SIAM J. MATH. ANAL.
Vol. 55, No. 3, pp. 1545–1595

STABILITY OF FRONT SOLUTIONS OF THE BIDOMAIN ALLEN–CAHN EQUATION ON AN INFINITE STRIP*

HIROSHI MATANO†, YOICHIRO MORI‡, AND MITSUNORI NARA§

Abstract. The bidomain model is the standard model for cardiac electrophysiology. In this paper, we study the bidomain Allen–Cahn equation, in which the Laplacian of the classical Allen–Cahn equation is replaced by the bidomain operator, a Fourier multiplier operator whose symbol is given by a homogeneous rational function of degree two. The bidomain Allen–Cahn equation supports planar front solutions much like the classical case. In contrast to the classical case, however, these fronts are not necessarily stable due to a lack of maximum principle; they can indeed become unstable depending on the parameters of the system. In this paper, we prove nonlinear stability and instability results for bidomain Allen–Cahn fronts on an infinite two-dimensional strip. We show that previously established spectral stability/instability results in L^2 imply stability/instability in the space of bounded uniformly continuous functions by establishing suitable decay estimates of the resolvent kernel of the linearized operator.

Key words. bidomain Allen–Cahn equation, traveling front solution, nonlinear stability, bifurcation of traveling fronts

MSC codes. 35Q92, 35B40, 92C30

DOI. 10.1137/21M1418095

1. Introduction. The bidomain model is the standard mathematical model to describe propagation of electrical signal in cardiac tissue [31, 19, 10, 14]. Cardiac tissue can be seen as consisting of an intracellular region connected via gap junctions together with the extracellular space. In the bidomain model, these two compartments are homogenized as inseparable continua [28, 29]. As a result, quantities in the intracellular compartment and the extracellular compartment and on the cell membrane are defined everywhere in space. Let u_i and u_e be intracellular and extracellular voltages of the cell membrane, respectively. In general, the bidomain model is given of the form

$$C_m \frac{\partial u}{\partial t} - f(u,s) = \operatorname{div}(A_i \nabla u_i) = -\operatorname{div}(A_e \nabla u_e), \quad \text{where} \quad u = u_i - u_e,$$
$$\frac{\partial s}{\partial t} = g(u,s),$$

where the constant $C_m > 0$ is the membrane capacitance, and A_i, A_e are the conductivity tensors, symmetric positive definite matrices that may be functions of position. The function $u = u_i - u_e$ represents the membrane potential, and $s \in \mathbb{R}^n (n \geq 1)$

*Received by the editors May 6, 2021; accepted for publication (in revised form) August 29, 2022; published electronically May 24, 2023.
https://doi.org/10.1137/21M1418095
Funding: The work of the first and third author was supported by JSPS KAKENHI grant 17K18732. The work of the second author was supported by National Science Foundation grant DMS 2042144 and a Math+X grant from the Simons Foundation. The work of the third author was also supported by JSPS KAKENHI grant 19K03556.
†Department of Mathematics, Meiji University, Tokyo 64-8525, Japan (matano@meiji.ac.jp).
‡Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104 USA (y1mori@sas.upenn.edu).
§Department of Mathematics, Iwate University, Iwate 020-8550, Japan (nara@iwate-u.ac.jp).

represent the gating variables, which describe the opening and closing of ion channels on the cell membrane. The first equation is a statement of conservation of electric current applied to the intracellular and extracellular spaces. The leftmost side of the first equation describes the transmembrane current, which consists of the capacitive current term $C_m \partial u/\partial t$ and the ion channel current term $f(u,s)$. The ion channel current $f(u,s)$ depends on the gating variables s which in turn obey the differential equation on the second line. The most important property of this model is that it supports propagating pulse solutions, which correspond to the cardiac electrical signal that coordinates the heart beat, the aberrations of which may cause cardiac arrhythmias [19, 10]. Thus, it is of great scientific interest to understand the properties of the traveling front and pulse solutions of the bidomain model.

There are many computational studies of the bidomain model [15, 6], but there are relatively few analytical studies. The well-posedness of the bidomain model has been studied in [32, 4, 11, 12]. A rigorous study of the homogenization limit can be found in [29], and the paper [13] constructs large amplitude periodic solutions under periodic forcing. We also mention a recent paper on stochastic forcing of the bidomain model [16].

As an initial step toward a qualitative understanding of the full bidomain model above, in [27] the authors studied the bidomain Allen–Cahn equation (see below) in $\mathbb{R}^2$, in which the gating variable dynamics are ignored. Much like the classical Allen–Cahn equation, the bidomain Allen–Cahn equation supports planar front solutions. However, in sharp contrast to the classical Allen–Cahn model, it was found that the planar fronts may become unstable as indicated by the study of the spectrum of the linearized operator around the planar front solution. The main goal of the present paper is to prove nonlinear stability and instability results for the planar front based on the spectral findings in [27].

1.1. Model formulation and well-posedness. We now introduce the the bidomain Allen–Cahn equation. Consider the following problem in $\mathbb{R}^2$ or an infinite strip (to be discussed shortly):

$$\operatorname{div}(A_i \nabla u_i) + \operatorname{div}(A_e \nabla u_e) = 0, \tag{1.1a}$$

$$\frac{\partial u}{\partial t} - f(u) = \operatorname{div}(A_i \nabla u_i), \text{ where } u = u_i - u_e. \tag{1.1b}$$

In the above, A_i and A_e are spatially constant 2×2 symmetric positive definite matrices. In (1.1b), the term $f(u)$ is an unbalanced bistable nonlinearity. To be more precise, we assume the following:

(i) f is smooth.
(ii) f has three zeros $u = 0, \alpha, 1$ with $0 < \alpha < 1$, such that $f'(0) < 0$, $f'(\alpha) > 0$, $f'(1) < 0$.
(iii) $f(s) > 0$ on $(-\infty, 0) \cup (a, 1)$ and $f(s) < 0$ on $(0, \alpha) \cup (1, \infty)$.
(iv) $\int_0^1 f(s)ds > 0$.

The last condition (iv) means that f is unbalanced. In the case of the standard semilinear diffusion equation $u_t = u_{xx} + f(u)$ with such a nonlinearity f, it is well known that the traveling front solution connecting 0 and 1 converges to 1 as $t \to \infty$.

By formally using the Fourier transform, the system (1.1a)–(1.1b) can be reduced to a scalar equation for $u = u_i - u_e$, which we call the *bidomain Allen–Cahn equation*, of the form

$$u_t = -\Lambda u + f(u), \tag{1.2}$$

where Λ, which we call the *bidomain operator*, is a Fourier multiplier operator defined on $\mathbb{R}^2$ as follows:

$$\begin{aligned}&\Lambda u = \mathcal{F}^{-1}Q\mathcal{F}u,\\ &Q(\mathbf{k}) = \frac{Q_i(\mathbf{k})Q_e(\mathbf{k})}{Q_i(\mathbf{k})+Q_e(\mathbf{k})}, \quad Q_{i,e}(\mathbf{k}) = \mathbf{k}^{\mathrm{T}}A_{i,e}\mathbf{k},\end{aligned} \tag{1.3}$$

where $\mathbf{k} = (k,l)^{\mathrm{T}} \in \mathbb{R}^2$. The symbols $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse on $\mathbb{R}^2$; namely, by letting $\mathbf{x} = (x,y)^{\mathrm{T}} \in \mathbb{R}^2$,

$$\hat{u}(\mathbf{k},t) = (\mathcal{F}u)(\mathbf{k},t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\mathbf{k}\cdot\mathbf{x}}u(\mathbf{x},t)\,d\mathbf{x},$$
$$u(\mathbf{x},t) = (\mathcal{F}^{-1}\hat{u})(\mathbf{x},t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{u}(\mathbf{k},t)\,d\mathbf{k}.$$

The Fourier multiplier symbol $Q(\mathbf{k})$ is rational function in $\boldsymbol{k}$ that is positive and homogeneous of degree two ($Q(a\mathbf{k}) = a^2Q(\mathbf{k})$ for $a > 0$). In this sense, the bidomain operator is similar to the (anisotropic) Laplacian, whose symbol is a positive second degree polynomial in $\mathbf{k}$.

As mentioned previously, our goal is to study the planar fronts of the bidomain Allen–Cahn equation. We shall study the stability of planar fronts in an infinite strip (see Figure 1). For this, it is necessary first to develop a solution theory on an unbounded domain in contrast to previous studies in which solutions were constructed on bounded domains [32, 4, 11, 12]. Here, we make use of the fact that, in our setting, the fundamental solution to the linear bidomain equation can be written explicitly using the Fourier transform. We now define the mild solution to the following initial value problem for the bidomain Allen–Cahn equation:

$$u(\mathbf{x},0) = u_0(\mathbf{x}),\ u_t = -\Lambda u + f(u) \quad \text{for } t > 0. \tag{1.4}$$

It will be technically convenient for us to construct our solution in the whole of $\mathbb{R}^2$ rather than in the infinite strip. Our solution is constructed in the space of bounded uniformly continuous functions on $\mathbb{R}^2$, which we denote by $BUC(\mathbb{R}^2)$, endowed with the topology of $L^\infty(\mathbb{R}^2)$:

$$\|u\|_{BUC(\mathbb{R}^2)} = \|u\|_{L^\infty(\mathbb{R}^2)}.$$

The space $BUC(\mathbb{R}^2)$ is a closed subspace of $L^\infty(\mathbb{R}^2)$. The reason we prefer to work in $BUC(\mathbb{R}^2)$ is that the mild solution, to be defined below, will fail to be continuous at $t = 0$ for initial data in $L^\infty(\mathbb{R}^2)$ (see the proof of Lemma 2.4 item (iii)). Define G_t to be the fundamental solution of the linear equation $u_t = -\Lambda u$:

$$G_t(\mathbf{x}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i\mathbf{k}\cdot\mathbf{x}}e^{-tQ(\mathbf{k})}d\mathbf{k}. \tag{1.5}$$

Definition 1.1 (mild solution to the bidomain Allen–Cahn equation). *Consider* (1.4)*, where $u_0 \in BUC(\mathbb{R}^2)$. For $T > 0$, we define $u(\mathbf{x},t) \in C^1((0,T];BUC(\mathbb{R}^2)) \cap C([0,T];BUC(\mathbb{R}^2))$ to be a mild solution of* (1.4) *if*

$$u(\mathbf{x},t) = (G_t * u_0)(t) + \int_0^t G_{t-s} * f(u(\cdot,s))ds, \quad \mathbf{x} \in \mathbb{R}^2, \quad t \in (0,T],$$

where $*$ denotes the convolution in $\mathbb{R}^2$.

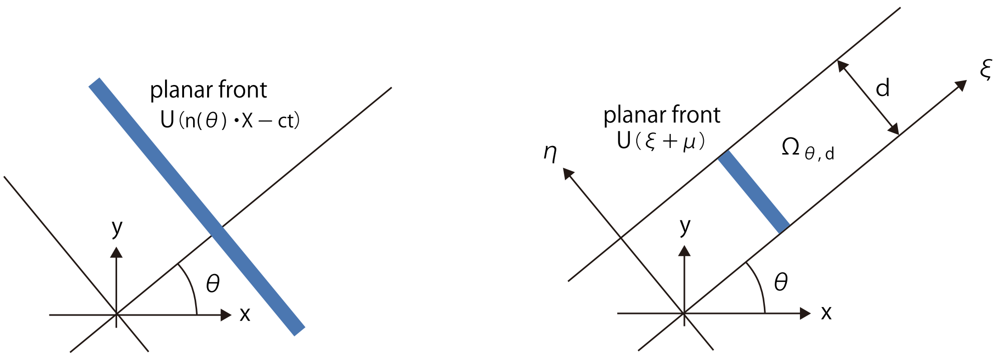

Fig. 1. *Planar fronts on* $\mathbb{R}^2$ *and* $\Omega_{\theta,d}$ *with the* (ξ,η)*-coordinate system, where the* ξ*-axis aligns with the direction of propagation* $\mathbf{n}=(\cos\theta,\sin\theta)$*.*

In Lemma 2.6, we shall establish the existence and uniqueness of mild solutions. In Lemma 2.8, we will see that mild solutions become immediately smooth in $\mathbf{x}\in\mathbb{R}^2$ for $t>0$. The proofs of these results are very similar to the proofs of the corresponding results for the semilinear heat equation in which Λ is replaced by the Laplacian. The fundamental solution G_t shares analytical similarities with the heat kernel. In fact, if there is a constant $\beta>0$ such that $A_i=\beta A_e$, then

$$Q(\mathbf{k})=\frac{\beta}{1+\beta}Q_e(\mathbf{k}). \tag{1.6}$$

In this case, $-\Lambda=\beta(1+\beta)^{-1}\nabla\cdot(A_i\nabla)$, which implies that (1.2), after a suitable affine transformation, reduces to the classical Allen–Cahn equation $u_t=\Delta u+f(u)$. This is known as the *monodomain* reduction. The important point is that the maximum principle does not hold for the bidomain Allen–Cahn equation (1.2) aside from the monodomain case when $A_i=\beta A_e$. That is, the fundamental solution (or kernel) G_t is positive if and only if $A_i=\beta A_e$. This fact is proved in the Appendix A. Thus, results based on the maximum principle for the classical Allen–Cahn equations may not hold in the bidomain case. One such result is the stability of planar fronts, which is the focus of this paper.

The mild solutions constructed in Lemma 2.6 satisfy (1.4) but with the operator Λ identified as the generator of the analytic semigroup defined by G_t acting on $BUC(\mathbb{R}^2)$. Indeed, it is not immediately clear whether the original definition of Λ as a Fourier multiplier operator makes sense even for sufficiently smooth functions in $BUC(\mathbb{R}^2)$, given that such functions do not decay as $|\mathbf{x}|\to\infty$. The semigroup properties of G_t are established in Proposition 2.5. The definition of Λ here is abstract; in Proposition 1.2, we prove that this solution is classical when considered on an infinite strip.

1.2. Planar fronts on an intfinite strip. We now consider the planar front solutions of the bidomain Allen–Cahn equation (1.2). We first make the following observation. If a solution $u(\mathbf{x},t)$ of (1.2) depends on the space variable of only one direction, then its behavior is identical to that of the standard Allen–Cahn equation. More precisely, if u is expressed in the form $u(\mathbf{x},t)=v(\mathbf{n}(\theta)\cdot\mathbf{x},t)$ with some unit vector $\mathbf{n}(\theta)=(\cos\theta,\sin\theta)^{\mathrm{T}}$ and a function $v(\widehat{\xi},t)$ $(\widehat{\xi}\in\mathbb{R},t\geq 0)$, then v satisfies the following one-dimensional reaction diffusion equation:

$$\frac{\partial v}{\partial t}=Q(\mathbf{n}(\theta))\frac{\partial^2 v}{\partial\widehat{\xi}^2}+f(v). \tag{1.7}$$

Thus, as long as we deal with solutions of (1.2) with such symmetry, their behavior is the same as that of the standard Allen–Cahn equation.

It is well known (see [5, 9], for instance) that, under the assumptions on $f(u)$ stated previously, the standard Allen–Cahn equation $\frac{\partial u}{\partial t} = \Delta u + f(u)$ on $\mathbb{R}^2$ has planar front solutions of the form

$$u(\mathbf{x},t) = \Phi(\mathbf{n}(\theta)\cdot\mathbf{x} - c_* t), \quad \mathbf{x} = (x,y)^{\mathrm{T}} \in \mathbb{R}^2,$$

for each direction of propagation $\mathbf{n}(\theta) \in S^1$, where Φ and c_* satisfy

$$\Phi''(\xi) + c_*\Phi'(\xi) + f(\Phi(\xi)) = 0, \tag{1.8}$$

$$\Phi(-\infty) = 1, \quad \Phi(+\infty) = 0, \quad \Phi(0) = \alpha. \tag{1.9}$$

Since f is a bistable nonlinearity, the speed c_* is uniquely determined, and the profile Φ is also unique under the condition $\Phi(0) = \alpha$ in (1.9).

Similarly, by substituting $u(\mathbf{x},t) = U(\mathbf{n}(\theta)\cdot\mathbf{x} - ct)$ into the bidomain Allen–Cahn equation (1.2), we obtain

$$Q(\mathbf{n}(\theta))U''(\xi) + c\,U'(\xi) + f(U(\xi)) = 0, \tag{1.10}$$

$$U(-\infty) = 1, \quad U(+\infty) = 0, \quad U(0) = \alpha. \tag{1.11}$$

Thus, we find that, for each $\mathbf{n}(\theta) \in S^1$, (1.2) on $\mathbb{R}^2$ has the planar front solutions of the form

$$u(\mathbf{x},t) = U(\mathbf{n}(\theta)\cdot\mathbf{x} - ct), \quad \mathbf{x} = (x,y)^{\mathrm{T}} \in \mathbb{R}^2, \tag{1.12}$$

where U and c are given by

$$U(\xi) = \Phi(\xi/K_\theta), \quad c = c_* K_\theta, \quad K_\theta = \sqrt{Q(\mathbf{n}(\theta))}. \tag{1.13}$$

We note that any translate $u(\mathbf{x},t) = U(\mathbf{n}(\theta)\cdot\mathbf{x} - ct + \xi_0), \xi_0 \in \mathbb{R}$ is also a planar front of the bidomain equations. We also note that the profile and the speed of the planar fronts depend on the value of K_θ. Such anisotropy may play a key role in the stability and instability of front solutions.

The objective of the present paper is to study the stability of the planar fronts given by (1.12)–(1.13) in the bidomain Allen–Cahn equation (1.2) on an infinite strip in $\mathbb{R}^2$ given by

$$\Omega_{\theta,d} = \left\{\mathbf{x} \in \mathbb{R}^2 \mid 0 \le \mathbf{x}\cdot(-\sin\theta,\cos\theta)^{\mathrm{T}} \le d\right\}.$$

This domain represents an infinite strip of width d that stretches in the direction $\mathbf{n} = (\cos\theta,\sin\theta)^{\mathrm{T}}$; see Figure 1. We consider the initial value problem of the form

$$u_t = -\Lambda u + f(u), \quad \mathbf{x} \in \Omega_{\theta,d},\ t > 0, \tag{1.14a}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega_{\theta,d}, \tag{1.14b}$$

under periodic boundary conditions. Here, by periodic boundary conditions, we mean that a function $w(\mathbf{x})$ defined on $\Omega_{\theta,d}$ can be extended over $\mathbb{R}^2$ that satisfies

$$w(\mathbf{x} + d(-\sin\theta,\cos\theta)^{\mathrm{T}}) = w(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2. \tag{1.15}$$

It is clear that if a function $u_0(\mathbf{x})$ on $\mathbb{R}^2$ satisfies the periodicity condition (1.15), then the solution $u(\mathbf{x},t)$ of (1.2) on $\mathbb{R}^2$ with the initial value u_0 satisfies (1.15) for all $t \ge 0$

so long as the solution is unique. This is because (1.2) is equivariant with respect to spatial translation (1.15). This means that the well-posedness and regularity of the solution for the problem (1.14) follow immediately from those for the problem on $\mathbb{R}^2$.

The main reason we consider the problem in an infinite strip rather than in $\mathbb{R}^2$ is that this will greatly facilitate the study of planar front stability, as we will discuss shortly. An added benefit of working in the infinite strip is that we can claim that mild solutions, defined in Definition 1.1, are classical in the sense to be specified below. Let $BUC(\Omega_{\theta,d})$ be the set of functions in $BUC(\mathbb{R}^2)$ which are periodic in the sense of (1.15).

PROPOSITION 1.2. *Let $u_0 \in BUC(\Omega_{\theta,d})$. The initial value problem* (1.14) *has a unique mild solution as defined in Definition* 1.1. *Furthermore, this solution is classical in the following sense. For $t>0$, there are smooth bounded functions $u_i(\mathbf{x},t)$ and $u_e(\mathbf{x},t)$ satisfying $u_i - u_e = u(\mathbf{x},t)$ such that* (1.1a) *and* (1.1b) *are satisfied. The functions u_i and u_e are uniquely determined up to an additive constant.*

The proof of this assertion uses a Fourier series decomposition of u with respect to the direction parallel to the planar front (η-direction defined below; see Figure 1) and thus depends on the fact that the solution is defined on an infinite strip and not the whole of $\mathbb{R}^2$. For general mild solutions defined on $\mathbb{R}^2$, there are fundamental difficulties associated with solving the second order elliptic equation (1.1a) satisfied by u_i and u_e.

1.3. Spectral stability of planar fronts. Let us now turn to the problem of the stability of planar fronts. We first introduce a coordinate system $(\widehat{\xi},\eta)$ where the $\widehat{\xi}$-axis coincides with the direction of propagation $\mathbf{n}$ and the η–axis is parallel to the planar front. In this coordinate system, (1.14) can be written as

$$\frac{\partial u}{\partial t} = -\Lambda_\theta u + f(u), \quad (\widehat{\xi},\eta) \in \mathbb{R}\times S^1_d, \quad t>0, \tag{1.16a}$$

$$u(\widehat{\xi},\eta,0) = u_0(\widehat{\xi},\eta), \quad (\widehat{\xi},\eta)\in\mathbb{R}\times S^1_d, \tag{1.16b}$$

where Λ_θ denotes the transformed operator $\Lambda_\theta u = \mathcal{F}^{-1}Q_\theta\mathcal{F}u$. Here, $\mathcal{F}$ is the Fourier transform in $(\widehat{\xi},\eta)\in\mathbb{R}^2$ and the Fourier multiplier symbol $Q_\theta(\mathbf{k})$ is given by

$$Q_\theta(\mathbf{k}) = \frac{Q_i^\theta(\mathbf{k})Q_e^\theta(\mathbf{k})}{Q_i^\theta(\mathbf{k})+Q_e^\theta(\mathbf{k})}, \quad Q_{i,e}^\theta(k,l) = \mathbf{k}^{\mathrm{T}}A_{i,e}^\theta\mathbf{k}, \tag{1.17}$$

$$A_{i,e}^\theta = R_\theta A_{i,e}R_{-\theta}, \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{1.18}$$

More precisely, Λ_θ should be seen as the generator of the analytic semigroup G_t (with Q replaced by Q_θ in (1.5)) acting on $BUC(\mathbb{R}^2)$ (or $BUC(\Omega_{\theta,d})$); see Lemma 2.6 and the discussion between (1.6) and (1.7). The traveling front solutions in this coordinate system are given by $U(\widehat{\xi}-ct+\xi_0)$ for any constant $\xi_0\in\mathbb{R}$. In order to facilitate our analysis of planar fronts, we will mostly make use of the moving coordinate system $(\xi,\eta)=(\widehat{\xi}-ct,\eta)$. Then (1.16a) in this new coordinate system is given by

$$\frac{\partial u}{\partial t} = -\Lambda_\theta u + c\frac{\partial u}{\partial \xi} + f(u). \tag{1.19}$$

In this moving coordinate system, the traveling front solutions $U(\xi+\xi_0), \xi_0\in\mathbb{R}$ are stationary solutions.

In order to study the stability of the planar fronts, consider the linearized equation around $U = U(\xi)$:

$$\frac{\partial v}{\partial t} = \mathcal{L}v, \quad \mathcal{L}v = -\Lambda_\theta v + c\frac{\partial v}{\partial \xi} + f'(U)v. \tag{1.20}$$

In [27], the authors studied the spectrum of $\mathcal{L}$ as a closed operator on $L^2(\mathbb{R}^2)$. First, take the Fourier transform in η. We have, for each $l \in \mathbb{R}$,

$$\begin{aligned} \frac{\partial v_l}{\partial t} &= \mathcal{L}_l v_l, \quad \mathcal{L}_l v_l = -\Lambda_l v_l + c\frac{\partial v_l}{\partial \xi} + f'(U)v_l, \\ \Lambda_l &= \mathcal{F}_\xi^{-1} Q_\theta(k,l)\mathcal{F}_\xi, \quad v_l(\xi,t) = \mathcal{F}_\eta v(\xi,\eta,t), \end{aligned} \tag{1.21}$$

where $\mathcal{F}_\xi$ and $\mathcal{F}_\eta$ are the Fourier transform in ξ and η, respectively. The operator $\mathcal{L}_l$ governs the growth of perturbations with wave number l in the η-direction. For each $l \in \mathbb{R}$, $\mathcal{L}_l$ is a closed operator on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. Let $\sigma_{L^2(\mathbb{R}^2)}(\mathcal{L})$ and $\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l)$ be the spectra of $\mathcal{L}$ and $\mathcal{L}_l$, respectively, as operators $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R})$. By Proposition 2.2 in [27], we have

$$\sigma_{L^2(\mathbb{R}^2)}(\mathcal{L}) = \bigcup_{l \in \mathbb{R}} \sigma_{L^2(\mathbb{R})}(\mathcal{L}_l).$$

The study of the spectrum of $\mathcal{L}$ thus reduces to the study of $\mathcal{L}_l$. Note that, in the case of the infinite strip, the relevant spectra will be those at $l = 2\pi k/d, k \in \mathbb{Z}$.

It is important to note that the operator $\mathcal{L}_0$ governs the growth of the solutions of (1.20) under perturbations that are independent of η. In other words, this operator coincides with the linearization of the classical Allen–Cahn equation:

$$\frac{\partial v}{\partial t} = \mathcal{L}_0 v, \quad \mathcal{L}_0 v := K_\theta^2 \frac{\partial^2 v}{\partial \xi^2} + c\frac{\partial v}{\partial \xi} + f'(U)v, \tag{1.22}$$

where K_θ is the constant that appears in (1.13). The spectrum of $\mathcal{L}_0$ on $L^2(\mathbb{R})$ is thus well known; namely, $\sigma_{L^2(\mathbb{R})}(\mathcal{L}_0)$ contains 0 as a simple eigenvalue, which comes from the translation equivariance of (1.13) in the direction ξ, and

$$\sigma(\mathcal{L}_0) \backslash \{0\} \subset \{z \in \mathbb{C} \mid \mathrm{Re} z \leq -\delta\} \tag{1.23}$$

for some constant $\delta > 0$ that depends only on f.

