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REVIEW ARTICLE

On the Vortex Sheets of Compressible Flows

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Abstract
This paper provides a review of the recent results on the stability of vortex sheets in compressible flows. Vortex sheets are contact discontinuities of the underlying flows. The vortex sheet problem is a free boundary problem with a characteristic boundary and is challenging in analysis. The formulation of the vortex sheet problem will be introduced. The linear stability and nonlinear stability for both the two-dimensional two-phase compressible flows and the two-dimensional elastic flows are summarized. The linear stability of vortex sheets for the three-dimensional elastic flows is also presented. The difficulties of the vortex sheet problems and the ideas of proofs are discussed.

Keywords Vortex sheets · Contact discontinuities · Stability and instability · Loss of derivatives · Two-phase flows · Elastic flows

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1 Introduction

Vortex sheets are contact discontinuities in fluid flows arising in many disciplines such as fluid mechanics, aerodynamics, astrophysics, oceanography, and so on. The flow velocity in vortex sheets is continuous in the normal direction but has a jump along the tangential direction. We refer the readers to [7, 8, 14, 15, 18, 24, 31, 38, 42, 43] and the references therein for more discussions on the physical background and applications of vortex sheets. The mathematical analysis of vortex sheets in fluid flows is challenging and has attracted wide attention, resulting in many significant works and progress. In this article, we provide a review of some recent results on the existence and stability of vortex sheet solutions for various compressible flows.

It is well known from the analysis in [18, 31, 32, 40] that the vortex sheets in the two-dimensional compressible flows governed by the Euler equations are unstable when the Mach number is less than $\sqrt{2}$, while the vortex sheets in the three-dimensional compressible Euler flows are always violently unstable. For the two-dimensional Euler flows

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = 0, \end{cases} \tag{1}$$

where ρ, $\mathbf{u}$, p are the density, velocity, and pressure, respectively, Coulombel and Secchi in their pioneer works [15, 16] established the local existence and stability of vortex sheet solutions when the Mach number is larger than $\sqrt{2}$. For the three-dimensional compressible flows, various stabilizing effects on the stability of vortex sheets have been discovered recently in the following works.

- Magnetic field: for the three-dimensional compressible magnetohydrodynamic flows, the nonlinear stability of vortex sheets was obtained in Chen and Wang [7] and Trakhinin [42] under a condition on the magnetic field using an energy method, which shows that the magnetic field can stabilize the three-dimensional vortex sheets.
- Surface tension: for the three-dimensional compressible Euler flows with surface tension, the local existence and structural stability of vortex sheets were proved in Stevens [41], which demonstrates that the surface tension provides a stabilizing effect on vortex sheets.
- Elasticity: for the three-dimensional compressible elastodynamic flows, the linear stability was established in Chen et al. [12] with the necessary and sufficient conditions for the linear stability and instability of vortex sheets, which indicates that the elasticity can provide stabilization on vortex sheets. In fact, the stabilizing effect on vortex sheets from elasticity was first revealed in the two-dimensional compressible isentropic elastic flows in [8–10, 21].

In addition, the linear and nonlinear stability of compressible vortex sheets have also been obtained in many other papers, mostly on the two-dimensional flows, such as [23, 38] for the two-dimensional two-phase flows, [45] for the two-dimensional magnetohydrodynamic flows, [5] for the relativistic flows, [34, 35] for the two-dimensional nonisentropic Euler flows, [11] for the two-dimensional nonisentropic elastic flows, [44, 46, 47] for the three-dimensional steady Euler flows, and so on. The stability of solutions with different discontinuities than vortex sheets in the two-dimensional compressible elastic flows was studied in [6, 33].

We want to point out that the incompressible vortex sheets have also been extensively studied, but this article will only focus on the results in the compressible flows.

Mathematically, the vortex sheet problem for the compressible Euler flows is a free boundary problem with a characteristic boundary, making it difficult to control the trace of the characteristic parts of the solutions. Since the Kreiss-Lopatinskiĭ condition does not hold uniformly, there is some loss of the tangential derivatives in the estimates of the solutions [15, 16]. There are extra difficulties in dealing with other compressible flows, for example, in elastic flows, the Lopatinskiĭ determinant has a more complicated distribution of roots, that is, the non-differentiable points of the eigenvalues may coincide with the roots of the Lopatinskiĭ determinant. The three-dimensional vortex sheet problems are often different from and much harder than the two-dimensional problems. As a result, novel techniques and ideas are required to establish both linear and nonlinear stability of vortex sheets; for example, the upper triangularization method was introduced to study the linear stability with constant coefficients of vortex sheets of elastic flows in [8] and later was adopted in the works of [5, 9, 10, 12]. Some standard techniques involved for the stability of vortex sheets (cf. [7, 15, 16, 34, 35, 42, 45]) include the normal mode analysis, symmetrization, para-linearization and microlocal analysis in the neighborhood of bicharacteristic curves [3, 30], Nash-Moser iteration, energy estimates in usual or anisotropic Sobolev spaces, and so on. The aim of this article is to review the recent results on the stability of compressible vortex sheets for the two-dimensional two-phase flows obtained in [23, 38], and for the two-dimensional and three-dimensional elastic flows in [8–10, 12]. In comparison with the compressible Euler flows, the two-phase flows have two different densities for the liquid and gas, making the analysis much more complicated (cf. [38]). For the elastic flows, the roots of the Lopatinskiĭ determinant exhibit substantial degeneracy, causing the Kreiss symmetrizer argument not applicable (cf. [8]). The works on vortex sheets in [8–12, 23, 38] investigate the linear stability with constant coefficients, the linear stability with variable coefficients, and the nonlinear stability, as in [15, 16].

The rest of the paper is organized as follows. In Sect. 2 , we present the stability results of vortex sheets in [23, 38] for the two-dimensional two-phase flows, including the linear stability with constant coefficients, the linear stability with variable coefficients, and the nonlinear stability. In Sect. 3, we summarize the stability results in [8–10] for the two-dimensional elastic flows, including the linear stability with constant coefficients, the linear stability with variable coefficients, and the nonlinear stability, and a brief remark on the nonisentropic flows. In Sect. 4, we review the linear stability results in [12] for the three-dimensional elastic flows.

2 Vortex Sheets in the Two-Phase Compressible Flows

In this section, we give a review of the stability results obtained in [23, 38] for the vortex sheets in the two-dimensional two-phase compressible flows, following closely their presentations. Consider the following equations of two-phase compressible flows of liquid-gas fluids in $\mathbb{R}^2$:

$$\begin{cases} \partial_t m + \nabla \cdot (m\mathbf{u}) = 0, \\ \partial_t n + \nabla \cdot (n\mathbf{u}) = 0, \\ \partial_t (n\mathbf{u}) + \nabla \cdot (n\mathbf{u} \otimes \mathbf{u}) + \nabla p(m,n) = 0, \end{cases} \tag{2}$$

where m is the gas mass, n is the liquid mass, $\mathbf{u}$ is the velocity, and

$$p(m,n) = (\gamma - 1)(m+n)^\gamma$$

is the pressure with $\gamma > 1$; see [4, 23, 26, 28, 29, 38] and the references therein for the physical background and applications of two-phase flows.

Denote by $x = (x_1, x_2)$ the spatial variable, $\mathbf{u} = (v, u)$ the velocity, and $\partial_1 = \partial_{x_1}$, $\partial_2 = \partial_{x_2}$. Let $(m, n, \mathbf{u})(t, x_1, x_2)$ be a piecewise smooth function across a smooth surface $\Gamma = \{x_2 = \varphi(t, x_1), t > 0, x_1 \in \mathbb{R}\}$, satisfying the Rankine-Hugoniot jump conditions on Γ:

$$\begin{cases} \partial_t \varphi[m] - [m\mathbf{u} \cdot \nu] = 0, \\ \partial_t \varphi[n] - [n\mathbf{u} \cdot \nu] = 0, \\ \partial_t \varphi[n\mathbf{u}] - [(n\mathbf{u} \cdot \nu)\mathbf{u}] - [p]\nu = 0, \end{cases} \tag{3}$$

where $\nu = (-\partial_1 \varphi, 1)$, $[q] = q^+ - q^-$ is the jump of a quantity q across the interface Γ, and q^+ and q^- denote the states in $\{x_2 > \varphi(t, x_1)\}$ and $\{x_2 < \varphi(t, x_1)\}$, respectively. A vortex sheet solution $(m, n, \mathbf{u})$ has continuous normal velocity and possible jump of tangential velocity, thus the Rankine-Hugoniot conditions (3) on Γ become the following:

$$\partial_t \varphi = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu, \quad m^+ + n^+ = m^- + n^-. \tag{4}$$

