THE KHINCHIN–KAHANE AND LÉVY INEQUALITIES FOR ABELIAN METRIC GROUPS, AND TRANSFER FROM NORMED (ABELIAN SEMI)GROUPS TO BANACH SPACES

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Abstract. The Khinchin–Kahane inequality is a fundamental result in the probability literature, with the most general version to date holding in Banach spaces. Motivated by modern settings and applications, we generalize this inequality to arbitrary metric groups which are abelian.

If instead of abelian one assumes the group's metric to be a norm (i.e., $\mathbb{Z}_{>0}$-homogeneous), then we explain how the inequality improves to the same one as in Banach spaces. This occurs via a "transfer principle" that helps carry over questions involving normed metric groups and abelian normed semigroups into the Banach space framework. This principle also extends the notion of the expectation to random variables with values in arbitrary abelian normed metric semigroups $\mathscr{G}$. We provide additional applications, including studying weakly ℓ_p $\mathscr{G}$-valued sequences and related Rademacher series.

On a related note, we also formulate a "general" Lévy inequality, with two features: (i) It subsumes several known variants in the Banach space literature; and (ii) We show the inequality in the minimal framework required to state it: abelian metric groups.

1. Introduction

The Khinchin–Kahane inequality is a classical inequality in the probability literature. It was initially studied by Khinchin [16] in the real case, and later extended by Kahane [10] to normed linear spaces. A detailed history of the inequality can be found in [18]. We begin by presenting a general version of the inequality for Banach spaces, as well as a sharp constant in some cases.

Definition 1.1. A *Rademacher variable* is a Bernoulli variable that equals ± 1 with probability $\frac{1}{2}$.

Theorem 1.2 (Kahane [10]; Latała and Oleszkiewicz [18]). *For all $p, q \in [1, \infty)$, there exists a universal constant $C_{p,q} > 0$ depending only on p, q, such that for all choices of Banach spaces $\mathbb{B}$, finite sets of vectors $x_1, \dots, x_n \in \mathbb{B}$, and independent Rademacher variables $r_1, \dots, r_n$,*

$$\mathbb{E}\left[\left\|\sum_{k=1}^n r_k x_k\right\|^q\right]^{1/q} \leqslant C_{p,q} \cdot \mathbb{E}\left[\left\|\sum_{k=1}^n r_k x_k\right\|^p\right]^{1/p}. \tag{1.3}$$

If moreover $p = 1 \leqslant q \leqslant 2$, then the constant $C_{1,q} = 2^{1-1/q}$ is optimal.

On this note, see also the work of Kalton [12, Section 6], involving (quasi-)Banach spaces with specified type.

Notice that to state the theorem, one only requires Rademacher – i.e., random symmetric – sums of vectors (see [11, §2.5] for more on Rademacher series). Thus, it is possible to state the result more generally than in a normed linear space: in fact, in any abelian group $\mathscr{G}$ equipped with a

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translation-invariant metric. Now it is natural to ask whether a variant of the Khinchin–Kahane inequality holds in this general (and strictly larger) setting. One of our main results provides a positive answer to this question; see Theorem 2.7.

In working outside Banach spaces, we are motivated by several reasons, both classical and modern. Traditionally, probability theory is now well-established in the Banach space setting; see the classic text [19]. In greater generality, Fourier analysis and Haar measure for locally compact abelian groups, and metric group-valued random variables have been well-studied, see e.g. [9, 25]. In this vein, it is of interest to prove stochastic inequalities in the most primitive framework required to state them. In [13], we showed such a extension of the Hoffmann-Jørgensen inequality for all metric semigroups, followed by applications and other extensions in [14]. The present paper is in a parallel vein, showing an extension of the Khinchin–Kahane inequality to abelian metric groups.

There are also modern reasons to work with metric groups. In modern-day settings, one often studies random variables with values in permutation groups, or more generally, abelian/compact Lie groups such as lattices/tori. Moreover, data can be manifold-valued, living in e.g. real or complex Lie groups rather than in linear spaces. Other frameworks arise from the study of large networks, e.g. the space of graphons with the cut-norm [20], or the space of labelled graphs [15]. The latter is a 2-torsion group, so cannot embed into a Banach space. Thus there is renewed modern interest in studying probability outside the Banach space paradigm. Our work lies firmly in this setting.

We now state our novel contributions. The first – see above – extends the Khinchin–Kahane inequality to abelian metric groups (Theorem 2.7 below). Second, in the course of proving Theorem 2.7, we also provide a two-fold extension of Lévy's inequality; see Theorem 2.10. In keeping with the above philosophy, this unifies at once several existing variants of the inequality in the literature, which to our knowledge had not been consolidated within a common framework. Moreover, we show the result in the minimal framework required to state it: for all abelian metric groups.

In addition to these two results for metric *groups*, we write down a useful "transfer principle" in Theorem 3.3, which holds more generally: for normed metric *semigroups*. Note, probability over metric (semi)groups $\mathscr{G}$ has a fundamental distinction from Banach spaces: the unavailability of an expectation function. It is thus natural to seek classes of metric semigroups $\mathscr{G}$ for which the expectation makes sense for every $\mathscr{G}$-valued random variable. Theorem 3.3 provides such a class: where $\mathscr{G}$ is in fact "normed" (defined below) and abelian – or merely normed if $\mathscr{G}$ is a group. In this case the expectation does not necessarily live in $\mathscr{G}$, but in its "Banach space closure" (below).

Theorem 3.3 enables defining linear functionals, operator spaces, and dual spaces over all abelian normed semigroups $\mathscr{G}$, and therefore extends the powerful theory of functional analysis to all such semigroups. Furthermore, we provide several applications of Theorem 3.3, including extending the notion of weakly ℓ_p-sequences to $\mathscr{G}$-valued random variables; as a consequence, several results in the probability literature, including those of Dilworth and Montgomery-Smith [6] (as well as prior results of Talagrand) extend to abelian normed semigroups and to all normed groups. See Section 3 for these applications of our results; the assertions about normed groups make use of a fact on bi-invariant metrics from geometry, which arose from the present body of work, and has been resolved in a recent Polymath project [24].

2. Khinchin–Kahane inequality and Lévy inequality for abelian metric groups

We begin by extending the Khinchin–Kahane inequality from Banach spaces to arbitrary abelian metric groups. We also prove a general version of Lévy's inequality, for abelian metric groups.

2.1. Metric semigroups.

Before the main results, we study basic properties of metric (semi)groups.

Definition 2.1. A *metric semigroup/monoid/group* is defined to be a semigroup/monoid/group $(\mathscr{G}, \cdot)$ equipped with a metric $d : \mathscr{G} \times \mathscr{G} \to [0, \infty)$ that is translation-invariant. In other words,

$$d(ac, bc) = d(a, b) = d(ca, cb), \qquad \forall a, b, c \in \mathscr{G}.$$

To begin, note that for such a semigroup $\mathscr{G}$ – for instance if $\mathscr{G}$ is abelian – the following "triangle inequality" is straightforward, and used below without further reference:

$$d(y_1y_2, z_1z_2) \leqslant d(y_1, z_1) + d(y_2, z_2), \quad \forall y_k, z_k \in \mathscr{G}. \tag{2.2}$$

We will also require the following preliminary result and its corollary.

Proposition 2.3. *Suppose $(\mathscr{G}, \cdot, d)$ is a metric semigroup, and $a, b \in \mathscr{G}$. Then*

$$d(a, ba) = d(b, b^2) = d(a, ab) \tag{2.4}$$

is independent of $a \in \mathscr{G}$. Moreover, $\mathscr{G}$ has at most one idempotent (i.e., $b \in \mathscr{G}$ such that $b^2 = b$). Such an element b is automatically the unique two-sided identity in $\mathscr{G}$, making it a metric monoid.