We now define the spectral stability of the planar front in $L^2(\mathbb{R}^2)$.

Definition 1.3 (spectral stability of planar fronts in $L^2(\mathbb{R}^2)$ [27]). *Let U and c be defined as in* (1.12)–(1.13), *and let $\mathcal{L}_l$ be the operator defined by* (1.21). *The planar front $U(\xi)$ in* (1.19) *on $\mathbb{R}^2$ is* spectrally stable *if*

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l) \subset \{z \in \mathbb{C} \mid \mathrm{Re}\, z < 0\} \quad \text{for all} \quad l \neq 0.$$

It is spectrally unstable *if there exists a value of l such that*

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l) \cap \{z \in \mathbb{C} \mid \mathrm{Re}\, z > 0\} \neq \emptyset.$$

Remark 1.4. We exclude the case $l = 0$ in the above definition since $\mathcal{L}_0$ does not play a role in the stability properties of the planar front. This is because $\mathcal{L}_0$ is concerned with perturbations that are independent of η, under which (1.2) reduces to the standard Allen–Cahn equation (1.7); hence the planar front is always stable (with asymptotic phase) under such perturbations. As mentioned above, the eigenvalue $0 \in \sigma_{L^2(\mathbb{R})}(\mathcal{L}_0)$ comes from the translation equivariance of (1.2) in the direction ξ, and its corresponding eigenfunction $U'(\xi)$ represents a phase shift of the traveling front.

We now summarize, without proof, the main results of [27] on the spectral stability of the planar fronts.

Proposition 1.5 (see [2, Proposition 2.3]).
One has

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l) \subset \left\{ z \in \mathbb{C} \mid \operatorname{Re} z \le f_{\max} - m_Q l^2 \right\},$$

where $f_{\max}$ and m_Q are positive constants defined by

$$f_{\max} = \max_{0 \le s \le 1} f'(s), \quad m_Q = \min_{s \in \mathbb{R}} Q_\theta(s, 1).$$

The above proposition shows that the wave is stable to short wave-length perturbations ($|l|$ large). The value of $f_{\max}$, however, is positive, and the above does not rule out the possibility that the planar front may be unstable to perturbations of longer wave-lengths. Planar fronts of the bidomain Allen–Cahn equation can indeed become spectrally unstable. To state the instability results, we introduce the notion of the Frank diagram.

Definition 1.6 (Frank diagram).
The Frank diagram $F \subset \mathbb{R}^2$ is the region enclosed by the Frank plot defined by

$$\partial F = \left\{ (\cos\theta, \sin\theta)^{\mathrm{T}} / K_\theta,\ 0 \le \theta \le 2\pi \right\},$$

which is equivalent to $\partial F = \{\mathbf{k} = (k, l)^{\mathrm{T}} \in \mathbb{R}^2 \mid Q(\mathbf{k}) = 1\}$.

In Figure 2, we plot the Frank diagram for the following choices of A_i and A_e:

$$A_i = \begin{pmatrix} 1+a & 0 \\ 0 & 1-a \end{pmatrix},\ A_e = \begin{pmatrix} 1-a & 0 \\ 0 & 1+a \end{pmatrix},\ |a| < 1. \tag{1.24}$$

The importance of the Frank diagram in the study of the bidomain model has been recognized in many studies [1, 2, 3]. In particular, [10] argues that the loss of convexity of the Frank diagram may play an important role in the electrophysiology of the heart after myocardial infarction.

We point out that the notion of the Frank diagram plays a central role in anisotropic growth models, which is important, for example, in crystal growth problems [20]. In this context, the most closely related to the bidomain Allen–Cahn equation may be the anisotropic Allen–Cahn equations studied, for example, in [7, 8, 23].

We now state the instability results relating the Frank diagram to the spectral instability of fronts.

Proposition 1.7 (see [27, Theorem 4.2 and Corollary 4.3]). *There is a $\delta > 0$ such that for $|l| < \delta$ there is an eigenvalue λ_l of $\mathcal{L}_l$ satisfying the following properties:*

1. *The eigenvalue λ_l is simple and is the principal eigenvalue of $\mathcal{L}_l$ in the following sense: there is a positive constant ν_δ independent of l such that*

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l) \backslash \{\lambda_l\} \subset \{z \in \mathbb{C} | \operatorname{Re} z < -\nu_\delta\}.$$

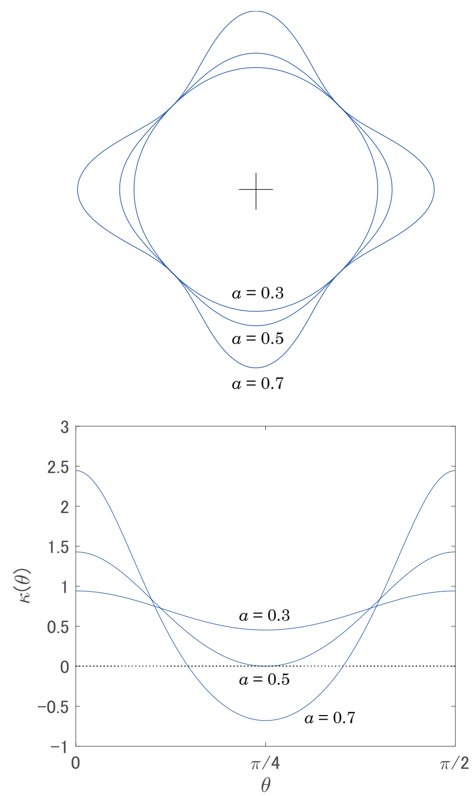

FIG. 2. *(Top) The Frank diagrams for three different parameter values when A_i and A_e are as in* (1.24)*. Here $a = 0.7$, 0.5, and 0.3. The value $a = 0.5$ is the threshold below which the Frank diagram is convex. The directions for which the Frank diagram is locally concave correspond to the directions where $\kappa(\theta) < 0$. (Bottom) Plot of the curvature $\kappa(\theta)$ of the Frank diagrams given in the top image.*

2. *The eigenvalue λ_l is a C^2 function of l and has the following expansion near $l = 0$:*

$$\lambda_l = i\alpha_1 cl - \alpha_0 l^2 + \mathcal{O}(l^3), \tag{1.25}$$

where c is the speed of planar front (see (1.13)*), and α_1, α_0 depend only on A_i, A_e, and θ.*

3. *Let $\kappa(\theta)$ be the curvature of the Frank plot ∂F at the point $(\cos\theta, \sin\theta)/K_\theta$. Then α_0 in* (1.25) *can be written as*

$$\alpha_0 = K_\theta (1 + \alpha_1^2)^{3/2} \kappa(\theta). \tag{1.26}$$

In particular, the planar front propagating in the direction $\mathbf{n}(\theta) = (\cos\theta, \sin\theta)$ is spectrally unstable in the sense of Definition 1.3 *if $\kappa(\theta) < 0$.*

As can be seen from Figure 2, for different choices of a and b in (1.24), $\kappa(\theta)$ can indeed be negative for a range of values of θ. For instance, if $b = 0$ and $|a| > 1/2$ in (1.24), $\kappa(\theta) < 0$ if θ satisfies the following condition:

$$|\cos(2\theta)| < \frac{1}{|a|}\sqrt{1 - \frac{2}{\sqrt{3}}\sqrt{1 - a^2}}. \tag{1.27}$$

In [27], a more detailed description, especially the explicit expressions for α_0, α_1 in terms of a, b, θ are given for the case (1.24).

We make some further remarks on the results of [27]:

(i) From Proposition 1.5, we find that, for every direction $\mathbf{n}$, the planar front is spectrally stable under short wave-length (i.e., $|l|$ is sufficiently large) perturbations.

(ii) From Proposition 1.7, we find that, for every direction $\mathbf{n}$, spectral stability under long wave-length (i.e., $|l|$ is sufficiently small) perturbations is determined by the convexity of the Frank diagram F in the direction $\mathbf{n}$. In particular, stability to long wave-length perturbations do not depend on the specific form of the bistable nonlinearity f.

(iii) Spectral stability for intermediate wave-length perturbations ($|l|$ is neither sufficiently large or sufficiently small) is largely unknown. However, in certain specific examples of f, the spectral stability can be studied in greater detail. There are choices of f for which planar fronts are spectrally unstable in every direction of propagation. See section 5 of [27] for details.

Note that, in (1.25), $\lambda_l \to 0$ as $|l| \to 0$. This implies that even if we exclude the translational mode $\lambda_0 = 0$, the spectral set $\sigma_{L^2(\mathbb{R}^2)}(\mathcal{L})$ comes arbitrarily close to the origin. This lack of spectral gap presents considerable difficulty in studying nonlinear stability of the front. Working in the infinite strip $\Omega_{\theta,d}$, the values of l are restricted to $l = 2\pi k/d, k \in \mathbb{Z}$, and we have a spectral gap. Even in the classical Allen–Cahn case, the stability of planar fronts on the whole of $\mathbb{R}^2$ is subtle and relies heavily on the maximum principle. Certain ergodicity conditions must be placed on the perturbations to ensure convergence to the planar front, and even when convergence can be proved, the rate is not necessarily exponential because of the lack of a spectral gap [17, 21, 25, 26, 33].

1.4. Nonlinear stability of planar fronts. We are now ready to state the definitions of nonlinear stability and instability of planar fronts on the infinite strip $\Omega_{\theta,d}$. Recall that $U(\widehat{\xi} - ct)$ (or $U(\xi)$) is the planar front solution to (1.16).

DEFINITION 1.8 (nonlinear stability of planar fronts on $\Omega_{\theta,d}$). *Let U and c be defined as in* (1.12)–(1.13). *The planar front U is* stable *if, for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that, for any solution u of* (1.16) *satisfying $\|u_0(\widehat{\xi}, \eta) - U(\widehat{\xi})\|_{BUC(\mathbb{R}\times S^1_d)} \le \delta$, it holds that*

$$\sup_{(\widehat{\xi},\eta)\in\mathbb{R}\times S^1_d} \left| u(\widehat{\xi}, \eta, t) - U(\widehat{\xi} - ct) \right| \le \varepsilon \quad \text{for all} \quad t > 0.$$

It is called stable with asymptotic phase *if it is stable and if there exists a constant $\delta_* > 0$ such that, for any solution u of* (1.16) *satisfying $\|u_0(\widehat{\xi}, \eta) - U(\widehat{\xi})\|_{BUC(\mathbb{R}\times S^1_d)} \le \delta_*$, it holds that*

$$\lim_{t\to\infty} \sup_{(\widehat{\xi},\eta)\in\mathbb{R}\times S^1_d} \left| u(\widehat{\xi}, \eta, t) - U(\widehat{\xi} - ct + \xi_0) \right| = 0 \tag{1.28}$$

for some constant $\xi_0 \in \mathbb{R}$. We say that it is exponentially stable with asymptotic phase *if the convergence* (1.28) *takes place exponentially:*

$$\sup_{(\widehat{\xi},\eta)\in\mathbb{R}\times S^1_d} \left| u(\widehat{\xi}, \eta, t) - U(\widehat{\xi} - ct + \xi_0) \right| \le C e^{-\nu t} \quad \text{for all} \quad t > 0$$

for some constants $\xi_0 \in \mathbb{R}$, $C > 0$, and $\nu > 0$.

DEFINITION 1.9 (nonlinear instability of planar fronts on $\Omega_{\theta,d}$).

Let U and c be defined as in (1.12)–(1.13). *The planar front U is* unstable *if it is not stable, namely if there exists a constant $\varepsilon_* > 0$ such that, for any $\delta > 0$, there exists a solution of* (1.16) *satisfying $\|u_0(\widehat{\xi},\eta) - U(\widehat{\xi})\|_{BUC(\mathbb{R}\times S^1_d)} \le \delta$ and*

$$\sup_{(\widehat{\xi},\eta)\in\mathbb{R}\times S^1_d} \left| u(\widehat{\xi},\eta,T) - U(\widehat{\xi}-ct)\right| \ge \varepsilon_*$$

for some $T > 0$. We say that U is orbitally unstable *if there exists a constant $\varepsilon_* > 0$ such that, for any $\delta > 0$, there exists a solution of* (1.16) *satisfying $\|u_0(\widehat{\xi},\eta) - U(\widehat{\xi})\|_{BUC(\mathbb{R}\times S^1_d)} \le \delta$ and*

$$\inf_{\xi'\in\mathbb{R}} \sup_{(\xi,\eta)\in\mathbb{R}\times S^1_d} \left| u(\widehat{\xi},\eta,T) - U(\widehat{\xi}-ct+\xi')\right| \ge \varepsilon_*$$

for some $T > 0$.

Note that the above definitions of nonlinear stability and instability are in the $BUC(\Omega_{\theta,d})$ topology and we thus allow perturbations that do not decay to 0 as ξ tends to infinity. In contrast, all spectral results quoted above were in $L^2(\mathbb{R}^2)$. In section 3, we first prove that the linearized operator $\mathcal{L}$ generates an analytic semigroup on BUC (Proposition 3.1), which follows from our earlier result on semigroup generation by the bidomain operator Λ (Proposition 2.5). To understand the decay and growth properties of the linear semigroup generated by $\mathcal{L}$, it is thus sufficient to study the spectrum of $\mathcal{L}$ in $BUC(\Omega_{\theta,d})$. The rest of section 3 is devoted to relating the spectral results in $L^2(\mathbb{R}^2)$ with the spectral properties of $\mathcal{L}$ acting on $BUC(\mathbb{R}^2)$ resolvent (Proposition 3.2). This hinges on a careful study of the resolvent kernel of $\mathcal{L}_l$ and its dependence on l. By proving that the resolvent kernel of $\mathcal{L}_l$ decays sufficiently fast as $|\xi| \to \infty$, we show that any point in the resolvent set of $\mathcal{L}_l$, considered as an operator in $L^2(\mathbb{R})$, is also in the resolvent set of $BUC(\mathbb{R})$. A further study on the dependence of the size of the resolvent kernel with respect to l allows us to obtain Proposition 3.2, stating that the spectrum of $\mathcal{L}$ as an operator on $BUC(\Omega_\theta, d)$ is contained in the $L^2(\mathbb{R})$ spectra of $\mathcal{L}_l, l = 2\pi\mathbb{Z}$. In section 3.2, the foregoing results on the resolvent set are combined with observations on the Fredholm properties of the operator $\mathcal{L}_l$ to prove that the nonnegative real parts of the BUC and L^2 spectra of $\mathcal{L}$ are identical. For details of the standard theory of analytic semigroup we applied, see [22], for instance.

In section 4, we prove our main results concerning the relationship between spectral stability (instability) and nonlinear stability (instability). Statement (i) in Theorem 1.10 is concerned with the case that the planar front is spectrally stable in the direction of the strip, while statement (ii) is concerned with the case that it is spectrally unstable in the direction of the strip.

THEOREM 1.10 (spectral stability and nonlinear stability).

The following hold:

(i) ***nonlinear stability:*** *if the planar front on $\mathbb{R}^2$ is spectrally stable in direction $\mathbf{n}(\theta) \in S^1$ in the sense of Definition* 1.3*, then, for any $d > 0$, the planar front on $\Omega_{\theta,d}$ is exponentially stable with asymptotic phase in the sense of Definition* 1.8.

(ii) ***nonlinear instability:*** *if the planar front on $\mathbb{R}^2$ is spectrally unstable in direction $\mathbf{n} \in S^1$ in the sense of Definition* 1.3 *and if $d > 0$ is sufficiently large, then the planar front on $\Omega_{\theta,d}$ is orbitally unstable in the sense of Definition* 1.9.

Our second result below states that the planar fronts on a sufficiently narrow strip are stable regardless of the direction of the strip. We note that, only short wave-length perturbations can exist in narrow strips due to the periodic boundary conditions.

THEOREM 1.11 (nonlinear stability on narrow strips). *If $d > 0$ is sufficiently small, then for every $\mathbf{n}(\theta) \in S^1$ the planar front on $\Omega_{\theta,d}$ is exponentially stable with asymptotic phase in the sense of Definition* 1.8.

To prove Theorems 1.10 and 1.11, in section 4.1, we first decompose the solution near a traveling front profile into a component that aligns with the traveling front and the component that is transverse to the family of front solutions. We then estimate the growth and decay of the component transverse to the front solutions by standard arguments using the variation of constants formula (see, for example, [22, 18]). For this, the decay and growth properties of the analytic semigroup generated by $\mathcal{L}$, implied by the spectral results in section 3, plays a crucial role.

Finally, in Appendix A, we prove that the bidomain operator does not satisfy the maximum principle unless A_i and A_e are proportional to each other.

1.5. Bifurcation of planar fronts. In concluding this introduction, let us briefly describe what happens to the front after it has been destabilized.

According to Proposition 1.7 mentioned above, the principal eigenvalue λ_l of $\mathcal{L}_l$ can become positive for sufficiently small $|l|$ when the planar front is propagating in a direction θ for which the curvature $\kappa(\theta)$ of the Frank plot ∂F is negative. On the other hand, when $|l|$ is sufficiently large, the spectrum of $\mathcal{L}_l$ is in the left half of the complex plane, as stated in Proposition 1.5. This indicates that, in such directions, a planar front can be destabilized as the width d of the strip becomes larger. This and other properties of the bidomain Allen–Cahn and FitzHugh–Nagumo systems were explored computationally in a recent paper [24]. Simulations indicate that, when a planar front is destabilized, the solution typically develops a front of saw-toothed profile; see Figure 3. Such front solutions rotate periodically along the direction η (right-bottom of Figure 3), except for special symmetric cases (for example, $\theta = \pi/4$ and $b = 0$ in (1.24)), in which the saw-tooth fronts do not rotate (right-top of Figure 3). A rigorous mathematical analysis of these and related phenomena will be a subject of future study.

2. Existence and regularity of the solution. The aim of this section is to show the existence and basic properties of the solution of the bidomain Allen–Cahn equation (1.2) on $\mathbb{R}^2$ and $\Omega_{\theta,d}$. The bidomain operator Λ introduced in the previous section is a self-adjoint operator on $L^2(\mathbb{R}^2)$. In this section, we shall first show that $-\Lambda$ generates an analytic semigroup on $BUC(\mathbb{R}^2)$ in a certain appropriate sense, where the explicit expression for the fundamental solutions of the linear bidomain equation plays an important role. Then we shall discuss the existence and uniqueness of the initial value problem of the bidomain Allen–Cahn equation on $\mathbb{R}^2$, where by the solution we mean mild solution; see Lemma 2.6. Finally, we discuss the problem on the infinite strip $\Omega_{\theta,d}$. Throughout this section, we let $\mathbf{x} = (x, y)^{\mathrm{T}} \in \mathbb{R}^2$, $\mathbf{x}' = (x', y')^{\mathrm{T}} \in \mathbb{R}^2$, and $\mathbf{k} = (k, l)^{\mathrm{T}} \in \mathbb{R}^2$.

2.1. Linear bidomain equation on $\mathbb{R}^2$. Let $BUC^k(\mathbb{R}^2)$ be the space of functions on $\mathbb{R}^2$ all of whose partial derivatives of order k and below are bounded and uniformly continuous with norm

$$\|u\|_{BUC^k(\mathbb{R}^2)} = \sum_{0 \le |\alpha| \le k} \|\partial_{\mathbf{x}}^{\alpha} u\|_{L^\infty(\mathbb{R}^2)}, \tag{2.1}$$

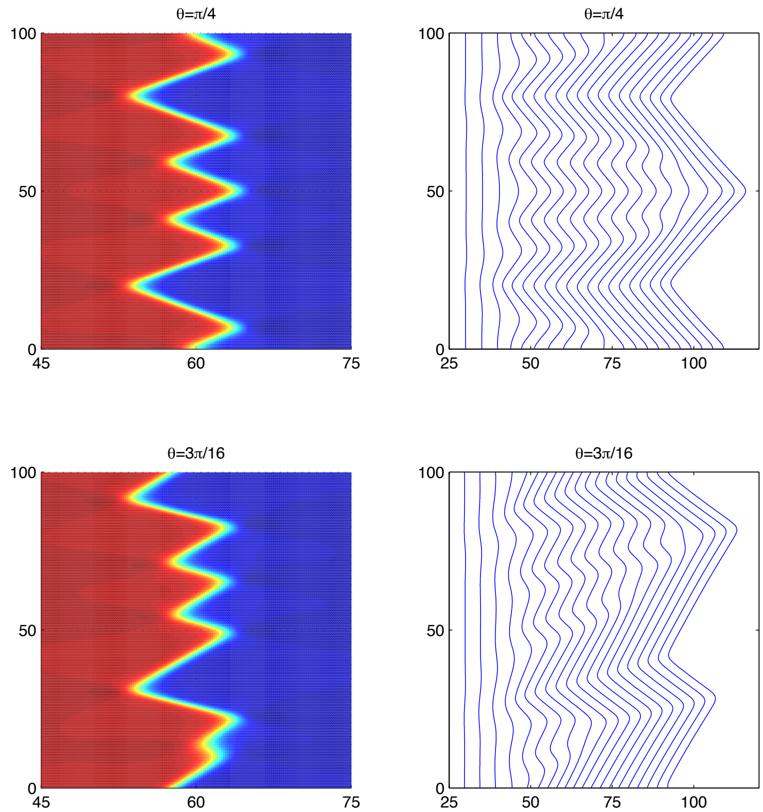

Fig. 3. *An unstable front for the bidomain Allen–Cahn model. Here* $a = 0.9$ *and* $b = 0$ *in* (1.24) *and the nonlinear bistable function* $f(u) = -u(u-0.4)(u-1)$*. Planar fronts propagating in the directions of* $\theta = \pi/4$ *and* $\theta = 3\pi/16$ *are shown. Left images are well-developed destabilizing fronts at a large time. Right images are the time sequences of the front location. When a planar front is destabilized, the solution typically develops a front of saw-toothed profile. For numerical procedure, see* [27, 24].

where $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i = 0, 1, 2, \dots$ is a multi-index and $|\alpha| = \alpha_1 + \alpha_2$. It is easily checked that $BUC^k(\mathbb{R}^2)$ is a Banach space under the above norm. Note that $BUC^0(\mathbb{R}^2) = BUC(\mathbb{R}^2)$. To study the analytic semigroup on $BUC^k(\mathbb{R}^2)$ generated by $-\Lambda$, we first consider basic properties of the solution to the Cauchy problem of the linear bidomain equation on $\mathbb{R}^2$. Namely, we consider the problem of the form

$$u_t = -\Lambda u, \quad \mathbf{x} \in \mathbb{R}^2, \quad t > 0, \tag{2.2a}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{2.2b}$$

where $u_0 \in BUC^k(\mathbb{R}^2)$.

Suppose u_0 is sufficiently smooth and decays sufficiently fast as $|\mathbf{x}| \to \infty$. We may then take the Fourier transform in $\mathbf{x} \in \mathbb{R}^2$ in the above equations to obtain

$$\hat{u}_t = -Q(\mathbf{k})\hat{u}, \quad \mathbf{k} \in \mathbb{R}^2, \quad t > 0, \tag{2.3a}$$

$$\hat{u}(\mathbf{k}, 0) = \hat{u}_0(\mathbf{k}), \quad \mathbf{k} \in \mathbb{R}^2. \tag{2.3b}$$

The solution $\hat{u}(\mathbf{k},t)$ of (2.3) is given by $\hat{u}(\mathbf{k},t) = e^{-tQ(\mathbf{k})}\hat{u}_0(\mathbf{k})$, and hence the solution $u(\mathbf{x},t)$ of (2.2) is expressed by using the inverse Fourier transform as

$$u(\mathbf{x},t) = \frac{1}{2\pi}\left(\mathcal{F}^{-1}e^{-tQ(\mathbf{k})}\right) * u_0(\mathbf{x}) = (G_t * u_0)(\mathbf{x}),$$

where G_t is given by

$$G_t(\mathbf{x}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i\mathbf{k}\cdot\mathbf{x}}e^{-tQ(\mathbf{k})}d\mathbf{k}.$$

Since $Q(\mathbf{k})$ is homogeneous of degree two, by setting $\mathbf{K} = \sqrt{t}\mathbf{k}$, it is rewritten as

$$G_t(\mathbf{x}) = \frac{1}{(2\pi)^2 t}\int_{\mathbb{R}^2} \exp\left(\frac{i\mathbf{K}\cdot\mathbf{x}}{\sqrt{t}}\right) e^{-Q(\mathbf{K})}d\mathbf{K} = \frac{1}{t}G_1\left(\frac{\mathbf{x}}{\sqrt{t}}\right). \tag{2.4}$$

Observe that the expression $G_t * u_0$ will make sense even when $u_0 \in BUC^k(\mathbb{R}^2)$ so long as G_t is in $L^1(\mathbb{R}^2)$. In what follows, we derive estimates for G_1 and G_t and establish some basic properties of the expression $(G_t * u_0)(\mathbf{x})$.

Before we state and prove our results, let us make some simple observations about the multiplier $Q(\mathbf{k})$. By the positive definiteness of the matrices A_i and A_e (see (1.3)), it is easily seen that there are positive constants $\lambda_{\min}$ and $\lambda_{\max}$ satisfying:

$$\lambda_{\min}|\mathbf{k}|^2 \le Q(\mathbf{k}) \le \lambda_{\max}|\mathbf{k}|^2. \tag{2.5}$$

Moreover, for any multi-index α, the function $\partial_{\mathbf{k}}^{\alpha}Q(\mathbf{k})$ is homogeneous of degree $2-|\alpha|$ and satisfies

$$|\partial_{\mathbf{k}}^{\alpha}Q(\mathbf{k})| \le C_\alpha|\mathbf{k}|^{2-|\alpha|} \tag{2.6}$$

for a positive constant C_α.

