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Holography and Irrelevant Operators

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Abstract

We explore the holographic proposal involving spacetimes with linear dilaton asymptotics in three dimensions from a gravity perspective. The holographic dual shares some properties with a symmetric product conformal field theory deformed by a single-trace analogue of the $T\overline{T}$ deformation. We present solutions of ten-dimensional supergravity which interpolate from BTZ black holes in the interior to either a linear dilaton spacetime near infinity, or to flat space. This allows a precise identification of field theory parameters with gravity parameters. The solutions manifestly exhibit the square root structure that is characteristic of $T\overline{T}$-deformed conformal field theories. We compute the mass of the spacetimes using the covariant phase space formalism and find agreement with the square root formula for the case of black holes without spin. We also discuss whether closed string tachyons might play a role when the deformation parameter becomes too large and the vacuum becomes unstable.

Contents

1 Introduction

Although holography for asymptotically AdS spacetimes has been extensively studied and tested in a variety of contexts, holography for spacetimes with non-AdS asymptotics is poorly understood. This certainly includes the cases of asymptotically flat and asymptotically de Sitter spacetimes. There are good reasons to suspect that new structures in quantum field theory are needed to define quantum gravity in such spacetimes, should such gravity theories exist. If one assumes a correspondence similar in spirit to the AdS/CFT correspondence then any holographic description is forced to have a high-energy density of states that grows faster than a local quantum field theory in order to match the spectrum of large black holes. If one is in low enough dimension, this obstruction might be avoided but it seems hard to avoid in theories with propagating gravitons.

It seems reasonable then to suspect that a field theory dual to a non-AdS spacetime,

with a potentially rich spectrum of black holes, might be controlled by some structure other than a local quantum field theory at high energies. On the other hand, we have recently learned that some special irrelevant deformations of local quantum field theories possess exactly the property of dramatically modifying the ultraviolet behavior of the theory. It is an exciting prospect that the new structures seen in quantum field theory by turning on controlled irrelevant deformations might play a role in defining quantum gravity on non-AdS spacetimes.

The most prominent family of controlled irrelevant deformations, which has generated considerable recent excitement, involves operators constructed from bilinears of conserved currents in two-dimensional quantum field theories. The first example in this family is the $T\overline{T}$ operator of [1]. This operator, which can be used to deform any $2d$ QFT, is given by the combination $\det(T)$,

$$T\overline{T}(x) = \lim_{y\to x} \left(T^{\mu\nu}(x)T_{\mu\nu}(y) - T^{\mu}{}_{\mu}(x)T^{\nu}{}_{\nu}(y) \right) , \tag{1.1}$$

where $T_{\mu\nu}$ is the stress tensor of the theory. Although this definition involves a coincident-point limit of local operators, all OPE divergences which arise in this limit are proportional to total derivatives. These total derivatives do not contribute to one-point functions in translationally invariant states. Therefore, up to fairly harmless total derivative ambiguities, one can always define a local irrelevant operator from the stress tensor using (1.1). Since any $2d$ QFT with translation invariance admits a stress-energy tensor, this operator is universal. We will call this the double-trace $T\overline{T}$ deformation to distinguish it from a different deformation of interest in this work.

We define a flow in the space of theories with tangent vector $\int T\overline{T}(x)$. The flow parameter λ has length dimension 2. The stress tensor must be recomputed at each step along the flow, and the composite operator (1.1) is constructed using this λ-dependent stress tensor. Quantizing this theory on a cylinder of radius L gives an energy spectrum which satisfies the inviscid Burgers' equation:

$$\frac{\partial}{\partial\lambda}E_n(L,\lambda) = \frac{1}{2}\frac{\partial}{\partial L}\left(E_n(L,\lambda)^2\right) + \frac{P_n(L)^2}{L} . \tag{1.2}$$

Here E_n are the energies and P_n are the quantized momenta [1–3].

There is considerable evidence that this deformation defines a theory at the quantum level for flat and even AdS_2 spacetimes [4], and leads to a new structure beyond local quantum field theory. For example, the high-energy density of states exhibits a Hagedorn growth which can be inferred from the explicit solution of (1.2) for a $T\overline{T}$-deformed CFT_2

on a cylinder of radius L,

$$E_n(\lambda) = \frac{L}{2\lambda}\left(\sqrt{1+\frac{4\lambda E_n}{L}+\frac{4\lambda^2 P_n^2}{L^2}}-1\right), \tag{1.3}$$

where $E_n(\lambda)$ is the deformed energy, E_n is the undeformed CFT_2 energy and P_n is the momentum of the n^{th} eigenstate.[1] Many of the basic questions that one might ask about this new structure, like the nature of physical observables, are still unsettled.

If one instead considers the symmetric product orbifold $\mathrm{Sym}^N(\mathrm{CFT}_2) = (\mathrm{CFT}_2)^N/S_N$ then we could deform by the single-trace operator

$$D(x) = \lim_{y\to x}\sum_{i=1}^{N}\left[(T_i(x))^{\mu\nu}\,(T_i(y))_{\mu\nu} - (T_i(x))^{\mu}{}_{\mu}\,(T_i(y))^{\nu}{}_{\nu}\right], \tag{1.4}$$

which exists at the orbifold point [6], instead of the usual double-trace operator

$$T\overline{T}(x) = \lim_{y\to x}\left[\Big(\sum_i T_i(x)\Big)^{\mu\nu}\Big(\sum_i T_i(y)\Big)_{\mu\nu} - \Big(\sum_i T_i(x)\Big)^{\mu}{}_{\mu}\Big(\sum_i T_i(y)\Big)^{\nu}{}_{\nu}\right]. \tag{1.5}$$

Here T_i denotes the stress tensor for the i^{th} copy of CFT_2 in the symmetric product.

There is a very intriguing holographic proposal for defining quantum gravity in spacetimes with linear dilaton asymptotics in three dimensions [6–8]. Such backgrounds include solutions that interpolate between AdS_3 spacetimes and linear dilaton spacetimes [9–11]. The proposal is that deforming by an operator like (1.4), although not precisely this operator,[2] provides the holographic description of this system [6–8]. The specific backgrounds studied arise as limits of the gravitational solution describing a collection of type II NS5-branes and fundamental strings. Such a gravitational solution has a holographic description controlled in the ultraviolet by the conjectured little string theory supported on the NS5-branes; for reviews, see [13, 14].

[1] All of our discussion is for the "good" sign of the deformation, which corresponds to $\lambda > 0$. For this sign, all of the energies are real for sufficiently small λ. For the "bad" sign $\lambda < 0$, most of the energies are complex. If one considers sequential flows by irrelevant operators then the bad sign behavior can be cured [5].

[2] The holographic definition of AdS_3 string theory with NS-flux cannot be the symmetric product orbifold, aside possibly from the stringy case of a single NS5-brane [12]. Rather the holographic theory is suspected to correspond to a marginal deformation of the orbifold theory. The existence of a well-defined irrelevant operator analogous to (1.4) away from the orbifold point is in no sense obvious. For our purpose, what is important is that an energy formula like (1.3) exists for at least some states in the holographic description.

This example is especially interesting because an asymptotically linear dilaton (ALD) spacetime is structurally quite different from an asymptotically AdS spacetime. For instance, AdS has a timelike boundary but an ALD spacetime has a null component in the boundary. We might therefore expect holography in the ALD setting to differ significantly from the familiar AdS/CFT correspondence. Much of the progress in realizing that deformations in the spirit of $D(x)$ play a role in holography has come from worldsheet considerations. In the specific case of the $M = 0$ BTZ background with pure NS-flux, there is a marginal worldsheet deformation that generates the ALD spacetime [9, 10]. The effect of this worldsheet deformation on the long string excitation spectrum has been argued to match the way $D(x)$ deforms the energies of the symmetric product orbifold [6]. This is compelling evidence that something like a single-trace $T\overline{T}$ operator, with a similar effect on the energy spectrum, should exist in the actual holographic CFT_2 which defines AdS_3 with pure NS-flux.

Our goal in the present work is to provide a complementary view of this holographic proposal: rather than using a worldsheet construction, we will work primarily in the target spacetime and perform an analysis using only classical general relativity. We find families of solutions which look like AdS_3 spacetimes in the interior but with either ALD asymptotics (3.5) or flat space asymptotics (3.8). What is striking about these solutions, which are parametrized by the mass and spin $(\widetilde{M}, \widetilde{J})$ of the interior BTZ black hole, is that the dilaton solution exhibits exactly the square root form seen in (1.3):

$$e^{2\Phi} = \frac{r_5^2\sqrt{1 + 8k_1\frac{\widetilde{M}r_5^2}{R^2} + 16\,(k_1)^2\,\frac{\widetilde{J}^2 r_5^4}{R^4}}}{\hat{r}_1^2 + k_1 r^2}\,. \tag{1.6}$$

The radial coordinate in (1.6) is r while the parameter k_1 corresponds to λ of (1.3) via the relation:

$$\lambda = \frac{(\alpha')^2|m_5|}{2R^2}k_1\,. \tag{1.7}$$

The constants r_5 and $\hat{r}_1$ in (1.6) are fixed length scales determined by the number of NS5-branes, m_5, and the number of fundamental strings, m_1. The string length is $\ell_s = \sqrt{\alpha'}$ and R is the size of an asymptotic circle which is finite in string-frame and characteristic of the ALD spacetime.

It is worth noting that the effect of the k_1 deformation is detectable even in the deep interior where r is very small. For example, the AdS_3 length scale is k_1-deformed in a mass and spin-dependent way. Alternately, the value of the dilaton at $r = 0$ in (1.6) is now mass and spin-dependent while it is simply constant in the case of AdS_3. Said differently: the

gravity solution 'knows' about the irrelevant deformation of the holographic CFT even in regions of spacetime normally associated to the deep infrared of the CFT.

Our conventions can be found in section 2 along with a review of the $\widetilde{M} = 0$ brane solutions. In section 3 we summarize the final form of the various families of solutions with mass and spin. A more detailed analysis leading to these solutions can be found in Appendix A. In section 4, we use the covariant phase space formalism to compute the mass of the ALD solutions. For the case without spin, we find a square root form for the mass of the spacetime (4.28) in accord with our expectations for a $T\bar{T}$-like deformation of the holographic description. The case with spin is more subtle because of issues with charge integrability which we discuss in section 4.2. Lastly, in section 4.3 we briefly explore whether a closed string tachyon might play a role in understanding the fate of the vacuum state when the deformation parameter becomes too large.

Some generalizations of the holography discussed here for the pure NS background to cases with (p, q) 5-branes, which really require the kind of gravity approach we have employed, will appear elsewhere.

Note Added: The results in this paper have been in a state of partial completion for quite a long time. During that period two interesting papers appeared with related results from different approaches [15, 16]. The long delay can, in part, be attributed to the arrival of a new baby during the COVID lockdown. After this paper appeared, we were informed of other potentially related work [17].

2 Type IIB Supergravity Brane Solutions

We first want to find general solutions of type II supergravity with NS5-brane and fundamental string charge that asymptote to flat space. There will be two decoupling limits that play a role in our discussion and we will want to write the solutions in convenient coordinates for exhibiting these limits.

As a starting point, in this section we review brane solutions of type IIB supergravity, which fully decouple to the Poincaré patch of the $M = 0$ BTZ background. In section 3, we will find the general class of solutions with non-zero mass and spin. Because the solutions involve a torus, we can dualize between type IIA or type IIB string theory; for concreteness, we will discuss solutions of the type IIB equations of motion.

2.1 Constraints from Flux Quantization

To fix conventions, let us begin with the bosonic action for ten-dimensional type IIB supergravity in string-frame:

$$S_{\rm IIB} = \frac{1}{(2\pi)^7\alpha'^4}\int d^{10}x\,\sqrt{-\hat{g}}\Big\{e^{-2\Phi}\left(\hat{R}+4(\partial\Phi)^2-\frac{1}{12}|H_3|^2\right)-\frac{1}{2}|F_1|^2 \\ -\frac{1}{12}|F_3-C_0H_3|^2-\frac{1}{4\cdot 5!}|\widetilde{F}_5|^2\Big\}, \tag{2.1}$$

where $\widetilde{\boldsymbol{F}}_5 = d\boldsymbol{C}_4 - \frac{1}{2}\boldsymbol{C}_2\wedge\boldsymbol{H}_3 + \frac{1}{2}\boldsymbol{B}_2\wedge\boldsymbol{F}_3$.[3] We define the gravitational constant

$$\kappa_{10}^2 := \frac{1}{2}(2\pi)^7\alpha'^4 e^{2\Phi_0}\,, \tag{2.2}$$

in terms of the string length, $\ell_s = \sqrt{\alpha'}$, and the asymptotic value of the string coupling $g_s = e^{\Phi_0}$. The hatted variables $\hat{g}$ and $\hat{R}$ refer to string-frame quantities.

The analysis of $T\overline{T}$-like deformations and holography in general flux backgrounds – when the Ramond fluxes $C_0, \boldsymbol{C}_2$, $\boldsymbol{C}_4$ may be nonzero – will be explored in future work. For our present purposes, we will restrict to the case of pure NS fluxes. In this case, it is consistent to set $C_0 = \boldsymbol{C}_2 = \boldsymbol{C}_4 = 0$.

