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GLOBAL EXISTENCE AND EXPONENTIAL DECAY OF STRONG SOLUTIONS FOR THE INHOMOGENEOUS INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH VACUUM*

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Dedicated to Professor Ling Hsiao on the Occasion of Her 80th Birthday

Abstract. The inhomogeneous incompressible Navier-Stokes equations with fractional Laplacian dissipations in the multi-dimensional whole space are considered. The existence and uniqueness of global strong solutions with vacuum are established for large initial data. The exponential decay-in-time of the strong solution is also obtained, which is different from the homogeneous case. The initial density may have vacuum and even compact support.

Key words. Navier-Stokes equations, vacuum, inhomogeneous, incompressible, exponential decay, global strong solution.

Mathematics Subject Classification. 35Q35, 35B65, 76N10, 76D05.

1. Introduction. In this paper, we are concerned with the Cauchy problem of the following fractional inhomogeneous incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \qquad x \in \mathbb{R}^n,\ t>0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \mu(-\Delta)^\alpha u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ \rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $\rho = \rho(x,t)$ denotes the density, $u = u(x,t) = (u_1(x,t), u_2(x,t), \cdots, u_n(x,t))$ denotes the fluid velocity, $p(x,t)$ is the scalar pressure, and $\mu > 0$ is the viscosity that is assumed to be one for simplicity; $\rho_0(x)$ and $u_0(x)$ are the prescribed initial data for the density and velocity with $\nabla \cdot u_0 = 0$. The fractional Laplacian operator $(-\Delta)^\alpha$ with $\alpha > 0$ is defined via the Fourier transform as

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\, \widehat{f}(\xi),$$

where $\widehat{f}$ is the Fourier transform of f. The fractional problems arise from many applications in fractional quantum mechanics [27], probability [4, 6], overdriven detonations in gases [10], anomalous diffusion in semiconductor growth [39], physics and chemistry [32], optimization and finance [11], and so on.

There have been a lot of studies on the fractional Laplace-type problems recently. When $\alpha = 1$, the system (1.1) becomes the classical inhomogeneous incompressible Navier-Stokes equations, describing fluids inhomogeneous in density. Typical examples of such fluids include the mixture of incompressible and non-reactant flows, flows with complex structure (e.g. blood flows or rivers), fluids containing a melted substance, etc. We refer to [30] for the detailed derivation of this system. Because of

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its physical importance, complexity, rich phenomena and mathematical challenges, there is a notablly large literature on the mathematical studies on the well-posedness of solutions to the classical inhomogeneous incompressible Navier-Stokes equations. For example, when the initial density is strictly positive, Kazhikov [25] proved that the system has at least one global weak solution in the energy space, the local (global if $n = 2$) existence and uniqueness of strong solutions were first obtained in [3, 26], and similar results were established recently in a series of works such as [1, 2, 7, 13, 14, 33, 34, 15, 16]. For the initial data with vacuum, there is a possible degeneracy near vacuum and hence the problem becomes much more complicated. The global weak solution with a finite energy was constructed first in Simon [36] and then by Lions [30] with the density-dependent viscosity. The global existence of two-dimensional strong solutions with general initial data was established in [21, 17, 31, 36] for the inhomogeneous Navier-Stokes equations with vacuum. The three-dimensional local strong solution was obtained in [9] under a compatibility condition, and the global strong small solution was proved in [12], as well as [22, 41, 20] for the case of density-dependent viscosity. For three or higher spatial dimensions, the existence of global strong solutions with general initial data is a well-known open problem. One difficulty is that the Laplacian dissipation is insufficient to control the nonlinearity when applying the standard techniques to establish global a priori bounds. Hence it is natural to explore the problem via replacing the Laplacian operator by the fractional Laplacian operators as in (1.1), motivated by the applications aforementioned, in order to obtain the global strong solution for the general initial data, which is the aim of this paper.

When the density ρ is a constant, the system (1.1) becomes the classical fractional homogeneous incompressible Navier-Stokes equations, which admit a unique global smooth solution as long as $\alpha \geq \frac{1}{2} + \frac{n}{4}$. This result dates back to J. Lions's book [29] in 1969, which is even true for some logarithmic corrections (see [37, 5] for details). These results were extended to the inhomogeneous system (1.1) in [18] for $\alpha \geq \frac{1}{2} + \frac{n}{4}$ and in [19] for the corresponding logarithmic case. It should be noted that both [18] and [19] require the initial density ρ_0 bounded away from zero, i.e., the flow has no vacuum. The goal of this paper is to relax this restriction. More precisely, we shall establish the global existence of strong solutions with vacuum to the system (1.1). Moreover, we shall also obtain the exponential decay-in-time of the strong solution. We recall that (ρ, u) is called a weak solution to the system (1.1) if it satisfies (1.1) in the sense of distributions, and a strong solution if the system (1.1) holds almost everywhere.

In this paper, we shall adopt the convention that C denotes a generic constant depending only on the initial data. For simplicity, we will frequently use the notation $\Lambda := (-\Delta)^{\frac{1}{2}}$. For $1 \leq r \leq \infty$ and integer $k \geq 0$, we use the following notations for the standard homogeneous and inhomogeneous Sobolev spaces:

$$L^r = L^r(\mathbb{R}^n), \quad \dot{W}^{k,r} = \{g \in L^1_{loc}(\mathbb{R}^n) : \ \|g\|_{\dot{W}^{k,r}} := \|\nabla^k g\|_{L^r} < \infty\}, \quad W^{k,r} := L^r \cap \dot{W}^{k,r},$$

$$\dot{H}^s = \left\{ g : \ \|g\|^2_{\dot{H}^s} = \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{f}(\xi)|^2 \, d\xi < \infty \right\}, \quad H^s := L^2 \cap \dot{H}^s.$$

Now we state our main result of this paper as follows.

THEOREM 1.1. *For the system (1.1) with* $\alpha = \frac{1}{2} + \frac{n}{4}$ *and* $n \geq 3$, *if the initial data* (ρ_0, u_0) *satisfies the following conditions:*

$$0 \leq \rho_0 \in L^{\frac{2n}{n+2}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \quad \nabla\rho_0 \in L^{\frac{4n}{n+6}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \tag{1.2}$$

$$\nabla \cdot u_0 = 0, \quad u_0 \in \dot{H}^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n), \quad \sqrt{\rho_0}u_0 \in L^2(\mathbb{R}^n), \tag{1.3}$$

then it has a unique global strong solution (ρ, u) *such that, for any given* $T > 0$ *and for any* $0 < \tau < T$,

$$\begin{aligned}
&0 \le \rho \in L^\infty(0,T; L^{\frac{2n}{n+2}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)), \quad \nabla\rho \in L^\infty(0,T; L^{\frac{4n}{n+6}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)),\\
&u \in L^\infty(0,T; \dot{H}^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n)) \cap L^2(0,T; \dot{H}^{1+\frac{n}{2}}(\mathbb{R}^n)), \quad \sqrt{\rho}\partial_t u \in L^\infty(\tau,T; L^2(\mathbb{R}^n)),\\
&\sqrt{\rho}\partial_{tt} u \in L^2(\tau,T; L^2(\mathbb{R}^n)), \quad \partial_t u \in L^2(\tau,T; \dot{H}^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n)) \cap L^\infty(\tau,T; \dot{H}^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n)),\\
&\Lambda^{1+\frac{n}{2}} u \in L^\infty(\tau,T; L^{\frac{4n}{n-2}}(\mathbb{R}^n)), \quad p \in L^\infty(\tau,T; H^1(\mathbb{R}^n)) \cap L^\infty(\tau,T; W^{1,\frac{4n}{n-2}}(\mathbb{R}^n)).
\end{aligned}$$

Moreover, there exists some positive constant γ *depending only on* $\|\rho_0\|_{L^{\frac{2n}{n+2}}}$ *such that, for all* $t \ge 1$,

$$\begin{aligned}
\|\Lambda^{\frac{1}{2}+\frac{n}{4}} u(t)\|_{L^2}^2 + \|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \|\Lambda^{1+\frac{n}{2}} u(t)\|_{L^2\cap L^{\frac{4n}{n-2}}}^2 + \|\Lambda^{\frac{1}{2}+\frac{n}{4}} \partial_t u(t)\|_{L^2}^2&\\
+ \|p(t)\|_{H^1\cap W^{1,\frac{4n}{n-2}}}^2 \le \widetilde{C} e^{-\gamma t},&
\end{aligned}$$

where $\widetilde{C}$ *depends only on* $\|\rho_0\|_{L^{\frac{2n}{n+2}}}$, $\|\rho_0\|_{L^\infty}$, $\|\nabla\rho_0\|_{L^2}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$ *and* $\|\Lambda^{\frac{1}{2}+\frac{n}{4}} u_0\|_{L^2}$.

Remark 1.1. For the exponential decay-in-time property of Theorem 1.1, the estimate of the density:

$$\|\rho(t)\|_{L^{\frac{2n}{n+2}}} \le \|\rho_0\|_{L^{\frac{2n}{n+2}}} \tag{1.4}$$

plays a crucial role. This estimate (1.4) does not hold for the homogeneous case (with constant density) in the whole space. In fact, only algebraic decay rate has been obtained for the homogeneous case in literature, e.g., [2, 8, 23, 24, 38, 35].

Remark 1.2. As a consequence of the proof of Theorem 1.1, the corresponding conclusions of the global existence and exponential decay of strong solutions are also valid for the system (1.1) with at least $\frac{1}{2} + \frac{n}{4} < \alpha < \frac{n}{2}$. We also remark that our arguments can be adopted to other similar systems with the same dissipations.

Remark 1.3. Under the assumption that the initial velocity is suitably small, the exponential decay-in-time of the strong solutions was obtained in [20] for the Cauchy problem of the three-dimensional classical inhomogeneous incompressible Navier-Stokes equations (i.e., the system (1.1) with $\alpha = 1$) with density-dependent viscosity and vacuum, which of course is valid for the constant viscosity case. We remark that Theorem 1.1 is proved without any smallness on the initial data. Moreover, the initial density is allowed to have vacuum. We also point out that the regularity assumption on the initial density $\nabla\rho_0 \in L^{\frac{4n}{n+6}}$ is used only to guarantee the uniqueness of the solution. As a matter of fact, it is not clear whether we can adopt the arguments used in [17, page 1373–page 1378] to remove the regularity assumption $\nabla\rho_0 \in L^{\frac{4n}{n+6}}$ for the system (1.1) with $\alpha = \frac{1}{2} + \frac{n}{4}$ and $n \ge 3$.

Remark 1.4. Finally, compared with the previous works [9, 12, 21, 22, 41], the following corresponding compatibility condition on the initial data is dropped from Theorem 1.1:

$$(-\Delta)^{\frac{1}{2}+\frac{n}{4}} u_0 + \nabla p_0 = \sqrt{\rho_0} g, \tag{1.5}$$

with $(p_0,\, g) \in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. However, without the compatibility condition, the price that we need to pay is that the parameter τ in Theorem 1.1 must be positive and can not be replaced by the initial time $\tau = 0$.

We now outline the main idea and make some comments on the proof of this theorem. The local existence of strong solutions to the system (1.1) can be derived easily following [9, 28] (see Lemma 2.1). In order to prove global existence we need to establish global a priori estimates on strong solutions of the system (1.1) in suitable higher-order norms. Since the density has no positive lower bound and the velocity has no smallness or compatibility conditions, the proof of Theorem 1.1 is much more involved compared with the related works in literature. Therefore, new ideas are needed to overcome these difficulties as explained below. First, taking the advantage of the estimate (1.4) on the density, we have the following key observation:

$$\|\sqrt{\rho}u\|_{L^2} \le \|\sqrt{\rho}\|_{L^{\frac{4n}{n+2}}}\|u\|_{L^{\frac{4n}{n-2}}} \le C\|\rho\|^{\frac12}_{L^{\frac{2n}{n+2}}}\|\Lambda^{\frac12+\frac n4}u\|_{L^2} \le C\|\Lambda^{\frac12+\frac n4}u\|_{L^2},$$

which implies that $\|\sqrt{\rho}u(t)\|^2_{L^2}$ decays with the rate of $e^{-\gamma t}$ for some $\gamma > 0$ depending only on $\|\rho_0\|_{L^{\frac{2n}{n+2}}}$ (see Lemma 2.2 for details). With the help of this key exponential decay-in-time rate, we can show that $\|\Lambda^{\frac12+\frac n4}u(t)\|^2_{L^2}$ decays at the same rate as $e^{-\gamma t}$ (see Lemma 2.3 for details). The next step is to derive the bound of $\|\sqrt{\rho}\partial_t u(t)\|^2_{L^2}$. However, it prevents us to achieve this goal due to the absence of the compatibility condition (1.5) for the initial velocity. To overcome this difficulty, we first derive the following crucial time-weighted estimate (see (2.25)):

$$t\|\sqrt{\rho}\partial_t u(t)\|^2_{L^2} + \int_0^t \tau\|\Lambda^{\frac12+\frac n4}\partial_\tau u(\tau)\|^2_{L^2}\,d\tau \le C, \quad \forall\, t\ge 0, \tag{1.6}$$

where the positive constant C is independent of the initial data of $\sqrt{\rho}\partial_t u$. In fact, the time-weighted estimate is crucial in dropping the compatibility condition on the initial data (see [20, 28, 31, 34] for example). As a result, (1.6) allows us to derive the desired exponential decay-in-time rate (see (2.26)):

$$e^{\gamma t}\|\sqrt{\rho}\partial_t u(t)\|^2_{L^2} + \int_1^t e^{\gamma\tau}\|\Lambda^{\frac12+\frac n4}\partial_\tau u(\tau)\|^2_{L^2}\,d\tau \le C, \quad \forall\, t\ge 1.$$

As a matter of fact, all these exponential decay-in-time rates and the time-weighted estimate (1.6) play an important role in obtaining the desired uniform-in-time bound of $\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau$ (see (2.34) for details). Next, by means of these a priori estimates, we can establish the time independent estimates on the gradient of the density. This further allows us to derive the time-weighted estimate (see (2.43)):

$$t^2\|\Lambda^{\frac12+\frac n4}\partial_t u(t)\|^2_{L^2} + \int_0^t \tau^2\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|^2_{L^2}\,d\tau \le C, \quad \forall\, t\ge 0. \tag{1.7}$$

Note that, thanks to the weighted factor t^2, the constant C in the above estimate is independent of the initial data of $\Lambda^{\frac12+\frac n4}\partial_t u$. With (1.7) in hand, we then can conclude the exponential decay-in-time rate (see (2.44)):

$$e^{\gamma t}\|\Lambda^{\frac12+\frac n4}\partial_t u(t)\|^2_{L^2} + \int_1^t e^{\gamma\tau}\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|^2_{L^2}\,d\tau \le C, \quad \forall\, t\ge 1.$$

Therefore, the higher regularity of the velocity and the pressure follow directly. The uniqueness is quite subtle as we only have the estimate $\int_0^t \tau\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau \le C$ rather than $\int_0^t \|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau \le C$. This means that the uniqueness can not be proved by the standard Gronwall's inequality, instead we use a new Gronwall type inequality in [28]. With all these a priori estimates obtained, we can finally establish the global existence and uniqueness as well as the exponential decay of global strong solution to the system (1.1) in Theorem 1.1.

As a byproduct, using the similar arguments of the proof for Theorem 1.1, we can also obtain the exponential decay of strong solutions to the two-dimensional Navier-Stokes equations with damping. We remark that without damping, only algebraic decay rate was obtained in [31].

The rest of the paper is organized as follows. In Section 2 we carry out the proof of Theorem 1.1. In the appendix, we present the byproduct on the exponential decay for the two-dimensional Navier-Stokes equations with damping and a sketch of the proof.

2. The Proof of Theorem 1.1. This section is devoted to the proof of Theorem 1.1. We shall prove Theorem 1.1 in several steps. In the first step, we state the local existence and uniqueness of strong solutions. The main part of the proof will focus on establishing a priori estimates for strong solutions. In the second step, we make use of the estimate on the density to derive the exponential decay-in-time: $e^{\gamma t}\|\sqrt{\rho}u(t)\|_{L^2}^2 \le C$ for some $\gamma > 0$, which also allows us to further establish the same exponential decay-in-time: $e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(t)\|_{L^2}^2 \le C$. In the third step, with the aid of the exponential decay estimates obtained above, we continue to derive the time-weighted estimates and the exponential decay of $\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2$ as well as some other quantities. With the above estimates at hand, the fourth step is devoted to obtaining the uniform-in-time bound of $\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau$ and thus establishing the estimate of the gradient of ρ. In the fifth step, we establish the time-weighted estimates and the exponential decay of $\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u(t)\|_{L^2}^2$ and some other quantities. Finally, combining all the above estimates, we prove Theorem 1.1. Now we present the details step by step.

2.1. Local well-posedness. Inspired by the works of [9, 28], one may construct the local existence and uniqueness of strong solutions.

