficiently different from dictatorships. Therefore, the same bound holds for CC&SP for any rule that satisfies CC. The $1 - \Omega(\frac{B}{\sqrt{n}})$ upper bound for CC&SP in our Theorem 4 also applies to all CC rules, which is stronger than the special case of (Mossel and Racz 2015), because our bound is lower and works for every $B \leq \sqrt{n}$ under more general models.

For possibility results (i.e., lower bound for optimal rules), as discussed in Section 1.1, our results imply that the bounds are tight for CC&PAR (when $B = 1$) and for CC&SP (when $B \leq \sqrt{n}$). We conjecture that they are tight for other axioms studied in this paper with all $B \leq \sqrt{n}$.

Quantitative and semi-random impossibilities. There is a large body of literature on quantitative impossibility theorems in social choice under IC. For example, quantitative

versions of Arrow's impossibility theorem (Arrow 1963) were proved (Kalai 2002; Mossel 2012; Keller 2012; Mossel, Oleszkiewicz, and Sen 2013). In judgement aggregation, Nehama (2013) and Filmus et al. (2020) developed quantitative characterizations of AND-homomorphism as oligarchy, whose worst-case version was due to List and Pettit (2002, 2004). Xia (2020) proved a semi-random version of the ANR impossibility theorem on *anonymity* and *neutrality*, whose worst-case version was proved by Moulin (1983).

Other smoothed/semi-random results. Semi-random models have been widely adopted to analyze the performance of algorithms in practice in combinatorial optimization (Blum and Spencer 1995), mathematical programming (Spielman and Teng 2004), machine learning (Blum and Dunagan 2002), and algorithmic game theory (Chung et al. 2008; Psomas, Schvartzman, and Weinberg 2019; Boodaghians et al. 2020; Blum and Gölz 2021), etc. We refer the readers to recent surveys on semi-random models (Feige 2021) and general approaches beyond worst-case analysis (Roughgarden 2021). In addition to the work discussed above, semi-random/smoothed analysis has been applied to other social choice problems, e.g., likelihood of ties (Xia 2021a), complexity of winner determination (Xia and Zheng 2021), judgement aggregation (Liu and Xia 2022), and fair division (Bai et al. 2022).

2 Preliminaries

For any $q \in \mathbb{N}$, we let $[q] = \{1, \ldots, q\}$. Let $\mathcal{A} = [m]$ denote the set of $m \geq 3$ *alternatives*. Let $\mathcal{L}(\mathcal{A})$ denote the set of all linear orders over $\mathcal{A}$. Let $n \in \mathbb{N}$ denote the number of voters (agents). Each voter uses a linear order $R \in \mathcal{L}(\mathcal{A})$ to represent his or her preferences, called a *vote*, where $a \succ_R b$ means that the agent prefers alternative a to alternative b. The vector of n voters' votes, denoted by P, is called a *(preference) profile*, sometimes called an n-profile. A voting rule r maps any profile to a single winner. For any profile P, let $\text{Hist}(P) \in \mathbb{R}_{\geq 0}^{m!}$ denote the anonymized version of P, also called the *histogram* of P, which contains the total number of each linear order in $\mathcal{L}(\mathcal{A})$ according to P.

Weighted majority graphs and the Condorcet winner. For any profile P and any pair of alternatives (a, b), let $P[a \succ b]$ denote the number of votes in P where a is preferred to b. Let $\text{WMG}(P)$ denote the *weighted majority graph* of P, whose vertices are $\mathcal{A}$ and whose weight on edge $a \rightarrow b$ is $w_P(a, b) = P[a \succ b] - P[b \succ a]$. The *Condorcet winner* of a profile P is the alternative whose outgoing edges in $\text{WMG}(P)$ are positively weighted.

Axioms. All axioms studied in this paper are *per-profile axioms* (Xia 2020), each of which is modeled as a function X that maps a voting rule r, a profile P, and a group size $B \geq 1$ to $\{0, 1\}$, where 0 (respectively 1) means that r violates (respectively, satisfies) the axiom at P w.r.t. group size B. Then, the classical (worst-case) satisfaction of the axiom under r becomes $\min_P X(r, P)$.