We will consider solutions of the form $\mathcal{M}_3\times S^3\times T^4$. The T^4 factor is a spectator which could also be replaced by K3 without changing the analysis. Although not visible in supergravity, in string theory flux quantization through the S^3 requires that

$$\frac{1}{4\pi^2\alpha'}\int_{S^3}\boldsymbol{H}_3 = m_5\,, \tag{2.3}$$

where m_5 is an integer corresponding to the NS5-brane charge.

There is another quantization condition on the dual field strength. If we define

$$\boldsymbol{H}_7 = e^{-2\Phi}*\boldsymbol{H}_3\,, \tag{2.4}$$

then this form satisfies the following quantization condition for any 7-cycle Σ_7,

$$\frac{1}{(2\pi)^6(\alpha')^3}\int_{\Sigma_7}\boldsymbol{H}_7 = m_1\in\mathbb{Z}\,. \tag{2.5}$$

[3] We use boldface to denote differential forms to conform with the notation often found in the gravity literature and used in section 4.

2.2 Review of NS5-Brane and F-String Bound State

There is a well-known class of solutions in type IIB supergravity which satisfy the flux quantization conditions outlined in the previous subsection and which form the prototype for the generalized solutions that we will consider in section 3. These much-studied solutions are interpreted as bound states of m_1 fundamental strings and m_5 NS5 branes, where m_1, m_5 are the integers appearing in the quantization conditions (2.5) and (2.3), respectively.

One way of presenting these solutions [18], which is nicely reviewed in [19], is

$$\begin{aligned} e^{-2\Phi} &= \frac{1}{g_s^2}\frac{f_1}{f_5}\,, \qquad B_{05} = \frac{1}{f_1} - 1\,, \qquad H_{mnp} = \epsilon_{mnp}{}^{q}\,\partial_q(\log f_5) \\ \widehat{ds}^{\,2} &= \frac{-dt^2 + dx_5^2}{f_1} + f_5\left(dx_1^2 + \cdots + dx_4^2\right) + \left(dx_6^2 + \cdots dx_9^2\right)\,, \end{aligned} \tag{2.6}$$

where $m, n, p, q = 1, \ldots, 4$, ϵ_{mnpq} is the volume form for $f_5 \sum_{i=1}^{4} dx_i^2$ and

$$f_1 = 1 + \frac{r_1^2}{r^2} = 1 + \frac{16\pi^4 g_s^2 \alpha'^3 m_1}{V_4 r^2}\,, \qquad f_5 = 1 + \frac{r_5^2}{r^2} = 1 + \frac{\alpha' m_5}{r^2}\,. \tag{2.7}$$

In these expressions, r represents a radial coordinate in the four-dimensional space transverse to the NS5-branes parameterized by x_1, x_2, x_3, x_4, while the coordinates x^6, x^7, x^8, x^9 parametrize a torus with volume V_4. The asymptotic region corresponds to large r. The constant g_s specifies the asymptotic value of the string coupling. We have a choice about how to treat x_5. To patch to AdS_3 in the interior, we will periodically identify

$$x_5 \sim x_5 + 2\pi R\,. \tag{2.8}$$

Note that the flux H_3 has two separate contributions, one arising from the explicit H_{mnp} components and one arising from $\boldsymbol{H} = d\boldsymbol{B}$ with $\boldsymbol{B} = B_{05}\, dx^0 \wedge dx^5$ and B_{05} as indicated in (2.6).

The solution (2.6) contains at least two qualitatively different asymptotic regions. For $r \gg r_1, r_5$, the solution approaches flat 10-dimensional Minkowski space. On the other hand, in the near-horizon limit $r \ll r_1, r_5$, the sphere and torus become spectators and we recover an effective three-dimensional gravity solution which is diffeomorphic to a Poincaré patch of AdS_3.

We will refer to the full spacetime with the asymptotically flat region retained as the undecoupled or totally undecoupled solution, and the near-horizon solution in the deep bulk as the decoupled or totally decoupled solution. The process of taking the full decoupling limit may be thought of as simultaneously sending the asymptotic string coupling $g_s \to 0$

and sending $\alpha' \to 0$, along with a re-scaling of coordinates. Alternatively, one may think of this decoupling limit as dropping the 1's in the harmonic functions f_1 and f_5 so that $f_i = 1 + \frac{r_i^2}{r^2} \;\to\; f_i = \frac{r_i^2}{r^2}$.

Partial Decoupling

There is also an intermediate regime in this spacetime which looks neither like AdS_3 in the Poincaré patch nor like flat space. To take this limit, define $r = g_s\hat{r}$ and take $g_s \to 0$ giving,

$$\begin{aligned} e^{-2\Phi} &= \frac{\hat{r}^2}{r_5^2}\left(1 + \frac{\hat{r}_1^2}{\hat{r}^2}\right), \qquad \widehat{ds}^2 = \frac{-dt^2 + dx_5^2}{f_1} + \frac{r_5^2}{\hat{r}^2}d\hat{r}^2 + r_5^2 d\Omega_3^2 + ds_{T^4}^2\,, \\ f_1 &= 1 + \frac{\hat{r}_1^2}{\hat{r}^2} = 1 + \frac{16\pi^4\alpha'^3 m_1}{V_4\,\hat{r}^2}, \qquad r_1 = g_s\hat{r}_1\,. \end{aligned} \tag{2.9}$$

This amounts to replacing $f_5 = 1 + \frac{r_5^2}{r^2}$ by $f_5 = \frac{r_5^2}{r^2}$, but retaining the full function f_1 without dropping the 1 there, since $\frac{r_1^2}{r^2}$ is not necessarily big.

Full Decoupling

The fully decoupled solution is found by making r very small and dropping the 1 in f_1 [20]. To put the resulting metric in a conventional AdS_3 form, we define dimensionless variables

$$\tilde{r}^2 = \hat{r}^2\frac{R^2}{r_5^2\,\hat{r}_1^2}, \qquad \tilde{\varphi} = \frac{x_5}{R} \sim \tilde{\varphi} + 2\pi, \qquad \tilde{t} = \frac{t}{R}. \tag{2.10}$$

In these variables the metric takes the form

$$\begin{aligned} \widehat{ds}^2 &= \ell^2\left(\tilde{r}^2\left(-d\tilde{t}^2 + d\tilde{\varphi}^2\right) + \frac{d\tilde{r}^2}{\tilde{r}^2} + d\Omega_3^2\right) + ds_{T^4}^2\,, \\ e^{-2\Phi} &= \frac{\tilde{r}^2 r_1^2}{R^2} + \frac{\hat{r}_1^2}{\ell^2} \xrightarrow{\text{decoupling}} \frac{\hat{r}_1^2}{\ell^2}\,, \end{aligned} \tag{2.11}$$

where $\ell = r_5$ is the AdS_3 length scale. We note that the size, R, of the x_5 circle at infinity is absorbed into a redefinition of time and therefore energy. This will play a role later when we compare energies in the partially decoupled theory to energies in the AdS_3 region. The last comments on the fully decoupled case concern the value of the gravitational coupling in the AdS_3 theory which is given by,

$$(\kappa_3)^2 = 8\pi G_3 = \frac{2\pi\ell_s}{|m_1|\sqrt{|m_5|}}, \tag{2.12}$$

and the central charge of the dual holographic CFT which is given by

$$c = \frac{3\ell}{2G_3} = 6|m_1 m_5|\,. \tag{2.13}$$

We can identify the radius L of the holographic CFT by noting that the mass of global AdS_3 is $-\frac{1}{8G_3}$. This corresponds to the ground state of the CFT with energy $-\frac{c}{12L}$ so we identify:

$$L = \ell = \sqrt{|m_5|}\ell_s \,. \tag{2.14}$$

This is independent of the radius R of the x_5 circle which appears in (2.8).

As we discussed in the introduction, gravity solutions of the partially-decoupled form have been argued to have field theory duals which are obtained by deforming a conformal field theory by an irrelevant operator related to $T\overline{T}$ [6–8]. These solutions have also been shown to arise from TsT transformations [15, 16, 21]. In the following section we will generalize these well-known solutions for the $M = 0$ BTZ case without spin to metrics which interpolate from a general spinning BTZ black hole in the interior to either an asymptotically linear dilaton spacetime or to an asymptotically flat spacetime.

3 Generalized NS5-Brane and F1-String Solutions

In this section, we will present a class of supergravity solutions with the desired quantized charges, symmetries and asymptotic behavior. These solutions can be interpreted as bound states of a system of F1 strings and NS5-branes that interpolate between an AdS_3 black hole in the deep interior with some mass and spin to either a linear dilaton solution in the asymptotic region, or to asymptotically flat six dimensions at large distances. We will separately present those two cases because the latter is a more complicated class of solutions.

The full spacetime equations of motion are quite difficult to solve in generality without some physically reasonable metric ansatz as an input. Our ansatz is spelled out in Appendix A along with the calculations of the metric and dilaton functions, and the details of the fairly involved asymptotic matching needed to identify the physical parameters of the solution. The aim of this section is to present clearly the assumptions going into the analysis and the final solutions. There are other classes of solutions in Appendix A, which are even more surprising: they appear to interpolate from an AdS_3 spacetime in the interior to an asymptotic 3-dimensional spacetime with positive curvature. Whether those solutions are fully sensible will be explored elsewhere.

3.1 Characterizing the General Solution

We will look for solutions to the equations of motion of type IIB supergravity, written using the same coordinates as (2.6), which satisfy the following assumptions:

1. Asymptotics. We assume that near infinity the family of solutions approach either the linear dilaton asymptotic form seen in (2.9), or the flat space asymptotic form seen in (2.6). We will insist that the isometries of T^4, S^3 and S^1 are preserved by the solutions. This collapses the unknown functions in the metric, fluxes and dilaton to functions of r and possibly time t. We assume these coordinates can be extended to global coordinates for our spacetime soution. Lastly, we assume the metric, dilaton, and NS-flux admit an expansion in powers of $\frac{1}{r}$ near $r \to \infty$.

   Near $r = 0$, we insist the solution takes the form $\mathcal{M}_3 \times S^3 \times T^4$ with the dilaton Φ tending to a constant. We need this form if we are to interpret the holographic theory as some kind of deformed CFT.

2. Symmetries. We assume that the solution is stationary and therefore only depends on the radial coordinate r. Because the spacetime is assumed to be stationary but not static, the timelike Killing vector ∂_t need not be irrotational. However, we will constrain the allowed rotation to occur only in the x_5 direction. Operationally, this means that a non-vanishing metric component g_{tx_5} is allowed – corresponding to a spinning BTZ black hole at small r – but no other off-diagonal metric components are permitted.

3. Fluxes. We assume that the Ramond potentials $\boldsymbol{C}_p$ are all vanishing and that the NS flux $\boldsymbol{H}_3 = d\boldsymbol{B}_2$ only threads the three-dimensional submanifolds S^3 and $\mathcal{M}_3$. So $\boldsymbol{H}_3$ is a sum of the volume forms $\boldsymbol{\epsilon}_{S^3}$ and $\boldsymbol{\epsilon}_{\mathcal{M}_3}$ on S^3 and $\mathcal{M}_3$, respectively, with coefficients that can in principle depend on r. Finally, we impose the usual flux quantization conditions (2.3) and (2.5) on H_3 required by string theory.

3.2 Solutions with ALD Asymptotics

The $\widetilde{J} = 0$ case

From the analysis presented in Appendix A, we find the following solutions in string-frame for the case where the black hole has no spin:

$$
\begin{aligned}
ds^2 &= -\frac{1-\frac{\alpha_e^2}{r^2}}{k_1+\frac{\hat{r}_1^2}{r^2}}\,dt^2 + \frac{1}{k_1+\frac{\hat{r}_1^2}{r^2}}\,dx_5^2 + \frac{r_5^2}{r^2-\alpha_e^2}\,dr^2 + r_5^2\,d\Omega_3^2 + ds_{T^4}^2\,, \\
e^{2\Phi} &= \frac{r_5^2\sqrt{1+8k_1\frac{\widetilde{M}r_5^2}{R^2}}}{\hat{r}_1^2+k_1 r^2}\,, \qquad \alpha_e^2 = 8\widetilde{M}\frac{\hat{r}_1^2 r_5^2}{R^2}\,, \quad r_5^2 = \alpha' m_5\,, \quad \hat{r}_1^2 = \frac{16\pi^4\alpha'^3 m_1}{V_4}\,.
\end{aligned}
\tag{3.1}
$$

These solutions depend on a dimensionless parameter k_1. The fully decoupled pure AdS_3 solutions correspond to $k_1 = 0$. We will momentarily identify k_1 with the deformation parameter λ of a $T\overline{T}$-deformed theory. It is important to point out that the $\widetilde{M} = 0$ solutions, which match (2.9), are sensitive to whether k_1 is zero or non-zero. However, the precise value of k_1 is not important and can be absorbed in a rescaling of coordinates. This reflects the fact that $E = 0$ states remain zero energy under a $T\overline{T}$ deformation: namely, zero energy does not flow. Any case with $\widetilde{M} \neq 0$, however, is sensitive to the value of k_1.