Then the problem of the existence and stability of vortex sheet solutions can be formulated as the following free boundary problem for $U^\pm = (m^\pm, n^\pm, v^\pm, u^\pm)$ and a free boundary $\Gamma = \{x_2 = \varphi(t, x_1), t > 0, x_1 \in \mathbb{R}\}$ such that

$$\begin{cases} \partial_t U^+ + A_1\left(U^+\right)\partial_1 U^+ + A_2\left(U^+\right)\partial_2 U^+ = 0, & x_2 > \varphi(t, x_1), \\ \partial_t U^- + A_1(U^-)\partial_1 U^- + A_2(U^-)\partial_2 U^- = 0, & x_2 < \varphi(t, x_1), \\ U(0, x_1, x_2) = \begin{cases} U_0^+(x_1, x_2), & x_2 > \varphi_0(x_1), \\ U_0^-(x_1, x_2), & x_2 < \varphi_0(x_1), \end{cases} \end{cases} \tag{5}$$

satisfying the jump conditions on Γ:

$$\partial_t \varphi = -v^+ \partial_1 \varphi + u^+ = -v^- \partial_1 \varphi + u^-, \quad m^+ + n^+ = m^- + n^-, \tag{6}$$

where $\varphi_0(x_1) = \varphi(0, x_1)$ and

$$A_1(U) = \begin{bmatrix} v & 0 & m & 0 \\ 0 & v & n & 0 \\ \frac{p_m}{n} & \frac{p_n}{n} & v & 0 \\ 0 & 0 & 0 & v \end{bmatrix}, \quad A_2(U) = \begin{bmatrix} u & 0 & 0 & m \\ 0 & u & 0 & n \\ 0 & 0 & u & 0 \\ \frac{p_m}{n} & \frac{p_n}{n} & 0 & u \end{bmatrix}.$$

We need to establish the local existence and stability of the problem (5)–(6) around a background piecewise constant vortex sheet solution.

To solve the vortex sheet problem we first reformulate the free-boundary into a fixed-boundary by straightening the interface using the standard partial hodograph transformation (cf. [19]):

$$t = \tilde{t}, \quad x_1 = \tilde{x}_1, \quad x_2 = \Phi^\pm\left(\tilde{t}, \tilde{x}_1, \tilde{x}_2\right) \tag{7}$$

with some smooth functions $\Phi^\pm$ satisfying

$$\pm\partial_{\tilde{x}_2} \Phi^\pm\left(\tilde{t}, \tilde{x}_1, \tilde{x}_2\right) \geqslant \kappa > 0, \quad \Phi^+\left(\tilde{t}, \tilde{x}_1, 0\right) = \Phi^-\left(\tilde{t}, \tilde{x}_1, 0\right) = \varphi\left(\tilde{t}, \tilde{x}_1\right) \tag{8}$$

for some constant $\kappa > 0$. Then the domain becomes the fixed domain $x_2 > 0$ (after dropping the tildes for simplicity of notation), the free boundary becomes the fixed boundary $x_2 = 0$, and the vortex sheet problem becomes the following problem for the smooth solutions $U^\pm = (m^\pm, n^\pm, v^\pm, u^\pm)^\top$ and $\Phi^\pm$:

$$\partial_t U^{\pm} + A_1(U^{\pm})\partial_1 U^{\pm} + \frac{1}{\partial_2 \Phi^{\pm}}\left[A_2(U^{\pm}) - \partial_t \Phi^{\pm} I_{4\times 4} - \partial_1 \Phi^{\pm} A_1(U^{\pm})\right]\partial_2 U^{\pm} = 0 \tag{9}$$

for $x_1 \in \mathbb{R},\ x_2 > 0$, with the boundary conditions on $x_2 = 0$:

$$\begin{cases} \Phi^+ = \Phi^- = \varphi, \\ (v^+ - v^-)\partial_1 \varphi - (u^+ - u^-) = 0, \\ \partial_t \varphi + v^+ \partial_1 \varphi - u^+ = 0, \\ (m^+ + n^+) - (m^- + n^-) = 0, \end{cases} \tag{10}$$

and the initial condition

$$(m^{\pm}, n^{\pm}, v^{\pm}, u^{\pm})|_{t=0} = (m_0^{\pm}, n_0^{\pm}, v_0^{\pm}, u_0^{\pm})(x_1, x_2), \quad \varphi|_{t=0} = \varphi_0(x_1). \tag{11}$$

We take the following simple vortex sheet solution with piecewise constants:

$$U_r = \left(m_r, n_r, v_r, 0\right)^{\top}, \ \ U_l = \left(m_l, n_l, v_l, 0\right)^{\top}, \ \ \Phi_{r,l}(t, x_1, x_2) \equiv \pm x_2, \ \ \varphi \equiv 0 \tag{12}$$

with $m_r + n_r = m_l + n_l,\ v_r + v_l = 0$, and $m_r, m_l, n_r, n_l > 0$ as well as $v_r > 0$ without loss of generality. We shall study the vortex sheet problem around the background solution defined by (12).

2.1 Linear Stability with Constant Coefficients

We now present the linear stability result of vortex sheets with constant coefficients in [38]. Denote by $\dot{U}_{\pm} = (\dot{m}_{\pm}, \dot{n}_{\pm}, \dot{v}_{\pm}, \dot{u}_{\pm})$ and $\Psi_{\pm}$ the small perturbation of the background solution (12), set

$$U^{\pm} = U_{r,l} + \dot{U}_{\pm}, \quad \Phi^{\pm} = \Phi_{r,l} + \Psi_{\pm},$$

and consider the linearized problem

$$\begin{cases} L\dot{U} = f, & \text{if } x_2 > 0, \\ B(\dot{U}, \psi) = g, & \text{if } x_2 = 0, \end{cases} \tag{13}$$

where $\dot{U} = (\dot{U}_+, \dot{U}_-)$, and

$$L\dot{U} = \partial_t \begin{bmatrix} \dot{U}_+ \\ \dot{U}_- \end{bmatrix} + \begin{bmatrix} A_1(U_r) & 0 \\ 0 & A_2(U_l) \end{bmatrix} \partial_1 \begin{bmatrix} \dot{U}_+ \\ \dot{U}_- \end{bmatrix} + \begin{bmatrix} A_1(U_r) & 0 \\ 0 & -A_2(U_l) \end{bmatrix} \partial_2 \begin{bmatrix} \dot{U}_+ \\ \dot{U}_- \end{bmatrix},$$

$$B(\dot{U}, \psi) = \begin{bmatrix} \left(v_r - v_l\right)\partial_1 \psi - \left(\dot{u}_+ - \dot{u}_-\right) \\ \partial_t \psi + v_r \partial_1 \psi - \dot{u}_+ \\ \left(\dot{m}_+ + \dot{n}_+\right) - \left(\dot{m}_- + \dot{n}_-\right) \end{bmatrix}.$$

We shall establish the estimates of $(\dot{U}, \psi)$ in terms of f and g in appropriate functional spaces.

Set $c_{r,l}^2 = \left(1 + \frac{m_{r,l}}{n_{r,l}}\right)p_n$ and perform the following changes of variables:

$$\begin{bmatrix} \dot{m}_+ \\ \dot{n}_+ \\ \dot{v}_+ \\ \dot{u}_+ \end{bmatrix} = \begin{bmatrix} 2n_r & 0 & -2m_r & 2m_r \\ -2n_r & 0 & -2n_r & 2n_r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2c_r & 2c_r \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix},$$

and

$$\begin{bmatrix} \dot{m}_- \\ \dot{n}_- \\ \dot{v}_- \\ \dot{u}_- \end{bmatrix} = \begin{bmatrix} 2n_l & 0 & -2m_l & 2m_l \\ -2n_l & 0 & -2n_l & 2n_l \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2c_l & 2c_l \end{bmatrix} \begin{bmatrix} W_5 \\ W_6 \\ W_7 \\ W_8 \end{bmatrix}.$$

Define $W = (W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_8)^\top$, and its "characteristic part" and "noncharacteristic part": $W^{\mathrm{c}} = (W_1, W_2, W_5, W_6)^\top$, $W^{\mathrm{nc}} = (W_3, W_4, W_7, W_8)^\top$.