LEMMA 2.1 (Local strong solution). *Under the conditions of Theorem 1.1, there exists a small time T^* and a unique strong solution $(\rho,\, u)$ defined on the time period $[0, T^*]$ to the system (1.1) with $\alpha = \frac{1}{2}+\frac{n}{4}$ and $n \ge 2$ such that, for any $0 < \tau < T^*$,*

$$
\begin{aligned}
0 \le \rho \in L^\infty(0, T^*; L^{\frac{2n}{n+2}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)), \quad \nabla\rho \in L^\infty(0, T^*; L^{\frac{4n}{n+6}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)),\\
p \in L^\infty(0, T^*; H^1(\mathbb{R}^n)), \quad u \in L^\infty(0, T^*; \dot H^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n)) \cap L^2(0, T^*; \dot H^{1+\frac{n}{2}}(\mathbb{R}^n)),\\
\sqrt{\rho}\partial_t u \in L^\infty(\tau, T^*; L^2(\mathbb{R}^n)), \quad \partial_t u \in L^2(\tau, T^*; \dot H^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n)).
\end{aligned}
$$

Proof. The proof can be performed via the Galerkin approximate approach. Firstly, assume that $(\rho_0,\, u_0)$ satisfies (1.2)-(1.3). We construct $(\rho_0^\delta,\, u_0^\delta)$ satisfying in addition to (1.2)-(1.3) and

$$
\begin{aligned}
0 < \delta \le \rho_0^\delta \le \varrho \quad \text{for some positive constant } \varrho,\\
\rho_0^\delta \to \rho_0 \ \text{ in } L^{\frac{2n}{n+2}}(\mathbb{R}^n), \quad \nabla\rho_0^\delta \to \nabla\rho_0 \ \text{ in } L^{\frac{4n}{n+6}}(\mathbb{R}^n) \quad \text{as } \delta \to 0,\\
u_0^\delta \to u_0 \ \text{ in } \dot H^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n), \quad \sqrt{\rho_0^\delta}u_0^\delta \to \sqrt{\rho_0}u_0 \ \text{ in } L^2(\mathbb{R}^n) \quad \text{as } \delta \to 0.
\end{aligned}
$$

Since $\rho_0^\delta \geq \delta > 0$ is strictly positive, by means of the classical theory of the ordinary differential equations and a fixed point theorem (see [9, 36]), one may construct a sequence of approximate solutions $(\rho^\delta,\, u^\delta)$ over the interval $(0,\, T^\delta)$ for some $T^\delta > 0$. Moreover, we derive

$$\frac{d}{dt}X(t) + C_1 Y(t) \leq C_2 X^\sigma(t)$$

for some $\sigma > 1$, where $X(t)$ and $Y(t)$ are given by

$$X(t) = \|\rho(t)\|_{L^{\frac{2n}{n+2}}\cap L^\infty} + \|\nabla\rho(t)\|_{L^{\frac{4n}{n+6}}\cap L^2} + \|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(t)\|_{L^2}^2 + t\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2,$$

$$Y(t) = \|\Lambda^{1+\frac{n}{2}}u(t)\|_{L^2}^2 + t\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u(t)\|_{L^2}^2.$$

Therefore, it yields that there exists a positive small time T^* independent of δ such that the solution $(\rho^\delta,\, u^\delta)$ satisfy all the estimates of Lemmas 2.2-2.6 over the interval $(0,\, T^*]$. In particular, we have

$$\begin{aligned}&\|\sqrt{\rho^\delta}\partial_t u^\delta\|_{L^2(0,T^*;\mathbb{R}^n)}^2 + \|p^\delta\|_{L^2(0,T^*;H^1(\mathbb{R}^n))}^2 + \|\Lambda^{\frac{1}{2}+\frac{n}{4}}u^\delta\|_{L^\infty(0,T^*;L^2(\mathbb{R}^n))}^2\\ &\quad + \|\Lambda^{1+\frac{n}{2}}u^\delta\|_{L^2(0,T^*;\mathbb{R}^n)}^2 \leq C_0,\end{aligned} \tag{2.1}$$

where C_0 is an absolute constant independent of t. By (2.25), it follows for any $t \in [0, T^*]$ that

$$t\|\sqrt{\rho^\delta}\partial_t u^\delta(t)\|_{L^2}^2 + \int_0^t \tau\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u^\delta(\tau)\|_{L^2}^2\, d\tau \leq C_0. \tag{2.2}$$

Thanks to (2.2), we may conclude for any $\gamma \in (0, \frac{1}{2})$ and for any $T \in [0, T^*]$

$$\|u^\delta\|_{H^\gamma(0,T;\dot{H}^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n))} \leq C_0(\gamma, T). \tag{2.3}$$

As a matter of fact, (2.3) can be deduced as follows

$$\begin{aligned}\|u^\delta\|_{H^\gamma(0,T;\dot{H}^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n))}^2 &= \|u^\delta\|_{L^2(0,T;\dot{H}^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n))}^2\\ &\quad + \int_0^T\int_0^{T-h} \frac{\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u^\delta(t+h) - \Lambda^{\frac{1}{2}+\frac{n}{4}}u^\delta(t)\|_{L^2}^2}{h^{1+2\gamma}}\, dtdh\\ &= \|u^\delta\|_{L^2(0,T;\dot{H}^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n))}^2\\ &\quad + \int_0^T\int_0^{T-h} \frac{\|\int_t^{t+h}\tau^{-\frac{1}{2}}\tau^{\frac{1}{2}}\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u^\delta(\tau)\, d\tau\|_{L^2}^2}{h^{1+2\gamma}}\, dtdh\\ &\leq \|u^\delta\|_{L^2(0,T;\dot{H}^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n))}^2\\ &\quad + \int_0^T\int_0^{T-h} \frac{\int_t^{t+h}\tau^{-1}\, d\tau \int_t^{t+h}\tau\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u^\delta(\tau)\|_{L^2}^2\, d\tau}{h^{1+2\gamma}}\, dtdh\\ &\leq \|u^\delta\|_{L^2(0,T;\dot{H}^{\frac{1}{2}+\frac{n}{4}}(\mathbb{R}^n))}^2\\ &\quad + \int_0^T \tau\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u^\delta(\tau)\|_{L^2}^2\, d\tau \int_0^T\int_0^{T-h}\frac{\int_t^{t+h}\tau^{-1}\, d\tau}{h^{1+2\gamma}}\, dtdh\end{aligned}$$

$$
\begin{aligned}
&\le C_0 + C_0 \int_0^T \int_0^{T-h} \frac{\int_t^{t+h} \tau^{-1}\, d\tau}{h^{1+2\gamma}}\, dt dh \\
&= C_0 + C_0 \int_0^T \frac{\ln h}{h^{2\gamma}} dh + C_0 \int_0^T \frac{T \ln T - (T-h)\ln(T-h)}{h^{1+2\gamma}} dh \\
&\le C_0(\gamma, T),
\end{aligned}
$$

where in the last line we used $0 < \gamma < \frac{1}{2}$. Thanks to the above estimates, by the Cantor diagonal argument, there is a subsequence of $(\rho^\delta,\, u^\delta)$ still denoted by $(\rho^\delta,\, u^\delta)$ and a pair $(\rho,\, u)$, such that for any $T \in [0, T^*]$ and for any $t_0 \in (0, T]$

$$
\begin{aligned}
&u^\delta \overset{\text{weak}\star}{\rightharpoonup} u \quad \text{in } L^\infty(0,T; \dot H^{\frac12+\frac n4}(\mathbb{R}^n)) \cap L^\infty(t_0,T; \dot H^{1+\frac n2}(\mathbb{R}^n));\\
&u^\delta \rightharpoonup u \quad \text{in } L^2(0,T; \dot H^{1+\frac n2}(\mathbb{R}^n)) \cap L^2(t_0,T; \dot H^{1+\frac n2}(\mathbb{R}^n) \cap \dot W^{1+\frac n2, \frac{4n}{n-2}}(\mathbb{R}^n));\\
&\partial_t u^\delta \rightharpoonup \partial_t u \quad \text{in } L^2(0,T; \dot H^{\frac12+\frac n4}(\mathbb{R}^n));\\
&\rho^\delta \overset{\text{weak}\star}{\rightharpoonup} \rho \quad \text{in } L^\infty(0,T; L^{\frac{2n}{n+2}}(\mathbb{R}^n) \cap \dot W^{1,\frac{4n}{4+6}}(\mathbb{R}^n));\\
&\partial_t \rho^\delta \rightharpoonup \partial_t \rho \quad \text{in } L^{\frac{2(n+2)}{n-2}}(0,T; L^{\frac{4n}{4+6}}(\mathbb{R}^n)).
\end{aligned}
$$

Moreover, the above estimates and the standard compact embedding imply that, up to subsequence, $u^\delta \to u$ in $L^2_{\mathrm{loc}}(0,T^*; \dot H^{\frac12+\frac n4}(\mathbb{R}^n))$ for some u that, in addition, satisfies (2.1), (2.2) and (2.3). For the density, we have $\rho^\delta \rightharpoonup \rho$ in $L^\infty(0,T^*;\mathbb{R}^n)$ and $0 \le \rho \le \varrho$. All those estimates are more than enough to justify that (ρ, u) is a weak solution to (1.1), precisely,

$$
\begin{aligned}
&< \rho(t)u(t), \chi(t) > - < \rho_0 u_0, \chi_0 > - \int_0^t < \rho u, \partial_\tau \chi > \, d\tau - \int_0^t < \rho u \otimes u, \nabla \chi > \, d\tau \\
&\quad + \int_0^t < \Lambda^{\frac12+\frac n4} u, \Lambda^{\frac12+\frac n4} \chi > \, d\tau = 0; \qquad (2.4)
\end{aligned}
$$

for all smooth compactly supported divergence-free vector function $\chi \in C^\infty([0,T^*) \times \mathbb{R}^n)$. Moreover, the continuity equation is fulfilled in a distributional meaning

$$
\partial_t \rho + \mathrm{div}(\rho u) = 0 \quad \text{in } \mathcal{S}'(0,T^*;\mathbb{R}^n). \qquad (2.5)
$$

Therefore, by (2.12) and (2.35) as well as the Aubin-Lions compactness lemma, we have

$$
\rho^\delta \to \rho \quad \text{in } C(0,T^*; L^p(\mathbb{R}^n)) \ \text{ for any } \frac{2n}{n+2} \le p < \infty. \qquad (2.6)
$$

As a result, (ρ, u) satisfies (2.1) and (2.2). Furthermore, combining (2.1), (2.5) and (2.4) yields that the momentum equation is fulfilled in the following strong sense

$$
\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + (-\Delta)^{\frac12+\frac n4} u + \nabla p = 0 \quad \text{in } L^2(0,T^*;\mathbb{R}^n)
$$

for some pressure function $\nabla p \in L^2(0,T^*;\mathbb{R}^n)$ satisfying (2.1). Next, we will show the time continuity of the solution (ρ, u), namely,

$$
\rho \in C([0,T^*]; L^q(\mathbb{R}^n)), \ \frac{2n}{n+2} \le q < \infty, \qquad (2.7)
$$

$$\rho u \in C([0,T^*];L^2(\mathbb{R}^n)). \tag{2.8}$$

Due to $\partial_t \rho^\delta = -u^\delta \cdot \nabla \rho^\delta$, one has

$$\begin{aligned}
\|\rho^\delta(t)-\rho_0^\delta\|_{L^{\frac{2n}{n+2}}} &= \left\| \int_0^t \partial_\tau \rho^\delta(\tau)\,d\tau \right\|_{L^{\frac{2n}{n+2}}} \\
&= \left\| \int_0^t u^\delta \cdot \nabla \rho^\delta(\tau)\,d\tau \right\|_{L^{\frac{2n}{n+2}}} \\
&\le \int_0^t \left\| u^\delta \cdot \nabla \rho^\delta(\tau) \right\|_{L^{\frac{2n}{n+2}}}\,d\tau \\
&\le \int_0^t \|u^\delta(\tau)\|_{L^{\frac{4n}{n-2}}} \|\nabla \rho^\delta(\tau)\|_{L^{\frac{4n}{n+6}}}\,d\tau \\
&\le C\int_0^t \|\Lambda^{\frac12+\frac n4} u^\delta(\tau)\|_{L^2} \|\nabla \rho^\delta(\tau)\|_{L^{\frac{4n}{n+6}}}\,d\tau \\
&\le C_0(t)t,
\end{aligned}$$

where in the last line we have used (2.16) and (2.35). By the Hölder inequality, one has

$$\|\rho^\delta(t)-\rho_0^\delta\|_{L^q} \le C\|\rho^\delta(t)-\rho_0^\delta\|_{L^{\frac{2n}{n+2}}}^{\frac{2n}{(n+2)q}} \|\rho^\delta(t)-\rho_0^\delta\|_{L^\infty}^{1-\frac{2n}{(n+2)q}} \le C_0(t)t^{\frac{2n}{(n+2)q}}.$$

By (2.6), we thus get for any $\epsilon > 0$

$$\begin{aligned}
\|\rho(t)-\rho_0\|_{L^q} &\le \|\rho(t)-\rho^\delta(t)\|_{L^q} + \|\rho^\delta(t)-\rho_0^\delta\|_{L^q} + \|\rho_0^\delta-\rho_0\|_{L^q} \\
&\le \frac{\epsilon}{3} + C_0(t)t^{\frac{2n}{(n+2)q}} + \frac{\epsilon}{3},
\end{aligned}$$

which implies that for t sufficiently small

$$\|\rho(t)-\rho_0\|_{L^q} \le \epsilon.$$

This yields that ρ continuous at the original time and satisfies the initial condition $\rho|_{t=0} = \rho_0$, which further leads to (2.7). To show (2.8), we first notice that $\Lambda^{\frac12+\frac n4}\partial_t u^\delta$ is bounded in $L^2([t_0,T^*]\times\mathbb{R}^n)$ for any $t_0 \in (0,T^*]$ due to (2.2). According to (2.16), we know that $\Lambda^{\frac12+\frac n4} u^\delta$ is bounded in $L^\infty([0,T^*];L^2(\mathbb{R}^n))$. Thus, one can conclude by means of Ascoli theorem that, up to extraction, $u^\delta \to u$ in $C([t_0,T^*];L^q(\mathbb{R}^n))$ for any $q < \frac{4n}{n-2}$ and for any $t_0 > 0$, which along with (2.6) further yields

$$\rho u \in C([t_0,T^*];L^2(\mathbb{R}^n)). \tag{2.9}$$

Consequently, it remains to verify the continuity of ρu at the original time. To this end, we first show that

$$\begin{aligned}
\|(\rho^\delta u^\delta)(t)-\rho_0^\delta u_0^\delta\|_{L^{\frac{4n}{3n+2}}} &= \left\| \int_0^t \partial_\tau(\rho^\delta u^\delta)(\tau)\,d\tau \right\|_{L^{\frac{4n}{3n+2}}} \\
&= \left\| \int_0^t (\partial_\tau \rho^\delta u^\delta)(\tau) + (\rho^\delta \partial_\tau u^\delta)(\tau)\,d\tau \right\|_{L^{\frac{4n}{3n+2}}} \\
&\le \int_0^t \left\|(\partial_\tau \rho^\delta u^\delta)(\tau)\right\|_{L^{\frac{4n}{3n+2}}}\,d\tau + \int_0^t \left\|(\rho^\delta \partial_\tau u^\delta)(\tau)\right\|_{L^{\frac{4n}{3n+2}}}\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^t \left\|(u^\delta\cdot\nabla\rho^\delta u^\delta)(\tau)\right\|_{L^{\frac{4n}{3n+2}}}\,d\tau \\
&\quad+\int_0^t \left\|(\sqrt{\rho^\delta}\sqrt{\rho^\delta}\partial_\tau u^\delta)(\tau)\right\|_{L^{\frac{4n}{3n+2}}}\,d\tau \\
&\le \int_0^t \|\nabla\rho^\delta(\tau)\|_{L^{\frac{4n}{n+6}}}\|u^\delta(\tau)\|^2_{L^{\frac{4n}{n-2}}}\,d\tau \\
&\quad+\int_0^t \|\rho^\delta(\tau)\|^{\frac12}_{L^{\frac{2n}{n+2}}}\|(\sqrt{\rho^\delta}\partial_\tau u^\delta)(\tau)\|_{L^2}\,d\tau \\
&\le C_0(t)t+C_0\int_0^t \|(\sqrt{\rho^\delta}\partial_\tau u^\delta)(\tau)\|_{L^2}\,d\tau \\
&\le C_0(t)t+C_0t^{\frac12}\left(\int_0^t \|(\sqrt{\rho^\delta}\partial_\tau u^\delta)(\tau)\|^2_{L^2}\,d\tau\right)^{\frac12} \\
&\le C_0(t)t+C_0t^{\frac12},
\end{aligned}
$$

where we have used (2.16) and (2.35) again. Using the Hölder inequality yields

$$
\begin{aligned}
\|(\rho^\delta u^\delta)(t)-\rho_0^\delta u_0^\delta\|_{L^2} &\le C\|(\rho^\delta u^\delta)(t)-\rho_0^\delta u_0^\delta\|^{\frac12}_{L^{\frac{4n}{3n+2}}}\|(\rho^\delta u^\delta)(t)-\rho_0^\delta u_0^\delta\|^{\frac12}_{L^{\frac{4n}{n-2}}} \\
&\le \left(C_0(t)t+C_0t^{\frac12}\right)^{\frac12}.
\end{aligned}
$$

We therefore derive

$$
\begin{aligned}
\|(\rho u)(t)-\rho_0u_0\|_{L^2} &\le \|(\rho u)(t)-(\rho^\delta u^\delta)(t)\|_{L^2}+\|(\rho^\delta u^\delta)(t)-\rho_0^\delta u_0^\delta\|_{L^2} \\
&\quad+\|\rho_0^\delta u_0^\delta-\rho_0u_0\|_{L^2} \\
&\le \|(\rho u)(t)-(\rho^\delta u^\delta)(t)\|_{L^2}+\left(C_0(t)t+C_0t^{\frac12}\right)^{\frac12} \\
&\quad+\|\rho_0^\delta u_0^\delta-\rho_0u_0\|_{L^2}.
\end{aligned}
\tag{2.10}
$$

Keeping in mind (2.9) and (2.8), we deduce for any $t\in(0,T^*]$ that

$$
\liminf_{\delta\to0}\|(\rho u)(t)-(\rho^\delta u^\delta)(t)\|_{L^2}=0. \tag{2.11}
$$

Combining (2.10), (2.11) and the fact $\sqrt{\rho_0^\delta}u_0^\delta\to\sqrt{\rho_0}u_0$ in $L^2(\mathbb{R}^n)$ as $\delta\to0$, one has for any $t\in(0,T^*]$

$$
\begin{aligned}
\|(\rho u)(t)-\rho_0u_0\|_{L^2} &\le \liminf_{\delta\to0}\|(\rho u)(t)-(\rho^\delta u^\delta)(t)\|_{L^2}+\left(C_0(t)t+C_0t^{\frac12}\right)^{\frac12} \\
&\quad+\liminf_{\delta\to0}\|\rho_0^\delta u_0^\delta-\rho_0u_0\|_{L^2} \\
&=\left(C_0(t)t+C_0t^{\frac12}\right)^{\frac12}.
\end{aligned}
$$

As a result, this implies that ρu continuous at the original time and satisfies the initial condition $\rho u|_{t=0}=\rho_0u_0$. This concludes the proof of the existence part of Theorem 1.1. Finally, the proof of the uniqueness of $(\rho,\,u)$ can be performed as the part of *Proof of Theorem 1.1* (see the end of this section). This finishes the proof of Lemma 2.1. □

2.2. Exponential decay of $\|\sqrt{\rho}u(t)\|_{L^2}^2$ and $\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(t)\|_{L^2}^2$. We begin with the basic energy estimates.