Note that the x_5 circle has periodicity $\frac{2\pi R}{\sqrt{k_1}}$ at $r = \infty$ in string-frame. In Einstein frame, the x_5 circle has a proper size that goes to infinity as $r \to \infty$. The dimensionless mass $\widetilde{M}$ is related to the dimensionful mass of the BTZ black hole near $r = 0$ via $\widetilde{M} = MG_3$, where G_3 is specified in (2.12).

On first glance, it might appear that the solutions (3.1) have the wrong large r asymptotics in Einstein-frame because the dilaton is mass-dependent. However a careful redefinition of the radial coordinate in the large r region, discussed around (A.27), shows that these solutions have the correct asymptotic behavior. Also notice that the value of the dilaton at $r = 0$ is now mass-dependent and differs from the pure AdS_3 case where $k_1 = 0$. As a consequence, the AdS_3 length scale is also mass-dependent in Einstein frame.

Global AdS_3 has mass $\widetilde{M} = -\frac{1}{8}$. This corresponds to the ground state of the holographic CFT which should have dimensionless energy $\widetilde{E} = EL = \widetilde{M}\frac{L}{G_3}$. Notice that the dilaton solution is only real if the condition,

$$
k_1 < \frac{R^2}{r_5^2}\,,
\tag{3.2}
$$

is satisfied. It is important to note that this condition is independent of m_1 but depends on the value of m_5 via r_5^2. As we increase m_5, the maximum value of k_1 decreases. It is this

asymmetry between m_1 and m_5 that singles out the single-trace $T\overline{T}$ deformation rather than the conventional double-trace deformation. In the latter case, we should have seen the maximum value of k_1 reduce as we increase the total central charge (2.13) which depends on both m_1 and m_5. We can then compare this bound with (1.3) which states that

$$\lambda < \frac{3L^2}{c_{\text{block}}} = \frac{3L^2}{6|m_5|}\,, \tag{3.3}$$

where $c_{\text{block}} = 6|m_5|$ is the central charge of a block that would appear if the holographic dual were actually a symmetric product [6]. Via this comparison we can identify,

$$\lambda = \frac{(\alpha')^2|m_5|}{2R^2}k_1\,, \qquad \tilde{\lambda} = \frac{\lambda}{L^2} = \frac{\alpha'}{2R^2}k_1\,, \tag{3.4}$$

where we have also defined the natural dimensionless $T\overline{T}$-deformation parameter $\tilde{\lambda}$. This makes clear that the $T\overline{T}$ deformation is controlled by the asymptotic size of the x_5 circle in string frame. We will see further evidence for this identification in the spinning case. It is striking that the square root solution of the $T\overline{T}$ flow equation for a CFT emerges directly from gravity in the structure of the dilaton solution.

The $\widetilde{J} \neq 0$ case

The solutions for spinning black holes are similar,

$$\begin{aligned} ds^2 &= -\frac{1-\frac{\alpha_e^2}{r^2}+\frac{\alpha_j^4}{r^4}}{k_1+\frac{\hat{r}_1^2}{r^2}}\,dt^2 + \frac{1}{k_1+\frac{\hat{r}_1^2}{r^2}}\left(dx_5 - \frac{\alpha_j^2}{r^2}dt\right)^2 + \frac{r_5^2}{r^2-\alpha_e^2+\frac{\alpha_j^4}{r^2}}\,dr^2 + r_5^2\,d\Omega_3^2 + ds_{T^4}^2\,, \\ e^{2\Phi} &= \frac{r_5^2\sqrt{1+8k_1\frac{\widetilde{M}r_5^2}{R^2}+16\,(k_1)^2\,\frac{\widetilde{J}^2r_5^4}{R^4}}}{\hat{r}_1^2+k_1r^2}\,, \qquad \alpha_e^2 = 8\widetilde{M}\frac{\hat{r}_1^2r_5^2}{R^2}\,, \qquad \alpha_j^2 = 4\widetilde{J}\,\frac{\hat{r}_1^2r_5^2}{R^2}. \end{aligned} \tag{3.5}$$

Here $\widetilde{J} = \frac{JG_3}{\ell}$ is a dimensionless spin, but this is not the natural quantity to study from the perspective of the dual CFT. The AdS/CFT correspondence maps the dimensionless parameter $J = \widetilde{J}\frac{\ell}{G_3}$ to the dimensionless momentum quantum number $\widetilde{P} = PL$ of the holographic CFT. With this identification, the form of the dilaton solution again beautifully matches the structure of the $T\overline{T}$-deformed energy formula (1.3),

$$\sqrt{1+8k_1\frac{\widetilde{M}r_5^2}{R^2}+16\,(k_1)^2\,\frac{\widetilde{J}^2r_5^4}{R^4}} \quad\longrightarrow\quad \sqrt{1+4\widetilde{\lambda}\widetilde{E}+4\widetilde{\lambda}^2\widetilde{P}^2}\,, \tag{3.6}$$

where we identify

$$\widetilde{P} = \frac{J}{m_1}\,, \qquad \widetilde{E} = 4m_5\widetilde{M} = \frac{1}{m_1}\left(\widetilde{M}\cdot\frac{\ell}{G_3}\right). \tag{3.7}$$

Here $\widetilde{E}$ and $\widetilde{P}$ are the energy and momentum, respectively, of a single block of the symmetric product.

3.3 Solutions with Flat Space Asymptotics

Lastly we turn to the fully undecoupled solutions that asymptote to a flat 6-dimensional spacetime. In string-frame, the metric and dilaton take the form:

$$
\begin{aligned}
ds^2 &= -\frac{1-\frac{\alpha_e^2}{r^2}+\frac{\alpha_j^4}{r^4}}{k_1+\frac{\hat{r}_1^2}{r^2}}\,dt^2+\frac{1}{k_1+\frac{\hat{r}_1^2}{r^2}}\left(dx_5-\frac{\alpha_j^2}{r^2}dt\right)^2 \\
&\quad+\frac{k_5 r^2-\frac{k_5\alpha_e^2}{2}+\sqrt{r_5^4-k_5^2\alpha_j^4+\frac{1}{4}k_5^2\alpha_e^4}}{r^4-r^2\alpha_e^2+\alpha_j^4}\,r^2\,dr^2+r^2\left(k_5+\frac{\alpha_5^2}{r^2}\right)d\Omega_3^2+ds_{T^4}^2\,, \\
e^{2\Phi} &= \frac{g_s^2+\frac{\gamma\alpha_5^2}{k_1 r^2}}{1+\frac{\hat{r}_1^2}{k_1 r^2}}\,, \qquad \alpha_e^2=8\widetilde{M}\frac{\hat{r}_1^2 r_5^2}{R^2}\,, \qquad \alpha_j^2=4\widetilde{J}\,\frac{\hat{r}_1^2 r_5^2}{R^2}\,, \qquad k_5=\frac{k_1 g_s^2}{\gamma}\,, \\
\alpha_5^2 &= \frac{1}{2}\left(\sqrt{k_5^2\alpha_e^4+4\left(r_5^4-k_5^2\alpha_j^4\right)}-k_5\alpha_e^2\right)\,, \qquad \gamma=\sqrt{1+\frac{k_1\alpha_e^2}{\hat{r}_1^2}+\frac{k_1^2\alpha_j^4}{\hat{r}_1^4}}\,.
\end{aligned}
\tag{3.8}
$$

This final form for the metric depends on the dimensionless parameters (g_s, k_1) where g_s is the $r=\infty$ value of the string coupling. The solution also depends on the flux quantum numbers via $(\hat{r}_1, r_5)$ defined in (3.1) and the black hole mass and spin $(\widetilde{M}, \widetilde{J})$. We note that the 3-sphere now grows as r becomes large which is why the asymptotic theory is 6-dimensional.

The square root structure of the solution to the $T\overline{T}$ flow equation is encoded in γ which now appears both in the metric via k_5 and the dilaton. Notice that the size of the S^3 at $r=0$ is also deformed from the pure AdS_3 case by an amount that depends on the mass and spin of the interior black hole. Lastly we note that the parameter k_1 still controls the size of the x_5 circle near infinity.

4 Mass Calculation

The supergravity solutions derived in the previous section appear to have something to do with a $T\overline{T}$-deformed theory. We made a tentative identification of the deformation parameter with k_1 of the spacetime solution based on the appearance of the square root in the dilaton solution of (3.5). This identification assumes that the holographic CFT is of symmetric product type and that the deformation is purely a single-trace $T\overline{T}$ deformation.

The only cases where we have formulae for the deformed energies are the single and double-trace $T\overline{T}$ deformations, and sequential flows by such irrelevant operators [5]. However, the undeformed holographic CFT is not a symmetric product though it is believed to be connected to a symmetric product by a marginal deformation. Regardless, we can provide further evidence for a correspondence with some kind of single-trace $T\overline{T}$-like deformation of that undeformed theory by directly computing the mass of the solutions (3.1).

The definition of mass in general relativity is not unique. There are a variety of proposals that depend on the asymptotics of the background. A way to uniformly define the mass is by using the covariant phase space formalism [22]. This will provide an independent way of confirming the parameter identification we made in section 3.

4.1 Surface Charges in Gravitational Theories

The covariant phase space formalism allows us to define conserved surface charges in generally covariant theories. In this section we will review this formalism before using it to compute the charges associated with the solutions in section 3. Our discussion of this formalism will be very brief and many facts will be stated without justification; for a more comprehensive treatment, we refer the reader to a review such as [23].

First consider a theory described by a general Lagrangian density $\boldsymbol{L}(\Phi^i)$ in a theory with n spacetime dimensions, where Φ^i represents an arbitrary collection of fields. Following common conventions in the gravity literature we denote differential forms in spacetime by boldface symbols, so $\boldsymbol{L} = L\, d^n x$ is the top form associated with the usual Lagrangian density L. In pure Einstein gravity, for instance, the only field Φ^i is the metric $g_{\mu\nu}$ and $L = \frac{1}{16\pi G}\sqrt{-g}R$. Upon variation of the fields Φ^i, one sees

$$\delta \boldsymbol{L} = \frac{\delta \boldsymbol{L}}{\delta \Phi^i}\, \delta\Phi^i - d\boldsymbol{\Theta}[\delta\Phi^i; \Phi^i]\,. \tag{4.1}$$

The form $\boldsymbol{\Theta}[\delta\Phi^i; \Phi^i]$ is called the presymplectic potential and depends both on the fields Φ^i and their variations $\delta\Phi^i$. One can think of $\boldsymbol{\Theta}$ as simultaneously being an $(n-1)$-form in spacetime and a 1-form in the space of field variations. In what follows, we will suppress the index i when writing the functional dependence on fields and simply write $\boldsymbol{\Theta}[\delta\Phi; \Phi]$. To define conserved charges, it is convenient to introduce a related object $\boldsymbol{\omega}$ called the presymplectic form via

$$\boldsymbol{\omega}[\delta_2\Phi, \delta_1\Phi; \Phi] = \delta_2\boldsymbol{\Theta}[\delta_1\Phi; \Phi] - \delta_1\boldsymbol{\Theta}[\delta_2\Phi; \Phi]\,. \tag{4.2}$$

Here we write the dependence of $\boldsymbol{\omega}$ on both variations $\delta_{1,2}\Phi$ to emphasize that $\boldsymbol{\Theta}$ itself depends on one variation $\delta_1\Phi$ of the fields, and $\boldsymbol{\omega}$ is defined by again performing a second

variation δ_2 of the fields and finding how $\boldsymbol{\Theta}$ varies (with appropriate antisymmetrization). From this perspective, the symbol δ acts as the exterior derivative in field space so that $\boldsymbol{\omega}$ is an $(n-1)$-form in spacetime and a 2-form in the space of field variations.

We will be especially interested in field variations $\delta_\xi \Phi$ that are associated with an infinitesimal diffeomorphism generated by $x^\mu \to x^\mu + \xi^\mu$. For instance, under such a transformation any tensor field $T_{\mu_1 \cdots \mu_p}$ transforms via the Lie derivative $\mathcal{L}_\xi$ as

$$\delta_\xi T_{\mu_1 \cdots \mu_p} = \mathcal{L}_\xi T_{\mu_1 \cdots \mu_p} \,. \tag{4.3}$$

We will consider the case where the field configuration Φ satisfies the equations of motion for the theory, the first variation $\delta_1 \Phi = \delta \Phi$ solves the linearized equations of motion about the solution Φ, and the second variation $\delta_2 \Phi = \delta_\xi \Phi$ is generated by a diffeomorphism of this form. In this case, one can show that $\boldsymbol{\omega}[\delta_\xi \Phi, \delta \Phi; \Phi]$ is exact, so that

$$\boldsymbol{\omega}[\delta_\xi \Phi, \delta \Phi; \Phi] = d \boldsymbol{k}_\xi [\delta \Phi; \Phi] \,. \tag{4.4}$$

This differential form $\boldsymbol{k}_\xi$ is an $(n-2)$-form in spacetime and a 1-form in the space of field variations. Note that we will also identify the $(n-2)$-form $\boldsymbol{k}_\xi$ with its Hodge dual, an anti-symmetric 2-tensor with components $k^{\mu\nu}$, which we write without boldface type. This $\boldsymbol{k}_\xi$ can also be expressed as

$$\boldsymbol{k}_\xi [\delta \Phi; \Phi] = -\delta \boldsymbol{Q}_\xi [\delta \Phi; \Phi] + i_\xi \boldsymbol{\Theta} [\delta \Phi; \Phi] \,, \tag{4.5}$$

up to the ambiguity of adding a total (spacetime) derivative term to $\boldsymbol{k}_\xi$. This ambiguity will not be relevant for us, since we are primarily interested in the integral

$$\delta Q_\xi = \oint_S \boldsymbol{k}_\xi [\delta \Phi; \Phi] \,, \tag{4.6}$$

where S is a closed codimension-2 surface. Typically we will think of a spacetime with a radial direction r and let the surface S be a sphere at a fixed (large) value of r and at a fixed time t. For this case, the total derivative ambiguity of $\boldsymbol{k}_\xi$ will not contribute to the integral in (4.6).

