GLOBAL WELL-POSEDNESS AND EXPONENTIAL DECAY FOR THE INHOMOGENEOUS NAVIER-STOKES EQUATIONS WITH LOGARITHMICAL HYPER-DISSIPATION

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Dedicated to Professor Constantine Dafermos on the Occasion of His 80th Birthday

ABSTRACT. We consider the Cauchy problem for the inhomogeneous incompressible logarithmical hyper-dissipative Navier-Stokes equations in higher dimensions. By means of the Littlewood-Paley techniques and new ideas, we establish the existence and uniqueness of the global strong solution with vacuum over the whole space $\mathbb{R}^n$. Moreover, we also obtain the exponential decay-in-time of the strong solution. Our result holds without any smallness on the initial data and the initial density is allowed to have vacuum.

1. INTRODUCTION

This paper is concerned with the unique global strong solution with vacuum to the generalized inhomogeneous incompressible Navier-Stokes equations of the form:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \qquad x \in \mathbb{R}^n,\, t > 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \mathcal{L}^2 u + \nabla p = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where $\rho = \rho(x,t)$ denotes the density, $u = u(x,t) \in \mathbb{R}^n$ the fluid velocity and $p(x,t)$ the scalar pressure; $\mathcal{L}$ is multiplier operator with the symbol $\frac{|\xi|^{\frac{1}{2}+\frac{n}{4}}}{g(\xi)}$, namely

$$\widehat{\mathcal{L}u}(\xi) = \frac{|\xi|^{\frac{1}{2}+\frac{n}{4}}}{g(\xi)} \widehat{u}(\xi), \tag{1.2}$$

where $g = g(\xi) > 0$ is a non-decreasing, radially symmetric function. We consider the Cauchy problem of (1.1) with (ρ, u) vanishing at infinity and satisfying the following initial condition:

$$\rho(x,0) = \rho_0(x), \quad \rho u(x,0) = \rho_0 u_0(x), \tag{1.3}$$

where $\rho_0(x)$ and $u_0(x)$ are the prescribed initial values for the density and velocity such that $\nabla \cdot u_0 = 0$.

When $\mathcal{L} = \sqrt{-\Delta}$, the system (1.1) becomes the standard inhomogeneous incompressible Navier-Stokes equations, which can describe the motion of two miscible and incompressible fluids with different densities and can also describe the motion of the fluid containing a melted substance. For detailed derivation and physical meaning of this system, we refer

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to [23]. On account of the physical importance and the mathematical challenges, many physicists and mathematicians have investigated the standard inhomogeneous incompressible Navier-Stokes equations. Let us recall some known results for this system. When the initial density is strictly positive, Kazhikov [19] proved that the system has at least one global weak solution in the energy space. Later, Antontesv-Kazhikov-Monakhov [3] and Ladyzhenskaya-Solonnikov [20] gave the first result on the local existence and uniqueness of strong solutions, while globally defined in two-dimensional case. Similar results were established in a series of works; see for example [1,2,6,9–12,25,26]. However, for the initial data that permits the region of vacuum, the problem becomes much more complicated, especially the higher regularity is difficult to derive since $\partial_t u$ in the momentum equations is multiplied by ρ possibly vanishing in some region. Simon [27] first proved the global existence of weak solutions with finite energy, which was later extended later by Lions [23] to the case of density-dependent viscosity. Under the initial compatibility assumption, Choe-Kim [7] successfully established the local existence of the strong solution in dimensions three, which was later improved by Craig-Huang-Wang [8] for global strong small solutions (see [15,17,33] for the case of density-dependent viscosity). The global existence of strong solution with the general initial data in dimension two was proved by [16, 24] for the initial-boundary value problem and the Cauchy problem. However, the global existence of strong or smooth solutions with general initial data in higher dimensions is full of challenges and remains an outstanding open problem. As a matter of fact, one notable difficulty is that the Laplacian dissipation is insufficient to control the nonlinearity when applying the standard techniques to establish global a priori bounds.

When the hyper-dissipation (1.2) is considered, the global strong solutions to the Navier-Stokes equations (1.1) have been studied in the literature on both homogeneous and inhomogeneous fluids. When ρ is a constant, the system (1.1) becomes the homogeneous incompressible Navier-Stokes equations, which admit a unique global smooth solution as long as $g(\xi) \equiv 1$ in (1.2). This result dates back to Lions's book [22] (see also [18, 31]). In Barbato-Morandin-Romito [5] and Tao [28] the global regularity of the Navier-Stokes equations with logarithmically supercritical hyper-dissipation was obtained. Recently, several works are devoted to generalizing these results of [5, 28] to the inhomogeneous case. More precisely, Fang-Zi [13] established the global well-posedness for the system (1.1) with $g(\xi) \equiv 1$ by using the arguments in [10]. Following the work [13], Han-Wei [14] attempted to prove the corresponding logarithmically improved result of the system (1.1) with $g(\xi) = \ln^{\frac{1}{4}}(e + |\xi|^2)$. It should be noted that both [13] and [14] require the restriction that the initial density ρ_0 is bounded away from zero, which implies that the density cannot contain vacuum state. Very recently, in Wang-Ye [30] we established the unique global strong solution with vacuum to the Cauchy problem of system (1.1) under the assumption $g(\xi) \equiv 1$ with $n \geq 3$. Moreover, the corresponding strong solution admits the exponential decay-in-time property. The goal of this paper is to improve the global existence result of [30] by reducing the dissipation $(-\Delta)^{\frac{1}{2}+\frac{n}{4}}$ through a logarithmic factor.

We first give the definition of weak and strong solutions to the system (1.1).

Definition 1.1. We call (ρ, u) a weak solution to the system (1.1) if (ρ, u) satisfies (1.1) in the sense of distributions. Moreover, a weak solution is called strong if all the derivatives involved in the system (1.1) are regular distributions and the system (1.1) holds almost everywhere in $\mathbb{R}^n \times (0, T)$.

Now our main result of this paper can be stated as follows.

Theorem 1.1. *Assume that $n \geq 3$ and the initial data (ρ_0, u_0) satisfies the following conditions:*

$$0 \leq \rho_0 \in L^{\frac{2n}{n+2}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \quad \nabla\rho_0 \in L^q(\mathbb{R}^n),$$

$$\nabla \cdot u_0 = 0, \quad \mathcal{L}u_0 \in L^2(\mathbb{R}^n), \quad \sqrt{\rho_0}u_0 \in L^2(\mathbb{R}^n),$$

with some $q > \frac{4n}{n+6}$. Let $g = g(\xi) > 0$ be a non-decreasing, radially symmetric function and satisfy

$$\int_1^\infty \frac{d\tau}{\tau g^4(\tau)} = \infty. \tag{1.4}$$

Then the Navier-Stokes system (1.1) has a unique global strong solution (ρ, u) satisfying, for any given $T > 0$ and for any $0 < \tau < T$,

$$\begin{cases} 0 \leq \rho \in L^\infty(0,T; L^{\frac{2n}{n+2}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)), \quad \nabla\rho \in L^\infty(0,T; L^q(\mathbb{R}^n)), \\ \sqrt{\rho}u, \ \mathcal{L}u \in L^\infty(0,T; L^2(\mathbb{R}^n)), \\ \sqrt{\rho}\partial_t u, \ \mathcal{L}^2 u \in L^\infty(\tau,T; L^2(\mathbb{R}^n)), \\ \partial_t\mathcal{L}u, \ \mathcal{L}^2 u \in L^2(\tau,T; L^2(\mathbb{R}^n)), \quad p \in L^\infty(\tau,T; H^1(\mathbb{R}^n)), \\ \mathcal{L}^2 u, \ \nabla p \in L^2(\tau,T; L^r(\mathbb{R}^n)), \quad \forall\, r \in \left(2, \frac{4n}{n-2}\right). \end{cases} \tag{1.5}$$

Moreover, there exists some positive constant γ depending only on $\|\rho_0\|_{L^{\frac{2n}{n+2}} \cap L^\infty}$ such that, for all $t \geq 1$,

$$\|\mathcal{L}u(t)\|_{L^2}^2 + \|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \|\mathcal{L}^2 u(t)\|_{L^2}^2 + \|p(t)\|_{H^1}^2 \leq C_0 e^{-\gamma t},$$

where C_0 depends only on $\|\rho_0\|_{L^{\frac{2n}{n+2}} \cap L^\infty}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$ and $\|\mathcal{L}u_0\|_{L^2}$.