Proof. Equation (2.4) is immediate using the translation-invariance of d:

$$d(a, ba) = d(ba, b^2a) = d(b, b^2) = d(ab, ab^2) = d(a, ab).$$

Next, if $\mathscr{G}$ has idempotents b, b', then using Equation (2.4),

$$d(b, b') = d(b^2, bb') = d(b^2, b^2b') = d(b, bb') = d(b', (b')^2) = 0.$$

Hence $b = b'$. Moreover, given such an idempotent $b \in \mathscr{G}$, compute using Equation (2.4):

$$d(a, ba) = d(a, ab) = d(b, b^2) = 0, \quad \forall a \in \mathscr{G}.$$

Hence b is automatically the unique two-sided identity in $\mathscr{G}$. □

Corollary 2.5. *A set $\mathscr{G}$ is a metric semigroup only if $\mathscr{G}$ is either a metric monoid, or the set of non-identity elements in a metric monoid $\mathscr{G}'$. This is if and only if the number of idempotents in $\mathscr{G}$ is one or zero, respectively. Moreover, the metric monoid $\mathscr{G}'$ is (up to a monoid isomorphism) the unique smallest element in the class of metric monoids containing $\mathscr{G}$ as a sub-semigroup. A finite metric semigroup is a metric group.*

Proof. Given a semigroup $\mathscr{G}$ that is not already a monoid, to adjoin an "identity" element $1'$ one defines: $1' \cdot a = a \cdot 1' := a \ \forall a \in \mathscr{G}' := \mathscr{G} \sqcup \{1'\}$. Also extend $d_{\mathscr{G}}$ to $d_{\mathscr{G}'} : \mathscr{G}' \times \mathscr{G}' \to [0, \infty)$ via: $d_{\mathscr{G}'}(1', 1') = 0$ and $d_{\mathscr{G}'}(1', b) = d_{\mathscr{G}'}(b, 1') := d_{\mathscr{G}}(b, b^2)$ for $b \in \mathscr{G}$. Then $\mathscr{G}'$ is a metric monoid. The next assertion follows from Proposition 2.3. It is clear that the monoid $\mathscr{G}' \supseteq \mathscr{G}$ is uniquely determined. The final part holds since left- and right- multiplication by any $a \in \mathscr{G}$ are bijections. □

Remark 2.6. We will denote the unique metric monoid containing a given metric semigroup $\mathscr{G}$ by $\mathscr{G}' := \mathscr{G} \cup \{1'\}$. Note that the idempotent $1'$ may already be in $\mathscr{G}$, in which case $\mathscr{G} = \mathscr{G}'$. One consequence of Corollary 2.5 is that instead of working with metric semigroups, one can use the associated monoid $\mathscr{G}'$ instead. (In other words, the (non)existence of the identity is not an issue in such cases.) This helps simplify other calculations. For instance, what would usually be a lengthy, inductive computation now becomes much simpler: by the triangle inequality (2.2), for all $k, l \geqslant 0$,

$$d_{\mathscr{G}}(z_0 \cdots z_k, z_0 \cdots z_{k+l}) = d_{\mathscr{G}'}(1', \prod_{i=1}^{l} z_{k+i}) \leqslant \sum_{i=1}^{l} d_{\mathscr{G}'}(1', z_{k+i}) = \sum_{i=1}^{l} d_{\mathscr{G}}(z_0, z_0 z_{k+i}), \forall z_0, \dots, z_{k+l} \in \mathscr{G}.$$

2.2. **Khinchin–Kahane and Lévy inequalities.** In the sequel, we will deal mostly with metric (semi)groups $\mathscr{G}$ that are abelian. Thus we mostly use additive notation – and only where we do not, should the reader assume that $\mathscr{G}$ need not *a priori* be abelian.

Our first main result is a Khinchin–Kahane (type) inequality for arbitrary abelian metric groups.

Theorem 2.7 (Khinchin–Kahane)**.** *Given $q \in [1, \infty)$, there exists a universal constant $K_q > 0$ depending only on q such that for all choices of abelian metric groups $\mathscr{G}$, finite sequences of elements $x_1, \dots, x_n \in \mathscr{G}$ (for any $n > 0$), independent Rademacher variables $r_1, \dots, r_n$, and scalar $p \in [1, \infty)$,*

$$\mathbb{E}_\mu \left[d\left(0, 2^l \sum_{k=1}^{n} r_k x_k\right)^q \right]^{1/q} \leqslant K_q \cdot \mathbb{E}_\mu \left[d\left(0, \sum_{k=1}^{n} r_k x_k\right)^p \right]^{1/p}, \tag{2.8}$$

where $l \in \mathbb{Z}_{>0}$ is such that $2^{l-1} \leqslant q < 2^l$. In fact we may choose $K_q = 64q^2(q/4)^{1/q}$.

Existing variants in the literature fall under the special case where $\mathscr{G} = \mathbb{B}$ is a Banach space (1.3). Note that the inequality (2.8) in this more general case does not compare the same quantities as the classical Khinchin–Kahane inequality (1.3) does, and to the best of our knowledge, is a novel inequality that does not follow from the Banach space version (1.3). (In Section 3, we will see how a "norm" on $\mathscr{G}$ updates both the expressions in (2.8) and the constant K_q to yield the Banach-space counterparts.) Also remark for completeness that a separability assumption on $\mathscr{G}$ is not required, since one may restrict to the subgroup generated by $x_1, \dots, x_n$.

Theorem 2.7 provides an example of "universal constants" which help compare L^p-norms of sums of independent $\mathscr{G}$-valued variables, across various $p > 0$. This theme is explored in greater detail and generality for abelian metric semigroups in related work [14].

The proof of Theorem 2.7 uses Lévy's inequality for abelian metric groups $\mathscr{G}$. To this end, we first define symmetric random variables and show a general version of Lévy's inequality over $\mathscr{G}$.

Definition 2.9. If $(\mathscr{G}, +, 0, d)$ is an abelian metric group and I is a totally ordered finite set, then a tuple $(X_i)_{i \in I}$ of random variables in $L^0(\Omega, \mathscr{G})$ is *symmetric* if for all subsets $J \subseteq I$ and all functions $\varepsilon : J \to \{\pm 1\}$, the $2^{|J|}$ ordered tuples of variables $(\varepsilon(j) X_j)_{j \in J}$ all have the same joint distribution – i.e., this is independent of ε.

Theorem 2.10 (Lévy's inequality)**.** *Fix an abelian metric group $(\mathscr{G}, +, 0, d)$, integers $m, n \in \mathbb{Z}_{>0}$, and symmetric random variables $X_1, \dots, X_n \in L^0(\Omega, \mathscr{G})$. Also fix subsets $B_1, \dots, B_m \subseteq \{1, \dots, n\}$, such that for all $1 \leqslant j \leqslant k \leqslant m$, $B_j \cap B_k$ is either B_j or empty. Set $X_B := \sum_{b \in B} X_b$ for all $B \subseteq \{1, \dots, n\}$. If $S_n = X_1 + \dots + X_n$, then for all $s, t > 0$,*

$$\mathbb{P}_\mu \left(\max_{1 \leqslant k \leqslant m} d(0, 2X_{B_k}) > s + t \right) \leqslant \mathbb{P}_\mu \left(d(0, S_n) > s \right) + \mathbb{P}_\mu \left(d(0, S_n) > t \right). \tag{2.11}$$

Note that if $\mathscr{G}$ is a normed linear space and $s = t$, then the left-hand side is concerned with the event that $\max_{1 \leqslant k \leqslant m} \|2X_{B_k}\| > 2t$, which is how the inequality usually appears in the literature.

It is the universal formulation and generalization of the result that is of note here. Indeed, Theorem 2.10 simultaneously strengthens several different variants in the literature, which to our knowledge had not previously been unified. See [19, Proposition 2.3] for two special cases where $\mathscr{G}$ is a Banach space, $s = t$, $m = n$, and $B_k = \{1, \dots, k\}$ or $B_k = \{k\}$ for all k. Theorem 2.10 also holds for other choices of subsets B_k, e.g. $\{1\}, \{1, 2\}, \{3, 4, 5\}, \{3, 4, 5, 6\}$; or $B_k = \{n - k + 1, \dots, n\}$ by reversing the order of summation; this last choice is used below. Moreover, Theorem 2.10 extends Lévy's inequality from Banach spaces to all abelian metric groups. We provide a formal proof as it is in a more general setting than what is available in the literature.

Proof of Theorem 2.10. Define the stopping time

$$\tau := \min\{1 \leqslant k \leqslant m : d(0, 2X_{B_k}) > s + t\}. \tag{2.12}$$

Now note that there is a smallest integer $1 \leqslant m_k \leqslant k$ such that $B_{m_k+1}, B_{m_k+2}, \dots, B_{k-1} \subseteq B_k$. By assumption, $B_1, \dots, B_{m_k}$ are all disjoint from B_k. Thus the event $\tau = k$, which denotes

$$d(0, 2X_{B_j}) \leqslant s + t < d(0, 2X_{B_k}) \quad \forall 1 \leqslant j \leqslant k - 1$$

can be represented also as the event

$$d(0, -2X_{B_j}) \leqslant s + t \quad \forall 1 \leqslant j \leqslant m_k, \qquad d(0, 2X_{B_j}) \leqslant s + t < d(0, 2X_{B_k}) \quad \forall m_k < j < k.$$

Thus let $\mathbf{X}_r := X_{\cup_{j=1}^r B_j}$ for all r. Then it follows from the symmetry assumption that

$$\begin{aligned} \mathbb{P}_\mu \left(d(0, S_n) > t, \tau = k \right) &= \mathbb{P}_\mu \left(d(0, (-\mathbf{X}_{m_k}) + X_{B_k} - (S_n - \mathbf{X}_{m_k} - X_{B_k})) > t, \tau = k \right) \\ &= \mathbb{P}_\mu \left(d(0, 2X_{B_k} - S_n) > t, \tau = k \right). \end{aligned}$$

We now prove the result. Note by the triangle inequality that

$$d(0, 2X_{B_k}) > s + t \quad \implies \quad d(0, S_n) > s \quad \text{or} \quad d(0, 2X_{B_k} - S_n) > t.$$

Thus by the above analysis, the result follows:

$$\begin{aligned}
&\mathbb{P}_\mu\left(\max_{1\leqslant k\leqslant m} d(0, 2X_{B_k}) > s + t\right) = \sum_{k=1}^m \mathbb{P}_\mu\left(\tau = k,\ d(0, 2X_{B_k}) > s + t\right) \\
&\leqslant \sum_{k=1}^m \left(\mathbb{P}_\mu\left(d(0, S_n) > s, \tau = k\right) + \mathbb{P}_\mu\left(d(0, 2X_{B_k} - S_n) > t, \tau = k\right)\right) \\
&= \mathbb{P}_\mu\left(d(0, S_n) > s,\ \tau \in [1, m]\right) + \mathbb{P}_\mu\left(d(0, S_n) > t,\ \tau \in [1, m]\right).
\end{aligned}$$

□

We next show the Khinchin–Kahane inequality (2.8).