Finally, we will specialize to the case where ξ is an exact Killing vector of the solution. In this case, the expression δQ_ξ in (4.6) can always be integrated in order to define a conserved charge Q_ξ. Operationally, this procedure always involves first computing the *change* in the charge Q_ξ associated with a variation $\delta \Phi$ around a particular solution Φ. In examples of interest, we usually think of a one-parameter family of solutions $\Phi(\alpha)$ which

solve the equations of motion for a continuous range of values of α. In this case, we can consider the variation

$$\delta_\alpha \Phi = \frac{\partial \Phi(\alpha)}{\partial \alpha} \,. \tag{4.7}$$

Since $\Phi(\alpha)$ is a solution for any α, the variation (4.7) always satisfies the linearized equations of motion. We can then compute

$$\frac{dQ_\xi}{d\alpha} = \oint_S \boldsymbol{k}_\xi[\delta_\alpha \Phi; \Phi] \,, \tag{4.8}$$

and integrate the resulting expression with respect to α (using a suitable initial condition) to find $Q_\xi(\alpha)$. For example, the BTZ black hole with mass M is a solution to the three-dimensional equations of motion for pure gravity (with negative cosmological constant) for any value of M. We can compute the change in the charge associated with the exact Killing vector $\xi = \partial_t$ in these backgrounds using the known expression for $\boldsymbol{k}_\xi$ in Einstein gravity. The resulting integral gives

$$\begin{aligned} \frac{dQ_\xi}{dM} &= \oint_S \boldsymbol{k}_\xi[\delta_M g; g] \,, \\ &= 1 \,. \end{aligned} \tag{4.9}$$

Using the initial condition $Q_\xi(M = 0) = 0$, we can then trivially integrate to find

$$Q_\xi = M \,, \tag{4.10}$$

which recovers the fact that the mass of the black hole is the conserved charge associated with the global timelike Killing vector ∂_t.

Charge Integrability

Given a family of spacetimes labeled by some parameter α, and possessing some exact Killing vector ξ, we have seen that the covariant phase space formalism allows us to compute the quantity $\frac{\partial Q_\xi}{\partial \alpha}$ which controls how the conserved charge associated with ξ varies with α. If α is the only parameter in our family of solutions, it is then trivial to integrate $\frac{\partial Q_\xi}{\partial \alpha}$ given an initial condition and recover the expression $Q_\xi(\alpha)$ for the conserved charge.

Now consider a family of solutions which depend on several parameters $\alpha_1, \ldots, \alpha_n$. The formalism permits us to compute a collection of partial derivatives $\frac{\partial Q_\xi}{\partial \alpha_i}$ for each i. We are not guaranteed that integrating any two of these quantities will yield equivalent charges.

For instance, in the simple case of a solution for a black hole which depends on a mass parameter M and a spin parameter J, we could obtain two separate derivatives

$$\begin{aligned} \frac{\partial Q_\xi}{\partial M} &= \oint_S \boldsymbol{k}_\xi[\partial_M g; g]\,, \\ \frac{\partial Q_\xi}{\partial J} &= \oint_S \boldsymbol{k}_\xi[\partial_J g; g]\,. \end{aligned} \tag{4.11}$$

If computing $\partial_J \partial_M Q_\xi$ using the first line of (4.11) yields the same result as computing $\partial_M \partial_J Q_\xi$ using the second line of (4.11), then we can unambiguously define the charge $Q_\xi(M, J)$ in a way which does not involve any choice of how to perform the integration. In this case, the charge $Q_\xi(M, J)$ is said to be integrable. More generally, the integrability condition will hold so long as

$$\delta_1 \oint_S \boldsymbol{k}_\xi[\delta_2\Phi; \Phi] = \delta_2 \oint_S \boldsymbol{k}_\xi[\delta_1\Phi; \Phi]\,, \tag{4.12}$$

for any pair of variations $\delta_1\Phi$, $\delta_2\Phi$.

Charge integrability holds in many cases of physical interest, such as the Kerr black hole in $D > 3$ dimensions or the spinning BTZ black hole in $D = 3$, both of which possess charges that are integrable in the space of M and J parameters. However, it was pointed out in [24] that solutions to the equations of motion for type IIB supergravity with multiple parameters α_i do not, in general, possess manifestly integrable charges. In such cases, if we are given only the information about the gravity solution, the definitions of conserved charges like mass and spin can be ambiguous; one obtains different answers depending on which quantity $\frac{\partial Q_\xi}{\partial \alpha_i}$ one integrates.

To resolve this ambiguity, one requires additional input which identifies a preferred notion of the surface charges. For instance, in the cases analyzed in [24], the additional input comes from holography: if a type IIB supergravity solution is dual to a conformal field theory, one can use the scale invariance of the field theory to identify particular choices of the gravitational charges that are natural from the perspective of the boundary theory.

However, for more general solutions that are not dual to CFTs, there does not appear to be a general principle for constructing unambiguous charges in the case of non-integrable solutions. We will see shortly that the supergravity solutions constructed in section 3 possess exactly this ambiguity when $\widetilde{J} \neq 0$. Because these bulk geometries are believed to be dual to field theories which are obtained through deforming a CFT by an irrelevant operator – namely, something akin to a single-trace $T\overline{T}$ operator – we cannot rely on the scale invariance of the dual theory to construct a preferred set of gravitational charges.

Surfaces Charges in Type IIB-Like Theories

The formalism reviewed above applies to any generally covariant Lagrangian $\boldsymbol{L}$. Next we will restrict attention to a class of Lagrangians which are relevant for type IIB supergravity theories in Einstein frame and with pure NS flux. Consider an action of the form

$$S = \frac{1}{16\pi G}\int d^D x\,\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}f(\Phi)|H|^2\right). \tag{4.13}$$

Here $\boldsymbol{H} = d\boldsymbol{B}$ is the field strength associated with a now general p-form potential $\boldsymbol{B} = B_{\mu_1\ldots\mu_p}\,dx^{\mu_1}\ldots dx^{\mu_p}$, $|H|^2 = \boldsymbol{H}\wedge *\boldsymbol{H}$, and $f(\Phi)$ is an arbitrary function which controls the scalar coupling to the p-form kinetic term. When $D = 10$, $p = 2$, and $f(\Phi) = e^{-\Phi}$, this gives the Einstein-frame action for type IIB supergravity. After performing the conformal mapping from string-frame to Einstein-frame, the general solutions which we derived in section 3 satisfy the equations of motion associated with this action.

The surface charges associated with the Lagrangian (4.13) were worked out in [25]; here we will recall the results which are relevant for our analysis.[4]

The contribution from the scalar is

$$\boldsymbol{k}^\Phi_\xi[\delta g,\delta\Phi; g;\Phi] = i_\xi\boldsymbol{\Theta}_\Phi\,, \qquad \boldsymbol{\Theta}_\Phi = *(d\Phi\,\delta\Phi)\,. \tag{4.14}$$

In components, (4.14) is

$$k^{\mu\nu,\Phi}_\xi[\delta g,\delta\Phi; g;\Phi] = 2\,(\delta\Phi)\cdot\xi^{[\nu}\partial^{\mu]}\Phi, \tag{4.15}$$

where $T^{[ab]} = \frac{1}{2}\left(T^{ab} - T^{ba}\right)$ denotes the usual antisymmetrization. Recall that the non-bolface $k^{\mu\nu}_\xi$ is the antisymmetric 2-tensor whose Hodge dual is the $(n-2)$-form $\boldsymbol{k}_\xi$.

The contribution from the p-form $\boldsymbol{B}$ is more complicated and can be expressed as

$$\boldsymbol{k}^B_\xi[\delta g,\delta B; g, B,\Phi] = -\delta\boldsymbol{Q}^B_\xi + i_\xi\boldsymbol{\Theta}_B - \boldsymbol{E}^B_{\mathcal{L}}[\mathcal{L}_\xi B,\delta B]\,, \tag{4.16}$$

where

$$\begin{aligned}
\boldsymbol{Q}^B_\xi &= f(\Phi)\,(i_\xi B)\wedge *H\,,\\
\boldsymbol{\Theta}^B &= f(\Phi)\,(\delta B)\wedge *H\,,\\
\boldsymbol{E}^B_{\mathcal{L}}[\mathcal{L}_\xi B,\delta B] &= f(\Phi)*\left(\frac{1}{2(p-1)!}\delta B_{\mu\alpha_1\cdots\alpha_{p-1}}\,(\mathcal{L}_\xi B)_\nu{}^{\alpha_1\ldots\alpha_{p-1}}\;dx^\mu\wedge dx^\nu\right).
\end{aligned} \tag{4.17}$$

[4]These formulas have also been implemented in Mathematica by the author of [25] in a convenient package which may be found here.

It is convenient to separate the component expression for $k_\xi^{B,\mu\nu}$ into its contributions from the tensor field variation $\delta\boldsymbol{B}$ (and its field strength variation $\delta\boldsymbol{H} = d(\delta\boldsymbol{B})$), the metric variation $\delta g_{\mu\nu}$, and the dilaton variation $\delta\Phi$:

$$\begin{aligned}
k_\xi^{B,\mu\nu} &= 2K_{\delta B}^{[\mu\nu]} + 2K_{\delta g}^{[\mu\nu]} + 2K_{\delta\Phi}^{[\mu\nu]}\,,\\
K_{\delta B}^{\mu\nu} &= -f(\Phi)\Bigg[\frac{2}{p}\xi^\mu H^{\nu\alpha_1\cdots\alpha_p}(\delta B)_{\alpha_1\cdots\alpha_p} - (\mathcal{L}_\xi B)^{\mu\alpha_1\cdots\alpha_{p-1}}(\delta B)^\nu{}_{\alpha_1\cdots\alpha_{p-1}}\\
&\qquad + H^{\mu\nu\alpha_1\cdots\alpha_{p-1}}\xi^\rho(\delta B)_{\rho\alpha_1\cdots\alpha_{p-1}} + (\delta H)^{\mu\nu\alpha_1\cdots\alpha_{p-1}}\xi^\rho B_{\rho\alpha_1\cdots\alpha_{p-1}}\Bigg]\,,\\
K_{\delta g}^{\mu\nu} &= f(\Phi)\Bigg[-\frac{1}{2}(\delta g)^\alpha{}_\alpha H^{\mu\nu\alpha_1\cdots\alpha_{p-1}}\xi^\rho B_{\rho\alpha_1\cdots\alpha_{p-1}} + (p-1)(\delta g)^{\mu\rho}H_\rho{}^{\nu\alpha_1\cdots\alpha_{p-1}}\xi^\sigma B_{\sigma\alpha_1\cdots\alpha_{p-1}}\\
&\qquad + 2(\delta g)^\mu{}_\rho H^{\rho\nu\alpha_1\cdots\alpha_{p-1}}\xi^\sigma B_{\sigma\alpha_1\cdots\alpha_{p-1}}\Bigg]\,,\\
K_{\delta\Phi}^{\mu\nu} &= -\frac{\partial f}{\partial\Phi}(\delta\Phi)H^{\mu\nu\alpha_1\cdots\alpha_{p-1}}\xi^\rho B_{\rho\alpha_1\cdots\alpha_{p-1}}\,.
\end{aligned} \tag{4.18}$$

Here we write $\delta g = g^{\mu\nu}\delta g_{\mu\nu}$ for the trace of the metric fluctuation.[5]

Finally, the contribution from the Einstein-Hilbert term $\sqrt{-g}R$ is

$$\boldsymbol{k}_\xi^g[\delta g; g] = -\delta\boldsymbol{Q}_\xi[g] - i_\xi\boldsymbol{\Theta}[\delta g; g]\,, \tag{4.19}$$

where

$$\begin{aligned}
\Theta^\mu &= \frac{\sqrt{-g}}{16\pi G}\left(\nabla_\nu\,\delta g^{\mu\nu} - \nabla^\mu\delta g\right)\,,\\
\boldsymbol{Q}_\xi &= \frac{\sqrt{-g}}{8\pi G}\nabla^\mu\xi^\nu\left(d^{D-2}x\right)_{\mu\nu}\,.
\end{aligned} \tag{4.20}$$

Here we follow the conventions of [23] for differential forms; for instance, $(d^{n-p}x)_{\mu_1\ldots\mu_p}$ is defined by $\frac{1}{p!(n-p)!}\varepsilon_{\mu_1\ldots\mu_p\nu_{p+1}\ldots\nu_n}dx^{\nu_{p+1}}\wedge\ldots\wedge dx^{\nu_n}$ where ε is the Levi-Civita symbol with entries $-1, 0$, or 1 (without the factor of $\sqrt{-g}$). In components, (4.19) can be written as

$$\begin{aligned}
k_\xi^{g,\mu\nu} =& \xi^{[\nu}\nabla^{\mu]}\delta g - \xi^{[\nu}\nabla_\alpha\delta g^{\mu]\alpha} + \xi_\alpha\nabla^{[\nu}\delta g^{\mu]\alpha} + \frac{1}{2}\delta g\nabla^{[\nu}\xi^{\mu]}\\
&- \frac{1}{2}\delta g^{\alpha[\nu}\nabla_\alpha\xi^{\mu]} + \frac{1}{2}\delta g^{\alpha[\nu}\nabla^{\mu]}\xi_\alpha\,.
\end{aligned} \tag{4.21}$$

[5] This differs from the usage in expressions like $\boldsymbol{k}_\xi^g[\delta g; g]$, where δg refers to the full metric fluctuation rather than its trace, but where we have suppressed indices for ease of notation.