LEMMA 2.2. *Under the assumptions of Theorem 1.1, the solution (ρ, u) of the system (1.1) admits the following bound for any $t \geq 0$,*

$$\|\rho(t)\|_{L^{\frac{2n}{n+2}}\cap L^\infty} \leq \|\rho_0\|_{L^{\frac{2n}{n+2}}\cap L^\infty}, \tag{2.12}$$

$$e^{\gamma t}\|\sqrt{\rho}u(t)\|_{L^2}^2 + \int_0^t e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2\,d\tau \leq \|\sqrt{\rho_0}u_0\|_{L^2}^2. \tag{2.13}$$

Proof. First, the non-negativeness of ρ is a direct consequence of the maximum principle and $\rho_0 \geq 0$. We multiply the equation $(1.1)_1$ by $|\rho|^{p-2}\rho$, integrate it over $\mathbb{R}^n$ and use $\nabla\cdot u = 0$ to conclude

$$\frac{d}{dt}\|\rho(t)\|_{L^p} = 0.$$

We then obtain $\|\rho(t)\|_{L^p} \leq \|\rho_0\|_{L^p}$. Letting $p \to \infty$ yields $\|\rho(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}$.

In order to show (2.13), we multiply equation $(1.1)_2$ by u, use the equation $(1.1)_1$ and integrate the resulting equation over $\mathbb{R}^n$ to show

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u(t)\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2 = 0. \tag{2.14}$$

Now it is easy to check that there exists some constant $C_\star = C_\star(n)$ such that

$$\begin{aligned}\|\sqrt{\rho}u\|_{L^2} \leq \|\sqrt{\rho}\|_{L^{\frac{4n}{n+2}}}\|u\|_{L^{\frac{4n}{n-2}}} &\leq C_\star\|\rho\|_{L^{\frac{2n}{n+2}}}^{\frac{1}{2}}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}\\ &\leq C_\star\|\rho_0\|_{L^{\frac{2n}{n+2}}}^{\frac{1}{2}}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2},\end{aligned} \tag{2.15}$$

where and henceforth the following embedding inequality will be used frequently:

$$\|u\|_{L^q} \leq C(q,n)\|\Lambda^{\frac{n}{2}-\frac{n}{q}}u\|_{L^2}, \quad 2 \leq q < \infty.$$

Thus, we conclude from (2.14) that

$$\frac{d}{dt}\|\sqrt{\rho}u(t)\|_{L^2}^2 + \gamma\|\sqrt{\rho}u(t)\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2 = 0,$$

where

$$\gamma = \frac{1}{C_\star^2\|\rho_0\|_{L^{\frac{2n}{n+2}}}}.$$

Integrating in time yields (2.13). This completes the proof of Lemma 2.2. □

Based on the estimate (2.13), we now derive the same exponential decay estimate for $\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(t)\|_{L^2}^2$.

LEMMA 2.3. *Under the assumptions of Theorem 1.1, the solution (ρ, u) of the system (1.1) admits the following bound for any $t \geq 0$,*

$$e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(t)\|_{L^2}^2 + \int_0^t e^{\gamma\tau}(\|\Lambda^{1+\frac{n}{2}}u(\tau)\|_{L^2}^2 + \|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2)\,d\tau \leq \widetilde{C_1}, \tag{2.16}$$

where $\widetilde{C_1}$ *depends only on* $\|\rho_0\|_{L^{\frac{2n}{n+2}}}$, $\|\rho_0\|_{L^\infty}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$ *and* $\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u_0\|_{L^2}$.

Proof. First, multiplying the equation $(1.1)_2$ by $\partial_t u$, using $\nabla\cdot u=0$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(t)\|_{L^2}^2+\|\sqrt{\rho}\partial_t u\|_{L^2}^2=-\int_{\mathbb{R}^n}\rho u\cdot\nabla u\cdot\partial_t u\,dx.$$

With the aid of the Gagliardo-Nirenberg inequality, one gets

$$\begin{aligned}-\int_{\mathbb{R}^n}\rho u\cdot\nabla u\cdot\partial_t u\,dx&\le\|u\cdot\nabla u\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\\&\le C\|\rho_0\|_{L^\infty}^{\frac{1}{2}}\|u\|_{L^{\frac{4n}{n-2}}}\|\nabla u\|_{L^{\frac{4n}{n+2}}}\|\sqrt{\rho}\partial_t u\|_{L^2}\\&\le C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2\|\sqrt{\rho}\partial_t u\|_{L^2}\\&\le\frac{1}{2}\|\sqrt{\rho}\partial_t u\|_{L^2}^2+C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2.\end{aligned}\tag{2.17}$$

We therefore conclude that

$$\frac{d}{dt}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(t)\|_{L^2}^2+\|\sqrt{\rho}\partial_t u\|_{L^2}^2\le C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2.$$

This implies

$$\begin{aligned}\frac{d}{dt}(e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(t)\|_{L^2}^2)+e^{\gamma t}\|\sqrt{\rho}\partial_t u\|_{L^2}^2&\le\gamma e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(t)\|_{L^2}^2\\&\quad+Ce^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2.\end{aligned}$$

Integrating in time and using (2.13) yield

$$\begin{aligned}&e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(t)\|_{L^2}^2+\int_0^t e^{\gamma\tau}\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau\\&\le\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u_0\|_{L^2}^2+\gamma\int_0^t e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2\,d\tau\\&\quad+C\int_0^t e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2\,d\tau\\&\le\widetilde{C}+C\int_0^t e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2\,d\tau.\end{aligned}$$

We thus get

$$\begin{aligned}&e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(t)\|_{L^2}^2+\int_0^t e^{\gamma\tau}\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau\\&\le\widetilde{C}+C\int_0^t e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2\,d\tau.\end{aligned}\tag{2.18}$$

Let us recall the classical Gronwall inequality: assume that $\phi(t)$, $g(t)$, $\alpha(t)$ be non-negative functions over $[0,\,T]$ and satisfy

$$\phi(t)\le\phi(0)+\int_0^t\alpha(\tau)\phi(\tau)\,d\tau+\int_0^t g(\tau)\,d\tau$$

or

$$\frac{d}{dt}\phi(t) \le \alpha(t)\phi(t) + g(t),$$

then, it holds for any $t \in [0, T]$ that

$$\phi(t) \le e^{\int_0^t \alpha(\tau)\,d\tau}\phi(0) + \int_0^t g(s)e^{\int_s^t \alpha(\tau)\,d\tau}\,ds.$$

By virtue of the above Gronwall inequality and (2.13), one deduces from (2.18) that

$$e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(t)\|_{L^2}^2 \le \widetilde{C_1}\exp\left[\int_0^t \|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2\,d\tau\right] \le \widetilde{C_1}, \tag{2.19}$$

which along with (2.18) also implies

$$\int_0^t e^{\gamma\tau}\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau \le \widetilde{C_1}. \tag{2.20}$$

Now let us recall the generalized Stokes equations

$$\begin{cases} (-\Delta)^{\frac{1}{2}+\frac{n}{4}}u + \nabla p = -\rho\partial_t u - \rho u\cdot\nabla u, \\ \nabla\cdot u = 0, \end{cases} \tag{2.21}$$

then we have

$$\nabla p = (-\Delta)^{-1}\nabla\nabla\cdot(\rho\partial_t u + \rho u\cdot\nabla u). \tag{2.22}$$

Thus, it follows from (2.21) and (2.22) that

$$\begin{aligned} \|\Lambda^{1+\frac{n}{2}}u\|_{L^2} &\le C\|\rho\partial_t u\|_{L^2} + C\|\rho u\cdot\nabla u\|_{L^2} \\ &\le C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\rho\|_{L^\infty}\|u\cdot\nabla u\|_{L^2} \\ &\le C\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|u\|_{L^{\frac{4n}{n-2}}}\|\nabla u\|_{L^{\frac{4n}{n+2}}} \\ &\le C\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2. \end{aligned} \tag{2.23}$$

This allows us to show

$$\begin{aligned} \int_0^t e^{\gamma\tau}\|\Lambda^{1+\frac{n}{2}}u(\tau)\|_{L^2}^2\,d\tau &\le \int_0^t e^{\gamma\tau}\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau + \int_0^t e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^4\,d\tau \\ &\le \widetilde{C_1}, \end{aligned} \tag{2.24}$$

where we have used (2.13), (2.19) and (2.20). We thus complete the proof of the lemma by combining (2.19), (2.20) and (2.24). □

2.3. Time-weighted estimates and exponential decay of $\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2$ and other quantities. The following lemma is crucial to derive the higher order estimates of the solutions.

LEMMA 2.4. *Under the assumptions of Theorem 1.1, the solution (ρ, u) of the system (1.1) admits the following bound for any $t \ge 0$,*

$$\begin{aligned} t\|\Lambda^{1+\frac{n}{2}}u(t)\|_{L^2}^2 + t\|p(t)\|_{H^1}^2 + t\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 \\ + \int_0^t \tau\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau \le \widetilde{C_1}. \end{aligned} \tag{2.25}$$

Moreover, for any $t \geq 1$, the following estimates hold true

$$e^{\gamma t}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \int_1^t e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau \leq \widetilde{C_1}, \tag{2.26}$$

$$e^{\gamma t}\|\Lambda^{1+\frac{n}{2}}u(t)\|_{L^2}^2 + \int_1^t e^{\gamma\tau}\|\Lambda^{1+\frac{n}{2}}u(\tau)\|_{L^2}^2\,d\tau \leq \widetilde{C_1}, \tag{2.27}$$

$$e^{\gamma t}\|p(t)\|_{H^1}^2 \leq \widetilde{C_1}, \tag{2.28}$$

where $\widetilde{C_1}$ depends only on $\|\rho_0\|_{L^{\frac{2n}{n+2}}}$, $\|\rho_0\|_{L^\infty}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$ and $\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u_0\|_{L^2}$.

Proof. First, applying the time derivative ∂_t to the equation $(1.1)_2$ gives

$$\rho\partial_{tt}u + \rho u\cdot\nabla\partial_t u + (-\Delta)^{\frac{1}{2}+\frac{n}{4}}\partial_t u + \nabla\partial_t p = -\partial_t\rho\partial_t u - \partial_t(\rho u)\cdot\nabla u. \tag{2.29}$$

Multiplying (2.29) by $\partial_t u$ and using the equation $(1.1)_1$, we derive that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}^2\\
&= -\int_{\mathbb{R}^n}\partial_t\rho\partial_t u\cdot\partial_t u\,dx - \int_{\mathbb{R}^n}\partial_t(\rho u)\cdot\nabla u\cdot\partial_t u\,dx\\
&= -2\int_{\mathbb{R}^n}\rho u\cdot\nabla\partial_t u\cdot\partial_t u\,dx - \int_{\mathbb{R}^n}\rho\partial_t u\cdot\nabla u\cdot\partial_t u\,dx - \int_{\mathbb{R}^n}\rho u\cdot\nabla(u\cdot\nabla u\cdot\partial_t u)\,dx\\
&:= N_1 + N_2 + N_3.
\end{aligned} \tag{2.30}$$

By means of the embedding inequalities, one shows

$$\begin{aligned}
N_1 &\leq C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\|\nabla\partial_t u\|_{L^{\frac{4n}{n+2}}}\|u\|_{L^{\frac{4n}{n-2}}}\\
&\leq C\|\sqrt{\rho}\partial_t u\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}\\
&\leq \frac{1}{8}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2\|\sqrt{\rho}\partial_t u\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
N_2 &\leq C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\|\nabla u\|_{L^{\frac{4n}{n+2}}}\|\partial_t u\|_{L^{\frac{4n}{n-2}}}\\
&\leq C\|\sqrt{\rho}\partial_t u\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}\\
&\leq \frac{1}{8}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2\|\sqrt{\rho}\partial_t u\|_{L^2}^2.
\end{aligned}$$

For the term N_3, it can be bounded by

$$
\begin{aligned}
N_3 \le & \left|\int_{\mathbb{R}^n} \rho u\cdot\nabla u\cdot\nabla u\cdot\partial_t u\,dx\right| + \left|\int_{\mathbb{R}^n} \rho u\cdot u\cdot\nabla^2 u\cdot\partial_t u\,dx\right| \\
& + \left|\int_{\mathbb{R}^n} \rho u\cdot u\cdot\nabla u\cdot\nabla\partial_t u\,dx\right| \\
\le & C\|\rho\|_{L^\infty}\|u\|_{L^{\frac{4n}{n-2}}}\|\nabla u\|^2_{L^{\frac{4n}{n+2}}}\|\partial_t u\|_{L^{\frac{4n}{n-2}}} + C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\|\nabla^2 u\|_{L^n}\|u\|^2_{L^{\frac{4n}{n-2}}} \\
& + C\|\rho\|_{L^\infty}\|u\|^2_{L^{\frac{4n}{n-2}}}\|\nabla u\|_{L^{\frac{4n}{n+2}}}\|\nabla\partial_t u\|_{L^{\frac{4n}{n+2}}} \\
\le & C\|\Lambda^{\frac12+\frac n4}u\|_{L^2}\|\Lambda^{\frac12+\frac n4}u\|^2_{L^2}\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2} + C\|\sqrt{\rho}\partial_t u\|_{L^2}\|\Lambda^{1+\frac n2}u\|_{L^2}\|\Lambda^{\frac12+\frac n4}u\|^2_{L^2} \\
& + C\|\Lambda^{\frac12+\frac n4}u\|^2_{L^2}\|\Lambda^{\frac12+\frac n4}u\|_{L^2}\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2} \\
\le & C\|\Lambda^{\frac12+\frac n4}u\|_{L^2}\|\Lambda^{\frac12+\frac n4}u\|^2_{L^2}\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2} + C\|\sqrt{\rho}\partial_t u\|^2_{L^2}\|\Lambda^{\frac12+\frac n4}u\|^2_{L^2} \\
& + C\|\sqrt{\rho}\partial_t u\|_{L^2}\|\Lambda^{\frac12+\frac n4}u\|^4_{L^2} + C\|\Lambda^{\frac12+\frac n4}u\|^2_{L^2}\|\Lambda^{\frac12+\frac n4}u\|_{L^2}\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2} \\
\le & \frac18\|\Lambda^{\frac12+\frac n4}\partial_t u\|^2_{L^2} + C\|\Lambda^{\frac12+\frac n4}u\|^6_{L^2} + C\|\Lambda^{\frac12+\frac n4}u\|^2_{L^2}\|\sqrt{\rho}\partial_t u\|^2_{L^2},
\end{aligned}
$$

where we have used the following fact due to (2.23)

$$
\|\Lambda^{1+\frac n2}u\|_{L^2} \le C\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\Lambda^{\frac12+\frac n4}u\|^2_{L^2}. \tag{2.31}
$$

Substituting the above estimates into (2.30) yields

$$
\begin{aligned}
& \frac{d}{dt}\|\sqrt{\rho}\partial_t u(t)\|^2_{L^2} + \|\Lambda^{\frac12+\frac n4}\partial_t u\|^2_{L^2} \\
& \le C\|\Lambda^{\frac12+\frac n4}u\|^2_{L^2}\|\sqrt{\rho}\partial_t u\|^2_{L^2} + C\|\Lambda^{\frac12+\frac n4}u\|^6_{L^2},
\end{aligned} \tag{2.32}
$$

which implies

$$
\begin{aligned}
& \frac{d}{dt}(t\|\sqrt{\rho}\partial_t u(t)\|^2_{L^2}) + t\|\Lambda^{\frac12+\frac n4}\partial_t u(t)\|^2_{L^2} \\
& \le C\|\Lambda^{\frac12+\frac n4}u\|^2_{L^2}(t\|\sqrt{\rho}\partial_t u\|^2_{L^2}) + \|\sqrt{\rho}\partial_t u(t)\|^2_{L^2} + Ct\|\Lambda^{\frac12+\frac n4}u\|^6_{L^2}.
\end{aligned}
$$

From (2.13) and (2.16), and by the Gronwall inequality, one has

$$
t\|\sqrt{\rho}\partial_t u(t)\|^2_{L^2} + \int_0^t \tau\|\Lambda^{\frac12+\frac n4}\partial_\tau u(\tau)\|^2_{L^2}\,d\tau \le \widetilde{C_1}. \tag{2.33}
$$

Moreover, we deduce from (2.32) that

$$
\begin{aligned}
& \frac{d}{dt}(e^{\gamma t}\|\sqrt{\rho}\partial_t u(t)\|^2_{L^2}) + e^{\gamma t}\|\Lambda^{\frac12+\frac n4}\partial_t u(t)\|^2_{L^2} \\
& \le C\|\Lambda^{\frac12+\frac n4}u\|^2_{L^2}(e^{\gamma t}\|\sqrt{\rho}\partial_t u\|^2_{L^2}) + \gamma e^{\gamma t}\|\sqrt{\rho}\partial_t u(t)\|^2_{L^2} + Ce^{\gamma t}\|\Lambda^{\frac12+\frac n4}u\|^6_{L^2}.
\end{aligned}
$$

Integrating it in time and making use of (2.16) as well as (2.33) lead to

$$
\begin{aligned}
& e^{\gamma t}\|\sqrt{\rho}\partial_t u(t)\|^2_{L^2} + \int_1^t e^{\gamma\tau}\|\Lambda^{\frac12+\frac n4}\partial_\tau u(\tau)\|^2_{L^2}\,d\tau \\
\le & \widetilde{C_1} + C\int_1^t \|\Lambda^{\frac12+\frac n4}u(\tau)\|^2_{L^2}(e^{\gamma\tau}\|\sqrt{\rho}\partial_\tau u(\tau)\|^2_{L^2})\,d\tau + \gamma\int_1^t e^{\gamma\tau}\|\sqrt{\rho}\partial_\tau u(\tau)\|^2_{L^2}\,d\tau \\
& + C\int_1^t (e^{\gamma\tau}\|\Lambda^{\frac12+\frac n4}u(\tau)\|^2_{L^2})\|\Lambda^{\frac12+\frac n4}u(\tau)\|^4_{L^2}\,d\tau \\
\le & \widetilde{C_1} + C\int_1^t \|\Lambda^{\frac12+\frac n4}u(\tau)\|^2_{L^2}(e^{\gamma\tau}\|\sqrt{\rho}\partial_\tau u(\tau)\|^2_{L^2})\,d\tau.
\end{aligned}
$$