LEMMA 2.1. *For any multi-index α, one has $\partial_{\mathbf{x}}^{\alpha}G_1 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$.*

Proof. $\partial_{\mathbf{x}}^{\alpha}G_1 \in L^\infty(\mathbb{R}^2)$ follows easily from (2.5). Indeed, we have

$$|\partial_{\mathbf{x}}^{\alpha}G_1(\mathbf{x})| \le \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}|\mathbf{k}|^\alpha e^{-Q(\mathbf{k})}d\mathbf{k} \le \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}|\mathbf{k}|^\alpha e^{-\lambda_{\min}|\mathbf{k}|^2}d\mathbf{k} < \infty.$$

To prove $\partial_{\mathbf{x}}^{\alpha}G_1 \in L^1(\mathbb{R}^2)$, we estimate $x^3\partial_{\mathbf{x}}^{\alpha}G_1(\mathbf{x})$. By integrating by parts, we have

$$\begin{aligned}\left|x^3\partial_{\mathbf{x}}^{\alpha}G_1(\mathbf{x})\right| &= \left|\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\mathbf{k}^\alpha\left(\partial_k^3 e^{i\mathbf{k}\cdot\mathbf{x}}\right)e^{-Q(\mathbf{k})}d\mathbf{k}\right| \\ &= \left|\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}e^{i\mathbf{k}\cdot\mathbf{x}}\partial_k^3\left(\mathbf{k}^\alpha e^{-Q(\mathbf{k})}\right)d\mathbf{k}\right| \\ &\le \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\left|\partial_k^3\left(\mathbf{k}^\alpha e^{-Q(\mathbf{k})}\right)\right|d\mathbf{k}.\end{aligned}$$

From (2.5) and (2.6), there exists a positive constant C such that

$$\begin{aligned}\left|\partial_k^3\left(\mathbf{k}^\alpha e^{-Q(\mathbf{k})}\right)\right| &= \Big|\partial_k^3\mathbf{k}^\alpha - 3\partial_k^2\mathbf{k}^\alpha\partial_k Q(\mathbf{k}) - 3\partial_k\mathbf{k}^\alpha\partial_k^2Q(\mathbf{k}) + 3\partial_k\mathbf{k}^\alpha(\partial_kQ(\mathbf{k}))^2 \\ &\quad + 3\mathbf{k}^\alpha\partial_k^2Q(\mathbf{k})\partial_kQ(\mathbf{k}) - \mathbf{k}^\alpha\partial_k^3Q(\mathbf{k}) - \mathbf{k}^\alpha(\partial_kQ(\mathbf{k}))^3\Big|e^{-Q(\mathbf{k})} \\ &\le C\left(|\mathbf{k}|^{3+|\alpha|} + |\mathbf{k}|^{-1}\right)e^{-\lambda_{\min}|\mathbf{k}|^2}.\end{aligned}$$

This implies $\partial_k^3(\mathbf{k}^\alpha e^{-Q(\mathbf{k})}) \in L^1(\mathbb{R}^2)$, and hence $\sup_{\mathbf{x}\in\mathbb{R}^2} |x^3\partial_{\mathbf{x}}^\alpha G_1(\mathbf{x})| < \infty$ holds. In the same way, we can prove $\sup_{\mathbf{x}\in\mathbb{R}^2} |y^3\partial_{\mathbf{x}}^\alpha G_1(\mathbf{x})| < \infty$. Consequently, we obtain

$$\sup_{\mathbf{x}\in\mathbb{R}^2} \left|(1+|\mathbf{x}|^3)\partial_{\mathbf{x}}^\alpha G_1(\mathbf{x})\right| < \infty.$$

This implies $\partial_{\mathbf{x}}^\alpha G_1 \in L^1(\mathbb{R}^2)$. ◻

Lemma 2.2. *There exists a positive constant C such that*

$$\|\partial_t G_t\|_{L^1(\mathbb{R}^2)} \le Ct^{-1}, \quad t>0. \tag{2.7}$$

Proof. By direct computations, we have

$$\begin{aligned} \partial_t G_t(\mathbf{x}) &= -\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} Q(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}e^{-tQ(\mathbf{k})}d\mathbf{k} = -\frac{1}{t^2}H\left(\frac{\mathbf{x}}{\sqrt{t}}\right),\\ H(\mathbf{x}) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} Q(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}e^{-Q(\mathbf{k})}d\mathbf{k}. \end{aligned}$$

In what follows, we shall prove $H \in L^1(\mathbb{R}^2)$. It suffices to prove (2.7) since $\|H\|_{L^1(\mathbb{R}^2)}$ is equal to $\|t^{-1}H(\cdot/\sqrt{t})\|_{L^1(\mathbb{R}^2)}$. For this purpose, we first estimate $x^3H(\mathbf{x})$. By integrating by parts, we have

$$\begin{aligned} \left|x^3H(\mathbf{x})\right| &= \left|\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} Q(\mathbf{k})\left(\partial_k^3 e^{i\mathbf{k}\cdot\mathbf{x}}\right)e^{-Q(\mathbf{k})}d\mathbf{k}\right|\\ &= \left|\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i\mathbf{k}\cdot\mathbf{x}}\partial_k^3\left(Q(\mathbf{k})e^{-Q(\mathbf{k})}\right)d\mathbf{k}\right|\\ &\le \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\left|\partial_k^3\left(Q(\mathbf{k})e^{-Q(\mathbf{k})}\right)\right|d\mathbf{k}. \end{aligned}$$

From (2.5) and (2.6), there exists a positive constant C such that

$$\begin{aligned} \left|\partial_k^3\left(Q(\mathbf{k})e^{-Q(\mathbf{k})}\right)\right| &= \Big|\partial_k^3 Q(\mathbf{k}) - 6\partial_k^2 Q(\mathbf{k})\partial_k Q(\mathbf{k}) + 3(\partial_k Q(\mathbf{k}))^3 - Q(\mathbf{k})\partial_k^3 Q(\mathbf{k})\\ &\quad + 3Q(\mathbf{k})\partial_k Q(\mathbf{k})\partial_k^2 Q(\mathbf{k}) - Q(\mathbf{k})(\partial_k Q(\mathbf{k}))^3\Big|e^{-Q(\mathbf{k})}\\ &\le C\left(|\mathbf{k}|^5 + |\mathbf{k}|^{-1}\right)e^{-\lambda_{\min}|\mathbf{k}|^2}. \end{aligned}$$

This implies $\partial_k^3(Q(\mathbf{k})e^{-Q(\mathbf{k})}) \in L^1(\mathbb{R}^2)$, and hence $\sup_{\mathbf{x}\in\mathbb{R}^2}|x^3H(\mathbf{x})| < \infty$ holds. In the same way, we can prove $\sup_{\mathbf{x}\in\mathbb{R}^2}|y^3H(\mathbf{x})| < \infty$. On the other hand, $H \in L^\infty(\mathbb{R}^2)$ follows easily because (2.5) and (2.6) imply that

$$|H(\mathbf{x})| \le \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} C|\mathbf{k}|^2 e^{-\lambda_{\min}|\mathbf{k}|^2}d\mathbf{k} < \infty$$

for some constant C. By combining these estimates, we have

$$\sup_{\mathbf{x}\in\mathbb{R}^2}|(1+|\mathbf{x}|^3)H(\mathbf{x})| < \infty.$$

This implies $H \in L^1(\mathbb{R}^2)$. ◻

Lemmas 2.1 and 2.2 lead to Lemma 2.3 below. It implies that the regularity of $G_t * u_0$ is basically the same as that of the solution of the linear heat equation $u_t = \Delta u$ on $\mathbb{R}^2$.

LEMMA 2.3 (regularity of $G_t * u_0$). *Let $u_0 \in BUC^k(\mathbb{R}^2)$. Then the solution $u(\mathbf{x},t) = (G_t * u_0)(\mathbf{x})$ of the linear problem* (2.2) *satisfies the following:*

(i) *For any multi-index α, one has*

$$\|\partial_{\mathbf{x}}^{\alpha}(G_t * u_0)\|_{BUC^k(\mathbb{R}^2)} \le t^{-\frac{|\alpha|}{2}} \|\partial_{\mathbf{x}}^{\alpha} G_1\|_{L^1(\mathbb{R}^2)} \|u_0\|_{BUC^k(\mathbb{R}^2)}, \quad t > 0. \tag{2.8}$$

(ii) *There exists a positive constant C such that*

$$\|\partial_t (G_t * u_0)\|_{BUC^k(\mathbb{R}^2)} \le t^{-1} C \|u_0\|_{BUC^k(\mathbb{R}^2)}, \quad t > 0. \tag{2.9}$$

Proof. We first note that if $w \in BUC(\mathbb{R}^2)$ and $g \in L^1(\mathbb{R}^2)$, then $g * w \in BUC(\mathbb{R}^2)$ and

$$\|g * w\|_{BUC(\mathbb{R}^2)} \le \|g\|_{L^1(\mathbb{R}^2)} \|w\|_{BUC(\mathbb{R}^2)}. \tag{2.10}$$

This is a consequence of Young's inequality and the fact that translation commutes with convolution:

$$\begin{aligned} \|g * w\|_{L^\infty(\mathbb{R}^2)} &\le \|g\|_{L^1(\mathbb{R}^2)} \|w\|_{L^\infty(\mathbb{R}^2)}, \\ \|\tau_{\mathbf{x}'}(g * w) - g * w\|_{L^\infty(\mathbb{R}^2)} = \|g * (\tau_{\mathbf{x}'} w - w)\|_{L^\infty(\mathbb{R}^2)} &\le \|g\|_{L^1(\mathbb{R}^2)} \|\tau_{\mathbf{x}'} w - w\|_{L^\infty(\mathbb{R}^2)}, \end{aligned} \tag{2.11}$$

where $(\tau_{\mathbf{x}'} w)(\mathbf{x}) = w(\mathbf{x} - \mathbf{x}')$.

Since $\partial_{\mathbf{x}}^{\alpha} G_1 \in L^1(\mathbb{R}^2)$ holds from Lemma 2.1, we have

$$\|\partial_{\mathbf{x}}^{\alpha} G_t\|_{L^1(\mathbb{R}^2)} = t^{-\left(1 + \frac{|\alpha|}{2}\right)} \left\| \partial_{\mathbf{x}}^{\alpha} G_1 \left(\frac{\cdot}{\sqrt{t}} \right) \right\|_{L^1(\mathbb{R}^2)} = t^{-\frac{|\alpha|}{2}} \|\partial_{\mathbf{x}}^{\alpha} G_1\|_{L^1(\mathbb{R}^2)}$$

for any $t > 0$. This gives (2.8) since, for any multi-index α and $|\beta| \le k$, we have

$$\|\partial_{\mathbf{x}}^{\alpha+\beta}(G_t * u_0)\|_{BUC(\mathbb{R}^2)} = \|\partial_{\mathbf{x}}^{\alpha} G_t * \partial_{\mathbf{x}}^{\beta} u_0\|_{BUC(\mathbb{R}^2)} \le \|\partial_{\mathbf{x}}^{\alpha} G_t\|_{L^1(\mathbb{R}^2)} \|\partial_{\mathbf{x}}^{\beta} u_0\|_{BUC(\mathbb{R}^2)}, \tag{2.12}$$

where we used (2.10). Similarly, (2.9) follows immediately from Lemma 2.2. ∎

LEMMA 2.4. *The family of linear operators $\{T(t)\}_{t>0}$ defined by $T(t)u_0 = G_t * u_0$ is an analytic semigroup on $BUC^k(\mathbb{R}^2)$.*

Proof. $T(t)$ satisfies the following:

(i) $T(t)T(s) = T(t+s)$ holds for any $t, s > 0$. This follows immediately from

$$\begin{aligned} G_t * G_s &= \frac{1}{2\pi} \left(\mathcal{F}^{-1} e^{-tQ(\mathbf{k})} \right) * \frac{1}{2\pi} \left(\mathcal{F}^{-1} e^{-sQ(\mathbf{k})} \right) \\ &= \frac{1}{2\pi} \mathcal{F}^{-1} \left[e^{-(t+s)Q(\mathbf{k})} \right] = G_{(t+s)}. \end{aligned}$$

(ii) It holds from Lemma 2.3 that there exists a positive constant C such that

$$\begin{aligned} \|T(t)u_0\|_{BUC^k(\mathbb{R}^2)} &\le \|G_1\|_{L^1(\mathbb{R}^2)} \|u_0\|_{BUC^k(\mathbb{R}^2)}, \\ \|tT'(t)u_0\|_{BUC^k(\mathbb{R}^2)} &\le C \|u_0\|_{BUC^k(\mathbb{R}^2)} \end{aligned}$$

hold for any $t > 0$, where $T'(t)$ is the derivative in the strong sense.

(iii) $\lim_{t \to +0} T(t)u_0 = u_0$ holds in the topology of $BUC^k(\mathbb{R}^2)$. This follows from the explicit expression of G_t given in (2.4).

Thus, $\{T(t)\}_{t>0}$ is an analytic semigroup in $BUC^k(\mathbb{R}^2)$; see Proposition 2.1.9 of [22], for instance. □

For functions that decay sufficiently fast at infinity, the bidomain operator Λ may be defined as a Fourier multiplier operator as in (1.3). However, this does not work in $BUC^k(\mathbb{R}^2)$. Functions that do not decay at infinity (such as the constant function $u \equiv 1$) have a Fourier transform that is not regular at the origin, making it difficult to define Λ as a Fourier multiplier since the symbol $Q(\mathbf{k})$ is discontinuous at the origin (see (1.3)). To avoid this technical difficulty, we make use of the above lemma and define Λ on $BUC^k(\mathbb{R}^2)$ as the generator of $T(t)$ as follows. Let $\mathcal{D}_k(\Lambda) \subset BUC^k(\mathbb{R}^2)$ be the domain of Λ:

$$-\Lambda u = \lim_{t \to +0} \frac{T(t)u - u}{t} \quad \text{in } BUC^k(\mathbb{R}^2), \tag{2.13}$$

$$\mathcal{D}_k(\Lambda) = \left\{ u \in BUC^k(\mathbb{R}^2) \mid \lim_{t \to +0} \frac{T(t)u - u}{t} \text{ exists in } BUC^k(\mathbb{R}^2) \right\}. \tag{2.13}$$

We may define the norm $\mathcal{D}_k(\Lambda)$ as follows:

$$\|u\|_{\mathcal{D}_k(\Lambda)} = \|u\|_{BUC^k(\mathbb{R}^2)} + \|\Lambda u\|_{BUC^k(\mathbb{R}^2)}. \tag{2.15}$$

We will not attempt to characterize $\mathcal{D}_k(\Lambda)$. We will, however, prove the following result. Let $C^{k,\gamma}(\mathbb{R}^2)$, $k = 0, 1, 2, \ldots$, $0 < \gamma < 1$, be the space of functions whose kth order partial derivatives are γ Hölder continuous. The $C^{k,\gamma}$ norm is defined as

$$\|u\|_{C^{k,\gamma}(\mathbb{R}^2)} = \|u\|_{BUC^k(\mathbb{R}^2)} + \sum_{|\alpha|=k} \|\partial_{\mathbf{x}}^{\alpha} u\|_{C^{\gamma}(\mathbb{R}^2)},$$
$$\|u\|_{C^{\gamma}(\mathbb{R}^2)} = \sup_{\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2} \frac{|u(\mathbf{x} + \mathbf{x}') - u(\mathbf{x})|}{|\mathbf{x}'|^{\gamma}}.$$

PROPOSITION 2.5. *Consider the operator Λ and its domain $\mathcal{D}_k(\Lambda)$ defined in (2.13) and (2.14). We have $\mathcal{D}_k(\Lambda) \subset C^{k+1,\gamma}(\mathbb{R}^2)$, $0 < \gamma < 1$. Furthermore, we have the following estimate for $u \in \mathcal{D}_k(\Lambda)$:*

$$\|u\|_{C^{k+1,\gamma}(\mathbb{R}^2)} \leq C_{k,\gamma} \|u\|_{\mathcal{D}_k(\Lambda)}, \quad 0 < \gamma < 1, \tag{2.16}$$

for a constant $C_{k,\gamma}$ that depends only on k and γ.

Proof. Using (2.8), we have

$$\begin{aligned} \|T(t)u\|_{BUC^k(\mathbb{R}^2)} &\leq C_0 \|u\|_{BUC^k(\mathbb{R}^2)}, \\ \|T(t)u\|_{BUC^{k+1}(\mathbb{R}^2)} &\leq C_1 \left(\frac{1}{\sqrt{t}} + 1 \right) \|u\|_{BUC^k(\mathbb{R}^2)}, \\ \|T(t)u\|_{BUC^{k+2}(\mathbb{R}^2)} &\leq C_2 \left(\frac{1}{t} + 1 \right) \|u\|_{BUC^k(\mathbb{R}^2)} \end{aligned} \tag{2.17}$$

for some positive constants C_0, C_1, and C_2. First, note that

$$\begin{aligned} \left\| \int_0^{\infty} \exp(-t) T(t) u dt \right\|_{BUC^k(\mathbb{R}^2)} &\leq \int_0^{\infty} \exp(-t) \|T(t)u\|_{BUC^k(\mathbb{R}^2)} dt \\ \leq C_0 \int_0^{\infty} \exp(-t) dt \|u\|_{BUC^k(\mathbb{R}^2)} &= C_0 \|u\|_{BUC^k(\mathbb{R}^2)}, \end{aligned} \tag{2.18}$$

where we used the first inequality in (2.17) in the second inequality. This implies that $(1+\Lambda)$ has a bounded inverse and, for $u \in BUC^k(\mathbb{R}^2)$, we have

$$(1+\Lambda)^{-1}u = \int_0^\infty \exp(-t)T(t)u dt. \tag{2.19}$$

Using the interpolation inequality on Hölder spaces, we have

$$\begin{aligned} \|T(t)u\|_{C^{k+1,\gamma}(\mathbb{R}^2)} &\leq C\|T(t)u\|_{BUC^{k+2}(\mathbb{R}^2)}^{1-\gamma}\|T(t)u\|_{BUC^{k+1}(\mathbb{R}^2)}^{\gamma} \\ &\leq C_3\left(\frac{1}{t^{(1+\gamma)/2}}+1\right)\|u\|_{BUC^k(\mathbb{R}^2)}, \end{aligned} \tag{2.20}$$

where we used (2.17) in the second inequality. We have

$$\begin{aligned} &\|(1+\Lambda)^{-1}u\|_{C^{k+1,\gamma}(\mathbb{R}^2)} \leq \int_0^\infty \exp(-t)\|T(t)u\|_{C^{k+1,\gamma}(\mathbb{R}^2)}dt \\ &\leq C_3\left(\int_0^\infty \left(\frac{1}{t^{(1+\gamma)/2}}+1\right)\exp(-t)dt\right)\|u\|_{BUC^k(\mathbb{R}^2)} \leq C_\gamma\|u\|_{BUC^k(\mathbb{R}^2)}, \end{aligned} \tag{2.21}$$

where we used the fact that $(1+\gamma)/2 < 1$ to conclude that the integral in the last line is finite. Any element $v \in \mathcal{D}_k(\Lambda)$ can be written as $v = (1+\Lambda)^{-1}u$. Thus, the above inequality can be written as

$$\|v\|_{C^{k+1,\gamma}(\mathbb{R}^2)} \leq C_\gamma\|(1+\Lambda)v\|_{BUC^k(\mathbb{R}^2)} \quad \text{for any } v \in \mathcal{D}_k(\Lambda). \tag{2.22}$$

The triangle inequality applied to the last expression yields the desired inequality. ☐

2.2. Bidomain Allen–Cahn equation on $\mathbb{R}^2$. In this subsection, by using the fundamental solution G_t and the analytic semigroup constructed in the previous subsection, we discuss the existence and basic properties of the solution $u(\mathbf{x},t)$ of the initial value problem of the bidomain Allen–Cahn equation on $\mathbb{R}^2$ of the form

$$u_t = -\Lambda u + f(u), \quad \mathbf{x} \in \mathbb{R}^2, \quad t > 0, \tag{2.23a}$$
$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2. \tag{2.23b}$$

The first lemma implies the local existence of the solution, which is derived immediately from Theorem 7.1.2 and Proposition 7.1.10 of [22]. Thus, we omit the proof.

LEMMA 2.6 (local existence of solutions of (2.23)). *Assume $u_0 \in BUC^k(\mathbb{R}^2)$. Then there exists a constant $T > 0$ such that the problem* (2.23) *has a unique mild solution $u(\mathbf{x},t) \in C^1((0,T];BUC^k(\mathbb{R}^2)) \cap C([0,T];BUC^k(\mathbb{R}^2))$ in the sense of Definition* 1.1. *Furthermore, $u(\mathbf{x},t)$ satisfies*

$$\frac{\partial u}{\partial t} = -\Lambda u + f(u), \quad t > 0,$$

where Λ is defined in (2.13).

We also state the following result on the continuity with respect to initial data, which is also standard (see [22]).

LEMMA 2.7 (continuity with respect to initial data). *For any $u_0 \in BUC^k(\mathbb{R}^2)$, there is a constant $\delta > 0$ such that the time T in Lemma 2.6 can be taken uniformly for initial data $\widetilde{u}_0$ satisfying $\|\widetilde{u}_0 - u_0\|_{BUC^k(\mathbb{R}^2)} \le \delta$. Let $\widetilde{u}$ be the mild solution corresponding to $\widetilde{u}_0$. Then $\|\widetilde{u}(t)\|_{BUC^k(\mathbb{R}^2)} \le M_0, 0 \le t \le T$, for some positive constant $M_0 > 0$ that does not depend on the choice of $\widetilde{u}_0$.*

The next lemma shows the regularity of the solution of (2.23) in $\mathbf{x} \in \mathbb{R}^2$, which is derived by estimating the fundamental solution G_t directly. Let $BUC^\infty(\mathbb{R}^2)$ consist of functions that belong to all $BUC^k(\mathbb{R}^2), k \in \mathbb{N}$.

LEMMA 2.8 (regularity of the solution of (2.23) in $\mathbf{x}$). *Let $u(\mathbf{x},t)$ be the solution of (2.23) defined on $\mathbb{R}^2 \times [0,T]$. Then $u(\cdot,t) \in BUC^\infty(\mathbb{R}^2)$ for each $t \in (0,T]$.*

Proof. Let any $t \in (0,T]$ be fixed, and let α be any given multi-index satisfying $|\alpha| = 1$. Differentiating the formula

$$u(\mathbf{x},t) = (G_t * u_0)(t) + \int_0^t G_{t-s} * f(u(\cdot,s))ds \tag{2.24}$$

and then applying (2.8) in Lemma 2.3, we have

$$\begin{aligned}
\|\partial_{\mathbf{x}}^\alpha u(\mathbf{x},t)\|_{L^\infty(\mathbb{R}^2)} &\le t^{-\frac{1}{2}} \|\partial_{\mathbf{x}}^\alpha G_1\|_{L^1(\mathbb{R}^2)} \|u_0\|_{L^\infty(\mathbb{R}^2)} \\
&\quad + \int_0^t (t-s)^{-\frac{1}{2}} \|\partial_{\mathbf{x}}^\alpha G_1\|_{L^1(\mathbb{R}^2)} \|f(u(\cdot,s))\|_{L^\infty(\mathbb{R}^2)} ds \\
&= t^{-\frac{1}{2}} \|\partial_{\mathbf{x}}^\alpha G_1\|_{L^1(\mathbb{R}^2)} \|u_0\|_{L^\infty(\mathbb{R}^2)} \\
&\quad + 2\sqrt{t} \|\partial_{\mathbf{x}}^\alpha G_1\|_{L^1(\mathbb{R}^2)} \|f(u)\|_{L^\infty(\mathbb{R}^2 \times [0,t])} \\
&< \infty.
\end{aligned}$$

This implies $u(\cdot,t) \in W^{1,\infty}(\mathbb{R}^2)$ for each $t \in (0,T]$.

Let any $t \in (0,T]$ be fixed, and let α be any given multi-index satisfying $|\alpha| = 2$. We choose multi-indexes α' and α'' that satisfy $|\alpha'| = 1$, $|\alpha''| = 1$, and $\partial_{\mathbf{x}}^\alpha = \partial_{\mathbf{x}}^{\alpha'} \partial_{\mathbf{x}}^{\alpha''}$. Setting $u_*(\mathbf{x}) := u(\mathbf{x}, t/2)$, we have

$$u(\mathbf{x},t) = (G_{t/2} * u_*)(t) + \int_{t/2}^t G_{t-s} * f(u(\cdot,s))ds. \tag{2.25}$$

Differentiating this formula and then applying (2.8) in Lemma 2.3, we have

$$\begin{aligned}
\|\partial_{\mathbf{x}}^\alpha u(\mathbf{x},t)\|_{L^\infty(\mathbb{R}^2)} &\le \left(\frac{t}{2}\right)^{-1} \|\partial_{\mathbf{x}}^\alpha G_1\|_{L^1(\mathbb{R}^2)} \|u_*\|_{L^\infty(\mathbb{R}^2)} \\
&\quad + \int_{t/2}^t (t-s)^{-\frac{1}{2}} \|\partial_{\mathbf{x}}^{\alpha''} G_1\|_{L^1(\mathbb{R}^2)} \|f'(u(\cdot,s)) \partial_{\mathbf{x}}^{\alpha'} u(\cdot,s)\|_{L^\infty(\mathbb{R}^2)} ds \\
&= \left(\frac{t}{2}\right)^{-1} \|\partial_{\mathbf{x}}^\alpha G_1\|_{L^1(\mathbb{R}^2)} \|u_*\|_{L^\infty(\mathbb{R}^2)} \\
&\quad + \sqrt{2t} \|\partial_{\mathbf{x}}^{\alpha''} G_1\|_{L^1(\mathbb{R}^2)} \|f'(u) \partial_{\mathbf{x}}^{\alpha'} u\|_{L^\infty(\mathbb{R}^2 \times [t/2,t])} \\
&< \infty.
\end{aligned}$$

This implies $u(\cdot,t) \in W^{2,\infty}(\mathbb{R}^2)$ for each $t \in (0,T]$. From the bootstrap argument, for any $k = 3,4,\dots,$ we obtain $u(\cdot,t) \in W^{k,\infty}$ for each $t \in (0,T]$. Consequently, $u(\cdot,t) \in BUC^\infty(\mathbb{R}^2)$ holds for each $t \in (0,T]$. ∎

2.3. Bidomain Allen-Cahn equation on the infinite strip $\Omega_{\theta,d}$. In this subsection, we consider well-posedness and the regularity of the solution of the bidomain Allen-Cahn equation on the infinite strip $\Omega_{\theta,d}$ as given in (1.14). There is a one-to-one correspondence between any function on $\Omega_{\theta,d}$ and its periodic extension to all of $\mathbb{R}^2$ using (1.15). Recall that $BUC(\Omega_{\theta,d})$ consist of functions of $BUC(\mathbb{R}^2)$ which are periodic in the sense of (1.15). The space $BUC(\Omega_{\theta,d})$ may thus be viewed as bounded uniformly continuous functions defined on the strip $\Omega_{\theta,d}$ with periodic boundary conditions.