Remark 1.1. Some typical examples of g, besides $g \equiv 1$, satisfying (1.4) include

$$g(\tau) = \left[\ln(1+\tau)\right]^{\frac{1}{4}};$$

$$g(\tau) = \left[\ln(1+\tau)\ln(1+\ln(1+\tau))\right]^{\frac{1}{4}};$$

$$g(\tau) = \left[\ln(1+\tau)\ln(1+\ln(1+\tau))\ln(1+\ln(1+\ln(1+\tau)))\right]^{\frac{1}{4}}.$$

We now explain the difficulties and strategy for the proof of Theorem 1.1. Since the local existence of strong solutions to the system (1.1) follows from the works in literature such as [7, 21, 30], our efforts are devoted to obtaining global a priori estimates on strong solutions in suitable higher-order norms. To this end, we may encounter several difficulties. The first one is that the density has no positive lower bound and the velocity has no smallness or compatibility conditions. Consequently, new ideas are needed to overcome these difficulties. First, thanks to the estimate on the density, we have the following observation: for $\varepsilon < \frac{n+2}{4}$,

$$\|\sqrt{\rho}u\|_{L^2} \leq \|\sqrt{\rho}\|_{L^{\frac{4n}{n+2-4\varepsilon}}} \|u\|_{L^{\frac{4n}{n-2+4\varepsilon}}} \leq C\|\rho_0\|_{L^{\frac{2n}{n+2}} \cap L^\infty}^{\frac{1}{2}} \|\mathcal{L}u\|_{L^2},$$

which implies that $\|\sqrt{\rho}u(t)\|_{L^2}^2$ decays with the rate of $e^{-\gamma t}$ for some $\gamma > 0$ depending only on $\|\rho_0\|_{L^{\frac{2n}{n+2}}\cap L^\infty}$. Unfortunately, the above energy estimate is insufficient to complete the proof of Theorem 1.1 as it is far from reaching the critical level

$$\int_0^t \|\Lambda^{1+\frac{n}{2}}u(\tau)\|_{L^2}^2\,d\tau < \infty. \tag{1.6}$$

In order to overcome this difficulty, the next natural step is to increase the regularity of u. More precisely, invoking the Littlewood-Paley technique, we are able to show the inequality of the following form

$$\frac{d}{dt}\|\mathcal{L}u(t)\|_{L^2}^2 + \|\sqrt{\rho}\partial_t u\|_{L^2}^2 \leq C\|\mathcal{L}u\|_{L^2}^2 g^4\left([e+\|\mathcal{L}u\|_{L^2}^2]^{\frac{1}{\sigma}}\right)\|\mathcal{L}u\|_{L^2}^2 + C\|\mathcal{L}u\|_{L^2}^4,$$

which along with (1.4) and the basic energy estimate yield the following key estimate:

$$\|\mathcal{L}u(t)\|_{L^2}^2 + \int_0^t (\|\mathcal{L}^2u(\tau)\|_{L^2}^2 + \|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2)\,d\tau < \infty. \tag{1.7}$$

We point out that the similar arguments in dealing with the logarithmic reduction type case were also used for tackling other fluid dynamic equations; we refer the readers to our recent papers [29, 32]. Moreover, based on the proof of (1.7), we show the following exponential decay estimate that improves (1.7),

$$e^{\gamma t}\|\mathcal{L}u(t)\|_{L^2}^2 + \int_0^t (e^{\gamma t}\|\mathcal{L}^2u(\tau)\|_{L^2}^2 + e^{\gamma t}\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2)\,d\tau < \infty.$$

At this stage, we are still not able to show that (1.6) is valid via (1.7), because (1.7) includes a logarithmic reduction. To solve this difficulty, we appeal to derive the bound of $\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2$. However, it is hard to achieve this goal due to the absence of the compatibility condition for the initial velocity u_0. To overcome this difficulty, we need to derive the following crucial time-weighted estimate:

$$t\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \int_0^t \tau\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau \leq C_0, \quad \forall\, t \geq 0, \tag{1.8}$$

where the positive constant C_0 is independent of the initial data of $\sqrt{\rho}\partial_t u$. Consequently, (1.8) allows us to derive that for any $t > 0$,

$$t\|\mathcal{L}^2u\|_{L^2}^2 + t\|p(t)\|_{H^1}^2 \leq C_0,$$

$$\int_0^t (\tau\|\mathcal{L}^2u(\tau)\|_{L^r}^2 + \tau\|\nabla p(\tau)\|_{L^r}^2)\,d\tau \leq C_0, \quad \forall\, r \in \Big[2,\, \frac{4n}{n-2}\Big).$$

Moreover, for any $t \geq 1$, the following estimates hold true,

$$e^{\gamma t}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \int_1^t e^{\gamma\tau}\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau \leq C_0,$$

$$e^{\gamma t}\|\mathcal{L}^2u\|_{L^2}^2 + e^{\gamma t}\|p(t)\|_{H^1}^2 \leq C_0,$$

$$\int_1^t (e^{\gamma\tau}\|\mathcal{L}^2u(\tau)\|_{L^r}^2 + e^{\gamma\tau}\|\nabla p(\tau)\|_{L^r}^2)\,d\tau \leq C_0, \quad \forall\, r \in \Big(2,\, \frac{4n}{n-2}\Big).$$

We remark that all these exponential decay-in-time estimates and the time-weighted estimate (1.8) allow us to derive the desired uniform-in-time bound of $\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau$.

Once this key bound is at our disposal, we can continue to complete the proof of Theorem 1.1.

The rest of the paper is organized as follows. In Section 2, we shall give the detailed proof of the main results in Theorem 1.1 through energy estimates. The Appendix will recall some basic information on the Besov spaces.

2. Energy Estimates and the Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. The proof consists of the local existence, basic energy estimates, a priori estimates that are uniform in time, exponential decay estimates, gradient estimates, and uniqueness.

We shall use C to denote a generic positive constant that may change from line to line. For any two quantities A and B, we use $A \approx B$ to denote the inequality $C^{-1}B \leq A \leq CB$ for a generic positive constant C.

2.1. **Local existence and basic energy estimates.** Inspired by the previous works [7,21], one may obtain the local existence and uniqueness of strong solution, and we omit the proof.

Lemma 2.1 (Local strong solution)**.** *Under the conditions in Theorem 1.1, there exists a small time T^* and a unique strong solution $(\rho,\, u)$ to the system (1.1) in $\mathbb{R}^n \times (0,\, T^*)$ satisfying (1.5).*

With the local well-posedness at hand, it suffices to establish a priori estimates for strong solutions for any given $t > 0$. We begin with the basic energy estimates.

Lemma 2.2. *Under the assumptions of Theorem 1.1, the solution (ρ, u) of the system (1.1) admits the following bound for any $t \geq 0$,*

$$e^{\gamma t}\|\sqrt{\rho}u(t)\|_{L^2}^2 + \int_0^t e^{\gamma\tau}\|\mathcal{L}u(\tau)\|_{L^2}^2\, d\tau \leq \|\sqrt{\rho_0}u_0\|_{L^2}^2, \tag{2.1}$$

$$\|\rho(t)\|_{L^{\frac{2n}{n+2}}\cap L^\infty} \leq \|\rho_0\|_{L^{\frac{2n}{n+2}}\cap L^\infty}, \tag{2.2}$$

where γ depends on $\|\rho_0\|_{L^{\frac{2n}{n+2}}\cap L^\infty}$.

Proof. First, the non-negativeness of ρ is a direct consequence of the maximum principle and $\rho_0 \geq 0$. Multiplying $(1.1)_1$ by $|\rho|^{p-2}\rho$ and integrating it over $\mathbb{R}^n$, one has

$$\frac{d}{dt}\|\rho(t)\|_{L^p} = 0,$$

which implies

$$\|\rho(t)\|_{L^p} \leq \|\rho_0\|_{L^p}.$$

Letting $p \to \infty$, it yields

$$\|\rho(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}.$$

Thus (2.2) follows. We multiply the equation $(1.1)_2$ by u, integrate it over $\mathbb{R}^n$ and use Plancherel's theorem to obtain

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u(t)\|_{L^2}^2 + \|\mathcal{L}u\|_{L^2}^2 = 0.$$

According to the assumptions on g (more precisely, g grows logarithmically), one may conclude that for any fixed $\varepsilon > 0$, there exists $C = C(\varepsilon)$ satisfying

$$g(\xi) \leq C|\xi|^{\varepsilon}. \tag{2.3}$$

As a matter of fact, throughout our arguments, $\varepsilon > 0$ can be arbitrarily small. In our proof, it actually needs to satisfy $\varepsilon < F(n)$, where the function $F(n) > 0$ depends only the space dimensions n. Without loss of generality, we may assume $\varepsilon < \frac{1}{8}$. Thanks to (2.3), it follows that

$$\begin{aligned}\|\Lambda^{\frac{1}{2}+\frac{n}{4}-\varepsilon}u\|_{L^2}^2 &= \int_{\mathbb{R}^n} \frac{g^2(\xi)}{|\xi|^{2\varepsilon}} \frac{|\xi|^{1+\frac{n}{2}}}{g^2(\xi)} |\widehat{u}(\xi)|^2 \, d\xi \\ &\leq C \int_{\mathbb{R}^n} \frac{|\xi|^{1+\frac{n}{2}}}{g^2(\xi)} |\widehat{u}(\xi)|^2 \, d\xi \\ &= C\|\mathcal{L}u\|_{L^2}^2,\end{aligned}$$

which implies that, for $\varepsilon < \frac{n+2}{4}$,

$$\begin{aligned}\|\sqrt{\rho}u\|_{L^2} &\leq \|\sqrt{\rho}\|_{L^{\frac{4n}{n+2-4\varepsilon}}} \|u\|_{L^{\frac{4n}{n-2+4\varepsilon}}} \\ &\leq C_\star \|\rho\|_{L^{\frac{2n}{n+2-4\varepsilon}}}^{\frac{1}{2}} \|\Lambda^{\frac{1}{2}+\frac{n}{4}-\varepsilon}u\|_{L^2} \\ &\leq C_\star \|\rho_0\|_{L^{\frac{2n}{n+2}}\cap L^\infty}^{\frac{1}{2}} \|\mathcal{L}u\|_{L^2},\end{aligned}$$

where $C_\star = C_\star(n) > 0$ is a constant. Taking γ as

$$\gamma = \frac{1}{C_\star^2 \|\rho_0\|_{L^{\frac{2n}{n+2}}\cap L^\infty}},$$

one has

$$\frac{d}{dt}\|\sqrt{\rho}u(t)\|_{L^2}^2 + \gamma\|\sqrt{\rho}u(t)\|_{L^2}^2 + \|\mathcal{L}u\|_{L^2}^2 = 0.$$

Then the Gronwall inequality yields the desired estimate (2.1). This completes the proof of Lemma 2.2. □

2.2. **Uniform estimates in time.** Next we will establish the time-independent estimate on the $L^\infty(0,T;L^2(\mathbb{R}^n))$-norm of $\mathcal{L}u$, which plays a key role in proving our main result.