By the same argument adopted in dealing with (2.19) and (2.20), we thus deduce

$$e^{\gamma t}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \int_1^t e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau \le \widetilde{C_1}.$$

By means of (2.31), (2.26) and (2.16), we have

$$\begin{aligned}
&e^{\gamma t}\|\Lambda^{1+\frac{n}{2}}u\|_{L^2}^2 + \int_1^t e^{\gamma\tau}\|\Lambda^{1+\frac{n}{2}}u(\tau)\|_{L^2}^2\,d\tau \\
&\le Ce^{\gamma t}(\|\sqrt{\rho}\partial_t u\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^4) + \int_1^t e^{\gamma\tau}(\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^4)\,d\tau \\
&= Ce^{\gamma t}\|\sqrt{\rho}\partial_t u\|_{L^2}^2 + Ce^{-\gamma t}(e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2)^2 \\
&\quad + \int_1^t \Big(e^{\gamma\tau}\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2 + e^{-\gamma\tau}(e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2)(e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2)\Big)\,d\tau \\
&\le \widetilde{C_1} + \widetilde{C_1}\int_1^t e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2\,d\tau \\
&\le \widetilde{C_1}.
\end{aligned}$$

We thus obtain (2.27). It follows from (2.22) that

$$\|\nabla p\|_{L^2} \le C\|\rho\partial_t u\|_{L^2} + C\|\rho u\cdot\nabla u\|_{L^2} \le C\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2,$$

where we have used the estimates in (2.17). Similarly, we obtain

$$\begin{aligned}
\|p\|_{L^2} &\le C\|\Lambda^{-1}(\rho\partial_t u\|_{L^2})\|_{L^2} + C\|\Lambda^{-1}(\rho u\cdot\nabla u)\|_{L^2} \\
&\le C\|\rho\partial_t u\|_{L^{\frac{2n}{n+2}}} + C\|\rho u\cdot\nabla u\|_{L^{\frac{2n}{n+2}}} \\
&\le C\|\sqrt{\rho}\|_{L^n}\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\rho\|_{L^n}\|u\cdot\nabla u\|_{L^2} \\
&\le C\|\sqrt{\rho_0}\|_{L^n}\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\rho_0\|_{L^n}\|u\cdot\nabla u\|_{L^2} \\
&\le C\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2.
\end{aligned}$$

As before, we therefore obtain for all $t \ge 1$,

$$e^{\gamma t}\|p\|_{H^1}^2 \le Ce^{\gamma t}\|\sqrt{\rho}\partial_t u\|_{L^2}^2 + Ce^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^4 \le \widetilde{C_1}.$$

This completes the proof of Lemma 2.4. □

2.4. Uniform in time bound of $\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau$ and gradient of ρ. The following estimates will be used to show the uniqueness of solutions and the exponential decay of other quantities.

LEMMA 2.5. *Under the assumptions of Theorem 1.1, the solution (ρ, u) of the system (1.1) admits the following bounds for any $t \ge 0$,*

$$\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau \le \widetilde{C_1}, \tag{2.34}$$

$$\|\nabla\rho(t)\|_{L^{\frac{4n}{n+6}}} \le \widetilde{C_1}\|\nabla\rho_0\|_{L^{\frac{4n}{n+6}}}, \tag{2.35}$$

$$\|\nabla\rho(t)\|_{L^2} \le \widetilde{C_1}\|\nabla\rho_0\|_{L^2}, \tag{2.36}$$

where $\widetilde{C_1}$ *depends only on* $\|\rho_0\|_{L^{\frac{2n}{n+2}}}$, $\|\rho_0\|_{L^\infty}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$ *and* $\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u_0\|_{L^2}$.

REMARK 2.1. We remark that the bound (2.35) will be used to show the uniqueness, while the bound (2.36) will be used to derive the exponential decay of $\|\Lambda^{1+\frac{n}{2}}\partial_t u(t)\|_{L^2}^2$ and other quantities.

Proof. First, it is easy to check that for any $2 < p < \frac{4n}{n-2}$,

$$\begin{aligned}
\|\rho\partial_t u\|_{L^p} &\le C\|\rho\partial_t u\|_{L^2}^{1-\frac{2n(p-2)}{(n+2)p}}\|\rho\partial_t u\|_{L^{\frac{4n}{n-2}}}^{\frac{2n(p-2)}{(n+2)p}} \\
&\le C\|\sqrt{\rho}\|_{L^\infty}^{1-\frac{2n(p-2)}{(n+2)p}}\|\sqrt{\rho}\partial_t u\|_{L^2}^{1-\frac{2n(p-2)}{(n+2)p}}\|\rho\|_{L^\infty}^{\frac{2n(p-2)}{(n+2)p}}\|\partial_t u\|_{L^{\frac{4n}{n-2}}}^{\frac{2n(p-2)}{(n+2)p}} \\
&\le C\|\sqrt{\rho}\partial_t u\|_{L^2}^{1-\frac{2n(p-2)}{(n+2)p}}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}^{\frac{2n(p-2)}{(n+2)p}} \\
&\le C\|\sqrt{\rho}\partial_t u\|_{L^2} + C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}, \qquad (2.37)
\end{aligned}$$

$$\begin{aligned}
\|\rho u\cdot\nabla u\|_{L^p} &\le C\|\rho u\cdot\nabla u\|_{L^2}^{1-\frac{2n(p-2)}{(n+2)p}}\|\rho u\cdot\nabla u\|_{L^{\frac{4n}{n-2}}}^{\frac{2n(p-2)}{(n+2)p}} \\
&\le C\|u\cdot\nabla u\|_{L^2}^{1-\frac{2n(p-2)}{(n+2)p}}\|u\cdot\nabla u\|_{L^{\frac{4n}{n-2}}}^{\frac{2n(p-2)}{(n+2)p}} \\
&\le C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^{2\left(1-\frac{2n(p-2)}{(n+2)p}\right)}\Big(\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}\|\Lambda^{1+\frac{n}{2}}u\|_{L^2}\Big)^{\frac{2n(p-2)}{(n+2)p}} \\
&\le C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2 + C\|\Lambda^{1+\frac{n}{2}}u\|_{L^2}^2, \qquad (2.38)
\end{aligned}$$

where we have used the following fact

$$\begin{aligned}
\|u\cdot\nabla u\|_{L^{\frac{4n}{n-2}}} &\le C\|u\|_{L^\infty}\|\nabla u\|_{L^{\frac{4n}{n-2}}} \\
&\le C(\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^{\frac{4}{n+2}}\|\Lambda^{1+\frac{n}{2}}u\|_{L^2}^{1-\frac{4}{n+2}})(\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^{1-\frac{4}{n+2}}\|\Lambda^{1+\frac{n}{2}}u\|_{L^2}^{\frac{4}{n+2}}) \\
&\le \|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}\|\Lambda^{1+\frac{n}{2}}u\|_{L^2}. \qquad (2.39)
\end{aligned}$$

Combining the estimates (2.13), (2.16), (2.26) and (2.27) allows us to show that, for any $2 < p < \frac{4n}{n-2}$ and for any $t \ge 0$,

$$\begin{aligned}
&\int_0^t (\|\rho\partial_t u(\tau)\|_{L^p} + \|\rho u\cdot\nabla u(\tau)\|_{L^p})\,d\tau \\
&\le C\int_0^t (\|\sqrt{\rho}\partial_\tau u\|_{L^2}^{1-\frac{2n(p-2)}{(n+2)p}}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u\|_{L^2}^{\frac{2n(p-2)}{(n+2)p}} + \|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2 + \|\Lambda^{1+\frac{n}{2}}u(\tau)\|_{L^2}^2)\,d\tau \\
&= C\int_0^1 \tau^{-\frac{1}{2}}(\tau^{\frac{1}{2}}\|\sqrt{\rho}\partial_\tau u\|_{L^2})^{1-\frac{2n(p-2)}{(n+2)p}}(\tau^{\frac{1}{2}}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u\|_{L^2})^{\frac{2n(p-2)}{(n+2)p}})\,d\tau \\
&\quad + C\int_1^t e^{-\frac{\gamma\tau}{2}}(e^{\frac{\gamma\tau}{2}}\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2} + e^{\frac{\gamma\tau}{2}}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u(\tau)\|_{L^2})\,d\tau \\
&\quad + C\int_0^t e^{-\gamma\tau}(e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2 + e^{\gamma\tau}\|\Lambda^{1+\frac{n}{2}}u(\tau)\|_{L^2}^2)\,d\tau \\
&\le C\int_0^1 \tau^{-\frac{1}{2}}(\tau^{\frac{1}{2}}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u\|_{L^2})^{\frac{2n(p-2)}{(n+2)p}})\,d\tau + \widetilde{C}\int_0^t e^{-\gamma\tau}\,d\tau
\end{aligned}$$

$$\le \widetilde{C} + C\left(\int_0^1 \tau\|\Lambda^{\frac12+\frac n4}\partial_\tau u\|_{L^2}^2\,d\tau\right)^{\frac{n(p-2)}{(n+2)p}}\left(\int_0^1 \tau^{-\frac{(n+2)p}{4(n+p)}}\,d\tau\right)^{1-\frac{n(p-2)}{(n+2)p}}$$
$$\le \widetilde{C}. \tag{2.40}$$

Using (2.22) and applying the L^p-estimate to (2.21) yield

$$\|\Lambda^{1+\frac n2}u\|_{L^p} \le C\|\rho\partial_t u\|_{L^p} + C\|\rho u\cdot\nabla u\|_{L^p},$$

which leads to

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\le C\|\nabla u\|_{L^{\frac{4n}{n+2}}}^{1-\frac{(n+2)p}{(3n+2)p-4n}}\|\Lambda^{1+\frac n2}u\|_{L^p}^{\frac{(n+2)p}{(3n+2)p-4n}}\\ &\le C\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{1-\frac{(n+2)p}{(3n+2)p-4n}}\|\Lambda^{1+\frac n2}u\|_{L^p}^{\frac{(n+2)p}{(3n+2)p-4n}}\\ &\le C\|\Lambda^{\frac12+\frac n4}u\|_{L^2} + C\|\Lambda^{1+\frac n2}u\|_{L^p}\\ &\le C\|\Lambda^{\frac12+\frac n4}u\|_{L^2} + C\|\rho\partial_t u\|_{L^p} + C\|\rho u\cdot\nabla u\|_{L^p}.\end{aligned} \tag{2.41}$$

Thanks to (2.13), (2.40) and (2.41), we immediately obtain

$$\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau \le \widetilde{C_1}. \tag{2.42}$$

Since ρ satisfies $\partial_t\rho + u\cdot\nabla\rho = 0$, direct computations yield

$$\frac{d}{dt}\|\nabla\rho(t)\|_{L^{\frac{4n}{n+6}}} \le \|\nabla u\|_{L^\infty}\|\nabla\rho(t)\|_{L^{\frac{4n}{n+6}}}, \quad \frac{d}{dt}\|\nabla\rho(t)\|_{L^2} \le \|\nabla u\|_{L^\infty}\|\nabla\rho(t)\|_{L^2},$$

The Gronwall inequality and (2.42) ensure that

$$\|\nabla\rho(t)\|_{L^{\frac{4n}{n+6}}} \le \|\nabla\rho_0\|_{L^{\frac{4n}{n+6}}}\exp\left[\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau\right] \le \widetilde{C_1}\|\nabla\rho_0\|_{L^{\frac{4n}{n+6}}},$$

$$\|\nabla\rho(t)\|_{L^2} \le \|\nabla\rho_0\|_{L^2}\exp\left[\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau\right] \le \widetilde{C_1}\|\nabla\rho_0\|_{L^2}.$$

We thus complete the proof of Lemma 2.5. □

REMARK 2.2. We remark that the estimates of the previous subsections would suffice to get already a satisfactory global well-posedness result with exponential decay. Here, thanks to the above obtained estimates, we want to show more regularities of the solution.

2.5. Time-weighted estimates and exponential decay of $\|\Lambda^{\frac12+\frac n4}\partial_t u(t)\|_{L^2}^2$.

LEMMA 2.6. *Under the assumptions of Theorem 1.1, the solution (ρ, u) of the system (1.1) admits the following bound for any $t \ge 0$,*

$$t^2\|\Lambda^{\frac12+\frac n4}\partial_t u(t)\|_{L^2}^2 + \int_0^t \tau^2\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|_{L^2}^2\,d\tau \le \widetilde{C}. \tag{2.43}$$

Moreover, for any $t \ge 1$, we have

$$e^{\gamma t}\|\Lambda^{\frac12+\frac n4}\partial_t u(t)\|_{L^2}^2 + \int_1^t e^{\gamma\tau}\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|_{L^2}^2\,d\tau \le \widetilde{C}, \tag{2.44}$$

where $\widetilde{C}$ *depends only on* $\|\rho_0\|_{L^{\frac{2n}{n+2}}}$, $\|\rho_0\|_{L^\infty}$, $\|\nabla\rho_0\|_{L^2}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$ *and* $\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u_0\|_{L^2}$.

Proof. Multiplying (2.29) by $\partial_{tt}u$ and integrating by parts imply that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u(t)\|_{L^2}^2+\|\sqrt{\rho}\partial_{tt}u\|_{L^2}^2\\
&=-\int_{\mathbb{R}^n}\partial_t\rho\partial_t u\cdot\partial_{tt}u\,dx-\int_{\mathbb{R}^n}\partial_t(\rho u)\cdot\nabla u\cdot\partial_{tt}u\,dx-\int_{\mathbb{R}^n}\rho u\cdot\nabla\partial_t u\cdot\partial_{tt}u\,dx\\
&=-\int_{\mathbb{R}^n}\partial_t\rho\partial_t u\cdot\partial_{tt}u\,dx-\int_{\mathbb{R}^n}\partial_t\rho u\cdot\nabla u\cdot\partial_{tt}u\,dx-\int_{\mathbb{R}^n}\rho\partial_t u\cdot\nabla u\cdot\partial_{tt}u\,dx\\
&\quad-\int_{\mathbb{R}^n}\rho u\cdot\nabla\partial_t u\cdot\partial_{tt}u\,dx\\
&:=H_1+H_2+H_3+H_4. \qquad (2.45)
\end{aligned}$$

We first bound H_3 and H_4 as

$$\begin{aligned}
|H_3|+|H_4|&\le C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_{tt}u\|_{L^2}(\|\partial_t u\|_{L^{\frac{4n}{n-2}}}\|\nabla u\|_{L^{\frac{4n}{n+2}}}+\|u\|_{L^{\frac{4n}{n-2}}}\|\partial_t\nabla u\|_{L^{\frac{4n}{n+2}}})\\
&\le C\|\sqrt{\rho_0}\|_{L^\infty}\|\sqrt{\rho}\partial_{tt}u\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}\\
&\le\frac{1}{2}\|\sqrt{\rho}\partial_{tt}u\|_{L^2}^2+C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^2\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}^2.
\end{aligned}$$

We rewrite H_1 as follows

$$\begin{aligned}
H_1&=-\frac{1}{2}\int_{\mathbb{R}^n}\partial_t\rho\partial_t|\partial_t u|^2\,dx\\
&=-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\partial_t\rho|\partial_t u|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^n}\partial_{tt}\rho|\partial_t u|^2\,dx\\
&=-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\partial_t\rho|\partial_t u|^2\,dx-\frac{1}{2}\int_{\mathbb{R}^n}\partial_t\mathrm{div}(\rho u)|\partial_t u|^2\,dx\\
&=-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\partial_t\rho|\partial_t u|^2\,dx+\int_{\mathbb{R}^n}\partial_t(\rho u_i)\partial_t u\cdot\partial_t\partial_i u\,dx\\
&=-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\partial_t\rho|\partial_t u|^2\,dx+\int_{\mathbb{R}^n}\partial_t\rho u_i\partial_t u\cdot\partial_t\partial_i u\,dx\\
&\quad+\int_{\mathbb{R}^n}\rho\partial_t u_i\partial_t u\cdot\partial_t\partial_i u\,dx. \qquad (2.46)
\end{aligned}$$

By the Hölder inequality and the embedding inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^n}\rho\partial_t u_i\partial_t u\cdot\partial_t\partial_i u\,dx&\le C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\|\partial_t u\|_{L^{\frac{4n}{n-2}}}\|\partial_t\nabla u\|_{L^{\frac{4n}{n+2}}}\\
&\le C\|\sqrt{\rho}\partial_t u\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}^2.
\end{aligned}$$

Similarly, using $\partial_t\rho=-u\cdot\nabla\rho$ gives

$$\begin{aligned}
\int_{\mathbb{R}^n}\partial_t\rho u_i\partial_t u\cdot\partial_t\partial_i u\,dx&\le C\|u\cdot\nabla\rho\|_{L^2}\|u\|_{L^\infty}\|\partial_t u\|_{L^{\frac{4n}{n-2}}}\|\partial_t\nabla u\|_{L^{\frac{4n}{n+2}}}\\
&\le C\|\nabla\rho\|_{L^2}\|u\|_{L^\infty}^2\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}^2\\
&\le C\|\nabla\rho_0\|_{L^2}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^{\frac{8}{n+2}}\|\Lambda^{1+\frac{n}{2}}u\|_{L^2}^{\frac{2(n-2)}{n+2}}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}^2\\
&\le C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^{\frac{8}{n+2}}\|\Lambda^{1+\frac{n}{2}}u\|_{L^2}^{\frac{2(n-2)}{n+2}}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}^2,
\end{aligned}$$

where and henceforth the following interpolation inequality will be used frequently:

$$\|u\|_{L^\infty} \le C(n)\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{\frac{4}{n+2}}\|\Lambda^{1+\frac n2}u\|_{L^2}^{\frac{n-2}{n+2}}, \quad n\ge 3.$$

