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DISTRIBUTED QUANTUM SENSING NETWORK WITH GEOGRAPHICALLY CONSTRAINED MEASUREMENT STRATEGIES

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ABSTRACT

Distributed quantum sensing network has the potential of enhancing the precision in estimating a global function of local parameters by utilizing an entangled probe, compared with that achieved with separable probes. This advantage is often characterized as a quadratic improvement of the quantum Cramér-Rao bound (QCRB). This argument is incomplete in that QCRB assumes a team of all-powerful sensors that can perform arbitrary joint measurements allowed by quantum mechanics. An immediate question arises as to whether such an advantage persists for isolated sensors with physically motivated constraints in their measurement strategies. In this paper, we first consider local operations and classical communication (LOCC) strategies and prove that the QCRB is indeed asymptotically attainable for arbitrary pure probe states, by extending previous work on single-parameter estimation [1]. We further numerically analyze a more restricted scenario where the sensors can only make independent local measurements, and provide evidence that the QCRB is not informative enough for comparing different probe states.

***Index Terms*—** Distributed Quantum Sensing, Cramér-Rao Bound, Local Operations and Classical Communication, Local Measurement

1. INTRODUCTION

The study of sensing and metrology through the lens of quantum mechanics was rooted in early investigations of uncertainty relations in quantum statistical parameter estimation [2–4], showing fundamental limitations of quantum measurements in the ultimate achievable precision. Complementary to the limitations was the discovery that by fully utilizing quantum resources such as entanglement in the sensing physical systems (called probes), it is possible to achieve higher precision than using separable probes [5–8], which is at the core of a variety of applications ranging from gravitational wave detection [9] to clock synchronization [10, 11].

Last decade has seen growing interest in understanding the limitations and advantages of multi-parameter quantum sensing [12–14]. Compared with the single-parameter case, the role of quantum entanglement in estimating multiple parameters is more elusive and depends on the exact setting of the problem [15, 16], mainly because of the difficulty in dealing with incompatibility of quantum measurements arising from the noncommutative nature of quantum mechanics [17] and the inconsistency in how people count resources.

Distributed quantum sensing emerges as a special scenario of multi-parameter quantum sensing, where a team of sensors are located at distant nodes in a quantum network and each parameter of interest is independently encoded through a quantum channel that acts nontrivially only on a single node. One usually assumes the existence of a centralized source that can prepare a global probe state under certain resource constraint and transmit the probes to individual sensors. This model of quantum sensor network was first introduced and analyzed by Proctor et al [18]. They showed that there is no advantage by using entangled probe states or global measurements if our goal is to estimate all the unknown parameters simultaneously. On the contrary, more interesting is the case where we only aim to estimate a global function of the parameters without having to know every one of them. It is in this case where quantum entanglement does help, as pointed out in [18] and manifested in later work that deals with similar and more concrete settings including atomic sensors [19, 20], linear optical network [21], Gaussian continuous-variable systems [22–24], etc.

Most work in this line of research relies on the mathematical tool of quantum Cramér-Rao bound (QCRB) [25, 26], a generalization of the Cramér-Rao bound (CRB) from classical statistical estimation theory [27]. The merit of using QCRB is that it provides a precision lower bound that only depends on the quantum state being measured, regardless of what measurement is performed, hence the optimization of the probe state and that of the measurement procedure can be decoupled. Moreover, in many cases including ours, it can be proven that this lower bound can actually be asymptotically attained by *some* quantum measurement [14, 28, 29]. However, there is no guarantee in the experimental feasibility of the measurement. In principle it could be the case that arbitrarily complicated measurement that projects onto highly entangled subspaces is needed to saturate QCRB. Previous work got around this issue in an ad-hoc way: first optimizing

QCRB over the probe state, then devising a practical measurement strategy, and finally checking that the measurement strategy indeed saturates the optimized QCRB. This approach worked because the sensing problems considered were well-behaved and shared nice symmetry. When the local parameters contribute unevenly to the global function, or when the probes are transmitted with noise [30, 31], it becomes unclear whether it is still reasonable to optimize QCRB anyway in the first step and then hope for finding a feasible saturating measurement.

Motivated by the above observation, it is important to understand whether QCRB is still an informative bound in distributed quantum sensing with geographically constrained measurement strategies. First, we look at measurements that can be realized by local operations and classical communication (LOCC) [32], which is usually considered as free resource in entanglement theory. In this case, we prove that QCRB is tight for any pure probe state. Next we consider the more restricted local measurement (LM) strategy. We resort to numerical methods to identify the looseness of QCRB, by a case study of two metrologically important probe states with the existence of local decoherence.