Lemma 2.3. *Under the assumptions of Theorem 1.1, the solution (ρ, u) of the system (1.1) admits the following bound for any $t \geq 0$,*

$$\|\mathcal{L}u(t)\|_{L^2}^2 + \int_0^t (\|\mathcal{L}^2 u(\tau)\|_{L^2}^2 + \|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2) \, d\tau \leq C_0, \tag{2.4}$$

where the constant C_0 depends only on the initial data.

Proof. Multiplying $(1.1)_2$ by $\partial_t u$, using $\nabla \cdot u = 0$ and integrating by parts, one has

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{L}u(t)\|_{L^2}^2 + \|\sqrt{\rho}\partial_t u\|_{L^2}^2 = -\int_{\mathbb{R}^n} \rho u \cdot \nabla u \cdot \partial_t u \, dx.$$

In view of the Gagliardo-Nirenberg inequality, it follows that

$$\begin{aligned}
-\int_{\mathbb{R}^n} \rho u\cdot\nabla u\cdot\partial_t u\,dx \leq& \|\sqrt{\rho}\|_{L^\infty}\|u\cdot\nabla u\|_{L^2}\|\sqrt{\rho}\partial_t u\|_{L^2}\\
\leq& C\|\rho_0\|_{L^\infty}^{\frac12}\|u\|_{L^{\frac{4n}{n-2}}}\|\nabla u\|_{L^{\frac{4n}{n+2}}}\|\sqrt{\rho}\partial_t u\|_{L^2}\\
\leq& C\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^2\|\sqrt{\rho}\partial_t u\|_{L^2}.
\end{aligned}$$

We thus get

$$\frac12\frac{d}{dt}\|\mathcal{L}u(t)\|_{L^2}^2+\|\sqrt{\rho}\partial_t u\|_{L^2}^2\leq C\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^2\|\sqrt{\rho}\partial_t u\|_{L^2}. \tag{2.5}$$

Now let us denote

$$A(t):=\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^2.$$

By the high-low frequency technique, we derive

$$A(t)\leq\|S_N\Lambda^{\frac12+\frac n4}u\|_{L^2}^2+\sum_{j\geq N}\|\Delta_j\Lambda^{\frac12+\frac n4}u\|_{L^2}^2,$$

where the operators S_j and Δ_j are defined in the Appendix and N will be specified later. By Plancherel's theorem and Sobolev's embedding, we obtain

$$\begin{aligned}
\|S_N\Lambda^{\frac12+\frac n4}u\|_{L^2}^2 =& C\|\chi(2^{-N}\xi)|\xi|^{\frac12+\frac n4}\widehat{u}(\xi)\|_{L^2}^2\\
=& C\left\|\chi(2^{-N}\xi)g(\xi)\frac{|\xi|^{\frac12+\frac n4}}{g(\xi)}\widehat{u}(\xi)\right\|_{L^2}^2\\
\leq& Cg^2(2^N)\|\mathcal{L}u\|_{L^2}^2,
\end{aligned}$$

where χ and φ are associated with the definition of Besov spaces (see Appendix for details). By Lemma A.1, it follows that, for $0<\sigma<\min\{\frac{n+2-8\varepsilon}{4},\frac{n+2-12\varepsilon}{8}\}$ with $0<\varepsilon<\frac{n+2}{12}$,

$$\begin{aligned}
\sum_{j\geq N}\|\Delta_j\Lambda^{\frac12+\frac n4}u\|_{L^2}^2 \leq& C\sum_{j\geq N}2^{-2j\sigma}\|\Delta_j\Lambda^{\frac12+\frac n4+\sigma}u\|_{L^2}^2\\
\leq& C\sum_{j\geq N}2^{-2j\sigma}\|\Lambda^{\frac12+\frac n4+\sigma}u\|_{L^2}^2\\
\leq& C2^{-2N\sigma}\int_{\mathbb{R}^n}|\xi|^{1+\frac n2+2\sigma}|\widehat{u}(\xi)|^2\,d\xi\\
\leq& C2^{-2N\sigma}\int_{|\xi|\leq r}|\xi|^{2\sigma}g^2(\xi)\frac{|\xi|^{1+\frac n2}}{g^2(\xi)}|\widehat{u}(\xi)|^2\,d\xi\\
&+C2^{-2N\sigma}\int_{|\xi|\geq r}\frac{g^4(\xi)}{|\xi|^{1+\frac n2-2\sigma}}\frac{|\xi|^{2+n}}{g^4(\xi)}|\widehat{u}(\xi)|^2\,d\xi\\
\leq& C2^{-2N\sigma}\left(r^{2\sigma}g^2(r)\|\mathcal{L}u\|_{L^2}^2+\frac{g^4(r)}{r^{1+\frac n2-2\sigma}}\|\mathcal{L}^2u\|_{L^2}^2\right)\\
\leq& C2^{-2N\sigma}\left(r^{2\sigma+2\varepsilon}\|\mathcal{L}u\|_{L^2}^2+\frac{1}{r^{1+\frac n2-2\sigma-4\varepsilon}}\|\mathcal{L}^2u\|_{L^2}^2\right)\\
\leq& C2^{-2N\sigma}\|\mathcal{L}u\|_{L^2}^{2-\lambda}\|\mathcal{L}^2u\|_{L^2}^{\lambda},
\end{aligned}$$

where we have fixed r as

$$r=\left(\frac{\|\mathcal{L}^2u\|_{L^2}}{\|\mathcal{L}u\|_{L^2}}\right)^{\frac{4}{2+n-4\varepsilon}},$$

and

$$\lambda=\frac{8(\sigma+\varepsilon)}{2+n-4\varepsilon}\in(0,\,1).$$

As a result, we arrive at

$$A(t)\leq Cg^2(2^N)\|\mathcal{L}u\|_{L^2}^2+C2^{-2N\sigma}\|\mathcal{L}u\|_{L^2}^{2-\lambda}\|\mathcal{L}^2u\|_{L^2}^{\lambda}. \tag{2.6}$$

Let us rewrite the equation $(1.1)_2$ as the Stokes type

$$\begin{cases}\mathcal{L}^2u+\nabla p=-\rho\partial_t u-\rho u\cdot\nabla u,\\ \nabla\cdot u=0.\end{cases} \tag{2.7}$$

It thus follows that

$$\begin{aligned}\|\mathcal{L}^2u\|_{L^2}\leq&C\|\rho\partial_t u\|_{L^2}+C\|\rho u\cdot\nabla u\|_{L^2}\\ \leq&C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}+C\|\rho\|_{L^\infty}\|u\cdot\nabla u\|_{L^2}\\ \leq&C\|\sqrt{\rho}\partial_t u\|_{L^2}+C\|u\|_{L^{\frac{4n}{n-2}}}\|\nabla u\|_{L^{\frac{4n}{n+2}}}\\ \leq&C\|\sqrt{\rho}\partial_t u\|_{L^2}+C\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^2,\end{aligned} \tag{2.8}$$

which further implies

$$\|\mathcal{L}^2u\|_{L^2}\leq C\|\sqrt{\rho}\partial_t u\|_{L^2}+CA(t). \tag{2.9}$$

Combining (2.6) and (2.9), we deduce

$$\begin{aligned}A(t)\leq&Cg^2(2^N)\|\mathcal{L}u\|_{L^2}^2+C2^{-2N\sigma}\|\mathcal{L}u\|_{L^2}^{2-\lambda}(\|\sqrt{\rho}\partial_t u\|_{L^2}^{\lambda}+A^{\lambda}(t))\\ \leq&\frac12A(t)+Cg^2(2^N)\|\mathcal{L}u\|_{L^2}^2+C2^{-2N\sigma}\|\mathcal{L}u\|_{L^2}^{2-\lambda}\|\sqrt{\rho}\partial_t u\|_{L^2}^{\lambda}+C2^{-\frac{2N\sigma}{1-\lambda}}\|\mathcal{L}u\|_{L^2}^{\frac{2-\lambda}{1-\lambda}},\end{aligned}$$

which implies

$$A(t)\leq Cg^2(2^N)\|\mathcal{L}u\|_{L^2}^2+C2^{-2N\sigma}\|\mathcal{L}u\|_{L^2}^{2-\lambda}\|\sqrt{\rho}\partial_t u\|_{L^2}^{\lambda}+C2^{-\frac{2N\sigma}{1-\lambda}}\|\mathcal{L}u\|_{L^2}^{\frac{2-\lambda}{1-\lambda}}. \tag{2.10}$$

This along with (2.9) gives

$$\begin{aligned}\|\mathcal{L}^2u\|_{L^2}\leq&C\|\sqrt{\rho}\partial_t u\|_{L^2}+Cg^2(2^N)\|\mathcal{L}u\|_{L^2}^2+C2^{-2N\sigma}\|\mathcal{L}u\|_{L^2}^{2-\lambda}\|\sqrt{\rho}\partial_t u\|_{L^2}^{\lambda}\\ &+C2^{-\frac{2N\sigma}{1-\lambda}}\|\mathcal{L}u\|_{L^2}^{\frac{2-\lambda}{1-\lambda}}.\end{aligned} \tag{2.11}$$