Proof of Theorem 2.7. For this proof, fix an abelian metric group $\mathscr{G}$, elements $x_1, \dots, x_n \in \mathscr{G}$, and Rademacher variables $r_1, \dots, r_n$. For ease of exposition we break the proof into two steps.

Step 1. We claim that *for all abelian metric groups $\mathscr{G}$ and $\mathscr{G}$-valued Rademacher sums $\sum_{k=1}^n r_k x_k$,*

$$\begin{aligned}
&\mathbb{P}_\mu\left(d(0, 2\sum_{k=1}^n r_k x_k) > s + t + u + v\right) \\
&\leqslant \left(\mathbb{P}_\mu\left(P_n > s\right) + \mathbb{P}_\mu\left(P_n > t\right)\right) \cdot \left(\mathbb{P}_\mu\left(P_n > u\right) + \mathbb{P}_\mu\left(P_n > v\right)\right)
\end{aligned} \tag{2.13}$$

for all $s, t, u, v > 0$, where $P_n := d\left(0, \sum_{k=1}^n r_k x_k\right)$.

Existing variants in the literature are usually special cases with $\mathscr{G} = \mathbb{B}$ a Banach space and $s = t = u = v$. While the proof uses familiar arguments, we include it for the reader's convenience, as it is in somewhat greater generality than can usually be found in the literature.

Define $S_k := \sum_{j=1}^k r_j x_j$ for $k \in [1, n]$. Similar to the proof of Lévy's inequality (Theorem 2.10), define the stopping time $\tau := \min\{k \in [1, n] : d(0, 2S_k) > s + t\}$. Also recall that $(r_1, \dots, r_n)$ and $(r_1, \dots, r_k, r_k r_{k+1}, \dots, r_k r_n)$ are identically distributed. Therefore (using that $d(0, g) \equiv d(0, -g)$),

$$\begin{aligned}
&\mathbb{P}_\mu\left(d(2S_{k-1}, 2S_n) > u + v, \tau = k\right) = \mathbb{P}_\mu\left(d(0, 2\sum_{j=k}^n r_j x_j) > u + v, \tau = k\right) \\
&= \mathbb{P}_\mu\left(d(0, 2\sum_{j=k}^n r_k r_j x_j) > u + v, \tau = k\right) \\
&= \mathbb{P}_\mu\left(d(0, 2x_k + 2\sum_{j=k+1}^n r_k r_j x_j) > u + v, \tau = k\right) \\
&= \mathbb{P}_\mu\left(d(0, 2x_k + 2\sum_{j=k+1}^n r_j x_j) > u + v, \tau = k\right) = \mathbb{P}_\mu\left(d(2x_k + 2S_n, 2S_k) > u + v, \tau = k\right).
\end{aligned}$$

The same argument without restricting to the event $\tau = k$ shows that:

$$\mathbb{P}_\mu\left(d(2S_{k-1}, 2S_n) > u + v\right) = \mathbb{P}_\mu\left(d(2x_k + 2S_n, 2S_k) > u + v\right).$$

Now note that if $d(0, 2S_n(\omega)) > s+t+u+v$ and $\tau(\omega) = k$, then $d(2S_{k-1}(\omega), 2S_n(\omega)) > u+v$. Since $\tau = k$ and $d(2S_k, 2x_k + 2S_n)$ are independent, we compute:

$$\begin{aligned}&\mathbb{P}_\mu\left(d(0,2S_n) > s+t+u+v, \tau = k\right) \leqslant \mathbb{P}_\mu\left(d(2S_{k-1}, 2S_n) > u+v, \tau = k\right)\\ &= \mathbb{P}_\mu\left(d(2x_k + 2S_n, 2S_k) > u+v\right)\mathbb{P}_\mu\left(\tau = k\right) = \mathbb{P}_\mu\left(d(2S_{k-1}, 2S_n) > u+v\right)\mathbb{P}_\mu\left(\tau = k\right)\\ &\leqslant \mathbb{P}_\mu\left(\tau = k\right)\left(\mathbb{P}_\mu\left(P_n > u\right) + \mathbb{P}_\mu\left(P_n > v\right)\right),\end{aligned}$$

by using Lévy's inequality (Theorem 2.10) with $m = 1, B_1 = \{k, \dots, n\}, X_l = 2r_l x_l\ \forall l \geqslant k$, and replacing (s,t) by (u,v). Now another application of Lévy's inequality with the same choice of parameters – except with $B_k = \{1, \dots, k\}$ – concludes the proof.

Step 2. We now prove the inequality (2.8) for $p, q \geqslant 1$. Repeatedly applying the inequality (2.13),

$$\mathbb{P}_\mu\left(d(0, 2^l S_n) > 4^l r\right) \leqslant 4^{2^l - 1}\mathbb{P}_\mu\left(d(0, S_n) > r\right)^{2^l}, \qquad \forall l \in \mathbb{Z}_{>0}. \tag{2.14}$$

Set l to be the unique positive integer such that $2^{l-1} \leqslant q < 2^l$, and change variables $t = 4^l r \in (0, \infty)$. Using that $\mathbb{E}_\mu[Z^q] = q\int_0^\infty t^{q-1}\mathbb{P}_\mu\left(Z > t\right)\ dt$ for an L^q random variable $Z \geqslant 0$, we compute:

$$\begin{aligned}\mathbb{E}_\mu[d(0, 2^l S_n)^q] &= q\int_0^\infty (4^l r)^{q-1}\mathbb{P}_\mu\left(d(0, 2^l S_n) > 4^l r\right)\cdot 4^l\ dr\\ &\leqslant q 4^{lq + 2^l - 1}\int_0^\infty r^{q-1}\mathbb{P}_\mu\left(d(0, S_n) > r\right)^{2^l}\ dr.\end{aligned}$$

Now $4^{lq} \leqslant (2q)^{2q}$ and $r\mathbb{P}_\mu\left(d(0, S_n) > r\right) \leqslant \mathbb{E}_\mu[d(0, S_n)]$ by Markov's inequality. Therefore,

$$\mathbb{E}_\mu[d(0, 2^l S_n)^q] \leqslant (2q)^{2q}4^{2q-1}q\int_0^\infty \mathbb{E}_\mu[d(0, S_n)]^{q-1}\cdot \mathbb{P}_\mu\left(d(0, S_n) > r\right)\ dr = \frac{(8q)^{2q+1}}{32}\mathbb{E}_\mu[d(0, S_n)]^q.$$

Taking qth roots and using Hölder's inequality now yields (2.8). $\square$

3. The transfer principle for normed (abelian semi)groups and its applications

In this section we formulate and prove the transfer principle promised above, which will allow one to take random variables and their probability/functional analysis from a subclass of abelian metric semigroups to Banach spaces. We begin by introducing the key notion required for this.

Definition 3.1. We say that a (possibly non-abelian) metric semigroup $(\mathscr{G}, \cdot, d)$ is *normed* if

$$d(g, g^{n+1}) = n d(g, g^2), \qquad \forall g \in \mathscr{G},\ n \in \mathbb{Z}_{\geqslant 0}. \tag{3.2}$$

Notice that if $\mathscr{G}$ is an abelian metric group, then (3.2) implies the following stronger version:

$$d(ng, nh) = |n| d(g, h), \qquad \forall g, h \in \mathscr{G},\ n \in \mathbb{Z}.$$

There is extensive literature on the analysis of topological semigroups with translation-invariant metrics and related structures. See [1, 3, 7] and the references therein for more on the subject. These references call any group with a metric (under which the inverse map is an isometry) a "normed" group, while the above condition (3.2) is termed $\mathbb{Z}_{>0}$-*homogeneity*. However, in Definition 3.1 we instead adopt the notation of [26], and define a norm to be more in the flavor of Banach spaces. The objects in Definition 2.1 will be called metric (semi)groups in this paper.

