INCOMPRESSIBLE LIMIT FOR COMPRESSIBLE VISCOELASTIC FLOWS WITH LARGE VELOCITY

XIANPENG HU, YAOBIN OU, DEHUA WANG, AND LU YANG

Abstract. We are concerned with the incompressible limit of global-in-time strong solutions with arbitrary large initial velocity for the three-dimensional compressible viscoelastic equations. The incompressibility is achieved by the large value of the volume viscosity, which is different from the low Mach number limit. To obtain the uniform estimates, we establish the estimates for the potential part and the divergence-free part of the velocity, respectively. As the volume viscosity goes to infinity the dispersion associated with the pressure waves tends to disappear, but the large volume viscosity provides a strong dissipation on the potential part of the velocity forcing the flow to be almost incompressible.

1. Introduction

Viscoelastic fluids combine the fluid characteristics with the solid characteristics, and have wide applications in engineering, biology, medicine and so on. The interaction of fluids and solids leads to complicated phenomena in viscoelastic fluids and causes many challenges in mathematical analysis. This paper is focused on the following three-dimensional compressible viscoelastic flows [7,19]:

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{v}) = 0, & \text{in} \quad [0,+\infty)\times\mathbb{R}^3, \\ (\rho\mathbf{v})_t + \mathrm{div}(\rho\mathbf{v}\otimes\mathbf{v}) - \mu\Delta\mathbf{v} - (\lambda+\mu)\nabla\mathrm{div}\mathbf{v} + \nabla p(\rho) = \mathrm{div}(\rho\mathbf{F}\mathbf{F}^\top), & \text{in} \quad [0,+\infty)\times\mathbb{R}^3, \\ \mathbf{F}_t + \mathbf{v}\cdot\nabla\mathbf{F} = \nabla\mathbf{v}\mathbf{F}, & \text{in} \quad [0,+\infty)\times\mathbb{R}^3, \\ (\rho,\mathbf{v},\mathbf{F})(x,0) = (\rho_0,\mathbf{v}_0,\mathbf{F}_0), & x\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

where ρ, $\mathbf{v}\in\mathbb{R}^3$, $\mathbf{F}\in M^{3\times3}$ (the set of 3×3 matrices with positive determinants) denote the density, the velocity field, and the deformation gradient, respectively. The shear and bulk viscosity coefficients μ and λ are constants and assumed to satisfy the following physical conditions:

$$\mu > 0,\ 2\mu + 3\lambda \geq 0.$$

The pressure $p(\rho)$ is a given function of ρ that is suitably smooth for $\rho > 0$ with $p'(\rho) > 0$. The symbol $\otimes$ denotes the Kronecker tensor product, $\mathbf{F}^\top$ stands for the transpose matrix of $\mathbf{F}$, and the notation $\mathbf{v}\cdot\nabla\mathbf{F}$ is understood as $(\mathbf{v}\cdot\nabla)\mathbf{F}$. In this paper, we assume that

$$\mathrm{div}(\rho\mathbf{F}^\top) = 0, \quad \mathbf{F}^{lk}\nabla_l\mathbf{F}^{ij} = \mathbf{F}^{lj}\nabla_l\mathbf{F}^{ik}. \tag{1.2}$$

It has been proved that (1.2) holds for all time $t > 0$ if it is satisfied initially (cf. Qian-Zhang [19], Hu-Wang [7], Hu-Zhao [11]).

As far as smooth solutions to the incompressible version of (1.1) are considered, authors in [3,14–16] showed a global-in-time existence of classical solutions to (1.1) in H^2 whenever the initial data is a small perturbation around the equilibrium $(0,I)$, where I is the identity matrix; see also [2] for the related Oldroyd-B model. The construction of global solutions in [14,15]

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heavily depends on the conserved quantities in (1.2) which decouple the relation between the pressure and the deformation gradient $\mathbf{F}$, see also [7, 8, 19] for the global existence of classical solutions near the equilibrium for the compressible model (1.1). It is remarkable to note that the global existence of classical solution in H^s $(s \geq 8)$ to the incompressible version of (1.1) with $\mu = 0$ has also been obtained in [13, 23, 24] via the vector field method by different approaches; see also [25] for a different approach via the space-time resonance method in two dimensions. It is also known that for the Oldroyd-B model with a finite relaxation time, the global existence of weak solutions with arbitrary initial data was verified in [1, 18]. The global existence of weak solutions for the FENE dumbbell model with arbitrary initial data was constructed by Masmoudi in [20] through a detailed analysis of the defect measure associated with the approximation. Moreover, for solutions with low regularity, with the help of (1.2), the global weak solutions of the incompressible version of (1.1) in two dimensions with the discontinuous initial data near the equilibrium have been established in [9] provided that $\|v_0\|_{L^4}$ is bounded. One of key observations in [9] is the *effective viscous flux* which admits a higher regularity than its components. This observation, combined with a similar structure for the density, further helps to construct a global weak solution with discontinuous initial data in [10] for the compressible model (1.1). However, the global existence of weak solutions, in the spirit of Leray, to the viscoelastic fluids with large initial data is still an outstanding open question.

In the compressible regime, different viscosities make different contributions to the flow of fluids. For example, the presence of the shear viscosity μ prevents the formation of shocks near the equilibrium in [7, 14, 15, 19], while as the shear viscosity vanishes, the shock formation is expected even for small initial perturbations of incompressible models, see references in [11]. However, for (1.1), the volume viscosity, which controls the volume change, plays a similar role in prohibiting the shock formation; see [11]. Moreover, Cui and Hu [4] studied the incompressible limit of the compressible viscoelastic system near the equilibrium when the shear viscosity converges to zero and the value of the volume viscosity is large. Similarly to the Navier-Stokes equations, the global existence of solutions to (1.1) with large initial data is still a challenging open problem. Note that Danchin and Mucha [5] proved the existence of global strong solutions to the compressible Navier-Stokes equations with arbitrary large initial velocity $\mathbf{v}_0$ for large volume viscosity in critical Besov spaces. Nevertheless, when dealing with the global existence of strong solutions with large initial data in the compressible viscoelasticity (1.1), the main difficulty lies in that there is no dissipative estimate for the deformation gradient.

The main purpose of this paper is to establish that for any fixed shear viscosity $\mu > 0$ and any initial velocity-field $\mathbf{v}_0$ with given regularity, the solution to (1.1) is global for λ sufficiently large and ρ_0 sufficiently close to some positive constant which we set to be one in this paper. In contrast with the results in [8], we are concerned with the global existence of strong solutions to (1.1) with large volume viscosity and large initial velocity. We apply the method developed by Danchin and Mucha [5] to derive the estimates for the potential part $Q\mathbf{v}$ and the "incompressible part" $P\mathbf{v}$ of the velocity, respectively. However, we adapt a different way in the Sobolev spaces, instead of, the composition of high frequency and low frequency in the Besov spaces in [5]. This result will strongly rely on the fact that, for $\lambda \to +\infty$, the limit velocity satisfies the incompressible viscoelastic flows:

$$
\begin{cases}
\operatorname{div}\mathbf{V} = 0, & \text{in} \quad [0, +\infty) \times \mathbb{R}^3, \\
\mathbf{V}_t + \mathbf{V} \cdot \nabla \mathbf{V} - \mu \Delta \mathbf{V} + \nabla P = \operatorname{div}(\widetilde{\mathbf{F}}\widetilde{\mathbf{F}}^\top), & \text{in} \quad [0, +\infty) \times \mathbb{R}^3, \\
\widetilde{\mathbf{F}}_t + \mathbf{V} \cdot \nabla \widetilde{\mathbf{F}} = \nabla \mathbf{V} \widetilde{\mathbf{F}}, & \text{in} \quad [0, +\infty) \times \mathbb{R}^3, \\
\mathbf{V}(x, 0) = \mathbf{V}_0, \ \widetilde{\mathbf{F}}(x, 0) = \widetilde{\mathbf{F}}_0, & x \in \mathbb{R}^3,
\end{cases}
\tag{1.3}
$$

where $\mathbf{V}_0$ is the Leray-Helmholtz projection of $\mathbf{v}_0$ on the divergence-free vector fields. We introduce the perturbation of deformation gradient defined by $\widetilde{\mathbf{E}} = \widetilde{\mathbf{F}} - \mathbf{I}$ for $\mathbf{I}$ being the identity matrix. The second conserved quantity in (1.2) also holds true and the first identity becomes $\mathrm{div}\widetilde{\mathbf{F}}^\top = 0$ for all $t \geq 0$. Thus (1.3) can be adapted into the following form:

$$\begin{cases} \mathrm{div}\mathbf{V} = 0, & \text{in} \quad [0,+\infty) \times \mathbb{R}^3, \\ \mathbf{V}_t + \mathbf{V}\cdot\nabla\mathbf{V} - \mu\Delta\mathbf{V} + \nabla P = \mathrm{div}\widetilde{\mathbf{E}} + \mathrm{div}(\widetilde{\mathbf{E}}\widetilde{\mathbf{E}}^\top), & \text{in} \quad [0,+\infty) \times \mathbb{R}^3, \\ \widetilde{\mathbf{E}}_t + \mathbf{V}\cdot\nabla\widetilde{\mathbf{E}} = \nabla\mathbf{V} + \nabla\mathbf{V}\widetilde{\mathbf{E}}, & \text{in} \quad [0,+\infty) \times \mathbb{R}^3, \\ \mathbf{V}(x,0) = \mathbf{V}_0, \ \widetilde{\mathbf{E}}(x,0) = \widetilde{\mathbf{E}}_0 := \widetilde{\mathbf{F}}_0 - \mathbf{I}, & x \in \mathbb{R}^3. \end{cases} \tag{1.4}$$

As the global existence of strong solution to (1.3) supplemented with general data is open in $\mathbb{R}^d$ $(d = 2, 3)$, we have to assume that $(\mathbf{V}_0, \widetilde{\mathbf{F}}_0)$ generates a global solution to (1.3), and then we analyze the stability of that solution in the setting of the compressible model (1.1) with large λ. The present work is to justify the incompressible limit of the global strong solutions to the system (1.1) around the incompressible state $(1, \mathbf{V}, \widetilde{\mathbf{F}})$ when the volume viscosity λ tends to infinity. The incompressibility follows from the large value of the volume viscosity. In this situation, the dispersion associated with the pressure waves tends to disappear as $\lambda \to +\infty$, but large λ provides a strong dissipation on the potential part of the velocity, and thus forces the flow to be almost incompressible. We remark that the mechanism for the incompressibility is different from the setting in [17, 24] where the incompressibility is achieved as the low Mach number limit.

We first introduce the perturbations a and $\mathbf{E}$ of the density and deformation gradient as the following:

$$\rho = 1 + a, \qquad \mathbf{F} = \mathbf{I} + \mathbf{E}. \tag{1.5}$$

Since $\mathrm{div}(\rho\mathbf{F}^\top) = 0$ holds for all $t \geq 0$, the *i-th* component of the vector $\mathrm{div}(\rho\mathbf{F}\mathbf{F}^\top)$ can be written as

$$\begin{aligned} \nabla_j(\rho\mathbf{F}^{ik}\mathbf{F}^{jk}) &= \rho\mathbf{F}^{jk}\nabla_j\mathbf{F}^{ik} + \mathbf{F}^{ik}\nabla_j(\rho\mathbf{F}^{jk}) = \rho\mathbf{F}^{jk}\nabla_j\mathbf{F}^{ik} \\ &= \rho\mathrm{div}\mathbf{E} + \rho\mathbf{E}^{jk}\nabla_j\mathbf{E}^{ik} = \mathrm{div}\mathbf{E} + a\mathrm{div}\mathbf{E} + (1+a)\mathbf{E}^{jk}\nabla_j\mathbf{E}^{ik}. \end{aligned}$$

Thus, the system (1.1) becomes equivalently

$$\begin{cases} a_t + \mathrm{div}(a\mathbf{v}) + \mathrm{div}\mathbf{v} = 0, & \text{in} \quad [0,+\infty) \times \mathbb{R}^3, \\ (1+a)\mathbf{v}_t + (1+a)\mathbf{v}\cdot\nabla\mathbf{v} + \nabla p(1+a) & \\ \quad - \mu\Delta\mathbf{v} - (\lambda+\mu)\nabla\mathrm{div}\mathbf{v} = \mathrm{div}\mathbf{E} + a\mathrm{div}\mathbf{E} + (1+a)\mathbf{E}^{jk}\nabla_j\mathbf{E}^{ik}, & \text{in} \quad [0,+\infty) \times \mathbb{R}^3, \\ \mathbf{E}_t + \mathbf{v}\cdot\nabla\mathbf{E} = \nabla\mathbf{v} + \nabla\mathbf{v}\mathbf{E}, & \text{in} \quad [0,+\infty) \times \mathbb{R}^3, \\ (a, \mathbf{v}, \mathbf{E})(x,0) = (a_0, \mathbf{v}_0, \mathbf{E}_0) = (\rho_0 - 1, \mathbf{v}_0, \mathbf{F}_0 - \mathbf{I}), & x \in \mathbb{R}^3. \end{cases} \tag{1.6}$$

For any vector field $\mathbf{v} = (v^1, v^2, v^3)$, we denote by P the orthogonal projection onto the subspace of incompressible vector fields, that is,

$$v = Pv + Qv, \quad \text{with} \quad \mathrm{div}(Pv) = 0, \ \mathrm{curl}(Qv) = 0,$$

where $Q := -(-\Delta)^{-1}\nabla\mathrm{div}$ stands for the projection operator on potential vector fields and $P := \mathrm{Id} + (-\Delta)^{-1}\nabla\mathrm{div}$. Indeed, from the results in [6], we know that the operators P and Q are linearly bounded operators in $W^{s,p}(\mathbb{R}^3)$ for all $s \geq 0$ and $1 < p < \infty$. Furthermore, we give the notation used throughout this paper. We denote the usual Sobolev space by $H^k(\mathbb{R}^3)$ endowed with the norm $||\cdot||_{H^k}$. We also use the following abbreviations for the Sobolev spaces involving time:

$$L^p_t(H^k) \equiv L^p(0,t;H^k(\mathbb{R}^3)), \quad C_t(H^k) \equiv C([0,t], H^k(\mathbb{R}^3)),$$

with

$$||\cdot||_{L^p_t(H^k)} \equiv ||\cdot||_{L^p(0,t;H^k)}, \quad ||\cdot||_{C_t(H^k)} \equiv ||\cdot||_{C([0,t],H^k)},$$

where $H^0(\mathbb{R}^3) \equiv L^2(\mathbb{R}^3)$. We will utilize the notation C to signify a generic positive constant. In addition, we use $f \lesssim g$ to represent the relation that $f \le Cg$.