We obtain that

$$\begin{aligned}H_1 \le{}& -\frac12\frac{d}{dt}\int_{\mathbb{R}^n}\partial_t\rho|\partial_t u|^2\,dx + C(\|\sqrt{\rho}\partial_t u\|_{L^2}+\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{\frac{8}{n+2}}\|\Lambda^{1+\frac n2}u\|_{L^2}^{\frac{2(n-2)}{n+2}})\\ &\times\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}^2.\end{aligned}$$

Thanks to $\partial_t\rho = -\mathrm{div}(\rho u)$, one obtains

$$\begin{aligned}H_2 ={}& -\frac{d}{dt}\int_{\mathbb{R}^n}\partial_t\rho u\cdot\nabla u\cdot\partial_t u\,dx + \int_{\mathbb{R}^n}\partial_{tt}\rho u\cdot\nabla u\cdot\partial_t u\,dx + \int_{\mathbb{R}^n}\partial_t\rho\partial_t(u\cdot\nabla u)\cdot\partial_t u\,dx\\ ={}& -\frac{d}{dt}\int_{\mathbb{R}^n}\partial_t\rho u\cdot\nabla u\cdot\partial_t u\,dx + \int_{\mathbb{R}^n}\partial_t(\rho u_i)\partial_i(u\cdot\nabla u)\cdot\partial_t u\,dx\\ &+\int_{\mathbb{R}^n}\partial_t(\rho u_i)u\cdot\nabla u\cdot\partial_t\partial_i u\,dx + \int_{\mathbb{R}^n}\partial_t\rho\partial_t(u\cdot\nabla u)\cdot\partial_t u\,dx\\ ={}& -\frac{d}{dt}\int_{\mathbb{R}^n}\partial_t\rho u\cdot\nabla u\cdot\partial_t u\,dx + \int_{\mathbb{R}^n}\rho\partial_t u_i[\partial_i(u\cdot\nabla u)\cdot\partial_t u + u\cdot\nabla u\cdot\partial_t\partial_i u]\,dx\\ &+\int_{\mathbb{R}^n}\partial_t\rho[u_i u\cdot\nabla u\cdot\partial_t\partial_i u + u_i\partial_i(u\cdot\nabla u)\cdot\partial_t u]\,dx + \int_{\mathbb{R}^n}\partial_t\rho\partial_t(u\cdot\nabla u)\cdot\partial_t u\,dx\\ ={}& -\frac{d}{dt}\int_{\mathbb{R}^n}\partial_t\rho u\cdot\nabla u\cdot\partial_t u\,dx + H_{21}+H_{22}+H_{23}.\end{aligned}\tag{2.47}$$

It follows from the Hölder inequality and the interpolation inequalities that

$$\begin{aligned}H_{21} \le{}& C\|\rho\|_{L^\infty}\|\partial_t u\|_{L^{\frac{4n}{n-2}}}\|\nabla u\nabla u\|_{L^{\frac{2n}{n+2}}}\|\partial_t u\|_{L^{\frac{4n}{n-2}}}\\ &+C\|\sqrt{\rho}\|_{L^\infty}\|\partial_t u\|_{L^{\frac{4n}{n-2}}}\|\sqrt{\rho}u\|_{L^2}\|\nabla^2 u\|_{L^n}\|\partial_t u\|_{L^{\frac{4n}{n-2}}}\\ &+C\|\rho\|_{L^\infty}\|\partial_t u\|_{L^{\frac{4n}{n-2}}}\|u\|_{L^{\frac{4n}{n-2}}}\|\nabla u\|_{L^{\frac{4n}{n+2}}}\|\nabla\partial_t u\|_{L^{\frac{4n}{n+2}}}\\ \le{}& C(\|\sqrt{\rho}u\|_{L^2}\|\Lambda^{1+\frac n2}u\|_{L^2}+\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^2)\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}H_{22} \le{}& C\|\partial_t\rho\|_{L^2}\|u\|_{L^\infty}^2\|\nabla u\|_{L^{\frac{4n}{n-2}}}\|\nabla\partial_t u\|_{L^{\frac{4n}{n+2}}}\\ &+C\|\partial_t\rho\|_{L^2}\|u\|_{L^\infty}(\|\nabla u\nabla u\|_{L^{\frac{4n}{n+2}}}+\|u\nabla^2 u\|_{L^{\frac{4n}{n+2}}})\|\partial_t u\|_{L^{\frac{4n}{n-2}}}\\ \le{}& C\|u\cdot\nabla\rho\|_{L^2}\|u\|_{L^\infty}^2\|\Lambda^{\frac32+\frac n4}u\|_{L^2}\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}\\ &+C\|u\cdot\nabla\rho\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^{\frac{8n}{n+2}}}^2\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}\\ \le{}& C\|\nabla\rho\|_{L^2}\|u\|_{L^\infty}^3\|\Lambda^{\frac32+\frac n4}u\|_{L^2}\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}\\ &+C\|\nabla\rho\|_{L^2}\|u\|_{L^\infty}^2\|\nabla u\|_{L^{\frac{8n}{n+2}}}^2\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}\\ \le{}& C\|\nabla\rho_0\|_{L^2}(\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{\frac{12}{n+2}}\|\Lambda^{1+\frac n2}u\|_{L^2}^{\frac{3(n-2)}{n+2}})(\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{\frac{n-2}{n+2}}\|\Lambda^{1+\frac n2}u\|_{L^2}^{\frac{4}{n+2}})\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}\\ &+C\|\nabla\rho_0\|_{L^2}(\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{\frac{8}{n+2}}\|\Lambda^{1+\frac n2}u\|_{L^2}^{\frac{2(n-2)}{n+2}})(\|\Lambda^{\frac12+\frac n4}u\|_{L^2}\|\Lambda^{1+\frac n2}u\|_{L^2})\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}\\ \le{}& C\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{\frac{n+10}{n+2}}\|\Lambda^{1+\frac n2}u\|_{L^2}^{\frac{3n-2}{n+2}}\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}
H_{23} &\le C\|\partial_t\rho\|_{L^2}\|\partial_t u\|_{L^{\frac{4n}{n-2}}}\|\nabla u\|_{L^n}\|\partial_t u\|_{L^{\frac{4n}{n-2}}} \\
&\quad + C\|\partial_t\rho\|_{L^2}\|u\|_{L^\infty}\|\nabla\partial_t u\|_{L^{\frac{4n}{n+2}}}\|\partial_t u\|_{L^{\frac{4n}{n-2}}} \\
&\le C\|u\cdot\nabla\rho\|_{L^2}(\|\nabla u\|_{L^n}+\|u\|_{L^\infty})\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}^2 \\
&\le C\|\nabla\rho\|_{L^2}(\|\nabla u\|_{L^n}^2+\|u\|_{L^\infty}^2)\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}^2 \\
&\le C\|\nabla\rho_0\|_{L^2}(\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{\frac{8}{n+2}}\|\Lambda^{1+\frac n2}u\|_{L^2}^{\frac{2(n-2)}{n+2}})\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}^2 \\
&\le C\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{\frac{8}{n+2}}\|\Lambda^{1+\frac n2}u\|_{L^2}^{\frac{2(n-2)}{n+2}}\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}^2.
\end{aligned}$$

Therefore, H_2 admits the following bound

$$\begin{aligned}
H_2 &\le -\frac{d}{dt}\int_{\mathbb{R}^n}\partial_t\rho u\cdot\nabla u\cdot\partial_t u\,dx + C(\|\sqrt{\rho}u\|_{L^2}\|\Lambda^{1+\frac n2}u\|_{L^2}+\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^2 \\
&\quad + \|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{\frac{8}{n+2}}\|\Lambda^{1+\frac n2}u\|_{L^2}^{\frac{2(n-2)}{n+2}})\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}^2 \\
&\quad + C\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{\frac{n+10}{n+2}}\|\Lambda^{1+\frac n2}u\|_{L^2}^{\frac{3n-2}{n+2}}\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}.
\end{aligned}$$

We finally get by collecting all the above estimates

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\Lambda^{\frac12+\frac n4}\partial_t u(t)\|_{L^2}^2+\phi(t)\Big)+\|\sqrt{\rho}\partial_{tt}u\|_{L^2}^2 \\
&\le A(t)\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}+B(t)\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}^2,
\end{aligned}\tag{2.48}$$

where

$$\phi(t) := -\frac12\int_{\mathbb{R}^n}\partial_t\rho|\partial_t u|^2\,dx-\int_{\mathbb{R}^n}\partial_t\rho u\cdot\nabla u\cdot\partial_t u\,dx,$$

$$A(t) := C\|\Lambda^{\frac12+\frac n4}u(t)\|_{L^2}^{\frac{n+10}{n+2}}\|\Lambda^{1+\frac n2}u(t)\|_{L^2}^{\frac{3n-2}{n+2}},$$

$$\begin{aligned}
B(t) &:= C(\|\sqrt{\rho}u(t)\|_{L^2}\|\Lambda^{1+\frac n2}u(t)\|_{L^2}+\|\Lambda^{\frac12+\frac n4}u(t)\|_{L^2}^2 \\
&\quad + \|\Lambda^{\frac12+\frac n4}u(t)\|_{L^2}^{\frac{8}{n+2}}\|\Lambda^{1+\frac n2}u(t)\|_{L^2}^{\frac{2(n-2)}{n+2}}+\|\sqrt{\rho}\partial_t u\|_{L^2}).
\end{aligned}$$

Hence, in view of $\partial_t\rho=-\mathrm{div}(\rho u)=-u\cdot\nabla\rho$ and the Hölder inequality along with the embedding inequality, we deduce

$$\begin{aligned}
|\phi(t)| &= \left|\frac12\int_{\mathbb{R}^n}\mathrm{div}(\rho u)|\partial_t u|^2\,dx-\int_{\mathbb{R}^n}\partial_t\rho u\cdot\nabla u\cdot\partial_t u\,dx\right| \\
&= \left|-\int_{\mathbb{R}^n}\rho u_i\partial_t u\cdot\partial_t\partial_i u\,dx-\int_{\mathbb{R}^n}\partial_t\rho u\cdot\nabla u\cdot\partial_t u\,dx\right| \\
&\le C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\|u\|_{L^{\frac{4n}{n-2}}}\|\nabla\partial_t u\|_{L^{\frac{4n}{n+2}}} \\
&\quad + C\|\partial_t\rho\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^{\frac{4n}{n+2}}}\|\partial_t u\|_{L^{\frac{4n}{n-2}}} \\
&\le C\|\sqrt{\rho_0}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\|\Lambda^{\frac12+\frac n4}u\|_{L^2}\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2} \\
&\quad + C\|\nabla\rho\|_{L^2}\|u\|_{L^\infty}^2\|\Lambda^{\frac12+\frac n4}u\|_{L^2}\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\sqrt{\rho_0}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\|\Lambda^{\frac12+\frac n4}u\|_{L^2}\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}\\
&\quad + C\|\nabla\rho_0\|_{L^2}(\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{\frac{8}{n+2}}\|\Lambda^{1+\frac n2}u\|_{L^2}^{\frac{2(n-2)}{n+2}})\|\Lambda^{\frac12+\frac n4}u\|_{L^2}\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}\\
&\le \frac12\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}^2 + C\|\sqrt{\rho}\partial_t u\|_{L^2}^2\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^2\\
&\quad + C\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{\frac{2n+20}{n+2}}\|\Lambda^{1+\frac n2}u\|_{L^2}^{\frac{4(n-2)}{n+2}}. \qquad (2.49)
\end{aligned}
$$

We first get from (2.48) that

$$
\begin{aligned}
&\frac{d}{dt}\Big(t^2\|\Lambda^{\frac12+\frac n4}\partial_t u(t)\|_{L^2}^2 + t^2\phi(t)\Big) + t^2\|\sqrt{\rho}\partial_{tt}u\|_{L^2}^2\\
&\le 2t\|\Lambda^{\frac12+\frac n4}\partial_t u(t)\|_{L^2}^2 + 2t\phi(t) + t^2A(t)\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2} + B(t)t^2\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}^2. \qquad (2.50)
\end{aligned}
$$

By (2.13), (2.16) and (2.25), we conclude

$$
\begin{aligned}
&\int_0^t \tau^2 A(\tau)\|\Lambda^{\frac12+\frac n4}\partial_\tau u(\tau)\|_{L^2}\,d\tau\\
&\le C\left(\int_0^t \tau^3A^2(\tau)\,d\tau\right)^{\frac12}\left(\int_0^t \tau\|\Lambda^{\frac12+\frac n4}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau\right)^{\frac12}\\
&\le C\left(\int_0^t \tau^3\|\Lambda^{\frac12+\frac n4}u(\tau)\|_{L^2}^{\frac{2(n+10)}{n+2}}\|\Lambda^{1+\frac n2}u(\tau)\|_{L^2}^{\frac{2(3n-2)}{n+2}}\,d\tau\right)^{\frac12}\\
&= C\left(\int_0^t \tau^{\frac{8}{n+2}}e^{-\frac{(n+10)\gamma\tau}{n+2}}(e^{\gamma\tau}\|\Lambda^{\frac12+\frac n4}u(\tau)\|_{L^2}^2)^{\frac{n+10}{n+2}}(\tau\|\Lambda^{1+\frac n2}u(\tau)\|_{L^2}^2)^{\frac{3n-2}{n+2}}\,d\tau\right)^{\frac12}\\
&\le C\left(\int_0^t \tau^{\frac{8}{n+2}}e^{-\frac{(n+10)\gamma\tau}{n+2}}\,d\tau\right)^{\frac12} \le \widetilde{C}, \qquad (2.51)
\end{aligned}
$$

where and in what follows, we use the following facts: for any $\sigma_1 \ge 0$, $\sigma_2 > 0$,

$$
\int_0^\infty \eta^{\sigma_1}e^{-\sigma_2\eta}\,d\eta < \infty \quad\text{and}\quad \tau^{\sigma_1}e^{-\sigma_2\tau} < \infty,\ \forall\,\tau \ge 0.
$$

Noticing the following estimate

$$
\begin{aligned}
\int_0^t \|\sqrt{\rho}\partial_t u(\tau)\|_{L^2}\,d\tau &= \int_0^t e^{-\gamma\frac{\tau}{2}}e^{\gamma\frac{\tau}{2}}\|\sqrt{\rho}\partial_t u(\tau)\|_{L^2}\,d\tau\\
&\le \left(\int_0^t e^{-\gamma\tau}\right)^{\frac12}\left(\int_0^t e^{\gamma\tau}\|\sqrt{\rho}\partial_t u(\tau)\|_{L^2}^2\,d\tau\right)^{\frac12} \le \widetilde{C} \qquad (2.52)
\end{aligned}
$$

and using the argument in dealing with (2.51), we show that

$$
\int_0^t B(\tau)\,d\tau \le \widetilde{C}. \qquad (2.53)
$$

According to (2.13), (2.16) and (2.25) again, one deduces from (2.49) that

$$\begin{aligned}
&\int_0^t \tau\phi(\tau)\,d\tau\\
&\le \frac{1}{2}\int_0^t \tau\|\Lambda^{\frac12+\frac n4}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau + C\int_0^t \tau\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2\|\Lambda^{\frac12+\frac n4}u(\tau)\|_{L^2}^2\,d\tau\\
&\quad + C\int_0^t \tau\|\Lambda^{\frac12+\frac n4}u(\tau)\|_{L^2}^{\frac{2n+20}{n+2}}\|\Lambda^{1+\frac n2}u(\tau)\|_{L^2}^{\frac{4(n-2)}{n+2}}\,d\tau\\
&\le \widetilde{C} + C\int_0^t \tau e^{-\frac{(n+10)\gamma\tau}{n+2}}\left(e^{\gamma\tau}\|\Lambda^{\frac12+\frac n4}u(\tau)\|_{L^2}^2\right)^{\frac{n+10}{n+2}}\|\Lambda^{1+\frac n2}u(\tau)\|_{L^2}^{\frac{4(n-2)}{n+2}}\,d\tau\\
&\le \widetilde{C} + C\int_0^t \tau e^{-\frac{(n+10)\gamma\tau}{n+2}}\|\Lambda^{1+\frac n2}u(\tau)\|_{L^2}^{\frac{4(n-2)}{n+2}}\,d\tau\\
&\le \widetilde{C} + C\chi_{\{n\ge 6\}}\int_0^t \tau^{\frac{8}{n+2}}e^{-\frac{(n+10)\gamma\tau}{n+2}}\left(\tau\|\Lambda^{1+\frac n2}u(\tau)\|_{L^2}^2\right)^{\frac{n-6}{n+2}}\|\Lambda^{1+\frac n2}u(\tau)\|_{L^2}^2\,d\tau\\
&\quad + C\chi_{\{3\le n<6\}}\left(\int_0^t \tau^{\frac{n+2}{6-n}}e^{-\frac{(n+10)\gamma\tau}{6-n}}\,d\tau\right)^{\frac{6-n}{n+2}}\left(\int_0^t\|\Lambda^{1+\frac n2}u(\tau)\|_{L^2}^2\,d\tau\right)^{\frac{2(n-2)}{n+2}}\\
&\le \widetilde{C} + C\chi_{\{n\ge 6\}}\int_0^t \|\Lambda^{1+\frac n2}u(\tau)\|_{L^2}^2\,d\tau\\
&\quad + C\chi_{\{3\le n<6\}}\left(\int_0^t\|\Lambda^{1+\frac n2}u(\tau)\|_{L^2}^2\,d\tau\right)^{\frac{2(n-2)}{n+2}}\\
&\le \widetilde{C}.
\end{aligned}\tag{2.54}$$

We get by integrating (2.50) in time and using (2.51) as well as (2.54)

$$\begin{aligned}
&t^2\|\Lambda^{\frac12+\frac n4}\partial_t u(t)\|_{L^2}^2 + t^2\phi(t) + \int_0^t \tau^2\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|_{L^2}^2\,d\tau\\
&\le \widetilde{C} + \int_0^t B(\tau)\tau^2\|\Lambda^{\frac12+\frac n4}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau.
\end{aligned}\tag{2.55}$$