The total contribution to the change in a charge Q_ξ as a parameter α of the solution is varied, therefore, is given by

$$\frac{\partial Q_\xi}{\partial \alpha} = \oint_S \boldsymbol{k}_\xi[\partial_\alpha g; g]\,,$$
$$\boldsymbol{k}_\xi = \boldsymbol{k}_\xi^\Phi + \boldsymbol{k}_\xi^B + \boldsymbol{k}_\xi^g\,. \tag{4.22}$$

We will take the surface S to lie at a fixed time and fixed large value R of the radial coordinate r in our solutions, and then take the limit as $R \to \infty$. The integral (4.22) then extracts the component k_ξ^{tr} of the antisymmetric 2-tensor which is Hodge dual to $\boldsymbol{k}_\xi$. For the purpose of computing the integral (4.22), the compact directions $S^3 \times T^4$ are merely spectators which are integrated over to yield an appropriate volume factor.

4.2 Evaluating the Charge

Our goal is to compute the Noether-Wald surface charge Q_ξ associated with the exact Killing vector $\xi = \partial_t$ of the solutions with ALD asymptotics ($k_5 = 0$) derived in section 3. We will restrict to the case given in (3.1) of a non-rotating black hole with $\widetilde{J} = 0$. We will comment on the case with both mass and spin at the close of this section. That case involves interesting subtleties related to the non-integrability discussed in section 4.1.

As is typical in the covariant phase space formalism, rather than computing the charge Q_ξ directly, we first find $\frac{\partial Q_\xi}{\partial \alpha}$ where α is one of the parameters in these solutions. At this stage we have a choice, since our spacetimes depend on the parameters $(k_1, \widetilde{M})$ with quantized fixed charges determined by (m_1, m_5). If we vary both $(k_1, \widetilde{M})$ then we find that the charge Q_ξ is *not* integrable in the space of these parameters; that is, we obtain different expressions for the mass if we, for instance, (1) compute $\frac{\partial Q_\xi}{\partial k_1}$ and integrate with respect to k_1, or (2) compute $\frac{\partial Q_\xi}{\partial \widetilde{M}}$ and integrate with respect to $\widetilde{M}$. In this case, we view the failure of integrability as reflecting the definition of an asymptotically linear dilaton spacetime. Namely, the parameter k_1 is a part of the definition of the asymptotic behavior and should not be varied.

For this reason, we will consider a one-parameter family of solutions where α_e, which controls $\widetilde{M}$, is varied but all other parameters are held fixed. Therefore we will use the variations of the fields

$$\delta_{\alpha_e} g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial \alpha_e}\,, \qquad \delta_{\alpha_e} B_{\mu\nu} = \frac{\partial B_{\mu\nu}}{\partial \alpha_e}\,, \qquad \delta_{\alpha_e} \Phi = \frac{\partial \Phi}{\partial \alpha_e}\,. \tag{4.23}$$

Under this linearized variation, the combined contribution from k_ξ^g, k_ξ^B, and k_ξ^Φ to the

change in the charge is

$$\delta_{\alpha_e} Q_\xi = \oint_S \left(\boldsymbol{k}_\xi^g + \boldsymbol{k}_\xi^B + \boldsymbol{k}_\xi^\Phi \right) = \frac{\hat{r}_1 \alpha_e}{4\sqrt{\hat{r}_1^2 + k_1 \alpha_e^2}} \,. \tag{4.24}$$

Here S is a fixed time slice at large r in $\mathcal{M}_3$ and includes an integral over all the compact directions. Because we have used the variations (4.23) associated with changing α_e, we have $\delta_{\alpha_e} Q_\xi = \frac{\partial Q_\xi}{\partial \alpha_e}$ so that

$$\frac{\partial Q_\xi}{\partial \alpha_e} = \frac{\hat{r}_1 \alpha_e}{4\sqrt{\hat{r}_1^2 + k_1 \alpha_e^2}} \,. \tag{4.25}$$

This equation can be trivially integrated to find

$$Q_\xi = \frac{\hat{r}_1}{4k_1} \sqrt{\hat{r}_1^2 + k_1 \alpha_e^2} + C \,, \tag{4.26}$$

where C is an integration constant which may be a function of k_1 and α_1 but not of α_e. We fix the integration constant by requiring that the conserved charge Q_ξ have a finite limit as $k_1 \to 0$. This is only possible if the constant C is chosen so that the charge takes the form

$$Q_\xi = \frac{\hat{r}_1^2}{4k_1} \left(\sqrt{1 + k_1 \frac{\alpha_e^2}{\hat{r}_1^2}} - 1 \right) \,. \tag{4.27}$$

To recover the undeformed black hole, we take $k_1 \to 0$. This is an easy way to fix the overall normalization. We will choose to normalize the charge, denoted $\widetilde{Q}_\xi$, so that $k_1 \to 0$ gives $\widetilde{M}$:

$$\widetilde{Q}_\xi = \frac{R^2}{4 r_5^2 k_1} \left(\sqrt{1 + k_1 \frac{8 \widetilde{M} r_5^2}{R^2}} - 1 \right) \,. \tag{4.28}$$

This square root is *identical* to the expression that appears in the dilaton solution found in (3.1). It is strong evidence for a holographic correspondence in the spirit proposed in [6].

A straightforward extension of this analysis to the case with mass and spin encounters the issue of non-integrability in the space of $(\widetilde{M}, \widetilde{J})$ solutions. Similar issues have been seen in past work [24, 26, 27]. It would be interesting to resolve the non-integrability and see if one can reproduce the mass and charge for these asymptotically linear dilaton spacetimes expected from the form of the dilaton solution seen in (3.5).

4.3 Winding Tachyon

We have seen that the solution which interpolates from global AdS_3 to ALD asymptotics, presented in section 3 with $\widetilde{M} = -\frac{1}{8}$, has a spacetime mass (4.28) which becomes complex when k_1 is made too large.

What might be happening in string theory when this ground state approaches this instability? Note that the string-frame metric sees an approximately constant circle for large r of size $\frac{R}{\sqrt{k_1}}$ and a very weak string coupling. Global AdS in the interior corresponds to anti-periodic boundary conditions for spacetime fermions on the x_5 circle. Boundary conditions of this type tend to produce tachyons for strings wrapping the circle if the circle becomes too small [28]. Qualitatively, this is the kind of phenomenon we might hope would happen prior to the spacetime becoming unstable, although the endpoint of condensing such a bulk closed string tachyon is often non-geometric [29]. If the holographic theory exists for k_1 beyond the value where the ground state energy becomes complex, we will see that the more likely resolution lies in the low-energy gravity theory.

To estimate where a tachyon might emerge, we note that a fundamental string on a circle of radius $\frac{R}{\sqrt{k_1}}$ with winding w and a twisted boundary condition breaking supersymmetry satisfies a mass-shell condition,

$$m^2 = \frac{w^2 R^2}{k_1 (\alpha')^2} + \frac{2}{\alpha'} \left(N_L + N_R \right) , \tag{4.29}$$

for bosonic excitations with no momentum on the circle. In the $(NS-, NS-)$ sector with odd winding and no oscillators, the zero point energy is given by $N_L = N_R = -\frac{1}{2}$. The first state to become tachyonic has $w = 1$ which happens for $k_1 > \frac{R^2}{2\alpha'}$. This is not quite right in our case because we have not taken the linear dilaton background into account. The linear dilaton slope is often denoted Q in the literature on worldsheet descriptions of linear dilaton backgrounds, where in this case $Q = \frac{2}{\sqrt{m_5 \alpha'}}$; see, for example [14]. For a string with no radial momentum, the mass-shell condition for a singly wound string is modified slightly from the simple circle of (4.29),

$$m^2 = \frac{R^2}{k_1 (\alpha')^2} + \frac{Q^2}{4} - \frac{2}{\alpha'} . \tag{4.30}$$

Now the condition to avoid a tachyon becomes

$$k_1 < \frac{R^2}{\alpha' \left(2 - \frac{1}{m_5} \right)} . \tag{4.31}$$

This should be contrasted with the condition (3.2) needed to avoid a complex dilaton solution, which we repeat here for convenience:

$$k_1 < \frac{R^2}{m_5 \alpha'} \,. \tag{4.32}$$

These conditions only agree for $m_5 = 1$. For large m_5 the gravity condition is violated first, long before one expects a closed string tachyon, suggesting a possible resolution in the low-energy theory without stringy ingredients. There is one final observation which is curious: if there were a 'long' string with tension reduced by a factor of m_5 so the tension $T_{\rm long} = \frac{1}{2\pi\alpha' m_5}$ then a tachyon for this string would emerge before the gravity solution becomes complex as k_1 is increased.[6]

5 The Two Parameter Family of Solutions

Now we will explore the family of solutions that depend on an additional parameter c_1, which are derived in Appendix A. These solutions are quite intriguing but exhibit some surprising features so our discussion is oriented around addressing the more basic issues with these solutions. At short distances and small black hole mass, the solutions again look like AdS_3 black holes. At large distances, these solutions exhibit positive curvature in string-frame, which is typically hard to engineer in string theory in stationary backgrounds.

With the details of the analysis again relegated to the Appendix, we first list the form of the string-frame metric for the case without spin:

$$\begin{aligned} ds^2 = &-\frac{1-\frac{\alpha_e^2}{r^2}}{k_1+\frac{\alpha_1^2}{r^2}}\,dt^2 + \frac{1}{k_1+\frac{\alpha_1^2}{r^2}}\,dx_5^2 + \frac{r_5^2(1+k_1\alpha_1^2c_1)(1+k_1c_1(\alpha_1^2+k_1\alpha_e^2))}{(r^2-\alpha_e^2)(1+k_1c_1(\alpha_1^2+k_1r^2))^2}\,dr^2 \\ &+ r_5^2\,d\Omega_3^2 + ds_{T^4}^2\,, \end{aligned} \tag{5.1}$$

with $\alpha_e^2 = 8\widetilde{M}\frac{\hat{r}_1^2 r_5^2}{R^2}$, and where the dilaton is given by

$$e^{2\Phi} = c_2^2\left(\frac{1+k_1c_1\,(\alpha_1^2+k_1r^2)}{\alpha_1^2+k_1r^2}\right)\,. \tag{5.2}$$

Notice that $e^{2\Phi}$ given in (5.2) always decreases as r ranges from 0 to ∞. Here the parameters c_1 and c_2 are related by the algebraic constraint

$$c_2^4 = \frac{r_5^4}{\hat{r}_1^4}\left\{\frac{\alpha_1^4+k_1\alpha_1^2\alpha_e^2}{(k_1c_1\alpha_1^2+1)(k_1c_1(\alpha_1^2+k_1\alpha_e^2)+1)}\right\}\,. \tag{5.3}$$

[6]Such long strings are expected from the low-energy physics of the D1-D5 system but are not expected to be visible in the weakly coupled perturbative string spectrum of the F1-NS5 system.

The curvatures for this metric are non-singular and the dilaton is well behaved as long as $k_1 > 0$ and $c_1 > 0$. Notice that the dilaton interpolates from a constant at small r to a different constant for very large r.

What is quite peculiar about the metric (5.1) is the behavior of f_r which determines the g_{rr} component of the metric. For large r,

$$f_r \sim \frac{1}{r^6} \,. \tag{5.4}$$

This means $r = \infty$ is now a point at finite distance. How should we treat this boundary? There are several possible approaches which might give physically interesting backgrounds:

- One can try to continue the metric beyond $r = \infty$.
- One can try to impose boundary conditions at $r = \infty$ or at some continuation of the metric. This is similar in spirit to the holographic proposal of [30].
- One can try to include a brane source at or beyond $r = \infty$. If such a source has a sensible string theory interpretation then the metric might admit a natural continuation which takes into account the stress-energy and charge of the brane.