The local existence and uniqueness of the strong solution to (1.1) can be obtained in the spirit of Matsumura and Nishida [21, 22], see also [7, 19]. Now we state our main result for (1.6).

Theorem 1.1. *Assume that the initial data $(\rho_0 - 1, \mathbf{v}_0, \mathbf{F}_0 - \mathbf{I}) \in H^2(\mathbb{R}^3)$ satisfies the constraint (1.2) and there exist constants $\delta_0, e_0 \ll 1$ such that $||\mathbf{E}_0 - \widetilde{\mathbf{E}}_0||^2_{H^2} \le \delta_0$ and $||\widetilde{\mathbf{E}}_0||^2_{H^4} \le e_0$. Suppose that the system (1.4) with initial data $(\mathbf{V}_0, \widetilde{\mathbf{E}}_0)$ generates a unique global solution $\mathbf{V} \in C([0,+\infty), H^4(\mathbb{R}^3)) \cap L^2(0,+\infty; H^5(\mathbb{R}^3))$, $\widetilde{\mathbf{E}} \in C([0,+\infty), H^4(\mathbb{R}^3)) \cap L^2(0,+\infty; H^4(\mathbb{R}^3))$ and denote*

$$M := ||\mathbf{V}||^2_{L^\infty(0,+\infty;H^4)} + ||\mathbf{V}_t||^2_{L^2(0,+\infty;H^3)} + ||\nabla\mathbf{V}||^2_{L^2(0,+\infty;H^4)}, \tag{1.7}$$

and

$$\delta' := ||\widetilde{\mathbf{E}}||^2_{L^\infty(0,T;H^4)} + ||\nabla\widetilde{\mathbf{E}}||^2_{L^2(0,T;H^3)}, \tag{1.8}$$

where M and δ' are positive constants with $\delta' \ll 1$. Assume in addition that there exists a large constant C such that if $\nu := \lambda + 2\mu$ satisfies $\nu \ge \mu$ and

$$Ce^{C(1+M^2)}(||\nu a_0||^2_{H^2} + ||Q\mathbf{v}_0||^2_{H^2} + M + \mu^2) \le \sqrt{\mu\nu}, \tag{1.9}$$

then (1.6) admits a unique global-in-time solution $(\rho, \mathbf{v}, \mathbf{E})$ satisfying

$$\begin{aligned} &\mathbf{v} \in L^\infty(0,+\infty; H^2(\mathbb{R}^3)), \quad \mathbf{v}_t \in L^2(0,+\infty; H^1(\mathbb{R}^3)), \quad \nabla\mathbf{v} \in L^2(0,+\infty; H^2(\mathbb{R}^3)), \\ &(a, \mathbf{E}) \in L^\infty(0,+\infty; H^2(\mathbb{R}^3)), \quad (\nabla a, \nabla\mathbf{E}) \in L^2(0,+\infty; H^1(\mathbb{R}^3)). \end{aligned} \tag{1.10}$$

In addition,

$$\begin{aligned} &||Q\mathbf{v}||^2_{L^\infty(0,+\infty;H^2)} + ||\nu\nabla Q\mathbf{v}||^2_{L^2(0,+\infty;H^2)} + ||\nu a||^2_{L^\infty(0,+\infty;H^2)} \\ &\le Ce^{C(1+M^2)}(||\nu a_0||^2_{H^2} + ||Q\mathbf{v}_0||^2_{H^2} + M + \mu^2) \end{aligned} \tag{1.11}$$

holds for any $t \ge 0$. Furthermore, if $a_0 = 0$ and $\mathbf{E}_0 = \widetilde{\mathbf{E}}_0$, then as λ tends to $+\infty$, $(\rho, \mathbf{v}, \mathbf{E})$ converges to $(1, \mathbf{V}, \widetilde{\mathbf{E}})$ as follows:

$$\begin{aligned} &\nu\left(||\rho - 1||^2_{L^\infty(0,+\infty;H^2)} + ||\nabla Q\mathbf{v}||^2_{L^2(0,+\infty;H^2)}\right) + ||(P\mathbf{v} - \mathbf{V}, \mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^\infty(0,+\infty;H^2)} \\ &\quad + ||P\mathbf{v}_t - \mathbf{V}_t||^2_{L^2(0,+\infty;H^1)} + ||\mu\nabla(P\mathbf{v} - \mathbf{V})||^2_{L^2(0,+\infty;H^2)} \lesssim \frac{1}{\nu}, \end{aligned} \tag{1.12}$$

where $(1, \mathbf{V}, \widetilde{\mathbf{E}})$ satisfies (1.4) with initial data $\mathbf{V}_0 = P\mathbf{v}_0$.

Remark 1.1. As justified in [14, 15], when $\|V_0\|_{H^4} + \|\tilde{E}_0\|_{H^4}$ is sufficiently small, the conditions (1.7)-(1.8) can be justified with (M, δ') replaced by a small parameter that depends on the initial data. However, for a large volume M, the global-in-time existence of (1.4) is still out of reach.

For the proof of the above result in Theorem 1.1, we are encountering two major mathematical difficulties in verifying the incompressible limit of the global strong solutions to the system (1.6) with large λ: the first one is to establish the uniform estimates for the potential part $Q\mathbf{v}$ and the "incompressible part" $P\mathbf{v}$ of the velocity, respectively; the second one is to deal with the deformation gradient $\mathbf{F}$ that lacks dissipative estimates. For the first obstacle, the uniform estimates are difficult to obtain due to the high-order term of the velocity in the momentum equation. Since large λ provides a strong dissipation on the potential part of the velocity $Q\mathbf{v}$, the uniform estimates for $Q\mathbf{v}$ and $P\mathbf{v}$ need to be established in different ways. More precisely, we first establish the uniform estimates for the "incompressible part" $P\mathbf{v}$ and the deformation gradient perturbation $\mathbf{E}-\widetilde{\mathbf{E}}$ by considering the momentum and deformation gradient equations together to eliminate the high-order terms of $P\mathbf{v}$. For the potential part $Q\mathbf{v}$ and the density perturbation a, the uniform estimates are developed by taking the momentum and continuity equations into account together to deal with the high-order terms of $Q\mathbf{v}$. This is one of the novelties in this paper. For the second difficulty, there is no dissipative estimates for the deformation gradient $\mathbf{F}$. Thus, we need to establish the estimates for the perturbation of the deformation gradient $\mathbf{E}=\mathbf{F}-\mathbf{I}$ by virtue of the property of Riesz potential. In more detail, we first utilize the equations for deformation gradient perturbations $\mathbf{E}$ and $\widetilde{\mathbf{E}}$ in the compressible system (1.6) and incompressible system (1.4) to obtain the equation for $\mathbf{E}-\widetilde{\mathbf{E}}$, namely (2.7). Then by virtue of the assumptions for $\widetilde{\mathbf{E}}$ and estimates for $\mathbf{E}-\widetilde{\mathbf{E}}$, we derive the estimates for the perturbation of deformation gradient $\mathbf{E}$, which is another novelty in our work.

The proof of Theorem 1.1 will be carried out in the remaining sections of this paper. In Section 2 we give the preliminary setup and derive the estimates for the deformation gradient. In Section 3, we prove the estimates for the divergence-free part of the velocity. In Section 4, we establish the estimates of the potential part of the velocity and the density. In Section 5, we close the estimates and then complete the proof of Theorem 1.1.

2. The Preliminary Setup and the Estimates for the Deformation Gradient

From now on, we assume that the shear viscosity $\mu=1$ and $p'(1)=1$ for simplicity. We aim to establish the global-in-time uniform estimates for the solution of (1.6), since the local existence issue has been well understood. To compare the solutions of (1.1) and (1.3), we set

$$\mathbf{u}:=\mathbf{v}-\mathbf{V}.$$

In the spirit of Danchin and Mucha's argument [5], we establish the estimates for the potential part $Q\mathbf{u}$ and divergence-free part $P\mathbf{u}$, respectively. To derive the equation for $Q\mathbf{u}$, we need to deal with the term $\mathrm{div}\mathbf{E}$ on the right-hand side of $(1.6)_2$. We first use the condition $\mathrm{div}(\rho\mathbf{F}^\top)=0$ for all $t\geq 0$ to derive that

$$\mathrm{divdiv}[(1+a)(\mathbf{I}+\mathbf{E})]=0,\quad \forall\ t\geq 0,$$

from which we obtain that

$$\begin{aligned}\mathrm{divdiv}(\mathbf{E}^\top)=\mathrm{divdiv}\mathbf{E}=\frac{\partial^2(\mathbf{E}^{ij})}{\partial x_i\partial x_j}&=\mathrm{divdiv}[(1+a)(\mathbf{I}+\mathbf{E})]-\mathrm{divdiv}(a\mathbf{I}+a\mathbf{E})\\&=-\Delta a-\mathrm{divdiv}(a\mathbf{E}).\end{aligned}\tag{2.1}$$

Then, by virtue of $Q\mathbf{u}=Q\mathbf{v}$ and $\mathbf{v}=Q\mathbf{u}+P\mathbf{u}+\mathbf{V}$, applying the operator Q to the velocity equation $(1.6)_2$ yields

$$(Q\mathbf{u})_t+Q((\mathbf{u}+\mathbf{V})\cdot\nabla Q\mathbf{u})-\nu\Delta Q\mathbf{u}+2\nabla a=-Q(a\mathbf{V}_t+a\mathbf{u}_t)-QR_2,\tag{2.2}$$

where

$$\begin{aligned}R_2:=&(1+a)(\mathbf{u}+\mathbf{V})\cdot\nabla P\mathbf{u}+(1+a)(\mathbf{u}+\mathbf{V})\cdot\nabla\mathbf{V}+a(\mathbf{u}+\mathbf{V})\cdot\nabla Q\mathbf{u}\\&+k(a)\nabla a+\mathbf{E}\cdot\nabla a-(1+a)\mathbf{E}^{jk}\nabla_j\mathbf{E}^{ik},\end{aligned}\tag{2.3}$$

for $k(a)=p'(1+a)-p'(1)=p'(1+a)-1$ and the initial data $Q\mathbf{u}|_{t=0}=Q\mathbf{v}_0$. From the first equation in (1.6), we find that a satisfies

$$a_t+(\mathbf{u}+\mathbf{V})\cdot\nabla a+\mathrm{div}Q\mathbf{u}=-a\mathrm{div}Q\mathbf{u}.\tag{2.4}$$

Noting that $P\mathbf{V}=\mathbf{V}$ and $P(Q\mathbf{u}\cdot\nabla Q\mathbf{u})=P(a\nabla a)=0$, we apply the operator P to the velocity equation $(1.6)_2$ and subtract the equation $(1.3)_2$ for $P\mathbf{V}=\mathbf{V}$ to derive that

$$(P\mathbf{u})_t+P((\mathbf{u}+\mathbf{V})\cdot\nabla P\mathbf{u})-\Delta P\mathbf{u}-P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})=-P(a\mathbf{V}_t+a\mathbf{u}_t+2a\nabla a)-PR_1, \tag{2.5}$$

where

$$\begin{aligned}R_1:=&(1+a)P\mathbf{u}\cdot\nabla(\mathbf{V}+Q\mathbf{u})+(1+a)\mathbf{V}\cdot\nabla Q\mathbf{u}+(1+a)Q\mathbf{u}\cdot\nabla\mathbf{V}+a(\mathbf{u}+\mathbf{V})\cdot\nabla P\mathbf{u}\\&+a\mathbf{V}\cdot\nabla\mathbf{V}+aQ\mathbf{u}\cdot\nabla Q\mathbf{u}-a\mathrm{div}\mathbf{E}-\mathbf{E}^{jk}\nabla_j(\mathbf{E}-\widetilde{\mathbf{E}})^{ik}\\&-(\mathbf{E}-\widetilde{\mathbf{E}})^{jk}\nabla_j\widetilde{\mathbf{E}}^{ik}-a\mathbf{E}^{jk}\nabla_j\mathbf{E}^{ik}.\end{aligned} \tag{2.6}$$

with the initial data $P\mathbf{u}|_{t=0}=0$. By subtracting $(1.4)_3$ from $(1.6)_3$, we have

$$(\mathbf{E}-\widetilde{\mathbf{E}})_t+\mathbf{v}\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}})+\mathbf{u}\cdot\nabla\widetilde{\mathbf{E}}-\nabla\mathbf{u}=\nabla\mathbf{v}(\mathbf{E}-\widetilde{\mathbf{E}})+\nabla\mathbf{u}\widetilde{\mathbf{E}}. \tag{2.7}$$

Noticing that the conditions $\mathbf{F}^{lk}\nabla_l\mathbf{F}^{ij}=\mathbf{F}^{lj}\nabla_l\mathbf{F}^{ik}$ and $\widetilde{\mathbf{F}}^{lk}\nabla_l\widetilde{\mathbf{F}}^{ij}=\widetilde{\mathbf{F}}^{lj}\nabla_l\widetilde{\mathbf{F}}^{ik}$ hold for all $t\geq 0$, we obtain that

$$\nabla_k\mathbf{E}^{ij}+\mathbf{E}^{lk}\nabla_l\mathbf{E}^{ij}=\nabla_j\mathbf{E}^{ik}+\mathbf{E}^{lj}\nabla_l\mathbf{E}^{ik},\ \nabla_k\widetilde{\mathbf{E}}^{ij}+\widetilde{\mathbf{E}}^{lk}\nabla_l\widetilde{\mathbf{E}}^{ij}=\nabla_j\widetilde{\mathbf{E}}^{ik}+\widetilde{\mathbf{E}}^{lj}\nabla_l\widetilde{\mathbf{E}}^{ik},\quad\forall\, t\geq 0.$$