Then we have the following system for W (see [38]):

$$\begin{cases} \mathcal{L}W = \mathcal{A}_0 \partial_t W + \mathcal{A}_1 \partial_1 W + \mathcal{A}_2 \partial_2 W = f, & \text{if } x_2 > 0, \\ \mathcal{B}(W^{\mathrm{nc}}, \psi) = \underline{M} W^{\mathrm{nc}}|_{x_2=0} + \underline{b}(\partial_t \psi, \partial_1 \psi)^\top = g, & \text{if } x_2 = 0, \end{cases} \tag{14}$$

where

$$\mathcal{A}_0 = \begin{bmatrix} \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 2c_r^2 & 0 \\ 0 & 0 & 0 & 2c_r^2 \end{matrix} & \mathbf{O} \\ \mathbf{O} & \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 2c_l^2 & 0 \\ 0 & 0 & 0 & 2c_l^2 \end{matrix} \end{bmatrix},$$

$$\mathcal{A}_1 = \begin{bmatrix} \begin{matrix} v_r & 0 & 0 & 0 \\ 0 & \frac{1}{4}v_r & -\frac{1}{2}c_r^2 & \frac{1}{2}c_r^2 \\ 0 & -\frac{1}{2}c_r^2 & 2c_r^2 v_r & 0 \\ 0 & \frac{1}{2}c_r^2 & 0 & 2c_r^2 v_r \end{matrix} & \mathbf{O} \\ \mathbf{O} & \begin{matrix} v_l & 0 & 0 & 0 \\ 0 & \frac{1}{4}v_l & -\frac{1}{2}c_l^2 & \frac{1}{2}c_l^2 \\ 0 & -\frac{1}{2}c_l^2 & 2c_l^2 v_l & 0 \\ 0 & \frac{1}{2}c_l^2 & 0 & 2c_l^2 v_l \end{matrix} \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2c_r^3 & 0 \\ 0 & 0 & 0 & 2c_r^3 \end{matrix} & \mathbf{O} \\ \mathbf{O} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2c_l^3 & 0 \\ 0 & 0 & 0 & -2c_l^3 \end{matrix} \end{bmatrix},$$

$$\underline{b} = \begin{bmatrix} 0 & 2v_r \\ 1 & v_r \\ 0 & 0 \end{bmatrix}, \quad \underline{M} = \begin{bmatrix} -2c_r & -2c_r & 2c_l & 2c_l \\ -2c_r & -2c_r & 0 & 0 \\ -2(m_r + n_r) & 2(m_r + n_r) & 2(m_l + n_l) & -2(m_l + n_l) \end{bmatrix}.$$

The problem (14) is a hyperbolic problem with characteristic boundary. We need to introduce some weighted Sobolev norms following [38]. Let $\Omega = \left\{(t, x_1, x_2) \in \mathbb{R}^3 : x_2 > 0\right\}$ be the half-space. For $s \in \mathbb{R}$ and $\lambda \geqslant 1$, define the weighted Sobolev space

$$H^s_\lambda(\mathbb{R}^2) = \left\{u \in \mathcal{D}'(\mathbb{R}^2):\ \exp(-\lambda t)u \in H^s(\mathbb{R}^2)\right\}$$

with the norm $\|u\|_{H^s_\lambda(\mathbb{R}^2)} = \|\exp(-\lambda t)u\|_{H^s(\mathbb{R}^2)}$. Set $\tilde{u} = \exp(-\lambda t)u$, then $\|u\|_{H^s_\lambda(\mathbb{R}^2)} \simeq \|\tilde{u}\|_{s,\lambda}$, where

$$\|v\|^2_{s,\lambda} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \left(\lambda^2 + |\xi|^2\right)^s |\hat{v}(\xi)|^2 \mathrm{d}\xi,$$

and $\hat{v}$ is the Fourier transform of v. Let

$$H^k_\lambda(\Omega) = \left\{u \in \mathcal{D}'(\Omega):\ \exp(-\lambda t)u \in H^k(\Omega)\right\}$$

for an integer k and $\lambda \geqslant 1$. For $s > r$, $H^s_\lambda\left(\mathbb{R}^2\right) \subset H^r_\lambda\left(\mathbb{R}^2\right)$ and $\|v\|_{r,\lambda} \leqslant \frac{1}{\lambda^{s-r}}\|v\|_{s,\lambda}$. The norm of the space $L^2\left(\mathbb{R}^+; H^s_\lambda(\mathbb{R}^2)\right)$ is given by

$$|||v|||^2_{L^2(H^s_\lambda)} = \int_0^\infty \|v(\cdot, x_2)\|^2_{H^s_\lambda(\mathbb{R}^2)} \mathrm{d}x_2.$$

We have the linear stability with constant coefficients stated as follows.

Theorem 1 [38, Theorem 3.1] *Let* $\left(U_{r,l}, \Phi_{r,l}\right)$ *be the background solution in* (12).

(i) *If*

$$v_r - v_l > \left(c_r^{\frac{2}{3}} + c_l^{\frac{2}{3}}\right)^{\frac{3}{2}} \quad \textit{and} \quad v_r - v_l \neq \sqrt{2}(c_r + c_l), \tag{15}$$

then there exists a positive constant C *such that for all* $\lambda \geqslant 1$ *and for all solutions* $(W, \psi) \in H^2_\lambda(\Omega) \times H^2_\lambda(\mathbb{R}^2)$ *to* (14), *the following estimate holds*:

$$\begin{aligned} &\lambda|||W|||^2_{L^2_\lambda(\Omega)} + \|W^{\mathrm{nc}}|_{x_2=0}\|^2_{L^2_\lambda(\mathbb{R}^2)} + \|\psi\|^2_{H^1_\lambda(\mathbb{R}^2)} \\ &\leqslant C\left(\tfrac{1}{\lambda^3}|||\mathcal{L}W|||^2_{L^2(H^1_\lambda)} + \tfrac{1}{\lambda^2}\|\mathcal{B}(W,\psi)\|^2_{H^1_\lambda(\mathbb{R}^2)}\right). \end{aligned} \tag{16}$$

(ii) *If*

$$v_r - v_l = \sqrt{2}(c_r + c_l), \tag{17}$$

then there exists a positive constant C *such that for all* $\lambda \geqslant 1$ *and for all solutions* $(W, \psi) \in H^3_\lambda(\Omega) \times H^3_\lambda(\mathbb{R}^2)$ *to* (14), *the following estimate holds*:

$$\begin{aligned} &\lambda|||W|||^2_{L^2_\lambda(\Omega)} + \|W^{\mathrm{nc}}|_{x_2=0}\|^2_{L^2_\lambda(\mathbb{R}^2)} + \|\psi\|^2_{H^1_\lambda(\mathbb{R}^2)} \\ &\leqslant C\Big(\frac{1}{\lambda^5}|||\mathcal{L}W|||^2_{L^2(H^2_\lambda)} + \frac{1}{\lambda^4}\|\mathcal{B}(W,\psi)\|^2_{H^2_\lambda(\mathbb{R}^2)}\Big). \end{aligned} \tag{18}$$

Theorem 1 can be proved using a normal mode analysis of (14) as in [15] with the elimination of the front, construction of the symmetrizer and energy estimates; see [38] for details.

2.2 Linear Stability with Variable Coefficients

For the linear stability with variable coefficients [38], we first linearize the equations around the state $U_{r,l}(t, x_1, x_2)$, $\Phi_{r,l}(t, x_1, x_2)$ as a perturbation of the constant solution in (12) given by the following:

$$\begin{cases} U_{r,l}(t, x_1, x_2) = \big(m_{r,l}, n_{r,l}, \pm v_r, 0\big)^\top + \dot{U}_{r,l}(t, x_1, x_2), \\ \Phi_{r,l}(t, x_1, x_2) = \pm x_2 + \dot{\Phi}_{r,l}(t, x_1, x_2), \end{cases} \tag{19}$$

such that $U_{r,l}, \nabla\Phi_{r,l} \in W^{2,\infty}(\Omega)$, $\|(U_r, U_l)\|_{W^{2,\infty}(\Omega)} + \|(\nabla\Phi_r, \nabla\Phi_l)\|_{W^{2,\infty}(\Omega)} \leqslant K_0$ for some constant $K_0 > 0$, where $\dot{U}_{r,l}$ have compact support. As in [38] we consider the linearized problem for the good unknown $\dot{V} = (\dot{V}_+, \dot{V}_-)^\top$ (see [1]) around the state $U_{r,l}$, $\Phi_{r,l}$:

$$\begin{cases} L_r'\dot{V}_+ = L(U_r, \nabla\Phi_r)\dot{V}_+ + C(U_r, \nabla U_r, \nabla\Phi_r)\dot{V}_+ = f_+, \\ L_l'\dot{V}_- = L(U_l, \nabla\Phi_l)\dot{V}_- + C(U_l, \nabla U_l, \nabla\Phi_l)\dot{V}_- = f_-, \\ \Psi^+(t, x_1, x_2)|_{x_2=0} = \Psi^-(t, x_1, x_2)|_{x_2=0} = \psi(t, x_1), \\ B'(U_{r,l}, \Phi_{r,l})\big(\dot{V}|_{x_2=0}, \psi\big) = b\nabla\psi + M\Big(\frac{\partial_2 U_r}{\partial_2\Phi_r}, \frac{\partial_2 U_l}{\partial_2\Phi_l}\Big)^\top\Big|_{x_2=0}\psi + M\dot{V}|_{x_2=0} = g, \end{cases} \tag{20}$$