2. DISTRIBUTED QUANTUM SENSING NETWORK AND QUANTUM CRAMÉR-RAO BOUND

We consider a network of s distributed sensors with identical underlying finite-dimensional Hilbert space $\mathcal{H}$. A centralized source prepares a multipartite quantum state $\rho_0 = |\psi_0\rangle\langle\psi_0|$ of s particles, then transmits them to the sensors. Each particle k undergoes a local quantum channel $\mathcal{E}_{\theta_k}^{(k)}$ before reaching the sensor k, hence the final probe state becomes

$$\rho_{\boldsymbol{\theta}} = \left(\bigotimes_{k=1}^{s} \mathcal{E}_{\theta_k}^{(k)}\right)(\rho_0)\,, \tag{1}$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_s) \in \mathbb{R}^s$ denotes the array of unknown parameters that are independent of each other, meaning there are no prior correlations between the values of the parameters.

We are interested in estimating a linear function $g(\boldsymbol{\theta}) := \boldsymbol{v}^\top \boldsymbol{\theta}$ of the parameters, where $\boldsymbol{v} \in \mathbb{R}^s$ is a known vector that is normalized such that $\|\boldsymbol{v}\|_2 = 1$ to fix irrelevant scales. There is no loss of generality by assuming linearity because we can approximate a general smooth function by a linear one if we already have sufficiently good *a priori* knowledge about the true values of the parameters [20].

In order to estimate $g(\boldsymbol{\theta})$, the sensors perform a positive operator-valued measure (POVM) $\mathcal{M} = \{E_x\}_x$ on $\rho_{\boldsymbol{\theta}}$, yielding classical statistical outcome x according to the probability distribution [32]

$$p_{\mathcal{M},\boldsymbol{\theta}}(x) = \mathrm{Tr}\left[\rho_{\boldsymbol{\theta}} E_x\right]. \tag{2}$$

We point out here that we limit our discussion to the asymptotic regime, which means that the above process is repeated ν times and we care about the asymptotic behavior of the protocol in the limit $\nu \gg 1$. From the observed data $X = \left(x^{(1)}, \ldots, x^{(\nu)}\right)$ we then construct an estimator $\hat{g}(X)$.

The figure of merit that we aim to minimize is the mean squared error

$$\mathrm{mse}(\hat{g}) := \mathbb{E}_{\boldsymbol{\theta}}\left[\left(\hat{g}(X) - g(\boldsymbol{\theta})\right)^2\right], \tag{3}$$

where $\mathbb{E}_{\boldsymbol{\theta}}$ is a short-hand notation for $\mathbb{E}_{X\sim p_{\mathcal{M},\boldsymbol{\theta}}^{\otimes\nu}}$. Note that normally in the design of a sensing protocol one needs to optimize over three ingredients: the initial probe state $|\psi_0\rangle$, the measurement $\mathcal{M}$, and the estimator $\hat{g}$. In this paper, we are primarily interested in the measurement strategy, hence we will assume $|\psi_0\rangle$ is given. As for the estimator, we follow the standard approach in statistical estimation theory by imposing additional constraint that $\hat{g}$ be locally unbiased (abbreviated l.u.). That is, the following should hold at the ground truth value of $\boldsymbol{\theta}$:

$$\begin{cases} \mathbb{E}_{\boldsymbol{\theta}}\left[\hat{g}(X)\right] = g(\boldsymbol{\theta}) \\ \nabla_{\boldsymbol{\theta}}\mathbb{E}_{\boldsymbol{\theta}}\left[\hat{g}(X)\right] = \nabla g(\boldsymbol{\theta}) \end{cases}. \tag{4}$$

It is known that for any statistical model $p_{\boldsymbol{\theta}}^{\otimes\nu}$, any l.u. estimator satisfies the classical Cramér-Rao bound (CRB):

$$\mathrm{mse}(\hat{g}) \geq \frac{1}{\nu}\left(\nabla g\right)^\top F_C(p_{\boldsymbol{\theta}})^{-1}\nabla g = \frac{1}{\nu}\boldsymbol{v}^\top F_C(p_{\boldsymbol{\theta}})^{-1}\boldsymbol{v}, \tag{5}$$

where F_C denotes the classical Fisher information matrix defined by $F_C(p_{\boldsymbol{\theta}})_{k\ell} := \mathbb{E}_{x\sim p_{\boldsymbol{\theta}}}\left[\left(\partial_k \ln p_{\boldsymbol{\theta}}(x)\right)\left(\partial_\ell \ln p_{\boldsymbol{\theta}}(x)\right)\right]$, $\forall k, \ell \in [s]$, where $\partial_k \equiv \partial/\partial\theta_k$. It is allowed that F_C be singular as long as its kernel space is orthogonal to $\boldsymbol{v}$, then F_C^{-1} means the pseudo-inverse in general. A noteworthy property of CRB is that for large ν it is asymptotically saturable, for example, by the maximum likelihood estimator [27]. Therefore, we are left with the minimization problem:

$$\underset{\mathcal{M}}{\text{minimize}} \quad \mathscr{F}(\mathcal{M}) := \frac{1}{\nu}\boldsymbol{v}^\top F_C(p_{\mathcal{M},\boldsymbol{\theta}})^{-1}\boldsymbol{v}. \tag{6}$$