3.1. The transfer principle and its applications to abelian normed semigroups.

We now return to our extension (2.8) of the traditional Khinchin–Kahane inequality. Notice that if the abelian metric group $\mathscr{G}$ is moreover normed, then the left-hand side of (2.8) equals a scalar times the original left-hand side in (1.3), and so it is natural to ask if Theorem 2.7 can be modified to yield the same (improved) constants as in the Banach space case (1.3).

It turns out that this is indeed possible, as we explain presently using our next result, Theorem 3.3, which says that $\mathscr{G}$ "is" a subspace/additive subgroup of a Banach space – and constructs a candidate for the "smallest" such Banach space. We show this result for all abelian normed *semigroups* $\mathscr{G}$. Of course, the Khinchin–Kahane inequality works with metric groups, so only the special case of our next result, wherein $\mathscr{G}$ is a(n abelian) normed group, is required to address our motivation in the preceding paragraph. This special case can be found in the literature, see below.

Theorem 3.3 (Transfer principle)**.** *Every (separable) abelian normed metric semigroup $\mathscr{G}$ canonically and isometrically embeds into a "smallest" (separable) real Banach space $\mathbb{B}(\mathscr{G})$. The same holds if $\mathscr{G}$ is a normed (but not a priori abelian) metric group. In particular, the theory of Bochner integration and expectations extends to all such (semi)groups $\mathscr{G}$: if X is $\mathscr{G}$-valued then $\mathbb{E}_\mu[X] \in \mathbb{B}(\mathscr{G})$.*

Remark 3.4. While the results in this work hold only for groups that are abelian, we stress that the abelian hypothesis is *not* required in the second assertion in Theorem 3.3. See Theorem 3.24.

The proof of Theorem 3.3 is related to previous constructions in the literature, and we defer it to later in this section. At present, we discuss some of its consequences and applications.

Example 3.5. The first application is that our Khinchin–Kahane inequality (2.8) can be refined for *normed* metric groups $\mathscr{G}$ to yield precisely the (sharp) constants in the Banach space setting:

Proposition 3.6. *For all normed groups $\mathscr{G}$, one has the "usual" Khinchin–Kahane inequality, with a universal constant $C_{p,q}$ (for fixed $p, q \geqslant 1$ but universal across all normed $\mathscr{G}$ and n, x_k, r_k):*

$$\mathbb{E}_\mu \left[d\left(0, \sum_{k=1}^n r_k x_k\right)^q \right]^{1/q} \leqslant C_{p,q} \cdot \mathbb{E}_\mu \left[d\left(0, \sum_{k=1}^n r_k x_k\right)^p \right]^{1/p}. \tag{3.7}$$

Moreover, for all p, q, the constant $C_{p,q}$ (universal over the category of all abelian normed groups) is equal to the universal constant when working only with the sub-category of all real Banach spaces.

Recall that in the classic paper [18], Latała–Oleszkiewicz obtained the optimal such universal constant across all Banach spaces in the regime $p = 1 \leqslant q \leqslant 2$, namely, $C_{1,q} = 2^{1-1/q}$. Proposition 3.6 shows that $C_{1,q}$ also works for the Khinchin–Kahane inequality in (abelian) normed metric groups. The proof is immediate: consider all $\mathscr{G}$-valued random variables to now be $\mathbb{B}(\mathscr{G})$-valued.

We now explain several other applications – all of which involve abelian normed *semigroups* (noting that such results/applications are not discussed even for normed groups in the literature):

Example 3.8. More broadly than Proposition 3.6, Theorem 3.3 provides a route to "transfer" problems from abelian normed metric semigroups to Banach spaces. For instance, Lévy's equivalences between modes of stochastic convergence of sums of independent $\mathscr{G}$-valued random variables immediately follow from their Banach space counterparts for $\mathbb{B}(\mathscr{G})$, e.g. [19, Theorem 2.4].

Example 3.9. A third – and more challenging – application of Theorem 3.3 is to extend to normed $\mathscr{G}$ the main result of [6], which provides universal constants that occur in bounding vector-valued Rademacher series. We now extend this theorem to arbitrary normed $\mathscr{G}$ (and the $K^w_{1,2}$ in the statement of the next result will be explained following Corollary 3.14). Note that such an extension result is not immediate as one has to first understand better the notion of "linear functionals" on $\mathscr{G}$. This is carried out below; in what follows, $\|g\|$ denotes $d(0, g)$.

Theorem 3.10. *Fix an i.i.d. sequence of Rademacher variables $\varepsilon_n \sim \mathrm{Unif}\{-1, 1\}$. Then there exists an absolute constant $c > 0$ such that for all choices of (a) separable abelian normed metric semigroups $\mathscr{G}$, (b) points $g_n \in \mathscr{G}$ such that the Rademacher series $X := \sum_n \varepsilon_n g_n$ is almost surely convergent (e.g. in $\mathbb{B}(\mathscr{G})$), and (c) scalars $t > 0$, we have:*

$$\mathbb{P}_\mu \left(\|X\| > 2\mathbb{E}\|X\| + 6K^w_{1,2}((g_n), t) \right) \leqslant 4e^{-t^2/8}, \tag{3.11}$$

$$\mathbb{P}_\mu \left(\|X\| > \frac{1}{2}\mathbb{E}\|X\| + cK^w_{1,2}((g_n), t) \right) \geqslant ce^{-t^2/c}. \tag{3.12}$$

Observe that the results of Talagrand [29] that are cited in [6] also extend to abelian normed semigroups, as does the observation that opens the proof of the main theorem in [6]:

Proposition 3.13. *All separable abelian normed semigroups are "isometric sub-semigroups" of ℓ_∞.*

Furthermore, the various applications of (the version of) Theorem 3.10 in [6] also hold in abelian normed semigroups. These include the following "semigroup-valued" precise form of the Khinchin–Kahane inequality, in a sense bringing us back full circle to Theorem 2.7.

Corollary 3.14 (see [6, Corollary 3])**.** *As above, let $X := \sum_n \varepsilon_n g_n$ be an almost surely convergent Rademacher series with all g_n in a separable abelian normed metric semigroup $\mathscr{G}$. Then there is an absolute constant $c > 0$ such that*

$$\frac{1}{c}\mathbb{E}[\|X\|^p]^{1/p} \leqslant \mathbb{E}\|X\| + K^w_{1,2}((g_n), \sqrt{p}) \leqslant c\mathbb{E}[\|X\|^p]^{1/p}, \quad \forall p \in [1, \infty).$$

Let us explain the preceding theorem (and hence its corollary), and in particular, why these results are not direct applications of the transfer principle in Theorem 3.3. In Theorem 3.10 and Corollary 3.14, the constant $K^w_{1,2}((g_n), t)$ was used for scalars $t > 0$. In the original setting of [6], defining this constant involves Banach space analysis and weakly ℓ_p sequences. We now extend this definition to all abelian normed semigroups $\mathscr{G}$. For $p \in [1, \infty)$, we say a sequence of points $(g_n)_n$ in $\mathscr{G}$ is *weakly ℓ_p* if $(g^*(g_n))_n$ is ℓ_p for every $g^* \in \mathscr{G}^*$, where $\mathscr{G}^*$ denotes the set of additive Lipschitz real-valued maps on $\mathscr{G}$. Note, this differs from the Banach space definition, which would require running over all functionals in $\mathbb{B}(\mathscr{G})^*$ (or the dual space to a larger Banach space), via Theorem 3.3.

Now define for a weakly ℓ_2 sequence (g_n) and a scalar sequence $(a_n) \in \ell_2$:

$$\begin{aligned} K_{1,2}((a_n), t) &:= \inf\{\|(a_{1,n})\|_1 + t\|(a_{2,n})\|_2 : a_n = a_{1,n} + a_{2,n}\ \forall n,\ (a_{j,n})_n \in \ell_j \text{ for } j = 1, 2\}, \\ K^w_{1,2}((g_n), t) &:= \sup\{K_{1,2}((g^*(g_n))_n, t) : g^* \in \mathscr{G}^*, \|g^*\| \leqslant 1\}. \end{aligned}$$

Then the key is that the $K^w_{1,2}$-value computed using $\mathscr{G}^*$ exactly matches the Banach space version that uses $\mathbb{B}(\mathscr{G})^*$ (hence the results of [6] extend to abelian normed semigroups), by our next result:

Proposition 3.15. *For $\mathscr{G}$ an abelian normed semigroup, let $\mathscr{G}^*$ denote the set of additive Lipschitz real-valued maps on $\mathscr{G}$. Then $\mathscr{G}^*$ is a Banach space, which coincides with the dual space construction if $\mathscr{G}$ is a Banach space. More generally if $\mathscr{G}$ is an abelian normed semigroup, then $\mathscr{G}^* \simeq \mathbb{B}(\mathscr{G})^*$.*

As this paper focuses on probability inequalities and analysis-related constructions, we defer the proof of Proposition 3.15 to a standalone appendix on a *category-theoretic* treatment of normed *modules* and their properties, for the interested reader; see Proposition A.7. In particular, as noted in [6], the assignment $t \mapsto K^w_{1,2}((g_n), t)$ is Lipschitz with Lipschitz constant at most

$$\ell^w_2((g_n)) := \sup_{\|g^*\| \leqslant 1} \|(g^*(g_n))\|_2$$

(where g^* runs over $\mathscr{G}^*$), and Theorem 3.10 holds over all abelian normed metric semigroups.