Substituting (2.10) into (2.5) ensures that

$$\begin{aligned}\frac12\frac{d}{dt}\|\mathcal{L}u(t)\|_{L^2}^2+\|\sqrt{\rho}\partial_t u\|_{L^2}^2\leq&Cg^2(2^N)\|\mathcal{L}u\|_{L^2}^2\|\sqrt{\rho}\partial_t u\|_{L^2}\\ &+C2^{-2N\sigma}\|\mathcal{L}u\|_{L^2}^{2-\lambda}\|\sqrt{\rho}\partial_t u\|_{L^2}^{1+\lambda}\\ &+C2^{-\frac{2N\sigma}{1-\lambda}}\|\mathcal{L}u\|_{L^2}^{\frac{2-\lambda}{1-\lambda}}\|\sqrt{\rho}\partial_t u\|_{L^2}.\end{aligned} \tag{2.12}$$

Now taking N such that

$$2^{2N\sigma}\approx e+\|\mathcal{L}u\|_{L^2}, \tag{2.13}$$

we deduce from (2.12) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathcal{L}u(t)\|_{L^2}^2+\|\sqrt{\rho}\partial_t u\|_{L^2}^2 \leq & \frac{1}{2}\|\sqrt{\rho}\partial_t u\|_{L^2}^2+Cg^4\left([e+\|\mathcal{L}u\|_{L^2}^2]^{\frac{1}{\sigma}}\right)\|\mathcal{L}u\|_{L^2}^4\\ &+C\|\mathcal{L}u\|_{L^2}^2. \end{aligned} \tag{2.14}$$

We therefore obtain

$$\begin{aligned}\frac{d}{dt}(e+\|\mathcal{L}u(t)\|_{L^2}^2)+\|\sqrt{\rho}\partial_t u\|_{L^2}^2 \leq & C\|\mathcal{L}u\|_{L^2}^2 g^4\left([e+\|\mathcal{L}u\|_{L^2}^2]^{\frac{1}{\sigma}}\right)(e+\|\mathcal{L}u\|_{L^2}^2)\\ &+C\|\mathcal{L}u\|_{L^2}^2. \end{aligned}$$

Making use of (2.1) and setting

$$X(t):=e+\|\mathcal{L}u(t)\|_{L^2}^2,$$

one derives

$$X(t)+\int_0^t\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau\leq C_0+C\int_0^t\|\mathcal{L}u(\tau)\|_{L^2}^2 g^4\big(X^{\frac{1}{\sigma}}(\tau)\big)X(\tau)\,d\tau, \tag{2.15}$$

where the constant C_0 depends only on initial data, independent of t. We denote

$$Z(t):=C_0+C\int_0^t\|\mathcal{L}u(\tau)\|_{L^2}^2 g^4\big(X^{\frac{1}{\sigma}}(\tau)\big)X(\tau)\,d\tau,\quad Z(0)=C_0\geq e,$$

then we obtain

$$\frac{d}{dt}Z(t)=C\|\mathcal{L}u(t)\|_{L^2}^2 g^4\big(X^{\frac{1}{\sigma}}(t)\big)X(t)\leq C\|\mathcal{L}u(t)\|_{L^2}^2 g^4\big(Z^{\frac{1}{\sigma}}(t)\big)Z(t).$$

Therefore, it follows that

$$\sigma\int_{Z^{\frac{1}{\sigma}}(0)}^{Z^{\frac{1}{\sigma}}(t)}\frac{d\tau}{\tau g^4(\tau)}=\int_{Z(0)}^{Z(t)}\frac{d\tau}{\tau g^4(\tau^{\frac{1}{\sigma}})}\leq C\int_0^t\|\mathcal{L}u(\tau)\|_{L^2}^2\,d\tau.$$

Recalling

$$\int_1^\infty\frac{d\tau}{\tau g^4(\tau)}=\infty,\qquad C\int_0^t\|\mathcal{L}u(\tau)\|_{L^2}^2\,d\tau\leq C_0,$$

one has,

$$Z(t)\leq C_0.$$

Thanks to (2.15), we also have

$$X(t)+\int_0^t\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau\leq C_0.$$

Then (2.13) and (2.11) yield

$$\int_0^t\|\mathcal{L}^2 u(\tau)\|_{L^2}^2\,d\tau\leq C_0.$$

We thus conclude the proof of Lemma 2.3. □

2.3. **Decay estimates.** Lemma 2.3 enables us to derive the following exponential decay estimate, which improves (2.4).

Lemma 2.4. *Under the assumptions of Theorem 1.1, the solution* (ρ, u) *of the system (1.1) admits the following bound for any* $t \geq 0$*,*

$$e^{\gamma t}\|\mathcal{L}u(t)\|_{L^2}^2 + \int_0^t (e^{\gamma t}\|\mathcal{L}^2 u(\tau)\|_{L^2}^2 + e^{\gamma t}\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2)\, d\tau \leq C_0, \tag{2.16}$$

where the constant C_0 *depends only on the initial data.*

Proof. Recalling (2.14), one has

$$\frac{d}{dt}\|\mathcal{L}u(t)\|_{L^2}^2 + \|\sqrt{\rho}\partial_t u\|_{L^2}^2 \leq Cg^4\left([e + \|\mathcal{L}u\|_{L^2}^2]^{\frac{1}{\sigma}}\right)\|\mathcal{L}u\|_{L^2}^4 + C\|\mathcal{L}u\|_{L^2}^2.$$

Using (2.3) and (2.4), we derive

$$\frac{d}{dt}\|\mathcal{L}u(t)\|_{L^2}^2 + \|\sqrt{\rho}\partial_t u\|_{L^2}^2 \leq C\|\mathcal{L}u\|_{L^2}^4 + C\|\mathcal{L}u\|_{L^2}^2.$$

This further gives

$$\begin{aligned}\frac{d}{dt}(e^{\gamma t}\|\mathcal{L}u(t)\|_{L^2}^2) + e^{\gamma t}\|\sqrt{\rho}\partial_t u\|_{L^2}^2 \leq& \gamma e^{\gamma t}\|\mathcal{L}u(t)\|_{L^2}^2 + C\|\mathcal{L}u\|_{L^2}^2(e^{\gamma t}\|\mathcal{L}u\|_{L^2}^2)\\ &+ Ce^{\gamma t}\|\mathcal{L}u\|_{L^2}^2.\end{aligned}$$

In view of (2.1), we get by integrating it in time

$$e^{\gamma t}\|\mathcal{L}u(t)\|_{L^2}^2 + \int_0^t e^{\gamma\tau}\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2\, d\tau \leq C_0 + C\int_0^t \|\mathcal{L}u(\tau)\|_{L^2}^2(e^{\gamma\tau}\|\mathcal{L}u(\tau)\|_{L^2}^2)\, d\tau.$$

By means of the Gronwall inequality, we have

$$e^{\gamma t}\|\mathcal{L}u(t)\|_{L^2}^2 + \int_0^t e^{\gamma\tau}\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2\, d\tau \leq C_0.$$

Noticing (2.13) and (2.11), we verify that

$$\int_0^t e^{\gamma t}\|\mathcal{L}^2 u(\tau)\|_{L^2}^2\, d\tau \leq C_0.$$

This completes the proof of Lemma 2.4. □

The following lemma is crucial to the derivation of the higher order estimates of the solutions.

Lemma 2.5. *Under the assumptions of Theorem 1.1, the solution* (ρ, u) *of the system (1.1) admits the following bounds for any* $t \geq 0$*,*

$$t\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \int_0^t \tau\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2}^2\, d\tau \leq C_0, \tag{2.17}$$

$$t\|\mathcal{L}^2 u\|_{L^2}^2 + t\|p(t)\|_{H^1}^2 \leq C_0, \tag{2.18}$$

$$\int_0^t (\tau\|\mathcal{L}^2 u(\tau)\|_{L^r}^2 + \tau\|\nabla p(\tau)\|_{L^r}^2)\, d\tau \leq C_0, \quad \forall\, r \in \Big[2,\, \frac{4n}{n-2}\Big). \tag{2.19}$$

Moreover, for any $t \geq 1$, the following estimates hold true

$$e^{\gamma t}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \int_1^t e^{\gamma\tau}\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau \leq C_0, \tag{2.20}$$

$$e^{\gamma t}\|\mathcal{L}^2 u\|_{L^2}^2 + e^{\gamma t}\|p(t)\|_{H^1}^2 \leq C_0, \tag{2.21}$$

$$\int_1^t (e^{\gamma\tau}\|\mathcal{L}^2 u(\tau)\|_{L^r}^2 + e^{\gamma\tau}\|\nabla p(\tau)\|_{L^r}^2)\,d\tau \leq C_0, \quad \forall\, r \in \Big(2,\, \frac{4n}{n-2}\Big), \tag{2.22}$$

where the constant C_0 depends only on the initial data.