We will explore one of these possibilities.

Continuing the metric

At large r, it is natural to change to the radial variable $\tilde{\rho} = \frac{1}{r^2}$.[7] The range of $\tilde{\rho}$ is $(\tilde{\rho}_0, 0)$ prior to any continuation. Here $\tilde{\rho}_0$ is a cutoff which is sufficiently small so that we can ignore $O(\tilde{\rho})$ corrections. We are interested in the physics for $\tilde{\rho} \to 0$ where the three-dimensional spacetime metric takes the simple form,

$$\begin{aligned} ds^2 &= -\frac{1}{k_1}\, dt^2 + \frac{1}{k_1}\, dx_5^2 + \frac{1}{k_{\tilde{\rho}}}\, d\tilde{\rho}^2 + O(\tilde{\rho}) \,, \\ k_{\tilde{\rho}} &= \frac{4k_1^4 c_1^2}{r_5^2(1 + k_1\alpha_1^2 c_1)\,(1 + k_1 c_1 \alpha_1^2 + k_1^2 c_1 \alpha_e^2)} \,, \end{aligned} \tag{5.5}$$

where $k_{\tilde{\rho}}$ is a positive constant. At this point, one could imagine imposing a boundary condition at $\tilde{\rho} = 0$ in the spirit of [30]. At this boundary, the string coupling has a finite value:

$$e^{2\Phi} = k_1 c_1 c_2^2 \,. \tag{5.6}$$

[7] We are denoting this radial variable by $\tilde{\rho}$ to avoid confusion with the ρ that appears in equation (A.27) and surrounding discussion.

Or we might attempt to continue the metric by taking $\tilde{\rho}$ negative. The string coupling takes the form,

$$e^{2\Phi} = c_2^2 \left(\frac{\tilde{\rho} + k_1 c_1 \left(\tilde{\rho} \alpha_1^2 + k_1 \right)}{\tilde{\rho} \alpha_1^2 + k_1} \right) . \tag{5.7}$$

The string coupling will either diverge or go to zero as we make $\tilde{\rho}$ more negative. If $k_1 c_1 \alpha_1^2 > 0$ then the string coupling always goes to zero before we hit a strong coupling singularity. We want k_1 positive otherwise the dx_5^2 term of (5.1) will flip sign at finite r introducing a closed time-like curve. If c_1 is not positive then the dilaton will not make sense for finite r so it appears we must encounter a point where gravity is shut off before we meet any potential strong coupling singularity.

This is quite suggestive. Holography for linear dilaton spacetimes is motivated, in part, by gravity shutting off at radial infinity because the string coupling goes to zero. Here we meet a finite value of $\tilde{\rho}$ at which the string coupling vanishes given by,

$$\tilde{\rho} = -\frac{k_1^2 c_1}{1 + k_1 c_1 \alpha_1^2} . \tag{5.8}$$

Finding a linear dilaton coordinate

After some experimentation and with some hindsight, it is useful to first answer the following question before continuing the metric (5.5): is there a new radial coordinate that makes the dilaton of (5.2) look like the expression with $c_1 = 0$? The answer turns out to be yes in some cases,

$$\mathfrak{r} = \sqrt{\frac{r^2(1 - c_1 k_1 \alpha_1^2) - c_1 \alpha_1^4}{1 + c_1 k_1 (k_1 r^2 + \alpha_1^2)}} , \tag{5.9}$$

with the dilaton taking the form,

$$e^{2\Phi} = \frac{c_2^2}{k_1 \mathfrak{r}^2 + \alpha_1^2} . \tag{5.10}$$

The expression (5.9) for $\mathfrak{r}$ makes sense for sufficiently large r and sufficiently small $c_1 k_1$; specifically $r^2 \geq \frac{c_1 \alpha_1^4}{1 - c_1 k_1 \alpha_1^2}$ and $c_1 k_1 \alpha_1^2 < 1$. At this point, we can ask what bounds we should impose on (c_1, k_1) so that the solution for the dilaton is real. There appears to be no upper bound on $c_1 > 0$ and the bound on k_1 from considering (5.2) for the case of $\alpha_e^2 < 0$ is the same as our earlier discussion with $c_1 = 0$.

Let us assume these conditions are satisfied. We can always choose r to be sufficiently large but making $c_1 k_1$ sufficiently small is not necessary based on what we have seen so far.

Note that the range of $\mathfrak{r}$ is $(0, \mathfrak{r}_\infty)$ where

$$\mathfrak{r}_\infty = \sqrt{\frac{(1 - c_1 k_1 \alpha_1^2)}{c_1 k_1^2}}\,, \tag{5.11}$$

is finite. Plotting $\mathfrak{r}$ as a function of r shows that it is monotonic asymptoting to the value $\mathfrak{r}_\infty$. The natural continuation in this variable is to allow $\mathfrak{r}$ to continue to infinity past the finite value $\mathfrak{r}_\infty$. The metric expressed in this radial variable takes the form,

$$\begin{aligned} ds^2 = &\frac{\alpha_e^2 - c_1(\alpha_1^4 + k_1\alpha_1^2\alpha_e^2) - \mathfrak{r}^2(1 + c_1k_1(\alpha_1^2 + k_1\alpha_e^2))}{\alpha_1^2 + k_1\mathfrak{r}^2}\, dt^2 + \left(c_1\alpha_1^2 + \frac{\mathfrak{r}^2}{\alpha_1^2 + k_1\mathfrak{r}^2}\right) dx_5^2 \\ &+ \frac{r_5^2\mathfrak{r}^2(1 + c_1k_1\alpha_1^2)(1 + c_1k_1(\alpha_1^2 + k_1\alpha_e^2))}{(c_1\alpha_1^4 + \mathfrak{r}^2(1 + c_1k_1\alpha_1^2))(c_1\alpha_1^4 + (c_1k_1\alpha_1^2 - 1)\alpha_e^2 + \mathfrak{r}^2(1 + c_1k_1(\alpha_1^2 + k_1\alpha_e^2)))}\, d\mathfrak{r}^2\,, \end{aligned} \tag{5.12}$$

which simplifies as $\mathfrak{r} \to \infty$ to

$$\begin{aligned} ds^2 = &\left(-\left\{\frac{1}{k_1} + c_1\left(\alpha_1^2 + k_1\alpha_e^2\right)\right\} + \frac{1}{k_1^2\mathfrak{r}^2}\left(\alpha_1^2 + k_1\alpha_e^2\right)\right) dt^2 + \left(\frac{1}{k_1} + c_1\alpha_1^2 - \frac{\alpha_1^2}{k_1^2\mathfrak{r}^2}\right) dx_5^2, \\ &+ \frac{r_5^2}{\mathfrak{r}^2}\, d\mathfrak{r}^2 \;+ O\left(\frac{1}{\mathfrak{r}^4}\right)\,. \end{aligned} \tag{5.13}$$

There are a couple of observations worth making concerning (5.13). First the $d\mathfrak{r}^2$ has precisely the form we expect for a linear dilaton theory and the dilaton (5.10) has the right radial dependence to define a linear dilaton spacetime. However the $\mathfrak{r}$-independent term in the dt^2 coefficient appearing in (5.13) now depends on α_e^2. Similarly the dx_5^2 leading metric is c_1 modified so the radius of the asymptotic circle in string-frame has changed.

Making the asymptotic radial metric canonical

We have just seen that finding a coordinate in which the dilaton has the same asymptotic form as the $c_1 = 0$ case is possible when $c_1k_1\alpha_1^2 < 1$. However, we have yet to see any reason to restrict c_1 to this particular case. There is another approach we can take. Let us see if we can find a coordinate in which the dr^2 coefficient of the metric (5.1) takes the asymptotic form

$$\frac{r_5^2 d\hat{\mathfrak{r}}^2}{\hat{\mathfrak{r}}^2 - \hat{\alpha}_e^2} = r_5^2 d\left(\operatorname{arctanh}(\sqrt{1 - \frac{\hat{\alpha}_e^2}{\hat{\mathfrak{r}}^2}})\right)^2 \sim \frac{r_5^2 d\hat{\mathfrak{r}}^2}{\hat{\mathfrak{r}}^2}\left(1 + O(\frac{\hat{\alpha}_e^2}{\hat{\mathfrak{r}}^2})\right)\,, \tag{5.14}$$

for a new coordinate $\hat{\mathfrak{r}}$ with a horizon at some potentially new location $\hat{\alpha}_e$. We can then see how the dilaton depends on this new coordinate.

As a first case, let us take $\alpha_e = 0$ and no horizon in the extension so $\hat{\alpha}_e = 0$ as well.[8] In this case we can take,

$$\frac{1}{\hat{\mathfrak{r}}} = \frac{\sqrt{1 + k_1 c_1 (\alpha_1^2 + k_1 r^2)}}{r} , \tag{5.15}$$

$$= k_1 \sqrt{c_1} + O\left(\frac{1}{r^2}\right) , \tag{5.16}$$

where the second line is expanded near $r = \infty$. We have picked the sign so that small r corresponds to small $\hat{\mathfrak{r}}$. This relation can be inverted giving,

$$r^2 = \frac{(1 + k_1 c_1 \alpha_1^2)\, \hat{\mathfrak{r}}^2}{1 - c_1 k_1^2\, \hat{\mathfrak{r}}^2} . \tag{5.17}$$

The point $r = \infty$ corresponds to $\hat{\mathfrak{r}} = \frac{1}{k_1\sqrt{c_1}}$. The continuation corresponds to $\hat{\mathfrak{r}}$ extending past this point to infinity. The dilaton can now be expressed in terms of $\hat{\mathfrak{r}}$,

$$\Phi = \frac{1}{2} \log \left[\frac{c_2^2\, (1 + k_1 c_1 \alpha_1^2)}{k_1 \hat{\mathfrak{r}}^2 + \alpha_1^2} \right] . \tag{5.18}$$

This is indeed an asymptotically linear dilaton spacetime with a slope that is superficially determined by r_5 and $c_1 k_1$. Beautifully, however, the constant $c_2^2\,(1 + k_1 c_1 \alpha_1^2)$ appearing in (5.18) becomes c_1-independent for this case of $\alpha_e = 0$ using the relation (5.3) and we recover the same dilaton as the $c_1 = 0$ case. We can now examine the full metric expressed in terms of $\hat{\mathfrak{r}}$,

$$ds^2 = \frac{1 + c_1 k_1 \alpha_1^2}{k_1 + \frac{\alpha_1^2}{\hat{\mathfrak{r}}^2}} \left(-dt^2 + dx_5^2\right) + \frac{r_5^2}{\hat{\mathfrak{r}}^2}\, d\hat{\mathfrak{r}}^2 . \tag{5.19}$$

Up to a rescaling of time and a redefinition of the asymptotic x_5 circle size, this solution is now identical to the $c_1 = 0$ case.

Now we can turn to the case of $\alpha_e > 0$ with $r > \alpha_e$. The change of variables in this case is given by,

$$\hat{\mathfrak{r}}^2 = \frac{\hat{\alpha}_e^2}{1 - \frac{(r^2 - \alpha_e^2)(1 + k_1 c_1 \alpha_1^2)}{r^2 (1 + k_1 c_1 (\alpha_1^2 + k_1 \alpha_e^2))}} . \tag{5.20}$$

This is invertible giving,

$$r^2 = \frac{\alpha_e^2 \hat{\mathfrak{r}}^2}{\hat{\alpha}_e^2 + \frac{\alpha_e^2 k_1^2 c_1 (\hat{\alpha}_e^2 - \hat{\mathfrak{r}}^2)}{(1 + k_1 c_1 \alpha_1^2)}} . \tag{5.21}$$

[8] We will assume that the α_e of (5.1) is finite. One might also consider the case where the coordinate patch parametrized by r is actually *inside* the black hole horizon with the horizon located in the extension past $r = \infty$.

Substituting into the dilaton expression (5.2) gives,

$$e^{2\phi} = c_2^2 \left(\frac{\hat{\alpha}_e^2 (1 + k_1 c_1 \alpha_1^2)(1 + k_1 c_1 (\alpha_1^2 + k_1 \alpha_e^2))}{k_1 \alpha_e^2 \, \hat{\mathfrak{r}}^2 + (1 + k_1 c_1 \alpha_1^2 + k_1^2 c_1 \alpha_e^2) \alpha_1^2 \hat{\alpha}_e^2} \right) . \tag{5.22}$$

If we impose the condition that the denominator of (5.22) take the form seen in (5.2) then we relate α_e and $\hat{\alpha}$ as follows:

$$\alpha_e = \hat{\alpha}_e \sqrt{\frac{1 + k_1 c_1 \alpha_1^2}{1 - k_1^2 c_1 \hat{\alpha}_e^2}} \, . \tag{5.23}$$

On substituting back into the dilaton we wonderfully find all factors conspire to give,

$$e^{2\Phi} = \frac{r_5^2}{\hat{r}_1^2} \left\{ \sqrt{\alpha_1^4 + k_1 \alpha_1^2 \hat{\alpha}_e^2} \right\} \cdot \frac{1}{\alpha_1^2 + k_1 \hat{\mathfrak{r}}^2} \, , \tag{5.24}$$

which is the form expected for a solution with $c_1 = 0$. At least for these cases, analytic continuation together with a suitable map of parameters recovers the solutions discussed earlier with $c_1 = 0$.