Example 3.16. As additional consequences of our "transfer principle" in Theorem 3.3, the main results in [21, 22] immediately extend to arbitrary abelian normed semigroups.

3.2. **Banach space embeddings.** We next return to (the proof of) Theorem 3.3, and discuss it vis-a-vis the question of embedding a given topological group into a Banach space. The theorem says that for a metric (semi)group $(\mathscr{G}, \cdot, d)$, the assumption of being (abelian and) normed is sufficient to embed $\mathscr{G}$ into a Banach space. Clearly, the assumption is also necessary. The next result provides additional equivalent conditions when $\mathscr{G}$ is a group, and also relates it to results in the literature.

Definition 3.17. Given $J \subseteq \mathbb{Z}_{>0}$, a (possibly non-abelian) metric semigroup $(\mathscr{G}, d)$ is *J-normed* if

$$d(z_0, z_0^{n+1}) = nd(z_0, z_0^2), \qquad \forall z_0 \in \mathscr{G}, n \in J. \tag{3.18}$$

Proposition 3.19. *Suppose $\mathscr{G}$ is a topological group, with a continuous map $\|\cdot\| : \mathscr{G} \to [0,\infty)$ satisfying: (a) $\|g\| = 0$ if and only if $g = 1$; (b) $\|g^{-1}\| = \|g\|$ for all $g \in \mathscr{G}$; and (c) the triangle inequality holds: $\|gh\| \leqslant \|g\| + \|h\|$ for $g, h \in \mathscr{G}$. Then the following are equivalent:*

1. *There exists a Banach space $\mathbb{B}$ and a group map $: \mathscr{G} \to (\mathbb{B}, +)$ that is an isometric embedding.*
2. *$\mathscr{G}$ is abelian and $d(g,h) := \|g^{-1}h\|$ is a translation-invariant metric for which $\mathscr{G}$ is normed.*
3. *$\mathscr{G}$ is $\{2\}$-normed and is* weakly commutative*, i.e., for all $g, h \in \mathscr{G}$ there exists $n = n(g,h) \in \mathbb{Z}_{>0}$ such that $(gh)^{2^n} = g^{2^n} h^{2^n}$.*
4. *$\mathscr{G}$ is $\{2\}$-normed and amenable.*

In fact there is a fifth (*a priori* weaker than (2) or (4), yet) equivalent condition – that $\mathscr{G}$ is $\{2\}$-normed *without* additional restrictions – which we explain in Theorem 3.24.

Proof. That (1) $\implies$ (2) $\implies$ (3) is immediate. That (3) or (4) implies (1) follows from [3, Proposition 4.12] via [7, Corollary 1]. This is a constructive proof, and the formula for the Banach space in question is discussed later in this subsection. Finally, that (1) $\implies$ (4) follows since every abelian group is amenable (see [5, 23, 30] for more on amenable groups). □

In this connection, the following result shows (as a special case) that even without requiring the semigroup to be abelian, the "normed" property of a translation-invariant metric on a semigroup:

$$d(z_0, z_0^{n+1}) = n\, d(z_0, z_0^2), \qquad \forall z_0 \in \mathscr{G},\ n \in \mathbb{Z}_{\geqslant 0}$$

already follows from – hence is equivalent to – the "doubling" property of being $\{2\}$-normed: $d(z_0, z_0^3) = 2d(z_0, z_0^2)$ for all $z_0 \in \mathscr{G}$. We omit the proof as it is a variant of [7, Lemma 1].

Lemma 3.20. *Given a nonempty subset $J \subseteq \mathbb{Z}_{>0}$, $J \neq \{1\}$, a metric semigroup $\mathscr{G}$ is J-normed if and only if $\mathscr{G}$ is $\mathbb{Z}_{>0}$-normed.*

We next constructively prove the "transfer principle" above.

Proof of Theorem 3.3. The first point is that every normed metric group is abelian by Theorem 3.24 below (see [24]), and so the second assertion reduces to the first. To show the first assertion, we will use additive notation throughout this proof as $\mathscr{G}$ is abelian. The proof is constructive, and carried out in stages; however, an outline is in the following equation:

$$\mathscr{G}_{\mathbb{N}} := \mathscr{G} \hookrightarrow \mathscr{G}_{\mathbb{N}\cup\{0\}} := \mathscr{G}' \hookrightarrow \mathscr{G}_{\mathbb{Z}} := \mathbb{Z} \otimes_{\mathbb{N}\cup\{0\}} \mathscr{G}_{\mathbb{N}\cup\{0\}} \hookrightarrow \mathscr{G}_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathscr{G}_{\mathbb{Z}} \hookrightarrow \mathbb{B}(\mathscr{G}) := \overline{\mathscr{G}_{\mathbb{Q}}}, \tag{3.21}$$

where $\mathbb{N} := \mathbb{Z}_{>0}$. We now explain these steps one by one.

1. Embed the semigroup into a metric monoid $\mathscr{G}'$ via Corollary 2.5. We label $\mathscr{G}_{\mathbb{N}} := \mathscr{G}$ and $\mathscr{G}_{\mathbb{N}\cup\{0\}} := \mathscr{G}'$ to denote that $\mathscr{G}, \mathscr{G}'$ are "modules" over $\mathbb{N}, \mathbb{N}\cup\{0\}$ respectively.
2. It is easily shown that $\mathscr{G}_{\mathbb{N}}$ and hence $\mathscr{G}_{\mathbb{N}\cup\{0\}}$ is cancellative. Therefore the monoid $\mathscr{G}_{\mathbb{N}\cup\{0\}}$ embeds into its Grothendieck group[1] $\mathscr{G}_{\mathbb{Z}}$ (which is a $\mathbb{Z}$-module) by attaching additive inverses and quotienting by an equivalence relation. Extend the metric $d_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}$ to $\mathscr{G}_{\mathbb{Z}}$ via: $d_{\mathscr{G}_{\mathbb{Z}}}(p - q, r - s) := d_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}(p+s, q+r)$, for all $p,q,r,s \in \mathscr{G}_{\mathbb{N}\cup\{0\}}$. Then $(\mathscr{G}_{\mathbb{Z}}, 0_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}, d_{\mathscr{G}_{\mathbb{Z}}})$ is an abelian metric group and $\mathscr{G}_{\mathbb{N}} \hookrightarrow \mathscr{G}_{\mathbb{N}\cup\{0\}} \hookrightarrow \mathscr{G}_{\mathbb{Z}}$ are isometric (hence injective) semigroup/monoid homomorphisms. $\mathscr{G}_{\mathbb{Z}}$ is also normed since for all $n \in \mathbb{Z}$ and all $p, q \in \mathscr{G}_{\mathbb{N}\cup\{0\}}$,
$$d_{\mathscr{G}_{\mathbb{Z}}}(0_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}, n(p-q)) = d_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}(|n|q, |n|p) = |n| d_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}(q,p) = |n| d_{\mathscr{G}_{\mathbb{Z}}}(0_{\mathscr{G}_{\mathbb{N}\cup\{0\}}}, p-q).$$
3. Note that $\mathscr{G}_{\mathbb{Z}}$ is a torsion-free $\mathbb{Z}$-module – i.e., if $ng = 0_{\mathscr{G}_{\mathbb{Z}}}$ for $n \in \mathbb{Z} \setminus \{0\}$ and $g \in \mathscr{G}_{\mathbb{Z}}$, then $g = 0_{\mathscr{G}_{\mathbb{Z}}}$. Now define $\mathscr{G}_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathscr{G}_{\mathbb{Z}}$; thus $\mathscr{G}_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space (i.e., a torsion-free divisible[2] group), and $\mathscr{G}_{\mathbb{Z}}$ embeds into $\mathscr{G}_{\mathbb{Q}}$. Moreover for every $g \in \mathscr{G}_{\mathbb{Q}}$ there exists $n_g \in \mathbb{N}$

[1]The *Grothendieck group* of an abelian monoid M is the abelian group Gr_M of equivalence classes in $M \times M$ under: $(a, a') \sim (b, b')$ if there exists $m \in M$ such that $a + b' + m = a' + b + m$. The element/class $[(m, m')]$ should be thought of as $m - m'$, the operation is coordinatewise addition, the zero is $[(m,m)]$ for any $m \in M$, and the inverse of $[(m, m')]$ is $[(m', m)]$. If M has the cancellation property then $M \hookrightarrow Gr_M$.