For any voting rule, any profile P, and any $B \geq 1$, Condorcet criterion is modeled as a function CC such that $\text{CC}(r, P, B) = 1$ if and only if either (1) there is no Con-
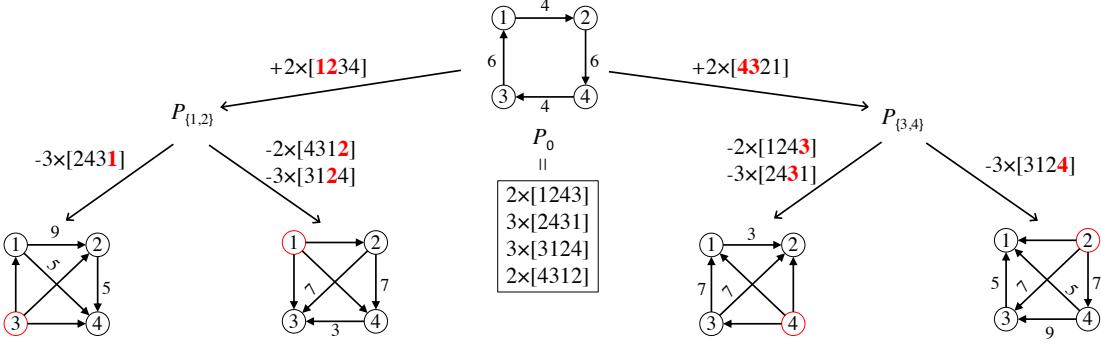
Figure 1: Proof diagram of CC&PAR impossibility (Peters 2019, Chapter 1). WMGs of the root and the leaves are shown (where all unweighted edges in the leaves have weight 1). Each edge represents a sequence of operations conditioned on the winner of the source profile being a highlighted alternative. Condorcet winners in the leaf nodes are highlighted.

dorcet winner under P, or (2) the Condorcet winner is the winner of P under r. Notice that technically CC does not depend on B, which is included for notational consistency. Participation is modeled by a function PAR such that $\text{PAR}(r, P, B) = 1$ if and only if no group of at most B voters have incentive to abstain from voting; otherwise $\text{PAR}(r, P, B) = 0$. Formally, $\text{PAR}(r, P, B) = 0$ if and only if there exists a subvector P' of P such that $|P'| \leq B$ and for all $R \in P'$, $r(P') \succ_R r(P)$. The simultaneous satisfaction of CC and PAR is denoted by CC&PAR, such that $\text{CC\&PAR}(r, P, B) = 1$ if and only if $\text{CC}(r, P, B) = 1$ and $\text{PAR}(r, P, B) = 1$.

Due to the space constraint, we focus on presenting the result about CC&PAR in the main text. The formal definitions of other axioms, i.e., half-way monotonicity (HM), Maskin monotonicity (MM), and strategy-proofness (SP), and the simultaneous satisfaction of CC and them, i.e., CC&HM, CC&MM, and CC&SP, are deferred to Appendix **??** of the full version of this paper (Xia 2023).

Proof diagram of the CC&PAR **impossibility.** We briefly recall the proof diagram by Peters (2019, Chapter 1) in Figure 1 to show that when $m = 4$, no voting rule r satisfies CC&PAR, which will play an important role in our proofs later. Each edge represents a sequence of operations, conditioned on the winner of the source profile being highlighted. Peters (2019, Chapter 1) proved that a violation of CC and/or PAR exists in the diagram. Take the leftmost branch for example. If $r(P_0) \in \{1, 2\}$, then two copies of [1234] are added one by one. If the winner is no longer 1 or 2 during this process, then PAR is violated. Otherwise, starting from $P_{\{1,2\}}$, if $r(P_{\{1,2\}}) = \{1\}$, then three votes of [2431] are subtracted one by one. If the winner is not 1 at any point, then PAR is violated. However, 3 is the Condorcet winner in the leaf node, which means that CC is violated if PAR has not been violated on the leftmost branch so far.

Semi-random satisfaction of axioms. Given a per-profile axiom X, a set Π of distributions over $\mathcal{L}(\mathcal{A})$, a voting rule r, $n \in \mathbb{N}$, and a group size B, the *semi-random satisfaction of X under r with n agents*, denoted by $\widetilde{X}_{\Pi}^{\min}(r, n, B)$, is defined in Equation (1) in the Introduction. We note that the

"min" in the superscript means that the adversary aims at minimizing the satisfaction of X.