Thus we have

$$\begin{aligned}&\nabla_k(\mathbf{E}-\widetilde{\mathbf{E}})^{ij}-\nabla_j(\mathbf{E}-\widetilde{\mathbf{E}})^{ik}\\&=\mathbf{E}^{lj}\nabla_l(\mathbf{E}-\widetilde{\mathbf{E}})^{ik}+(\mathbf{E}-\widetilde{\mathbf{E}})^{lj}\nabla_l\widetilde{\mathbf{E}}^{ik}-\mathbf{E}^{lk}\nabla_l(\mathbf{E}-\widetilde{\mathbf{E}})^{ij}-(\mathbf{E}-\widetilde{\mathbf{E}})^{lk}\nabla_l\widetilde{\mathbf{E}}^{ij}.\end{aligned} \tag{2.8}$$

We assume that the maximal solution $(\rho,\mathbf{v},\mathbf{E})$ of (1.6) corresponding to data $(\rho_0,\mathbf{v}_0,\mathbf{E}_0)$ is defined on the time interval $[0,T_*)$. In order to bound a, $\mathbf{E}$, $P\mathbf{u}$ and $Q\mathbf{u}$, we need to perform suitable energy estimates. It is natural to estimate $Q\mathbf{u}_t+2\nabla a$ rather than just $Q\mathbf{u}_t$ from the expression of (2.2). In what follows, for $T<T_*$ we set

$$\begin{aligned}&X(T):=||(Q\mathbf{u},\nu a)||^2_{L^\infty(0,T;H^2)},\\&Y(T):=||(Q\mathbf{u}_t+2\nabla a,\nu^{1/2}\nabla a)||^2_{L^2(0,T;H^1)}+||\nu\nabla Q\mathbf{u}||^2_{L^2(0,T;H^2)},\\&Z(T):=||(P\mathbf{u},\mathbf{E}-\widetilde{\mathbf{E}})||^2_{L^\infty(0,T;H^2)},\\&W(T):=||(P\mathbf{u}_t,P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}))||^2_{L^2(0,T;H^1)}+||\nabla P\mathbf{u}||^2_{L^2(0,T;H^2)}.\end{aligned} \tag{2.9}$$

Furthermore, we fix some $M>0$ and small $0<\delta'\leq 1$ so that the incompressible solution $(\mathbf{V},\widetilde{\mathbf{E}})$ to (1.4) satisfies the following:

$$V(T):=||\mathbf{V}||^2_{L^\infty(0,T;H^4)}+||\mathbf{V}_t||^2_{L^2(0,T;H^3)}+||\nabla\mathbf{V}||^2_{L^2(0,T;H^4)}\leq M,\ \text{for all } T\geq 0, \tag{2.10}$$

and

$$E(T):=||\widetilde{\mathbf{E}}||^2_{L^\infty(0,T;H^4)}+||\nabla\widetilde{\mathbf{E}}||^2_{L^2(0,T;H^3)}\leq\delta',\ \text{for all } T\geq 0. \tag{2.11}$$

We claim that if ν is large enough, then one may find some large D and small δ so that for all $T<T_*$, the following bounds are valid:

$$X(T)+Y(T)\leq D\quad\text{and}\quad Z(T)+W(T)\leq\delta. \tag{2.12}$$

Then by a standard continuation principle, we may extend the solution beyond T_*. For the purpose of (2.12), we first assume that for some large constant D and small constant $\delta\in(0,1)$,

$$X(T)+Y(T)\leq 2D,\quad Z(T)+W(T)\leq\delta^{\frac12}, \tag{2.13}$$

which indicates that

$$||a||^2_{L^\infty(0,T;H^2)} \ll 1. \tag{2.14}$$

Throughout this paper, we will use frequently the following interpolation inequality.

Lemma 2.1. (Interpolation inequality, see Galdi [6]) *Let $u \in L^q(\mathbb{R}^n)$, with $D^\alpha u \in L^r(\mathbb{R}^n)$, $|\alpha| = m > 0$, $1 \le q, r \le \infty$. Then $D^\alpha u \in L^s(\mathbb{R}^n)$, $|\alpha| = j$, and the following inequality holds for $0 \le j < m$ and some $C = C(n, m, j, q, r, \theta)$:*

$$\|D^j u\|_{L^s(\mathbb{R}^n)} \le C\|u\|^\theta_{W^{m,r}(\mathbb{R}^n)}\|u\|^{1-\theta}_{L^q(\mathbb{R}^n)}, \tag{2.15}$$

where

$$\frac{1}{s} = \frac{j}{n} + \theta\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\theta)\frac{1}{q},$$

for all θ in the interval

$$\frac{j}{m} \le \theta \le 1,$$

with the following exceptional cases:
(i) If $j = 0$, $rm < n$, $q = \infty$, then we make the additional assumption that either $u(x) \to 0$ as $|x| \to \infty$, or $u \in L^{\bar{q}}(\mathbb{R}^n)$ for some $\bar{q} \in (0, \infty)$.
(ii) If $1 < r < \infty$, and $m - j - n/r$ is a nonnegative integer, then (2.15) *only holds for $j/m \le \theta < 1$.*

We now derive the estimates on $\mathbf{E}$. By virtue of (2.1) we have

$$\begin{aligned}\Delta\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}}) &= \nabla\mathrm{div}\mathrm{div}\mathbf{E} - \mathrm{curl}\mathrm{curl}\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}}) \\ &= -\triangle\nabla a - \nabla\mathrm{div}\mathrm{div}(a\mathbf{E}) - \mathrm{curl}\mathrm{curl}\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}}).\end{aligned} \tag{2.16}$$

Thus applying Δ^{-1} to (2.16) and using the property of Riesz potential, we arrive at

$$\begin{aligned}||\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2} &\lesssim ||\nabla a||^2_{L^2} + ||\nabla(a\mathbf{E})||^2_{L^2} + ||P\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2} \\ &\lesssim ||\nabla a||^2_{L^2} + ||a||^2_{H^2}||\nabla\mathbf{E}||^2_{L^2} + ||P\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2},\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}||\nabla\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2} &\lesssim ||\nabla^2 a||^2_{L^2} + ||\nabla^2(a\mathbf{E})||^2_{L^2} + ||\nabla P\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2} \\ &\lesssim ||\nabla^2 a||^2_{L^2} + ||a||^2_{H^2}||\nabla^2\mathbf{E}||^2_{L^2} + ||\mathbf{E}||^2_{H^2}||\nabla^2 a||^2_{L^2} \\ &\quad + ||\nabla P\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2}.\end{aligned} \tag{2.18}$$

With the help of the above estimates and (2.8), we can deduce that

$$\begin{aligned}||\nabla(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2} &\le ||\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2} + ||\mathrm{curl}(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2} \\ &\lesssim ||\nabla a||^2_{L^2} + ||a||^2_{H^2}||\nabla\mathbf{E}||^2_{L^2} + ||P\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2} \\ &\quad + (||\mathbf{E}||_{H^2} + ||\widetilde{\mathbf{E}}||_{H^2})||\nabla(\mathbf{E} - \widetilde{\mathbf{E}})||_{L^2},\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}||\nabla^2(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2} &\le ||\nabla\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2} + ||\nabla\mathrm{curl}(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2} \\ &\lesssim ||\nabla^2 a||^2_{L^2}(1 + ||\mathbf{E}||^2_{H^2}) + ||a||^2_{H^2}||\nabla^2\mathbf{E}||^2_{L^2} + ||\nabla P\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2} \\ &\quad + (||\mathbf{E}||_{H^2} + ||\widetilde{\mathbf{E}}||_{H^2})||\nabla^2(\mathbf{E} - \widetilde{\mathbf{E}})||_{L^2}.\end{aligned} \tag{2.20}$$

With sufficiently small δ' in (2.11) and δ in (2.13), we can obtain that

$$||\nabla(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2_T(H^1)} \lesssim ||\nabla a||^2_{L^2_T(H^1)} + ||a||^2_{L^\infty_t(H^2)}||\nabla\mathbf{E}||^2_{L^2_T(H^1)} + ||P\mathrm{div}(\mathbf{E} - \widetilde{\mathbf{E}})||^2_{L^2_T(H^1)}. \tag{2.21}$$

Thus from (2.21) we derive that

$$\begin{aligned}||\nabla \mathbf{E}||^2_{L^2_T(H^1)} &\le ||\nabla(\mathbf{E}-\widetilde{\mathbf{E}})||^2_{L^2_T(H^1)} + ||\nabla\widetilde{\mathbf{E}}||^2_{L^2_T(H^1)} \\ &\lesssim ||\nabla a||^2_{L^2_T(H^1)} + ||a||^2_{L^\infty_T(H^2)}||\nabla \mathbf{E}||^2_{L^2_T(H^1)} \\ &\quad + ||P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})||^2_{L^2_T(H^1)} + ||\nabla\widetilde{\mathbf{E}}||^2_{L^2_T(H^1)}.\end{aligned} \tag{2.22}$$

With the assumption (2.14), we have

$$\begin{aligned}||\nabla \mathbf{E}||^2_{L^2_T(H^1)} &\lesssim ||\nabla a||^2_{L^2_T(H^1)} + ||P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})||^2_{L^2_T(H^1)} + ||\nabla\widetilde{\mathbf{E}}||^2_{L^2_T(H^1)} \\ &\le \nu^{-1}Y(T) + W(T) + E(T),\end{aligned} \tag{2.23}$$

and

$$||\nabla(\mathbf{E}-\widetilde{\mathbf{E}})||^2_{L^2_T(H^1)} \lesssim ||\nabla \mathbf{E}||^2_{L^2_T(H^1)} + ||\nabla\widetilde{\mathbf{E}}||^2_{L^2_T(H^1)} \le \nu^{-1}Y(T) + W(T) + E(T). \tag{2.24}$$

3. Estimates for the Divergence-Free Part of the Velocity

We first establish the estimates for $P\mathbf{u}$ and $\mathbf{E}-\widetilde{\mathbf{E}}$ by virtue of (2.5) and (2.7).

Step 1: L^2-estimates for $P\mathbf{u}$ and $\mathbf{E}-\widetilde{\mathbf{E}}$. Noticing the fact that $P^2 = P$, we take the inner product of (2.5) with $P\mathbf{u}$ to deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||P\mathbf{u}||^2_{L^2} + ||\nabla P\mathbf{u}||^2_{L^2} - \int_{\mathbb{R}^3} \mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})\cdot P\mathbf{u}dx \\ &= \int_{\mathbb{R}^3} -[(\mathbf{u}+\mathbf{V})\cdot\nabla P\mathbf{u} + a(\mathbf{V}_t + P\mathbf{u}_t + (Q\mathbf{u}_t + 2\nabla a) + R_1]\cdot P\mathbf{u}dx \\ &= \int_{\mathbb{R}^3} [\frac{1}{2}\mathrm{div}\mathbf{u}|P\mathbf{u}|^2 - (a(\mathbf{V}_t + P\mathbf{u}_t + (Q\mathbf{u}_t + 2\nabla a)) + R_1)\cdot P\mathbf{u}]dx \\ &\lesssim ||P\mathbf{u}||_{L^2}[||P\mathbf{u}||_{L^2}||\mathrm{div}\mathbf{u}||_{H^2} + ||a(\mathbf{V}_t + P\mathbf{u}_t + (Q\mathbf{u}_t + 2\nabla a))||_{L^2} + ||R_1||_{L^2}],\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}||R_1||_{L^2} \lesssim\ & (1+||a||_{H^2})(||P\mathbf{u}||_{L^2}||\nabla(\mathbf{V}+Q\mathbf{u})||_{H^2} + ||\mathbf{V}||_{H^2}||\nabla Q\mathbf{u}||_{L^2} + ||\nabla \mathbf{V}||_{H^1}||\nabla Q\mathbf{u}||_{L^2} \\ &+ ||a||_{H^2}(||\mathbf{u}+\mathbf{V}||_{L^2}||\nabla P\mathbf{u}||_{H^2} + ||\mathbf{V}||_{L^2}||\nabla\mathbf{V}||_{H^2} + ||Q\mathbf{u}||_{L^2}||\nabla Q\mathbf{u}||_{H^2} \\ &+ ||\nabla\mathbf{E}||_{L^2} + ||\mathbf{E}||_{H^2}||\nabla\mathbf{E}||_{L^2}) + (||\mathbf{E}||_{H^2} + ||\widetilde{\mathbf{E}}||_{H^2})||\nabla(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}.\end{aligned} \tag{3.2}$$

Multiplying (2.7) by $(\mathbf{E}-\widetilde{\mathbf{E}})$ yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\mathbf{E}-\widetilde{\mathbf{E}}||^2_{L^2} - \int_{\mathbb{R}^3} \nabla P\mathbf{u}\cdot(\mathbf{E}-\widetilde{\mathbf{E}})dx \\ &= \int_{\mathbb{R}^3} (\nabla\mathbf{v}(\mathbf{E}-\widetilde{\mathbf{E}}) + \nabla\mathbf{u}\widetilde{\mathbf{E}} - \mathbf{u}\cdot\nabla\widetilde{\mathbf{E}} - \mathbf{v}\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}}) + \nabla Q\mathbf{u})\cdot(\mathbf{E}-\widetilde{\mathbf{E}})dx \\ &\lesssim ||\mathbf{E}-\widetilde{\mathbf{E}}||_{L^2}(||\mathbf{E}-\widetilde{\mathbf{E}}||_{L^2}||\nabla\mathbf{v}||_{H^2} + ||\nabla\mathbf{u}||_{L^2}||\widetilde{\mathbf{E}}||_{H^2} + ||\nabla\mathbf{u}||_{L^2}||\nabla\widetilde{\mathbf{E}}||_{H^1} + ||\nabla Q\mathbf{u}||_{L^2}).\end{aligned} \tag{3.3}$$

Hence combining (3.1) and (3.3) we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(||P\mathbf{u}||^2_{L^2} + ||\mathbf{E}-\widetilde{\mathbf{E}}||^2_{L^2}) + ||\nabla P\mathbf{u}||^2_{L^2} \\ &\le ||P\mathbf{u}||_{L^2}[||P\mathbf{u}||_{L^2}||\mathrm{div}\mathbf{u}||_{H^2} + ||a(\mathbf{V}_t + P\mathbf{u}_t + (Q\mathbf{u}_t + 2\nabla a))||_{L^2} + ||R_1||_{L^2}] \\ &\quad + ||\mathbf{E}-\widetilde{\mathbf{E}}||_{L^2}(||\mathbf{E}-\widetilde{\mathbf{E}}||_{L^2}||\nabla\mathbf{v}||_{H^2} + ||\nabla\mathbf{u}||_{L^2}||\widetilde{\mathbf{E}}||_{H^2} + ||\nabla\mathbf{u}||_{L^2}||\nabla\widetilde{\mathbf{E}}||_{H^1} + ||\nabla Q\mathbf{u}||_{L^2}).\end{aligned} \tag{3.4}$$