DEFINITION 2.9 (mild solution of the bidomain Allen–Cahn equation in $\Omega_{\theta,d}$). *Consider the problem* (1.14) *where* $u_0(\mathbf{x}) \in BUC(\Omega_{\theta,d})$. *The function* $u(\mathbf{x},t) \in C^1((0,T]; BUC(\Omega_{\theta,d})) \cap C([0,T]; BUC(\Omega_{\theta,d})), T > 0$, *is a mild solution to* (1.14) *if the periodic extension of* $u(x,t)$ *via* (1.15) *is a mild solution in the sense of Definition* 1.1 *with initial data given by the periodic extension of* $u_0(x)$.

The well-posedness and the regularity of the solution $u(\mathbf{x},t)$ of (1.14) follows immediately from those in the previous subsection. This establishes the first half of Proposition 1.2.

We now show that this solution is classical. To proceed further, we first introduce a coordinate system on $\Omega_{\theta,d}$. We let

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \xi + \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \eta. \tag{2.26}$$

In this coordinate system, $(\xi,\eta) \in \mathbb{R} \times \mathbb{S}^1_d = \Omega_{\theta,d}$. The definition of the coordinate ξ is what we referred to as $\widehat{\xi}$ in section 1. In this section only, we shall use ξ instead of $\widehat{\xi}$ to avoid cumbersome notation.

We shall often expand a function $v(\xi,\eta) \in BUC(\mathbb{R} \times \mathbb{S}^1_d)$ in terms of its Fourier series in the η-direction:

$$v(\xi,\eta) = \sum_{k=-\infty}^{\infty} v_{2\pi k/d}(\xi) \exp(2\pi i k\eta/d), \quad v_l(\xi) = \frac{1}{d}\int_0^d v(\xi,\eta)\exp(-il\eta)d\eta. \tag{2.27}$$

We first prove the following result. Let $BUC^k(\Omega_{\theta,d}), k \in \mathbb{N}$, be the subset of $BUC(\Omega_{\theta,d})$ with bounded continuous kth order derivatives in $BUC(\Omega_{\theta,d})$. A function in $BUC^\infty(\Omega_{\theta,d})$ belongs to all function $BUC^k(\Omega_{\theta.d}), k \in \mathbb{N}$.

The operator Λ_θ, the matrices A_i^θ, A_e^θ, the symbol Q_θ, and other mathematical objects all depend on θ. In what follows, however, we will mostly be working with a fixed value of θ, and we will thus omit this dependence to avoid cluttered notation.

PROPOSITION 2.10. *Consider a function* $v \in BUC^\infty(\Omega_{\theta,d})$. *Then there exist* $v_i, v_e \in BUC^\infty(\Omega_{\theta,d})$ *satisfying*

$$\nabla\cdot(A_i\nabla v_i) + \nabla\cdot(A_e\nabla v_e) = 0, \quad v_i - v_e = v. \tag{2.28}$$

The functions v_i *and* v_e *are uniquely determined up to a constant.*

It is possible to relax the assumption on the regularity of v with corresponding changes in the regularity of v_i and v_e, as will be clear in the proof. We will only need the smooth case, and we will thus not pursue this here.

Proof. Equation (2.28) may be rewritten as the following equation for v_i:

$$\nabla\cdot((A_i + A_e)\nabla v_i) = \nabla\cdot(A_e\nabla v). \tag{2.29}$$

Let $q_i(k) = Q_i(k,1)$ and $q_e(k) = Q_e(k,1)$. Note that q_i and q_e are both quadratic in k. Let

$$\frac{q_i(k)}{q_i(k)+q_e(k)} = c_1 + \frac{c_2(k)}{q_i(k)+q_e(k)}, \tag{2.30}$$

where c_1 is a constant and $c_2(k)$ is first order in k. Let

$$K(x) = \mathcal{F}^{-1}\left(\frac{c_2(k)}{q_i(k)+q_e(k)}\right), \tag{2.31}$$

where $\mathcal{F}^{-1}$ is the one-dimensional inverse Fourier transform. Given that $q_i + q_e$ is a positive quadratic function of k, $K(x)$ is a function that decays exponentially as $|x| \to \infty$.

Expand the $v(\mathbf{x})$ in terms of Fourier series as in (2.27). Given that $v \in BUC^\infty$, $v_l(x)$ are smooth functions and satisfy the bound

$$\|d^m v_l/dx^m\|_{L^\infty} \le \frac{C_{n,m}}{l^n} \quad \text{for any } n, m \in \mathbb{N}, \tag{2.32}$$

where the constant $C_{n,m}$ depends on n, m but not on l. Define

$$w = \sum_{l\in(2\pi/d)\mathbb{Z}} w_l(x)\exp(il\eta),\ w_l(x) = c_1 v_l(x) + \int_{\mathbb{R}} |l| K(l(x-z)) v_l(z)dz. \tag{2.33}$$

It can be checked that $v_i = w$ is a smooth function that satisfies (2.29). Indeed, direct computation shows that

$$\nabla\cdot((A_i + A_e)\nabla(w_l\exp(il\eta))) = \nabla\cdot(A_e\nabla(v_l\exp(il\eta))). \tag{2.34}$$

Given the decay estimate (2.32), the claim follows. To prove uniqueness, suppose $\widetilde{v}_i$ is another solution. We see that $v_i - \widetilde{v}_i$ satisfies

$$\nabla\cdot((A_i + A_e)\nabla(v_i - \widetilde{v}_i)) = 0. \tag{2.35}$$

Given that $v_i - \widetilde{v}_i$ must be bounded, we see that $v_i - \widetilde{v}_i$ is spatially constant by Liouville's theorem. ◻

We now consider the action of Λ on sufficiently smooth functions $v(\xi,\eta)$ defined on $\Omega_{\theta,d}$. As was introduced in (1.21), using the expansion (2.27), formal computations indicate that

$$-\Lambda v = \sum_{l\in(2\pi/d)\mathbb{Z}} (\Lambda_l v_l)(\xi)\exp(il\eta), \quad (\Lambda_l v_l)(\xi) = \mathcal{F}_\xi^{-1} Q(k,l)\mathcal{F}_\xi v_l(\xi). \tag{2.36}$$

This expression, however, does not make sense even for sufficiently smooth functions in $BUC(\Omega_{\theta,d})$ since $v_l(\xi)$ does not, in general, belong to $L^2(\mathbb{R})$. This can be remedied by rewriting Λ_l in the following way, as was first done in [27]. Let

$$q_{BD}(k) = \frac{Q_i(k,1)Q_e(k,1)}{Q_i(k,1)+Q_e(k,1)} = p(k) + \frac{c_q(k)}{Q_i(k,1)+Q_e(k,1)} = p(k) + q(k), \tag{2.37}$$

where $c_q(k)$ is linear in k and $p(k)$ is a second degree polynomial in k obtained as a quotient by performing polynomial division with remainder. Let

(2.38)
$$\Lambda_l w(\xi) = l^2 p(-i\partial_\xi/l)w(\xi) + (|l|^3\check{q}(l\xi) * w)(\xi), \quad \check{q}(\xi) = (\mathcal{F}^{-1}q(k))(\xi) \quad \text{for } l \neq 0,$$
$$\Lambda_0 w(\xi) = c_p\frac{d^2w}{dx^2},$$

where $*$ denotes the convolution on $\mathbb{R}$ and c_p is the coefficient of k^2 in the quadratic polynomial $p(k)$. It is easily seen that $\check{q}(\xi)$ decays exponentially as $|\xi| \to \infty$, and thus the above convolution is well-defined for bounded functions $w(\xi)$. For sufficiently smooth functions that decay sufficiently fast as $|\xi| \to \infty$, by construction, the above definition of Λ_l coincides with (2.36). We shall henceforth take (2.38) to be the definition of Λ_l. This has the advantage that it is well-defined for any bounded function that is sufficiently smooth. We now prove the following result on the operator Λ.

PROPOSITION 2.11. *Suppose* $v \in BUC^4(\Omega_{\theta,d})$. *Then*

$$(\Lambda v)(\xi,\eta) = \sum_{l\in(2\pi/d)\mathbb{Z}} (\Lambda_l v_l)(\xi)\exp(il\eta), \tag{2.39}$$

where Λ *was defined in* (2.13) *and* Λ_l *is given as in* (2.36). *Suppose* $v \in BUC^\infty(\Omega_{\theta,d})$, *and let* v_i *be a solution to* (2.28), *whose existence is guaranteed by Proposition* 2.10. *Then*

$$-\Lambda v = \nabla \cdot (A_i \nabla v_i). \tag{2.40}$$

Proof. We first consider the operator $-\Lambda$ acting on a single term in the Fourier expansion $w(\xi)\exp(il\eta)$:

$$\begin{aligned} &-\Lambda(w(\xi)\exp(il\eta)) = \lim_{t\to 0+} t^{-1}(T(t)(w(\xi)\exp(il\eta)) - w(\xi)\exp(il\eta)) \\ &= \lim_{t\to 0+} t^{-1}\left(\int_{\mathbb{R}^2} G_t(\xi',\eta')w(\xi-\xi')\exp(il(\eta-\eta'))d\xi'd\eta' - w(\xi)\exp(il\eta)\right). \end{aligned} \tag{2.41}$$

By the definition of G_t, we see that

$$\begin{aligned} &\int_{\mathbb{R}^2} G_t(\xi',\eta')w(\xi-\xi')\exp(il(\eta-\eta'))d\xi'd\eta' \\ &= \int_{\mathbb{R}} G_{t,l}(\xi')w(\xi-\xi')d\xi'\exp(il\eta), \quad G_{t,l}(x) = \mathcal{F}^{-1}(\exp(-tQ(k,l))). \end{aligned} \tag{2.42}$$

We first consider the case $l = 0$. In this case,

$$G_{t,0}(x) = \mathcal{F}^{-1}(\exp(-tQ(k,0))) = \mathcal{F}^{-1}(\exp(-c_p t k^2)). \tag{2.43}$$

This is nothing other than a scaled heat kernel, and it is well known that if $w \in BUC^2(\mathbb{R})$, we have

$$\Lambda w = -\lim_{t\to 0+} t^{-1}((G_{t,0} * w)(\xi) - w(\xi)) = -c_p \frac{d^2 w}{d\xi^2} = (\Lambda_0 w)(\xi), \tag{2.44}$$

where the convergence above is in $BUC(\mathbb{R}^2)$.

We next consider the case $l \neq 0$. Define the kernel

$$K_l(\xi) = \frac{1}{|l|}K_{BD}(l\xi), \quad K_{BD}(\xi) = (\mathcal{F}^{-1}q_{BD}^{-1}(k))(\xi). \tag{2.45}$$

We note that $K_{BD}(x)$ is exponentially decaying as $|\xi| \to \infty$. It is clear from the above construction that $K_l(x)$ is the fundamental solution to the operator Λ_l. Indeed, for a function $w(\xi) \in BUC^2(\mathbb{R})$, it can be shown that

$$K_l * (\Lambda_l w) = w. \tag{2.46}$$

Combining the above, we have

$$\begin{aligned} &-\Lambda(w(\xi)\exp(il\eta)) \\ &= \lim_{t\to 0+} t^{-1}\left(G_{t,l} * (K_l * (\Lambda_l w)) - K_l * (\Lambda_l w)\right)\exp(il\eta) \\ &= \lim_{t\to 0+} t^{-1}\left((G_{t,l} * K_l - K_l) * (\Lambda_l w)\right)\exp(il\eta). \end{aligned} \tag{2.47}$$

Given the definition of K_l and of $G_{t,l}$, we see that

$$\begin{aligned} G_{t,l} * K_l - K_l &= \mathcal{F}^{-1}\left(\frac{\exp(-tQ(k,l)) - 1}{Q(k,l)}\right) = -\mathcal{F}^{-1}\left(\int_0^t \exp(-sQ(k,l))ds\right) \\ &= -\int_0^t G_{s,l}(\xi)ds, \end{aligned} \tag{2.48}$$

where we used Fubini's theorem in the last equality. Note that

$$\begin{aligned} &\int_{\mathbb{R}^2}\int_0^t G_s(\xi-\xi',\eta-\eta')ds\,w(\xi-\xi')\exp(il(\eta-\eta'))d\xi' d\eta' \\ &= \int_0^t \int_{\mathbb{R}^2} G_s(\xi-\xi',\eta-\eta')w(\xi-\xi')\exp(il(\eta-\eta'))d\xi' d\eta' \\ &= \int_{\mathbb{R}}\left(\int_0^t G_{s,l}(\xi')ds\right) w(\xi-\xi')d\xi' \exp(il\eta), \end{aligned} \tag{2.49}$$

where we used Fubini's theorem in the above equalities. We thus see that

$$\begin{aligned} &\lim_{t\to 0+} t^{-1}\left(\left(\int_0^t G_{s,l}ds\right) * w\right)(x)\exp(il\eta) \\ &= \lim_{t\to 0+} t^{-1}\int_0^t \int_{\mathbb{R}^2} G_s(\xi-\xi',\eta-\eta')w(\xi-\xi')\exp(il(\eta-\eta'))d\xi' d\eta' = w(\xi)\exp(il\eta), \end{aligned} \tag{2.50}$$

where the above limit is valid in the $BUC(\mathbb{R}^2)$ topology. Furthermore, we have

$$\begin{aligned} &\left|t^{-1}\left(\left(\int_0^t G_{s,l}ds\right) * w\right)(x)\exp(il\eta)\right| \\ &= \left|t^{-1}\int_0^t \int_{\mathbb{R}^2} G_s(\xi-\xi',\eta-\eta')w(\xi-\xi')\exp(il(\eta-\eta'))d\xi' d\eta'\right| \\ &\leq \|G_1\|_{L^1(\mathbb{R}^2)}\|w\|_{L^\infty(\mathbb{R})}. \end{aligned} \tag{2.51}$$

Combining the above, we have

$$-\Lambda(w(\xi)\exp(il\eta)) = -(\Lambda_l w)(\xi)\exp(il\eta), \tag{2.52}$$

$$\left\|t^{-1}(T(t)(w(\xi)\exp(il\eta)) - w(\xi)\exp(il\eta))\right\|_{L^\infty} \leq \|G_1\|_{L^1(\mathbb{R}^2)}\|\Lambda_l w\|_{L^\infty(\mathbb{R})}. \tag{2.53}$$

Finally, let us consider the general case (2.39). First, note that there is a constant $c_{\mathcal{K}}$ that does not depend on l such that

$$\|\Lambda_l w\|_{L^\infty(\mathbb{R})} \leq c_{\mathcal{K}}(\|w\|_{BUC^2(\mathbb{R})} + l^2\|w\|_{BUC(\mathbb{R})}). \tag{2.54}$$

For $v \in BUC^4(\mathbb{R}^2)$, we have the estimate

$$\|v_l\|_{BUC^2(\mathbb{R})} + l^2\|v_l\|_{BUC(\mathbb{R})} \leq \frac{C_v}{l^2} \tag{2.55}$$

for a constant C_v that does not depend on l. Let

$$\mathcal{A}_N = \{l = 2\pi k/d,\, k \in \mathbb{Z},\, |k| \leq N\}, \quad \mathcal{B}_N = ((2\pi/d)\mathbb{Z})\backslash \mathcal{A}_N. \tag{2.56}$$

For $v \in BUC^4(\mathbb{R}^2)$, we have, for positive integers N,

$$\begin{aligned}(2.57)\quad & \left\| \sum_{l \in \mathcal{B}_N} (\Lambda_l v_l)(\xi) \exp(il\eta) \right\|_{L^\infty(\mathbb{R}^2)} \\ & \leq \sum_{l \in \mathcal{B}_N} c_{\mathcal{K}}(\|v_l\|_{BUC^2(\mathbb{R})} + l^2 \|v_l\|_{BUC(\mathbb{R})}) \leq \sum_{l \in \mathcal{B}_N} c_{\mathcal{K}} C_v l^{-2} \leq \frac{c_{\mathcal{K}} C_v (2\pi)^2}{d^2 N}.\end{aligned}$$

Likewise, we have

$$\begin{aligned}(2.58)\quad & \left\| t^{-1}(T(t) - I) \sum_{l \in \mathcal{B}_N} v_l(\xi) \exp(il\eta) \right\|_{L^\infty(\mathbb{R}^2)} \\ & \leq \sum_{l \in \mathcal{B}_N} c_{\mathcal{K}} \|G_1\|_{L^1(\mathbb{R}^2)} (\|v_l\|_{BUC^2(\mathbb{R})} + l^2 \|v_l\|_{BUC(\mathbb{R})}) \\ & \leq \sum_{l \in \mathcal{B}_N} \|G_1\|_{L^1(\mathbb{R}^2)} c_{\mathcal{K}} C_v l^{-2} \leq \frac{\|G\|_{L^1(\mathbb{R}^2)} c_{\mathcal{K}} C_v (2\pi)^2}{d^2 N}.\end{aligned}$$

Fix $\epsilon > 0$, and let $N > 1/\epsilon$. Then

(2.59)

$$\begin{aligned} & \left\| -\Lambda v - \sum_{l \in (2\pi/d)\mathbb{Z}} (\Lambda_l v_l)(\xi) \exp(il\eta) \right\|_{BUC(\mathbb{R}^2)} \\ & \leq \left\| \lim_{t \to 0+} t^{-1}(T(t) - I) \left(\sum_{l \in \mathcal{A}_N} v_l(\xi) \exp(il\eta) \right) - \sum_{l \in \mathcal{A}_N} (\Lambda_l v_l(\xi)) \exp(il\eta) \right\| \\ & \quad + \left\| \lim_{t \to 0+} t^{-1}(T(t) - I) \sum_{\mathcal{B}_N} v_l(\xi) \exp(il\eta) \right\|_{L^\infty(\mathbb{R}^2)} + \left\| \sum_{l \in \mathcal{B}_N} (\Lambda_l v_l)(\xi) \exp(il\eta) \right\|_{L^\infty(\mathbb{R}^2)} \\ & \leq \frac{c_{\mathcal{K}} C_v (2\pi)^2}{d^2} (1 + \|G\|_{L^1(\mathbb{R}^2)}) \epsilon, \end{aligned}$$

where (2.52), (2.57), and (2.58) were used in the last inequality. Since $\epsilon > 0$ was arbitrary, we have (2.39) for $v \in BUC^4(\mathbb{R}^2)$.

Finally, we must show that

$$- \sum_{l \in (2\pi/d)\mathbb{Z}} (\Lambda_l v_l)(x) \exp(il\eta) = \nabla \cdot (A_i \nabla v_i). \tag{2.60}$$

It can be directly checked that this is true for finite Fourier sums. The general case follows given the decay estimate (2.32). ∎

We may now prove the rest of Proposition 1.2.

Proof of Proposition 1.2. The existence and uniqueness for the mild solution for problem (1.14) is an immediate consequence of Lemma 2.6. By Lemma 2.8, $u(x, t) \in BUC^\infty(\Omega_{\theta,d})$ for $t > 0$. By Propositions 2.10 and 2.11, one may obtain functions u_i and u_e belonging to $BUC^\infty(\Omega_{\theta,d})$ that satisfy the system (1.1). ∎

3. Spectrum of linearized operators. In this section, we study the properties of the spectrum of the linearized operator $\mathcal{L}$ for later discussion. Throughout this section, we consider (1.20), namely the linearized equation around the front U:

$$\frac{\partial v}{\partial t} = \mathcal{L}v, \quad \mathcal{L}v = -\Lambda_\theta v + c\frac{\partial v}{\partial \xi} + f'(U)v,$$

where $v = v(\xi,\eta,t)$ and $(\xi,\eta) \in \mathbb{R} \times S_d^1$. Note here that the ξ coordinate moves with the front, and we thus have a term in the expression above that is proportional to the front speed c. Let us first prove that $\mathcal{L}$, like Λ, is a generator of an analytic semigroup on $BUC^k(\Omega_{\theta,d})$.

PROPOSITION 3.1. *The operator $\mathcal{L}$ generates an analytic semigroup on $BUC^k(\Omega_{\theta,d})$ where the domain of $\mathcal{L}$, denoted by $\mathcal{D}_k(\mathcal{L})$, is the same as $\mathcal{D}_k(\Lambda)$. Furthermore, for u in the domain of $\mathcal{L}$, we have the estimate*

$$\|u\|_{C^{k+1,\gamma}(\Omega_{\theta,d})} \leq C_{\mathcal{L}} \left(\|u\|_{BUC^k(\Omega_{\theta,d})} + \|\mathcal{L}u\|_{BUC^k(\Omega_{\theta,d})} \right), \tag{3.1}$$

where $C_{\mathcal{L}}$ is a positive constant.

Proof. For any $u \in \mathcal{D}_k(\Lambda)$, by Proposition 2.5, we have, for any $\epsilon > 0$,

$$\begin{aligned} \left\| \frac{\partial u}{\partial \xi} \right\|_{BUC^k(\mathbb{R}^2)} &\leq C\|u\|_{BUC^k(\mathbb{R}^2)}^{\gamma/(1+\gamma)} \|u\|_{C^{k+1,\gamma}(\mathbb{R}^2)}^{1/(1+\gamma)} \\ &\leq \frac{C\gamma}{(1+\gamma)\epsilon^\gamma} \|u\|_{BUC^k(\mathbb{R}^2)} + \frac{C\epsilon}{1+\gamma} \|u\|_{C^{k+1,\gamma}(\mathbb{R}^2)} \\ &\leq \frac{C\gamma}{(1+\gamma)\epsilon^\gamma} \|u\|_{BUC^k(\mathbb{R}^2)} + \frac{CC_\gamma \epsilon}{1+\gamma} \|u\|_{\mathcal{D}_k(\Lambda)}. \end{aligned} \tag{3.2}$$

Thus, for $u \in \mathcal{D}_k(\Lambda)$, we have

$$\left\| c\frac{\partial v}{\partial \xi} + f'(U)v \right\|_{BUC^k(\mathbb{R}^2)} \leq a\|u\|_{BUC^k(\mathbb{R}^2)} + \delta\|\Lambda u\|_{BUC^k(\mathbb{R}^2)} \tag{3.3}$$

for a constant $a > 0$ and a constant $\delta > 0$ can be made arbitrarily small. This implies the desired result. ∎

To study the spectrum of $\mathcal{L}$, we expand v in a Fourier series in η as in (2.27). Each Fourier coefficient v_l is acted upon by the following operator $\mathcal{L}_l$:

$$\mathcal{L}_l v_l = -\Lambda_l v_l + c\frac{\partial v_l}{\partial \xi} + f'(U)v_l, \tag{3.4}$$

where Λ_l was defined in (2.38). The results in section 3.1 are used mainly for proving nonlinear stability of planar fronts stated in Theorems 1.10 and 1.11, and the results in section 3.2 are used for proving nonlinear instability stated in Theorem 1.10.

3.1. Stability criteria. The aim of this subsection is to prove Proposition 3.2 below, which allows us to apply the results in [27] concerning the spectral properties of $\mathcal{L}_l$ as an operator on $L^2(\mathbb{R})$ to prove our main theorems in the topology of $BUC(\mathbb{R} \times S_d^1)$.

PROPOSITION 3.2. *For the operator $\mathcal{L}$ and $\mathcal{L}_l$ defined in* (1.20) *and* (3.4)*, one has*

$$\mathbb{C}\backslash\sigma_{BUC(\mathbb{R}\times S_d^1)}(\mathcal{L}) \supset \bigcap_{k\in\mathbb{Z}} \mathbb{C}\backslash\sigma_{BUC(\mathbb{R})}(\mathcal{L}_{2\pi k/d}) \supset \bigcap_{k\in\mathbb{Z}} \mathbb{C}\backslash\sigma_{L^2(\mathbb{R})}(\mathcal{L}_{2\pi k/d}). \tag{3.5}$$

To prove Proposition 3.2, we decompose $\mathcal{L}_l$ defined in (3.4) as $\mathcal{L}_l = \mathcal{L}_l^- + A^-$ or $\mathcal{L}_l = \mathcal{L}_l^+ + A^+$, where

$$\mathcal{L}_l^- u = -\Lambda_l u + c\frac{\partial u}{\partial \xi} + f'(1)u, \quad A^- u = f'(U)u - f'(1)u, \tag{3.6}$$

$$\mathcal{L}_l^+ u = -\Lambda_l u + c\frac{\partial u}{\partial \xi} + f'(0)u, \quad A^+ u = f'(U)u - f'(0)u. \tag{3.7}$$

In what follows, we provide some auxiliary lemmas. The first one is concerned with the relation of the spectrum of $\mathcal{L}_l$ and $\mathcal{L}_l^\pm$ as operators on $L^2(\mathbb{R})$.