Proof. Applying ∂_t to the equation $(1.1)_2$, one has

$$\rho\partial_{tt}u + \rho u\cdot\nabla\partial_t u + \mathcal{L}^2\partial_t u + \nabla\partial_t p = -\partial_t\rho\partial_t u - \partial_t(\rho u)\cdot\nabla u. \tag{2.23}$$

We get by multiplying (2.23) by $\partial_t u$ and using the equation $(1.1)_1$ that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \|\mathcal{L}\partial_t u\|_{L^2}^2\\
&= -\int_{\mathbb{R}^n}\partial_t\rho\partial_t u\cdot\partial_t u\,dx - \int_{\mathbb{R}^n}\partial_t(\rho u)\cdot\nabla u\cdot\partial_t u\,dx\\
&= -2\int_{\mathbb{R}^n}\rho u\cdot\nabla\partial_t u\cdot\partial_t u\,dx - \int_{\mathbb{R}^n}\rho\partial_t u\cdot\nabla u\cdot\partial_t u\,dx - \int_{\mathbb{R}^n}\rho u\cdot\nabla(u\cdot\nabla u\cdot\partial_t u)\,dx\\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{2.24}$$

By (2.3), we have

$$\begin{aligned}
\|\Lambda^{\frac12+\frac n4-\varepsilon}\partial_t u\|_{L^2}^2 =& \int_{\mathbb{R}^n}\frac{g^2(\xi)}{|\xi|^{2\varepsilon}}\frac{|\xi|^{1+\frac n2}}{g^2(\xi)}|\widehat{\partial_t u}(\xi)|^2\,d\xi\\
\leq& C\int_{\mathbb{R}^n}\frac{|\xi|^{1+\frac n2}}{g^2(\xi)}|\widehat{\partial_t u}(\xi)|^2\,d\xi\\
=& C\|\mathcal{L}\partial_t u\|_{L^2}^2.
\end{aligned} \tag{2.25}$$

Similarly, it follows that

$$\|\Lambda^{1+\frac n2-2\varepsilon}u\|_{L^2}^2 \leq C\|\mathcal{L}^2 u\|_{L^2}^2.$$

By means of (2.25), we deduce for $0 < \varepsilon < \frac{n-2}{4}$,

$$\begin{aligned}
J_1 \leq& C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\|\nabla\partial_t u\|_{L^{\frac{4n}{n+2+4\varepsilon}}}\|u\|_{L^{\frac{4n}{n-2-4\varepsilon}}}\\
\leq& C\|\sqrt{\rho}\partial_t u\|_{L^2}\|\Lambda^{\frac12+\frac n4-\varepsilon}\partial_t u\|_{L^2}\|\Lambda^{\frac12+\frac n4+\varepsilon}u\|_{L^2}\\
\leq& C\|\sqrt{\rho}\partial_t u\|_{L^2}\|\mathcal{L}\partial_t u\|_{L^2}\|\Lambda^{\frac12+\frac n4+\varepsilon}u\|_{L^2}\\
\leq& \frac{1}{16}\|\mathcal{L}\partial_t u\|_{L^2}^2 + C\|\Lambda^{\frac12+\frac n4+\varepsilon}u\|_{L^2}^2\|\sqrt{\rho}\partial_t u\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
J_2 \leq& C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\|\nabla u\|_{L^{\frac{4n}{n+2-4\varepsilon}}}\|\partial_t u\|_{L^{\frac{4n}{n-2+4\varepsilon}}}\\
\leq& C\|\sqrt{\rho}\partial_t u\|_{L^2}\|\Lambda^{\frac12+\frac n4+\varepsilon}u\|_{L^2}\|\Lambda^{\frac12+\frac n4-\varepsilon}\partial_t u\|_{L^2}\\
\leq& C\|\sqrt{\rho}\partial_t u\|_{L^2}\|\mathcal{L}\partial_t u\|_{L^2}\|\Lambda^{\frac12+\frac n4+\varepsilon}u\|_{L^2}
\end{aligned}$$

$$\leq \frac{1}{16}\|\mathcal{L}\partial_t u\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}+\frac{n}{4}+\varepsilon}u\|_{L^2}^2\|\sqrt{\rho}\partial_t u\|_{L^2}^2.$$

Similarly, one can check that

$$\begin{aligned}
J_3 \leq & \left|\int_{\mathbb{R}^n} \rho u\cdot\nabla u\cdot\nabla u\cdot\partial_t u\,dx\right| + \left|\int_{\mathbb{R}^n} \rho u\cdot u\cdot\nabla^2 u\cdot\partial_t u\,dx\right| \\
& + \left|\int_{\mathbb{R}^n} \rho u\cdot u\cdot\nabla u\cdot\nabla\partial_t u\,dx\right| \\
\leq & C\|\rho\|_{L^\infty}\|u\|_{L^{\frac{4n}{n-2-4\varepsilon}}}\|\nabla u\|_{L^{\frac{4n}{n+2}}}^2\|\partial_t u\|_{L^{\frac{4n}{n-2+4\varepsilon}}} \\
& + C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\|\nabla^2 u\|_{L^{\frac{n}{1+2\varepsilon}}}\|u\|_{L^{\frac{4n}{n-2-4\varepsilon}}}^2 \\
& + C\|\rho\|_{L^\infty}\|u\|_{L^{\frac{4n}{n-2}}}^2\|\nabla u\|_{L^{\frac{4n}{n+2-4\varepsilon}}}\|\nabla\partial_t u\|_{L^{\frac{4n}{n+2+4\varepsilon}}} \\
\leq & C\|\Lambda^{\frac{1}{2}+\frac{n}{4}+\varepsilon}u\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2\|\Lambda^{\frac{1}{2}+\frac{n}{4}-\varepsilon}\partial_t u\|_{L^2} \\
& + C\|\sqrt{\rho}\partial_t u\|_{L^2}\|\Lambda^{1+\frac{n}{2}-2\varepsilon}u\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{n}{4}+\varepsilon}u\|_{L^2}^2 \\
\leq & C\|\Lambda^{\frac{1}{2}+\frac{n}{4}+\varepsilon}u\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2\|\mathcal{L}\partial_t u\|_{L^2} + C\|\sqrt{\rho}\partial_t u\|_{L^2}\|\mathcal{L}^2 u\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{n}{4}+\varepsilon}u\|_{L^2}^2 \\
\leq & \frac{1}{16}\|\mathcal{L}\partial_t u\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^4\|\Lambda^{\frac{1}{2}+\frac{n}{4}+\varepsilon}u\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}+\frac{n}{4}+\varepsilon}u\|_{L^2}^2\|\sqrt{\rho}\partial_t u\|_{L^2}^2,
\end{aligned}$$

where we have used the following fact due to (2.8),

$$\|\mathcal{L}^2 u\|_{L^2} \leq C\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2.$$

Substituting the above estimates into (2.24) yields

$$\begin{aligned}
\frac{d}{dt}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \|\mathcal{L}\partial_t u\|_{L^2}^2 \leq & C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^4\|\Lambda^{\frac{1}{2}+\frac{n}{4}+\varepsilon}u\|_{L^2}^2 \\
& + C\|\Lambda^{\frac{1}{2}+\frac{n}{4}+\varepsilon}u\|_{L^2}^2\|\sqrt{\rho}\partial_t u\|_{L^2}^2.
\end{aligned}$$

Noticing the following facts:

$$\begin{aligned}
\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2} \leq & C\|\Lambda^{\frac{1}{2}+\frac{n}{4}-\varepsilon}u\|_{L^2}^{\frac{n+2-8\varepsilon}{n+2-4\varepsilon}}\|\Lambda^{1+\frac{n}{2}-2\varepsilon}u\|_{L^2}^{\frac{4\varepsilon}{n+2-4\varepsilon}} \\
\leq & C\|\mathcal{L}u\|_{L^2}^{\frac{n+2-8\varepsilon}{n+2-4\varepsilon}}\|\mathcal{L}^2 u\|_{L^2}^{\frac{4\varepsilon}{n+2-4\varepsilon}},
\end{aligned} \tag{2.26}$$

$$\begin{aligned}
\|\Lambda^{\frac{1}{2}+\frac{n}{4}+\varepsilon}u\|_{L^2} \leq & C\|\Lambda^{\frac{1}{2}+\frac{n}{4}-\varepsilon}u\|_{L^2}^{\frac{n+2-12\varepsilon}{n+2-4\varepsilon}}\|\Lambda^{1+\frac{n}{2}-2\varepsilon}u\|_{L^2}^{\frac{8\varepsilon}{n+2-4\varepsilon}} \\
\leq & C\|\mathcal{L}u\|_{L^2}^{\frac{n+2-12\varepsilon}{n+2-4\varepsilon}}\|\mathcal{L}^2 u\|_{L^2}^{\frac{8\varepsilon}{n+2-4\varepsilon}},
\end{aligned}$$

we conclude that

$$\begin{aligned}
\frac{d}{dt}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \|\mathcal{L}\partial_t u\|_{L^2}^2 \leq & C\|\mathcal{L}u\|_{L^2}^{\frac{6n+12-56\varepsilon}{n+2-4\varepsilon}}\|\mathcal{L}^2 u\|_{L^2}^{\frac{32\varepsilon}{n+2-4\varepsilon}} \\
& + C\|\mathcal{L}u\|_{L^2}^{\frac{2n+4-24\varepsilon}{n+2-4\varepsilon}}\|\mathcal{L}^2 u\|_{L^2}^{\frac{16\varepsilon}{n+2-4\varepsilon}}\|\sqrt{\rho}\partial_t u\|_{L^2}^2.
\end{aligned}$$

Taking $0 < \varepsilon \leq \frac{n+2}{20}$, we get

$$\frac{d}{dt}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \|\mathcal{L}\partial_t u\|_{L^2}^2 \leq C\|\mathcal{L}u\|_{L^2}^4(\|\mathcal{L}u\|_{L^2}^2 + \|\mathcal{L}^2 u\|_{L^2}^2)$$

$$+ C(\|\mathcal{L}u\|_{L^2}^2 + \|\mathcal{L}^2 u\|_{L^2}^2)\|\sqrt{\rho}\partial_t u\|_{L^2}^2. \tag{2.27}$$

This implies

$$\begin{aligned}\frac{d}{dt}(t\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2) + t\|\mathcal{L}\partial_t u\|_{L^2}^2 \leq &\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + Ct\|\mathcal{L}u\|_{L^2}^4(\|\mathcal{L}u\|_{L^2}^2 + \|\mathcal{L}^2 u\|_{L^2}^2)\\ &+ C(\|\mathcal{L}u\|_{L^2}^2 + \|\mathcal{L}^2 u\|_{L^2}^2)(t\|\sqrt{\rho}\partial_t u\|_{L^2}^2).\end{aligned} \tag{2.28}$$