We apply "$P\mathrm{div}$" to (2.7) to obtain that

$$P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})_t - \Delta P\mathbf{u} = P\mathrm{div}h, \tag{3.5}$$

where

$$h = \nabla\mathbf{v}(\mathbf{E}-\widetilde{\mathbf{E}}) + \nabla\mathbf{u}\widetilde{\mathbf{E}} - \mathbf{u}\cdot\nabla\widetilde{\mathbf{E}} - \mathbf{v}\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}}). \tag{3.6}$$

Then taking the inner product of (3.5) with $P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})$ gives that

$$\frac{1}{2}\frac{d}{dt}||P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}^2 - \int_{\mathbb{R}^3}\Delta P\mathbf{u}\cdot P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx = \int_{\mathbb{R}^3} P\mathrm{div}h\cdot P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx. \tag{3.7}$$

To eliminate the high-order term $\int_{\mathbb{R}^3}\Delta P\mathbf{u}\cdot P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx$, we multiply (2.5) by $-P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})$ to deduce that

$$\begin{aligned}&\int_{\mathbb{R}^3} -P\mathbf{u}_t\cdot P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx + ||P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}^2 + \int_{\mathbb{R}^3}\Delta P\mathbf{u}\cdot P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx \\ &= \int_{\mathbb{R}^3} -[(\mathbf{u}+\mathbf{V})\cdot\nabla P\mathbf{u} + a(\mathbf{V}_t + P\mathbf{u}_t + (Q\mathbf{u}_t + 2\nabla a)) + R_1]\cdot P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx.\end{aligned} \tag{3.8}$$

With the help of (3.5), we have

$$\begin{aligned}&\int_{\mathbb{R}^3} -P\mathbf{u}_t\cdot P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx \\ &= -\frac{d}{dt}\int_{\mathbb{R}^3} P\mathbf{u}\cdot P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx + \int_{\mathbb{R}^3} P\mathbf{u}\cdot P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})_t dx \\ &= -\frac{d}{dt}\int_{\mathbb{R}^3} P\mathbf{u}\cdot P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx - ||\nabla P\mathbf{u}||_{L^2}^2 + \int_{\mathbb{R}^3} P\mathbf{u}\cdot P\mathrm{div}h dx.\end{aligned} \tag{3.9}$$

By direct calculations, we have

$$\begin{aligned}&\int_{\mathbb{R}^3}(P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}) + P\mathbf{u})\cdot P\mathrm{div}h dx \\ &= \int_{\mathbb{R}^3}(P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}) + P\mathbf{u})\cdot P\mathrm{div}(\nabla\mathbf{v}(\mathbf{E}-\widetilde{\mathbf{E}}) + \nabla\mathbf{u}\widetilde{\mathbf{E}} - \mathbf{u}\cdot\nabla\widetilde{\mathbf{E}} - \mathbf{v}\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}}))dx \\ &:= \sum_1^4 I_i,\end{aligned}$$

with

$$\begin{aligned}I_1 &\le \int_{\mathbb{R}^3}(|P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})| + |P\mathbf{u}|)(|\nabla^2\mathbf{v}||\mathbf{E}-\widetilde{\mathbf{E}}| + |\nabla\mathbf{v}||\nabla(\mathbf{E}-\widetilde{\mathbf{E}})|)dx \\ &\le ||(P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}), P\mathbf{u})||_{L^2}||\nabla^2\mathbf{v}||_{L^2}||\mathbf{E}-\widetilde{\mathbf{E}}||_{H^2},\end{aligned} \tag{3.10}$$

$$\begin{aligned}I_2 &= \int_{\mathbb{R}^3}(|P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})| + |P\mathbf{u}|)(|\nabla^2\mathbf{u}\widetilde{\mathbf{E}}| + |\nabla\mathbf{u}||\nabla\widetilde{\mathbf{E}}|)dx \\ &\le ||(P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}), P\mathbf{u})||_{L^2}||\nabla^2\mathbf{u}||_{L^2}||\widetilde{\mathbf{E}}||_{H^2},\end{aligned} \tag{3.11}$$

$$\begin{aligned}I_3 &= \int_{\mathbb{R}^3}(|P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})| + |P\mathbf{u}|)(|\mathbf{u}\cdot\nabla^2\widetilde{\mathbf{E}}| + |\nabla\mathbf{u}||\nabla\widetilde{\mathbf{E}}|)dx \\ &\le ||(P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}), P\mathbf{u})||_{L^2}(||\nabla\mathbf{u}||_{L^2}||\nabla^2\widetilde{\mathbf{E}}||_{H^1} + ||\nabla^2\mathbf{u}||_{L^2}||\widetilde{\mathbf{E}}||_{H^2}),\end{aligned} \tag{3.12}$$

and

$$
\begin{aligned}
I_4 &= \int_{\mathbb{R}^3} [P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}) \cdot P\mathrm{div}(\mathbf{v}\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}})) + P\mathbf{u}(|\mathbf{v}\cdot\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})| + |\nabla\mathbf{v}||\nabla(\mathbf{E}-\widetilde{\mathbf{E}})|)]dx \\
&\le ||P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}^2 ||\nabla(\mathbf{u}+\mathbf{V})||_{H^2} + ||P\mathbf{u}||_{H^1}||\nabla\mathbf{v}||_{L^2}||\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}.
\end{aligned} \tag{3.13}
$$

Notice that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (|P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})|^2 - 2P\mathbf{u}\cdot P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}) + 2|P\mathbf{u}|^2 + 2|\mathbf{E}-\widetilde{\mathbf{E}}|^2)dx \\
&= \int_{\mathbb{R}^3} (|P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}) - P\mathbf{u}|^2 + |P\mathbf{u}|^2 + 2|\mathbf{E}-\widetilde{\mathbf{E}}|^2)dx \approx ||(P\mathbf{u}, P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}), \mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}^2.
\end{aligned}
$$

Using (3.9) and $P\mathbf{u}|_{t=0} = 0$, we integrate the summation of twice (3.4) and (3.8) in time to arrive at

$$
\begin{aligned}
&||(P\mathbf{u}, P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}), \mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}^2(t) + ||\nabla P\mathbf{u}||_{L_T^2(L^2)}^2 + ||P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})||_{L_T^2(L^2)}^2 \\
&\lesssim ||(P\mathrm{div}(\mathbf{E}_0-\widetilde{\mathbf{E}}_0), \mathbf{E}_0-\widetilde{\mathbf{E}}_0)||_{L^2}^2 + \int_0^T [||(P\mathbf{u}, P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}), \mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}^2 ||\nabla(\mathbf{u}+\mathbf{V})||_{H^2} \\
&\quad + ||\mathbf{E}-\widetilde{\mathbf{E}}||_{L^2}(||\nabla\mathbf{u}||_{L^2}||\widetilde{\mathbf{E}}||_{H^2} + ||\mathbf{u}||_{H^2}||\nabla\widetilde{\mathbf{E}}||_{L^2} + ||\nabla Q\mathbf{u}||_{L^2}) \\
&\quad + ||(P\mathbf{u}, P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}))||_{L^2}(||a(\mathbf{V}_t + P\mathbf{u}_t + (Q\mathbf{u}_t + 2\nabla a))||_{L^2} + ||R_1||_{L^2} \\
&\quad + ||\nabla^2\mathbf{v}||_{L^2}||\mathbf{E}-\widetilde{\mathbf{E}}||_{H^2} + ||\nabla^2\mathbf{u}||_{L^2}||\widetilde{\mathbf{E}}||_{H^2} + ||\nabla\mathbf{u}||_{L^2}||\nabla^2\widetilde{\mathbf{E}}||_{H^1}) \\
&\quad + ||P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}||(\mathbf{u}+\mathbf{V})\cdot\nabla P\mathbf{u}||_{L^2} + ||P\mathbf{u}||_{H^1}||\nabla(\mathbf{u}+\mathbf{V})||_{L^2}||\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}]ds,
\end{aligned} \tag{3.14}
$$

where we have used the estimates (3.10)-(3.13).

Step 2: Low-order estimates for $P\mathbf{u}$ and $\mathbf{E}-\widetilde{\mathbf{E}}$. Next, we multiply $\nabla(2.5)$ by $\nabla P\mathbf{u}$ to obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla P\mathbf{u}||_{L^2}^2 + ||\nabla^2 P\mathbf{u}||_{L^2}^2 - \int_{\mathbb{R}^3} \nabla\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})\cdot\nabla P\mathbf{u}dx \\
&= \int_{\mathbb{R}^3} [-\nabla((\mathbf{u}+\mathbf{V})\cdot\nabla P\mathbf{u}) - \nabla(a(\mathbf{V}_t + P\mathbf{u}_t + (Q\mathbf{u}_t + 2\nabla a))) - \nabla R_1]\cdot\nabla P\mathbf{u}dx \\
&\lesssim ||\nabla P\mathbf{u}||_{L^2}[||\nabla P\mathbf{u}||_{L^2}||\nabla(\mathbf{u}+\mathbf{V})||_{H^2} + ||\nabla(a(\mathbf{V}_t + P\mathbf{u}_t + Q\mathbf{u}_t + 2\nabla a))||_{L^2} + ||\nabla R_1||_{L^2}],
\end{aligned} \tag{3.15}
$$

where

$$
\begin{aligned}
||\nabla R_1||_{L^2} \lesssim\ & (1+||a||_{H^2})(||P\mathbf{u}||_{H^1}||\nabla(\mathbf{V}+Q\mathbf{u})||_{H^2} + ||\mathbf{V}||_{H^2}||\nabla^2 Q\mathbf{u}||_{L^2} + ||\nabla\mathbf{V}||_{H^2}||\nabla Q\mathbf{u}||_{L^2}) \\
& + ||a||_{H^2}(||\mathbf{u}+\mathbf{V}||_{H^1}||\nabla P\mathbf{u}||_{H^2} + ||\mathbf{V}||_{H^1}||\nabla\mathbf{V}||_{H^2} + ||Q\mathbf{u}||_{H^1}||\nabla Q\mathbf{u}||_{H^2} + ||\nabla^2\mathbf{E}||_{L^2} \\
& + ||\mathbf{E}||_{H^2}||\nabla^2\mathbf{E}||_{L^2}) + (||\mathbf{E}||_{H^2} + ||\widetilde{\mathbf{E}}||_{H^2})||\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2} + ||\mathbf{E}-\widetilde{\mathbf{E}}||_{H^2}||\nabla^2\widetilde{\mathbf{E}}||_{L^2}.
\end{aligned} \tag{3.16}
$$

Applying ∇ to (2.7) and multiplying by $\nabla(\mathbf{E}-\widetilde{\mathbf{E}})$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}^2-\int_{\mathbb{R}^3}\nabla^2P\mathbf{u}\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}})dx\\
&=\int_{\mathbb{R}^3}(\nabla^2Q\mathbf{u}+\nabla^2\mathbf{v}(\mathbf{E}-\widetilde{\mathbf{E}})+\nabla\mathbf{v}\nabla(\mathbf{E}-\widetilde{\mathbf{E}})+\nabla^2\mathbf{u}\widetilde{\mathbf{E}}+\nabla\mathbf{u}\nabla\widetilde{\mathbf{E}}\\
&\qquad-\mathbf{u}\cdot\nabla^2\widetilde{\mathbf{E}}-\nabla\mathbf{u}\cdot\nabla\widetilde{\mathbf{E}}-\mathbf{v}\cdot\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})-\nabla\mathbf{v}\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}}))\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}})dx\\
&\lesssim||\nabla(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}(||\nabla(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}||\nabla\mathbf{v}||_{H^2}+||\nabla\mathbf{u}||_{L^2}||\nabla^2\widetilde{\mathbf{E}}||_{H^1}\\
&\quad+||\nabla^2\mathbf{u}||_{L^2}||\widetilde{\mathbf{E}}||_{H^2}+||\nabla^2Q\mathbf{u}||_{L^2}).
\end{aligned}\tag{3.17}$$

Hence combining (3.15) and (3.17), we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||\nabla P\mathbf{u}||_{L^2}^2+||\nabla(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}^2)+||\nabla^2P\mathbf{u}||_{L^2}^2\\
&\le||\nabla P\mathbf{u}||_{L^2}[||\nabla P\mathbf{u}||_{L^2}||\nabla(\mathbf{u}+\mathbf{V})||_{H^2}+||\nabla(a(\mathbf{V}_t+P\mathbf{u}_t+Q\mathbf{u}_t+2\nabla a))||_{L^2}\\
&\quad+||\nabla R_1||_{L^2}]+||\nabla(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}(||\nabla(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}||\nabla\mathbf{v}||_{H^2}+||\nabla\mathbf{u}||_{L^2}||\nabla^2\widetilde{\mathbf{E}}||_{H^1}\\
&\quad+||\nabla^2\mathbf{u}||_{L^2}||\widetilde{\mathbf{E}}||_{H^2}+||\nabla^2Q\mathbf{u}||_{L^2}).
\end{aligned}\tag{3.18}$$

Next, we apply "∇" to (3.5) to obtain that

$$\nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})_t-\nabla\Delta P\mathbf{u}=\nabla P\mathrm{div}h.\tag{3.19}$$

Thus we multiply (3.19) by $\nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})$ to derive that

$$\frac{1}{2}\frac{d}{dt}||\nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}^2-\int_{\mathbb{R}^3}\nabla\Delta P\mathbf{u}\cdot\nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx=\int_{\mathbb{R}^3}\nabla P\mathrm{div}h\cdot\nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx.\tag{3.20}$$