where

$$L(U_r, \nabla\Phi_r) = \partial_t + A_1(U_r)\partial_1 + \frac{1}{\partial_2\Phi_r}\big[A_2(U_r) - \partial_t\Phi_r I_{4\times4} - \partial_1\Phi_r A_1(U_r)\big]\partial_2,$$

and

$$C(U_r, \nabla U_r, \nabla\Phi_r)\dot{V}_+ = \big(dA_1(U_r)\dot{V}_+\big)\partial_1 U_r + \frac{1}{\partial_2\Phi_r}\big\{dA_2(U_r)\dot{V}_+ - \partial_1\Phi_r[dA_1(U_r)\dot{V}_+]\big\}\partial_2 U_r$$

with the similar formulas for $L(U_l, \nabla\Phi_l)$ and $C(U_l, \nabla U_l, \nabla\Phi_l)\dot{V}_-$, and

$$b(t, x_1) = \begin{bmatrix} 0 & \big(v_r - v_l\big)|_{x_2=0} \\ 1 & v_r|_{x_2=0} \\ 0 & 0 \end{bmatrix}, \quad M(t, x_1) = \begin{bmatrix} 0 & 0 & \partial_1\varphi & -1 & 0 & 0 & -\partial_1\varphi & 1 \\ 0 & 0 & \partial_1\varphi & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \end{bmatrix}.$$

The linear stability with variable coefficients reads as follows.

Theorem 2 [38, Theorem 5.1] *Assume that the particular solution defined by* (19) *satisfies*

$$v_r - v_l > \Big(c_r^{\frac{2}{3}} + c_l^{\frac{2}{3}}\Big)^{\frac{3}{2}}, \quad v_r - v_l \neq \sqrt{2}\big(c_r + c_l\big), \tag{21}$$

and that the perturbations $\dot{U}_{r,l}$, $\nabla\dot{\Phi}_{r,l}$ *have compact support and are small enough in* $W^{2,\infty}(\Omega)$. *Then there exist some constants* C_1 *and* $\lambda_1 \geqslant 1$, *such that, for all* $\lambda \geqslant \lambda_1$, *the solution* $(\dot{V}, \psi) \in H^2_\lambda(\Omega) \times H^2_\lambda(\mathbb{R}^2)$ *to the linearized problem* (20) *satisfies the following estimates*:

$$
\begin{aligned}
&\lambda|||\dot{V}|||^2_{L^2_\lambda(\Omega)} + \|\dot{V}^{\rm nc}|_{x_2=0}\|^2_{L^2_\lambda(\mathbb{R}^2)} + \|\psi\|^2_{H^1_\lambda(\mathbb{R}^2)} \\
&\leqslant C_1\Big(\frac{1}{\lambda^3}|||L'\dot{V}|||^2_{L^2(H^1_\lambda)} + \frac{1}{\lambda^2}|||B'(\dot{V},\psi)|||^2_{H^1_\lambda(\mathbb{R}^2)}\Big) \\
&= C_1\Big(\frac{1}{\lambda^3}|||(f_+,f_-)|||^2_{L^2(H^1_\lambda)} + \frac{1}{\lambda^2}|||g|||^2_{H^1_\lambda(\mathbb{R}^2)}\Big).
\end{aligned}
\tag{22}
$$

The idea for proving the above theorem is to turn the variable-coefficient problem into the constant-coefficient problem by freezing the coefficients. The proof of Theorem 2 includes the transformations of the interior equations by the Friedrichs symmetrization, para-linearization and elimination of the front, energy estimates for the para-linearized problem and microlocalization as in [15], and the details can be found in [38].

2.3 Nonlinear Stability

To prove the existence and stability of vortex sheet solutions for the free-boundary problem (9)–(11), we need to find a solution $U(t,x_1,x_2)$ and $\varphi(t,x_1)$ locally in time, which was achieved in [23] from the linearization of (9)–(11) around the piecewise constant background vortex sheet solution (12). For the convenience of notation we also denote by $\bar{U}^\pm$ the background solution $U_{r,l}$, i.e., $\bar{U}^\pm = U_{r,l}$. Then we have the existence and nonlinear stability of vortex sheet solutions as follows.

Theorem 3 [23, Theorem 2.1] *Let* $T>0$, $\alpha\in\mathbb{N}, \alpha\geqslant 15$, *and the background solution defined by* (12) *satisfy the "supersonic" condition:*

$$
v_r - v_l > \left(c_r^{\frac{2}{3}} + c_l^{\frac{2}{3}}\right)^{\frac{3}{2}}, \quad v_r - v_l \neq \sqrt{2}(c_r + c_l),
\tag{23}
$$

where $c_{r,l} = \sqrt{(1+\frac{m_{r,l}}{n_{r,l}})p_n(m_{r,l},n_{r,l})}$. *Assume that the initial data* $(U_0^\pm,\varphi_0)$ *has the form* $U_0^\pm = \bar{U}^\pm + \dot{U}_0^\pm$, *with* $\dot{U}_0^\pm \in H_*^{2\alpha+15}(\mathbb{R}^2_+)$ *and* $\varphi_0\in H^{2\alpha+16}(\mathbb{R})$ *compatible up to order* $\alpha+7$ *and compactly supported. Then, there exists* $\delta>0$, *such that, if* $[\dot{U}_0^\pm]_{2\alpha+15,*,T} + ||\varphi_0||_{H^{2\alpha+16}} \leqslant \delta$, *the problem* (9)–(11) *has a unique solution* $U^\pm = \bar{U}^\pm + \dot{U}^\pm, \Phi^\pm = \pm x_2 + \dot{\Phi}^\pm, \varphi$ *on* $[0,\ T]$, *satisfying* $(\dot{U}^\pm,\dot{\Phi}^\pm)\in H_*^{\alpha-1}((0,T)\times\mathbb{R}^2_+)$, *and* $\varphi\in H^\alpha((0,T)\times\mathbb{R})$.

The proof of Theorem 3 can be found in [23]. In the proof, we apply the Nash-Moser procedure as in [16], but we need to use the anisotropic Sobolev spaces (cf. [13, 39]) instead of the usual Sobolev spaces to deal with the jump of the normal derivatives of the densities in order to derive the derivative estimates and the tame estimates. The major steps of the proof include deriving the a priori estimates of the tangential derivatives and normal derivatives, and the tame estimates in the anisotropic Sobolev spaces. A different symmetrization for the two-phase flows was used in [37] for the local existence of shock waves and vortex sheets based on the result in [35].

3 Vortex Sheets in the Two-Dimensional Compressible Elastodynamics

In this section, we summarize the works on the stability of vortex sheets for elastic flows in [8–10] (see also [21]). We consider the vortex sheets for the two-dimensional isentropic compressible flows in elastodynamics (cf. [17]) of the form

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div}(\rho \mathtt{F}\mathtt{F}^\top), \\ \mathtt{F}_t + \mathbf{u} \cdot \nabla \mathtt{F} = \nabla \mathbf{u} \mathtt{F}, \end{cases} \tag{24}$$

where ρ denotes the density, $\mathbf{u} = (v, u) \in \mathbb{R}^2$ the velocity, $\mathtt{F} = (F_{ij}) \in \mathbf{M}^{2\times 2}$ the deformation gradient, and $p = p(\rho)$ the pressure. Using the intrinsic property $\operatorname{div}(\rho \mathtt{F}^\top) = 0$ one can rewrite the system (24) as the following conservative form:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - \operatorname{div}(\rho \mathtt{F}\mathtt{F}^\top) = 0, \\ (\rho \mathtt{F}_j)_t + \operatorname{div}(\rho \mathtt{F}_j \otimes \mathbf{u} - \mathbf{u} \otimes \rho \mathtt{F}_j) = 0, \end{cases} \tag{25}$$

where $\mathtt{F}_j$ is the jth column of the deformation gradient $\mathtt{F}$, $j = 1, 2$. We refer the readers to [2, 22, 24, 25, 27, 36] for the physical background and motivation of the vortex sheets in elastic flows. Let $U(t, x_1, x_2) = (\rho, \mathbf{u}, \mathtt{F})(t, x_1, x_2)$ be a solution to the system (25) which is smooth on each side of a smooth interface $\varGamma = \{x_2 = \psi(t, x_1)\}$:

$$U(t, x_1, x_2) = \begin{cases} U^+(t, x_1, x_2), & \text{when } x_2 > \psi(t, x_1), \\ U^-(t, x_1, x_2), & \text{when } x_2 < \psi(t, x_1), \end{cases}$$

where $U^\pm = (\rho^\pm, \mathbf{u}^\pm, \mathtt{F}^\pm)$. Setting $\partial_i = \partial_{x_i}$, $i = 1, 2$, we denote by $\nu = (-\partial_1 \psi, 1)$ a normal vector on $\varGamma$. For a vortex sheet in the elastic flow, the Rankine-Hugoniot jump conditions become

$$\rho^+ = \rho^-, \quad \psi_t = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu, \quad \mathtt{F}_j^+ \cdot \nu = \mathtt{F}_j^- \cdot \nu = 0, \quad j = 1, 2 \quad \text{on } \varGamma. \tag{26}$$