Direct computations also yield

$$\begin{aligned}
t^2\,|\phi(t)| &\le \frac12 t^2\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}^2 + Ct^2\|\sqrt{\rho}\partial_t u\|_{L^2}^2\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^2\\
&\quad + Ct^2\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^{\frac{2n+20}{n+2}}\|\Lambda^{1+\frac n2}u\|_{L^2}^{\frac{4(n-2)}{n+2}}\\
&= \frac12 t^2\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}^2 + Cte^{-\gamma t}(t\|\sqrt{\rho}\partial_t u\|_{L^2}^2)(e^{\gamma t}\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^2)\\
&\quad + Ct^{\frac{8}{n+2}}e^{-\frac{(n+10)\gamma t}{n+2}}(e^{\gamma t}\|\Lambda^{\frac12+\frac n4}u\|_{L^2}^2)^{\frac{n+10}{n+2}}(t\|\Lambda^{1+\frac n2}u\|_{L^2}^2)^{\frac{2(n-2)}{n+2}}\\
&\le \frac12 t^2\|\Lambda^{\frac12+\frac n4}\partial_t u\|_{L^2}^2 + \widetilde{C}.
\end{aligned}\tag{2.56}$$

Inserting (2.56) into (2.55) implies

$$t^2\|\Lambda^{\frac12+\frac n4}\partial_t u(t)\|_{L^2}^2 + \int_0^t \tau^2\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|_{L^2}^2\,d\tau \le \widetilde{C} + \int_0^t B(\tau)\tau^2\|\Lambda^{\frac12+\frac n4}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau.$$

This along with the Gronwall inequality and (2.53) yields

$$t^2\|\Lambda^{\frac12+\frac n4}\partial_t u(t)\|_{L^2}^2 + \int_0^t \tau^2\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|_{L^2}^2\,d\tau \le \widetilde{C},$$

which is (2.43). With the help of (2.43), we are in the position to derive the exponential decay of $\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u(t)\|_{L^2}$. To this end, we multiply (2.48) by $e^{\gamma t}$ to obtain

$$\begin{aligned}&\frac{d}{dt}\Big(e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u(t)\|_{L^2}^2+e^{\gamma t}\phi(t)\Big)+e^{\gamma t}\|\sqrt{\rho}\partial_{tt}u\|_{L^2}^2\\ &\le \gamma e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u(t)\|_{L^2}^2+\gamma e^{\gamma t}\phi(t)+A(t)e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}\\ &\quad +B(t)e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}^2. \end{aligned}\tag{2.57}$$

Now integrating (2.57) on the time interval $[1,\, t]$ yields

$$\begin{aligned}&e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u(t)\|_{L^2}^2+e^{\gamma t}\phi(t)+\int_1^t e^{\gamma\tau}\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|_{L^2}^2\,d\tau\\ &\le \widetilde{C}+\gamma\int_1^t e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau+\gamma\int_1^t e^{\gamma\tau}\phi(\tau)\,d\tau\\ &\quad+\int_1^t A(\tau)e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u(\tau)\|_{L^2}\,d\tau+\int_1^t B(\tau)e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau\\ &\le \widetilde{C}+2\gamma\int_1^t e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau+\gamma\int_1^t e^{\gamma\tau}\phi(\tau)\,d\tau\\ &\quad+C\int_1^t A^2(\tau)e^{\gamma\tau}\,d\tau+\int_1^t B(\tau)e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau. \end{aligned}\tag{2.58}$$

According to the estimates (2.16), (2.26) and (2.27), it follows from (2.49) that

$$e^{\gamma t}\,|\phi(t)|\le \frac{1}{2}e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u(t)\|_{L^2}^2+\widetilde{C},\tag{2.59}$$

$$\gamma\int_1^t e^{\gamma\tau}\phi(\tau)\,d\tau\le\widetilde{C}.\tag{2.60}$$

Appealing to the estimates (2.16), (2.26) and (2.27) again, we can also show

$$\gamma\int_1^t e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau\le\widetilde{C},\tag{2.61}$$

$$\begin{aligned}C\int_1^t A^2(\tau)e^{\gamma\tau}\,d\tau&=C\int_1^t (e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u(\tau)\|_{L^2}^2)^{\frac{n+10}{n+2}}(e^{\gamma\tau}\|\Lambda^{1+\frac{n}{2}}u(\tau)\|_{L^2}^2)^{\frac{3n-2}{n+2}}e^{-3\gamma\tau}\,d\tau\\ &\le C\int_1^t e^{-3\gamma\tau}\,d\tau\le\widetilde{C}. \end{aligned}\tag{2.62}$$

Inserting the above estimates (2.59)-(2.62) into (2.58) yields

$$\begin{aligned}&e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u(t)\|_{L^2}^2+\int_1^t e^{\gamma\tau}\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|_{L^2}^2\,d\tau\\ &\le\widetilde{C}+\int_1^t B(\tau)e^{\gamma\tau}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau. \end{aligned}$$

Similarly, it follows from the estimates (2.16), (2.26) and (2.27) that

$$\int_1^t B(\tau)\,d\tau\le\widetilde{C}.$$

As a result, we have by the Gronwall inequality

$$e^{\gamma t}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u(t)\|_{L^2}^2+\int_1^t e^{\gamma\tau}\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|_{L^2}^2\,d\tau\le\widetilde{C}.$$

Consequently, we complete the proof of Lemma 2.6. □

With the estimates of Lemma 2.6 at hand, we can obtain the following estimate.

LEMMA 2.7. *Under the assumptions of Theorem 1.1, the solution (ρ,u) of the system (1.1) admits the following bound for any $t\ge 1$,*

$$e^{\gamma t}\|\Lambda^{1+\frac{n}{2}}u(t)\|_{L^{\frac{4n}{n-2}}}^2+e^{\gamma t}\|p(t)\|_{W^{1,\frac{4n}{n-2}}}^2\le\widetilde{C},$$

where $\widetilde{C}$ depends only on $\|\rho_0\|_{L^{\frac{2n}{n+2}}}$, $\|\rho_0\|_{L^\infty}$, $\|\nabla\rho_0\|_{L^2}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$ and $\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u_0\|_{L^2}$.

Proof. Thanks to (2.21) and (2.22), we get

$$\begin{aligned}\|\Lambda^{1+\frac{n}{2}}u\|_{L^{\frac{4n}{n-2}}}+\|\nabla p\|_{L^{\frac{4n}{n-2}}}&\le C\|\rho\partial_t u\|_{L^{\frac{4n}{n-2}}}+C\|\rho u\cdot\nabla u\|_{L^{\frac{4n}{n-2}}}\\&\le C\|\rho\|_{L^\infty}\|\partial_t u\|_{L^{\frac{4n}{n-2}}}+C\|\rho\|_{L^\infty}\|u\nabla u\|_{L^{\frac{4n}{n-2}}}\\&\le C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}+C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}\|\Lambda^{1+\frac{n}{2}}u\|_{L^2},\end{aligned}$$

where we have used (2.39) in the last line. Recalling the estimates obtained in the previous lemmas, we see that for any $t\ge 1$,

$$e^{\gamma t}\|\Lambda^{1+\frac{n}{2}}u(t)\|_{L^{\frac{4n}{n-2}}}^2+e^{\gamma t}\|\nabla p(t)\|_{L^{\frac{4n}{n-2}}}^2\le\widetilde{C}.$$

Similar argument also implies

$$\begin{aligned}\|p\|_{L^{\frac{4n}{n-2}}}&\le C\|\Lambda^{-1}(\rho\partial_t u\|_{L^2})\|_{L^{\frac{4n}{n-2}}}+C\|\Lambda^{-1}(\rho u\cdot\nabla u)\|_{L^{\frac{4n}{n-2}}}\\&\le C\|\rho\partial_t u\|_{L^{\frac{4n}{n+2}}}+C\|\rho u\cdot\nabla u\|_{L^{\frac{4n}{n+2}}}\\&\le C\|\rho\partial_t u\|_{L^2}^{\frac{4}{n+2}}\|\rho\partial_t u\|_{L^{\frac{4n}{n-2}}}^{\frac{n-2}{n+2}}+C\|\rho\|_{L^\infty}\|u\|_{L^\infty}\|\nabla u\|_{L^{\frac{4n}{n+2}}}\\&\le C(\|\sqrt{\rho}\partial_t u\|_{L^2}+\|\rho\partial_t u\|_{L^{\frac{4n}{n-2}}}+\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^{\frac{4}{n+2}}\|\Lambda^{1+\frac{n}{2}}u\|_{L^2}^{\frac{n-2}{n+2}}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2})\\&\le C(\|\sqrt{\rho}\partial_t u\|_{L^2}+\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t u\|_{L^2}+\|\Lambda^{\frac{1}{2}+\frac{n}{4}}u\|_{L^2}^{\frac{n+6}{n+2}}\|\Lambda^{1+\frac{n}{2}}u\|_{L^2}^{\frac{n-2}{n+2}}).\end{aligned}$$

Consequently, it gives that for any $t\ge 1$,

$$e^{\gamma t}\|p(t)\|_{L^{\frac{4n}{n-2}}}^2\le\widetilde{C}.$$

Moreover, we also deduce that for any $t\le 1$,

$$t^2\|\Lambda^{1+\frac{n}{2}}u(t)\|_{L^{\frac{4n}{n-2}}}^2+t^2\|p(t)\|_{W^{1,\frac{4n}{n-2}}}^2\le\widetilde{C}.$$

Hence, we obtain the desired estimates and thus complete the proof of the lemma. □

2.6. The proof of Theorem 1.1. We need the following Gronwall type inequality which will be used to guarantee the uniqueness of strong solutions (see [28, Lemma 2.5]).

LEMMA 2.8. *Let $X_1(t)$, $X_2(t)$, $Y(t)$, $\beta(t)$ and $\gamma(t)$ be non-negative functions. In addition, $\beta(t)$ and $t\gamma(t)$ are two integrable functions over $[0, T]$. Let $X_1(t)$ and $X_2(t)$ be absolutely continuous over $[0, T]$ and satisfy*

$$\begin{cases} \frac{d}{dt}X_1(t) \le AY^{\frac{1}{2}}(t), \\ \frac{d}{dt}X_2(t) + Y(t) \le \beta(t)X_2(t) + \gamma(t)X_1^2(t) \\ X_1(0) = 0, \end{cases}$$

where A is a positive constant. Then, the following estimates hold

$$X_1(t) \le AX_2^{\frac{1}{2}}(0)t^{\frac{1}{2}}e^{\frac{1}{2}\int_0^t(\beta(s)+A^2 s\gamma(s))\,ds},$$

$$X_2(t) + \int_0^t Y(s)\,ds \le X_2(0)e^{\int_0^t(\beta(s)+A^2 s\gamma(s))\,ds}.$$

In particular, if $X_2(0) = 0$, we have

$$X_1(t) = X_2(t) = Y(t) \equiv 0.$$

We continue to prove our theorem. According to Lemma 2.1, there exists a $T^* > 0$ such that the system (1.1) has a unique local strong solution (ρ, u) on the time period $[0, T^*]$. We may follow the standard argument to show that this local solution can be extended to a global one. To this end, we set

$$\widetilde{T} = \sup\{T :\ (\rho, u) \text{ is a strong solution on } [0, T]\}.$$

Now we claim that

$$\widetilde{T} = \infty.$$

Otherwise, if $\widetilde{T} < \infty$, it follows from the estimates of the above Lemmas 2.2-2.7 that

$$(\rho, u)(x, \widetilde{T}) = \lim_{t\to\widetilde{T}}(\rho, u)(x, t)$$

satisfies the initial conditions (1.2) and (1.3) at time $t = \widetilde{T}$. Thus, taking $(\rho, u)(x, \widetilde{T})$ as the initial data, Lemma 2.1 allows us to extend the local strong solutions beyond $\widetilde{T}$. This contradicts the assumption of $\widetilde{T}$ above. The proof of the existence of the global solution is completed. Furthermore the decay properties of the solution are implied in the proof of the above Lemmas 2.2-2.7. Thus it remains to show the uniqueness. To this end, we make use of the following two momentum conservation equations

$$\rho\partial_t u + \rho u\cdot\nabla u + (-\Delta)^{\frac{1}{2}+\frac{n}{4}}u + \nabla p = 0, \quad \widetilde{\rho}\partial_t\widetilde{u} + \widetilde{\rho}\widetilde{u}\cdot\nabla\widetilde{u} + (-\Delta)^{\frac{1}{2}+\frac{n}{4}}\widetilde{u} + \nabla\widetilde{p} = 0,$$

to obtain

$$\begin{aligned} &\rho\partial_t(u-\widetilde{u}) + \rho u\cdot\nabla(u-\widetilde{u}) + (-\Delta)^{\frac{1}{2}+\frac{n}{4}}(u-\widetilde{u}) + \nabla(p-\widetilde{p}) \\ &= -(\rho-\widetilde{\rho})(\partial_t\widetilde{u} + \widetilde{u}\cdot\nabla\widetilde{u}) - \rho(u-\widetilde{u})\cdot\nabla\widetilde{u}. \end{aligned}$$

Now we deduce by multiplying the above identity by $u-\widetilde{u}$ and integrating it over $\mathbb{R}^n$,

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}(u-\widetilde{u})(t)\|_{L^2}^2+\|\Lambda^{\frac{1}{2}+\frac{n}{4}}(u-\widetilde{u})\|_{L^2}^2=J_1+J_2,$$

where

$$J_1:=-\int_{\mathbb{R}^n}(\rho-\widetilde{\rho})(\partial_t\widetilde{u}+\widetilde{u}\cdot\nabla\widetilde{u})\cdot(u-\widetilde{u})\,dx,$$

$$J_2:=-\int_{\mathbb{R}^n}\rho(u-\widetilde{u})\cdot\nabla\widetilde{u}\cdot(u-\widetilde{u})\,dx.$$

The term J_2 can be bounded by

$$J_2\leq C\|\nabla\widetilde{u}\|_{L^\infty}\|\sqrt{\rho}(u-\widetilde{u})\|_{L^2}^2.$$

For the term J_1, we have by (2.39),

$$\begin{aligned}
J_1&\leq C\|\rho-\widetilde{\rho}\|_{L^{\frac{2n}{n+2}}}(\|\partial_t\widetilde{u}\|_{L^{\frac{4n}{n-2}}}+\|\widetilde{u}\cdot\nabla\widetilde{u}\|_{L^{\frac{4n}{n-2}}})\|u-\widetilde{u}\|_{L^{\frac{4n}{n-2}}}\\
&\leq C\|\rho-\widetilde{\rho}\|_{L^{\frac{2n}{n+2}}}(\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t\widetilde{u}\|_{L^2}+\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\widetilde{u}\|_{L^2}\|\Lambda^{1+\frac{n}{2}}\widetilde{u}\|_{L^2})\|\Lambda^{\frac{1}{2}+\frac{n}{4}}(u-\widetilde{u})\|_{L^2}\\
&\leq\frac{1}{2}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}(u-\widetilde{u})\|_{L^2}^2+C(\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t\widetilde{u}\|_{L^2}^2+\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\widetilde{u}\|_{L^2}^2\|\Lambda^{1+\frac{n}{2}}\widetilde{u}\|_{L^2}^2)\|\rho-\widetilde{\rho}\|_{L^{\frac{2n}{n+2}}}^2.
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho}(u-\widetilde{u})(t)\|_{L^2}^2+\|\Lambda^{\frac{1}{2}+\frac{n}{4}}(u-\widetilde{u})\|_{L^2}^2\\
&\leq C(\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t\widetilde{u}\|_{L^2}^2+\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\widetilde{u}\|_{L^2}^2\|\Lambda^{1+\frac{n}{2}}\widetilde{u}\|_{L^2}^2)\|\rho-\widetilde{\rho}\|_{L^{\frac{2n}{n+2}}}^2\\
&\quad+C\|\nabla\widetilde{u}\|_{L^\infty}\|\sqrt{\rho}(u-\widetilde{u})\|_{L^2}^2.
\end{aligned}$$

Using the following two density equations

$$\partial_t\rho+u\cdot\nabla\rho=0,\quad \partial_t\widetilde{\rho}+\widetilde{u}\cdot\nabla\widetilde{\rho}=0,$$

we deduce

$$\partial_t(\rho-\widetilde{\rho})+u\cdot\nabla(\rho-\widetilde{\rho})=-(u-\widetilde{u})\cdot\nabla\widetilde{\rho}.$$

It implies that

$$\begin{aligned}
\frac{n+2}{2n}\frac{d}{dt}\|(\rho-\widetilde{\rho})(t)\|_{L^{\frac{2n}{n+2}}}^{\frac{2n}{n+2}}&\leq C\|\rho-\widetilde{\rho}\|_{L^{\frac{2n}{n+2}}}^{\frac{2n}{n+2}-1}\|(u-\widetilde{u})\cdot\nabla\widetilde{\rho}\|_{L^{\frac{2n}{n+2}}}\\
&\leq C\|\rho-\widetilde{\rho}\|_{L^{\frac{2n}{n+2}}}^{\frac{2n}{n+2}-1}\|u-\widetilde{u}\|_{L^{\frac{4n}{n-2}}}\|\nabla\widetilde{\rho}\|_{L^{\frac{4n}{n+6}}}\\
&\leq C\|\rho-\widetilde{\rho}\|_{L^{\frac{2n}{n+2}}}^{\frac{2n}{n+2}-1}\|\Lambda^{\frac{1}{2}+\frac{n}{4}}(u-\widetilde{u})\|_{L^2}\|\nabla\widetilde{\rho}\|_{L^{\frac{4n}{n+6}}}.
\end{aligned}$$

We may conclude

$$\frac{d}{dt}\|(\rho-\widetilde{\rho})(t)\|_{L^{\frac{2n}{n+2}}}\leq C\|\Lambda^{\frac{1}{2}+\frac{n}{4}}(u-\widetilde{u})\|_{L^2}\|\nabla\widetilde{\rho}\|_{L^{\frac{4n}{n+6}}}.$$

Now let us denote

$$X_1(t) := \|(\rho-\widetilde{\rho})(t)\|_{L^{\frac{2n}{n+2}}},\quad X_2(t) := \|\sqrt{\rho}(u-\widetilde{u})(t)\|_{L^2}^2,\quad Y(t) := \|\Lambda^{\frac{1}{2}+\frac{n}{4}}(u-\widetilde{u})(t)\|_{L^2}^2,$$

$$\beta(t) := C\|\nabla\widetilde{u}(t)\|_{L^\infty},\quad \gamma(t) := C(\|\Lambda^{\frac{1}{2}+\frac{n}{4}}\partial_t\widetilde{u}(t)\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}+\frac{n}{4}}\widetilde{u}(t)\|_{L^2}^2\|\Lambda^{1+\frac{n}{2}}\widetilde{u}(t)\|_{L^2}^2),$$

which satisfy

$$\begin{cases} \frac{d}{dt}X_1(t) \le AY^{\frac{1}{2}}(t), \\ \frac{d}{dt}X_2(t) + Y(t) \le \beta(t)X_2(t) + \gamma(t)X_1^2(t), \\ X_1(0) = 0. \end{cases}$$

Recalling (2.16), (2.26), (2.34) and (2.35), we know that

$$\int_0^t \beta(\tau)\,d\tau \le C_0(t), \qquad \int_0^t \tau\gamma(\tau)\,d\tau \le C_0(t).$$

Due to $u(x,0) = \widetilde{u}(x,0)$, we have $X_2(0) = 0$. Making use of the Gronwall type inequality in Lemma 2.8, we immediately have the uniqueness, namely,

$$u(x,t) = \widetilde{u}(x,t), \quad \rho(x,t) = \widetilde{\rho}(x,t).$$

This completes the proof of Theorem 1.1.