LEMMA 3.3 (spectrum of $\mathcal{L}_l^\pm$). *For each $l = 2\pi k/d$ with $k \in \mathbb{Z}$, one has*

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l^+) \subset \sigma_{L^2(\mathbb{R})}(\mathcal{L}_l), \tag{3.8}$$

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l^-) \subset \sigma_{L^2(\mathbb{R})}(\mathcal{L}_l). \tag{3.9}$$

Proof. Let $z \in \sigma_{L^2(\mathbb{R})}(\mathcal{L}_l^+)$. To show that $z \in \sigma_{L^2(\mathbb{R})}(\mathcal{L}_l)$, it is sufficient to show the following:

$$^\forall \varepsilon > 0,\ ^\exists v \in L^2(\mathbb{R}) \text{ such that } \|v\|_{L^2(\mathbb{R})} = 1,\ \|(z - \mathcal{L}_l)v\| \le \varepsilon. \tag{3.10}$$

Note that z can be expressed as

$$z = -Q(k_*, l) + ick_* + f'(0)$$

for some $k_* \in \mathbb{R}$. Consider the function v_δ for $\delta > 0$:

$$v_\delta(\xi) = (\delta/\pi)^{-1/4} \exp(-\delta\xi^2/2) \exp(ik_*\xi).$$

It is easily seen that $\|v_\delta\|_{L^2(\mathbb{R})} = 1$. Note that

$$\left\|(z - \mathcal{L}_l^+)v_\delta\right\|_{L^2(\mathbb{R})} = \|(Q(k,l) - Q(k_*,l) + ic(k - k_*))(\mathcal{F}_\xi v_\delta)(k)\|_{L^2(\mathbb{R})}.$$

We may compute the Fourier transform of v_δ as

$$(\mathcal{F}_\xi v_\delta)(k) = (\pi\delta)^{-1/4} \exp(-(k - k_*)^2/(2\delta)).$$

The function $(\mathcal{F}_\xi v_\delta)(k)$ is sharply peaked at $k = k_*$ as $\delta \to 0$, and it is thus readily seen that

$$\lim_{\delta \to 0} \left\|(z - \mathcal{L}_l^+)v_\delta\right\|_{L^2(\mathbb{R})} = 0. \tag{3.11}$$

Let

$$v_{\delta,L}(\xi) = v_\delta(\xi - L).$$

Since $v_{\delta,L}$ is merely a translation of v_δ, we have

$$\|v_\delta\|_{L^2(\mathbb{R})} = \|v_{\delta,L}\|_{L^2(\mathbb{R})} = 1, \quad \left\|(z - \mathcal{L}_l^+)v_\delta\right\|_{L^2(\mathbb{R})} = \left\|(z - \mathcal{L}_l^+)v_{\delta,L}\right\|_{L^2(\mathbb{R})}. \tag{3.12}$$

We also have, for each fixed $\delta > 0$,

$$\left\|A^+ v_{\delta,L}\right\|_{L^2(\mathbb{R})} = \|(f'(U) - f'(0))v_\delta(\xi - L)\|_{L^2(\mathbb{R})} \to 0 \quad \text{as } L \to \infty, \tag{3.13}$$

where we used the fact that $f'(U) - f'(0)$ decays exponentially to 0 as $\xi \to \infty$. Now,

$$\|(z-\mathcal{L}_l)v_{\delta,L}\|_{L^2(\mathbb{R})} \leq \left\|(z-\mathcal{L}_l^+)v_\delta\right\|_{L^2(\mathbb{R})} + \left\|A^+ v_{\delta,L}\right\|_{L^2(\mathbb{R})},$$

where we used the triangle inequality and (3.12). By (3.11) and (3.13), the right-hand side of the above can be made arbitrarily small by taking δ sufficiently small and L sufficiently large. Together with (3.12), this establishes (3.10).

We may show that $z \in \sigma_{L^2(\mathbb{R})}(\mathcal{L}_l^-)$ implies $z \in \sigma_{L^2(\mathbb{R})}(\mathcal{L}_l)$ in a similar fashion. ∎

LEMMA 3.4.

$$\mathbb{C}\backslash\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l) \subset \mathbb{C}\backslash\sigma_{BUC(\mathbb{R})}(\mathcal{L}_l). \tag{3.14}$$

Proof. Step 1. Fix $z \in \mathbb{C}\backslash\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l)$ and $v \in BUC(\mathbb{R})$ arbitrarily. We decompose v into a sum of $L^2(\mathbb{R})$-functions by a partition of unity. Choose a function $\varphi_0 \in C_0^\infty(\mathbb{R})$ supported on $[-1,1]$ such that

$$\sum_{n=-\infty}^{\infty} \varphi_n(\xi) \equiv 1, \quad \varphi_n(\xi) := \varphi_0(\xi - n),$$

and decompose v as

$$v(\xi) = \sum_{n=-\infty}^{\infty} v_n(\xi), \quad v_n(\xi) := \varphi_n(\xi) v(\xi).$$

Since $z \in \mathbb{C}\backslash\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l)$, we have $(z-\mathcal{L}_l)^{-1}$ as a resolvent on $L^2(\mathbb{R})$. From Lemma 3.3, we also have $(z-\mathcal{L}_l^\pm)^{-1}$ as resolvents on $L^2(\mathbb{R})$. In what follows, we shall estimate $(z-\mathcal{L}_l)^{-1}v_n$, which we rewrite as

$$(z-\mathcal{L}_l)^{-1} v_n = \begin{cases} \left(z-\mathcal{L}_l^+\right)^{-1} v_n + (z-\mathcal{L}_l)^{-1} A^+ \left(z-\mathcal{L}_l^+\right)^{-1} v_n & \text{for } n \geq 1, \\ \left(z-\mathcal{L}_l^-\right)^{-1} v_n + (z-\mathcal{L}_l)^{-1} A^- \left(z-\mathcal{L}_l^-\right)^{-1} v_n & \text{for } n < 0. \end{cases} \tag{3.15}$$

Step 2. We estimate $w_n := (z-\mathcal{L}_l^+)^{-1} v_n$ with $n \geq 1$. We have

$$w_n = g * v_n, \quad g(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{ik\xi}}{q(k)} dk, \; q(k) = (z + Q(k,l) - ick - f'(0)). \tag{3.16}$$

Here there exists a positive constant C_1 such that

$$|g(\xi)| \leq \frac{C_1}{1+\xi^2}, \quad \xi \in \mathbb{R}. \tag{3.17}$$

To prove (3.17), we estimate $q(k)$ in (3.16). Since $z \in \mathbb{C}\backslash\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l^+)$ from Lemma 3.3, we find that $q(k) \neq 0$ holds for any $k \in \mathbb{R}$. Combining this fact with (2.5), it is easily found that there exists a positive constant C_2 that depends on z and l such that

$$|q(k)| \geq C_2(1+k^2), \quad k \in \mathbb{R}. \tag{3.18}$$

This implies that $1/|q| \in L^1(\mathbb{R})$ and hence that $g \in L^\infty(\mathbb{R})$. In addition, integrating by parts, we have

$$\left|\xi^2 g(\xi)\right| = \left|\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\partial_k^2(e^{ik\xi})}{q(k)} dk\right| \leq \frac{1}{2\pi}\int_{\mathbb{R}} \left|\partial_k^2\left(\frac{1}{q(k)}\right)\right| dk.$$

From (3.18) and (2.6), there exists a positive constant C_3 such that

$$\left|\partial_k^2\left(\frac{1}{q(k)}\right)\right| = \left|-\frac{\partial_k^2 Q(k,l)}{q(k)^2} + \frac{2(\partial_k Q(k,l)-ic)^2}{q(k)^3}\right| \le \frac{C_3}{(1+k^2)^2}.$$

This implies that $\partial_k^2(1/q) \in L^1(\mathbb{R})$ and hence that $\sup_{\xi\in\mathbb{R}}|\xi^2 g(\xi)| < \infty$ holds. Thus, we obtain (3.17). Consequently, from (3.16), w_n with $n \ge 1$ is estimated as

$$|w_n(\xi)| = |(g * v_n)(\xi)| \le \int_{n-1}^{n+1} \frac{C_1\|v\|_{L^\infty(\mathbb{R})}}{1+(\xi-s)^2} ds. \tag{3.19}$$

Thus, $\widetilde{u}_+ = \sum_{n=1}^\infty (z-\mathcal{L}_l^+)^{-1} v_n = \sum_{n=1}^\infty w_n$ is estimated as

$$\|\widetilde{u}_+\|_{L^\infty(\mathbb{R})} \le \sup_{\xi\in\mathbb{R}} \int_0^\infty \frac{2C_1\|v\|_{L^\infty(\mathbb{R})}}{1+(\xi-s)^2} ds \le \pi C_1 \|v\|_{L^\infty(\mathbb{R})}. \tag{3.20}$$

Given that $v \in BUC(\mathbb{R})$, we see that $\widetilde{u}_+ \in BUC(\mathbb{R})$.

Step 3. We estimate $(z-\mathcal{L}_l)^{-1}A^+(z-\mathcal{L}_l^+)^{-1}v_n$ with $n \ge 1$. By the definition of A^+, there exist positive constants C_f and ν depending only on the nonlinearity f such that

$$\begin{aligned}\left|\sum_{n=1}^\infty A^+(z-\mathcal{L}_l^+)^{-1}v_n\right| &\le \left|\sum_{n=1}^\infty (f'(U)-f'(0))w_n\right| \\ &\le C_f \min\left\{1, e^{-\nu\xi}\right\} \int_0^\infty \frac{2C_1\|v\|_{L^\infty(\mathbb{R})}}{1+(\xi-s)^2} ds \\ &\le C_f \min\left\{1, e^{-\nu\xi}\right\} \cdot 2C_1\|v\|_{L^\infty(\mathbb{R})}\left(\frac{\pi}{2} + \arctan\xi\right)\end{aligned}$$

for each $\xi \in \mathbb{R}$. Thus, we find that there exists a positive constant C_4 such that

$$\left\|\sum_{n=1}^\infty A^+(z-\mathcal{L}_l^+)^{-1}v_n\right\|_{L^2(\mathbb{R})} \le C_4\|v\|_{L^\infty(\mathbb{R})}.$$

Since $\mathcal{L}_l$ is a closed operator on $L^2(\mathbb{R})$ with a domain $H^2(\mathbb{R})$, there exists a positive constant C_5 such that

$$\left\|\sum_{n=1}^\infty (z-\mathcal{L}_l)^{-1}A^+(z-\mathcal{L}_l^+)^{-1}v_n\right\|_{H^2(\mathbb{R})} \le C_5\|v\|_{L^\infty(\mathbb{R})}.$$

By the Sobolev inequality, there exists a positive constant C_6 such that

$$\left\|\sum_{n=1}^\infty (z-\mathcal{L}_l)^{-1}A^+(z-\mathcal{L}_l^+)^{-1}v_n\right\|_{L^\infty(\mathbb{R})} \le C_6\|v\|_{L^\infty(\mathbb{R})} \tag{3.21}$$

Consider

$$u_+ = \widetilde{u}_+ + \sum_{n=1}^\infty (z-\mathcal{L}_l)^{-1}A^+(z-\mathcal{L}_l^+)^{-1}v_n = \sum_{n=1}^\infty (z-\mathcal{L}_l)^{-1}v_n, \tag{3.22}$$

where we used (3.15). Given (3.20) and (3.21), $u_+ \in BUC(\mathbb{R})$ and

$$\|u_+\|_{BUC(\mathbb{R})} \le C_7\|v\|_{BUC(\mathbb{R})}. \tag{3.23}$$

In the same way, we may also conclude that

$$u_- = \sum_{n=-\infty}^{0} (z - \mathcal{L}_l)^{-1} v_n \in BUC(\mathbb{R}), \quad \|u_-\|_{BUC(\mathbb{R})} \leq C_8 \|v\|_{BUC(\mathbb{R})}. \tag{3.24}$$

Step 4. We define the function u by

$$u = u_+ + u_- = \sum_{n=-\infty}^{\infty} (z - \mathcal{L}_l)^{-1} v_n. \tag{3.25}$$

By (3.23) and (3.24), the above function u is in $BUC(\mathbb{R})$ and satisfies

$$(z - \mathcal{L}_l) u = v \quad \text{and} \quad \|u\|_{BUC(\mathbb{R})} \leq C \|v\|_{BUC(\mathbb{R})}. \tag{3.26}$$

It is also clear, by construction, that the above u is the unique element in $BUC(\mathbb{R})$ that satisfies the above. Therefore,

$$u = (z - \mathcal{L}_l)^{-1} v, \ \left\|(z - \mathcal{L}_l)^{-1} v\right\|_{BUC(\mathbb{R})} \leq C \|v\|_{BUC(\mathbb{R})}. \tag{3.27}$$

This completes the proof. ∎

LEMMA 3.5. *Suppose*

$$z \in \bigcap_{l \in (2\pi/d)\mathbb{Z}} \left(\mathbb{C} \backslash \sigma_{BUC(\mathbb{R})}(\mathcal{L}_l)\right). \tag{3.28}$$

Then there is a constant C that does not depend on l such that

$$\left\|(z - \mathcal{L}_l)^{-1}\right\|_{\mathcal{L}(L^\infty(\mathbb{R}))} \leq \frac{C}{1 + l^2}, \quad l \in (2\pi/d)\mathbb{Z}. \tag{3.29}$$

Proof. Step 1. In what follows, we let $Q^{-1} = Q^{-1}(\cdot, l)$ and $Q_{i,e}^{-1} = Q_{i,e}^{-1}(\cdot, l)$. For any $u \in L^\infty(\mathbb{R})$, we have $\Lambda_l^{-1} u = (\mathcal{F}_\xi^{-1} Q^{-1}) * u$. Then, since $Q^{-1} = Q_i^{-1} + Q_e^{-1}$, we have

$$\Lambda_l^{-1} u = \left(\mathcal{F}_\xi^{-1} Q_i^{-1}\right) * u + \left(\mathcal{F}_\xi^{-1} Q_e^{-1}\right) * u. \tag{3.30}$$

Here we compute $\mathcal{F}_\xi^{-1} Q_i^{-1}$ as

$$\begin{aligned}
(\mathcal{F}_\xi^{-1} Q_i^{-1})(\xi, l) &= \frac{1}{2\pi} \int_{\mathbb{R}} Q_i^{-1}(k, l) \exp(ik\xi) dk \\
&= \frac{1}{2\pi |l|} \int_{\mathbb{R}} Q_i^{-1}(k, 1) \exp(ikl\xi) dk = \frac{1}{|l|} K_{Q_i}(l\xi), \\
K_{Q_i}(\xi) &= \frac{1}{2\pi} \int_{\mathbb{R}} Q_i^{-1}(k, 1) \exp(ik\xi) dk.
\end{aligned}$$

The function $Q_i(k, 1)$ is a positive quadratic polynomial in k, and thus its Fourier transform K_{Q_i} decays exponentially as $|\xi| \to \infty$. In particular, the L^1 norm of K_{Q_i} is finite:

$$\left\|\mathcal{F}_\xi^{-1} Q_i^{-1}\right\|_{L^1(\mathbb{R})} = \frac{1}{l^2} \|K_{Q_i}\|_{L^1(\mathbb{R})}.$$

Likewise, we have

$$\left\|\mathcal{F}_\xi^{-1} Q_e^{-1}\right\|_{L^1(\mathbb{R})} = \frac{1}{l^2}\|K_{Q_e}\|_{L^1(\mathbb{R})}, \quad K_{Q_e}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} Q_e^{-1}(k,1)\exp(ik\xi)dk. \tag{3.31}$$

We thus have

$$\|\Lambda_l^{-1}u\|_{L^\infty(\mathbb{R})} \le \frac{C_1}{l^2}\|u\|_{L^\infty(\mathbb{R})}, \quad C_1 = \|K_{Q_i}\|_{L^1(\mathbb{R})} + \|K_{Q_e}\|_{L^1(\mathbb{R})}. \tag{3.32}$$

Step 2. Define $\mathcal{R} := z - \mathcal{L}_l - \Lambda_l$, namely

$$\mathcal{R}\Lambda_l^{-1}u = \left(z - \frac{\partial}{\partial\xi} - f'(U)\right)\Lambda_l^{-1}u.$$

Note that

$$\left\|\frac{\partial}{\partial\xi}\mathcal{F}_\xi^{-1}Q_{i,e}^{-1}\right\|_{L^1(\mathbb{R})} = \frac{1}{|l|}\left\|\frac{\partial K_{Q_{i,e}}}{\partial\xi}\right\|_{L^1(\mathbb{R})}.$$

We thus have

$$\begin{aligned}\left\|\frac{\partial}{\partial\xi}\Lambda_l^{-1}u\right\|_{L^\infty(\mathbb{R})} &= \left\|\frac{\partial}{\partial\xi}\mathcal{F}_\xi^{-1}Q_i^{-1} * u + \frac{\partial}{\partial\xi}\mathcal{F}_\xi^{-1}Q_e^{-1} * u\right\|_{L^\infty(\mathbb{R})} \\ &\le \frac{1}{|l|}\left(\left\|\frac{\partial K_{Q_i}}{\partial\xi}\right\|_{L^1(\mathbb{R})} + \left\|\frac{\partial K_{Q_e}}{\partial\xi}\right\|_{L^1(\mathbb{R})}\right)\|u\|_{L^\infty(\mathbb{R})}.\end{aligned}$$

Thus, by using (3.32), we have

$$\begin{aligned}&\|\mathcal{R}\Lambda_l^{-1}u\|_{L^\infty(\mathbb{R})} \\ &\quad\le \left(\frac{C_1\|z - f'(U)\|_{L^\infty(\mathbb{R})}}{l^2} + \frac{1}{|l|}\left(\left\|\frac{\partial K_{Q_i}}{\partial\xi}\right\|_{L^1(\mathbb{R})} + \left\|\frac{\partial K_{Q_e}}{\partial\xi}\right\|_{L^1(\mathbb{R})}\right)\right)\|u\|_{L^\infty(\mathbb{R})} \\ &\quad\le \frac{C_2}{|l|}\|u\|_{L^\infty(\mathbb{R})},\end{aligned} \tag{3.33}$$

where C_2 is a positive constant depending only on $A_{i,e}$, f, and z.
Step 3. Since we have

$$\Lambda_l^{-1} - (z - \mathcal{L}_l)^{-1} = (z - \mathcal{L}_l)^{-1}\mathcal{R}\Lambda_l^{-1},$$

it holds from (3.32) and (3.33) that

$$\begin{aligned}\|(z - \mathcal{L}_l)^{-1}u\|_{L^\infty(\mathbb{R})} &\le \|\Lambda_l^{-1}u\|_{L^\infty(\mathbb{R})} + \|(z - \mathcal{L}_l)^{-1}\mathcal{R}\Lambda_l^{-1}u\|_{L^\infty(\mathbb{R})} \\ &\le \left(\frac{C_1}{l^2} + \frac{C_2}{|l|}\left\|(z - \mathcal{L}_l)^{-1}\right\|_{\mathcal{L}(L^\infty(\mathbb{R}))}\right)\|u\|_{L^\infty(\mathbb{R})}.\end{aligned} \tag{3.34}$$

We thus see that

$$\left\|(z - \mathcal{L}_l)^{-1}\right\|_{\mathcal{L}(L^\infty(\mathbb{R}))} \le \left(\frac{C_1}{l^2} + \frac{C_2}{|l|}\left\|(z - \mathcal{L}_l)^{-1}\right\|_{\mathcal{L}(L^\infty(\mathbb{R}))}\right). \tag{3.35}$$

If $|l| \geq 2C_2$, we have

$$\left\|(z-\mathcal{L}_l)^{-1}\right\|_{\mathcal{L}(L^\infty(\mathbb{R}))} \leq \frac{2C_1}{l^2}. \tag{3.36}$$

The result follows since there are only a finite number of l such that $|l| < 2C_2$. ◻

Now we are ready to prove Proposition 3.2.

Proof of Proposition 3.2. The second set inclusion in (3.5) follows immediately from Lemma 3.4. We now show the first set inclusion:

$$\mathbb{C}\backslash\sigma_{BUC(\mathbb{R}\times S^1_d)}(\mathcal{L}) \supset \bigcap_{k\in\mathbb{Z}} \mathbb{C}\backslash\sigma_{BUC(\mathbb{R})}(\mathcal{L}_{2\pi k/d}). \tag{3.37}$$

Fix any $z \in \mathbb{C}$ satisfying

$$z \in \bigcap_{k\in\mathbb{Z}} \mathbb{C}\backslash\sigma_{BUC(\mathbb{R})}(\mathcal{L}_{2\pi k/d}).$$

For any $v \in BUC(\mathbb{R})$, decompose v in terms of a Fourier series as in (2.27). Define the operator

$$\mathcal{A}v = \sum_{l\in(2\pi/d)\mathbb{Z}} \left((z-\mathcal{L}_l)^{-1}v_l\right)\exp(il\eta). \tag{3.38}$$

By Lemma 3.5, we see as follows that the above operator $\mathcal{A}$ is a bounded operator on $BUC(\mathbb{R}\times S^1_d)$:

$$\begin{aligned}\|\mathcal{A}v\|_{BUC(\mathbb{R}\times S^1_d)} &\leq \sum_{l\in(2\pi/d)\mathbb{Z}} \left\|(z-\mathcal{L}_l)^{-1}v_l\right\|_{BUC(\mathbb{R})} \\ &\leq \left(\sum_{l\in(2\pi/d)\mathbb{Z}} \frac{C}{1+l^2}\right) \sup_{l\in(2\pi/d)\mathbb{Z}} \|v_l\| \leq C_1\|v\|_{BUC(\mathbb{R}\times S^1_d)}.\end{aligned}$$

We now show that $\mathcal{A}$ is the inverse of $z-\mathcal{L}$.

First, we show that if $v_l \in BUC^\infty(\mathbb{R})$, then $w_l = (z-\mathcal{L}_l)^{-1}v_l \in BUC^\infty(\mathbb{R})$. To see this, note that

$$(z-\mathcal{L}_l)w_l = zv_l + \Lambda_l v_l - c\frac{\partial v_l}{\partial \xi} - f'(U)v_l = w_l. \tag{3.39}$$

Using (2.38), we see that

$$\frac{\partial^2 w_l}{\partial \xi^2} + \beta\frac{\partial w_l}{\partial \xi} = \mathcal{B}w_l + \delta v_l, \tag{3.40}$$

where β and δ are constants and $\mathcal{B}$ is an operator that maps $BUC^k(\mathbb{R}), k = 0, 1, 2, \ldots,$ to itself. Given that $w_l \in BUC(\mathbb{R})$, $\partial w_l/\partial\xi + \beta w_l$ is in $BUC^1(\mathbb{R})$, which implies that w_l is in $BUC^2(\mathbb{R})$. Repeating this argument, we see that $w_l \in BUC^k(\mathbb{R})$ for all $k \in \mathbb{N}$.

Take any $v \in BUC(\mathbb{R}\times S^1_d)$. Now consider a sequence of smooth functions v_n that converges to v in $BUC(\mathbb{R}\times S^1_d)$ as $n \to \infty$. We may even take v_n to be band limited, in the sense that each v_n has a finite Fourier series expansion:

$$v_n(\xi,\eta) = \sum_{l=2\pi k/d, |k|\leq N_n} v_{n,l}(\xi)\exp(il\eta), \tag{3.41}$$

where N_n is a finite number that depends on n. We have

$$\mathcal{A}v_n = \sum_{l=2\pi k/d, |k|\leq N_n} \left((z-\mathcal{L}_l)^{-1} v_{n,l}\right) \exp(il\eta). \tag{3.42}$$

Note that, since $v_l \in BUC^\infty(\mathbb{R})$, $(z-\mathcal{L}_l)^{-1}v_l \in BUC^\infty(\mathbb{R})$. Thus, (2.39) of Proposition 2.11 implies that

$$(z-\mathcal{L})\mathcal{A}v_n = \sum_{l=2\pi k/d, |k|\leq N_n} (z-\mathcal{L}_l)\left((z-\mathcal{L}_l)^{-1} v_{n,l}\right) \exp(il\eta) = v_n. \tag{3.43}$$

Let $n \to \infty$ in the above. Since $\mathcal{A}$ is a bounded operator from $BUC(\mathbb{R}\times S^1_d)$ to $BUC(\mathbb{R}\times S^1_d)$, $\mathcal{A}v_n$ converges to $\mathcal{A}v$ in $BUC(\mathbb{R}\times S^1_d)$. Since $\mathcal{L}$ is a closed operator on $BUC(\mathbb{R}\times S^1_d)$ by Proposition 3.1, we see that

$$(z-\mathcal{L})\mathcal{A}v = v \quad \text{for all } v \in BUC(\mathbb{R}\times S^1_d). \tag{3.44}$$

Next, take any $w \in \mathcal{D}(\mathcal{L})$, the domain of $\mathcal{L}$. Note first that there is a sequence of smooth functions w_n such that $w_n \to w$ in $\mathcal{D}(\mathcal{L})$. This follows from the properties of $T(t)$, the semigroup generated by Λ. Since $T(t)$ is an analytic semigroup on $BUC(\mathbb{R}\times S^1_d)$, it is also an analytic semigroup on $\mathcal{D}(\Lambda) = \mathcal{D}(\mathcal{L})$ (see Proposition 3.1). Thus, $T(t)u \to u$ as $t \to 0$ in $\mathcal{D}(\Lambda)$ if $u \in \mathcal{D}(\Lambda)$. Given (2.8) of Lemma 2.3, $T(t)u$ is smooth. Expand w_n as follows:

$$w_n(\xi,\eta) = \sum_{l=(2\pi k/d)\mathbb{Z}} w_{n,l}(\xi) \exp(il\eta). \tag{3.45}$$

By Proposition 2.11, we have

$$(z-\mathcal{L})w_n = \sum_{l=(2\pi k/d)\mathbb{Z}} ((z-\mathcal{L}_l)w_{n,l}) \exp(il\eta). \tag{3.46}$$

By construction, $(z-\mathcal{L})w_n \in BUC(\mathbb{R}\times S^1_d)$. Thus,

$$\mathcal{A}(z-\mathcal{L})w_n = \sum_{l=(2\pi k/d)\mathbb{Z}} \left((z-\mathcal{L}_l)^{-1}(z-\mathcal{L}_l)w_{n,l}\right) \exp(il\eta) = w_n. \tag{3.47}$$

Taking the limit as $n \to \infty$ in the above, we see that:

$$\mathcal{A}(z-\mathcal{L})w = w \quad \text{for all } w \in \mathcal{D}(\mathcal{L}). \tag{3.48}$$

From (3.44) and (3.48), we see that $\mathcal{A} = (z-\mathcal{L})^{-1}$ and that z is in the resolvent set of $\mathcal{L}$ as an operator on $BUC(\mathbb{R}\times S^1_d)$. ∎

3.2. Instability criteria. We now prove some results on the point spectrum which will be useful in proving planar front instabilities. First, we show some results estimating the spectrum and the essential spectrum of $\mathcal{L}_l$. We let

$$\sigma^+_{L^2(\mathbb{R})}(\mathcal{L}_l) = \sigma_{L^2(\mathbb{R})}(\mathcal{L}_l) \cap \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}, \tag{3.49}$$

$$\sigma^+_{BUC(\mathbb{R})}(\mathcal{L}_l) = \sigma_{BUC(\mathbb{R})}(\mathcal{L}_l) \cap \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\} \tag{3.50}$$

and define the constants

$$f_{\min} = \min_{0\leq s\leq 1} f'(s), \quad f_{\max} = \max_{0\leq s\leq 1} f'(s),$$
$$\bar{f} = \frac{f_{\min}+f_{\max}}{2}, \quad f_\Delta = \frac{f_{\max}-f_{\min}}{2}.$$

Then the following holds, which is presented as Proposition 2.3 in [27].