As for the two-phase flows in Sect. 2, we introduce the change of variables (cf. [19]) to straighten the free boundary $\varGamma$ through the functions $\Phi^\pm(t, x_1, x_2) = \Phi(t, x_1, \pm x_2)$ with $\inf\{\partial_2 \Phi\} > 0$ and $\Phi(t, x_1, 0) = \psi(t, x_1)$, and $\partial_t \Phi^\pm + v^\pm \partial_1 \Phi^\pm - u^\pm = 0$ for $x_2 \geqslant 0$. Then we arrive at the following problem:

$$\partial_t U^\pm + A_1(U^\pm)\partial_1 U^\pm + \frac{1}{\partial_2 \Phi^\pm}\big[A_2(U^\pm) - \partial_t \Phi^\pm I - \partial_1 \Phi^\pm A_3(U^\pm)\big]\partial_2 U^\pm = 0 \tag{27}$$

for $x_2 > 0$ with the boundary conditions on $x_2 = 0$:

$$\begin{cases} (v^+ - v^-)\partial_1 \psi - (u^+ - u^-) = 0, \\ \partial_t \psi + v^+ \partial_1 \psi - u^+ = 0, \\ \rho^+ - \rho^- = 0, \end{cases} \tag{28}$$

where the precise formulas of the matrices A_1, A_2, A_3 can be found in [8]. We remark that the condition $\mathtt{F}_j^\pm \cdot \nu = 0$ in (26) holds for $t > 0$ if it is satisfied at $t = 0$ due to the transport equation $\partial_t(\mathrm{F}_j \cdot \nu) + \mathbf{u} \cdot \nabla(\mathrm{F}_j \cdot \nu) = 0$, thus $\mathrm{F}_j^{\pm} \cdot \nu = 0$ is considered as the restriction on the initial data and hence does not appear in the boundary conditions (28). The above system (27)–(28) has a piecewise constant solution of the following form:

$$\mathring{U}^{+} = (\mathring{\rho}, v^r, 0, F_{11}^r, 0, F_{12}^r, 0)^{\top},\ \mathring{U}^{-} = (\mathring{\rho}, v^l, 0, F_{11}^l, 0, F_{12}^l, 0)^{\top},\ \mathring{\Phi}^{\pm}(t, x_1, x_2) = \pm x_2, \tag{29}$$

where the constants $\mathring{\rho}$, v^r, v^l, F_{11}^r, F_{11}^l, F_{12}^r, and F_{12}^l satisfy

$$v^r + v^l = F_{11}^r + F_{11}^l = F_{12}^r + F_{12}^l = 0 \text{ and } v^r > 0, F_{11}^r, F_{12}^r \neq 0.$$

We want to construct the vortex sheet solutions of (27)–(28) which is a perturbation of the background solution (29) and prove the stability.

3.1 Linear Stability with Constant Coefficients

Now we linearize the system (27)–(28) around the above constant states (29). Let $\dot{U}^{\pm} = (\dot{\rho}^{\pm}, \dot{\mathbf{u}}^{\pm}, \dot{\mathrm{F}}^{\pm}) = U^{\pm} - \mathring{U}^{\pm}$ and $\dot{\Phi}^{\pm} = \Phi^{\pm} - \mathring{\Phi}^{\pm}$ be the small perturbation of the constant solution. As in [8], we consider the following change of variables:

$$W = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} \dot{U}^{+} \\ \dot{U}^{-} \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2\mathring{\rho}} & 0 & \frac{1}{2c} & 0 & 0 & 0 & 0 \\ \frac{1}{2\mathring{\rho}} & 0 & \frac{1}{2c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{30}$$

with $c = \sqrt{p'(\mathring{\rho})}$. Denote

$$W = (W_1, W_2, W_3, \cdots, W_{14})^{\top},$$

and consider the following linearized problem:

$$\begin{cases} \mathcal{L}W = \mathcal{A}_0 \partial_t W + \mathcal{A}_1 \partial_1 W + \mathcal{A}_2 \partial_2 W = 0, & x_2 > 0, \\ \mathcal{B}(W^{\mathrm{nc}}, \varphi) = \underline{M} W^{\mathrm{nc}} + \underline{b}(\partial_t \varphi, \partial_1 \varphi)^{\top} = 0, & x_2 = 0, \end{cases} \tag{31}$$

where $W^{\mathrm{nc}} = (W_2, W_3, W_9, W_{10})^{\top}$ is the non-characteristic part of W, and the precise formulas of $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{A}_2$, $\underline{M}$, $\underline{b}$ can be found in [8].

Using the notation of the weighted Sobolev spaces and norms defined in Sect. 2.1, we have the following linear stability with constant coefficients.

Theorem 4 [8, *Theorem* 2.1]

(i) *If the particular solution defined by* (29) *satisfies*

$$\begin{aligned} &(v^r)^2 > 2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2, \quad \textit{or} \\ &(v^r)^2 < (F_{11}^r)^2 + (F_{12}^r)^2 \textit{ and } (v^r)^2 \neq \frac{\left((F_{11}^r)^2 + (F_{12}^r)^2\right)\left(2c^2 + (F_{11}^r)^2 + (F_{12}^r)^2\right)}{4\left((F_{11}^r)^2 + (F_{12}^r)^2 + c^2\right)}, \end{aligned} \tag{32}$$

then there is a positive constant C such that for all $\lambda \geqslant 1$, $W \in H^2_\lambda(\mathbb{R}^3_+)$ *and* $\varphi \in H^2_\lambda(\mathbb{R}^2)$, *the following estimate holds*:

$$\begin{aligned}&\lambda|||W|||^2_{L^2(H^0_\lambda)} + \|W^{\mathrm{nc}}|_{x_2=0}\|^2_{L^2_\lambda(\mathbb{R}^2)} + \|\varphi\|^2_{H^1_\lambda(\mathbb{R}^2)}\\&\leqslant C\Big(\frac{1}{\lambda^3}|||\mathcal{L}W|||^2_{L^2(H^1_\lambda)} + \frac{1}{\lambda^2}\|\mathcal{B}(W^{\mathrm{nc}}|_{x_2=0}, \varphi)\|^2_{H^1_\lambda(\mathbb{R}^2)}\Big).\end{aligned} \tag{33}$$

(ii) *If the particular solution defined by* (29) *satisfies*

$$\begin{aligned}&(v^r)^2 = \frac{\big((F^r_{11})^2 + (F^r_{12})^2\big)\big(2c^2 + (F^r_{11})^2 + (F^r_{12})^2\big)}{4\big((F^r_{11})^2 + (F^r_{12})^2 + c^2\big)}, \quad or\\&(v^r)^2 = (F^r_{11})^2 + (F^r_{12})^2,\end{aligned} \tag{34}$$

then there is a positive constant C such that for all $\lambda \geqslant 1$, $W \in H^3_\lambda(\mathbb{R}^3_+)$ *and* $\varphi \in H^3_\lambda(\mathbb{R}^2)$, *the following estimate holds*:

$$\begin{aligned}&\lambda|||W|||^2_{L^2(H^0_\lambda)} + \|W^{\mathrm{nc}}|_{x_2=0}\|^2_{L^2_\lambda(\mathbb{R}^2)} + \|\varphi\|^2_{H^1_\lambda(\mathbb{R}^2)}\\&\leqslant C\Big(\frac{1}{\lambda^5}|||\mathcal{L}W|||^2_{L^2(H^2_\lambda)} + \frac{1}{\lambda^4}\|\mathcal{B}(W^{\mathrm{nc}}|_{x_2=0}, \varphi)\|^2_{H^2_\lambda(\mathbb{R}^2)}\Big).\end{aligned} \tag{35}$$