Appendix A. The Case of Dimension $n = 2$. As a byproduct of the approach in the proof of Theorem 1.1, we also obtain the exponential decay-in-time of the strong solution in dimension $n = 2$ provided that a damping term u is added in the momentum equation. More precisely, we have the following result.

THEOREM A.1. *Consider the following system*

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \qquad x\in\mathbb{R}^2,\ t>0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) - \Delta u + u + \nabla p = 0, \\ \nabla\cdot u = 0, \\ \rho(x,0) = \rho_0(x), \quad u(x,0) = u_0(x). \end{cases} \tag{A.1}$$

Assume that the initial data $(\rho_0,\, u_0)$ *satisfies the following conditions:*

$$0 \le \rho_0 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2), \quad \nabla\rho_0 \in L^q(\mathbb{R}^2), \quad q > 2,$$

$$\nabla\cdot u_0 = 0, \quad u_0 \in H^1(\mathbb{R}^2), \quad \sqrt{\rho_0}u_0 \in L^2(\mathbb{R}^2).$$

Then the system (A.1) has a unique global strong solution (ρ, u) *satisfying, for any given* $T > 0$ *and for any* $0 < \tau < T$,

$$0 \le \rho \in L^\infty(0,T;L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)), \quad \nabla\rho \in L^\infty(0,T;L^q(\mathbb{R}^2),$$

$$u \in L^\infty(0,T;H^1(\mathbb{R}^2))\cap L^2(0,T;H^2(\mathbb{R}^2))\cap L^\infty(\tau,T;\dot{W}^{2,m}(\mathbb{R}^2)),$$

$$\sqrt{\rho}\partial_t u \in L^\infty(\tau,T;L^2(\mathbb{R}^2)), \quad \partial_t u \in L^2(\tau,T;H^1(\mathbb{R}^2))\cap L^\infty(\tau,T;H^1(\mathbb{R}^2)),$$

$$\nabla p \in L^\infty(\tau, T; L^2(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)),$$

for any $m \in (2, \infty)$. Moreover, there exists some positive constant γ depending only on $\|\rho_0\|_{L^1}$ and $\|\rho_0\|_{L^\infty}$ such that, for all $t \geq 1$,

$$\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \|u(t)\|_{H^2}^2 + \|\Delta u(t)\|_{L^m}^2 + \|\partial_t u(t)\|_{H^1}^2 + \|\nabla p(t)\|_{L^2\cap L^m}^2 \leq \widetilde{C} e^{-\gamma t},$$

where $\widetilde{C}$ depends only on $\|\rho_0\|_{L^1}$, $\|\rho_0\|_{L^\infty}$, $\|\nabla\rho_0\|_{L^q}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$ and $\|u_0\|_{H^1}$.

Remark A.1. When the damping term u is absent from the system (A.1), it seems difficult to obtain the exponential decay of the strong solution as in Theorem A.1. The key obstacle is that the classical Sobolev embedding inequality is critical in dimension $n = 2$. However, if the initial density decays not too slowly at infinity, then it is proved in [31] that the corresponding system admits a unique global strong solution. Moreover, the following large-time decay rates were obtained: $\|\nabla u(t)\|_{L^2} + \|\nabla^2 u(t)\|_{L^2} + \|\nabla p(t)\|_{L^2} \leq \widetilde{C} t^{-1}$.

Remark A.2. It should be mentioned that once we have the following key estimate (see (A.20))

$$\int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau \leq \widetilde{C_1},$$

we are able to reformulate (A.1) in Lagrangian coordinates without requiring the additional regularity on the initial density, namely, $\nabla\rho_0 \in L^q(\mathbb{R}^2)$ for $q > 2$. For more details, we refer to [17, page 1373–page 1378]. In this sense, the condition $\nabla\rho_0 \in L^q(\mathbb{R}^2)$ for $q > 2$ can be dropped and we still obtain the corresponding global existence, uniqueness as well as exponential decay-in-time results as stated in Theorem A.1.

As the proof of Theorem A.1 can be carried out as that of Theorem 1.1 with some suitable modifications, we only give a sketch of the proof in this appendix. First, the basic energy estimates read as follows.

Lemma A.1. *Under the assumptions of Theorem A.1, the solution (ρ, u) of the system (A.1) admits the following bound for any $t \geq 0$,*

$$\begin{aligned} \|\rho(t)\|_{L^1\cap L^\infty} &\leq \|\rho_0\|_{L^1\cap L^\infty}, \\ e^{\gamma t}\|\sqrt{\rho}u(t)\|_{L^2}^2 + \int_0^t e^{\gamma\tau}\|u(\tau)\|_{H^1}^2\, d\tau &\leq \|\sqrt{\rho_0}u_0\|_{L^2}^2. \end{aligned} \tag{A.2}$$

Proof. The first part of the estimate (A.2) and the non-negativeness of ρ can be deduced as in Lemma 2.2. To show the second part of (A.2), we multiply equation $(A.1)_2$ by u and integrate the resulting equation over $\mathbb{R}^2$ to get

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u(t)\|_{L^2}^2 + \|u\|_{H^1}^2 = 0.$$

Fixing $r \in (1, \infty)$, we see that

$$\|\sqrt{\rho}u\|_{L^2} \leq C\|\sqrt{\rho}\|_{L^{2r}}\|u\|_{L^{\frac{2r}{r-1}}} \leq C\|\rho_0\|_{L^1\cap L^\infty}^{\frac{1}{2}}\|u\|_{H^1} \leq C\|u\|_{H^1},$$

which is crucial for the exponential decay estimate, but different from (2.15). It follows that

$$\frac{d}{dt}\|\sqrt{\rho}u(t)\|_{L^2}^2+\gamma\|\sqrt{\rho}u(t)\|_{L^2}^2+\|u\|_{H^1}^2=0.$$

By the Gronwall inequality, one can prove

$$e^{\gamma t}\|\sqrt{\rho}u(t)\|_{L^2}^2+\int_0^t e^{\gamma\tau}\|u(\tau)\|_{H^1}^2\,d\tau\le\|\sqrt{\rho_0}u_0\|_{L^2}^2.$$

This completes the proof of Lemma A.1. □

LEMMA A.2. *Under the assumptions of Theorem A.1, the solution (ρ,u) of the system (A.1) admits the following bound for any $t\ge 0$,*

$$e^{\gamma t}\|u(t)\|_{H^1}^2+\int_0^t e^{\gamma\tau}(\|u(\tau)\|_{H^2}^2+\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2+\|\sqrt{\rho}\dot{u}(\tau)\|_{L^2}^2)\,d\tau\le\widetilde{C_1},\qquad\text{(A.3)}$$

where $\dot{u}:=\partial_t u+u\cdot\nabla u$ is the material derivatives of the velocity u, and $\widetilde{C_1}$ depends only on $\|\rho_0\|_{L^1}$, $\|\rho_0\|_{L^\infty}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$ and $\|u_0\|_{H^1}$.

Proof. We first rewrite the equation $(\text{A.1})_2$ as

$$\rho\dot{u}=\Delta u-u-\nabla p.\qquad\text{(A.4)}$$

Multiplying the equation (A.4) by $\dot{u}$ and integrating it over $\mathbb{R}^2$ lead to

$$\|\sqrt{\rho}\dot{u}\|_{L^2}^2=\int_{\mathbb{R}^2}\dot{u}\cdot\Delta u\,dx-\int_{\mathbb{R}^2}\dot{u}\cdot u\,dx-\int_{\mathbb{R}^2}\dot{u}\cdot\nabla p\,dx.\qquad\text{(A.5)}$$

On the one hand, one has

$$\int_{\mathbb{R}^2}\dot{u}\cdot\Delta u\,dx=\int_{\mathbb{R}^2}\partial_t u\cdot\Delta u\,dx+\int_{\mathbb{R}^2}(u\cdot\nabla u)\cdot\Delta u\,dx=-\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2,$$

where we have used the following fact due to $\nabla\cdot u=0$ (see [40, (3.3)] for details):

$$\int_{\mathbb{R}^2}(u\cdot\nabla u)\cdot\Delta u\,dx=0.$$

On the other hand, we have

$$-\int_{\mathbb{R}^2}\dot{u}\cdot u\,dx=-\int_{\mathbb{R}^2}\partial_t u\cdot u\,dx-\int_{\mathbb{R}^2}(u\cdot\nabla u)\cdot u\,dx=-\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2.$$

Due to [31, (3.8)], the last term in (A.5) can be bounded by

$$\begin{aligned}-\int_{\mathbb{R}^2}\dot{u}\cdot\nabla p\,dx=\int_{\mathbb{R}^2}\partial_j u_i\partial_i u_j p\,dx&\le C\|p\|_{\mathrm{BMO}}\|\partial_j u\cdot\nabla u_j\|_{\mathcal{H}^1}\\&\le C\|\nabla p\|_{L^2}\|\nabla u\|_{L^2}^2.\end{aligned}\qquad\text{(A.6)}$$

We rewrite (A.4) as the Stokes system

$$\begin{cases}-\Delta u+u+\nabla p=-\rho\dot{u},\\ \nabla\cdot u=0.\end{cases}\qquad\text{(A.7)}$$

Then, it gives

$$\nabla p = (-\Delta)^{-1}\nabla\nabla\cdot(\rho\dot{u}), \tag{A.8}$$

which yields

$$\|\nabla p\|_{L^2} \le C\|\rho\dot{u}\|_{L^2} \le C\|\sqrt{\rho}\dot{u}\|_{L^2}. \tag{A.9}$$

Combining all the above estimates implies that

$$\frac{d}{dt}\|u(t)\|_{H^1}^2 + \|\sqrt{\rho}\dot{u}\|_{L^2}^2 \le C\|u(t)\|_{H^1}^4.$$

This allows us to show

$$\frac{d}{dt}(e^{\gamma t}\|u(t)\|_{H^1}^2) + e^{\gamma t}\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2 \le \gamma e^{\gamma t}\|u(t)\|_{H^1}^2 + C\|u(t)\|_{H^1}^2(e^{\gamma t}\|u(t)\|_{H^1}^2).$$

By the estimate (A.2) and the Gronwall inequality, we get

$$e^{\gamma t}\|u(t)\|_{H^1}^2 + \int_0^t e^{\gamma\tau}\|\sqrt{\rho}\dot{u}(\tau)\|_{L^2}^2\,d\tau \le \widetilde{C_1}.$$

It follows from the regularity properties of Stokes system (A.7) that

$$\int_0^t e^{\gamma\tau}\|u(\tau)\|_{H^2}^2\,d\tau \le \int_0^t e^{\gamma\tau}\|\rho\dot{u}(\tau)\|_{L^2}^2\,d\tau \le \int_0^t e^{\gamma\tau}\|\sqrt{\rho}\dot{u}(\tau)\|_{L^2}^2\,d\tau \le \widetilde{C_1}.$$

We can also verify, by (A.2) for ρ and $H^s(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$ with $s>1$ for u,

$$\begin{aligned}\int_0^t e^{\gamma\tau}\|\sqrt{\rho}\partial_\tau u(\tau)\|_{L^2}^2\,d\tau &\le \int_0^t e^{\gamma\tau}(\|\sqrt{\rho}\dot{u}\|_{L^2} + \|\sqrt{\rho}u\cdot\nabla u\|_{L^2})\,d\tau\\ &\le C\int_0^t e^{\gamma\tau}(\|\sqrt{\rho}\dot{u}\|_{L^2} + \|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^\infty}\|\nabla u\|_{L^2})\,d\tau\\ &\le C\int_0^t e^{\gamma\tau}(\|\sqrt{\rho}\dot{u}\|_{L^2} + \|u\|_{H^2}^2)\,d\tau \le \widetilde{C_1}.\end{aligned} \tag{A.10}$$

We thus complete the proof of the lemma. □

LEMMA A.3. *Under the assumptions of Theorem A.1, the solution (ρ,u) of the system (A.1) admits the following bound for any $t\ge 0$,*

$$t\|\nabla p(t)\|_{L^2}^2 + t\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2 + \int_0^t \tau\|\dot{u}(\tau)\|_{H^1}^2\,d\tau \le \widetilde{C_1}, \tag{A.11}$$

moreover, for any $t_0>0$ and any $t\ge t_0$, the following holds true

$$\begin{aligned}&e^{\gamma t}\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2 + e^{\gamma t}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + e^{\gamma t}\|u(t)\|_{H^2}^2 + e^{\gamma t}\|\nabla p(t)\|_{L^2}^2\\ &\quad+\int_{t_0}^t e^{\gamma\tau}\|\dot{u}(\tau)\|_{H^1}^2\,d\tau \int_{t_0}^t e^{\gamma\tau}\|\partial_t u(\tau)\|_{H^1}^2\,d\tau \le \frac{e^{\gamma t_0}}{t_0}\widetilde{C_1} := C_{t_0},\end{aligned} \tag{A.12}$$

where $\widetilde{C_1}$ depends only on $\|\rho_0\|_{L^1}$, $\|\rho_0\|_{L^\infty}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$ and $\|u_0\|_{H^1}$.

Proof. According to the proof of [31, Lemma 3.3], we have

$$\frac{d}{dt}\left(\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2+\varphi(t)\right)+\|\dot{u}(t)\|_{H^1}^2\leq C(\|\nabla u\|_{L^4}^4+\|p\|_{L^4}^4),$$

where $\varphi(t):=-\int_{\mathbb{R}^2}p\partial_j u_i\partial_i u_j\,dx$. The following estimate is an easy consequence of (A.6) and (A.9)

$$|\varphi(t)|\leq C\|\sqrt{\rho}\dot{u}\|_{L^2}\|\nabla u\|_{L^2}^2\leq\frac{1}{2}\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C\|\nabla u\|_{L^2}^4. \tag{A.13}$$

According to (A.7) and (A.8), we have

$$u=-(\mathbb{I}-\Delta)^{-1}\left(\rho\dot{u}+(-\Delta)^{-1}\nabla\nabla\cdot(\rho\dot{u})\right), \tag{A.14}$$

where $\mathbb{I}$ is an identity operator. Therefore, one concludes

$$\|\nabla u\|_{L^4}^4+\|p\|_{L^4}^4\leq C(\|\Delta u\|_{L^{\frac{4}{3}}}^4+\|\nabla p\|_{L^{\frac{4}{3}}}^4)\leq C\|\rho\dot{u}\|_{L^{\frac{4}{3}}}^4\leq C\|\rho\|_{L^2}^2\|\sqrt{\rho}\dot{u}\|_{L^2}^4,$$

which along with (A.14) gives

$$\frac{d}{dt}\left(\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2+\varphi(t)\right)+\|\dot{u}(t)\|_{H^1}^2\leq C\|\sqrt{\rho}\dot{u}\|_{L^2}^4, \tag{A.15}$$

which then implies

$$\frac{d}{dt}\left(t\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2+t\varphi(t)\right)+t\|\dot{u}(t)\|_{H^1}^2\leq\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2+Ct\|\sqrt{\rho}\dot{u}\|_{L^2}^4. \tag{A.16}$$

By (A.13) and the Gronwall inequality, one has

$$t\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2+\int_0^t\tau\|\dot{u}(\tau)\|_{H^1}^2\,d\tau\leq\widetilde{C_1}. \tag{A.17}$$

Multiplying (A.15) by $e^{\gamma t}$ yields

$$\begin{aligned}\frac{d}{dt}\left(e^{\gamma t}\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2+e^{\gamma t}\varphi(t)\right)+e^{\gamma t}\|\dot{u}(t)\|_{H^1}^2\leq{}&\gamma e^{\gamma t}\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2+\gamma e^{\gamma t}\varphi(t)\\&+Ce^{\gamma t}\|\sqrt{\rho}\dot{u}\|_{L^2}^4.\end{aligned} \tag{A.18}$$

We thus have by integrating (A.18) in time and using (A.3), (A.13) as well as (A.17),

$$\begin{aligned}&e^{\gamma t}\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2+\int_{t_0}^t e^{\gamma\tau}\|\dot{u}(\tau)\|_{H^1}^2\,d\tau\\&\leq C_{t_0}+e^{\gamma t}|\varphi(t)|+\gamma\int_{t_0}^t e^{\gamma\tau}\|\sqrt{\rho}\dot{u}(\tau)\|_{L^2}^2\,d\tau+\gamma\int_{t_0}^t e^{\gamma\tau}\varphi(\tau)\,d\tau+\int_{t_0}^t e^{\gamma\tau}\|\sqrt{\rho}\dot{u}(\tau)\|_{L^2}^4\,d\tau\\&\leq C_{t_0}+\frac{1}{2}e^{\gamma t}\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2+C\int_{t_0}^t\|\sqrt{\rho}\dot{u}(\tau)\|_{L^2}^2(e^{\gamma\tau}\|\sqrt{\rho}\dot{u}(\tau)\|_{L^2}^2)\,d\tau.\end{aligned}$$