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A Derivation of General Solutions

In this Appendix we derive a general class of type IIB supergravity solutions which satisfy the assumptions laid out in section 3.1. We will work in string frame with the action (2.1), for which the associated equations of motion for the metric, dilaton, and flux are

$$R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}|H|^2_{\mu\nu} = 0\,, \tag{A.1}$$

$$R + 4\nabla^2\Phi - 4|\nabla\Phi|^2 - \frac{1}{12}|H|^2 = 0\,, \tag{A.2}$$

$$\nabla^\mu\left(e^{-2\Phi}H_{\mu\nu\rho}\right) = 0\,, \tag{A.3}$$

respectively. A combination of the first two equations gives a nice equation for the dilaton. See, for example [31]:

$$\nabla^2 e^{-2\Phi} = e^{-2\Phi}|H|^2. \tag{A.4}$$

We will begin with the following general ansatz consistent with the symmetry and flux assumptions stated in section 3.1:

$$ds^2 = -\frac{f_e(r)}{f_1(r)}\,dt^2 + \frac{1}{f_x(r)}\left\{dx_5 + f_j(r)\,dt\right\}^2 + f_r(r)\,dr^2 + r^2 f_5(r)\,d\Omega_3^2 + f_{T^4}(r)ds^2_{T^4}\,,$$
$$\Phi = \Phi(r)\,, \qquad H_3 = \frac{2m_5}{\ell_s}\epsilon_{S^3} + f_{\mathcal{M}_3}(r)e^{2\Phi}\epsilon_3^{\mathcal{M}_3}\,. \tag{A.5}$$

Here ϵ_{S^3} is the volume form for S^3 with $\int\epsilon_{S^3} = 2\pi^2\ell_s^3$. Flux quantization through S^3 is automatically satisfied. Flux quantization through $\mathcal{M}_3$ requires that

$$f_{\mathcal{M}_3}(r) = \frac{32m_1\pi^4\alpha'^3 g_s^2}{V_4 r^3 f_5(r)^{3/2}}. \tag{A.6}$$

That fixes one unknown function. This form for H_3 also solves (A.3). One way to make $R_{\mu\nu}$ vanish in the T^4 directions is by setting $f_{T^4}(r)$ to a constant. It is not clear that any other solution exists so will set $f_{T^4}(r) = 1$. This is in accord with the torus playing no significant role in the physics.

The choice of f_r is the freedom to parametrize the radial direction in a convenient way. At this point, we need to understand what asymptotic conditions to impose on our spacetime at small r and large r.

A.1 The partially decoupled case $k_5 = 0$

Interpolating ansatz

It will simplify our analysis to take the following ansatz for the ten-dimensional metric,

$$\begin{aligned} f_1(r) &= k_1 + \frac{\alpha_1^2}{r^2}, \qquad f_e(r) = 1 - \frac{\alpha_e^2}{r^2}, \qquad f_5(r) = \frac{\alpha_5^2}{r^2}, \\ f_x(r) &= k_x + \frac{\alpha_x^2}{r^2}\,, \qquad f_j(r) = 0\,, \end{aligned} \tag{A.7}$$

with f_j set to 0 because $J = 0$. We could have considered the more general $f_5(r) = k_5 + \frac{\alpha_5^2}{r^2}$ but for this partially decoupled case, we set $k_5 = 0$. We also demand that $k_1 > 0$ and $k_x > 0$ for a sensible metric at large r. We have chosen to absorb the constant k_e that might have appeared in f_e into a redefinition of the other parameters. Because this is the string-frame metric rather than the Einstein-frame metric, we will leave k_1 in the ansatz until later when we determine how to normalize the time direction at large r.

Constraints from equations of motion

With this assumption, plugging our ansatz into the Einstein and dilaton equations gives constraints,

$$\alpha_x^2 = \frac{k_x}{k_1}\alpha_1^2\,, \qquad \alpha_5 = r_5\,, \tag{A.8}$$

and the following solution for $e^{2\Phi}$,

$$e^{2\Phi} = c_2^2 \left(\frac{1 + k_1 c_1 \left(\alpha_1^2 + k_1 r^2\right)}{\alpha_1^2 + k_1 r^2} \right)\,, \tag{A.9}$$

where c_1 and c_2 are unknown constants. Imposing the dilaton equation of motion relates c_1 and c_2 as follows,

$$\frac{r_5^4}{\hat{r}_1^4} - \frac{(k_1 c_1 \alpha_1^2 + 1)(k_1 c_1 (\alpha_1^2 + k_1 \alpha_e^2) + 1)}{\alpha_1^4 + k_1 \alpha_1^2 \alpha_e^2} c_2^4 = 0\,. \tag{A.10}$$

Note that c_2 cannot be set to zero. At this juncture, we have not imposed either small r or large r asymptotics on the dilaton. The constants (c_1, c_2) determine the dilaton and f_r. Everything else is fixed by both matching onto BTZ at small r and imposing large r asymptotics, which we have yet to examine.

If we now demand that $g_s \to 0$ as $r \to \infty$ then we need to set $c_1 = 0$ in (A.9) to get a dilaton:

$$e^{2\Phi} = \left(\frac{c_2^2}{\alpha_1^2 + k_1 r^2} \right)\,. \tag{A.11}$$

We also require $\frac{c_2^2}{k_1} \geq 0$ since it is the coefficient of the large r string coupling of (A.9) so $c_2^2 \geq 0$. For a sensible value of $e^{2\Phi}$ in a neighborhood of $r = 0$ we must also then require $\alpha_1^2 > 0$. This also fixes c_2 in terms of the other parameters of the ansatz using (A.10):

$$c_2^4 = \frac{r_5^4}{\hat{r}_1^4}(\alpha_1^4 + k_1\alpha_1^2\alpha_e^2)\,. \tag{A.12}$$

With the dilaton determined, we can now solve for f_r:

$$f_r(r) = \frac{r_5^2}{r^2 - \alpha_e^2}\,. \tag{A.13}$$

Small r asymptotics

At very small r we demand that the solution in Einstein-frame look like a three-dimensional BTZ black hole. This expectation is at least reasonable for low mass black holes whose structure should not care very much about the large r asymptotic behavior of the metric. For simplicity let us first consider a black hole with mass but no spin. In the notation of the full decoupling limit presented in equation (2.11), the metric of a three-dimensional BTZ black hole of mass $\widetilde{M}$ and spin $J = 0$ is

$$\widehat{ds}^2 = \ell^2\left(-\left(-8\widetilde{M} + \tilde{r}^2\right)d\tilde{t}^2 + \tilde{r}^2 d\tilde{\varphi}^2 + \frac{d\tilde{r}^2}{-8\widetilde{M} + \tilde{r}^2}\right)\,, \tag{A.14}$$

where the mass parameter $\widetilde{M}$ has been made dimensionless using the effective three-dimensional Newton constant.

We will initially be noncommittal about the relation between these dimensionless variables and the dimensionful variables of our ansatz (A.7). Because $\alpha_5 = r_5$ by the equations of motion, the reduction on S^3 and T^4 to a three-dimensional theory is unchanged from the discussion around (2.11). There are no mass-dependent factors from integrating over those volumes in string-frame.

The three-dimensional Einstein-frame metric is given by,

$$ds_E^2 = e^{-4\Phi}ds_{\text{string}}^2\,. \tag{A.15}$$

The most commonly studied backgrounds involve a constant dilaton so moving between string-frame and Einstein-frame is straightforward. This is no longer immediately the case in backgrounds where the dilaton varies so we will want to keep careful track of dilaton factors. For this $c_1 = 0$ case we can expand $e^{-4\Phi}$ at small r to find,

$$e^{-4\Phi} = \frac{\alpha_1^4}{c_2^4}\left(1 + \frac{2k_1 r^2}{\alpha_1^2}\right) + O(r^4)\,, \tag{A.16}$$

where c_2^4 is given in (A.12).

How shall we fix the small r asymptotics? Let us first note that there is no singularity in $e^{2\Phi}$ which must be positive for all r so the conformal factor in (A.15) does not introduce any potential singularities. On the other hand, f_r appearing in (A.13) does have a singularity at $r = \alpha_e$. Comparing with (A.14) tells us that

$$\alpha_e^2 \sim \widetilde{M}\,. \tag{A.17}$$

We still need to relate the coordinates of our ansatz to the dimensionless coordinates of the BTZ solution. There is a natural place to begin such an identification with,

$$e^{-4\Phi}\frac{1}{f_x(r)}dx_5^2 \;\to\; \ell^2\tilde{r}^2 d\tilde{\varphi}^2\,, \tag{A.18}$$

for small r. There is no freedom to redefine the angular coordinate which allows us to unambiguously identify:

$$e^{-4\Phi}r^2 = \ell^2\frac{\alpha_x^2}{R^2}\tilde{r}^2 = \ell^2\frac{k_x}{k_1}\frac{\alpha_1^2}{R^2}\tilde{r}^2 \quad\Rightarrow\quad r^2 = \ell^2\frac{k_x}{k_1}\frac{c_2^4}{\alpha_1^2 R^2}\tilde{r}^2 + O(\tilde{r}^4)\,. \tag{A.19}$$

Once we have this relation between the dimensionless $\tilde{r}$ and r, we can determine a small $\tilde{r}$ regime where the dilaton (A.11) is approximately constant. This requires that $\frac{d\Phi}{d\tilde{r}}$ be small:

$$k_x\frac{\ell^2 c_2^4}{\alpha_1^4 R^2}\tilde{r} = \sqrt{k_1 k_x}\frac{\ell c_2^2}{\alpha_1^3 R}r \ll 1. \tag{A.20}$$

In this regime we demand that the solution looks like a BTZ black hole in approximately AdS space.

Now we can examine the metric coefficient of dr^2 in (A.5),

$$e^{-4\Phi}f_r(r)dr^2 = \left(\frac{\alpha_1^4}{c_2^4}\left(1+\frac{2k_1 r^2}{\alpha_1^2}\right)+\ldots\right)\left(\frac{r_5^2}{r^2-\alpha_e^2}\right)dr^2\,. \tag{A.21}$$

Comparison with (A.14) teaches us that

$$\ell^2 = \frac{\alpha_1^4}{c_2^4}r_5^2, \qquad \alpha_e^2 = 8\widetilde{M}\frac{k_x}{k_1}\frac{\alpha_1^2 r_5^2}{R^2}\,. \tag{A.22}$$

As expected, α_e^2 is related to the mass of the black hole. Interestingly the definition of ℓ looks mass-dependent when $k_1 \neq 0$ unlike the case of constant dilaton. We can also simplify (A.19),

$$\frac{r^2}{\tilde{r}^2} = \frac{k_x}{k_1}\frac{\alpha_1^2 r_5^2}{R^2}, \tag{A.23}$$

and the regime where the solution will look approximately like AdS:

$$\sqrt{k_1 k_x}\frac{r_5 r}{\alpha_1 R} \ll 1. \tag{A.24}$$

The remaining task is to relate t and $\tilde{t}$, which follows by requiring the singularity at $r^2 = \alpha_e^2$ be a coordinate singularity. Let us recall that $e^{-4\Phi}\frac{1}{f_1(r)}$ is positive and introduces neither singularities nor changes to the signature of the time coordinate. On the other hand, $f_r(r)$ does change sign at $r = \alpha_e$ and this change must be matched by the behavior of the dt^2 metric coefficient. This constrains the dt^2 metric coefficient to match BTZ,

$$\frac{\alpha_1^2}{c_2^4}\left(r^2 - \alpha_e^2\right)\left\{1 + O(r^4)\right\} dt^2 = \ell^2\left(\tilde{r}^2 - 8\widetilde{M}\right) d\tilde{t}^2\,. \tag{A.25}$$

From the $O(r^2)$ term we conclude:

$$d\tilde{t}^2 = \frac{r^2}{\tilde{r}^2}\frac{\alpha_1^2}{c_2^4\ell^2}\, dt^2 \qquad \Longrightarrow \qquad \tilde{t} = \frac{t}{R}\sqrt{\frac{k_x}{k_1}}\,. \tag{A.26}$$

The other r-independent condition from (A.25) is then automatically satisfied.

Matching large and small r asymptotics

To go further we need to examine the large r asymptotics. In essence, the issue is what large r criteria define metrics which are asymptotically linear dilaton? To understand the large r asymptotics, we need an appropriate large r coordinate. From the $e^{-4\Phi}f_r(r)dr^2$ term with the explicit form for f_r given in (A.13), we see that the natural radial coordinate for large r takes the form,

$$\rho = \frac{r_5 k_1}{2c_2^2} r^2 = \frac{r^2}{\alpha_\rho}\,, \qquad \alpha_\rho = \frac{2c_2^2}{r_5 k_1}\,. \tag{A.27}$$

In terms of ρ, the three-dimensional asymptotic metric takes the form:

$$ds^2 = -\frac{4\rho^2}{r_5^2}\left(\frac{1}{k_1} + O\left(\frac{1}{\rho}\right)\right) dt^2 + \frac{4\rho^2}{r_5^2}\left(\frac{1}{k_x} + O\left(\frac{1}{\rho}\right)\right) dx_5^2 + d\rho^2\,. \tag{A.28}$$

What is important is that the leading term at large ρ is completely independent of the mass $\widetilde{M}$. This defines the vacuum to which each solution should asymptote with mass-dependent subleading corrections.