[2]These terms – as also tensor products – are defined and studied in standard algebra textbooks, e.g. [17].

such that $n_g g \in \mathscr{G}_{\mathbb{Z}}$. Define $d_{\mathscr{G}_{\mathbb{Q}}}$ on $\mathscr{G}_{\mathbb{Q}}^2$ via:

$$d_{\mathscr{G}_{\mathbb{Q}}}(g,h) := \frac{1}{n_g n_h} d_{\mathscr{G}_{\mathbb{Z}}}(n_h(n_g g), n_g(n_h h)).$$

It is not hard to check that $d_{\mathscr{G}_{\mathbb{Q}}}$ is well-defined and induces a "$\mathbb{Q}$-norm" on $\mathscr{G}_{\mathbb{Q}}$ that extends $d_{\mathscr{G}_{\mathbb{Z}}}$ on $\mathscr{G}_{\mathbb{Z}}$. In particular, it induces a translation-invariant metric on $\mathscr{G}_{\mathbb{Q}}$, so that we have embedded the normed semigroup $\mathscr{G}_{\mathbb{N}}$ isometrically into a normed $\mathbb{Q}$-vector space.

(4) Define $\mathbb{B}(\mathscr{G})$ to be the set of equivalence classes of $d_{\mathscr{G}_{\mathbb{Q}}}$-Cauchy sequences (i.e., the topological completion) of $\mathscr{G}_{\mathbb{Q}}$. One shows using algebraic and topological arguments that $\mathbb{B}(\mathscr{G})$ is an abelian group and $\mathscr{G}_{\mathbb{Q}}$ embeds isometrically into $\mathbb{B}(\mathscr{G})$. Moreover, if $x \in \mathbb{R}$ and $(g_n)_n$ is Cauchy in $\mathbb{B}(\mathscr{G})$, then choose any sequence $x_n \in \mathbb{Q}$ converging to x, and define $x \cdot [(g_n)_n] := [(x_n g_n)_n]$. It is easy to verify that $(x_n g_n)_n$ is also a Cauchy sequence in $\mathscr{G}_{\mathbb{Q}}$, and the resulting operation makes $\mathbb{B}(\mathscr{G})$ into an $\mathbb{R}$-vector space.

Now define $d_{\mathbb{B}(\mathscr{G})}([(g_n)_n], [(h_n)_n]) := \lim_{n\to\infty} d_{\mathscr{G}_{\mathbb{Q}}}(g_n, h_n)$ (this exists and is well-defined by applying topological arguments). It is easily verified that $d_{\mathbb{B}(\mathscr{G})}$ induces a norm on $\mathbb{B}(\mathscr{G})$, making $\mathbb{B}(\mathscr{G})$ a complete normed linear space, and proving (3.21).

To conclude the proof, observe that if any of the steps starts with a separable metric space, then the subsequent constructions also yield separable metric spaces. The final assertion about extending Bochner integration to $\mathscr{G}$ now follows; note the Bochner integral (or expectation) of $\mathscr{G}$-valued random variables lives in $\mathbb{B}(\mathscr{G})$ and not necessarily in $\mathscr{G}$. □

Remark 3.22. If one restricts Theorem 3.3 to groups instead of semigroups, Proposition 3.19 from [3, 7] is related, as it embeds an abelian normed group into *some* Banach space, via the "double dual". To our knowledge, an explicit construction of a *minimal* Banach space "envelope" was not recorded to date in the literature. We briefly discuss this and other aspects of our proof:

(1) There was no mention of separability in the proof in [3]. This is useful for applications (see Section 3.1) and hence is addressed by our proof.
(2) The construction in [3, Proposition 4.12] is that of the "double-dual" $\mathbb{B} := \mathrm{Hom}_{gp,bdd}(\mathscr{G}, \mathbb{R})^*$, i.e., the dual space to the set of bounded/Lipschitz real-valued group maps : $\mathscr{G} \to \mathbb{R}$. Thus, if $\mathscr{G}$ is an infinite-dimensional Banach space, then the double-dual construction $\mathscr{G}^{**}$ is strictly larger than $\mathscr{G}$ – and hence, does not yield the "minimal" Banach space envelope of $\mathscr{G}$ for "most" real Banach spaces – whereas the above proof does. One of the referees mentioned to us that the construction can be refined to yield the minimal Banach space; however, to our knowledge this refinement is not written down. This was one reason to write the above argument in full detail – especially given that our constructive proof is along *different* lines.
(3) To the best of our knowledge, we could not find references to embedding abelian normed *semigroups*. For this a little more work is needed to embed into an abelian normed group (via the monoid-extension and then its Grothendieck group, as above). This semigroup-extension also features in several applications (in Section 3.1), hence its proof above.

We end with two further remarks. First, each step in (3.21) is canonical (and optimal), in the sense that it uses only the given information without any additional structure (and it constructs the "minimal" larger structure containing the structure at each step). The natural way to encode this information is via category theory. In other words, every further step/extension in (3.21) is the smallest possible – hence *universal* – "enveloping" object in some category. For the interested reader, we defer these categorical discussions to Appendix A.

Second, given Corollary 2.5, it is natural to ask in the non-abelian situation if every (cancellative) metric semigroup embeds into a metric group. This question is harder to tackle; see [4, Chapter 1] for a sufficient condition involving right reversibility.

3.3. **Non-abelian normed groups.** We end this section with a geometric question that clarifies the second assertion in Theorem 3.3: *Do non-commutative normed metric groups exist?* Or: find a non-abelian topological group $\mathscr{G}$ with a bi-invariant metric d, such that $d(1, g^n) = |n| d(1, g)$ for all $g \in \mathscr{G}$ and $n \in \mathbb{Z}$. To our knowledge (and that of experts including [8, 28] and others), the answer to this question was not known until recent work [24], whose main result we now describe.

As a possible approach to answering the aforementioned question, a first step is to ask if certain prototypical examples of non-commutative groups with a bi-invariant metric are normed. This is now shown to be false for a well-studied example: the *free group* F_2 on two generators. Recall, this is simply the set of words in a, b, a^{-1}, b^{-1}, modulo the relations $aa^{-1} = a^{-1}a = bb^{-1} = b^{-1}b = 1$.

Lemma 3.23. *Let $\mathscr{G} = F_2$ and let $d_{\mathscr{G}}$ denote the bi-invariant word metric in the generators $a^{\pm 1}, b^{\pm 1}$ and their conjugates. Then $(\mathscr{G}, \cdot, 1, d_{\mathscr{G}})$ is not normed.*

Note that we work with $d_{\mathscr{G}}$ and not the usual word metric in the four semigroup generators $a^{\pm 1}, b^{\pm 1}$ of $\mathscr{G}$. For more on the metric $d_{\mathscr{G}}$ and related structures, see [2] and the references therein.

Proof. First compute: $[a, b]^3 = aba^{-1} \cdot b^{-1}ab \cdot a^{-1}b^{-1}a \cdot ba^{-1}b^{-1}$. Examining the word lengths, the right-hand side yields at most 4, while $l_{\mathscr{G}}([a, b]) \neq 1$. Hence $(\mathscr{G}, \cdot, 1, d_{\mathscr{G}})$ is not normed, as

$$l_{\mathscr{G}}([a, b]^3) \leqslant 4 < 6 \leqslant 3 l_{\mathscr{G}}([a, b]). \qquad \square$$

We conclude with a solution to the above question, obtained by the first author in recent joint work [24] with T. Fritz, S. Gadgil, P. Nielsen, L. Silberman, and T. Tao. It turns out that non-commutative normed metric groups do not exist! Namely:

Theorem 3.24 ([24])**.** *Given a group $\mathscr{G}$, the following are equivalent:*

(1) *$\mathscr{G}$ is a metric group (with a bi-invariant metric) that is $\{2\}$-normed (equivalently, normed).*
(2) *$\mathscr{G}$ is abelian and torsion-free.*
(3) *$\mathscr{G}$ is an additive subgroup of (i.e., embeds isometrically and additively into) a Banach space.*

This yields a novel characterization – from analysis – of a fundamental class of algebraic objects: abelian torsion-free groups. Now given Theorem 3.24, many of the above results (e.g. our transfer principle in Theorem 3.3) are stated for normed metric groups $\mathscr{G}$, as a norm on $\mathscr{G}$ implies $\mathscr{G}$ is abelian. Note however that this last can fail if "group" is replaced by "semigroup" or even "monoid", since non-abelian (free) monoids with norms – i.e., homogeneous length functions – indeed exist; see [24] for details. Another consequence is that the four assertions in Proposition 3.19 are further equivalent to an *a priori* weaker assertion than (2) or (4): namely, that $\mathscr{G}$ is merely $\{2\}$-normed.