The semi-random analysis generalizes the classical quantitative analysis in social choice (under IC). To see this, let π_{uni} denote the uniform distribution over $\mathcal{L}(\mathcal{A})$ and let $\Pi_{\text{IC}} = \{\pi_{\text{uni}}\}$. Then, $\widetilde{X}_{\Pi_{\text{IC}}}^{\min}$ becomes the likelihood of satisfaction of X under IC. Throughout the paper, we make the following assumptions on Π.

Assumption 1 *We assume that Π is*

- strictly positive, *which means that there exists $\epsilon > 0$ such that for every $\pi \in \Pi$ and every $R \in \mathcal{L}(\mathcal{A})$, $\pi(R) \geq \epsilon$;*
- closed, *which means that Π is a closed subset of the probability simplex in $\mathbb{R}^{m!}$; and*
- $\pi_{uni} \in CH(\Pi)$, *where $CH(\Pi)$ is the convex hull of Π.*

The first part of Assumption 1 requires that no distribution in Π is too "deterministic". The second part is a mild technical assumption. The first two parts guarantee that the semi-random analysis using Π is sufficiently different from the worst-case analysis. The third part requires that the uniform distribution π_{uni} is in the convex hull of Π, though π_{uni} itself may not be in Π.

We believe that Assumption 1 is mild, because it is satisfied by many classical models for preferences. For example, it is satisfied by IC, which corresponds to $\Pi = \{\pi_{\text{uni}}\}$, and the models in the following example, which is taken from (Xia 2020).

Example 1 *In the* single-agent Mallows with bounded dispersion, *given $\underline{\varphi} > 0$, each distribution is parameterized by a central ranking $W \in \mathcal{L}(\mathcal{A})$ and a dispersion $\varphi \in [\underline{\varphi}, 1]$, such that the probability for $R \in \mathcal{L}(\mathcal{A})$ is proportional to $\varphi^{KT(R,W)}$, where $KT(R, W)$ is the total number of pairwise differences between R and W, i.e., the* Kendall-Tau distance.

In the single-agent Plackett-Luce with bounded parameters, *given $\underline{\varphi} > 0$, each distribution is parameterized by a vector $\vec{\theta} \in [\underline{\varphi}, 1]^m$ such that $\vec{\theta} \cdot \vec{1} = 1$. The probability for $R = a_1 \succ a_2 \succ \cdots \succ a_m$ is $\prod_{i=1}^{m-1} (\theta_{a_i} / \sum_{\ell=i}^{m} \theta_{a_\ell})$.*

If $\underline{\varphi} = 0$ is allowed in Example 1, then the semi-random analysis degenerates to worst-case analysis, which trivializes the question.
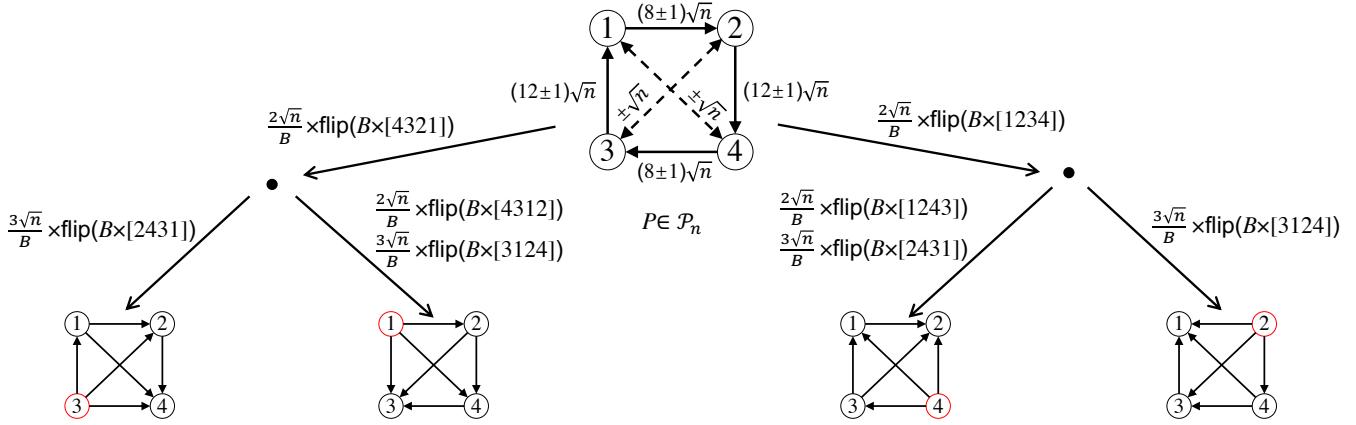
Figure 2: The violation template. "$\pm$" represents the upper and lower bounds on the weight. $\ell \times \text{flip}(B \times R)$ represents ℓ operations, each of which flips B votes that are R. In the leaves, weights are not shown and Condorcet winners are highlighted.