The mere requirement that $\mathcal{M}$ be a valid POVM leads to the famous matrix inequality $F_C(p_{\mathcal{M},\boldsymbol{\theta}}) \preccurlyeq F_Q(\rho_{\boldsymbol{\theta}})$. Here F_Q is the quantum Fisher information matrix [26] defined by $F_Q(\rho_{\boldsymbol{\theta}})_{k\ell} := \frac{1}{2}\mathrm{Tr}\left[\rho_{\boldsymbol{\theta}}\{L_k, L_\ell\}\right]$, $\forall k, \ell \in [s]$, and L_k is the kth symmetric logarithmic derivative (SLD) operator implicitly defined by $\partial_k \rho_{\boldsymbol{\theta}} = \frac{1}{2}\{L_k, \rho_{\boldsymbol{\theta}}\}$, where $\{A, B\} := AB + BA$ denotes the anticommutator of two operators. This gives rise to a measurement-independent lower bound of the minimization problem in (6), the QCRB:

$$\min_{\mathcal{M}} \mathscr{F}(\mathcal{M}) \geq \frac{1}{\nu}\boldsymbol{v}^\top F_Q(\rho_{\boldsymbol{\theta}})^{-1}\boldsymbol{v}. \tag{7}$$

In the noiseless case where the parameter-imprinting channels $\mathcal{E}_{\theta_k}^{(k)}$ are unitary, $\rho_{\boldsymbol{\theta}} = |\psi_{\boldsymbol{\theta}}\rangle\langle\psi_{\boldsymbol{\theta}}|$ is a pure state, then one can show that the above QCRB is actually tight[1] [14, 28].

[1]In fact in the pure model one does not need the assumption that θ_k's are encoded locally to show the tightness of QCRB.

However, if we restrict $\mathcal{M}$ to range only over LOCC or LM, then the gap between the left- and right-hand sides of (7) has not been well-studied.

3. RELATED WORK

As mentioned above, our work is motivated by literature showing the quantum advantage of estimating a linear function of local parameters [18–23]. Rubio et al [33] considered the more general task of simultaneously estimating multiple linear functions, in both the asymptotic and non-asymptotic regime, but with their analysis limited to qubit sensors that are correlated in a symmetric way. Other works also exist that discuss the effects of noise [34] and remedies through error correction [35]. We do not attempt to reach such generality in this paper but stick with the simplest model of estimating a single linear function with many repetitions of the protocol.

Our result involving LOCC measurement strategies directly builds upon previous work by Zhou et al [1]. There the authors proved, among others, that one-way LOCC POVM is always QCRB-saturating for pure states parametrized by a *single* parameter. We essentially generalize the "single parameter" assumption to "single linear function". On the other hand, for mixed states, one-way LOCC POVM falls short of attaining the QCRB, even in the single-parameter case, with the Werner state being a counterexample [36].

It is also worthwhile to mention another thread of research [37, 38] that introduced a hierarchy of measurement-induced Fisher information, which is essentially the maximum classical Fisher information achievable by a restricted family of POVMs. Compared with our work, theirs focused on bipartite single-parameter quantum states and put special emphasis on the distribution and transfer of Fisher information among subsystems as well as its relation to other measures of nonclassical correlations such as quantum discord.

4. DISTRIBUTED QUANTUM SENSING VIA LOCC

Let's consider the LOCC measurement strategies. In this model, each sensor implements a local POVM whose elements may depend on the measurement results broadcast from other sensors in the previous rounds. A further restriction is the so-called one-way LOCC measurement, which takes a simple form

$$E_{\vec{x}} = E^{(1)}_{x_1} \otimes E^{(2)}_{x_2|x_1} \otimes \cdots \otimes E^{(s)}_{x_s|x_1,\ldots,x_{s-1}}, \tag{8}$$

where $\left\{E^{(k)}_{x_k|x_1,\ldots,x_{k-1}}\right\}_{x_k}$ is a POVM acting exclusively on the kth sensor, for each k and each value of $x_1, \ldots, x_{k-1}$. We are able to show the following result by a reparameterization argument.