According to (2.16), we conclude

$$\begin{aligned}&\int_0^t \tau\|\mathcal{L}u(\tau)\|_{L^2}^4(\|\mathcal{L}u(\tau)\|_{L^2}^2 + \|\mathcal{L}^2 u(\tau)\|_{L^2}^2)\,d\tau\\ &= \int_0^t \tau e^{-2\gamma\tau}(e^{\gamma\tau}\|\mathcal{L}u(\tau)\|_{L^2}^2)^2(\|\mathcal{L}u(\tau)\|_{L^2}^2 + \|\mathcal{L}^2 u(\tau)\|_{L^2}^2)\,d\tau\\ &\leq C\int_0^t (\|\mathcal{L}u(\tau)\|_{L^2}^2 + \|\mathcal{L}^2 u(\tau)\|_{L^2}^2)\,d\tau \leq C_0.\end{aligned} \tag{2.29}$$

Combining (2.28), (2.29) and the Gronwall inequality, we deduce for any $t \geq 0$,

$$t\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \int_0^t \tau\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau \leq C_0. \tag{2.30}$$

It follows from the Stokes system (2.7) that

$$\begin{aligned}\|\mathcal{L}^2 u\|_{L^2} + \|\nabla p\|_{L^2} \leq &C\|\rho\partial_t u\|_{L^2} + C\|\rho u\cdot\nabla u\|_{L^2}\\ \leq &C\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2\\ \leq &C\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\mathcal{L}u\|_{L^2}^{\frac{2(n+2-8\varepsilon)}{n+2-4\varepsilon}}\|\mathcal{L}^2 u\|_{L^2}^{\frac{8\varepsilon}{n+2-4\varepsilon}}\\ \leq &\frac{1}{2}\|\mathcal{L}^2 u\|_{L^2} + C\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\mathcal{L}u\|_{L^2}^{\frac{2(n+2-8\varepsilon)}{n+2-12\varepsilon}},\end{aligned} \tag{2.31}$$

where we have used the estimate (2.26). It thus implies

$$t\|\mathcal{L}^2 u(t)\|_{L^2}^2 + t\|\nabla p(t)\|_{L^2}^2 \leq Ct\|\sqrt{\rho}\partial_t u\|_{L^2}^2 + Ct\|\mathcal{L}u\|_{L^2}^{\frac{4(n+2-8\varepsilon)}{n+2-12\varepsilon}} \leq C_0. \tag{2.32}$$

Moreover, it follows from (2.26) and (2.31) that

$$\begin{aligned}t\|\nabla p\|_{L^2}^2 \leq &Ct\|\sqrt{\rho}\partial_t u\|_{L^2}^2 + Ct\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^4\\ \leq &Ct\|\sqrt{\rho}\partial_t u\|_{L^2}^2 + Ct\|\mathcal{L}u\|_{L^2}^{\frac{4(n+2-8\varepsilon)}{n+2-4\varepsilon}}\|\mathcal{L}^2 u\|_{L^2}^{\frac{16\varepsilon}{n+2-4\varepsilon}}\\ \leq &Ct\|\sqrt{\rho}\partial_t u\|_{L^2}^2 + Ct\|\mathcal{L}u\|_{L^2}^2(\|\mathcal{L}u\|_{L^2}^2 + \|\mathcal{L}^2 u\|_{L^2}^2),\end{aligned}$$

which implies

$$\int_0^t \tau\|\nabla p(\tau)\|_{L^2}^2\,d\tau \leq C_0.$$

Similarly, we obtain

$$\begin{aligned}\|p\|_{L^2} \leq &C\|\Lambda^{-1}(\rho\partial_t u\|_{L^2})\|_{L^2} + C\|\Lambda^{-1}(\rho u\cdot\nabla u)\|_{L^2}\\ \leq &C\|\rho\partial_t u\|_{L^{\frac{2n}{n+2}}} + C\|\rho u\cdot\nabla u\|_{L^{\frac{2n}{n+2}}}\\ \leq &C\|\sqrt{\rho}\|_{L^n}\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\rho\|_{L^n}\|u\cdot\nabla u\|_{L^2}\\ \leq &C\|\sqrt{\rho_0}\|_{L^n}\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\rho_0\|_{L^n}\|u\cdot\nabla u\|_{L^2}\end{aligned}$$

$$\leq C\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\Lambda^{\frac{1}{2}+\frac{n}{4}} u\|_{L^2}^2,$$

which gives

$$t\|p(t)\|_{L^2}^2 \leq C_0.$$

Again, we deduce from the Stokes system (2.7) that

$$\begin{aligned}\|\mathcal{L}^2 u\|_{L^{\frac{4n}{n-2+4\varepsilon}}} + \|\nabla p\|_{L^{\frac{4n}{n-2+4\varepsilon}}} &\leq C\|\rho\partial_t u\|_{L^{\frac{4n}{n-2+4\varepsilon}}} + C\|\rho u\cdot\nabla u\|_{L^{\frac{4n}{n-2+4\varepsilon}}} \\ &\leq C\|\partial_t u\|_{L^{\frac{4n}{n-2+4\varepsilon}}} + C\|u\|_{L^\infty}\|\nabla u\|_{L^{\frac{4n}{n-2+4\varepsilon}}} \\ &\leq C\|\mathcal{L}\partial_t u\|_{L^2} + C\|\mathcal{L}u\|_{L^2}\|\mathcal{L}^2 u\|_{L^2},\end{aligned}$$

where and in what follows we use the following facts:

$$\begin{aligned}\|u\|_{L^\infty} &\leq C\|\Lambda^{\frac{1}{2}+\frac{n}{4}-\varepsilon} u\|_{L^2}^{\frac{4-8\varepsilon}{n+2-4\varepsilon}} \|\Lambda^{1+\frac{n}{2}-2\varepsilon} u\|_{L^2}^{\frac{n-2+4\varepsilon}{n+2-4\varepsilon}} \\ &\leq C\|\mathcal{L}u\|_{L^2}^{\frac{4-8\varepsilon}{n+2-4\varepsilon}} \|\mathcal{L}^2 u\|_{L^2}^{\frac{n-2+4\varepsilon}{n+2-4\varepsilon}},\end{aligned}$$

$$\begin{aligned}\|\nabla u\|_{L^{\frac{4n}{n-2+4\varepsilon}}} &\leq C\|\Lambda^{\frac{1}{2}+\frac{n}{4}-\varepsilon} u\|_{L^2}^{\frac{n-2+4\varepsilon}{n+2-4\varepsilon}} \|\Lambda^{1+\frac{n}{2}-2\varepsilon} u\|_{L^2}^{\frac{4}{n+2-4\varepsilon}} \\ &\leq C\|\mathcal{L}u\|_{L^2}^{\frac{n-2+4\varepsilon}{n+2-4\varepsilon}} \|\mathcal{L}^2 u\|_{L^2}^{\frac{4}{n+2-4\varepsilon}}.\end{aligned}$$

Recalling (2.30) and (2.32), one has

$$\int_0^t (\tau\|\mathcal{L}^2 u(\tau)\|_{L^{\frac{4n}{n-2+4\varepsilon}}}^2 + \tau\|\nabla p(\tau)\|_{L^{\frac{4n}{n-2+4\varepsilon}}}^2)\, d\tau \leq C_0,$$

or equivalently

$$\int_0^t (\tau\|\mathcal{L}^2 u(\tau)\|_{L^r}^2 + \tau\|\nabla p(\tau)\|_{L^r}^2)\, d\tau \leq C_0, \quad \forall\, r\in\Big(2,\, \frac{4n}{n-2}\Big).$$

Next, we show the estimates (2.20)-(2.22). To this end, we deduce from (2.27) that

$$\begin{aligned}&\frac{d}{dt}(e^{\gamma t}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2) + e^{\gamma t}\|\mathcal{L}\partial_t u\|_{L^2}^2 \\ &\leq \gamma e^{\gamma t}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + Ce^{\gamma t}\|\mathcal{L}u\|_{L^2}^4(\|\mathcal{L}u\|_{L^2}^2 + \|\mathcal{L}^2 u\|_{L^2}^2) \\ &\quad + C(\|\mathcal{L}u\|_{L^2}^2 + \|\mathcal{L}^2 u\|_{L^2}^2)(e^{\gamma t}\|\sqrt{\rho}\partial_t u\|_{L^2}^2).\end{aligned}$$

Making use of the Gronwall inequality, (2.16) and (2.17), we may deduce

$$e^{\gamma t}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \int_1^t e^{\gamma\tau}\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2}^2\, d\tau \leq C_0. \tag{2.33}$$

With the estimate (2.33) in hand, the desired estimates (2.21)-(2.22) can be proved via the same arguments in proving (2.18) and (2.19), and hence we omit the details. This completes the proof of Lemma 2.5. □

2.4. **Gradient estimates.** The following estimates will be used to show the uniqueness of the solutions.