3 Semi-random Impossibility of CC and PAR

Theorem 1 (CC+Participation) *For any fixed $m \geq 4$, any Π that satisfies Assumption 1, any voting rule r, any $n \geq 12$, and any $1 \leq B \leq \sqrt{n}$,*

$$\widetilde{\text{CC\&PAR}}_{\Pi}^{\min}(r, n, B) = 1 - \Omega\left(\frac{B}{\sqrt{n}}\right)$$

The theorem is more general than its classical, worst-case counterpart, as the likelihood is strictly smaller than 1. It is also more general than its quantitative counterparts (under IC), because the latter is a special case of the former, where $\Pi = \{\pi_{\text{uni}}\}$, as discussed in the last section.

3.1 Proof sketch of Theorem 1

Overview. To illustrate the idea, we make the following assumptions in the proof sketch: (i) $m = 4$, (ii) $\sqrt{n}$ is an integer, (iii) $B \mid \sqrt{n}$, and (iv) $m! \mid n$. In addition, it suffices to prove the theorem when $n \geq 12$ and is sufficiently large, because the (worst-case) impossibility theorem holds for every $n \geq 12$ (Brandt, Geist, and Peters 2017; Peters 2019). The proof for the general case can be found in Appendix **??**.

Let PAR_B denote the group version of participation with size B. Instead of upper-bounding $\widetilde{\text{CC\&PAR}}_{\Pi}^{\min}$, we will lower-bound its complement $\neg(\widetilde{\text{CC\&PAR}})_{\Pi}^{\max}$ as $\Omega(\frac{B}{\sqrt{n}})$, which is the max-semi-random likelihood for CC or PAR_B to be violated and is defined similarly to $\widetilde{X}_{\Pi}^{\min}$ in (1), except that inf is replaced by sup. The theorem then follows after noticing

$$\widetilde{\text{CC\&PAR}}_{\Pi}^{\min}(r, n, B) = 1 - \neg(\widetilde{\text{CC\&PAR}})_{\Pi}^{\max}(r, n, B)$$

As discussed in Section 1.1, the proof proceeds in two steps. In step 1, we identify a set of n-profiles, denoted by $\mathcal{V}_{n,B}$, where CC or PAR_B is violated, and prove that $\mathcal{V}_{n,B}$ contains sufficiently many profiles. This will be achieved by first scaling the (worst-case) proof diagram in (Peters 2019, Chapter 1), i.e., Figure 1 in Section 2, by a factor of $\sqrt{n}$ to define a *violation template*, and then implementing it at profiles whose histograms are in an $O(\sqrt{n})$ neighborhood of

$\frac{n}{m!} \cdot \vec{1}$. Each implementation leads to a *violation tree*, which contains at least one violation of CC or PAR_B. Then, we upper-bound the number of violation trees any profile $P^* \in \mathcal{V}_{n,B}$ can be on, by considering the *rotated trees* generated by the *rotated template* rooted at P^*.

Then in step 2, we prove that there exists $\vec{\pi} \in \Pi^n$ so that the likelihood of $\mathcal{V}_{n,B}$ found in step 1 is lower-bounded by $\Omega(\frac{B}{\sqrt{n}})$. This is achieved by starting with a $\vec{\pi} \in \Pi^n$ such that $\sum_{j=1}^{n} \pi_j$ is $O(1)$ away from $\frac{n}{m!} \cdot \vec{1}$, and then considering the sum of likelihood of $\mathcal{V}_{n,B}$ under all $n!$ permutations of components in $\vec{\pi}$. This converts the likelihood of $\mathcal{V}_{n,B}$ to the likelihood about the histogram of a randomly-generated profile. Finally, we apply the point-wise concentration bound (Xia 2021a, Lemma 1) to derive the desired lower bound.