By virtue of (3.19), taking the inner product of $\nabla(2.5)$ with $-\nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})$ to eliminate the high-order term $\int_{\mathbb{R}^3}\nabla\Delta P\mathbf{u}\cdot P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx$ yields that

$$\begin{aligned}
&-\frac{d}{dt}\int_{\mathbb{R}^3}\nabla P\mathbf{u}\cdot\nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx-||\nabla^2P\mathbf{u}||_{L^2}^2+||\nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}^2\\
&\quad+\int_{\mathbb{R}^3}\nabla\Delta P\mathbf{u}\cdot\nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx\\
&=\int_{\mathbb{R}^3}-\nabla P[(\mathbf{u}+\mathbf{V})\cdot\nabla P\mathbf{u}+a(\mathbf{V}_t+P\mathbf{u}_t+(Q\mathbf{u}_t+2\nabla a))+R_1]\cdot\nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})dx\\
&\quad-\int_{\mathbb{R}^3}\nabla P\mathbf{u}\cdot\nabla P\mathrm{div}hdx.
\end{aligned}\tag{3.21}$$

Integrating the summation of twice (3.18) and (3.21) in time gives that

$$
\begin{aligned}
&||(\nabla P\mathbf{u}, \nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}), \nabla(\mathbf{E}-\widetilde{\mathbf{E}}))||_{L^2}^2(t) + ||\nabla^2 P\mathbf{u}||^2_{L^2_T(L^2)} + ||\nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})||^2_{L^2_T(L^2)}\\
&\lesssim ||(\nabla P\mathrm{div}(\mathbf{E}_0-\widetilde{\mathbf{E}}_0), \nabla(\mathbf{E}_0-\widetilde{\mathbf{E}}_0))||_{L^2}^2\\
&\quad + \int_0^T [||(\nabla P\mathbf{u}, \nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}), \nabla(\mathbf{E}-\widetilde{\mathbf{E}}))||_{L^2}^2 ||\nabla(\mathbf{u}+\mathbf{V})||_{H^2}\\
&\quad + ||\nabla(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}(||\mathbf{u}||_{H^2}||\nabla^2\widetilde{\mathbf{E}}||_{L^2} + ||\nabla^2\mathbf{u}||_{L^2}||\widetilde{\mathbf{E}}||_{H^2} + ||\nabla^2 Q\mathbf{u}||_{L^2})\\
&\quad + ||(\nabla P\mathbf{u}, \nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}))||_{L^2}(||\nabla(a(\mathbf{V}_t + P\mathbf{u}_t + (Q\mathbf{u}_t + 2\nabla a)))||_{L^2} + ||\nabla R_1||_{L^2}\\
&\quad + ||\nabla\mathbf{u}||_{L^2}||\nabla^3\widetilde{\mathbf{E}}||_{H^1} + ||\nabla\mathbf{u}||_{H^2}||\widetilde{\mathbf{E}}||_{H^2} + ||\nabla(\mathbf{u}+\mathbf{V})||_{H^2}||\mathbf{E}-\widetilde{\mathbf{E}}||_{H^2})\\
&\quad + ||\nabla P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}||(\mathbf{u}+\mathbf{V})\cdot\nabla P\mathbf{u}||_{H^1}\\
&\quad + ||\nabla^2 P\mathbf{u}||_{H^1}||\nabla(\mathbf{u}+\mathbf{V})||_{L^2}||\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}]ds,
\end{aligned}
\tag{3.22}
$$

where we have used

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \nabla P\mathbf{u}\cdot\nabla P\mathrm{div}(\mathbf{v}\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}}))dx = -\int_{\mathbb{R}^3}\nabla^2 P\mathbf{u}\cdot P\mathrm{div}(\mathbf{v}\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}}))dx\\
&\lesssim ||\nabla^2 P\mathbf{u}||_{L^3}||P\mathrm{div}(\mathbf{v}\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}}))||_{L^{3/2}}\\
&\lesssim ||\nabla^2 P\mathbf{u}||_{H^1}||\nabla(\mathbf{u}+\mathbf{V})||_{L^2}||\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}.
\end{aligned}
$$

Step 3: Higher-order estimates for $P\mathbf{u}$ and $\mathbf{E}-\widetilde{\mathbf{E}}$. Next, applying ∇^2 to (2.5) and multiplying the resulting equality by $\nabla^2 P\mathbf{u}$ yield

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla^2 P\mathbf{u}||_{L^2}^2 + ||\nabla\Delta P\mathbf{u}||_{L^2}^2 - \int_{\mathbb{R}^3}\nabla^2\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})\cdot\nabla^2 P\mathbf{u}dx\\
&= \int_{\mathbb{R}^3}[-\nabla^2((\mathbf{u}+\mathbf{V})\cdot\nabla P\mathbf{u}) - \nabla(a(\mathbf{V}_t + P\mathbf{u}_t + (Q\mathbf{u}_t + 2\nabla a))) - \nabla R_1]\cdot\nabla^2 P\mathbf{u}dx\\
&\lesssim ||\nabla^2 P\mathbf{u}||_{L^2}^2||\nabla(\mathbf{u}+\mathbf{V})||_{H^2}\\
&\quad + ||\nabla\Delta P\mathbf{u}||_{L^2}\left(||\nabla(a(\mathbf{V}_t + P\mathbf{u}_t + Q\mathbf{u}_t + 2\nabla a))||_{L^2} + ||\nabla R_1||_{L^2}\right)\\
&\lesssim ||\nabla^2 P\mathbf{u}||_{L^2}^2||\nabla(\mathbf{u}+\mathbf{V})||_{H^2} + \frac{1}{2}||\nabla\Delta P\mathbf{u}||_{L^2}^2\\
&\quad + \frac{1}{2}\left(||\nabla(a(\mathbf{V}_t + P\mathbf{u}_t + Q\mathbf{u}_t + 2\nabla a))||_{L^2}^2 + ||\nabla R_1||_{L^2}^2\right).
\end{aligned}
\tag{3.23}
$$

Multiplying $\nabla^2(2.7)$ by $\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}^2 - \int_{\mathbb{R}^3}\nabla^3 P\mathbf{u}\cdot\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})dx\\
&= \int_{\mathbb{R}^3}[\nabla^3\mathbf{v}(\mathbf{E}-\widetilde{\mathbf{E}}) + \nabla^2\mathbf{v}\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}}) + \nabla\mathbf{v}\cdot\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}}) - \nabla\mathbf{v}\cdot\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})\\
&\qquad - \mathbf{v}\cdot\nabla^3(\mathbf{E}-\widetilde{\mathbf{E}}) - \nabla^2\mathbf{v}\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}}) + \nabla^3\mathbf{u}\widetilde{\mathbf{E}} + \nabla^2\mathbf{u}\nabla\widetilde{\mathbf{E}} + \nabla\mathbf{u}\nabla^2\widetilde{\mathbf{E}}\\
&\qquad - \nabla\mathbf{u}\cdot\nabla^2\widetilde{\mathbf{E}} - \mathbf{u}\cdot\nabla^3\widetilde{\mathbf{E}} - \nabla^2\mathbf{u}\cdot\nabla\widetilde{\mathbf{E}} - \mathbf{v}\cdot\nabla^3(\mathbf{E}-\widetilde{\mathbf{E}}) - \nabla\mathbf{v}\cdot\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})\\
&\qquad - \nabla^2\mathbf{v}\cdot\nabla(\mathbf{E}-\widetilde{\mathbf{E}}) + \nabla^3 Q\mathbf{u}]\cdot\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})dx\\
&\lesssim ||\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}(||(\mathbf{E}-\widetilde{\mathbf{E}})||_{H^2}||\nabla(\mathbf{u}+\mathbf{V})||_{H^2} + ||\nabla\mathbf{u}||_{L^2}||\nabla^3\widetilde{\mathbf{E}}||_{H^1}\\
&\quad + ||\nabla\mathbf{u}||_{H^2}||\widetilde{\mathbf{E}}||_{H^2} + ||\nabla^3 Q\mathbf{u}||_{L^2}).
\end{aligned}
\tag{3.24}
$$

Hence combining (3.23) and (3.24), we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||\nabla^2 P\mathbf{u}||_{L^2}^2+||\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}^2)+\frac{1}{2}||\nabla\Delta P\mathbf{u}||_{L^2}^2\\
&\le ||\nabla^2 P\mathbf{u}||_{L^2}^2||\nabla(\mathbf{u}+\mathbf{V})||_{H^2}+||\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}||(\mathbf{E}-\widetilde{\mathbf{E}})||_{H^2}||\nabla(\mathbf{u}+\mathbf{V})||_{H^2}\\
&\quad+||\nabla(a(\mathbf{V}_t+P\mathbf{u}_t+Q\mathbf{u}_t+2\nabla a))||_{L^2}^2+||\nabla R_1||_{L^2}^2\\
&\quad+||\nabla^2(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2}(||\nabla\mathbf{u}||_{L^2}||\nabla^3\widetilde{\mathbf{E}}||_{H^1}+||\nabla\mathbf{u}||_{H^2}||\widetilde{\mathbf{E}}||_{H^2}+||\nabla^3 Q\mathbf{u}||_{L^2}).
\end{aligned}\tag{3.25}$$

Then by virtue of (3.22), we can bound $||P\mathbf{u}_t||_{L^2_T(H^1)}$ directly from (2.5) by the estimates of Stokes equation as follows:

$$\begin{aligned}
&||P\mathbf{u}_t||_{L^2_T(H^1)}^2+||\Delta P\mathbf{u}||_{L^2_T(H^1)}^2\\
&\lesssim ||(\mathbf{u}+\mathbf{V})\cdot\nabla P\mathbf{u}||_{L^2_T(H^1)}^2+||a(\mathbf{V}_t+P\mathbf{u}_t+(Q\mathbf{u}_t+2\nabla a))||_{L^2_T(H^1)}^2\\
&\quad+||R_1||_{L^2_T(H^1)}^2+||P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2_T(H^1)}^2\\
&\lesssim ||P\mathbf{u}||_{L^\infty_T(H^1)}^2||\nabla P\mathbf{u}||_{L^2_T(H^2)}^2+\int_0^T(||\nabla(Q\mathbf{u}+\mathbf{V})||_{H^1}^2+||Q\mathbf{u}+\mathbf{V}||_{L^\infty}^2)||\nabla^2 P\mathbf{u}||_{L^2}^2ds\\
&\quad+\int_0^T||\nabla(Q\mathbf{u}+\mathbf{V})||_{L^2}^2||\nabla P\mathbf{u}||_{H^1}^2ds+||a(\mathbf{V}_t+P\mathbf{u}_t+(Q\mathbf{u}_t+2\nabla a))||_{L^2_T(H^1)}^2\\
&\quad+||R_1||_{L^2_T(H^1)}^2+||P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}})||_{L^2_T(H^1)}^2.
\end{aligned}\tag{3.26}$$

To control the term $\int_0^T||Q\mathbf{u}+\mathbf{V}||_{L^\infty}^2||\nabla^2 P\mathbf{u}||_{L^2}^2ds$, we apply the interpolation inequality (2.15) to obtain that

$$\begin{aligned}
\int_0^T||Q\mathbf{u}+\mathbf{V}||_{L^\infty}^2||\nabla^2 P\mathbf{u}||_{L^2}^2ds&\le\int_0^T||Q\mathbf{u}+\mathbf{V}||_{L^\infty}^2||\nabla P\mathbf{u}||_{L^2}||\nabla^3 P\mathbf{u}||_{L^2}ds\\
&\le\frac{1}{2}||\nabla^3 P\mathbf{u}||_{L^2_T(L^2)}^2+\frac{1}{2}\int_0^T||Q\mathbf{u}+\mathbf{V}||_{L^\infty}^4||\nabla P\mathbf{u}||_{L^2}^2ds.
\end{aligned}$$

Step 4: Collection of the estimates for $P\mathbf{u}$ and $\mathbf{E}-\widetilde{\mathbf{E}}$. By virtue of $P\mathbf{u}_0=0$, (2.20) and small enough δ' in (2.11), integrating (3.25) with respect to time and combining with (3.22) and (3.26), we can obtain that

$$\begin{aligned}
&||(P\mathbf{u},\mathbf{E}-\widetilde{\mathbf{E}})||_{L^\infty_T(H^2)}^2+||(P\mathbf{u}_t,P\mathrm{div}(\mathbf{E}-\widetilde{\mathbf{E}}))||_{L^2_T(H^1)}^2+||\nabla P\mathbf{u}||_{L^2_T(H^2)}^2\\
&\lesssim ||\mathbf{E}_0-\widetilde{\mathbf{E}}_0||_{H^2}^2+\int_0^T(||P\mathbf{u}||_{H^2}^2+||\mathbf{E}-\widetilde{\mathbf{E}}||_{H^2}^2)(||(\nabla\mathbf{V},\nabla P\mathbf{u},\nabla Q\mathbf{u})||_{H^2}^2+||(Q\mathbf{u},\mathbf{V})||_{L^\infty}^4)ds\\
&\quad+||a||_{L^\infty_T(H^2)}^2(||\mathbf{V}_t||_{L^2_T(H^1)}^2+||P\mathbf{u}_t||_{L^2_T(H^1)}^2+||Q\mathbf{u}_t+2\nabla a||_{L^2_T(H^1)}^2)+||R_1||_{L^2_T(H^1)}^2\\
&\quad+||\nabla Q\mathbf{u}||_{L^2_T(H^2)}^2(1+||\widetilde{\mathbf{E}}||_{L^\infty_T(H^2)}^2)+||\nabla Q\mathbf{u}||_{L^2_T(L^2)}||\nabla\widetilde{\mathbf{E}}||_{L^\infty_T(H^3)}.
\end{aligned}\tag{3.27}$$

By assuming (2.13) with small enough δ, the first term of the right-hand side in (3.26) can be absorbed by the left-hand side of (3.27). For the third term of the right-hand side of (3.27), one has

$$\begin{aligned}
&||a||_{L^\infty_T(H^2)}^2(||\mathbf{V}_t||_{L^2_T(H^1)}^2+||P\mathbf{u}_t||_{L^2_T(H^1)}^2+||Q\mathbf{u}_t+2\nabla a||_{L^2_T(H^1)}^2)\\
&\lesssim \nu^{-2}X(T)(V(T)+W(T)+Y(T)).
\end{aligned}$$