PROPOSITION 3.6 (see [27]). *Define the set* $\widehat{S}_l \subset \mathbb{C}$ *by*

$$\widehat{S}_l = \left\{ z \in \mathbb{C} \mid z = ics - Q_\theta(s,l) + \bar{f},\ s \in \mathbb{R} \right\}.$$

Then the spectrum $\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l)$ *satisfies*

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l) \subset \left\{ z \in \mathbb{C} \mid \operatorname{dist}(z, \widehat{S}_l) \le f_\Delta \right\}, \tag{3.51}$$

where $\operatorname{dist}(z, \widehat{S}_l)$ *is the distance between the point* $z \in \mathbb{C}$ *and the set* $\widehat{S}_l$.

Remark 3.7. From Proposition 3.6, we find (i) $\sigma^+_{L^2(\mathbb{R})}(\mathcal{L}_l)$ is uniformly bounded in $l \in \mathbb{R}$; (ii) $\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l)$ lies in the left-half of the complex plane when $|l|$ is sufficiently large, or, equivalently, $\sigma^+_{L^2(\mathbb{R})}(\mathcal{L}_l)$ is empty if $|l|$ is sufficiently large. Indeed, since $Q_\theta(s,l) \ge \lambda(s^2 + l^2)$ holds for some positive λ, we have

$$\widehat{S}_l \subset \left\{ x + iy \in \mathbb{C} \mid y = cs,\ x \le -\lambda(s^2 + l^2) + \bar{f},\ s \in \mathbb{R} \right\}.$$

Outline of the proof of Proposition 3.6. The operator $\mathcal{L}_l$ can be written as

$$\mathcal{L}_l v = \overline{\mathcal{L}_l} v + (f'(U) - \bar{f})v, \quad \overline{\mathcal{L}_l} v = -\Lambda_l v + c\frac{\partial v}{\partial \xi} + \bar{f} v. \tag{3.52}$$

Using the Fourier transform, it is easily seen that the spectral set of $\overline{\mathcal{L}_l}$ is $\widehat{S}_l$ and the norm of the resolvent of $\overline{\mathcal{L}_l}$ are given by

$$\left\| (z - \overline{\mathcal{L}_l})^{-1} \right\|_{\mathcal{L}(L^2(\mathbb{R}))} = \operatorname{dist}\left(z, \widehat{S}_l\right). \tag{3.53}$$

The operator $\mathcal{L}_l$ is a perturbation of $\overline{\mathcal{L}_l}$ by the multiplication operator $f'(U) - \bar{f}$ whose operator norm is given by f_Δ. The result follows by a standard Neumann series argument. ∎

Next, we examine the essential and point spectra of $\mathcal{L}_l$ as an operator on $L^2(\mathbb{R})$. For our purposes, it is convenient to define the essential and point spectra as follows.

DEFINITION 3.8 (essential and point spectra). *Let* $\mathcal{A}$ *be a densely defined closed operator on a Hilbert space* $\mathcal{H}$ *with domain* $D(\mathcal{A})$, *and let* $\sigma(\mathcal{A}) \subset \mathbb{C}$ *denote the spectrum of* $\mathcal{A}$. *A complex value* $z \in \sigma(\mathcal{A})$ *belongs to the point spectrum of* $\mathcal{A}$, *if* $z - \mathcal{A}$ *is a Fredholm operator with index* 0 *as a bounded operator from* $D(\mathcal{A})$ *to* $\mathcal{H}$. *Otherwise,* $z \in \sigma(\mathcal{A})$ *is in the essential spectrum of* $\mathcal{A}$.

We shall denote the essential and point spectra of $\mathcal{A}$ by $\sigma^{\mathrm{ess}}_{L^2(\mathbb{R})}(\mathcal{A})$ and $\sigma^{\mathrm{pt}}_{L^2(\mathbb{R})}(\mathcal{A})$, respectively.

To estimate the essential spectrum of $\mathcal{L}_l$ on $L^2(\mathbb{R})$, we define the constants

$$m_{\min} = \min\{f'(0), f'(1)\}, \quad m_{\max} = \max\{f'(0), f'(1)\},$$
$$\overline{m} = \frac{m_{\max} + m_{\min}}{2}, \quad m_\Delta = \frac{m_{\max} - m_{\min}}{2},$$

where we note that $m_{\max}, m_{\min}, \overline{m}$ are all negative and m_Δ is positive.

LEMMA 3.9. *Define the set*

$$S_l = \{ z \in \mathbb{C} \mid z = ics - Q_\theta(s,l) + \overline{m},\ s \in \mathbb{R} \}.$$

Then the essential spectrum of $\mathcal{L}_l$ *on* $L^2(\mathbb{R})$, *which is denoted by* $\sigma^{\mathrm{ess}}_{L^2(\mathbb{R})}(\mathcal{L}_l)$, *satisfies*

$$\sigma^{\mathrm{ess}}_{L^2(\mathbb{R})}(\mathcal{L}_l) \subset \{ z \in \mathbb{C} \mid \operatorname{dist}(z, S_l) \le m_\Delta \}. \tag{3.54}$$

Proof. Define the following function g_0 that interpolates monotonically between $f'(0)$ and $f'(1)$:

$$g_0(\xi) = \overline{m} + m_\Delta \tanh(\xi).$$

Now let

$$f'(U(\xi)) = g_0(\xi) + g_1(\xi).$$

Since $U(\xi)$ decays exponentially to 0 and 1 as $\xi \to -\infty$ and $\xi \to \infty$, the function $g_1(\xi)$ is a smooth function that decays exponentially to 0 as $\xi \to \pm\infty$. With this, we have

$$\mathcal{L}_l v = \mathcal{Q}v + \mathcal{R}v, \quad \mathcal{Q}v = -\Lambda_l u + c\frac{\partial u}{\partial \xi} + g_0 u, \quad \mathcal{R}u = g_1 u. \tag{3.55}$$

Let us first examine the spectrum of $\mathcal{Q}$. Decompose $\mathcal{Q}$ further as follows:

$$\mathcal{Q} = \mathcal{Q}_0 + \mathcal{Q}_1, \quad \mathcal{Q}_0 u = -\Lambda_l u + c\frac{\partial u}{\partial \xi} + \overline{m}u, \quad \mathcal{Q}_1 u = (g_0 - \overline{m})u.$$

We may now apply exactly the same argument that we used in proving Proposition 3.6 to show that

$$\sigma_{L^2(\mathbb{R})}(\mathcal{Q}) \subset \{z \in \mathbb{C} \mid \operatorname{dist}(z, S_l) \le m_\Delta\}. \tag{3.56}$$

Now we consider the Fredholm index of the operator

$$zI - \mathcal{L}_l = zI - \mathcal{Q} - \mathcal{R} \quad \text{with } z \in \mathbb{C}\backslash\sigma_{L^2(\mathbb{R})}(\mathcal{Q}),$$

viewed as an operator from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Since z is in the resolvent set of $\mathcal{Q}$, $(zI - \mathcal{Q})^{-1}$ exists, and the Fredholm index of the above is equal to that of

$$I - \mathcal{R}(zI - \mathcal{Q})^{-1},$$

seen as a bounded operator on $L^2(\mathbb{R})$. Since $(zI - \mathcal{Q})^{-1}$ maps $L^2(\mathbb{R})$ to $H^2(\mathbb{R})$ and the function g_1 decays exponentially as $\xi \to \pm\infty$, we see that $\mathcal{R}(zI - \mathcal{Q})^{-1}$ is a compact operator on $L^2(\mathbb{R})$. Therefore, the Fredholm index of the above, and hence of $zI - \mathcal{L}_l$, is 0 when z is in the resolvent set of $\mathcal{Q}$. From this, we conclude that $\sigma^{\text{ess}}_{L^2(\mathbb{R})}(\mathcal{L}_l) \subset \sigma_{L^2(\mathbb{R})}(\mathcal{Q})$, and hence (3.54) follows. □

Now we are ready to prove Proposition 3.10.

The aim of this subsection is to prove Proposition 3.10 below. We introduce the notion

$$\sigma^{>}_{L^2(\mathbb{R})}(\mathcal{L}_l) = \sigma_{L^2(\mathbb{R})}(\mathcal{L}_l) \cap \{z \in \mathbb{C} \mid \operatorname{Re} z > m_{\max}\}, \tag{3.57}$$

$$\sigma^{>}_{BUC(\mathbb{R})}(\mathcal{L}_l) = \sigma_{BUC(\mathbb{R})}(\mathcal{L}_l) \cap \{z \in \mathbb{C} \mid \operatorname{Re} z > m_{\max}\}, \tag{3.58}$$

where we note that these sets are possibly empty. Then the following hold.

PROPOSITION 3.10. *For each $l = 2\pi k/d$ with $k \in \mathbb{Z}$, one has*

$$\sigma^{>}_{BUC(\mathbb{R})}(\mathcal{L}_l) = \sigma^{>}_{L^2(\mathbb{R})}(\mathcal{L}_l). \tag{3.59}$$

Moreover, $\sigma^{>}(\mathcal{L}_l)$ consists of a finite number of points, and each point in $\sigma^{>}(\mathcal{L}_l)$ is an eigenvalue in $L^2(\mathbb{R})$ and in $BUC(\mathbb{R})$.

Proof of Proposition 3.10. Note first that Proposition 3.2 gives

$$\sigma^{>}_{BUC(\mathbb{R})}(\mathcal{L}_l) \subset \sigma^{>}_{L^2(\mathbb{R})}(\mathcal{L}_l). \tag{3.60}$$

From Lemma 3.9, we find that

$$\sigma^{\text{ess}}_{L^2(\mathbb{R})}(\mathcal{L}_l) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z \leq \overline{m} + m_\Delta\}, \tag{3.61}$$

where $\overline{m} + m_\Delta = m_{\max} < 0$. Thus, $\sigma^{>}_{L^2(\mathbb{R})}(\mathcal{L}_l)$ consists of point spectra, and thus all points in $\sigma^{>}_{L^2(\mathbb{R})}(\mathcal{L}_l)$ are eigenvalues. For any $z \in \sigma^{>}_{L^2(\mathbb{R})}(\mathcal{L}_l)$, therefore, we have:

$$\mathcal{L}_l v = zv,$$

where $v \in H^2(\mathbb{R})$. Then the Sobolev embedding theorem gives $v \in BUC(\mathbb{R})$ and hence $z \in \sigma^{>}_{BUC(\mathbb{R})}(\mathcal{L}_l)$. Thus, we obtain

$$\sigma^{>}_{L^2(\mathbb{R})}(\mathcal{L}_l) \subset \sigma^{>}_{BUC(\mathbb{R})}(\mathcal{L}_l).$$

This implies (3.59).

Since $\sigma^{>}_{L^2(\mathbb{R})}(\mathcal{L}_l)$ is bounded by Proposition 3.6, if it consists of an infinite number of points, there is an accumulation point z_*. By (3.61), z_* must belong to the point spectrum, but any point in the point spectrum must be an isolated point (Theorem 7 of [30]; see also [18]). Thus, $\sigma^{>}_{L^2(\mathbb{R})}(\mathcal{L}_l)$ consists of a finite number of points. ∎

3.3. Auxiliary result for the spectral properties of $\mathcal{L}$ and $\mathcal{L}_l$. In this subsection, we provide some auxiliary results for later discussions. Let $\sigma^{+}_{L^2(\mathbb{R})}(\mathcal{L}_l)$ and $\sigma^{+}_{BUC(\mathbb{R})}(\mathcal{L}_l)$ be defined as in (3.49)–(3.50). Note by Proposition 3.10 that the two sets coincide.

LEMMA 3.11. *Suppose that, for a direction* $\mathbf{n} \in S^1$*, the planar front on* $\mathbb{R}^2$ *is spectrally unstable in the sense of Definition* 1.3*, namely that there exists a constant* $l_* \in \mathbb{R}$ *such that*

$$\sigma^{+}_{L^2(\mathbb{R})}(\mathcal{L}_{l_*}) \neq \emptyset.$$

Then there exists a positive constant δ *such that*

$$\sigma^{+}_{L^2(\mathbb{R})}(\mathcal{L}_l) \neq \emptyset$$

holds for all $l \in [l_* - \delta, l_* + \delta]$.

Proof. Take any point $z_* \in \sigma^{+}_{L^2(\mathbb{R})}(\mathcal{L}_{l_*})$. Given Proposition 3.10, $\sigma^{+}_{L^2(\mathbb{R})}(\mathcal{L}_{l_*})$ is a finite set and thus z_* is an isolated point of the spectrum of $\mathcal{L}_{l_*}$. There is thus a circular contour $\mathcal{C}$ of radius r centered at z_* such that

$$\mathcal{C} \subset \{z \in \mathbb{C} | \operatorname{Re} z > 0\} \backslash \sigma_{L^2(\mathbb{R})}(\mathcal{L}_{l_*}).$$

We may even take r small enough so that z_* is the only point of the spectrum inside of $\mathcal{C}$. Consider the Dunford integral

$$\mathcal{P}_l = \frac{1}{2\pi i} \int_{\mathcal{C}} (z - \mathcal{L}_l)^{-1} dz.$$

When $l = l_*$, $\mathcal{P}_{l_*}$ is the spectral projection for $z = z_*$, and thus $\mathcal{P}_{l_*} \neq 0$. Since $\mathcal{L}_l$ depends continuously as an operator from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$, $\mathcal{P}_l \neq 0$ for l sufficiently close to l_*. This implies that $\mathcal{L}_l$ must have a point in the spectrum inside $\mathcal{C}$ for all values of l sufficiently close to l_*. ∎

We now study the simplicity of eigenvalues.

LEMMA 3.12. *Let $z \in \sigma^{>}_{L^2(\mathbb{R})}(\mathcal{L}_l)$ be an algebraically simple eigenvalue of $\mathcal{L}_l$ in $L^2(\mathbb{R})$. Then z is an algebraically simple eigenvalue in $BUC(\mathbb{R})$.*

Proof. Let z be as in the statement of the lemma, and let φ be the corresponding eigenfunction. Define the operator $\mathcal{L}_l^*$ as

$$\mathcal{L}_l^* u = -\Lambda_l u - c\frac{\partial u}{\partial \xi} + f'(U)u. \tag{3.62}$$

If $z \in \sigma^{>}_{L^2(\mathbb{R})}(\mathcal{L}_l)$, then $\bar{z}$ is in the spectrum of $\mathcal{L}_l^*$ and is an algebraically simple eigenvalue in $L^2(\mathbb{R})$ with eigenfunction φ^*. The assumption of simplicity implies that the $L^2(\mathbb{R})$ inner product of φ and φ^* satisfies $\langle \varphi, \varphi^* \rangle_{\mathbb{R}} \neq 0$, which we may normalize so that $\langle \varphi, \varphi^* \rangle_{\mathbb{R}} = 1$. Note that the φ and φ^* satisfy a 4th order differential equation in ξ and that $f'(U)$ converges exponentially to a constant as $|\xi| \to \infty$. Since φ and φ^* are in $L^2(\mathbb{R})$, φ and φ^* must decay exponentially.

We first prove that z is algebraically simple in $BUC(\mathbb{R})$: it suffices to show that, for any given $v \in BUC(\mathbb{R})$, the equation

$$(z - \mathcal{L}_l)u = v - \langle v, \varphi^* \rangle_{\mathbb{R}} \varphi \tag{3.63}$$

has a solution $u \in BUC(\mathbb{R})$. Note that $\langle v, \varphi^* \rangle$ is well-defined for $v \in BUC(\mathbb{R})$ given that φ^* is exponentially decaying as $|\xi| \to \infty$.

We decompose $\mathcal{L}_l$ as $\mathcal{L}_l = \mathcal{L}_l^- + A^-$ or $\mathcal{L}_l = \mathcal{L}_l^+ + A^+$, where

$$\mathcal{L}_l^- u = -\Lambda_l u + c\frac{\partial u}{\partial x} + f'(1)u, \quad A^- u = f'(U)u - f'(1)u,$$
$$\mathcal{L}_l^+ u = -\Lambda_l u + c\frac{\partial u}{\partial x} + f'(0)u, \quad A^+ u = f'(U)u - f'(0)u.$$

We note that the spectra of $\mathcal{L}_l^\pm$ are easily determined by the Fourier transform as

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l^-) = \{z \in \mathbb{C} \mid ick - Q(k,l) + f'(1),\, k \in \mathbb{R}\},$$
$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_l^+) = \{z \in \mathbb{C} \mid ick - Q(k,l) + f'(0),\, k \in \mathbb{R}\}.$$

Since $z \in \sigma^{+}_{L^2(\mathbb{R})}$ and thus $\mathrm{Re} z > m_{\max} = \max(f'(0), f'(1))$, $(z - \mathcal{L}_l^\pm)^{-1}$ are well-defined on $L^2(\mathbb{R})$.

Let $g := v - \langle v, \varphi^* \rangle \varphi$, and decompose $g \in BUC(\mathbb{R})$ into a sum of $L^2(\mathbb{R})$-functions by a partition of unity. Choose a function $\psi_0 \in C_0^\infty(\mathbb{R})$ supported on $[-1,1]$ such that

$$\sum_{n=-\infty}^{\infty} \psi_n(x) \equiv 1, \quad \psi_n(x) := \psi_0(x-n),$$

and decompose g as

$$g(x) = \sum_{n=-\infty}^{\infty} g_n(x), \quad g_n(x) := \psi_n(x) g(x).$$

In the framework of $L^2(\mathbb{R})$, we consider the equation

$$(z - \mathcal{L}_l)u_n = g_n. \tag{3.64}$$

For $n \geq 1$, we rewrite g_n as

$$g_n = (z - \mathcal{L}_l)(z - \mathcal{L}_l^+)^{-1} g_n + A^+(z - \mathcal{L}_l^+)^{-1} g_n.$$

Define $\Pi w = w - \langle w, \varphi^* \rangle \varphi$. Since $z - \mathcal{L}_l$ and Π commute, we have

$$g_n = \Pi g_n = (z - \mathcal{L}_l)\Pi(z - \mathcal{L}_l^+)^{-1} g_n + \Pi\left(A^+(z - \mathcal{L}_l^+)^{-1} g_n\right). \tag{3.65}$$

By the assumptions on z, we find that $z - \mathcal{L}_l$ is invertible on the range of Π and hence that $(z - \mathcal{L}_l)^{-1} g_n$ is well-defined. Thus, from (3.65), the solution u_n of (3.64) is given as

$$\begin{aligned} u_n &= (z - \mathcal{L}_l)^{-1} g_n \\ &= \Pi(z - \mathcal{L}_l^+)^{-1} g_n + (z - \mathcal{L}_l)^{-1} \Pi(A^+(z - \mathcal{L}_l^+)^{-1} g_n). \end{aligned}$$

By applying the same argument for the case $n < 0$, we consider

$$u_n = \begin{cases} \Pi(z - \mathcal{L}_l^+)^{-1} g_n + (z - \mathcal{L}_l)^{-1} \Pi(A^+(z - \mathcal{L}_l^+)^{-1} g_n) & \text{for } n \geq 1, \\ \Pi(z - \mathcal{L}_l^-)^{-1} g_n + (z - \mathcal{L}_l)^{-1} \Pi(A^-(z - \mathcal{L}_l^-)^{-1} g_n) & \text{for } n < 0. \end{cases}$$

In what follows, we estimate u_n and $u = \sum_{n=-\infty}^{\infty} u_n$. For this purpose, we provide some auxiliary lemmas.

Then, by similar computations to those in Step 2 of the proof of Lemma 3.4, we find that there exists a positive constant C_1 such that

$$\left\| \sum_{n=1}^{\infty} \left(z - \mathcal{L}_l^+\right)^{-1} g_n \right\|_{L^\infty(\mathbb{R})} \leq C_1 \|g\|_{L^\infty(\mathbb{R})}.$$

Moreover, by computations similar to those in Step 3 of the proof of Lemma 3.4, we find that there exist positive constants C_2 and C_3 such that

$$\left\| \sum_{n=1}^{\infty} A^+(z - \mathcal{L}_l^+)^{-1} g_n \right\|_{L^2(\mathbb{R})} \leq C \|g\|_{L^\infty(\mathbb{R})}$$

and that

$$\left\| \sum_{n=1}^{\infty} (z - \mathcal{L}_l)^{-1} \Pi A^+(z - \mathcal{L}_l^+)^{-1} g_n \right\|_{L^\infty(\mathbb{R})} \leq C \|g\|_{L^\infty(\mathbb{R})}.$$

For $\mathcal{L}_l^-$ and A^-, similar results also hold. Consequently, we can define the function u by

$$u = \sum_{n=-\infty}^{\infty} u_n,$$

where $u \in BUC(\mathbb{R})$, since it is a limit of function series in the topology of $L^\infty(\mathbb{R})$. This is the desired solution of (3.63). ∎

LEMMA 3.13. *Suppose $z \in \sigma^{>}_{L^2(\mathbb{R})}(\mathcal{L}_{l_0})$, $l_0 \in (2\pi/d)\mathbb{Z}$, is an algebraically simple eigenvalue and that $z \in \mathbb{C} \backslash \sigma_{L^2(\mathbb{R})}(\mathcal{L}_l)$ for all $l \in (2\pi/d)\mathbb{Z}$ such that $l \neq l_0$. Then z is an algebraically simple eigenvalue of $\mathcal{L}$ in $BUC(\mathbb{R} \times S^1_d)$.*

Proof. Let φ the eigenfunction corresponding to $z \in \sigma^{>}_{L^2(\mathbb{R})}(\mathcal{L}_{l_0})$ and φ^* be the eigenfunction corresponding to the adjoint $\mathcal{L}^*_{l_0}$ as in the proof of the previous lemma. As before, we normalize so that $\langle \varphi, \varphi^* \rangle_{\mathbb{R}} = 1$. Then $\varphi(\xi)\exp(il_0\eta)$ is an eigenfunction of $\mathcal{L}$ in $BUC(\mathbb{R} \times S^1_d)$. To show that this eigenvalue is simple in $BUC(\mathbb{R} \times S^1_d)$, we must show that the following equation can be solved for any $v \in BUC(\mathbb{R} \times S^1_d)$:

$$(z - \mathcal{L})u = v - \langle v, \varphi^* \exp(il_0\eta)\rangle_{\mathbb{R}\times S^1_d}\, \varphi \exp(il_0\eta)/d^2,$$

$$\langle v, w\rangle_{\mathbb{R}\times S^1_d} = \int_{\mathbb{R}\times S^1_d} v(\xi,\eta) w(\xi,\eta) d\xi d\eta.$$

Using the previous lemma and its proof, together with Lemma 3.5, we obtain the desired result in the same way as in the proof of Proposition 3.2. We omit the details. ☐

4. Nonlinear stability and instability of planar fronts. In this section, we complete the proof of Theorems 1.10 and 1.11 by using the estimates on the spectrum of the linearized operator $\mathcal{L}$ obtained in the previous section.

4.1. Spectral decomposition due to translation invariance. To prove nonlinear stability and instability, we make some preparations. In the remainder of this paper, we write $\langle u, w\rangle$ in the sense of

$$\langle u, w\rangle = \int_{\mathbb{R}\times S^1_d} u(\xi,\eta)\overline{w(\xi,\eta)}\, d\xi d\eta, \quad u \in BUC(\mathbb{R}\times S^1_d),\, w \in L^1(\mathbb{R}\times S^1_d).$$

We define the functions $\varphi(\xi)$ and $\varphi^*(\xi)$ by

$$\varphi = \frac{\partial U}{\partial \xi}, \quad \varphi^* = \frac{\exp(c\xi/K_\theta^2)\varphi}{\langle \varphi, \exp(c\xi/K_\theta^2)\varphi\rangle},$$

where K_θ is the constant that appears in (1.13). We note that $\langle \varphi, \varphi^*\rangle = 1$, that $\mathcal{L}\varphi = 0$, and that $\mathcal{L}^*\varphi^* = 0$, where $\mathcal{L}^*$ is the "adjoint" of $\mathcal{L}$; namely,

$$\mathcal{L}^* v = -\Lambda_\theta v - c\frac{\partial v}{\partial \xi} + f'(U)v.$$

It is well known that $\varphi(\xi)$ and $\varphi^*(\xi)$ are both negative and decay exponentially to zero as $\xi \to \pm\infty$ and hence $\varphi, \varphi^* \in L^1(\mathbb{R})$. We define the projection

$$\Pi u = u - \langle u, \varphi^*\rangle \varphi, \tag{4.1}$$

which commutes with $\mathcal{L}$; namely, $\mathcal{L}\Pi = \Pi\mathcal{L}$. We also define the space of functions

$$\mathcal{K} = \left\{ u \in BUC(\mathbb{R}\times S^1_d) \mid u = \Pi u \right\}.$$

Note that $v \in \mathcal{K}$ implies $\langle v, \varphi^*\rangle = 0$.