Step 1. We first formally define the violation template illustrated in Figure 2.

Definition 1 (Violation template) *Given any n-profile P with at least $7\sqrt{n}$ copies of $\mathcal{L}(\mathcal{A})$ and any $1 \leq B \leq \sqrt{n}$, a violation template is defined by modifying the proof diagram (Figure 1) as follows, where $Rev(R)$ denote the reverse ranking of R, also called the flip of R:*

- *every $+R$ operation on an edge in Figure 1 is replaced by a sequence of $\frac{\sqrt{n}}{B}$ operations, each of which flips $B \times Rev(R)$ votes and is denoted by $\text{flip}(B \times Rev(R))$;*
- *every $-R$ operation on an edge in Figure 1 is replaced by a sequence of $\frac{\sqrt{n}}{B}$ operations, each of which flips $B \times R$ votes and is denoted by $\text{flip}(B \times R)$.*

The violation template will be implemented multiple times, by letting its root to be n-profiles whose WMGs are similar to the WMG at the root in Figure 1 (scaled by a factor of $\sqrt{n}$) and whose histograms are close to $\frac{n}{m!} \cdot \vec{1}$. Formally, we define the set of such profiles P, denoted by $\mathcal{P}_n$, as follows. Let $w(e)$ denote the weight on edge e in the root of Figure 1.

Definition 2 *Let $\mathcal{P}_n$ denote the set of n-profiles P such that*

- *for every edge $e \in [4] \times [4]$, $|w_P(e) - \sqrt{n} \cdot w(e)| \leq \sqrt{n}$;*
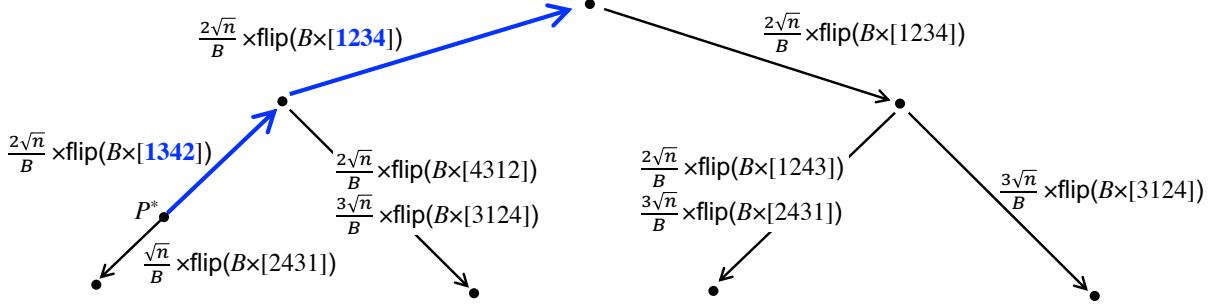
Figure 3: A rotated template rooted at P^*. Reversed edges and rankings are highlighted.

- $|Hist(P) - \frac{n}{m!} \cdot \vec{1}|_\infty \leq 4\sqrt{n}$.

We note that any flip operation in the violation template does not completely specify the set of voters whose votes should be flipped (except that their votes must be the same as the designated ranking). Consequently, different combinations of votes when implementing the violation template lead to different *violation trees*, formally defined as follows.

Definition 3 (Violation trees) *For any $P \in \mathcal{P}_n$ and any $1 \leq B \leq \sqrt{n}$, a violation tree is a tree of $\frac{20\sqrt{n}}{B} + 1$ profiles obtained from implementing the violation template rooted at P. Let $\mathcal{T}_{P,B}$ denote the set of all violation trees rooted at P.*

For example, in a violation tree rooted at $P \in \mathcal{P}_n$, the leftmost branch of Figure 2 consists of profiles $P, P_1, \ldots, P_{\frac{2\sqrt{n}}{B}}, \ldots, P_1^*, \ldots, P_{\frac{3\sqrt{n}}{B}}^*$, such that each P_j is obtained from P_{j-1} by flipping B votes of $[4321]$ (where $P_0 = P$); and each P_j^* is obtained from P_{j-1}^* by flipping B votes of $[2431]$ (where $P_0^* = P_{\frac{2\sqrt{n}}{B}}$). The following claim lower-bounds the size of $\mathcal{T}_{P,B}$. All missing proofs can be found in the appendix.