From (3.2) and (3.16), we can estimate the terms in $||R_1||^2_{L^2_T(H^1)}$ in the right-hand side of (3.27) as follows:

$$\begin{aligned}&(1+||a||^2_{L^\infty_T(H^2)})(||\mathbf{V}||^2_{L^\infty_T(H^2)}||\nabla Q\mathbf{u}||^2_{L^2_T(H^1)}+||\nabla\mathbf{V}||^2_{L^\infty_T(H^2)}||\nabla Q\mathbf{u}||^2_{L^2_T(L^2)})\\ &\quad\lesssim(1+\nu^{-2}X(T))\nu^{-2}Y(T)V(T),\end{aligned}$$

$$||a||^2_{L^\infty_T(H^2)}||\mathbf{u}+\mathbf{V}||^2_{L^\infty_T(H^1)}||\nabla P\mathbf{u}||^2_{L^2_T(H^2)}\lesssim\nu^{-2}X(T)W(T)(X(T)+Z(T)+V(T)),$$

$$\begin{aligned}&||a||^2_{L^\infty_T(H^2)}(||\mathbf{V}||^2_{L^\infty_T(H^1)}||\nabla\mathbf{V}||^2_{L^2_T(H^2)}+||Q\mathbf{u}||^2_{L^\infty_T(H^1)}||\nabla Q\mathbf{u}||^2_{L^2_T(H^2)})\\ &\quad\lesssim\nu^{-2}X(T)(V^2(T)+\nu^{-2}X(T)Y(T)),\end{aligned}$$

$$\begin{aligned}&||a||^2_{L^\infty_T(H^2)}(||\nabla\mathbf{E}||^2_{L^2_T(H^1)}+||\mathbf{E}||^2_{L^\infty_T(H^2)}||\nabla\mathbf{E}||^2_{L^2_T(H^1)})\\ &\quad\lesssim\nu^{-2}X(T)(\nu^{-1}Y(T)+W(T)+E(T))(1+Z(T)+E(T)),\end{aligned}$$

and

$$\begin{aligned}&(||\mathbf{E}||^2_{L^\infty_T(H^2)}+||\widetilde{\mathbf{E}}||^2_{L^\infty_T(H^2)})||\nabla(\mathbf{E}-\widetilde{\mathbf{E}})||^2_{L^2_T(H^1)}\\ &\quad\lesssim(Z(T)+E(T))(\nu^{-1}Y(T)+W(T)+E(T)),\end{aligned}$$

where we have used (2.23) and (2.24). For the last term, we have

$$||\nabla Q\mathbf{u}||^2_{L^2_T(H^2)}(1+||\widetilde{\mathbf{E}}||^2_{L^\infty_T(H^2)})+||\nabla Q\mathbf{u}||_{L^2_T(L^2)}||\nabla\widetilde{\mathbf{E}}||_{L^\infty_T(H^3)}\lesssim\nu^{-2}Y(T)(1+E(T)).$$

Recall the assumption that $X(T)\ll\nu^2$, we obtain that

$$\begin{aligned}&Z(T)+W(T)\\ &\lesssim Z(0)+\int_0^T(||(\nabla\mathbf{V},\nabla P\mathbf{u},\nabla Q\mathbf{u})||^2_{H^2}+||(Q\mathbf{u},\mathbf{V})||^4_{L^\infty})Z(t)ds\\ &\quad+\nu^{-2}X(T)(V(T)+W(T)+Y(T))+\nu^{-2}Y(T)(1+V(T)+E(T)+Z(T))\\ &\quad+\nu^{-2}X(T)W(T)(X(T)+Z(T)+V(T))+\nu^{-2}X(T)V^2(T)\\ &\quad+\nu^{-2}X(T)(\nu^{-1}Y(T)+W(T)+E(T))(1+E(T))\\ &\quad+(Z(T)+E(T))(\nu^{-1}Y(T)+W(T)+E(T)).\end{aligned}\tag{3.28}$$

Using Grönwall's inequality, we conclude that

$$\begin{aligned}&Z(T)+W(T)\\ &\le C_1e^{C_1\int_0^T(||(\nabla\mathbf{V},\nabla P\mathbf{u},\nabla Q\mathbf{u})||^2_{H^2}+||(Q\mathbf{u},\mathbf{V})||^4_{L^\infty})ds}[Z(0)+\nu^{-2}X(T)(V(T)+W(T)+Y(T))\\ &\quad+\nu^{-2}Y(T)(1+V(T)+E(T)+Z(T))+\nu^{-2}X(T)W(T)(X(T)+Z(T)+V(T))\\ &\quad+\nu^{-2}X(T)V^2(T)+\nu^{-2}X(T)(\nu^{-1}Y(T)+W(T)+E(T))(1+E(T))\\ &\quad+(Z(T)+E(T))(\nu^{-1}Y(T)+W(T)+E(T))],\end{aligned}\tag{3.29}$$

for some constant $C_1>1$.

4. Estimates for the Potential Part of the Velocity and the Density

To estimate the potential part of the velocity, we consider the momentum and continuity equations together.

Step 1: Low-order estimates for $Q\mathbf{u}$ and a. Now, we first multiply (2.2) and (2.4) by $Q\mathbf{u}$ and $2a$, respectively, and integrate by part to discover that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||Q\mathbf{u}||_{L^2}^2+\nu||\nabla Q\mathbf{u}||_{L^2}^2+2\int_{\mathbb{R}^3}\nabla a\cdot Q\mathbf{u}dx\\
&=\int_{\mathbb{R}^3}-((\mathbf{u}+\mathbf{V})\cdot\nabla Q\mathbf{u})\cdot Q\mathbf{u}-(a\mathbf{V}_t+a\mathbf{u}_t+R_2)\cdot Q\mathbf{u}dx\\
&:=\int_{\mathbb{R}^3}\frac{1}{2}\mathrm{div}\mathbf{u}|Q\mathbf{u}|^2dx+\int_{\mathbb{R}^3}f_1\cdot Q\mathbf{u}dx,
\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}
\frac{d}{dt}||a||_{L^2}^2+2\int_{\mathbb{R}^3}a\mathrm{div}Q\mathbf{u}dx&=\int_{\mathbb{R}^3}[-(\mathbf{u}+\mathbf{V})\cdot\nabla a-a\mathrm{div}Q\mathbf{u}]\cdot 2adx\\
&:=\int_{\mathbb{R}^3}\mathrm{div}\mathbf{u}a^2dx+2\int_{\mathbb{R}^3}g_1\cdot adx,
\end{aligned}\tag{4.2}$$

where

$$f_1:=-(a\mathbf{V}_t+a\mathbf{u}_t+R_2),\quad\text{and}\quad g_1:=-a\mathrm{div}Q\mathbf{u}.$$

Applying ∇ to (2.4) gives that

$$\nabla a_t+(\mathbf{u}+\mathbf{V})\cdot\nabla^2a+\nabla\mathrm{div}Q\mathbf{u}=\nabla g_1-\nabla(\mathbf{u}+\mathbf{V})\cdot\nabla a.\tag{4.3}$$

Thus we take the L^2 inner product of (4.3) with ∇a to discover that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla a||_{L^2}^2+\int_{\mathbb{R}^3}((\mathbf{u}+\mathbf{V})\cdot\nabla^2a)\cdot\nabla adx+\int_{\mathbb{R}^3}\nabla\mathrm{div}Q\mathbf{u}\cdot\nabla adx\\
&=\int_{\mathbb{R}^3}[\nabla g_1-\nabla(\mathbf{u}+\mathbf{V})\cdot\nabla a]\cdot\nabla adx.
\end{aligned}\tag{4.4}$$

In order to eliminate the high-order term $\int_\Omega\nabla\mathrm{div}Q\mathbf{u}\cdot\nabla adx$, we test (4.3) by $Q\mathbf{u}$ and (2.2) by ∇a, respectively, and summarize the resulting equalities to derive that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}Q\mathbf{u}\cdot\nabla adx+\int_{\mathbb{R}^3}(\mathbf{u}+\mathbf{V})\cdot\nabla(Q\mathbf{u}\cdot\nabla a)dx-\nu\int_{\mathbb{R}^3}\Delta Q\mathbf{u}\cdot\nabla adx\\
&\quad+2||\nabla a||_{L^2}^2+\int_{\mathbb{R}^3}\nabla\mathrm{div}Q\mathbf{u}\cdot Q\mathbf{u}dx\\
&=\int_{\mathbb{R}^3}[\nabla g_1-\nabla(\mathbf{u}+\mathbf{V})\cdot\nabla a]\cdot Q\mathbf{u}dx+\int_{\mathbb{R}^3}f_1\cdot\nabla adx.
\end{aligned}\tag{4.5}$$

Noting that $\Delta Q\mathbf{u}\equiv\nabla\mathrm{div}Q\mathbf{u}$, we add ν times (4.4) to (4.5) and use integration by parts to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\nu|\nabla a|^2+2Q\mathbf{u}\cdot\nabla a)dx+(2||\nabla a||_{L^2}^2-||\nabla Q\mathbf{u}||_{L^2}^2)\\
&=\int_{\mathbb{R}^3}(\frac{\nu}{2}|\nabla a|^2+Q\mathbf{u}\cdot\nabla a)\mathrm{div}\mathbf{u}dx+\nu\int_{\mathbb{R}^3}[\nabla g_1-\nabla(\mathbf{u}+\mathbf{V})\cdot\nabla a]\cdot\nabla adx\\
&\quad+\int_{\mathbb{R}^3}[\nabla g_1-\nabla(\mathbf{u}+\mathbf{V})\cdot\nabla a]\cdot Q\mathbf{u}dx+\int_{\mathbb{R}^3}f_1\cdot\nabla adx.
\end{aligned}\tag{4.6}$$

Multiplying the above equality by ν, adding up twice (4.1) and (4.2), and integrating the resulting equality in time, we get

$$\begin{aligned}
&||(Q\mathbf{u},a,\nu\nabla a)||_{L^2}^2(t)+\nu(2||\nabla a||_{L^2_T(L^2)}^2+||\nabla Q\mathbf{u}||_{L^2_T(L^2)}^2)\\
&\le ||(Q\mathbf{u}_0,a_0,\nu\nabla a_0)||_{L^2}^2+\int_0^T||(Q\mathbf{u},a,\nu\nabla a)||_{L^2}^2||\mathrm{div}\mathbf{u}||_{H^2}ds\\
&\quad+\int_0^T[||\nabla(\mathbf{u}+\mathbf{V})||_{H^2}||\nu\nabla a||_{L^2}||(\nu\nabla a,Q\mathbf{u})||_{L^2}+||(g_1,\nu\nabla g_1,f_1)||_{L^2}||(Q\mathbf{u},a,\nu\nabla a)||_{L^2}]ds,
\end{aligned} \tag{4.7}$$

where we have used that

$$\int_{\mathbb{R}^3}(4a^2+|\nu\nabla a|^2+2\nu Q\mathbf{u}\cdot\nabla a+2|Q\mathbf{u}|^2)dx\gtrsim||(Q\mathbf{u},a,\nu\nabla a)||_{L^2}^2. \tag{4.8}$$

Moreover, from (4.2) we have

$$\frac{d}{dt}||a||_{L^2}^2\lesssim||a||_{L^2}(||\mathrm{div}Q\mathbf{u}||_{L^2}+||a||_{L^2}||\mathrm{div}\mathbf{u}||_{H^2}). \tag{4.9}$$

Multiplying (4.9) by ν^2 and integrating in time give that

$$||\nu a||_{L^2}^2(t)\lesssim||\nu a_0||_{L^2}^2+\frac{1}{2}||\nu a||_{L^\infty_T(L^2)}^2+\frac{1}{2}\int_0^T(||\nu\mathrm{div}Q\mathbf{u}||_{L^2}^2+||\nu a||_{L^2}^2||\mathrm{div}\mathbf{u}||_{H^2}^2)ds. \tag{4.10}$$

We multiply (4.1) by ν to derive that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\nu||Q\mathbf{u}||_{L^2}^2\big)+||\nu\nabla Q\mathbf{u}||_{L^2}^2\\
&\lesssim||\nu^{1/2}Q\mathbf{u}||_{L^2}||\nu^{1/2}\nabla a||_{L^2}+||\nu\nabla Q\mathbf{u}||_{L^2}(||Q\mathbf{u}||_{L^2}||\mathrm{div}\mathbf{u}||_{H^1}+\mathcal{M}_0)\\
&\lesssim||\nu^{1/2}Q\mathbf{u}||_{L^2}||\nu^{1/2}\nabla a||_{L^2}+\frac{1}{2}||\nu\nabla Q\mathbf{u}||_{L^2}^2+\frac{1}{2}(||Q\mathbf{u}||_{L^2}^2||\mathrm{div}\mathbf{u}||_{H^1}^2+\mathcal{M}_0^2),
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
\mathcal{M}_0=&(1+||a||_{H^2})||\mathbf{u}+\mathbf{V}||_{L^2}(||\nabla P\mathbf{u}||_{H^1}+||\nabla\mathbf{V}||_{H^1})+||a||_{H^2}||\mathbf{u}+\mathbf{V}||_{L^2}||\nabla Q\mathbf{u}||_{H^1}\\
&+||a||_{H^1}||\nabla a||_{L^2}+||a||_{H^1}||\nabla\mathbf{E}||_{L^2}+(1+||a||_{H^2})||\mathbf{E}||_{H^1}||\nabla\mathbf{E}||_{L^2}.
\end{aligned}$$

Thus integrating (4.11) in time gives that

$$\begin{aligned}
&\nu||Q\mathbf{u}||_{L^\infty_T(L^2)}^2+||\nu\nabla Q\mathbf{u}||_{L^2_T(L^2)}^2\\
&\lesssim\nu||Q\mathbf{u}_0||_{L^2}^2+\int_0^T(\nu||\nabla a||_{L^2}^2+||Q\mathbf{u}||_{L^2}^2||\mathrm{div}\mathbf{u}||_{H^1}^2+\mathcal{M}_0^2)ds.
\end{aligned} \tag{4.12}$$