Lemma 2.6. *Under the assumptions of Theorem 1.1, the solution* (ρ, u) *of the system (1.1) admits the following bounds for any* $t \geq 0$*,*

$$\int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau \leq C_0,$$

$$\|\nabla \rho(t)\|_{L^q} \leq C_0,$$

where the constant C_0 *depends only on the initial data.*

Proof. For any $2 < p < \frac{4n}{n-2}$, one may conclude that

$$\begin{aligned}
\|\rho\partial_t u\|_{L^p} \leq& C\|\rho\partial_t u\|_{L^2}^{1-\vartheta}\|\rho\partial_t u\|_{L^{\frac{4n}{n-2+4\varepsilon}}}^{\vartheta} \\
\leq& C\|\sqrt{\rho}\partial_t u\|_{L^2}^{1-\vartheta}\|\partial_t u\|_{L^{\frac{4n}{n-2+4\varepsilon}}}^{\vartheta} \\
\leq& C\|\sqrt{\rho}\partial_t u\|_{L^2}^{1-\vartheta}\|\Lambda^{\frac{1}{2}+\frac{n}{4}-\varepsilon}\partial_t u\|_{L^2}^{\vartheta} \\
\leq& C\|\sqrt{\rho}\partial_t u\|_{L^2}^{1-\vartheta}\|\mathcal{L}\partial_t u\|_{L^2}^{\vartheta},
\end{aligned}$$

where

$$\vartheta = \frac{2n(p-2)}{(n-2+4\varepsilon)} \in (0,\, 1).$$

Similarly, we have

$$\begin{aligned}
\|\rho u\cdot\nabla u\|_{L^p} \leq& C\|\rho u\cdot\nabla u\|_{L^2} + C\|\rho u\cdot\nabla u\|_{L^{\frac{4n}{n-2+4\varepsilon}}} \\
\leq& C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2 + C\|u\|_{L^\infty}\|\nabla u\|_{L^{\frac{4n}{n-2+4\varepsilon}}} \\
\leq& C\|\mathcal{L}u\|_{L^2}^2 + C\|\mathcal{L}^2 u\|_{L^2}^2.
\end{aligned}$$

Applying the L^p-estimate to (2.7) yields

$$\|\mathcal{L}^2 u\|_{L^p} \leq C\|\rho\partial_t u\|_{L^p} + C\|\rho u\cdot\nabla u\|_{L^p}.$$

This allows us to show for some $2 < p < \frac{4n}{n-2}$ that

$$\begin{aligned}
\|\nabla u\|_{L^\infty} \leq& C\|\nabla u\|_{L^{\frac{4n}{n+2+4\varepsilon}}}^{1-\sigma}\|\mathcal{L}^2 u\|_{L^p}^{\sigma} \\
\leq& C\|\mathcal{L}u\|_{L^2}^{1-\sigma}(\|\sigma\partial_t u\|_{L^p} + \|\sigma u\cdot\nabla u\|_{L^p})^{\sigma} \\
\leq& C\|\mathcal{L}u\|_{L^2}^{1-\sigma}\|\sqrt{\sigma}\partial_t u\|_{L^2}^{(1-\vartheta)\sigma}\|\mathcal{L}\partial_t u\|_{L^2}^{\sigma\vartheta} + C\|\mathcal{L}u\|_{L^2} + C\|\mathcal{L}u\|_{L^2}^2 + C\|\mathcal{L}^2 u\|_{L^2}^2,
\end{aligned}$$

where $\sigma \in (0,\, 1)$. Thanks to the estimates of Lemma 2.2-Lemma 2.5, we immediately obtain

$$\begin{aligned}
\int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau \leq& C\int_0^t \|\mathcal{L}u(\tau)\|_{L^2}^{1-\sigma}\|\sqrt{\sigma}\partial_\tau u(\tau)\|_{L^2}^{(1-\vartheta)\sigma}\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2}^{\sigma\vartheta}\, d\tau \\
&+ C\int_0^t (\|\mathcal{L}u(\tau)\|_{L^2} + \|\mathcal{L}u(\tau)\|_{L^2}^2 + \|\mathcal{L}^2 u(\tau)\|_{L^2}^2)\, d\tau \\
=& C\int_0^1 \|\mathcal{L}u(\tau)\|_{L^2}^{1-\sigma}\|\sqrt{\sigma}\partial_\tau u(\tau)\|_{L^2}^{(1-\vartheta)\sigma}\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2}^{\sigma\vartheta}\, d\tau
\end{aligned}$$

$$
\begin{aligned}
&+C\int_1^t \|\mathcal{L}u(\tau)\|_{L^2}^{1-\sigma}\|\sqrt{\sigma}\partial_\tau u(\tau)\|_{L^2}^{(1-\vartheta)\sigma}\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2}^{\sigma\vartheta}\,d\tau\\
&+C\int_0^t (\|\mathcal{L}u(\tau)\|_{L^2}+\|\mathcal{L}u(\tau)\|_{L^2}^2+\|\mathcal{L}^2u(\tau)\|_{L^2}^2)\,d\tau\\
\le& C_0+C\int_0^1 \|\mathcal{L}u(\tau)\|_{L^2}^{1-\sigma}\|\sqrt{\sigma}\partial_\tau u(\tau)\|_{L^2}^{(1-\vartheta)\sigma}\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2}^{\sigma\vartheta}\,d\tau\\
\le& C_0,
\end{aligned}
$$

where we have used the following fact

$$
\begin{aligned}
&C\int_0^1 \|\mathcal{L}u(\tau)\|_{L^2}^{1-\sigma}\|\sqrt{\sigma}\partial_\tau u(\tau)\|_{L^2}^{(1-\vartheta)\sigma}\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2}^{\sigma\vartheta}\,d\tau\\
&\le C\int_0^1 \tau^{-\frac{\sigma}{2}}\|\mathcal{L}u(\tau)\|_{L^2}^{1-\sigma}(\tau^{\frac12}\|\sqrt{\sigma}\partial_\tau u(\tau)\|_{L^2})^{(1-\vartheta)\sigma}(\tau^{\frac12}\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2})^{\sigma\vartheta}\,d\tau\\
&\le C\int_0^1 \tau^{-\frac{\sigma}{2}}(\tau^{\frac12}\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2})^{\sigma\vartheta}\,d\tau\\
&\le C\left(\int_0^1 \tau^{-\frac{\sigma}{2-\sigma\vartheta}}\,d\tau\right)^{\frac{2-\sigma\vartheta}{2}}\left(\int_0^1 \tau\|\mathcal{L}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau\right)^{\frac{\sigma\vartheta}{2}}\\
&\le C_0.
\end{aligned}
$$

Sincere ρ satisfies

$$\partial_t\rho+u\cdot\nabla\rho=0,$$

we have

$$\partial_t\nabla\rho+u\cdot\nabla(\nabla\rho)=-\nabla u\cdot\nabla\rho.$$

Thanks to $\nabla\cdot u=0$, we get by direct computations

$$\frac{d}{dt}\|\nabla\rho(t)\|_{L^q}\le\|\nabla u\|_{L^\infty}\|\nabla\rho(t)\|_{L^q}.$$

Appealing to the Gronwall inequality, we have

$$\|\nabla\rho(t)\|_{L^q}\le\|\nabla\rho_0\|_{L^q}\exp\left[\int_0^t\|\nabla u(\tau)\|_{L^\infty}\,d\tau\right]\le C_0.$$

We thus complete the proof of the lemma. □

2.5. **Proof of Theorem 1.1.** Now we are ready to prove Theorem 1.1. The desired bounds of Theorem 1.1 follow directly by combining together all the estimates of the above Lemmas 2.2-2.6. Thus our main objective is to prove the uniqueness. To explain the ideas clearly, we shall present a formal argument which can be made rigorous by appropriate regularizations. To this end, we make use of the following two momentum equations:

$$\rho\partial_t u+\rho u\cdot\nabla u+\mathcal{L}^2u+\nabla p=0,\quad \widetilde{\rho}\partial_t\widetilde{u}+\widetilde{\rho}\widetilde{u}\cdot\nabla\widetilde{u}+\mathcal{L}^2\widetilde{u}+\nabla\widetilde{p}=0.$$

Then the following holds,

$$\rho\partial_t(u-\widetilde{u})+\rho u\cdot\nabla(u-\widetilde{u})+\mathcal{L}^2(u-\widetilde{u})+\nabla(p-\widetilde{p})=-(\rho-\widetilde{\rho})(\partial_t\widetilde{u}+\widetilde{u}\cdot\nabla\widetilde{u})-\rho(u-\widetilde{u})\cdot\nabla\widetilde{u}.$$

Multiplying the above identity by $u-\widetilde{u}$ and integrating it over $\mathbb{R}^n$ lead to

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}(u-\widetilde{u})(t)\|_{L^2}^2+\|\mathcal{L}(u-\widetilde{u})\|_{L^2}^2=K_1+K_2,$$

where

$$K_1:=-\int_{\mathbb{R}^n}\rho(u-\widetilde{u})\cdot\nabla\widetilde{u}\cdot(u-\widetilde{u})\,dx,$$

$$K_2:=-\int_{\mathbb{R}^n}(\rho-\widetilde{\rho})(\partial_t\widetilde{u}+\widetilde{u}\cdot\nabla\widetilde{u})\cdot(u-\widetilde{u})\,dx,$$

We first have

$$K_1\le C\|\nabla\widetilde{u}\|_{L^\infty}\|\sqrt{\rho}(u-\widetilde{u})\|_{L^2}^2.$$