Claim 1 *For any $P \in \mathcal{P}_n$ and any $1 \leq B \leq \sqrt{n}$, $|\mathcal{T}_{P,B}| = \Omega((\frac{n}{m! \sqrt[B]{B!}})^{20\sqrt{n}})$.*

The next claim states that each violation tree contains a violation of CC or PAR_B.

Claim 2 *For every $P \in \mathcal{P}_n$ and every $T \in \mathcal{T}_{P,B}$, CC or PAR_B is violated in T.*

Let $\mathcal{V}_{n,B}$ denote the set of all profiles on violation trees $\bigcup_{P \in \mathcal{P}_n} \mathcal{T}_{P,B}$, where CC or PAR_B is violated. The next claim upper-bounds the number of violation trees that each profile in $\mathcal{V}_{n,B}$ can possibly be on.

Claim 3 *Every $P^* \in \mathcal{V}_{n,B}$ is on no more than $O(\frac{\sqrt{n}}{B} \cdot (\frac{n}{m! \sqrt[B]{B!}})^{20\sqrt{n}})$ violation trees rooted in $\mathcal{P}_n$.*

Proof sketch. The proof is done by defining a *rotated template* rooted at every node V in the violation template, which reverses all edges along the path from the root to V. That is, an edge $V_1 \to V_2$ along the path that flips $B \times R$ becomes $V_1 \leftarrow V_2$ in the rotated template that flips $B \times \mathrm{Rev}(R)$. Consequently, the rotated template is a diagram rooted at V. Because each violation template has $\frac{20\sqrt{n}}{B} + 1$ nodes, there

are $\frac{20\sqrt{n}}{B} + 1$ rotated templates. Figure 3 illustrates a rotated template rooted at a node in the leftmost branch. Then, Claim 3 is proved by upper-bounding the number of *rotated trees* obtained by applying the rotated template, which provides an upper bound on the violation trees rooted in $\mathcal{P}_n$ that contains P^*. □

We are now ready to lower-bound $|\mathcal{V}_{n,B}|$. To this end, we count the number of (profile, violation tree) pairs, denoted by (P, T), where T is rooted at a profile in $\mathcal{P}_n$, P is on T, and CC and/or PAR_B is violated at P. By Claim 1 and Claim 2, the total number of such (profile, violation tree) pairs is at least $|\mathcal{P}_n| \times \Omega\left(\left(\frac{n}{m! \sqrt[B]{B!}}\right)^{20\sqrt{n}}\right)$. By Claim 3, the total number of such (profile, violation tree) pairs is at most $|\mathcal{V}_{n,B}| \times O\left(\frac{\sqrt{n}}{B} \cdot \left(\frac{n}{m! \sqrt[B]{B!}}\right)^{20\sqrt{n}}\right)$. Therefore,

$$\frac{|\mathcal{V}_{n,B}|}{|\mathcal{P}_n|} \geq \frac{\Omega\left(\left(\frac{n}{m! \sqrt[B]{B!}}\right)^{20\sqrt{n}}\right)}{O\left(\frac{\sqrt{n}}{B} \cdot \left(\frac{n}{m! \sqrt[B]{B!}}\right)^{20\sqrt{n}}\right)} = \Omega\left(\frac{B}{\sqrt{n}}\right) \quad (2)$$

Step 2. Let $\vec{\pi} \in \Pi^n$ be such that $\sum_{j=1}^n \pi_j$ is $O(1)$ away from $\frac{n}{m!} \cdot \vec{1}$, which can be defined by rounding as shown in (Xia 2021a). Let $\mathcal{S}_n$ denote the set of all permutations over $[n]$. For any permutation $\eta \in \mathcal{S}_n$, let $\eta(\vec{\pi})$ denote the vector of distributions where the indices are permuted according to η. That is, $\eta(\vec{\pi}) = (\pi_{\eta(1)}, \ldots, \pi_{\eta(n)})$. We prove that there exists a permutation η over $[n]$, such that

$$\Pr_{P \sim \eta(\vec{\pi})}(P \in \mathcal{V}_{n,B}) = \Omega(\frac{B}{\sqrt{n}}) \quad (3)$$