Theorem 1 *For any pure probe state $\rho_{\boldsymbol{\theta}} = |\psi_{\boldsymbol{\theta}}\rangle\langle\psi_{\boldsymbol{\theta}}|$, the QCRB in* (7) *for estimating a linear function $g(\boldsymbol{\theta}) := \boldsymbol{v}^\top\boldsymbol{\theta}$ of the unknown parameters $\boldsymbol{\theta}$ can be saturated by a one-way LOCC measurement.*

5. DISTRIBUTED QUANTUM SENSING VIA LM

Sometimes even LOCC is unrealistic due to the lack of reliable quantum memory—the quantum state stored in the subsystem quickly decoheres before the classical message from other parties has been received. This motivates us to study the most restrictive scenario where the sensors can only perform local measurements whose basis choices are independent of each other. Mathematically this means we can write the POVM elements as

$$E_{\vec{x}} = E^{(1)}_{x_1} \otimes E^{(2)}_{x_2} \otimes \cdots \otimes E^{(s)}_{x_s}. \tag{9}$$

Even though we cannot identify a general criterion for LM strategies to saturate the QCRB, we give numerical evidence that when the noise in the transmission channel is non-negligible, the QCRB can be misleading and it may be nontrivial to find a good LM strategy.

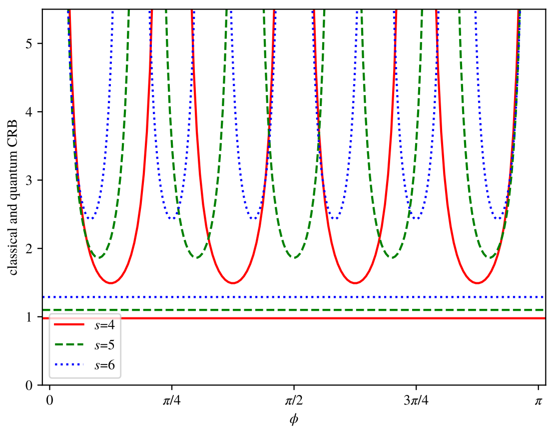
To simplify our discussion, we consider qubit probes—each particle is a two-level system—that are originally prepared in a given state $|\psi_0\rangle$ and then each suffers from a local depolarizing noise channel with efficiency η before reaching the corresponding sensor. The unknown parameter θ_k is imprinted onto the state via a local unitary rotation that we take to be along the z axis. We assume that noise only exists in the resource distribution process but not during the measurement. Hence, we can calculate the final pre-measurement state:

$$\rho_{\boldsymbol{\theta}} = U_{\boldsymbol{\theta}} \mathcal{N}_\eta^{\otimes s}\left(|\psi_0\rangle\langle\psi_0|\right) U_{\boldsymbol{\theta}}^\dagger, \tag{10}$$

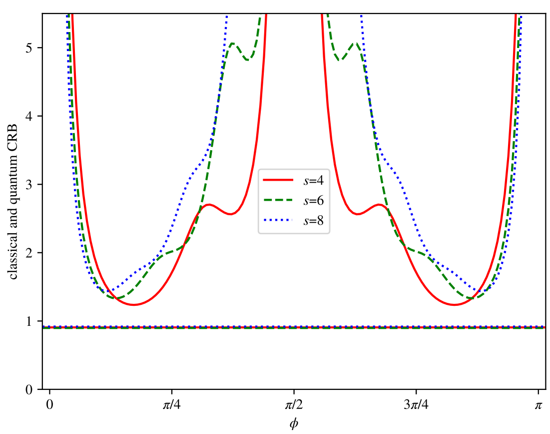
where $U_{\boldsymbol{\theta}} = \bigotimes_{k=1}^s e^{-i\sigma_z^{(k)}\theta_k/2}$ and $\mathcal{N}_\eta(\varrho) := \eta\varrho + \frac{1-\eta}{2}I$. Since the parameters are encoded locally, we can always offset our prior knowledge to be centered at $\boldsymbol{\theta} = 0$, and hence we will only consider local estimation at that point. Moreover, the global function is chosen to be the (scaled) average of the parameters, i.e., $\boldsymbol{v} = \left(\frac{1}{\sqrt{s}}, \ldots, \frac{1}{\sqrt{s}}\right)^\top$.