To analyze the asymptotic behavior of the solution $u(\xi,\eta,t)$, we introduce the translation operator τ_σ with $\sigma \in \mathbb{R}$ by

$$\tau_\sigma u(\xi,\eta,t) = u(\xi+\sigma,\eta,t),$$

and decompose $u(\xi,\eta,t)$ as

$$u = \tau_{\sigma(t)}(U + v) = U(\xi - \sigma(t)) + v(\xi - \sigma(t), \eta, t), \tag{4.2}$$

with $v(\cdot,t) \in \mathcal{K}$ by virtue of Lemma 4.1 below.

LEMMA 4.1. *If $w \in BUC(\mathbb{R}\times S_d^1)$ is sufficiently small, then there exists a unique pair $(v,\sigma)\in\mathcal{K}\times\mathbb{R}$ such that*

$$U+w=\tau_\sigma(U+v). \tag{4.3}$$

Proof. Equation (4.3) is equivalent to

$$\tau_{-\sigma}(U+w)-U=v. \tag{4.4}$$

Since $v\in\mathcal{K}$ means $\langle v,\varphi^*\rangle=0$, we consider $\langle \tau_{-\sigma}(U+w)-U,\varphi^*\rangle=0$, which is equivalent to

$$\langle U+w-\tau_\sigma U,\tau_\sigma\varphi^*\rangle=0. \tag{4.5}$$

Then the implicit function theorem implies that, for any given $w\in BUC(\mathbb{R}\times S_d^1)$ that is sufficiently small, there exists a unique constant $\sigma=\sigma(w)$ such that (4.5) holds. Indeed, we have

$$\frac{\partial}{\partial\sigma}\langle U+w-\tau_\sigma U,\tau_\sigma\varphi^*\rangle|_{(w,\sigma)=(0,0)}=\left\langle\frac{\partial U}{\partial\xi},\varphi^*\right\rangle=1\neq 0.$$

Consequently, by determining v from (4.4), we obtain a unique pair $(v,\sigma)\in\mathcal{K}\times\mathbb{R}$ that satisfies (4.3). ∎

We now derive the equations that v and $\sigma(t)$ satisfy. By substituting (4.2) into the original equation $u_t=-\Lambda u+c\frac{\partial u}{\partial\xi}+f(u)$, we have

$$-\sigma'\tau_\sigma\varphi-\sigma'\tau_\sigma v_\xi+\tau_\sigma\frac{\partial v}{\partial t}=-\Lambda\tau_\sigma(U+v)+c\tau_\sigma(\varphi+v_\xi)+f(\tau_\sigma(U+v)).$$

By applying $\tau_{-\sigma}$, we have

$$\frac{\partial v}{\partial t}=\sigma'\varphi+\sigma' v_\xi-\Lambda(U+v)+c(\varphi+v_\xi)+f(U+v).$$

By using the equality $-\Lambda U+c\varphi+f(U)=0$, we obtain

$$\begin{aligned}\frac{\partial v}{\partial t}&=-\Lambda v+\sigma'\varphi+(\sigma'+c)v_\xi+f(U+v)-f(U)\\ &=\mathcal{L}v+\sigma'\varphi+\sigma' v_\xi+H(v),\end{aligned} \tag{4.6}$$

$$H(v)=f(U+v)-f(U)-f'(U)v. \tag{4.7}$$

Since $v(\cdot,t),v_t(\cdot,t)\in\mathcal{K}$, we have

$$\langle\mathcal{L}v,\varphi^*\rangle=\langle v,\mathcal{L}^*\varphi^*\rangle=0,\quad\left\langle\frac{\partial v}{\partial t},\varphi^*\right\rangle=0,$$

and hence

$$0=\sigma'\langle\varphi,\varphi^*\rangle+\sigma'\langle v_\xi,\varphi^*\rangle+\langle H(v),\varphi^*\rangle.$$

Thus, we obtain

$$\sigma'=-\frac{\langle H(v),\varphi^*\rangle}{1+\langle v_\xi,\varphi^*\rangle}. \tag{4.8}$$

Finally, by applying Π to (4.6), we have

$$\frac{\partial v}{\partial t}=\mathcal{L}v+N(v),\ N(v)=-\frac{\langle H(v),\varphi^*\rangle}{1+\langle v_\xi,\varphi^*\rangle}\Pi v_\xi+\Pi H(v). \tag{4.9}$$

LEMMA 4.2. *For $\delta > 0$ sufficiently small, there exists a constant M such that, for any v, w satisfying $\|v\|_{BUC^1} \le \delta, \|w\|_{BUC^1} \le \delta$, we have*

$$\|N(v) - N(w)\|_{BUC} \le M\,(\|v\|_{BUC^1} + \|w\|_{BUC^1})\,\|v - w\|_{BUC^1}. \tag{4.10}$$

Proof. Elementary calculus estimates yield the following pointwise bound for H:

$$|H(v) - H(w)| \le M_0\,(|v| + |w|)\,|v - w|, \quad M_0 = \frac{1}{2}\max_{-\delta \le s \le 1+\delta} |f''(s)|. \tag{4.11}$$

Noting that $\varphi^* \in L^1$, it is easily seen that if $\delta > 0$ is small enough, there is a constant M_1 such that

$$\left\| \frac{v_\xi}{1 + \langle v_\xi, \varphi^* \rangle} - \frac{w_\xi}{1 + \langle w_\xi, \varphi^* \rangle} \right\|_{BUC} \le M_1 \|v_\xi - w_\xi\|_{BUC}. \tag{4.12}$$

The above two inequalities, together with the definition of Π in (4.1), yield the desired estimate. ∎

4.2. Nonlinear stability of planar fronts. In this subsection, we complete the proof of the statement (i) of Theorem 1.10 and the proof of Theorem 1.11. The lemma below is used to prove statement (i) in Theorem 1.10. In what follows, $\mathcal{L}|_{\mathcal{K}}$ denotes the restriction of $\mathcal{L}$ on $\mathcal{K}$.

LEMMA 4.3 (spectral gap). *For a direction $\mathbf{n} \in S^1$, if the planar front on $\mathbb{R}^2$ is spectrally stable in the sense of Definition* 1.3*, for any $d > 0$, there exists a positive constant ω such that*

$$\sigma_{BUC(\mathbb{R}\times S^1_d)}(\mathcal{L}|_{\mathcal{K}}) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z \le -\omega\}. \tag{4.13}$$

Proof. Let any $d > 0$ be fixed. From Proposition 3.6, there exist a positive constant ω_1 and a positive integer k_* such that

$$\bigcup_{k\in\mathbb{Z},\, |k|>k_*} \sigma_{L^2(\mathbb{R})}(\mathcal{L}_{2\pi k/d}) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z \le -\omega_1\}.$$

On the other hand, from the assumption of the lemma, we have

$$\bigcup_{k\in\mathbb{Z},\, 0<|k|\le k_*} \sigma_{L^2(\mathbb{R})}(\mathcal{L}_{2\pi k/d}) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}.$$

Moreover, Proposition 3.6 implies that the set

$$\bigcup_{k\in\mathbb{Z},\, 0<|k|\le k_*} \sigma_{L^2(\mathbb{R})}(\mathcal{L}_{2\pi k/d}) \cap \{z \in \mathbb{C} \mid -1 \le \operatorname{Re} z\}$$

(which is possibly empty) is compact. Thus, there exists a positive constant ω_2 such that

$$\bigcup_{k\in\mathbb{Z},\, 0<|k|\le k_*} \sigma_{L^2(\mathbb{R})}(\mathcal{L}_{2\pi k/d}) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z \le -\omega_2\}.$$

For $k = 0$, it is well known (see Proposition 3.1 in [27], for instance) that there exists a positive constant ω_3 such that

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_0)\backslash\{0\} \subset \{z \in \mathbb{C} \mid \operatorname{Re} z \le -\omega_3\}.$$

By combining the above estimates, we obtain

$$\begin{aligned}\sigma_{BUC(\mathbb{R}\times S^1_d)}(\mathcal{L})\backslash\{0\} &\subset \bigcup_{k\in\mathbb{Z}} \sigma_{BUC(\mathbb{R})}(\mathcal{L}_{2\pi k/d})\backslash\{0\}\\ &\subset \bigcup_{k\in\mathbb{Z}} \sigma_{L^2(\mathbb{R})}(\mathcal{L}_{2\pi k/d})\backslash\{0\}\\ &\subset \{z\in\mathbb{C}\,|\,\mathrm{Re}\, z\le -\min\{\omega_1,\omega_2,\omega_3\}\}\,,\end{aligned}$$

where we used Proposition 3.2 to obtain the first and second lines of the right-hand side. By Lemma 3.13, 0 is an algebraically simple eigenvalue of $\mathcal{L}$ in $BUC(\mathbb{R}\times S^1_d)$. We thus obtain

$$\sigma_{BUC(\mathbb{R}\times S^1_d)}(\mathcal{L}|_{\mathcal{K}}) = \sigma_{BUC(\mathbb{R}\times S^1_d)}(\mathcal{L})\backslash\{0\}.$$

This completes the proof. ∎

The lemma below is used to prove Theorem 1.11.

LEMMA 4.4 (spectral gap). *If $d>0$ is sufficiently small, there exists a positive constant ω such that*

$$\sigma_{BUC(\mathbb{R}\times S^1_d)}(\mathcal{L}|_{\mathcal{K}}) \subset \{z\in\mathbb{C}\,|\,\mathrm{Re}\, z\le -\omega\}\,. \tag{4.14}$$

Proof. From Proposition 3.6, there exist positive constants ω_1 and d_* such that if $d<d_*$, then

$$\bigcup_{k\in\mathbb{Z},\,k\neq 0} \sigma_{L^2(\mathbb{R})}(\mathcal{L}_{2\pi k/d}) \subset \{z\in\mathbb{C}\,|\,\mathrm{Re}\, z\le -\omega_1\}\,.$$

For $k=0$, it is well known (see Proposition 3.1 in [27], for instance) that there exists a positive constant ω_2 such that

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_0)\backslash\{0\} \subset \{z\in\mathbb{C}\,|\,\mathrm{Re}\, z\le -\omega_2\}\,.$$

By combining these estimates, we obtain the desired result in a way similar to the end of the proof of Lemma 4.3. ∎

Now we are ready to prove the nonlinear stability of planar fronts in the sense of Definition 1.8 stated in Theorems 1.10 and 1.11.

Proof of the statement (i) *of Theorem* 1.10 *and the proof of Theorem* 1.11. From Lemmas 4.3 and 4.4, there exists a positive constant ω such that for any $0<\eta<\omega$ and $u\in\mathcal{K}$,

$$\|\exp(t\mathcal{L})u\|_{BUC(\mathbb{R}\times S^1_d)} \le C_\eta \exp(-\eta t)\|u\|_{BUC(\mathbb{R}\times S^1_d)}, \quad t>0. \tag{4.15}$$

We also see from Proposition 3.1 that there is a constant C_1 such that

$$\begin{aligned}&\|\exp(t\mathcal{L})u\|_{BUC^k} \le C_1\|u\|_{BUC^k} \quad \text{for } k=0,1,\ 0\le t\le 1,\\ &\|\mathcal{L}\exp(t\mathcal{L})u\|_{BUC} \le \frac{C_1}{t}\|u\|_{BUC} \quad \text{for } 0\le t\le 1.\end{aligned} \tag{4.16}$$

Note that, for $u\in\mathcal{D}_1(\mathcal{L})$, we have

$$\begin{aligned}&\|u\|_{BUC^1(\mathbb{R}\times S^1_d)} \le C_2\|u\|_{BUC}^{\gamma/(1+\gamma)}\|u\|_{C^{1,\gamma}}^{1/(1+\gamma)}\\ &\le C_2\|u\|_{BUC}^{\gamma/(1+\gamma)}\left(C_{\mathcal{L}}\left(\|u\|_{BUC}+\|\mathcal{L}u\|_{BUC}\right)\right)^{1/1+\gamma}\\ &\le C_3\left(\|u\|_{BUC}+\|u\|_{BUC}^{\gamma/(1+\gamma)}\|\mathcal{L}u\|_{BUC}^{1/(1+\gamma)}\right).\end{aligned} \tag{4.17}$$

Combining the above with (4.16), we have

$$\|\exp(t\mathcal{L})u\|_{BUC^1} \leq C_4 \left(1 + \frac{1}{t^{1/(1+\gamma)}}\right) \|u\|_{BUC}. \tag{4.18}$$

For $t > 1$, we have, for $u \in \mathcal{K}$,

$$\begin{aligned} &\|\exp(t\mathcal{L})u\|_{BUC^1} = \|\exp((t-1)\mathcal{L})\exp(\mathcal{L})u\|_{BUC^1} \\ &\leq 2C_4 \|\exp((t-1)\mathcal{L})u\|_{BUC} \leq 2C_4 C_\eta \exp(1)\exp(-\eta t)\|u\|_{BUC}. \end{aligned} \tag{4.19}$$

The above, combined with (4.16), yields, for $u \in \mathcal{K}$,

$$\begin{aligned} &\|\exp(t\mathcal{L})u\|_{BUC^1} \leq C_5 \left(1 + \frac{1}{t^{1/(1+\gamma)}}\right) \exp(-\eta t)\|u\|_{BUC}, \\ &\|\exp(t\mathcal{L})u\|_{BUC^1} \leq C_5 \exp(-\eta t)\|u\|_{BUC^1}. \end{aligned} \tag{4.20}$$

Given $v_0 \in BUC^1 \cap \mathcal{K}$, consider the integral equation corresponding to (4.9):

$$v(t) = \exp(t\mathcal{L})v_0 + \int_0^t \exp((t-s)\mathcal{L})N(v(s))ds. \tag{4.21}$$

We seek a solution to this equation for v in the following function space X, so long as $\|v_0\|_{BUC^1}$ is small enough:

$$\begin{aligned} &X = \{u \in C([0,\infty); BUC^1(\mathbb{R} \times S^1_d) \cap \mathcal{K}) | \exp(t\eta)\|u(t)\|_{BUC^1(\mathbb{R}\times S^1_d)} < \infty\}, \\ &\|u\|_X = \sup_{t \geq 0} \exp(t\eta)\|u(t)\|_{BUC^1}. \end{aligned} \tag{4.22}$$

It is easily checked that X is a Banach space. Define the operator

$$\Psi(v) = \exp(t\mathcal{L})v_0 + \int_0^t \exp((t-s)\mathcal{L})N(v(s))ds. \tag{4.23}$$

Consider the set

$$B_\delta = \{v \in X | \|v\|_X \leq \delta\}. \tag{4.24}$$

We shall prove that Ψ is a contraction on B_δ if δ is small enough. Take $\delta > 0$ small enough so that Lemma 4.2 applies. Using Lemma 4.2 and (4.20), we have

$$\begin{aligned} &\exp(t\eta)\|\Psi(v)\|_{BUC^1} \\ &\leq C_5\|v_0\|_{BUC^1} + \exp(t\eta)\int_0^t C_5 \left(1 + \frac{1}{(t-s)^{1/(1+\gamma)}}\right) \exp(-\eta(t-s))\|N(v(s))\|_{BUC} ds \\ &\leq C_5\|v_0\|_{BUC^1} + \int_0^t C_5 \left(1 + \frac{1}{(t-s)^{1/(1+\gamma)}}\right) \exp(-\eta s) M ds \|v\|_X^2. \end{aligned} \tag{4.25}$$

In the above, we used the fact that $N(v) \in \mathcal{K}$. Note that

$$\begin{aligned} &\int_0^t \left(1 + \frac{1}{(t-s)^{1/(1+\gamma)}}\right) \exp(-\eta s) ds \\ &\leq \int_0^1 \left(1 + \frac{1}{(t-s)^{1/(1+\gamma)}}\right) ds + \int_1^\infty \exp(-\eta s) ds < \infty. \end{aligned} \tag{4.26}$$

Thus,

$$\|\Psi(v)\|_X \leq C_5\|v_0\|_{BUC^1} + C_6\|v\|_X^2. \tag{4.27}$$

This shows that, by taking $\|v_0\|_{BUC^1} \leq \delta/(2C_5)$ and taking δ smaller if necessary, Ψ maps B_δ to itself. Furthermore, we have

$$\begin{aligned}
&\|\Psi(u) - \Psi(v)\|_X \\
&\leq \sup_{t\geq 0} \exp(t\eta) \int_0^t C_5 \left(1 + \frac{1}{(t-s)^{1/(1+\gamma)}}\right) \exp(-\eta(t-s)) \|N(u(s)) - N(v(s))\|_{BUC}\, ds \\
&\leq C_6 (\|u\|_X + \|v\|_X) \|u - v\|_X \leq 2C_6\delta\|u - v\|_X,
\end{aligned} \tag{4.28}$$

where we used Lemma 4.2 and (4.20). Taking δ small enough, Ψ is a contraction on B_δ. Substituting this v back into (4.8), we see immediately that $\sigma(t)$ converges exponentially to a constant as $t \to \infty$. This demonstrates stability in the sense of Definition 1.8 provided that the initial data is in BUC^1.

Finally, we must show stability for initial data in BUC. Pick some $\delta_0 > 0$, and let $u_0 \in BUC$ satisfy $\|u_0 - U\|_{BUC} \leq \delta_0$. By Lemma 2.7, if we take $\delta_0 > 0$ small enough, there exist $T > 0$ and $M_0 > 0$ that do not depend on the choice of u_0 so that the corresponding mild solution $u(t)$ satisfies

$$u(t) = \exp(t\Lambda)u_0 + \int_0^t \exp((t-s)\Lambda)f(u(s))ds, \quad \|u(t)\|_{BUC} \leq M_0, \quad 0 \leq t \leq T. \tag{4.29}$$

Let us estimate the difference between $u(t)$ and U in the BUC^1 norm. We have

$$\begin{aligned}
&\|u(t) - U\|_{BUC^1} \\
&\quad\leq \|\exp(t\Lambda)(u_0 - U)\|_{BUC^1} + \|\exp(t\Lambda)U - U\|_{BUC^1} \\
&\qquad + \int_0^t \|\exp((t-s)\Lambda)f(u(s))\|_{BUC^1}\, ds \\
&\quad\leq C_7 \left(1 + \frac{1}{t^{1/(1+\gamma)}}\right) \|u_0 - U\|_{BUC} + \|\exp(t\Lambda)U - U\|_{BUC^1} \\
&\qquad + C_7 \int_0^t \left(1 + \frac{1}{(t-s)^{1/(1+\gamma)}}\right) \|f(u(s)\|_{BUC} ds \\
&\quad\leq C_8 \left(\frac{1}{t^{1/(1+\gamma)}} \|u_0 - U\|_{BUC} + \|\exp(t\Lambda)U - U\|_{BUC^1} + t^{\gamma/(1+\gamma)}\right),
\end{aligned} \tag{4.30}$$

where we used (4.20) to obtain the second inequality. In the above, we used estimates on $\exp(t\Lambda)$ similar to (4.18), which can be derived in exactly the same way. Letting $t_* = \|u_0 - U\|_{BUC}$, we have

$$\begin{aligned}
\|u(t_*) - U\|_{BUC^1} &\leq C_8 \left(\|u_0 - U\|_{BUC}^{\gamma/(1+\gamma)} + Q(\|u_0 - U\|_{BUC})\right), \\
Q(t) &= \sup_{0\leq s\leq t} \|\exp(t\Lambda)U - U\|_{BUC^1}.
\end{aligned} \tag{4.31}$$

Given that U is smooth, function $Q(t)$ is a monotone continuous function that tends to 0 as $t \to 0$. Thus, $\|u(t_*) - U\|_{BUC^1}$ can be made arbitrary small by taking $\|u_0 - U\|_{BUC}$ sufficiently small. Applying the BUC^1 nonlinear stability result for the initial data $u(t_*)$, we obtain the desired result. ∎

4.3. Nonlinear instability of planar fronts. In this subsection, we complete the proof of the statement (ii) of Theorem 1.10. Define the sets by

$$\sigma_u = \{z \in \sigma_{BUC(\mathbb{R}\times S^1_d)}(\mathcal{L}|_{\mathcal{K}}) \mid \mathrm{Re}\ z > 0\},$$
$$\sigma_{cs} = \{z \in \sigma_{BUC(\mathbb{R}\times S^1_d)}(\mathcal{L}|_{\mathcal{K}}) \mid \mathrm{Re}\ z \le 0\}.$$

Then the following holds.

LEMMA 4.5 (spectral gap).
For a direction $\mathbf{n} \in S^1$, *if the planar front on* $\mathbb{R}^2$ *is spectrally unstable in the sense of Definition* 1.3 *and if* $d > 0$ *is sufficiently large, then the set* σ_u *is not empty, is bounded, and satisfies*

$$\sigma_u \subset \{z \in \mathbb{C} \mid \mathrm{Re}\, z \ge \omega\} \tag{4.32}$$

for a positive constant ω.

Proof. We here use the notation $\sigma^+_{L^2(\mathbb{R})}(\mathcal{L}_l)$, etc., given in (3.49) and (3.50). Since the planar front on $\mathbb{R}^2$ is spectrally unstable in the sense of Definition 1.3 by the assumption of the lemma, there exists a constant $l_* \in \mathbb{R}$ such that

$$\sigma^+_{L^2(\mathbb{R})}(\mathcal{L}_{l_*}) \ne \emptyset.$$

Then Lemma 3.11 implies that there exists a constant $\delta > 0$ such that, for all $l \in [l_* - \delta, l_* + \delta]$, it holds that

$$\sigma^+_{L^2(\mathbb{R})}(\mathcal{L}_l) \ne \emptyset.$$

We choose $d > 0$ sufficiently large to satisfy $2\pi/d \le 2\delta$. Then there exists a $k_* \in \mathbb{R}$ such that $2\pi k_*/d \in [l_* - \delta, l_* + \delta]$ and hence that

$$\sigma^+_{L^2(\mathbb{R})}(\mathcal{L}_{2\pi k_*/d}) \ne \emptyset.$$

Thus, Proposition 3.10 gives that

$$\sigma^+_{BUC(\mathbb{R})}(\mathcal{L}_{2\pi k_*/d}) \ne \emptyset$$

and that any point in this set is an eigenvalue. Let any $z \in \sigma^+_{BUC(\mathbb{R})}(\mathcal{L}_{2\pi k_*/d})$ be fixed; then there exists a function $v \in BUC(\mathbb{R})$ such that

$$\mathcal{L}_l v = zv.$$

Then we have

$$\mathcal{L}v^* = zv^*, \quad v^*(\xi, \eta) = v(\xi) e^{i(2\pi k/d)\eta},$$

which means that z is an eigenvalue of $\mathcal{L}$ in $BUC(\mathbb{R} \times S^1_d)$. Thus, we find that

$$\sigma_{BUC(\mathbb{R}\times S^1_d)}(\mathcal{L}) \cap \{z \in \mathbb{C} \mid \mathrm{Re}\, z > 0\} \ne \emptyset.$$

Next, we show boundedness. From Proposition 3.6, there exists a positive integer k_* such that $\sigma_{L^2(\mathbb{R})}(\mathcal{L}_{2\pi k/d})$ is empty if $|k| \ge k_*$. On the other hand, Proposition 3.10 implies that

$$\bigcup_{k\in\mathbb{Z},\, |k|\le k_*} \sigma^+_{L^2(\mathbb{R})}(\mathcal{L}_{2\pi k/d})$$

consists of a finite number of points. Thus, since Propositions 3.2 and 3.10 imply

$$\begin{aligned}\sigma_{BUC(\mathbb{R}\times S^1_d)}(\mathcal{L})\cap\{z\in\mathbb{C}\mid \operatorname{Re} z>0\}&\subset\bigcup_{k\in\mathbb{Z}}\sigma^+_{BUC(\mathbb{R})}(\mathcal{L}_{2\pi k/d})\\&=\bigcup_{k\in\mathbb{Z}}\sigma^+_{L^2(\mathbb{R})}(\mathcal{L}_{2\pi k/d}),\end{aligned}$$

we find that

$$\sigma_{BUC(\mathbb{R}\times S^1_d)}(\mathcal{L})\cap\{z\in\mathbb{C}\mid \operatorname{Re} z>0\}$$

consists of a finite number of points. Thus, this set is bounded and satisfies

$$\sigma_{BUC(\mathbb{R}\times S^1_d)}(\mathcal{L})\cap\{z\in\mathbb{C}\mid \operatorname{Re} z>0\}\subset\{z\in\mathbb{C}\mid \operatorname{Re} z\geq\omega\}$$

for some positive constant ω. ∎

Now we are ready to prove the nonlinear instability stated in Theorem 1.10. The proof is based on the contraction mapping theorem.