For the term K_2, one may deduce

$$\begin{aligned}J_1\le& C\|\rho-\widetilde{\rho}\|_{L^{\frac{2n}{n+2-4\varepsilon}}}(\|\partial_t\widetilde{u}\|_{L^{\frac{4n}{n-2+4\varepsilon}}}+\|\widetilde{u}\cdot\nabla\widetilde{u}\|_{L^{\frac{4n}{n-2+4\varepsilon}}})\|u-\widetilde{u}\|_{L^{\frac{4n}{n-2+4\varepsilon}}}\\
\le& C\|\rho-\widetilde{\rho}\|_{L^{\frac{2n}{n+2-4\varepsilon}}}(\|\Lambda^{\frac{1}{2}+\frac{n}{4}-\varepsilon}\partial_t\widetilde{u}\|_{L^2}+\|\widetilde{u}\|_{L^\infty}\|\nabla\widetilde{u}\|_{L^{\frac{4n}{n-2+4\varepsilon}}})\|\Lambda^{\frac{1}{2}+\frac{n}{4}-\varepsilon}(u-\widetilde{u})\|_{L^2}\\
\le& C\|\rho-\widetilde{\rho}\|_{L^{\frac{2n}{n+2-4\varepsilon}}}(\|\mathcal{L}\partial_t\widetilde{u}\|_{L^2}+\|\mathcal{L}\widetilde{u}\|_{L^2}\|\mathcal{L}^2\widetilde{u}\|_{L^2})\|\mathcal{L}(u-\widetilde{u})\|_{L^2}\\
\le&\frac{1}{2}\|\mathcal{L}(u-\widetilde{u})\|_{L^2}^2+C(\|\mathcal{L}\partial_t\widetilde{u}\|_{L^2}^2+\|\mathcal{L}\widetilde{u}\|_{L^2}^2\|\mathcal{L}^2\widetilde{u}\|_{L^2}^2)\|\rho-\widetilde{\rho}\|^2_{L^{\frac{2n}{n+2-4\varepsilon}}}.\end{aligned}$$

This implies

$$\begin{aligned}\frac{d}{dt}\|\sqrt{\rho}(u-\widetilde{u})(t)\|_{L^2}^2+\|\mathcal{L}(u-\widetilde{u})\|_{L^2}^2\le& C(\|\mathcal{L}\partial_t\widetilde{u}\|_{L^2}^2+\|\mathcal{L}\widetilde{u}\|_{L^2}^2\|\mathcal{L}^2\widetilde{u}\|_{L^2}^2)\|\rho-\widetilde{\rho}\|^2_{L^{\frac{2n}{n+2-4\varepsilon}}}\\
&+C\|\nabla\widetilde{u}\|_{L^\infty}\|\sqrt{\rho}(u-\widetilde{u})\|_{L^2}^2.\end{aligned}$$

Using the difference of the density equations yields

$$\partial_t(\rho-\widetilde{\rho})+u\cdot\nabla(\rho-\widetilde{\rho})=-(u-\widetilde{u})\cdot\nabla\widetilde{\rho}.$$

One can check that

$$\begin{aligned}\frac{n+2-4\varepsilon}{2n}\frac{d}{dt}\|(\rho-\widetilde{\rho})(t)\|^{\frac{2n}{n+2-4\varepsilon}}_{L^{\frac{2n}{n+2-4\varepsilon}}}\le& C\|\rho-\widetilde{\rho}\|^{\frac{2n}{n+2-4\varepsilon}-1}_{L^{\frac{2n}{n+2-4\varepsilon}}}\|(u-\widetilde{u})\cdot\nabla\widetilde{\rho}\|_{L^{\frac{2n}{n+2-4\varepsilon}}}\\
\le& C\|\rho-\widetilde{\rho}\|^{\frac{2n}{n+2-4\varepsilon}-1}_{L^{\frac{2n}{n+2-4\varepsilon}}}\|u-\widetilde{u}\|_{L^{\frac{4n}{n-2+4\varepsilon}}}\|\nabla\widetilde{\rho}\|_{L^{\frac{4n}{n+6-8\varepsilon}}}\\
\le& C\|\rho-\widetilde{\rho}\|^{\frac{2n}{n+2-4\varepsilon}-1}_{L^{\frac{2n}{n+2-4\varepsilon}}}\|\mathcal{L}(u-\widetilde{u})\|_{L^2}\|\nabla\widetilde{\rho}\|_{L^{\frac{4n}{n+6-8\varepsilon}}}.\end{aligned}$$

This in particular implies that

$$\frac{d}{dt}\|(\rho-\widetilde{\rho})(t)\|_{L^{\frac{2n}{n+2-4\varepsilon}}}\le C\|\mathcal{L}(u-\widetilde{u})\|_{L^2}\|\nabla\widetilde{\rho}\|_{L^{\frac{4n}{n+6-8\varepsilon}}}.$$

Now let us denote

$$X_1(t):=\|(\rho-\widetilde{\rho})(t)\|_{L^{\frac{2n}{n+2-4\varepsilon}}},\quad X_2(t):=\|\sqrt{\rho}(u-\widetilde{u})(t)\|_{L^2}^2,$$

$$A:=C\|\nabla\widetilde{\rho}\|_{L^{\frac{4n}{n+6-8\varepsilon}}},\quad Y(t):=\|\mathcal{L}(u-\widetilde{u})(t)\|_{L^2}^2,$$

$$\beta(t):=C\|\nabla\widetilde{u}(t)\|_{L^\infty},\quad\gamma(t):=C(\|\mathcal{L}\partial_t\widetilde{u}(t)\|_{L^2}^2+\|\mathcal{L}\widetilde{u}(t)\|_{L^2}^2\|\mathcal{L}^2\widetilde{u}(t)\|_{L^2}^2),$$

which satisfy

$$\begin{cases} \frac{d}{dt}X_1(t) \le AY^{\frac{1}{2}}(t), \\ \frac{d}{dt}X_2(t) + Y(t) \le \beta(t)X_2(t) + \gamma(t)X_1^2(t), \\ X_1(0) = 0. \end{cases}$$

Recalling the estimates of Lemma 2.2-Lemma 2.5, one has

$$\int_0^t \beta(\tau)\,d\tau \le C_0, \qquad \int_0^t \tau\gamma(\tau)\,d\tau \le C_0.$$

Due to $u(x,0) = \widetilde{u}(x,0)$, we have $X_2(0) = 0$. According to the Gronwall type inequality (see [21, Lemma 2.5]), it is clear that

$$\|\sqrt{\rho}(u-\widetilde{u})(t)\|_{L^2}^2 + \|(\rho-\widetilde{\rho})(t)\|_{L^{\frac{2n}{n+2-4\varepsilon}}}^2 + \|\mathcal{L}(u-\widetilde{u})(t)\|_{L^2}^2 \equiv 0,$$

which guarantees the uniqueness. Therefore, we complete the proof of Theorem 1.1.

Appendix A. Besov spaces

This Appendix recalls the inhomogeneous Besov spaces. We begin with the so-called Littlewood-Paley theory. We choose a smooth radial non-increasing function $\chi \in [0,1]$ such that χ is supported in the ball $\mathcal{B} := \{\xi \in \mathbb{R}^n, |\xi| \le \frac{4}{3}\}$ and with value 1 on $\{\xi \in \mathbb{R}^n, |\xi| \le \frac{3}{4}\}$. Now we set $\varphi(\xi) = \chi\big(\frac{\xi}{2}\big) - \chi(\xi)$, which is supported in the annulus $\mathcal{C} := \{\xi \in \mathbb{R}^n, \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ and satisfies

$$\chi(\xi) + \sum_{j\ge 0}\varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n.$$

Let $h = \mathcal{F}^{-1}(\varphi)$ and $\widetilde{h} = \mathcal{F}^{-1}(\chi)$, then the dyadic blocks Δ_j of our decomposition can be defined by

$$\Delta_j u = 0, \quad j \le -2; \qquad \Delta_{-1}u = \chi(D)u = \int_{\mathbb{R}^n} \widetilde{h}(y)u(x-y)\,dy;$$

$$\Delta_j u = \varphi(2^{-j}D)u = 2^{jn}\int_{\mathbb{R}^n} h(2^j y)u(x-y)\,dy, \quad \forall j \in \mathbb{N}.$$

The following operator S_j reads as the low-frequency cut-off

$$S_j u = \chi(2^{-j}D)u = \sum_{-1\le k\le j-1}\Delta_k u = 2^{jn}\int_{\mathbb{R}^n}\widetilde{h}(2^j y)u(x-y)\,dy, \quad \forall j \in \mathbb{N}.$$

Now the inhomogeneous Besov spaces are defined through the dyadic decomposition.

Definition A.1. Let $s \in \mathbb{R}, (p,r) \in [1,+\infty]^2$. The inhomogeneous Besov space $B^s_{p,r}$ is defined as a space of $f \in S'(\mathbb{R}^n)$ such that

$$B^s_{p,r} = \{f \in S'(\mathbb{R}^n); \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}} = \begin{cases} \Big(\sum_{j\geq -1} 2^{jrs} \|\Delta_j f\|^r_{L^p} \Big)^{\frac{1}{r}}, & r < \infty, \\ \sup_{j\geq -1} 2^{js} \|\Delta_j f\|_{L^p}, & r = \infty. \end{cases}$$

The following lemma provides the Bernstein type inequalities for fractional derivatives.

Lemma A.1 (see [4]). *Assume $1 \leq a \leq b \leq \infty$. If the integer $j \geq -1$, then it holds that*

$$\|\Lambda^k \Delta_j f\|_{L^b} \leq C_1\, 2^{jk+jn(\frac{1}{a}-\frac{1}{b})} \|\Delta_j f\|_{L^a}, \quad k \geq 0.$$

If the integer $j \geq 0$, then we have

$$C_2\, 2^{jk} \|\Delta_j f\|_{L^b} \leq \|\Lambda^k \Delta_j f\|_{L^b} \leq C_3\, 2^{jk+jn(\frac{1}{a}-\frac{1}{b})} \|\Delta_j f\|_{L^a}, \quad k \in \mathbb{R},$$

where C_1, C_2 and C_3 are constants depending on k, a and b only.

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