Thus summarizing (4.7), (4.10) and (4.12), we can obtain the following inequality

$$\begin{aligned}
&||(Q\mathbf{u},\nu a,\nu\nabla a)||_{L^\infty_T(L^2)}^2+\nu||\nabla a||_{L^2_T(L^2)}^2+||\nu\nabla Q\mathbf{u}||_{L^2_T(L^2)}^2\\
&\lesssim||(Q\mathbf{u}_0,\nu a_0,\nu\nabla a_0)||_{L^2}^2+\int_0^T||(Q\mathbf{u},\nu a,\nu\nabla a)||_{L^2}^2||\nabla(\mathbf{u}+\mathbf{V})||_{H^2}^2ds\\
&\quad+||(g_1,f_1,\nu\nabla g_1)||_{L^2_T(L^2)}^2,
\end{aligned} \tag{4.13}$$

where

$$||(g_1,\nu\nabla g_1)||_{L^2_T(L^2)}^2\lesssim||\mathrm{div}Q\mathbf{u}||_{L^2_T(H^2)}^2(||a||_{L^\infty_T(L^2)}^2+||\nu\nabla a||_{L^\infty_T(L^2)}^2),$$

and

$$||f_1||_{L^2_T(L^2)}^2\lesssim||a(\mathbf{V}_t+P\mathbf{u}_t+(Q\mathbf{u}_t+2\nabla a))||_{L^2_T(L^2)}^2+||a||_{L^\infty_T(H^2)}^2||\nabla a||_{L^2_T(L^2)}^2+||R_2||_{L^2_T(L^2)}^2,$$

with

$$
\begin{aligned}
\|R_2\|^2_{L^2_T(L^2)} \lesssim\ & (1+\|a\|^2_{L^\infty_T(H^2)})\|\mathbf{u}+\mathbf{V}\|^2_{L^\infty_T(L^2)}(\|\nabla P\mathbf{u}\|^2_{L^2_T(H^2)}+\|\nabla \mathbf{V}\|^2_{L^2_T(H^2)}) \\
&+\|a\|^2_{L^\infty_T(H^2)}\|\mathbf{u}+\mathbf{V}\|^2_{L^\infty_T(L^2)}\|\nabla Q\mathbf{u}\|^2_{L^2_T(H^2)}+\|a\|^2_{L^\infty_T(H^2)}\|\nabla a\|^2_{L^2_T(L^2)} \\
&+\|\mathbf{E}\|^2_{L^\infty_T(H^2)}\|\nabla a\|^2_{L^2_T(L^2)}+(1+\|a\|^2_{L^\infty_T(H^2)})\|\mathbf{E}\|^2_{L^\infty_T(H^2)}\|\nabla \mathbf{E}\|^2_{L^2_T(L^2)}.
\end{aligned}
\tag{4.14}
$$

Step 2: Higher-order estimates for $Q\mathbf{u}$ and a. We multiply ∇(2.2) by $\nabla Q\mathbf{u}$ to discover that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla Q\mathbf{u}\|^2_{L^2}+\nu\|\nabla^2 Q\mathbf{u}\|^2_{L^2}+2\int_{\mathbb{R}^3}\nabla^2 a\cdot\nabla Q\mathbf{u}dx \\
&=\frac{1}{2}\int_{\mathbb{R}^3}\mathrm{div}\mathbf{u}|\nabla Q\mathbf{u}|^2dx+\int_{\mathbb{R}^3} f_2\cdot\nabla Q\mathbf{u}dx,
\end{aligned}
\tag{4.15}
$$

where $f_2=-\nabla(\mathbf{u}+\mathbf{V})\cdot\nabla Q\mathbf{u}-\nabla Q(a\mathbf{u}_t+a\mathbf{V}_t)-\nabla QR_2$. Next, testing ∇^2(2.4) by $\nabla^2 a$ yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^2 a\|^2_{L^2}+\int_{\mathbb{R}^3}\nabla^2\mathrm{div}Q\mathbf{u}\cdot\nabla^2 adx \\
&=\int_{\mathbb{R}^3}\nabla g_2\cdot\nabla^2 adx-\int_{\mathbb{R}^3}((\mathbf{u}+\mathbf{V})\cdot\nabla^3 a)\cdot\nabla^2 adx,
\end{aligned}
\tag{4.16}
$$

where $g_2=-\nabla(\mathbf{u}+\mathbf{V})\cdot\nabla a-\nabla(a\mathrm{div}Q\mathbf{u})$. To eliminate the highest-order term $\int_{\mathbb{R}^3}\nabla^2\mathrm{div}Q\mathbf{u}\cdot\nabla^2 adx$, we test ∇^2(2.4) by $\nabla Q\mathbf{u}$ and ∇(2.2) by $\nabla^2 a$ to get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\nabla Q\mathbf{u}\cdot\nabla^2 adx-\nu\int_{\mathbb{R}^3}\nabla\Delta Q\mathbf{u}\cdot\nabla^2 adx+2\|\nabla^2 a\|^2_{L^2}+\int_{\mathbb{R}^3}\nabla^2\mathrm{div}Q\mathbf{u}\cdot\nabla Q\mathbf{u}dx \\
&=\int_{\mathbb{R}^3}\nabla g_2\cdot\nabla Q\mathbf{u}dx+\int_{\mathbb{R}^3} f_2\cdot\nabla^2 adx-\int_{\mathbb{R}^3}(\mathbf{u}+\mathbf{V})\cdot\nabla(\nabla Q\mathbf{u}\cdot\nabla^2 a)dx.
\end{aligned}
\tag{4.17}
$$

Then we add ν times (4.16) to (4.17) to cancel the highest term and use integration by parts to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\nu|\nabla^2 a|^2+2\nabla Q\mathbf{u}\cdot\nabla^2 a)dx+(2\|\nabla^2 a\|^2_{L^2}-\|\nabla^2 Q\mathbf{u}\|^2_{L^2}) \\
&=\int_{\mathbb{R}^3}(\frac{\nu}{2}|\nabla^2 a|^2+\nabla Q\mathbf{u}\cdot\nabla^2 a)\mathrm{div}\mathbf{u}dx+\nu\int_{\mathbb{R}^3}\nabla g_2\cdot\nabla^2 adx \\
&\quad+\int_{\mathbb{R}^3}\nabla g_2\cdot\nabla Q\mathbf{u}dx+\int_{\mathbb{R}^3} f_2\cdot\nabla^2 adx.
\end{aligned}
\tag{4.18}
$$

Multiplying the above equality by ν, adding up twice (4.15) and 4 times (4.4) and integrating in time, we obtain

$$
\begin{aligned}
&\|(\nabla Q\mathbf{u},\nabla a,\nu\nabla^2 a)\|^2_{L^2}(t)+\nu(2\|\nabla^2 a\|^2_{L^2_T(L^2)}+\|\nabla^2 Q\mathbf{u}\|^2_{L^2_T(L^2)}) \\
&\le\|(\nabla Q\mathbf{u}_0,\nabla a_0,\nu\nabla^2 a_0)\|^2_{L^2}+\int_0^T\|(\nabla Q\mathbf{u},\nabla a,\nu\nabla^2 a)\|^2_{L^2}\|\mathrm{div}\mathbf{u}\|_{H^2}ds \\
&\quad+\int_0^T\|(g_2,f_2,\nu\nabla g_2)\|_{L^2}\|(\nabla Q\mathbf{u},\nabla a,\nu\nabla^2 a)\|_{L^2}ds,
\end{aligned}
\tag{4.19}
$$

by noticing that

$$
\int_{\mathbb{R}^3}(4|\nabla a|^2+|\nu\nabla^2 a|^2+2\nu\nabla Q\mathbf{u}\cdot\nabla^2 a+2|\nabla Q\mathbf{u}|^2)dx\gtrsim\|(\nabla Q\mathbf{u},\nabla a,\nu\nabla^2 a)\|^2_{L^2}.
\tag{4.20}
$$

We can obtain the following estimate from multiplying (4.15) by ν that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\nu\|\nabla Q\mathbf{u}\|_{L^2}^2\right)+\|\nu\nabla^2 Q\mathbf{u}\|_{L^2}^2\\
&\lesssim \|\nu\nabla^2 Q\mathbf{u}\|_{L^2}(\|\nabla a\|_{L^2}+\|\nabla Q\mathbf{u}\|_{L^2}\|\mathrm{div}\mathbf{u}\|_{H^1}+\|\nabla Q\mathbf{u}\|_{L^2}\|\nabla(\mathbf{u}+\mathbf{V})\|_{H^1}+\|f_1\|_{L^2})\\
&\lesssim \frac{1}{2}\|\nu\nabla^2 Q\mathbf{u}\|_{L^2}^2+\frac{1}{2}[\|\nabla a\|_{L^2}^2+\|\nabla Q\mathbf{u}\|_{L^2}^2(\|\mathrm{div}\mathbf{u}\|_{H^1}^2+\|\nabla(\mathbf{u}+\mathbf{V})\|_{H^1}^2)+\|f_1\|_{L^2}^2],
\end{aligned}\tag{4.21}$$

whence integrating in time gives

$$\begin{aligned}
&\nu\|\nabla Q\mathbf{u}\|_{L_T^\infty(L^2)}^2+\|\nu\nabla^2 Q\mathbf{u}\|_{L_T^2(L^2)}^2\\
&\lesssim \nu\|\nabla Q\mathbf{u}_0\|_{L^2}^2+\int_0^T[\|\nabla a\|_{L^2}^2+\|\nabla Q\mathbf{u}\|_{L^2}^2(\|\mathrm{div}\mathbf{u}\|_{H^1}^2+\|\nabla(\mathbf{u}+\mathbf{V})\|_{H^1}^2)+\|f_1\|_{L^2}^2]ds.
\end{aligned}\tag{4.22}$$

Thus with (4.19) and (4.22) we have

$$\begin{aligned}
&\|(\nabla Q\mathbf{u},\nabla a,\nu\nabla^2 a)\|_{L_T^\infty(L^2)}^2+\nu\|\nabla^2 a\|_{L_T^2(L^2)}^2+\|\nu\nabla^2 Q\mathbf{u}\|_{L_T^2(L^2)}^2\\
&\lesssim \|(\nabla Q\mathbf{u}_0,\nabla a_0,\nu\nabla^2 a_0)\|_{L^2}^2+\int_0^T\|\nabla(\mathbf{u}+\mathbf{V})\|_{H^2}^2\|(\nabla Q\mathbf{u},\nabla a,\nu\nabla^2 a)\|_{L^2}^2 ds\\
&\quad+\|(g_2,f_2,\nu\nabla g_2)\|_{L_T^2(L^2)}^2,
\end{aligned}\tag{4.23}$$

where

$$\|(g_2,\nu\nabla g_2)\|_{L_T^2(L^2)}^2\lesssim\left(\|\nabla(\mathbf{u}+\mathbf{V})\|_{L_T^2(H^2)}^2+\|\mathrm{div}Q\mathbf{u}\|_{L_T^2(H^2)}^2\right)\left(\|\nabla a\|_{L_T^\infty(L^2)}^2+\|\nu a\|_{L_T^\infty(H^2)}^2\right),$$

and

$$\begin{aligned}
\|f_2\|_{L_T^2(L^2)}^2\lesssim{}&\|(\mathbf{u}+\mathbf{V})\cdot\nabla Q\mathbf{u}\|_{L_T^2(H^1)}^2+\|\nabla(a(\mathbf{V}_t+P\mathbf{u}_t+(Q\mathbf{u}_t+2\nabla a)))\|_{L_T^2(L^2)}^2\\
&+\|a\|_{L_T^\infty(H^2)}^2\|\nabla^2 a\|_{L_T^2(L^2)}^2+\|\nabla R_2\|_{L_T^2(L^2)}^2,
\end{aligned}$$

with

$$\begin{aligned}
\|\nabla R_2\|_{L_T^2(L^2)}^2\lesssim{}&(1+\|a\|_{L_T^\infty(H^2)}^2)\|\mathbf{u}+\mathbf{V}\|_{L_T^\infty(H^1)}^2(\|\nabla P\mathbf{u}\|_{L_T^2(H^2)}^2+\|\nabla\mathbf{V}\|_{L_T^2(H^2)}^2)\\
&+\|a\|_{L_T^\infty(H^2)}^2\|\mathbf{u}+\mathbf{V}\|_{L_T^\infty(H^1)}^2\|\nabla Q\mathbf{u}\|_{L_T^2(H^2)}^2+\|a\|_{L_T^\infty(H^2)}^2\|\nabla^2 a\|_{L_T^2(L^2)}^2\\
&+\|\mathbf{E}\|_{L_T^\infty(H^2)}^2\|\nabla^2 a\|_{L_T^2(L^2)}^2+(1+\|a\|_{L_T^\infty(H^2)}^2)\|\mathbf{E}\|_{L_T^\infty(H^2)}^2\|\nabla^2\mathbf{E}\|_{L_T^2(L^2)}^2.
\end{aligned}\tag{4.24}$$

Next, we take the inner product of $\nabla^2(2.2)$ and $\nabla^2 Q\mathbf{u}$ to discover that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^2 Q\mathbf{u}\|_{L^2}^2+\nu\|\nabla\Delta Q\mathbf{u}\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}\nabla^2[-Q((\mathbf{u}+\mathbf{V})\cdot\nabla Q\mathbf{u})-2\nabla a-Q(a\mathbf{V}_t+a\mathbf{u}_t)-QR_2]\cdot\nabla^2 Q\mathbf{u}dx\\
&\lesssim \|\nabla^2 Q\mathbf{u}\|_{L^2}^2\|\nabla(\mathbf{u}+\mathbf{V})\|_{H^2}+\|\nabla\Delta Q\mathbf{u}\|_{L^2}[\|\nabla^2 a\|_{L^2}+\|\nabla(a\mathbf{V}_t+a\mathbf{u}_t)\|_{L^2}+\|\nabla R_2\|_{L^2}]\\
&\lesssim \|\nabla^2 Q\mathbf{u}\|_{L^2}^2\|\nabla(\mathbf{u}+\mathbf{V})\|_{H^2}+\frac{1}{2}\|\nabla\Delta Q\mathbf{u}\|_{L^2}^2\\
&\quad+\frac{1}{2}[\|\nabla^2 a\|_{L^2}^2+\|\nabla(a\mathbf{V}_t+a\mathbf{u}_t)\|_{L^2}^2+\|\nabla R_2\|_{L^2}^2].
\end{aligned}\tag{4.25}$$

Similarly to (4.22), we have

$$\begin{aligned}
&\nu||\nabla^2 Q\mathbf{u}||^2_{L^\infty_T(L^2)}+||\nu\nabla\triangle Q\mathbf{u}||^2_{L^2_T(L^2)}\\
&\lesssim \nu||\nabla^2 Q\mathbf{u}_0||^2_{L^2}+\int_0^T(||\nabla^2 a||^2_{L^2}+||\nabla(a\mathbf{V}_t+a\mathbf{u}_t)||^2_{L^2}+||\nabla R_2||^2_{L^2})ds.
\end{aligned}\tag{4.26}$$