It suffices to prove that the sum of the left hand side of (3) for all $\eta \in \mathcal{S}_n$ is at least $n!$ times the right hand side of (3), that is,

$$\sum_{\eta \in \mathcal{S}_n} \Pr_{P \sim \eta(\vec{\pi})}(P \in \mathcal{V}_{n,B}) = \Omega(n! \cdot \frac{B}{\sqrt{n}}) \quad (4)$$

Nevertheless, (4) is still hard to prove due to the lack of information about the profiles in $\mathcal{V}_{n,B}$. The key insight of our proof is to convert the left hand side of (4) to probabilities for the histogram of P to be the histograms of profiles in

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$\mathcal{V}_{n,B}$. Notice that the left hand side of (4) is equivalent to

$$\sum_{\eta \in \mathcal{S}_n} \sum_{P^* \in \mathcal{V}_{n,B}} \Pr_{P \sim \eta(\vec{\pi})}(P = P^*)$$

$$= \sum_{P^* \in \mathcal{V}_{n,B}} \sum_{\eta \in \mathcal{S}_n} \Pr_{P \sim \vec{\pi}}(P = \eta^{-1}(P^*))$$

$$= \sum_{P^* \in \mathcal{V}_{n,B}} \prod_{R \in \mathcal{L}(\mathcal{A})} (n_R^*!) \times$$
$$\Pr_{P \sim \vec{\pi}}(\text{Hist}(P) = \text{Hist}(P^*)), \qquad (5)$$

where n_R^* is the number of R votes in P^*. (5) follows the following claim, whose proof can be found in Appendix **??**.

Claim 4 *For any $P^* \in \mathcal{V}_{n,B}$,*

$$\sum_{\eta \in \mathcal{S}_n} \Pr_{P \sim \vec{\pi}}(P = \eta^{-1}(P^*)) =$$

$$\prod_{R \in \mathcal{L}(\mathcal{A})} (n_R^*!) \times \Pr_{P \sim \vec{\pi}}(Hist(P) = Hist(P^*))$$

Because $\text{Hist}(P^*)$ is $7\sqrt{n}$ away from $\frac{n}{m!} \cdot \vec{1}$, we have $\prod_{R \in \mathcal{L}(\mathcal{A})} (n_R^*!) = \Theta\left(\left(\frac{n}{m!}\right)!\right)$. By the point-wise concentration lemma (Xia 2021a, Lemma 1), the likelihood for the histogram of P to be any $\vec{x}$ in an $O(\sqrt{n})$ neighborhood of $\sum_{j=1}^n \pi_j$ is $\Omega(n^{(1-m!)/2})$. Recall that $\sum_{j=1}^n \pi_j$ is $O(1)$ away from $\frac{n}{m!} \cdot \vec{1}$. Therefore, for every $P^* \in \mathcal{V}_{n,B}$,

$$\Pr_{P \sim \vec{\pi}}(\text{Hist}(P) = \text{Hist}(P^*)) = \Omega(n^{(1-m!)/2}) \quad (6)$$

Combining (2), (5), and (6), the left hand side of (4) becomes

$$\Omega\left(|\mathcal{V}_{n,B}| \cdot \left(\frac{n}{m!}\right)! \cdot n^{(1-m!)/2}\right)$$
$$= \Omega\left(\frac{B}{\sqrt{n}} \cdot \left(\frac{n}{m!}\right)! \cdot |\mathcal{P}_n| \cdot n^{(1-m!)/2}\right) \qquad (7)$$

Next, we lower-bound $|\mathcal{P}_n|$ in the following claim, whose proof is in Appendix **??**.

Claim 5 $|\mathcal{P}_n| = \Omega(\frac{n!}{(\frac{n}{m!})!} \cdot n^{(m!-1)/2})$.

Finally, (4) follows after (7) and Claim 5, which completes the proof for the special case of Theorem 1.

4 Other Semi-random Impossibilities

The proof of Theorem 1 can leverage any proof diagram like Figure 1, that has the following three high-level features:

1. The diagram consists of constantly many nodes.
2. The diagram works all slightly perturbed root profiles.
3. X is violated in each violation tree (scaled by $\sqrt{n}$).