Appendix A. Categories of normed metric modules

We now construct "dual spaces" to abelian normed metric groups, in order to prove Proposition 3.15 (see the discussion after Theorem 3.10). This construction is similar – with minor adjustments – for abelian normed metric semigroups and their refinements: (i) semigroups, (ii) monoids, (iii) groups, (iv) torsion-free divisible groups, (v) real vector spaces, and (vi) Banach spaces. To study all of these constructions systematically, we use the language of category theory. (For basics of categories and functors, see [17].) We will show in Proposition A.7 below that "dual space constructions" are covariant endofunctors – more generally, so are spaces of linear Lipschitz operators.

Using categories has additional advantages. Recall that the proof of Theorem 3.3 showed that every abelian normed semigroup (respectively, group) embeds into a smallest abelian normed group (respectively, Banach space). We now make these statements precise using category theory. Briefly, we will show in a unified way that the above constructions are instances of "universal objects", and provide examples of pairs of adjoint "induction-restriction functors".

To proceed, we first propose a unifying framework in which to simultaneously study abelian normed metric semigroups of types (i)–(vi) above: normed metric modules. In the sequel, $\mathbb{N} = \mathbb{Z}_{>0}$.

Definition A.1. Suppose a subset $S \subseteq \mathbb{R}$ is closed under addition and multiplication.

(1) An *S-module* is defined to be an abelian semigroup $(G,+)$ together with an action map $\cdot : S \times G \to G$, satisfying the following properties for $s, s' \in S$ and $g, g' \in G$:[3]

$$s \cdot (g+g') = s \cdot g + s \cdot g', \quad (s+s') \cdot g = (s \cdot g) + (s' \cdot g), \quad (ss') \cdot g = s \cdot (s' \cdot g), \quad 1 \cdot g = g \text{ if } 1 \in S.$$

(2) A *metric S-module* is an S-module $(G,+)$ together with a translation-invariant metric d. We say $(G,+,d)$ is *normed* if $d(s \cdot g, s \cdot g') = |s| d(g,g')$ for all $s \in S$ and $g, g' \in G$.

(3) Let $\mathscr{C}_S$ denote the category whose objects are normed metric S-modules G_S, and morphisms are S-module maps that are moreover Lipschitz. For each such morphism $\varphi : G_S \to G'_S$, define $\|\varphi\|$ to be the smallest constant $K \geqslant 0$ such that $\|\varphi(g)\| \leqslant K \|g\|$ for all $g \in G_S$. Also denote by $\overline{\mathscr{C}}_S$ the full sub-category of all objects in $\mathscr{C}_S$ that are complete metric spaces.

Now $\mathbb{N}$-modules are semigroups and $(\mathbb{N} \cup \{0\})$-modules are monoids. Using this notation, Theorem 3.3 discusses the objects in the categories $\mathscr{C}_S$ for $S = \mathbb{N}, \mathbb{N} \cup \{0\}, \mathbb{Z}, \mathbb{Q}$, as well as $\overline{\mathscr{C}}_{\mathbb{R}}$, the category of Banach spaces and bounded operators. Note that normed linear spaces (i.e., $\mathscr{C}_{\mathbb{R}}$) are missing from Theorem 3.3; however, our next result also produces a similar "universal" normed linear space containing a(n abelian) normed group. Thus, the constructions in (3.21) possess functorial properties and therefore are universal in the above categories:

Theorem A.2. *Suppose each of S, T, U is either $\mathbb{N}, \mathbb{N} \cup \{0\}$, or a unital subring of $\mathbb{R}$, with $S \subseteq T$ or $S \supseteq T$. Suppose also that G_S is an object of $\mathscr{C}_S$. Now define*

$$\mathscr{G}_T(G_S) := \begin{cases} G_S \text{ (viewed as an object of } \mathscr{C}_T), & \text{if } S \supseteq T; \\ \text{the unique object of } \mathscr{C}_T \text{ defined as in (3.21)}, & \text{if } S = \mathbb{N}, \mathbb{N} \cup \{0\},\ T \supseteq S; \\ T \otimes_S G_S, & \text{if } \mathbb{Z} \subseteq S \subseteq T. \end{cases} \tag{A.3}$$

(1) *$\mathscr{G}_T(G_S)$ is an object of $\mathscr{C}_S \cap \mathscr{C}_T$ with the following universal property: given an object G_T in $\mathscr{C}_S \cap \mathscr{C}_T$, together with a morphism $\iota : G_S \to G_T$ in $\mathscr{C}_S$, ι extends via the unique isometric monomorphism $G_S \hookrightarrow \mathscr{G}_T(G_S)$ to a unique morphism $\iota_T : \mathscr{G}_T(G_S) \to G_T$ in $\mathscr{C}_T$.*

(2) *In particular, $(\mathscr{G}_T(G_S), \iota_T)$ is unique up to a unique isomorphism in $\mathscr{C}_T$.*

(3) *Given G_S, define $\overline{\mathscr{G}}_T(G_S)$ to be the Cauchy completion of $\mathscr{G}_T(G_S)$ (as a metric space). Then $\overline{\mathscr{G}}_T(G_S)$ is an object of $\overline{\mathscr{C}}_T$ and satisfies the same properties as in the previous parts.*

(4) *Suppose $\mathbb{N} \subseteq S \subseteq T \subseteq U \subseteq \mathbb{R}$, with S, T, U of the form $\mathbb{N}, \mathbb{N} \cup \{0\}$, or a unital subring of $\mathbb{R}$. For all objects G_S in $\mathscr{G}_S$, there exist unique isomorphisms:*

$$\mathscr{G}_U(\mathscr{G}_T(G_S)) \cong \mathscr{G}_U(G_S), \qquad \overline{\mathscr{G}}_U(\mathscr{G}_T(G_S)) \cong \overline{\mathscr{G}}_U(\overline{\mathscr{G}}_T(G_S)) \cong \overline{\mathscr{G}}_U(G_S).$$

(5) *Given a unital subring $S \subseteq \mathbb{R}$, S is dense in $\mathbb{R}$ if and only if $\overline{G_S} = \overline{\mathscr{G}}_T(G_S) = \mathbb{B}(G_S)$ for all objects G_S of $\mathscr{C}_S$ and all subrings $S \subseteq T \subseteq \mathbb{R}$. (Here, $\mathbb{B}(G_S)$ is as in Theorem 3.3.)*

For the above reason, if $S \subseteq T$ or $S \supseteq T$ then we call $\mathscr{G}_T(G_S), \overline{\mathscr{G}}_T(G_S)$ the *universal envelopes* of G_S in $\mathscr{C}_T$ and $\overline{\mathscr{C}}_T$ respectively.[4] Also observe that $\mathbb{B}(G_S)$ is the completion of the smallest normed linear space containing G_S, for all $S \supseteq \mathbb{Q}$ and objects G_S in $\mathscr{C}_S$.

Proof of Theorem A.2. The proof involves (sometimes standard) category-theoretic arguments, and is included for the convenience of the reader.

(1) The first part is immediate if $S \supseteq T$; we now show it assuming that $S \subseteq T$. Given an object G_S in $\mathscr{C}_S$, note $\mathscr{G}_T(G_S) \subseteq \mathbb{B}(G_S)$. This immediately shows $\mathscr{G}_T(G_S)$ is an object of $\mathscr{C}_T$. Now given a morphism $\iota : G_S \to G_T$ in $\mathscr{C}_S$, if $S = \mathbb{N}$ then first define $\iota_T(0_{\mathscr{G}_T(G_S)}) := 0_{G_T}$. If $S = \mathbb{N} \cup \{0\}$ then define $\iota_T(-g) := -\iota(g)$ for $g \in G_S$. Finally, if S is a unital subring of

[3]Note that if $0 \in S$ then G is necessarily a monoid.

[4]Such "minimal envelopes" are ubiquitous in mathematics; examples include the universal enveloping algebra of a Lie algebra, the convex hull of a set (in a real vector space), and the σ-algebra generated by a set of subsets.

$\mathbb{R}$ and $x := \sum_{j=1}^n t_j g_j \in T \otimes_S G_S$ (with $g_j \in G_S\ \forall i$), then define $\iota_T(x) := \sum_{j=1}^n t_j \iota(g_j)$. These conditions are necessary to extend ι to ι_T; moreover, it is not hard to show using Theorem 3.3 that they are also sufficient to uniquely extend ι to ι_T. Also using Theorem 3.3, one verifies that ι_T is Lipschitz, with $\|\iota_T\| = \|\iota\|$.

(2) This is a standard categorical consequence of universality.

(3) This is clear if $S \supseteq T$, so say $S \subseteq T$, $G_S \in \mathscr{C}_S$. Given $\iota : G_S \to G_T$ with $G_T \in \mathscr{C}_S \cap \overline{\mathscr{C}}_T$, by (1) ι extends uniquely to $\iota_T : \mathscr{G}_T(G_S) \to G_T$, which in turn extends uniquely to $\overline{\iota_T} : \overline{\mathscr{G}}_T(G_S) \to G_T$ by uniform continuity. Now verify $\overline{\iota_T}$ is a morphism in $\overline{\mathscr{C}}_T$, with $\|\overline{\iota_T}\| = \|\iota_T\|$.