In view of (2.2), we obtain that

$$\begin{aligned}
&||Q\mathbf{u}_t+2\nabla a||^2_{L^2_T(H^1)}\\
&\lesssim ||a(\mathbf{V}_t+P\mathbf{u}_t+(Q\mathbf{u}_t+2\nabla a))||^2_{L^2_T(H^1)}+||a||^2_{L^\infty_T(H^2)}||\nabla a||^2_{L^2_T(H^1)}\\
&\quad+||R_2||^2_{L^2_T(H^1)}+||(\mathbf{u}+\mathbf{V})\cdot\nabla Q\mathbf{u}||^2_{L^2_T(H^1)}+||\nu\triangle Q\mathbf{u}||^2_{L^2_T(H^1)}.
\end{aligned}\tag{4.27}$$

Bounding $||(\mathbf{u}+\mathbf{V})\cdot\nabla Q\mathbf{u}||^2_{L^2_T(H^1)}$ as in (3.26), we have

$$\begin{aligned}
&||(\mathbf{u}+\mathbf{V})\cdot\nabla Q\mathbf{u}||^2_{L^2_T(H^1)}\\
&\le ||P\mathbf{u}+Q\mathbf{u}||^2_{L^\infty_T(H^1)}||\nabla Q\mathbf{u}||^2_{L^2_T(H^2)}+\int_0^T(||\nabla\mathbf{V}||^2_{H^1}+||\mathbf{V}||^2_{L^\infty})||\nabla^2 Q\mathbf{u}||^2_{L^2}ds\\
&\quad+\int_0^T||\nabla\mathbf{V}||^2_{L^2}||\nabla Q\mathbf{u}||^2_{H^1}ds\\
&\le ||P\mathbf{u}+Q\mathbf{u}||^2_{L^\infty_T(H^1)}||\nabla Q\mathbf{u}||^2_{L^2_T(H^2)}+\int_0^T||\nabla\mathbf{V}||^2_{H^1}||\nabla^2 Q\mathbf{u}||^2_{L^2}ds\\
&\quad+\frac{1}{2}||\nabla^3 Q\mathbf{u}||^2_{L^2_T(L^2)}+\frac{1}{2}\int_0^T||\mathbf{V}||^4_{L^\infty}||\nabla Q\mathbf{u}||^2_{L^2}ds+\int_0^T||\nabla\mathbf{V}||^2_{L^2}||\nabla Q\mathbf{u}||^2_{H^1}ds.
\end{aligned}$$

With small enough δ' in (2.11) and δ in (2.13) and the assumption (2.14), we combine (4.13), (4.23), (4.25), (4.26), and (4.27) to derive that

$$\begin{aligned}
&||(Q\mathbf{u},\nu a)||^2_{L^\infty_T(H^2)}+\nu||\nabla a||^2_{L^2_T(H^1)}+||Q\mathbf{u}_t+2\nabla a||^2_{L^2_T(H^1)}+||\nu\nabla Q\mathbf{u}||^2_{L^2_T(H^2)}\\
&\lesssim ||(Q\mathbf{u}_0,\nu a_0)||^2_{H^2}+\int_0^T(||(\nabla\mathbf{V},\nabla P\mathbf{u},\nabla Q\mathbf{u})||^2_{H^2}+||\mathbf{V}||^4_{L^\infty})||(Q\mathbf{u},\nu a)||^2_{H^2}ds\\
&\quad+(||P\mathbf{u}||^2_{L^\infty_T(H^1)}+||\mathbf{V}||^2_{L^\infty_T(H^1)})(||\nabla P\mathbf{u}||^2_{L^2_T(H^2)}+||\nabla\mathbf{V}||^2_{L^2_T(H^2)})+||\nabla\mathbf{E}||^2_{L^2_T(H^1)}||\mathbf{E}||^2_{L^\infty_T(H^2)}\\
&\quad+||a||^2_{L^\infty_T(H^2)}(||\mathbf{V}_t||^2_{L^2_T(H^1)}+||P\mathbf{u}_t||^2_{L^2_T(H^1)})+||a||^2_{L^\infty_T(H^2)}||P\mathbf{u}+\mathbf{V}||^2_{L^\infty_T(H^1)}||\nabla Q\mathbf{u}||^2_{L^2_T(H^2)},
\end{aligned}\tag{4.28}$$

where we used $||\mathbf{E}||^2_{L^\infty_T(H^2)}||\nabla a||^2_{L^2_T(H^1)}\le(\delta^{\frac{1}{2}}+\delta')||\nabla a||^2_{L^2_T(H^1)}$.

Recalling the setting of $X(T)$, $Y(T)$, $W(T)$ and $Z(T)$ in (2.9), we have

$$(||P\mathbf{u}||^2_{L^\infty_T(H^1)}+||\mathbf{V}||^2_{L^\infty_T(H^1)})(||\nabla P\mathbf{u}||^2_{L^2_T(H^2)}+||\nabla\mathbf{V}||^2_{L^2_T(H^2)})\lesssim(Z(T)+V(T))(W(T)+V(T)),$$

$$||a||^2_{L^\infty_T(H^2)}(||\mathbf{V}_t||^2_{L^2_T(H^1)}+||P\mathbf{u}_t||^2_{L^2_T(H^1)})\lesssim\nu^{-2}X(T)(V(T)+W(T)),$$

$$||a||^2_{L^\infty_T(H^2)}||P\mathbf{u}+\mathbf{V}||^2_{L^\infty_T(H^1)}||\nabla Q\mathbf{u}||^2_{L^2_T(H^2)}\lesssim\nu^{-4}X(T)(Z(T)+V(T))Y(T),$$

and using (2.24) we get

$$||\nabla\mathbf{E}||^2_{L^2_T(H^1)}||\mathbf{E}||^2_{L^\infty_T(H^2)}\lesssim(\nu^{-1}Y(T)+W(T)+E(T))(Z(T)+E(T)).$$

Therefore, we obtain that

$$
\begin{aligned}
& X(T)+Y(T) \\
& \lesssim X(0)+\int_0^T (\|(\nabla \mathbf{V}, \nabla P\mathbf{u}, \nabla Q\mathbf{u})\|_{H^2}^2+\|\mathbf{V}\|_{L^\infty}^4) X(t) ds+\nu^{-2} X(T)(V(T)+W(T)) \\
& \quad +(Z(T)+V(T))(W(T)+V(T))+\nu^{-4} X(T)(Z(T)+V(T)) Y(T) \\
& \quad +(\nu^{-1} Y(T)+W(T)+E(T))(Z(T)+E(T)).
\end{aligned} \tag{4.29}
$$

Using Grönwall's inequality, we conclude that

$$
\begin{aligned}
& X(T)+Y(T) \\
& \leq C_2 e^{C_2 \int_0^T (\|(\nabla \mathbf{V}, \nabla P\mathbf{u}, \nabla Q\mathbf{u})\|_{H^2}^2+\|\mathbf{V}\|_{L^\infty}^4) ds}[X(0)+\nu^{-2} X(T)(V(T)+W(T)) \\
& \quad +(Z(T)+V(T))(W(T)+V(T))+\nu^{-4} X(T)(Z(T)+V(T)) Y(T) \\
& \quad +(\nu^{-1} Y(T)+W(T)+E(T))(Z(T)+E(T))],
\end{aligned} \tag{4.30}
$$

for some constant $C_2>1$.

5. Global-in-Time Closure of the Estimates and the Proof of Theorem 1.1

By applying $\|Q\mathbf{u}\|_{L^\infty}^4 \leq \|Q\mathbf{u}\|_{L^2}\|\nabla^2 Q\mathbf{u}\|_{L^2}^3$, we can obtain that

$$
\int_0^T \|Q\mathbf{u}\|_{L^\infty}^4 ds \lesssim \|Q\mathbf{u}\|_{L_T^\infty(L^2)}\|\nabla^2 Q\mathbf{u}\|_{L_T^\infty(L^2)}\|\nabla^2 Q\mathbf{u}\|_{L_T^2(L^2)}^2 \lesssim \nu^{-2} D^2,
$$

which can be also applied to $\mathbf{V}$. We suppose that

$$
\nu^{-1} D \leq [4(1+M^2)]^{-1}, \tag{5.1}
$$

and

$$
\delta^{1/2} \leq 1/2, \quad C_2 e^{2C_2(1+M^2)} \delta^{1/2} \leq 1/2, \tag{5.2}
$$

where C_2 is the same as in (4.30). Thus inequality (3.29) implies that

$$
\begin{aligned}
X(T)+Y(T) \leq & C_1 e^{C_1(M+M^2+\nu^{-2} D+\delta^{1/2})}[X(0)+\nu^{-2} X(T)(M+\delta^{1/2})+(M+\delta^{1/2})^2 \\
& +2\nu^{-4} D(M+\delta^{1/2}) X(T)+\nu^{-1} Y(T)(\delta^{1/2}+\delta')+\delta^{1/2}(\delta^{1/2}+\delta')] \\
\leq & C_1 e^{C_1(1+2M^2+\frac{1}{2}\nu^{-1}+\frac{1}{2})}[X(0)+\nu^{-2} X(T)(M+1/2)+M^2+M+1 \\
& +\nu^{-3}(M+1/2) X(T)+\nu^{-1} Y(T)],
\end{aligned} \tag{5.3}
$$

which gives that

$$
X(T)+Y(T) \leq C_1 e^{2C_1(1+M^2)}(X(0)+2M^2+2). \tag{5.4}
$$

Then we choose δ_0 and δ' satisfying

$$
2C_2^3 e^{6C_2(1+M^2)} \delta_0 \leq 1/16, \quad 2C_2^3 e^{6C_2(1+M^2)}(\delta'+\delta'^2) \leq 1/16. \tag{5.5}
$$

From (4.30), (5.2) and (5.5), we obtain that

$$\begin{aligned} Z(T)+W(T) \leq\ & C_2 e^{C_2(M+M^2+\nu^{-2}D+\nu^{-2}D^2+\delta^{1/2})}[\delta_0+2\nu^{-2}D(2D+M+\delta^{1/2})W(T) \\ & +\delta' W(T)+2\nu^{-2}D(M+M^2+2D+2\nu^{-1}D+\delta^{1/2}+\delta'+1) \\ & +Z(T)(2\nu^{-1}D+\delta^{1/2}+\delta')+\delta'(2\nu^{-1}D+\delta')] \\ \leq\ & C_2 e^{C_2(1+2M^2+\frac{1}{2}\nu^{-1}+\frac{1}{4}+\frac{1}{2})}[\delta_0+2\nu^{-2}D(2D+M+1/2)W(T) \\ & +\delta' W(T)+2\nu^{-2}D(1+2M^2+2D+2\nu^{-1}D+2) \\ & +Z(T)(2\nu^{-1}D+\delta^{1/2}+\delta')+\delta'(2\nu^{-1}D+\delta')], \end{aligned} \tag{5.6}$$

which implies that

$$Z(T)+W(T) \leq C_2 e^{2C_2(1+M^2)}[\delta_0+2\nu^{-2}D(2M^2+2D+4)+\delta'(2\nu^{-1}D+\delta')]. \tag{5.7}$$

Next we take ν satisfying

$$8C_1C_2^2 e^{(2C_1+3C_2)(1+M^2)}(X(0)+2M^2+2) \leq \sqrt{\nu}, \tag{5.8}$$

then set

$$D := C_1 e^{2C_1(1+M^2)}(X(0)+2M^2+2). \tag{5.9}$$

Thus we can derive that

$$2C_2^3 e^{6C_2(1+M^2)}D(2M^2+2D+4) \leq \nu^2/16. \tag{5.10}$$

Then we choose

$$\delta := C_2 e^{2C_2(1+M^2)}[\delta_0+2\nu^{-2}D(2M^2+2D+4)+\delta'(2\nu^{-1}D+\delta')]. \tag{5.11}$$

Hence we end up with

$$X(T)+Y(T) \leq D, \tag{5.12}$$

and

$$Z(T)+W(T) \leq \delta. \tag{5.13}$$

From (5.8), (5.9) and (5.11), we see that the assumption (2.13) is recovered. The choices for δ_0 and δ' give that

$$\begin{aligned} \nu^{-1}D &\leq [64C_1C_2^4 e^{(2C_1+6C_2)(1+M^2)}(X(0)+2M^2+2)]^{-1} \leq [128(M^2+1)]^{-1}, \\ \delta &\leq C_2 e^{2C_2(1+M^2)}[\delta_0+2\nu^{-2}D(2M^2+2D+4)+\delta'(1+\delta')] \leq 3/16, \end{aligned}$$

and

$$[C_2 e^{2C_2(1+M^2)}\delta^{1/2}]^2 \leq C_2^3 e^{6C_2(1+M^2)}[\delta_0+2\nu^{-2}D(2M^2+2D+4)+\delta'(1+\delta')] \leq 3/16,$$

from which we see that the assumptions (5.1) and (5.2) are recovered. This also ensures that $||a(T)||_{H^2}$ is small for all $T \in [0, T_*)$, as we have

$$||a(T)||_{H^2}^2 \leq C\nu^{-2}(||\nu a(T)||_{H^2}^2) \leq C\nu^{-2}D. \tag{5.14}$$

If ν and the compressible part of the data fulfill (5.8), then defining D and δ according to (5.9) and (5.11) ensures that (2.12) is fulfilled for all $T < T_*$. Then, combining with the continuation criterion, one can conclude that $T_* = +\infty$ and that (2.12) is satisfied for all time. In fact, if we assume that $T_* < +\infty$, applying (2.12) for all $T < T_*$ yields

$$||a(T)||_{L_T^\infty(H^2)}^2 + ||\mathbf{v}||_{L_T^\infty(H^2)}^2 \leq C < +\infty. \tag{5.15}$$

Then for all $T_0 \in [0, T_*)$, one can solve (1.6) starting with data $(a_0, \mathbf{v}_0, \mathbf{E}_0)$ at time $T = T_0$ and obtain a solution according to Theorem 1.1 on the interval $[T_0, T_1 + T_0]$ with T_1 independent of T_0. Choosing $T_0 > T_* - T_1$ shows that the solution can be continued beyond T_*, which leads to a contradiction. This completes the proof of Theorem 1.1.

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Conflict of Interest Statement

X. Hu and D. Wang are members of the Editorial Board of ANONA, but this did not affect the final decision for the article.

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