(iii) *If the particular solution defined by* (29) *satisfies*

$$(v^r)^2 = (F^r_{11})^2 + (F^r_{12})^2 + 2c^2, \tag{36}$$

then there is a positive constant C such that for all $\lambda \geqslant 1$, $W \in H^4_\lambda(\mathbb{R}^3_+)$ *and* $\varphi \in H^4_\lambda(\mathbb{R}^2)$, *the following estimate holds*:

$$\begin{aligned}&\lambda|||W|||^2_{L^2(H^0_\lambda)} + \|W^{\mathrm{nc}}|_{x_2=0}\|^2_{L^2_\lambda(\mathbb{R}^2)} + \|\varphi\|^2_{H^1_\lambda(\mathbb{R}^2)}\\&\leqslant C\Big(\frac{1}{\lambda^7}|||\mathcal{L}W|||^2_{L^2(H^3_\lambda)} + \frac{1}{\lambda^6}\|\mathcal{B}(W^{\mathrm{nc}}|_{x_2=0}, \varphi)\|^2_{H^3_\lambda(\mathbb{R}^2)}\Big).\end{aligned} \tag{37}$$

(iv) *If the particular solution defined by* (29) *satisfies*

$$(F^r_{11})^2 + (F^r_{12})^2 < (v^r)^2 < 2c^2 + (F^r_{11})^2 + (F^r_{12})^2, \tag{38}$$

the constant vortex sheets (29) *is linearly unstable*, *in the sense that the Lopatinskiĭ condition is violated.*

As remarked in [8], the above theorem provides a sufficient and necessary condition on the stability of the linearized problem, that is, one has the linear stability under the conditions (i)–(iii), and the instability under the remaining condition (iv). The elasticity gives a stable subsonic region in (32), which shows the stabilization effect. To prove this theorem, one major difficulty is that there is some degeneracy at the roots of the Lopatinskiĭ determinant, which make it hard to apply the Kreiss symmetrizer argument (cf. [15]) because it does not separate the incoming and outgoing modes at those points of degeneracy. Instead an upper triangularization method was developed in [8] to separate only the outgoing modes that were shown to be zero, thus one only needs to derive the estimates for the incoming modes directly from the Lopatinskiĭ determinant. The upper triangularization procedure simplifies greatly the computations and estimates for the outgoing modes and can apply to other compressible flows [6, 9, 10, 12].

3.2 Linear Stability with Variable Coefficients

Next we discuss the linear stability of compressible vortex sheets with variable coefficients in [9]. For the linear stability of the variable coefficients, we consider the following background state:

$$\begin{cases} U^{r,l} = (\rho^{r,l}, v^{r,l}, u^{r,l}, F_{11}^{r,l}, F_{21}^{r,l}, F_{12}^{r,l}, F_{22}^{r,l})^\top = \bar{U}^{r,l} + \dot{U}^{r,l} \\ \qquad = (\bar{\rho}, \pm\bar{v}, 0, \pm\bar{F}_{11}, 0, \pm\bar{F}_{12}, 0)^\top + (\dot{\rho}^{r,l}, \dot{v}^{r,l}, \dot{u}^{r,l}, \dot{F}_{11}^{r,l}, \dot{F}_{21}^{r,l}, \dot{F}_{12}^{r,l}, \dot{F}_{22}^{r,l})^\top, \\ \Phi^{r,l}(t, x_1, x_2) = \pm x_2 + \dot{\Phi}^{r,l}, \end{cases} \tag{39}$$

where $U^{r,l}$ and $\Phi^{r,l}$ are states on the both sides of the vortex sheet; $\bar{\rho} > 0$, $\bar{v}$, $\bar{F}_{11}$, and $\bar{F}_{12}$ are constants; $\dot{U}^{r,l}$ and $\dot{\Phi}^{r,l}$ are functions which are the perturbation around the constant states, such that, $\dot{U}^{r,l} \in W^{2,\infty}(\Omega), \dot{\Phi}^{r,l} \in W^{3,\infty}(\Omega)$, $\|(\dot{U}^r, \dot{U}^l)\|_{W^{2,\infty}(\Omega)} + \|(\dot{\Phi}^r, \dot{\Phi}^l)\|_{W^{3,\infty}(\Omega)} \leqslant K$ for some constant $K > 0$, and $\dot{U}^{r,l}$ and $\dot{\Phi}^{r,l}$ have compact support, where $\Omega = \{(t, x_1, x_2) \in \mathbb{R}^3 : x_2 > 0\}$.

Now we linearize (27) and the boundary conditions (28) around the states (39) and denote by $(V^\pm, \Psi^\pm)$ the perturbation of the states $(U^{r,l}, \Phi^{r,l})$. We define the operator

$$\begin{aligned} &L(U^{r,l}, \nabla\Phi^{r,l})V^\pm \\ &= \partial_t V^\pm + A_1(U^{r,l})\partial_1 V^\pm + \frac{1}{\partial_2\Phi^{r,l}}\big(A_2(U^{r,l}) - \partial_t\Phi^{r,l} - \partial_1\Phi^{r,l}A_1(U^{r,l})\big)\partial_2 V^\pm, \end{aligned}$$

set the good unknowns [1]

$$\dot{V}^\pm = \big(\dot{\rho}^\pm, \dot{v}^\pm, \dot{u}^\pm, \dot{F}_{11}^\pm, \dot{F}_{21}^\pm, \dot{F}_{12}^\pm, \dot{F}_{22}^\pm\big)^\top = V^\pm - \frac{\Psi^\pm}{\partial_2\Phi^{r,l}}\partial_2 U^{r,l}$$

and then consider the following linearized problem:

$$\begin{cases} L'_{r,l}\dot{V}^\pm = f^{r,l}, & x_2 > 0, \\ B'(\dot{V}, \psi) = g, & x_2 = 0, \end{cases} \tag{40}$$

where

$$\begin{aligned} &L'_{r,l}\dot{V}^\pm = L(U^{r,l}, \nabla\Phi^{r,l})\dot{V}^\pm + C(U^{r,l}, \nabla U^{r,l}, \nabla\Phi^{r,l})\dot{V}^\pm = f^{r,l}, \\ &C(U^{r,l}, \nabla U^{r,l}, \nabla\Phi^{r,l})\dot{V}^\pm \\ &= [dA_1(U^{r,l})\dot{V}^\pm]\partial_1 U^{r,l} + \frac{1}{\partial_2\Phi^{r,l}}[dA_2(U^{r,l})\dot{V}^\pm - \partial_1\Phi^{r,l}dA_1(U^{r,l})\dot{V}^\pm]\partial_2 U^{r,l}, \\ &B'(\dot{V}, \psi) = \underline{b}\nabla\psi + \underline{M}\begin{bmatrix}\partial_2 U^r/\partial_2\Phi^r \\ \partial_2 U^l/\partial_2\Phi^l\end{bmatrix}\psi + \underline{M}\dot{V}|_{x_2=0}, \end{aligned}$$

where $A_1, A_2, \underline{b}, \underline{M}$ can be found in [9].

Then the main result on the linear stability with variable coefficients can be stated as follows.

Theorem 5 [9, Theorem 2.1] *Suppose that the particular solution defined by* (39) *satisfies one of the following two conditions*:

(i) $\bar{v}^2 > 2c(\bar{\rho})^2 + \bar{F}_{11}^2 + \bar{F}_{12}^2$, *or*
(ii) $\bar{v}^2 < \bar{F}_{11}^2 + \bar{F}_{12}^2$ *but*

$$\bar{v}^2 \neq \frac{\bar{F}_{11}^2 + \bar{F}_{12}^2}{4}, \quad \bar{v}^2 \neq \frac{\left(\sqrt{\bar{F}_{11}^2 + \bar{F}_{12}^2 + c(\bar{\rho})^2} - \sqrt{\bar{F}_{11}^2 + \bar{F}_{12}^2}\right)^2}{4},$$
$$\bar{v}^2 \neq \frac{\bar{F}_{11}^2 + \bar{F}_{12}^2 + c(\bar{\rho})^2}{4}, \quad \bar{v}^2 \neq \frac{\left(\bar{F}_{11}^2 + \bar{F}_{12}^2\right)\left(2c(\bar{\rho})^2 + \bar{F}_{11}^2 + \bar{F}_{12}^2\right)}{4\left(\bar{F}_{11}^2 + \bar{F}_{12}^2 + c(\bar{\rho})^2\right)};$$

and moreover, the perturbations $\dot{U}^{r,l}$ and $\dot{\Phi}^{r,l}$ have compact support, and K is small enough. Then there are two constants C_0 and λ_0 which are determined by the particular solution, such that for all $\dot{V}$ and ψ and all $\lambda \geqslant \lambda_0$ the following estimate holds:

$$\lambda |||\dot{V}|||^2_{L^2(H^0_\lambda)} + \left\|\dot{V}^{\mathrm{nc}}|_{x_2=0}\right\|^2_{L^2_\lambda(\mathbb{R}^2)} + \|\psi\|^2_{H^1_\lambda(\mathbb{R}^2)}$$
$$\leqslant C_0\left(\frac{1}{\lambda^3}|||L'\dot{V}|||^2_{L^2(H^1_\lambda)} + \frac{1}{\lambda^2}\|B'(\dot{V},\psi)\|^2_{H^1_\lambda(\mathbb{R}^2)}\right),$$

where $L'\dot{V} = (L'_r\dot{V}^+, L'_l\dot{V}^-)$.