Proof of the statement (ii) *of Theorem* 1.10. Let $\omega>0$ be the constant defined in Lemma 4.5, and choose a positive constant η such that $3\eta<\omega$. From Lemma 4.5, we can choose a simple closed curve γ_u enclosing σ_u and define the projection operators

$$\Pi_u=\frac{1}{2\pi i}\int_{\gamma_u}(zI-\mathcal{L})^{-1}dz,\quad \Pi_{cs}=I-\Pi_u.$$

We first list two estimates on the semigroup $\exp(t\mathcal{L})$. For any $w\in\Pi_{cs}\mathcal{K}$, and $0<\gamma<1$, we have

$$\|\exp(t\mathcal{L})w\|_{BUC^1}\leq M_\gamma\left(1+\frac{1}{t^{1/(1+\gamma)}}\right)\exp(\eta t)\|w\|_{BUC},\quad t>0, \tag{4.33}$$

where M_γ is a constant that depends only on γ. The above estimate can be derived in the same way as (4.20) using the assumption on the spectrum. For any $w\in\Pi_u\mathcal{K}$, we have

$$\|\exp(t\mathcal{L})w\|_{BUC^1}\leq M\exp(3\eta t)\|w\|_{BUC},\quad t\leq 0, \tag{4.34}$$

where M is a positive constant. We have here used the fact that $\exp(t\mathcal{L})$, $t<0$, is well- defined on $\Pi_u\mathcal{K}$ since it is a finite dimensional invariant subspace of $\mathcal{L}$. This also follows easily from the spectral condition and the fact that $(zI-\mathcal{L})^{-1}w\in BUC^1$ for $w\in BUC$ with z in the resolvent set.

Take any $v_0\in\Pi_u\mathcal{K}$ such that $v_0\neq 0$. Define

$$Y=\{v\in C((-\infty,0];BUC^1\cap\mathcal{K})\,|\,\|v\|_Y:=\sup_{t<0}e^{-2\eta t}\|v(\cdot,t)\|_{BUC^1}<\infty\}. \tag{4.35}$$

It is easily checked that Y is a Banach space. We consider the map Ψ:
(4.36)

$$\Psi(v)=\exp(t\mathcal{L})v_0+\int_{-\infty}^t\exp((t-s)\mathcal{L})\Pi_{cs}N(v(s))ds+\int_0^t\exp((t-s)\mathcal{L})\Pi_uN(v(s))ds.$$

If we take $\|v_0\|_{BUC^1}$ small enough, the above map Ψ is a contraction on the set $\|v\|_Y\leq R$ for sufficiently small R. This can be shown in essentially the same way as in the proof of statement (i) of Theorem 1.10 and the proof of Theorem 1.11, using estimates (4.33), (4.34) and Lemma 4.2.

Let $v_* \in Y$ be this fixed point of Ψ. Then, given that $v_* \in Y$, we have

$$\lim_{t\to\infty} \|v_*(t)\|_{BUC^1} = 0. \tag{4.37}$$

Furthermore, since v_* is the fixed point of the map Ψ, we have

$$v_*(0) = v_0 + \int_{-\infty}^{0} \exp((t-s)\mathcal{L})\Pi_{cs} N(v_*(s))ds. \tag{4.38}$$

Since $\Pi_u(v_*(0)) = v_0 \neq 0$, $\|v_*(0)\|_{BUC^1} > 0$. This implies the nonlinear instability of planar fronts in the sense of Definition 1.9. ☐

Appendix A. Nonpositivity of the fundamental solution G_t. As stated in the discussion surrounding (1.6), the bidomain operator does not satisfy the maximum principle unless $A_i = \beta A_e$ for some $\beta > 0$ (the monodomain case). We present a proof of this result in $\mathbb{R}^n, n = 2,3$. We will prove the nonpositivity of the fundamental solution G_t given in (1.5), which we reproduce here to include the case $n = 3$:

$$\begin{aligned} G_t(\mathbf{x}) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \exp(-tQ(\mathbf{k}))\exp(i\mathbf{k}\cdot\mathbf{x})d\mathbf{k}, \\ Q(\mathbf{k}) &= \frac{Q_i(\mathbf{k})Q_e(\mathbf{k})}{Q_i(\mathbf{k})+Q_e(\mathbf{k})}, \quad Q_{i,e}(\mathbf{k}) = \mathbf{k}^{\mathrm{T}} A_{i,e}\mathbf{k}, \end{aligned} \tag{A.1}$$

where A_i and A_e are $n \times n$ symmetric positive definite matrices.

Proposition A.1. *Consider the function $G_t(\mathbf{x})$, $t > 0$, given in* (A.1)*, $n = 2,3$. Suppose A_i and A_e are not proportional to each other, in the sense that there is no $\beta > 0$ such that $A_i = \beta A_e$. Then, for every $t > 0$, there is an $\mathbf{x} \in \mathbb{R}^n$ such that $G_t(\mathbf{x}) < 0$.*

Remark A.2. We have not been able to locate a proof of the above nonpositivity result in the literature. Indeed, a recent paper [16] states that it is unknown whether the bidomain operator satisfies the maximum principle. We note, however, that numerical computations plotting G_t (see, for example, Chapter 12 of [19]) clearly show places where $G_t(\mathbf{x})$ is negative for specific examples of A_i and A_e. We also remark that in section 5 of [27] the authors prove the following. In the nonmonodomain case, it is always possible to find a bistable nonlinearity f so that the corresponding bidomain Allen–Cahn equation has a planar front solution that is spectrally unstable. This implies that the nonmonodomain bidomain operator in dimension 2 violates the maximum principle. We emphasize that our proof only applies to the constant coefficient case in free space. Failure of the maximum principle is almost certainly true in the general case of the variable coefficient bidomain operator in a bounded domain with suitable boundary conditions, but this question is likely open.

Proof. We first consider the case $n = 2$. Consider the initial value problem

$$u_t = -\Lambda u, \quad u(\mathbf{x},0) = w(\mathbf{x}), \tag{A.2}$$

where we take $w(\mathbf{x})$ to be a smooth compactly supported function. The solution u can be written as follows:

$$u(\mathbf{x},t) = \int_{\mathbb{R}^2} G_t(\mathbf{x}-\mathbf{y})w(\mathbf{y})d\mathbf{y}. \tag{A.3}$$

Since $w(\mathbf{x})$ is smooth and compactly supported, it is easily seen that the above solution u satisfies (A.2) pointwise for $t \geq 0$, where Λ in the sense of a Fourier multiplier operator as in (1.3). Our goal here is to show that if there is no $\beta > 0$ such that $A_i = \beta A_e$, then there are values of $\mathbf{x}$ for which $G_t(\mathbf{x}) < 0$. We will exhibit a smooth compactly supported function $w(\boldsymbol{x}) \geq 0$ such that

$$w(\mathbf{x}) \geq 0,\; w(0) = 0,\; -(\Lambda w)(0) < 0. \tag{A.4}$$

If we can find such a function, this implies that, for sufficiently small $t > 0$, we have

$$u(0,t) = \int_{\mathbb{R}^2} G_t(0-\mathbf{y})w(\mathbf{y})d\mathbf{y} = \int_{\mathbb{R}^2} G_t(\mathbf{y})w(\mathbf{y})d\mathbf{y} < 0. \tag{A.5}$$

Since $w(\mathbf{x}) \geq 0$, this is sufficient to show that $G_t(\mathbf{x})$ is negative for some values of $\mathbf{x}$. Given that $G_t(\mathbf{x}) = t^{-1}G_1(\mathbf{x}/\sqrt{t})$, this shows that, for any $t > 0$, there is an $\mathbf{x}$ for which $G_t(\mathbf{x}) < 0$.

We first make a suitable affine change of coordinates to reduce A_i and A_e to a simple form. Let P be a 2×2 invertible matrix, and let $\mathbf{x}' = P^{-1}\mathbf{x}$ be the new coordinate system. Then from (A.1) we see that

$$G_t(P\mathbf{x}') = \frac{1}{(2\pi)^n \det P} \int_{\mathbb{R}^n} \exp(-tQ(P^{\mathrm{T}}\mathbf{k})) \exp(i\mathbf{k}\cdot\mathbf{x}')d\mathbf{k}. \tag{A.6}$$

Let $P_1 = A_i^{-1/2}$, which exists given that A_i is symmetric positive definite. Furthermore, choose an orthogonal matrix P_2 so that $P_1 A_e P_1^{\mathrm{T}}$ is diagonalized. Then

$$P_2P_1A_iP_1^{\mathrm{T}}P_2^{\mathrm{T}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_2P_1A_eP_1^{\mathrm{T}}P_2^{\mathrm{T}} = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, \tag{A.7}$$

where $\beta_1 > 0$ and $\beta_2 > 0$. If we let

$$P_3 = \begin{pmatrix} \sqrt{2/(1+\beta_1)} & 0 \\ 0 & \sqrt{2/(1+\beta_2)} \end{pmatrix} \tag{A.8}$$

and $P = P_3P_2P_1$, we have

$$PA_iP^{\mathrm{T}} = \begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & 1+\alpha_2 \end{pmatrix}, \quad PA_eP^{\mathrm{T}} = \begin{pmatrix} 1-\alpha_1 & 0 \\ 0 & 1-\alpha_2 \end{pmatrix}, \text{ where}$$
$$\alpha_k = \frac{1-\beta_k}{1+\beta_k}, \quad k = 1,2. \tag{A.9}$$

Note that $|\alpha_1| < 1$ and $|\alpha_2| < 1$. (Note that this parametrization is slightly different from (1.24) but equivalent.) If $\alpha_1 \neq \alpha_2$, A_i and A_e are not proportional to each other. Given expression (A.6), we thus only have to prove the desired statement when A_i and A_e have the form given above and $\alpha_1 \neq \alpha_2$. For the above A_i and A_e, the symbol of $Q(\mathbf{k})$, $\mathbf{k} = (k_1,k_2)^{\mathrm{T}}$, is given by (see (A.6))

$$\begin{aligned} Q(\mathbf{k}) &= \frac{((1+\alpha_1)k_1^2 + (1+\alpha_2)k_2^2)((1-\alpha_1)k_1^2 + (1-\alpha_2)k_2^2)}{2(k_1^2+k_2^2)} \\ &= \frac{1}{2}(1-\alpha_1^2)k_1^2 + \frac{1}{2}(1-\alpha_2^2)k_2^2 + (\alpha_1-\alpha_2)^2\frac{k_1^2k_2^2}{2(k_1^2+k_2^2)}. \end{aligned} \tag{A.10}$$

Thus, for any compactly supported smooth function $w(\mathbf{x})$, we have

$$\begin{aligned} -(\Lambda w)(\mathbf{x}) =& \frac{1}{2}(1-\alpha_1^2)\frac{\partial^2 w}{\partial {x_1}^2} + \frac{1}{2}(1-\alpha_2^2)\frac{\partial^2 w}{\partial {x_2}^2} \\ &+ \frac{(\alpha_1-\alpha_2)^2}{4\pi}\int_{\mathbb{R}^2} \log(|\mathbf{x}-\mathbf{y}|)\frac{\partial^4 w}{\partial y_1^2 \partial y_2^2}d\mathbf{y}, \end{aligned} \tag{A.11}$$

where $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$ and $\mathbf{y} = (y_1, y_2)^{\mathrm{T}}$. Let us now apply this to the following function w. Let ϕ be the following smooth radial cutoff function:

$$\phi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ 0 & \text{if } r \ge 2 \end{cases} \tag{A.12}$$

and takes values between 0 and 1 when $1 < r < 2$. Let $\phi_\epsilon(r) = \phi(r/\epsilon)$ for any $\epsilon > 0$. Let w be the following function expressed in polar coordinates:

$$w(r,\theta) = \phi(r)(1 - \phi_\epsilon(r))(1 - \cos(4\theta)), \tag{A.13}$$

where we let $0 < \epsilon < 1/2$. This function is clearly nonnegative. Now let us evaluate (A.11) at $\mathbf{x} = 0$ for the above function w. We have

$$-(\Lambda w)(0) = \frac{(\alpha_1 - \alpha_2)^2}{4\pi} \int_{\mathbb{R}^2} \log|\mathbf{y}| \frac{\partial^4 w}{\partial y_1^2 \partial y_2^2} d\mathbf{y}, \tag{A.14}$$

where we used the fact that w is identically equal to 0 in the neighborhood of $\mathbf{x} = 0$. Noting again that w is identically equal to 0 near the origin and that it is compactly supported, we may integrate by parts to obtain

$$\begin{aligned} -(\Lambda w)(0) &= \frac{(\alpha_1 - \alpha_2)^2}{4\pi} \int_{\epsilon \le |\mathbf{y}| \le 2} K(\mathbf{y}) w(\mathbf{y}) d\mathbf{y}, \\ K(\mathbf{y}) &= \frac{\partial^4}{\partial y_1^2 \partial y_2^2} \log|\mathbf{y}| = 6\frac{(y_1^4 - 6y_1^2 y_2^2 + y_2^4)}{(y_1^2 + y_2^2)^4} = \frac{6\cos(4\theta)}{r^4}. \end{aligned} \tag{A.15}$$

Let us now evaluate the above integral:

$$\begin{aligned} &\frac{1}{4\pi} \int_{\epsilon < |y| < 2} K(\mathbf{y}) w(\mathbf{y}) d\mathbf{y} \\ &= \int_\epsilon^2 \int_0^{2\pi} \frac{3\cos(4\theta)}{2\pi r^4} \phi(r)(1 - \phi_\epsilon(r))(1 - \cos(4\theta)) d\theta r dr \\ &= -\int_\epsilon^2 \frac{3}{2r^3} \phi(r)(1 - \phi_\epsilon(r)) dr \le -\int_{2\epsilon}^1 \frac{3}{2r^3} dr = \frac{3}{4}\left(1 - \frac{1}{4\epsilon^2}\right) < 0, \end{aligned} \tag{A.16}$$

where we used $\epsilon < 1/2$ in the two inequalities above. We thus see from (A.15) that $-(\Lambda w)(0) < 0$ if $\alpha_1 \ne \alpha_2$.

We now turn to the case $n = 3$. This is a direct consequence of the $n = 2$ result. Suppose A_i and A_e are not proportional to each other, and consider the 2×2 submatrices of the 3×3 matrices A_i and A_e. Of the three pairs of 2×2 submatrices extracted from A_i and A_e, at least one pair is not proportional. Without loss of generality, suppose the principal 2×2 submatrices are not proportional, and let them be $\widehat{A}_i$ and $\widehat{A}_e$. Let $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{R}^3, \widehat{\mathbf{x}} = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2$, and likewise for $\mathbf{k}$ and $\widehat{\mathbf{k}}$. Define $\widehat{G}_t(\widehat{\mathbf{x}})$ to be the fundamental solution associated with the two dimensional bidomain operator defined by the matrices $\widehat{A}_i$ and $\widehat{A}_e$:

$$\begin{aligned} \widehat{G}_t(\widehat{\mathbf{x}}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp(-t\widehat{Q}(\widehat{\mathbf{k}})) \exp(i\mathbf{k} \cdot \widehat{\mathbf{x}}) d\widehat{\mathbf{k}}, \\ \widehat{Q}(\widehat{\mathbf{k}}) &= \frac{\widehat{Q}_i(\widehat{\mathbf{k}})\widehat{Q}_e(\widehat{\mathbf{k}})}{\widehat{Q}_i(\widehat{\mathbf{k}}) + \widehat{Q}_e(\widehat{\mathbf{k}})}, \quad \widehat{Q}_{i,e}(\widehat{\mathbf{k}}) = \widehat{\mathbf{k}}^{\mathrm{T}} \widehat{A}_{i,e} \widehat{\mathbf{k}}. \end{aligned} \tag{A.17}$$

By the Fourier inversion formula applied to the variable x_3 and k_3, we have

$$\int_{\mathbb{R}} G_t(\mathbf{x})dx_3 = G_t(\widehat{\mathbf{x}}). \tag{A.18}$$

By the proof of the $n = 2$ case, we know that there is a point $\widehat{\mathbf{x}} \in \mathbb{R}^2$ such that $G_t(\widehat{\mathbf{x}}) < 0$. The above identity implies that there is an $\mathbf{x} \in \mathbb{R}^3$ such that $G_t(\mathbf{x}) < 0$. ◻

Finally, we give an alternate proof of the nonpositivity of G_t which applies when $\sqrt{Q(\mathbf{k})}$ is a nonconvex function. Note that $\sqrt{Q(\mathbf{k})}$ can be convex even when $A_i \neq \beta A_e$, and thus the following proof does not apply to all nonmonodomain cases. (For example, if $|a| \leq 1/2, a \neq 0$, in (1.24), $\sqrt{Q(\mathbf{k})}$ is convex but A_i and A_e are not proportional.) The argument, however, may be of independent interest. We prove the following proposition, which was kindly communicated to us by Professor Yoshikazu Giga.

PROPOSITION A.3 (Giga). *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a function that is a nonconvex positive function and homogeneous of degree one. Then $\mathcal{F}^{-1}[\exp(-p(\mathbf{k})^2)]$ is not nonnegative, where $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^n$.*

Proposition A.3 follows immediately from Bochner's theorem and Lemma A.4 below. Recall that a function $f : \mathbb{R}^n \to \mathbb{C}$ is called a positive definite function if, for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ and any $k \in \mathbb{N}$, the matrix

$$A = \{f_{ij}\} \quad \text{with} \quad f_{ij} = f(\mathbf{x}_i - \mathbf{x}_j),\ 1 \leq i, j \leq k,$$

is positive semidefinite.

LEMMA A.4. *Suppose that $p : \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous of degree one. Then if p is nonconvex, $\exp(-p(x)^2)$ is not a positive definite function.*

Proof. Let $\mathbf{x}_1 = \mathbf{0} \in \mathbb{R}^2$. Since p is not a convex function, the set $\Omega_1 = \{\mathbf{x} \in \mathbb{R}^2 \mid p(\mathbf{x} = 1)\}$ is not a convex set. Thus, we can choose $\mathbf{x}_2, \mathbf{x}_3 \in \Omega_1$ and $\delta > 0$ that satisfy

$$p(\mathbf{x}_2 - \mathbf{x}_3) > p(\mathbf{x}_2) + p(\mathbf{x}_3) + \delta = 2 + \delta. \tag{A.19}$$

Let L be any positive constant. We consider the matrix

$$A_L = \begin{pmatrix} 1 & a & a \\ a & 1 & c \\ a & c & 1 \end{pmatrix},$$

where

$$a = \exp\left(-p(L\mathbf{x}_2)^2\right) = \exp\left(-p(L\mathbf{x}_3)^2\right) = \exp\left(-L^2\right),$$
$$c = \exp\left(-p(L\mathbf{x}_2 - L\mathbf{x}_3)^2\right) < 1.$$

To complete the proof of the lemma, it suffices to show that A_L cannot be positive semidefinite.

Since p is positively homogeneous of degree one, it follows from (A.19) that

$$p(L\mathbf{x}_2 - L\mathbf{x}_3) > (2 + \delta)L.$$

Thus, by using the Taylor series, we have

$$a^2 = \exp(-2L^2) = 1 - 2L^2 + o(L^2),$$
$$c = \exp\left(-p(L\mathbf{x}_2 - L\mathbf{x}_3)^2\right) = 1 - p(L\mathbf{x}_2 - L\mathbf{x}_3)^2 + o(L^2) > 1 - (2 + \delta)^2 L^2 + o(L^2)$$

as $L \to +0$. Consequently, we have

$$|A_L| = (1-c)(1+c-2a^2) > (1-c)(-(4\delta+\delta^2)L^2 + o(L^2))$$

as $L \to +0$. Since $1-c$ is obviously positive, $|A| < 0$ holds when L is sufficiently small. Since A_L is a 3×3 matrix, this means that A is not positive semidefinite. ☐

Acknowledgments. The authors thank the University of Minnesota and the IMA, where some of the work was performed. YM and NM thank Arnd Scheel for helpful discussions. YM thanks Yoshikazu Giga for helpful discussions, who in particular kindly allowed us to include his proof of Proposition A.3 in this paper.

REFERENCES

[1] L. Ambrosio, P. C. Franzone, and G. Savaré, *On the asymptotic behaviour of anisotropic energies arising in the cardiac bidomain model*, Interface Free Bound., 2 (2000), pp. 213–266.
[2] G. Bellettini, P. Colli Franzone, and M. Paolini, *Convergence of front propagation for anisotropic bistable reaction-diffusion equations*, Asymptot. Anal., 15 (1997), pp. 325–358.
[3] G. Bellettini, M. Paolini, and F. Pasquarelli, *Nonconvex mean curvature flow as a formal singular limit of the nonlinear bidomain model*, Adv. Differerential Equations, 18 (2013), pp. 895–934.
[4] Y. Bourgault, Y. Coudière, and C. Pierre, *Existence and uniqueness of the solution for the bidomain model used in cardiac electrophysiology*, Nonlinear Anal. Real World Appl., 10 (2009), pp. 458–482.
[5] X. Chen, *Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations*, Adv. Differential Equations, 2 (1997), pp. 125–160.
[6] P. Colli Franzone, L. Pavarino, and B. Taccardi, *Simulating patterns of excitation, repolarization and action potential duration with cardiac bidomain and monodomain models*, Math. Biosci., 197 (2005), pp. 35–66.
[7] C. M. Elliott, M. Paolini, and R. Schätzle, *Interface estimates for the fully anisotropic Allen-Cahn equation and anisotropic mean-curvature flow*, Math. Models Methods Appl. Sci., 6 (1996), pp. 1103–1118.
[8] C. M. Elliott and R. Schätzle, *The limit of the anisotropic double-obstacle Allen-Cahn equation*, Proc. Roy. Soc. Edinburgh Sect. A, 126 (1996), pp. 1217–1234.
[9] P. C. Fife and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Rational Mech. Anal., 65 (1977), pp. 335–361.
[10] P. C. Franzone, L. F. Pavarino, and S. Scacchi, *Mathematical Cardiac Electrophysiology*, MS&A. Model. Simul. Appl. 13, Springer, Cham, 2014.
[11] P. C. Franzone and G. Savaré, *Degenerate evolution systems modeling the cardiac electric field at micro-and macroscopic level*, in Evolution Equations, Semigroups and Functional Analysis, Springer, Basel, 2002, pp. 49–78.
[12] Y. Giga and N. Kajiwara, *On a resolvent estimate for bidomain operators and its applications*, J. Math. Anal. Appl., 459 (2018), pp. 528–555.
[13] Y. Giga, N. Kajiwara, and K. Kress, *Strong time-periodic solutions to the bidomain equations with arbitrary large forces*, Nonlinear Anal. Real World Appl., 47 (2019), pp. 398–413.
[14] B. E. Griffith and C. S. Peskin, Electrophysiology, Comm. Pure Appl. Math., 66 (2013), pp. 1837–1913.
[15] C. S. Henriquez, *Simulating the electrical behavior of cardiac tissue using the bidomain model*, Crit. Rev. Biomed. Eng., 21 (1992), pp. 1–77.
[16] M. Hieber, A. Hussein, and M. Saal, *Global Strong Well-Posedness of the Stochastic Bidomain Equations with FitzHugh-Nagumo Transport*, preprint, https://arxiv.org/abs/2002.03960, 2020.
[17] T. Kapitula, *Multidimensional stability of planar travelling waves*, Trans. Amer. Math. Soc., 349 (1997), pp. 257–269.
[18] T. Kapitula and K. Promislow, *Spectral and Dynamical Stability of Nonlinear Waves*, Appl. Math. Sci. 185, Springer, New York, 2013.
[19] J. Keener and J. Sneyd, *Mathematical Physiology,* 2nd ed., Springer, New York, 2009.
[20] R. Kobayashi and Y. Giga, *On anisotropy and curvature effects for growing crystals*, Japan J. Indust. Appl. Math., 18 (2001), pp. 207–230.

[21] C. Levermore and J. Xin, *Multidimensional stability of traveling waves in a bistable reaction-diffusion equation* II, Commun. Partial Differ. Equ., 17 (1992), pp. 1901–1924.

[22] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Springer, New York, 2012.

[23] H. Matano, Y. Mori, and M. Nara, *Asymptotic behavior of spreading fronts in the anisotropic Allen-Cahn equation on* $\boldsymbol{R}^n$, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 36 (2019), pp. 585–626.

[24] H. Matano, Y. Mori, M. Nara, and K. Sakakibara, *Asymptotic behavior of fronts and pulses of the bidomain model*, SIAM J. Appl. Dyn. Syst., 21 (2022), pp. 616–649.

[25] H. Matano and M. Nara, *Large time behavior of disturbed planar fronts in the Allen-Cahn equation*, J. Differential Equations, 251 (2011), pp. 3522–3557.

[26] H. Matano, M. Nara, and M. Taniguchi, *Stability of planar waves in the Allen-Cahn equation*, Comm. Partial Differential Equatons, 34 (2009), pp. 976–1002.

[27] Y. Mori and H. Matano, *Stability of front solutions of the bidomain equation*, Comm. Pure Appl. Math., 69 (2016), pp. 2364–2426.

[28] J. Neu and W. Krassowska, *Homogenization of syncytial tissues*, Crit. Rev. Biomed. Eng., 21 (1993), pp. 137–199.

[29] M. Pennacchio, G. Savaré, and P. C. Franzone, *Multiscale modeling for the bioelectric activity of the heart*, SIAM J. Math. Anal., 37 (2006), pp. 1333–1370, https://doi.org/10.1137/040615249.

[30] M. Schechter, *Fredholm Operators and the Essential Spectrum*, report, Courant Institute of Mathematical Sciences, New York, NY, 1965.

[31] L. Tung, *A Bi-domain Model for Describing Ischemic Myocardial D-C Potentials*, Ph.D. thesis, Massachusetts Institute of Technology, Cambridge, MA, 1978.

[32] M. Veneroni, *Reaction-diffusion systems for the macroscopic bidomain model of the cardiac electric field*, Nonlinear Anal. Real World Appl., 10 (2009), pp. 849–868.

[33] J. X. Xin, *Multidimensional stability of traveling waves in a bistable reaction-diffusion equation* I, Comm. Partial Differential Equations, 17 (1992), pp. 1889–1899.