This implies

$$e^{\gamma t}\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2+\int_{t_0}^t e^{\gamma\tau}\|\dot{u}(\tau)\|_{H^1}^2\,d\tau\leq C_{t_0}+C\int_{t_0}^t\|\sqrt{\rho}\dot{u}(\tau)\|_{L^2}^2(e^{\gamma\tau}\|\sqrt{\rho}\dot{u}(\tau)\|_{L^2}^2)\,d\tau.$$

By means of (A.3) again and the Gronwall inequality, one obtains

$$e^{\gamma t}\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2+\int_{t_0}^t e^{\gamma\tau}\|\dot{u}(\tau)\|_{H^1}^2\,d\tau\le C_{t_0}.$$

Thanks to (A.9) and (A.14), we get

$$e^{\gamma t}\|\nabla p(t)\|_{L^2}^2+e^{\gamma t}\|u(t)\|_{H^2}^2\le Ce^{\gamma t}\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2\le C_{t_0}.$$

The following estimate follows immediately from (A.10)

$$e^{\gamma t}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2\le e^{\gamma t}\|\sqrt{\rho}\dot{u}(t)\|_{L^2}^2+e^{\gamma t}\|u(t)\|_{H^2}^2\le C_{t_0}.$$

Finally, we have

$$\begin{aligned}\int_{t_0}^t e^{\gamma\tau}\|\partial_t u(\tau)\|_{H^1}^2\,d\tau&\le\int_{t_0}^t e^{\gamma\tau}\|\dot{u}(\tau)\|_{H^1}^2\,d\tau+\int_{t_0}^t e^{\gamma\tau}\|(u\cdot\nabla u)(\tau)\|_{H^1}^2\,d\tau\\&\le C\int_{t_0}^t e^{\gamma\tau}\|\dot{u}(\tau)\|_{H^1}^2\,d\tau+C\int_{t_0}^t e^{\gamma\tau}\|u(\tau)\|_{H^2}^4\,d\tau\le C_{t_0},\end{aligned}$$

from the estimate

$$\|u\cdot\nabla u\|_{H^1}\le C\|uu\|_{H^2}\le C\|u\|_{H^2}^2. \tag{A.19}$$

Therefore, we complete the proof of the lemma. □

LEMMA A.4. *Under the assumptions of Theorem A.1, the solution (ρ,u) of the system (A.1) admits the following bounds for any $t\ge 0$,*

$$\int_0^t\|\nabla u(\tau)\|_{L^\infty}\,d\tau\le\widetilde{C_1},\qquad\|\nabla\rho(t)\|_{L^q}\le\widetilde{C_1}\|\nabla\rho_0\|_{L^q}, \tag{A.20}$$

where $\widetilde{C_1}$ depends only on $\|\rho_0\|_{L^1}$, $\|\rho_0\|_{L^\infty}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$ and $\|u_0\|_{H^1}$.

Proof. For any $2<p<\infty$, we get $\|\rho\dot{u}\|_{L^p}\le C\|\rho\|_{L^\infty}\|\dot{u}\|_{H^1}\le C\|\dot{u}\|_{H^1}$. Applying the L^p-estimate to (A.7) gives

$$\|\nabla u\|_{L^\infty}\le C\|\nabla u\|_{L^2}^{\frac{p-2}{2p-2}}\|\Delta u\|_{L^p}^{\frac{p}{2p-2}}\le C\|\nabla u\|_{L^2}^{\frac{p-2}{2p-2}}\|\rho\dot{u}\|_{L^p}^{\frac{p}{2p-2}}\le C\|\nabla u\|_{L^2}^{\frac{p-2}{2p-2}}\|\dot{u}\|_{H^1}^{\frac{p}{2p-2}}.$$

According to (A.3), (A.11) and (A.12), we obtain the first estimate of (A.20). The second estimate of (A.20) is a direct consequence of the first estimate. The proof of the lemma is completed. □

LEMMA A.5. *Under the assumptions of Theorem A.1, the solution (ρ,u) of the system (A.1) admits the following bound for any $m\in(2,\infty)$,*

$$e^{\gamma t}\|\partial_t u(t)\|_{H^1}^2+e^{\gamma t}\|\Delta u(t)\|_{L^m}^2+e^{\gamma t}\|\nabla p(t)\|_{L^m}^2+\int_1^t e^{\gamma\tau}\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|_{L^2}^2\,d\tau\le\widetilde{C},$$

where $\widetilde{C}$ depends only on $\|\rho_0\|_{L^1}$, $\|\rho_0\|_{L^\infty}$, $\|\nabla\rho_0\|_{L^q}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$ and $\|u_0\|_{H^1}$.

Proof. The proof can be performed by modifying that proof of Lemma 2.6. We first have by (2.45) that

$$\frac{1}{2}\frac{d}{dt}\|\partial_t u(t)\|_{H^1}^2+\|\sqrt{\rho}\partial_{tt}u\|_{L^2}^2=H_1+H_2+H_3+H_4.$$

The H_3 and H_4 can be easily bounded by

$$
\begin{aligned}
|H_3|+|H_4| &\le C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_{tt}u\|_{L^2}(\|\partial_t u\|_{L^4}\|\nabla u\|_{L^4}+\|\nabla\partial_t u\|_{L^2}\|u\|_{L^\infty})\\
&\le C\|\sqrt{\rho_0}\|_{L^\infty}\|\sqrt{\rho}\partial_{tt}u\|_{L^2}\|\partial_t u\|_{H^1}\|u\|_{H^2}\\
&\le \frac{1}{16}\|\sqrt{\rho}\partial_{tt}u\|_{L^2}^2+C\|u\|_{H^2}^2\|\partial_t u\|_{H^1}^2.
\end{aligned}
$$

Recalling (2.46), we thus obtain

$$
\begin{aligned}
H_1 =& -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\partial_t\rho|\partial_t u|^2\,dx+\int_{\mathbb{R}^2}\partial_t\rho u_i\partial_t u\cdot\partial_t\partial_i u\,dx+\int_{\mathbb{R}^2}\rho\partial_t u_i\partial_t u\cdot\partial_t\partial_i u\,dx\\
\le& -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\partial_t\rho|\partial_t u|^2\,dx+C\|u\cdot\nabla\rho\|_{L^q}\|u\|_{L^\infty}\|\partial_t u\|_{L^{\frac{2q}{q-2}}}\|\partial_t\nabla u\|_{L^2}\\
&+C\|\rho\|_{L^\infty}\|\partial_t u\|_{L^4}^2\|\partial_t\nabla u\|_{L^2}\\
\le& -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\partial_t\rho|\partial_t u|^2\,dx+C\|\nabla\rho\|_{L^q}\|u\|_{L^\infty}^2\|\partial_t u\|_{L^{\frac{2q}{q-2}}}\|\partial_t\nabla u\|_{L^2}\\
&+C\|\rho\|_{L^\infty}\|\partial_t u\|_{L^4}^2\|\partial_t\nabla u\|_{L^2}\\
\le& -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\partial_t\rho|\partial_t u|^2\,dx+C\|u\|_{H^2}^2\|\partial_t u\|_{H^1}\|\partial_t\nabla u\|_{L^2}+C\|\partial_t u\|_{H^1}^2\|\partial_t\nabla u\|_{L^2}\\
\le& -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\partial_t\rho|\partial_t u|^2\,dx+C\|u\|_{H^2}^2\|\partial_t u\|_{H^1}^2+C\|\partial_t u\|_{H^1}\|\partial_t u\|_{H^1}^2.
\end{aligned}
$$

In view of (2.47), one has

$$
H_2=-\frac{d}{dt}\int_{\mathbb{R}^2}\partial_t\rho u\cdot\nabla u\cdot\partial_t u\,dx+H_{21}+H_{22}+H_{23}.
$$

Now we may deduce that

$$
\begin{aligned}
H_{21} &\le C\|\rho\|_{L^\infty}\|\partial_t u\|_{L^4}(\|\nabla u\|_{L^4}^2\|\partial_t u\|_{L^4}+\|u\|_{L^\infty}\|\Delta u\|_{L^2}\|\partial_t u\|_{L^4}\\
&\quad+\|u\|_{L^\infty}\|\nabla u\|_{L^4}\|\nabla\partial_t u\|_{L^2})\\
&\le C\|\rho_0\|_{L^\infty}\|\partial_t u\|_{H^1}(\|u\|_{H^2}^2\|\partial_t u\|_{H^1}+\|u\|_{H^2}^2\|\partial_t u\|_{H^1}+\|u\|_{H^2}^2\|\nabla\partial_t u\|_{L^2})\\
&\le C\|u\|_{H^2}^2\|\partial_t u\|_{H^1}^2,
\end{aligned}
$$

$$
\begin{aligned}
H_{22} &\le C\|u\cdot\nabla\rho\|_{L^q}(\|u\|_{L^\infty}^2\|\nabla u\|_{L^{\frac{2q}{q-2}}}\|\nabla\partial_t u\|_{L^2}+\|u\|_{L^\infty}\|\nabla u\|_{L^4}^2\|\partial_t u\|_{L^{\frac{2q}{q-2}}}\\
&\quad+\|u\|_{L^\infty}^2\|\Delta u\|_{L^2}\|\partial_t u\|_{L^{\frac{2q}{q-2}}})\\
&\le C\|\nabla\rho\|_{L^q}(\|u\|_{L^\infty}^3\|\nabla u\|_{L^{\frac{2q}{q-2}}}\|\nabla\partial_t u\|_{L^2}+\|u\|_{L^\infty}^2\|\nabla u\|_{L^4}^2\|\partial_t u\|_{L^{\frac{2q}{q-2}}}\\
&\quad+\|u\|_{L^\infty}^3\|\Delta u\|_{L^2}\|\partial_t u\|_{L^{\frac{2q}{q-2}}})\\
&\le C\|u\|_{H^2}^4\|\partial_t u\|_{H^1},
\end{aligned}
$$

$$
\begin{aligned}
H_{23} &\le C\|u\cdot\nabla\rho\|_{L^q}(\|\partial_t u\|_{L^4}^2\|\nabla u\|_{L^{\frac{2q}{q-2}}}+\|u\|_{L^\infty}\|\nabla u\|_{L^{\frac{2q}{q-2}}}\|\nabla\partial_t u\|_{L^2})\\
&\le C\|\nabla\rho\|_{L^q}(\|u\|_{L^\infty}\|\partial_t u\|_{L^4}^2\|\nabla u\|_{L^{\frac{2q}{q-2}}}+\|u\|_{L^\infty}^2\|\nabla u\|_{L^{\frac{2q}{q-2}}}\|\nabla\partial_t u\|_{L^2})\\
&\le C(\|u\|_{H^2}^2\|\partial_t u\|_{H^1}^2+\|u\|_{H^2}^3\|\partial_t u\|_{H^1}).
\end{aligned}
$$

One thus deduces

$$H_2 \le -\frac{d}{dt}\int_{\mathbb{R}^2} \partial_t\rho u\cdot\nabla u\cdot\partial_t u\,dx + C(\|u\|_{H^2}^2\|\partial_t u\|_{H^1}^2 + \|u\|_{H^2}^4\|\partial_t u\|_{H^1} + \|u\|_{H^2}^3\|\partial_t u\|_{H^1}).$$

Putting all the above estimates together implies that

$$\frac{d}{dt}\left(\|\partial_t u(t)\|_{H^1}^2 + \phi(t)\right) + \|\sqrt{\rho}\partial_{tt}u(t)\|_{L^2}^2 \le C\|\partial_t u\|_{H^1}\|\partial_t u\|_{H^1}^2 + R(t), \quad \text{(A.21)}$$

where

$$R(t) := C(\|u(t)\|_{H^2}^2\|\partial_t u(t)\|_{H^1}^2 + \|u(t)\|_{H^2}^4\|\partial_t u(t)\|_{H^1} + \|u(t)\|_{H^2}^3\|\partial_t u(t)\|_{H^1}),$$

$$\phi(t) := -\frac{1}{2}\int_{\mathbb{R}^2}\partial_t\rho|\partial_t u|^2\,dx - \int_{\mathbb{R}^2}\partial_t\rho u\cdot\nabla u\cdot\partial_t u\,dx.$$

By the embedding inequality, we also get

$$\begin{aligned}
|\phi(t)| &= \left|-\int_{\mathbb{R}^2}\rho u_i\partial_t u\cdot\partial_t\partial_i u\,dx - \int_{\mathbb{R}^2}\partial_t\rho u\cdot\nabla u\cdot\partial_t u\,dx\right| \\
&\le C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\|u\|_{L^\infty}\|\nabla\partial_t u\|_{L^2} + C\|u\cdot\nabla\rho\|_{L^q}\|u\|_{L^\infty}\|\nabla u\|_{L^4}\|\partial_t u\|_{L^4} \\
&\le C\|u\|_{H^2}\|\sqrt{\rho}\partial_t u\|_{L^2}\|\partial_t u\|_{H^1} + C\|u\|_{H^2}^3\|\partial_t u\|_{H^1} \\
&\le \frac{1}{2}\|\partial_t u(t)\|_{H^1}^2 + C\|u\|_{H^2}^6 + \|u\|_{H^2}^2\|\sqrt{\rho}\partial_t u\|_{L^2}^2.
\end{aligned}$$

This leads to

$$|\phi(t)| \le \frac{1}{2}\|\partial_t u(t)\|_{H^1}^2 + C\|u(t)\|_{H^2}^6 + \|u(t)\|_{H^2}^2\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2. \quad \text{(A.22)}$$

Now we multiply (A.21) by $e^{\gamma t}$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\left(e^{\gamma t}\|\partial_t u(t)\|_{H^1}^2 + e^{\gamma t}\phi(t)\right) + e^{\gamma t}\|\sqrt{\rho}\partial_{tt}u\|_{L^2}^2 \\
&\le \gamma e^{\gamma t}\|\partial_t u(t)\|_{H^1}^2 + \gamma e^{\gamma t}\phi(t) + Ce^{\gamma t}\|\partial_t u\|_{H^1}\|\partial_t u\|_{H^1}^2 + e^{\gamma t}R(t). \quad \text{(A.23)}
\end{aligned}$$

For any $t \ge 1$, by (A.11) and (A.12), there exists $\sigma \in (\frac{1}{2},\, 1)$ such that

$$e^{\gamma\sigma}\|\partial_t u(\sigma)\|_{H^1}^2 + e^{\gamma\sigma}\phi(\sigma) := \widetilde{C}.$$

It follows from (A.12) again

$$\int_{\frac{1}{2}}^t e^{\gamma\tau}\|\partial_\tau u(\tau)\|_{H^1}^2\,d\tau + \int_{\frac{1}{2}}^t e^{\gamma\tau}\phi(\tau)\,d\tau + \int_{\frac{1}{2}}^t e^{\gamma\tau}R(\tau)\,d\tau \le \widetilde{C}.$$

Noticing the above estimate, we integrate (A.23) on the time interval $[\sigma,\, t]$ to show

$$\begin{aligned}
&e^{\gamma t}\|\partial_t u(t)\|_{H^1}^2 + e^{\gamma t}\phi(t) + \int_\sigma^t e^{\gamma\tau}\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|_{L^2}^2\,d\tau \\
&\le \widetilde{C} + C\int_\sigma^t \|\partial_\tau u(\tau)\|_{H^1}(e^{\gamma\tau}\|\partial_\tau u(\tau)\|_{H^1}^2)\,d\tau. \quad \text{(A.24)}
\end{aligned}$$

From (A.22) and (A.12), it follows that for any $t \geq \frac{1}{2}$,

$$e^{\gamma t}\,|\phi(t)| \leq \frac{1}{2}e^{\gamma t}\|\partial_t u(t)\|_{H^1}^2 + \widetilde{C}. \tag{A.25}$$

Combining (A.24) and (A.25) ensures

$$e^{\gamma t}\|\partial_t u(t)\|_{H^1}^2 + \int_\sigma^t e^{\gamma\tau}\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|_{L^2}^2\,d\tau \leq \widetilde{C} + C\int_\sigma^t \|\partial_\tau u(\tau)\|_{H^1}(e^{\gamma\tau}\|\partial_\tau u(\tau)\|_{H^1}^2)\,d\tau.$$

The Gronwall inequality and (A.12) allow us to conclude that for any $t \geq \sigma$,

$$e^{\gamma t}\|\partial_t u(t)\|_{H^1}^2 + \int_\sigma^t e^{\gamma\tau}\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|_{L^2}^2\,d\tau \leq \widetilde{C}.$$

Since $\sigma \leq 1$, we further have for any $t \geq 1$,

$$e^{\gamma t}\|\partial_t u(t)\|_{H^1}^2 + \int_1^t e^{\gamma\tau}\|\sqrt{\rho}\partial_{\tau\tau}u(\tau)\|_{L^2}^2\,d\tau \leq \widetilde{C}. \tag{A.26}$$

By (A.19), (A.8), (A.14) and (A.26), we derive that for any $t \geq 1$,

$$\begin{aligned} e^{\gamma t}\|\Delta u(t)\|_{L^m}^2 + e^{\gamma t}\|\nabla p(t)\|_{L^m}^2 &\leq Ce^{\gamma t}(\|\rho\partial_t u(t)\|_{L^m} + \|\rho u\cdot\nabla u(t)\|_{L^m})^2 \\ &\leq Ce^{\gamma t}(\|\partial_t u(t)\|_{H^1} + \|u\cdot\nabla u(t)\|_{H^1})^2 \\ &\leq Ce^{\gamma t}(\|\partial_t u(t)\|_{H^1}^2 + \|u(t)\|_{H^2}^4) \leq \widetilde{C}. \end{aligned}$$

This finishes the proof of Lemma A.5. ∎

Therefore, Theorem A.1 follows immediately from Lemmas A.1-A.5.

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