Therefore, any existing proof diagram for CC&PAR, e.g., the one used in (Brandt, Geist, and Peters 2017), can be used to prove Theorem 1. We chose the diagram in (Peters 2019, Chapter 1) for its simplicity.

In this section, we prove semi-random impossibilities for the other three combinations of axioms, i.e., CC&HM, CC&MM, and CC&SP, by leveraging existing proof diagrams that have the three aforementioned features.

Theorem 2 (CC+half-way monotonicity) *For any fixed $m \geq 4$, any Π that satisfies Assumption 1, any voting rule r, any $n \geq 24$, and any $1 \leq B \leq \sqrt{n}$,*
$$\widetilde{\text{CC\&HM}}_{\Pi}^{\min}(r, n, B) = 1 - \Omega\left(\frac{B}{\sqrt{n}}\right)$$

The proof leverages the same diagram (Peters 2019, Chapter 1) as in the proof of Theorem. The only difference is to verify that it has the third feature above. The full proof can be found in Appendix **??**.

Theorem 3 (CC+Maskin monotonicity) *For any fixed $m \geq 3$, any Π that satisfies Assumption 1, any voting rule r, any $n \in \mathbb{N}$, and any $1 \leq B \leq \sqrt{n}$,*
$$\widetilde{\text{CC\&MM}}_{\Pi}^{\min}(r, n, B) = 1 - \Omega\left(\frac{B}{\sqrt{n}}\right)$$

The theorem is proved by leveraging a simple proof diagram rooted at profiles that have a (Condorcet) cycle over 3 alternatives in the WMG, where votes are changed to make certain alternatives the Condorcet winner in the leaves. The full proof can be found in Appendix **??**.

Theorem 4 (CC+strategy-proofness) *For any fixed $m \geq 3$, any Π that satisfies Assumption 1, any voting rule r, any $n \in \mathbb{N}$, and any $1 \leq B \leq \sqrt{n}$,*
$$\widetilde{\text{CC\&SP}}_{\Pi}^{\min}(r, n, B) = 1 - \Omega\left(\frac{B}{\sqrt{n}}\right)$$

When $m \geq 4$, Theorem 4 follows after Theorem 2, as SP is stronger than HM. The proof (for all $m \geq 3$) uses the same diagram for Theorem 3 and can be found in Appendix **??**.

Recall that IC corresponds to $\Pi = \{\pi_{\text{uni}}\}$. Therefore, all semi-random impossibilities in this paper hold for IC.

Corollary 1 (Quantitative Impossibilities under IC)
For any $X \in \{\text{CC\&PAR}, \text{CC\&HM}, \text{CC\&MM}, \text{CC\&SP}\}$, any $m \geq 4$ ($m \geq 3$ for CC&MM and CC&SP), any voting rule r, any sufficiently large $n \in \mathbb{N}$, and any $1 \leq B \leq \sqrt{n}$,
$$\Pr_{P \sim IC} X(r, P, B) = 1 - \Omega\left(\frac{B}{\sqrt{n}}\right)$$

Corollary 1 also implies that for any voting rule that satisfies CC, the likelihood for r to satisfy PAR, HM, MM, or SP, respectively, is $\Omega\left(\frac{B}{\sqrt{n}}\right)$ under IC.

5 Summary and Future Work

We prove the first set of semi-random impossibility results involving CC, showing that many existing voting rules are already optimal for CC&PAR (for $B = 1$) and CC&SP (for every $B \leq \sqrt{n}$). The proof technique has potential to strengthen other worst-case impossibilities to their semi-random variants. For future work, we conjecture that all bounds for the axioms are tight and can be achieved by many rules that satisfy CC. How to strengthen the theorems by studying variable m and allowing ϵ to depend on n are natural open questions. Other promising directions include addressing the limitations discussed in Section 1.2 and proving semi-random variants of other worst-case impossibility results, such as Arrow's, Gibbard-Satterthwait (for non-CC rules), and various impossibility theorems in judgement aggregation. The proof technique developed in this paper (see Section 4) does not seem to be directly applicable, because existing proofs use diagrams that contains $\Theta(n)$ nodes. Studying the empirical likelihood of the impossibility results in practice, for example on Preflib data (Mattei and Walsh 2013) and considering probabilistic versions of domain restrictions are promising future directions as well.

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