For $|\psi_0\rangle$, we pick two metrologically important states, the Greenberger–Horne–Zeilinger (GHZ) state $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes s} + |1\rangle^{\otimes s}\right)$ and the symmetric Dicke state $\left|D_s^{s/2}\right\rangle = \frac{1}{\sqrt{\binom{s}{s/2}}}\sum_{\mathrm{perm}} |+\rangle^{\otimes s/2}|-\rangle^{\otimes s/2}$ (for even s) where the sum is over all permutations of the particles and $|\pm\rangle$ are the eigenstates of the Pauli X operator [8]. These two states have been shown to offer an advantage in estimating the average of the parameters, yielding a mean squared error that is s-times smaller than separable probes. However, it is also established that under local independent noise, this advantage quickly degrades to at most a constant-factor improvement [30, 31].

Before showing our numerical results, it is important to point out that in the lossless case, both the GHZ state and the Dicke state admit a simple QCRB-saturating LM strategy, where each sensor makes a projective measurement in the $\{|+\rangle, |-\rangle\}$ basis. In fact, this measurement basis is optimal not only at $\boldsymbol{\theta} = 0$ but also simultaneously for any $\boldsymbol{\theta}$.



(a) GHZ state



(b) Dicke state

Fig. 1. Classical CRB (curved lines) with respect to the azimuthal angle ϕ of local projective measurement basis on the equator of the Bloch sphere, for (a) GHZ state with 4, 5, 6 qubits and (b) symmetric Dicke state with 4, 6, 8 qubits. The noise level is taken to be $\eta = 0.8$. The corresponding QCRBs are also plotted as horizontal lines.

Thanks to the permutation symmetry of $|\psi_0\rangle$, $\boldsymbol{v}$, and our noise model, we can focus on LM strategies that are uniform among different sensors, i.e., $\left\{E_{x_1}^{(1)}\right\}_{x_1} = \cdots = \left\{E_{x_s}^{(s)}\right\}_{x_s}$. Since the classical Fisher information matrix $F_C(p_{\mathcal{M},\boldsymbol{\theta}})$ as a matrix function of the POVM $\mathcal{M}$ is monotonic under post-processing of the measurement outcome [26], it follows that we can restrict to rank-one POVMs, i.e., where each element is of rank 1 $E_{x_k}^{(k)} \propto \left|e_{x_k}^{(k)}\right\rangle \left\langle e_{x_k}^{(k)}\right|$. Another useful observation is that Pauli Z measurement on both the GHZ state and the Dicke state is insensitive to unitary rotation around the z axis, hence we make the further restriction that the projection basis $\left|e_{x_k}^{(k)}\right\rangle$ lies on the equator of the Bloch sphere. Indeed, we numerically find that the sensitivity decays very quickly when the direction of $\left|e_{x_k}^{(k)}\right\rangle$ deviates the equator.

In Fig 1 we plot the classical CRB $\boldsymbol{v}^\top F_C(p_{\mathcal{M},\boldsymbol{\theta}})^{-1}\boldsymbol{v}\big|_{\boldsymbol{\theta}=0}$ with respect to the azimuthal angle ϕ of the projection basis when we perform local projective measurement $\{\Pi_0, I - \Pi_0\}$ where $\Pi_0 = |e_0\rangle\langle e_0|$ and $|e_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right)$. Notably, the standard $\{|+\rangle, |-\rangle\}$ basis employed in the lossless case becomes useless, and we need to carefully adjust the azimuthal angle ϕ for achieving the optimal sensitivity. This is more obvious in the GHZ state, where we find that the optimal value of ϕ is an odd multiple of $\frac{\pi}{2s}$. When s becomes large, it becomes essential to ensure high precision on ϕ for the measurement apparatus to achieve meaningful estimation performance. From this perspective, we find that Dicke state is more advantageous than GHZ state for noisy sensing in that the former has a larger span of values of ϕ that are useful. Nevertheless, in both cases, for the class of measurements we consider, we observe that the QCRB cannot be saturated and we need to look into more detail to understand the metrological usefulness of a general mixed probe state. Our numerical results are generated using the QuTip Python package [39].

6. CONCLUSION

We have considered the task of estimating a linear function of local parameters in a distributed quantum sensing network and questioned the achievability of QCRB for a group of geographically separated sensors. We answered the question in the positive for any pure probe state when the sensors can cooperate using LOCC. However, when taking into account noise in the resource distribution process and allowing only independent local measurements, we gave numerical evidence that QCRB is not an informative measure of the metrological usefulness of the probe states. This calls into the search for a tighter estimation bound that is inherent to localized measurement strategies. More realistic noise models including those that occur in the measurement should also be considered in a practical setting. Another future research direction is to analyze the same question in the non-asymptotic regime, where it is also interesting to understand the achievability of the Bayesian versions of the QCRB.

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