From the above theorem, we can see the stabilization effect due to elasticity in (ii), which was also observed in the constant-coefficient case.

One of the difficulties to prove Theorem 5 is that the standard bicharacteristic extension method (cf. [15]) does not work here since the roots of the Lopatinskiĭ determinant coincide with the poles for the system of elastic flows. Fortunately the upper triangularization method developed in [8] for the constant coefficients can be extended to the para-linearized system to separate the outgoing mode. The main steps in the proof of Theorem 5 are reducing the system by the para-differential calculus and then using the microlocalization to derive the desired energy estimates in the theorem. See [9] for the details.

3.3 Nonlinear Stability

As in [10], take the background solution of piecewise constants for the trivial vortex sheet as

$$\overline{U}^{\pm} = \left(\bar{\rho}, \pm\bar{v}, 0, \pm\overline{F}_{11}, 0, \pm\overline{F}_{12}, 0\right)^{\top}, \quad \overline{\varphi} = 0, \quad \overline{\Phi}^{\pm} = \pm x_2. \tag{41}$$

After straightening the free boundary, we need to solve the following initial-boundary value problem for $U^{\pm}$ in a fixed domain:

$$\begin{cases} L(U^{\pm}, \Phi^{\pm})U^{\pm} = 0, \quad x_2 > 0, \\ \mathbb{B}(U^+, U^-, \varphi)|_{x_2=0} = 0, \\ (U^+, U^-, \varphi)|_{t=0} = (U_0^+, U_0^-, \varphi_0), \end{cases} \tag{42}$$

where

$$L(U, \Phi) = \partial_t + A_1(U)\partial_1 + \widetilde{A}_2(U, \Phi)\partial_2, \tag{43}$$

$$\mathbb{B}(U^+, U^-, \varphi) = \begin{bmatrix} [v_1]\partial_1\varphi - [v_2] \\ \partial_t\varphi + v_1^+|_{x_2=0}\partial_1\varphi - v_2^+|_{x_2=0} \\ [\rho] \end{bmatrix}, \tag{44}$$

$$\widetilde{A}_2(U, \boldsymbol{\Phi}) = \frac{1}{\partial_2\boldsymbol{\Phi}}\big(A_2(U) - \partial_t\boldsymbol{\Phi}I_7 - \partial_1\boldsymbol{\Phi}A_1(U)\big).$$

For the nonlinear stability of elastic vortex sheets, the main result is as follows.

Theorem 6 [10, Theorem 1.1] *Let $T > 0$ and $s_0 \geqslant 14$ be an integer. Suppose that the background state (41) satisfies one of the following stability conditions*:

$$\bar{v}^2 > 2c(\bar{\rho})^2 + \overline{F}_{11}^2 + \overline{F}_{12}^2, \tag{45}$$

or

$$\begin{cases} 0 < \bar{v}^2 < \overline{F}_{11}^2 + \overline{F}_{12}^2, \\ \bar{v}^2 \neq \dfrac{\overline{F}_{11}^2 + \overline{F}_{12}^2}{4},\ \bar{v}^2 \neq \dfrac{\Big((\overline{F}_{11}^2 + \overline{F}_{12}^2 + c(\bar{\rho})^2)^{1/2} - (\overline{F}_{11}^2 + \overline{F}_{12}^2)^{1/2}\Big)^2}{4}, \\ \bar{v}^2 \neq \dfrac{\overline{F}_{11}^2 + \overline{F}_{12}^2 + c(\bar{\rho})^2}{4},\ \bar{v}^2 \neq \dfrac{(\overline{F}_{11}^2 + \overline{F}_{12}^2)(\overline{F}_{11}^2 + \overline{F}_{12}^2 + 2c(\bar{\rho})^2)}{4(\overline{F}_{11}^2 + \overline{F}_{12}^2 + c(\bar{\rho})^2)}. \end{cases} \tag{46}$$

Suppose further that the initial data $U_0^\pm$ and φ_0 with certain constraints and compatibility conditions satisfy that $(U_0^\pm - \overline{U}^\pm, \varphi_0) \in H^{s_0+1/2}(\mathbb{R}_+^2) \times H^{s_0+1}(\mathbb{R})$ has a compact support. Then there exists a positive constant ϵ such that, if

$$\|U_0^\pm - \overline{U}^\pm\|_{H^{s_0+1/2}(\mathbb{R}_+^2)} + \|\varphi_0\|_{H^{s_0+1}(\mathbb{R})} \leqslant \epsilon,$$

then problem (42) admits a solution $(U^\pm, \boldsymbol{\Phi}^\pm, \varphi)$ on the time interval $[0, T]$ satisfying

$$(U^\pm - \overline{U}^\pm, \boldsymbol{\Phi}^\pm - \overline{\boldsymbol{\Phi}}^\pm) \in H^{s_0-8}((0, T) \times \mathbb{R}_+^2), \quad \varphi \in H^{s_0-7}((0, T) \times \mathbb{R}).$$

Theorem 6 indicates that the deformation gradient in elasticity stabilizes the elastic system even in the subsonic zone. Again, to understand the spectrum of the para-linearized system the upper triangularization method is very useful to treat the degeneracy of the Kreiss-Lopatinskiĭ condition and the characteristic boundary. The nonlinear stability can be proved through the Nash-Moser iteration scheme inspired by [16], and then show the convergence of the scheme using the tame estimates derived for the variable coefficient linearized problem. See [10] for the details.

We conclude this section by a remark on the nonisentropic case. Some stability results of vortex sheets for the two-dimensional nonisentropic Euler flows were obtained in [34, 35]. We also studied the stability of vortex sheets for the the two-dimensional nonisentropic elastic flows in [11], where the linear stability was derived in some supersonic and subsonic regions by an analysis of the roots of the Lopatinskiĭ determinant for the linearized problem, and the nonlinear stability was obtained for the small perturbation of entropy. Our results in [11] also confirm that the elasticity can provide stabilization in vortex sheets in the nonisentropic flows.

4 Vortex Sheets in the Three-Dimensional Compressible Elastodynamics

In this section, we review the stability result of vortex sheets for the three-dimensional compressible elastic flows following the presentation of [12]. This work shows that the elasticity can also stabilize the compressible fluid flows in three dimensions, however the linear stability problem is much more difficult and the spectrum analysis is much more complicated in the three-dimensional case than in the two-dimensional case.

Consider the three-dimensional compressible elastic flows governed by the following equations [17]:

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \mathrm{div}(\rho \mathrm{F}\mathrm{F}^\top), \\ (\rho\mathrm{F}_j)_t + \mathrm{div}(\rho\mathrm{F}_j\otimes\mathbf{u} - \mathbf{u}\otimes\rho\mathrm{F}_j) = 0, \end{cases} \tag{47}$$

where ρ denotes the density, $\mathbf{u} = (u, v, w) \in \mathbb{R}^3$ the velocity, F_j the jth column of the deformation gradient $\mathrm{F} = (F_{ij}) \in \mathbf{M}^{3\times3}$, and $p = p(\rho)$ the pressure.

Set $U = (\rho, \mathbf{u}, \mathrm{F})$. A vortex sheet solution to system (47) is a piecewise smooth function:

$$U(t, x_1, x_2, x_3) = \begin{cases} U^+(t, x_1, x_2, x_3), \ x_3 > \psi(t, x_1, x_2), \\ U^-(t, x_1, x_2, x_3), \ x_3 < \psi(t, x_1, x_2), \end{cases}$$

across the interface $\Gamma\colon x_3 = \psi(t, x_1, x_2)$ satisfying the Rankine-Hugoniot jump conditions

$$\rho^+ = \rho^-, \quad \psi_t = \mathbf{u}^+\cdot\nu = \mathbf{u}^-\cdot\nu, \quad \mathrm{F}_j^+\cdot\nu = \mathrm{F}_j^-\cdot\nu = 0, j = 1, 2, 3, \tag{48}$$

where $U^\pm = (\rho^\pm, \mathbf{u}^\pm, \mathrm{F}^\pm)$ and $\nu = (-\partial_1\psi, -\partial_2\psi, 1)$ with $\partial_i = \partial_{x_i}, i = 1, 2, 3$.