(4) This part is standard from above using universal properties, and is omitted for brevity.

(5) First if S is not dense in $\mathbb{R}$, i.e. $S = \mathbb{Z}$, then choose $G_S = \mathbb{Z}$. Now $\overline{G_S} = \mathbb{Z} \neq \mathbb{R} = \mathbb{B}(G_S)$, as asserted. Conversely, suppose S is dense in $\mathbb{R}$ and G_S is in $\mathscr{C}_S$. Repeat the construction in step (4) of the proof of Theorem 3.3, to show that the embedding : $G_S \hookrightarrow \mathbb{B}(G_S)$ uniquely extends to an isometric isomorphism : $\overline{G_S} \to \mathbb{B}(G_S)$ of Banach spaces.

Finally, given $S \subseteq T \subseteq \mathbb{R}$, note that $\mathscr{G}_T(G_S) = T \otimes_S G_S \subseteq \mathbb{R} \otimes_S G_S \subseteq \mathbb{B}(G_S)$. Hence by universality of completions, $\overline{\mathscr{G}}_T(G_S) \subseteq \mathbb{B}(G_S)$. Moreover, by the previous paragraph $\overline{\mathscr{G}}_T(G_S)$ is a Banach space containing G_S. This shows the reverse inclusion. □

Having discussed *universality*, we now study *functoriality*. The following result shows that the assignments $\mathscr{G}_S$ provide examples of induction and restriction functors.

Theorem A.4. *Suppose each of $S \subsetneq T$ is either $\mathbb{N}, \mathbb{N} \cup \{0\}$, or a unital subring of $\mathbb{R}$.*

(1) *Then $\mathscr{G}_S : \mathscr{C}_T \to \mathscr{C}_S$ is a covariant "restriction" (of scalars) functor which is fully faithful but not essentially surjective. If S is a ring then $\mathscr{G}_S$ is faithfully exact.*

(2) *Moreover, $\mathscr{G}_T : \mathscr{C}_S \to \mathscr{C}_T$ is a covariant "extension" (of scalars) functor which is faithful and essentially surjective but not full. If S is a ring, then $\mathscr{G}_T$ is additive, right-exact, and left adjoint to $\mathscr{G}_S$.*

(3) *If S is dense in $\mathbb{R}$, then $\mathscr{G}_S, \mathscr{G}_T$ yield an equivalence of categories : $\overline{\mathscr{C}}_S \leftrightarrow \overline{\mathscr{C}}_T$.*

In other words, the module-theoretic correspondence involving extension-restriction of scalars also holds for the categories $\mathscr{C}_S, \overline{\mathscr{C}}_S$ of normed metric modules.

Proof. Assume henceforth that G_S, G'_S are objects in $\mathscr{C}_S$, and G_T, G'_T are objects in $\mathscr{C}_T$.

(1) It is immediate that $\mathscr{G}_S : \mathscr{C}_T \to \mathscr{C}_S$ is a faithful, covariant functor. It is not essentially surjective because $S \subsetneq T$ is not a T-module. We now show $\mathscr{G}_S$ is full – in fact we show more strongly that all S-module maps are in fact T-linear. Note, every S-module map between objects G_T, G'_T in $\mathscr{C}_T$ gives rise to a unique $\mathbb{Z}$-module map between them. Given such a map φ, we only use the continuity and additivity of φ to show that φ is in fact T-linear. Thus, fix $g \in G_T$ and consider the function $f : T \to G'_T$ given by $f(t) := \varphi(tg)$. Clearly f is continuous and additive, so given a sequence of rationals $m_k/n_k \to t$, we compute:

$$0 \leftarrow f(m_k - tn_k) = m_k f(1) - n_k f(t) = m_k \varphi(g) - n_k \varphi(tg).$$

It follows that $\varphi(tg) = t\varphi(g)$, showing that φ is in fact T-linear and hence $\mathscr{G}_S$ is full. Finally if S is a ring, the restriction functor $\mathscr{G}_S$ is easily seen to be faithfully exact (i.e., it takes a short sequence to a short exact sequence if and only if the short sequence is exact).

(2) That $\mathscr{G}_T : \mathscr{C}_S \to \mathscr{C}_T$ is a faithful, covariant functor is trivial. It is also essentially surjective because $G_T \cong \mathscr{G}_T(\mathscr{G}_S(G_T))$ for all objects G_T in $\mathscr{C}_T$. Now fix $t_0 \in T \setminus S$. To show that $\mathscr{G}_T$ is not full, set $G_S = G'_S := S$ and define $\varphi_T : \mathscr{G}_T(G_S) = T \to \mathscr{G}_T(G'_S) = T$ via: $\varphi_T(t) = t_0 t$. Then there does not exist a map $\varphi_S : G_S = S \to G'_S = S$ such that $\varphi_T = \mathscr{G}_T(\varphi_S)$. The assertions in the case when S is a ring are also standard.

(3) This part follows from straightforward verifications using the last part of Theorem A.2. □

Remark A.5. The above results continue to hold upon replacing the categories $\mathscr{C}_S, \overline{\mathscr{C}}_S$ by the larger categories with the same objects, but where the morphisms are allowed to be uniformly continuous rather than Lipschitz.

We now construct dual spaces, as promised in the discussion following Theorem 3.10 above. More generally, we will study the structure of the spaces $\mathrm{Hom}_{\mathscr{C}_T}(\mathscr{G}_T(G_S), G_T)$ for $S \subseteq T$. We begin with an elementary observation, which helps define norms of Lipschitz maps.

Lemma A.6. *Suppose S is either $\mathbb{N}, \mathbb{N} \cup \{0\}$, or a unital subring of $\mathbb{R}$. Fix a morphism $\varphi : G_S \to G'_S$ in $\mathscr{C}_S$, and consider the following assertions:*

1. *φ is Lipschitz on G_S.*
2. *φ is (uniformly) continuous.*
3. *(If $0 \in S$:) φ is continuous at 0.*

Then $(2), (3)$ are equivalent and implied by (1). The converse holds if and only if S is dense in $\mathbb{R}$.

Proof. We only show the very last assertion, as the rest is standard. If $S = \mathbb{Z}$ then consider $G_S = G'_S$ to be the functions from $\mathbb{N}$ to S with finite support. Let $\{e_n : n \in \mathbb{N}\}$ denote the "standard basis" of G_S, and define $\varphi(e_n) := ne_n$. Then φ is continuous but not Lipschitz. Conversely, suppose S is dense in $\mathbb{R}$, and $\|\varphi\| = \infty$. Then there exist $g_n \in G_S$ such that $\|\varphi(g_n)\| > 2n\|g_n\|$ for all n. Choose $s_n \in (n, 2n)$ such that $(s_n\|g_n\|)^{-1} \in S$. Then $\|\varphi(h_n)\| > 1 \ \forall n$, where $h_n := (s_n\|g_n\|)^{-1} g_n \in G_S$. Since $h_n \to 0$, it follows that φ is not continuous at 0. □

Proposition A.7. *Suppose $S \subseteq T$ are both of the form $\mathbb{N}, \mathbb{N} \cup \{0\}$, or a unital subring of $\mathbb{R}$, and $G_S \in \mathscr{C}_S, G'_T \in \mathscr{C}_T$. Identifying G'_T with $\mathscr{G}_S(G'_T)$, the set $\mathrm{Hom}_{\mathscr{C}_S}(G_S, G'_T)$ is itself an object of $\mathscr{C}_T$. It is moreover an object of $\overline{\mathscr{C}}_T$ (i.e., complete) for all $G_S \in \mathscr{C}_S$, if and only if G'_T is complete.*

In particular for $T = \mathbb{R}$, the above construction yields a Banach space of "linear functionals", which we called the *dual space* G_S^* above (see Proposition 3.15). More generally, the assignment $\mathrm{Hom}_{\mathscr{C}_S}(G_S, -)$ defines a covariant additive functor : $\mathscr{C}_S \to \mathscr{C}_T$ and : $\overline{\mathscr{C}}_S \to \overline{\mathscr{C}}_T$. This result (together with Lemma A.6) explains why we chose the category $\mathscr{C}_S$ to have linear morphisms that were also bounded/Lipschitz, and not merely uniformly continuous.

Proof. We only sketch why if G'_T is complete, then so is $H := \mathrm{Hom}_{\mathscr{C}_S}(G_S, G'_T)$ for any fixed G_S. Suppose $\varphi_n \in H$ is a Cauchy sequence. Then so is $\varphi_n(g)$ for any $g \in G_S$, and hence one defines $\varphi : G_S \to G'_T$ via: $\varphi(g) := \lim_n \varphi_n(g)$. One checks φ is S-linear. Moreover $\|\varphi\| \leqslant \sup_n \|\varphi_n\| < \infty$, so $\varphi \in H$. A standard argument now shows $d_H(\varphi_n, \varphi) := \|\varphi_n - \varphi\| \to 0$ as $n \to \infty$. □

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