Take the function $\Phi(t, x_1, x_2, x_3)$ with $\inf\{\partial_3\Phi\} > 0$ and $\Phi(t, x_1, x_2, 0) = \psi(t, x_1, x_2)$, define $U^\pm(t, x_1, x_2, x_3) = U(t, x_1, x_2, \Phi(t, x_1, x_2, \pm x_3))$ and $\Phi^\pm(t, x_1, x_2, x_3) = \Phi(t, x_1, x_2, \pm x_3)$ such that $\partial_t\Phi^\pm + u^\pm\partial_1\Phi^\pm + v^\pm\partial_2\Phi^\pm - w^\pm = 0$ for $x_3 \geqslant 0$, and consider the following initial-boundary value problem for (47):

$$\begin{cases} \mathcal{L}(U^\pm, \Phi^\pm) = 0, \quad x_3 > 0, \\ \mathcal{B}(U^\pm, \psi)|_{x_3=0} = 0, \\ (U^\pm, \psi)|_{t=0} = (U_0^\pm, \psi_0), \end{cases} \tag{49}$$

where

$$\mathcal{L}(U,\Phi) = \partial_t U + A_1(U)\partial_1 U + A_2(U)\partial_2 U + \tilde{A}_3(U,\Phi)\partial_3 U,$$
$$\tilde{A}_3(U,\Phi) = \frac{1}{\partial_3\Phi}[A_3(U) - \partial_t\Phi I - \partial_1\Phi A_1(U) - \partial_2\Phi A_2(U)],$$
$$\mathcal{B}(U^\pm,\psi) = \begin{bmatrix} (u^+ - u^-)\partial_1\psi + (v^+ - v^-)\partial_2\psi - (w^+ - w^-) \\ \partial_t\psi + u^+\partial_1\psi + v^+\partial_2\psi - w^+ \\ \rho^+ - \rho^- \end{bmatrix},$$

and A_1, A_2, A_3 are some 13×13 matrices (see [12] for the precise forms).

Without loss of generality we take the piecewise constant background solution to the system (49) as the following:

$$\begin{cases} \bar{U}^+ = (\bar{\rho}, u^r, 0, 0, F^r_{11}, F^r_{21}, 0, F^r_{12}, F^r_{22}, 0, F^r_{13}, F^r_{23}, 0)^\top, \\ \bar{U}^- = (\bar{\rho}, -u^r, 0, 0, -F^r_{11}, -F^r_{21}, 0, -F^r_{12}, -F^r_{22}, 0, -F^r_{13}, -F^r_{23}, 0)^\top, \\ \bar{\Phi}^\pm(t, x_1, x_2, x_3) = \pm x_3 \end{cases} \tag{50}$$

with $u^r \neq 0, F^r_{ij} \neq 0$ for $i = 1, 2,\ j = 1, 2, 3$. We linearize the problem (49) around the background solution (50) and let

$$\dot{U}^\pm = (\dot{\rho}^\pm, \dot{\mathbf{u}}^\pm, \dot{\mathrm{F}}^\pm) = U^\pm - \bar{U}^\pm, \quad \dot{\Phi}^\pm = \Phi^\pm - \bar{\Phi}^\pm$$

be the small perturbation of the constant solution. As in the two-dimensional case, we change variables by the following transformation:

$$W = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}\begin{bmatrix} \dot{U}^+ \\ \dot{U}^- \end{bmatrix}, \tag{51}$$

where the precise formula of the 13×13 matrix T can be found in [12]. In terms of the new variables $W = (W_1, W_2, \cdots, W_{26})^\top$, the linearized problem of (49) at the constant background solution (50) can be written as

$$\begin{cases} \mathcal{L}W = \mathcal{A}_0\partial_t W + \mathcal{A}_1\partial_1 W + \mathcal{A}_2\partial_2 W + \mathcal{A}_3\partial_3 W = 0,\ x_3 > 0, \\ \mathcal{B}(W^{\mathrm{nc}},\varphi) = \underline{M}W^{\mathrm{nc}}|_{x_3=0} + \underline{b}\big(\partial_t\varphi, \partial_1\varphi, \partial_2\varphi\big)^\top = 0, \end{cases} \tag{52}$$

where $W^{\mathrm{nc}} = (W_1, W_2, W_{14}, W_{15})^\top$, and $\mathcal{A}_i,\ i = 0, 1, 2, 3,\ \underline{M}, \underline{b}$ can be found in [12].

Let F_j be the jth row of the deformation matrix F^r for $j = 1, 2, 3$. Denote by $\Pi_b(a)$ the parallel projection of a onto b and $\Pi_b^\perp(a) = a - \Pi_b(a)$ the perpendicular projection of a onto b. Using the weighted Sobolev spaces and norms for the three-dimensional space that are analogous to those defined in Sect. 2.1 for the two-dimensional space, we can state the linear stability with constant coefficients as follows.

Theorem 7 [12, *Theorem* 3.1]

(i) *Assume that the background solution defined by* (50) *satisfies* $\mathrm{F}_1 \times \mathrm{F}_2 \neq \mathbf{0}$. *If*

$$0 < (u^r)^2 < \mathcal{F}(\mathrm{F}_1, \mathrm{F}_2), \tag{53}$$

then there is a positive constant C such that for all $\lambda > 1$, $W \in H^3_\lambda(\mathbb{R}^4_+)$ and $\varphi \in H^3_\lambda(\mathbb{R}^3)$, the following estimate holds:

$$
\begin{aligned}
&\lambda|||W|||^2_{L^2(H^0_\lambda)} + ||W^{\mathrm{nc}}|_{x_3=0}||^2_{0,\lambda} + ||\varphi||^2_{0,\lambda} \\
&\leqslant C\Big(\frac{1}{\lambda^5}|||\mathcal{L}^\lambda W|||^2_{L^2(H^2_\lambda)} + \frac{1}{\lambda^4}||\mathcal{B}^\lambda(W^{\mathrm{nc}}|_{x_3=0},\varphi)||^2_{H^2_\lambda(\mathbb{R}^3)}\Big).
\end{aligned} \tag{54}
$$

(ii) *Assume that the background solution defined by* (50) *satisfies* $\mathrm{F}_1 \times \mathrm{F}_2 \neq \mathbf{0}$. *If*

$$
\mathcal{F}(\mathrm{F}_1,\mathrm{F}_2) \leqslant (u^r)^2 \leqslant \Big|\Pi^\perp_{\mathrm{F}_2}(\mathrm{F}_1)\Big|^2, \tag{55}
$$

then there is a positive constant C such that for all $\lambda > 1$, $W \in H^4_\lambda(\mathbb{R}^4_+)$ and $\varphi \in H^4_\lambda(\mathbb{R}^3)$, the following estimate holds:

$$
\begin{aligned}
&\lambda|||W|||^2_{L^2(H^0_\lambda)} + ||W^{\mathrm{nc}}|_{x_3=0}||^2_{0,\lambda} + ||\varphi||^2_{0,\lambda} \\
&\leqslant C\Big(\frac{1}{\lambda^7}|||\mathcal{L}^\lambda W|||^2_{L^2(H^3_\lambda)} + \frac{1}{\lambda^6}||\mathcal{B}^\lambda(W^{\mathrm{nc}}|_{x_3=0},\varphi)||^2_{H^3_\lambda(\mathbb{R}^3)}\Big).
\end{aligned} \tag{56}
$$

(iii) *Assume that the background solution defined by* (50) *satisfies*

$$
(u^r)^2 > \Big|\Pi^\perp_{\mathrm{F}_2}(\mathrm{F}_1)\Big|^2, \tag{57}
$$

then the constant vortex sheet solutions (50) *are linearly unstable*.

We remark that the function $\mathcal{F}(\mathrm{F}_1,\mathrm{F}_2)$ in the above theorem is complicated and can be found in [12], satisfying $\Big|\Pi^\perp_{\mathrm{F}_2}(\mathrm{F}_1)\Big|^2 \leqslant 4\mathcal{F}(\mathrm{F}_1,\mathrm{F}_2) \leqslant 2\Big|\Pi^\perp_{\mathrm{F}_2}(\mathrm{F}_1)\Big|^2$.

The sufficient and necessary conditions for the linear stability of the background solution (50) are provided in Theorem 7, which are obtained by a spectral analysis [15] and the upper triangulation method [8]. Different from the two-dimensional case [8], the linear stability holds only in a subsonic bubble of the three-dimensional elastic flows. In contrast with the three-dimensional compressible Euler flows for which the vortex sheets are violently unstable, Theorem 7 shows the stabilizing effect from elasticity which allows a subsonic region for the linear stability. The proof of Theorem 7 includes the following main steps after the linearization: the normal mode analysis, the upper triangularization to separate only the outgoing modes, a delicate analysis of the Lopatinskiǐ determinant and the estimates in the neighborhood of the zeros of the Lopatinskiǐ determinant, and the desired energy estimates in the theorem. See [12] for details. We remark that the linear stability with variable coefficients and the nonlinear stability for the vortex sheets of the three-dimensional elastic flows are very challenging and still part of the ongoing studies.

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