Our remaining task is to identify α_1^2. We have one example of an asymptotically linear dilaton metric in (2.9) which corresponds to $\widetilde{M} = 0$ BTZ. This case turns out to be

somewhat special when compared with black holes with $\widetilde{M} \neq 0$. For example, any non-zero value of k_1 can be set to 1 by a redefinition of the radial coordinate.[9] This is not the case when $\widetilde{M} \neq 0$. Regardless, we can still try to use this case to normalize α_1^2 in our metric ansatz (A.7). Let us examine the large ρ behavior of our dilaton:

$$\begin{aligned}\Phi &= -\frac{1}{2}\log\rho + \frac{1}{2}\log\frac{r_5}{2} - \frac{1}{2}\log\left(1 + \frac{\alpha_1^2 r_5}{2c_2^2}\frac{1}{\rho}\right) \\ &= -\frac{1}{2}\log\rho + \frac{1}{2}\log\frac{r_5}{2} - \frac{\alpha_1^2 r_5}{4c_2^2}\frac{1}{\rho} + O\left(\frac{1}{\rho^2}\right) .\end{aligned} \tag{A.29}$$

All the potential mass-dependence, which is hidden in c_2^2, is ρ-suppressed. If we set $\alpha_e = 0$ and compare these expressions for the dilaton and metric with the expressions found in (2.9), we can identify α_1^2:

$$\alpha_1^2 = \hat{r}_1^2 . \tag{A.30}$$

At this point we could redefine time to absorb the $\frac{1}{k_1}$ factor with the replacement $\alpha_1^2 \to \frac{\alpha_1^2}{k_1}$. However the AdS limit where $k_1 \to 0$ is more transparent with the explicit k_1. On the other hand, redefining the periodicity of x_5 so that $k_x \to k_1$ does look natural. With this redefinition, $\alpha_x \to \alpha_1$ using (A.8). The final string-frame metric and dilaton take the form,

$$\begin{aligned} ds^2 &= -\frac{1 - \frac{\alpha_e^2}{r^2}}{k_1 + \frac{\hat{r}_1^2}{r^2}}\, dt^2 + \frac{1}{k_1 + \frac{\hat{r}_1^2}{r^2}}\, dx_5^2 + \frac{r_5^2}{r^2 - \alpha_e^2}\, dr^2 + r_5^2\, d\Omega_3^2 + ds_{T^4}^2 , \\ e^{2\Phi} &= \frac{r_5^2\sqrt{1 + 8k_1\frac{\widetilde{M} r_5^2}{R^2}}}{\hat{r}_1^2 + k_1 r^2} , \qquad \alpha_e^2 = 8\widetilde{M}\frac{\hat{r}_1^2 r_5^2}{R^2} . \end{aligned} \tag{A.31}$$

Note that the x_5 circle has periodicity $\frac{2\pi R}{\sqrt{k_1}}$ at $r = \infty$ in string-frame.

The case of $c_1 \neq 0$

In solving for the dilaton, we set $c_1 = 0$ in (A.9). In this subsection we will examine whether there are any sensible solutions with $c_1 > 0$. The case of $c_1 < 0$ is ruled out because there would be a singularity in the dilaton at some finite r. If a solution exists with $c_1 > 0$, it would be quite curious because it interpolates from one approximately constant value of the dilaton near $r = 0$ to a different approximately constant value near $r = \infty$.

[9] There is a similar special feature for $E = 0$ states under a $T\overline{T}$ deformation; namely the energy does not flow. This is readily visible from (1.3).

The first step is to solve for c_2^4 using (A.10). This is just algebraic and gives:

$$c_2^4 = \frac{r_5^4}{\hat{r}_1^4}\left\{\frac{\alpha_1^4 + k_1\alpha_1^2\alpha_e^2}{(k_1c_1\alpha_1^2+1)(k_1c_1(\alpha_1^2+k_1\alpha_e^2)+1)}\right\}. \tag{A.32}$$

The $r \to \infty$ UV value of the dilaton is determined by $e^{2\Phi} = k_1c_1c_2^2$ while the $r \to 0$ IR value,

$$e^{2\Phi} = \left(\frac{1+k_1c_1\alpha_1^2}{\alpha_1^2}\right)c_2^2 > k_1c_1c_2^2\,, \tag{A.33}$$

is strictly larger. If the asymptotic solutions are both approximately AdS solutions with the same string-frame length scale ℓ then this relation would be in accord with the two-dimensional c-theorem. This suggests a potential holographic interpretation of this class of solutions as describing a flow from a UV two-dimensional CFT to an IR CFT. What is particularly interesting about this possibility is that by taking c_1 very small, this background would have a long intermediate region that looks like a linear dilaton but which eventually becomes AdS_3 again.

When $c_1 > 0$, f_r is constrained to be

$$f_r(r) = \frac{r_5^2(1+k_1\alpha_1^2c_1)(1+k_1c_1(\alpha_1^2+k_1\alpha_e^2))}{(r-\alpha_e)(r+\alpha_e)(1+k_1c_1(\alpha_1^2+k_1r^2))^2}\,, \tag{A.34}$$

which diverges at $r = \alpha_e$ just like the $c_1 = 0$ solution. Again it appears that this is only a coordinate singularity and not a genuine curvature singularity. The full Ricci scalar associated with this ten-dimensional solution is given by

$$R = \frac{2k_1}{r_5^2}\times \\ \frac{(3k_1r^4 + 10r^2\alpha_1^2 + 2k_1r^2\alpha_e^2 - 5\alpha_1^2\alpha_e^2 + 5k_1c_1(k_1r^2+\alpha_1^2)(r^2(2\alpha_1^2+k_1\alpha_e^2) - \alpha_1^2\alpha_e^2))}{(k_1r^2+\alpha_1^2)^2(1+k_1c_1\alpha_1^2)\,(1+k_1c_1(\alpha_1^2+k_1\alpha_e^2))}\,. \tag{A.35}$$

If $k_1 > 0$ and $\alpha_1^2 > 0$, the Ricci scalar is finite for all values of r. It appears to interpolate from a constant negative value at small r to a constant positive value at large r. One finds

$$\lim_{r\to 0} R = -\frac{10k_1\alpha_e^2}{r_5^2\alpha_1^2(1+k_1\alpha_1^2c_1+k_1^2\alpha_e^2c_1)}\,, \tag{A.36}$$

while

$$\lim_{r\to\infty} R = \frac{2\,(3+10k_1\alpha_1^2c_1+5c_1k_1^2\alpha_e^2)}{r_5^2(1+k_1c_1\alpha_1^2)(1+k_1\alpha_1^2c_1+k_1^2\alpha_e^2c_1)}\,. \tag{A.37}$$

We can also compute the Ricci scalar $R^{(3)}$ associated with the three-dimensional spacetime $\mathcal{M}_3$ parameterized by t, r, x_5. This is given by

$$R^{(3)} = 2\left(1 + k_1 c_1 (k_1 r^2 + \alpha_1^2)\right) \times \frac{(4k_1 r^2 \alpha_1^2 - 3\alpha_1^4 + 2k_1^2 r^2 \alpha_e^2 - 5k_1 \alpha_1^2 \alpha_e^2 - 3k_1 c_1 \alpha_1^2 (k_1 r^2 + \alpha_1^2)(\alpha_1^2 + k_1 \alpha_e^2))}{r_5^2 (k_1 r^2 + \alpha_1^2)^2 (1 + k_1 c_1 \alpha_1^2)(1 + k_1 c_1 (\alpha_1^2 + k_1 \alpha_e^2))} \,. \tag{A.38}$$

Again we can look at the small r and large r limits,

$$\begin{aligned} \lim_{r\to 0} R^{(3)} &= -\frac{2\left(3\alpha_1^2 + 5k_1\alpha_e^2 + 3k_1 c_1 \alpha_1^2 (\alpha_1^2 + k_1 \alpha_e^2)\right)}{r_5^2 \alpha_1^2 (1 + k_1 c_1 (\alpha_1^2 + k_1 \alpha_e^2))} \,, \\ \lim_{r\to \infty} R^{(3)} &= \frac{2c_1 \left(4k_1\alpha_1^2 + 2k_1^2 \alpha_e^2 - 3k_1^2 c_1 \alpha_1^2 (\alpha_1^2 + k_1 \alpha_e^2)\right)}{r_5^2 (1 + k_1 c_1 \alpha_1^2)(1 + k_1 c_1 (\alpha_1^2 + k_1 \alpha_e^2))} \,. \end{aligned} \tag{A.39}$$

We see that the small r limit of $R^{(3)}$ is negative-definite, corresponding to an AdS_3 in the deep interior, but that the large r value of $R^{(3)}$ has competing contributions with opposite signs. In order to have negative curvature at large r, we require

$$c_1 > \frac{2}{3k_1}\left(\frac{1}{\alpha_1^2} + \frac{1}{\alpha_1^2 + k_1 \alpha_e^2}\right) \,. \tag{A.40}$$

Notice that this condition is easy to violate by taking k_1 small. The surprising result is that we seem to be able to construct a three-dimensional spacetime $\mathcal{M}_3$ with *positive* scalar curvature at least in string-frame!

Our remaining task with these solutions is to determine the parameter map following the analysis of the $c_1 = 0$ case. We will highlight the differences. For simplicity, we assume $k_x = k_1$. The relation between r^2 and $\tilde{r}^2$ given in (A.19) now becomes,

$$r^2 = \ell^2 \frac{c_2^4}{\alpha_1^2 R^2} \frac{\tilde{r}^2}{(1 + k_1 c_1 \alpha_1^2)^2} + O(\tilde{r}^4) \,. \tag{A.41}$$

Also from,

$$e^{-4\Phi} f_r(r) dr^2 = e^{-4\Phi} \frac{(1 + k_1 \alpha_1^2 c_1)(1 + k_1 c_1 (\alpha_1^2 + k_1 \alpha_e^2))}{(1 + k_1 c_1 (\alpha_1^2 + k_1 r^2))^2} \frac{r_5^2 \, dr^2}{r^2 - \alpha_e^2} \,, \tag{A.42}$$

we can read off

$$\ell^2 = \frac{\alpha_1^4 r_5^2}{c_2^4} \frac{(1 + k_1 c_1 (\alpha_1^2 + k_1 \alpha_e^2))}{(1 + k_1 c_1 \alpha_1^2)^3} \,, \tag{A.43}$$

which has additional mass dependence beyond the $c_1 = 0$ case. The last step is to relate α_e to $\widetilde{M}$ which now gives the following relations:

$$r^2 = \frac{\alpha_1^2 r_5^2}{R^2} \frac{(1 + k_1 c_1 (\alpha_1^2 + k_1 \alpha_e^2))}{(1 + k_1 c_1 \alpha_1^2)^5} \tilde{r}^2 \,, \tag{A.44}$$

$$\alpha_e^2 = \frac{8\widetilde{M} r_5^2 \alpha_1^2 (1 + c_1 k_1 \alpha_1^2)}{R^2 (1 + c_1 k_1 \alpha_1^2)^5 - 8 c_1 k_1^2 \widetilde{M} r_5^2 \alpha_1^2} \,. \tag{A.45}$$

Adding spin $\widetilde{J}$

Now let us return to the case $c_1 = 0$ and extend our solutions (A.31) to include spin. This will also help us identify the relation between the deformation parameter k_1 and λ of (1.3). The BTZ solution with spin takes the form,

$$\begin{aligned}\widehat{ds}^2 =&\ell^2 \left(-\left(-8\widetilde{M} + \tilde{r}^2 + \frac{16\widetilde{J}^2}{\tilde{r}^2}\right) d\tilde{t}^2 + \tilde{r}^2 \left(d\tilde{\varphi} - \frac{4\widetilde{J}}{\tilde{r}^2} d\tilde{t}\right)^2 \right. \\ &\left. + \frac{d\tilde{r}^2}{-8\widetilde{M} + \tilde{r}^2 + \frac{16\widetilde{J}^2}{\tilde{r}^2}} \right) . \end{aligned} \tag{A.46}$$

In parallel with the mass $\widetilde{M} = MG_3$, we have absorbed factors of the gravitational coupling into the angular momentum via $\widetilde{J} = \frac{JG_3}{\ell}$ with G_3 defined in (2.12). This redefinition is quite important because we have already seen that ℓ is mass-dependent in (A.22). Soon we will see that it also becomes spin-dependent when $k_1 \neq 0$. We will use the following ansatz for f_j and f_e which have the expected behavior at small r,

$$f_j(r) = -\frac{\alpha_j^2}{r^2}, \qquad f_e(r) = 1 - \frac{\alpha_e^2}{r^2} + \frac{\tilde{\alpha}_e^4}{r^4}. \tag{A.47}$$

We can then try solving the equations of motion for the dilaton, which again gives

$$e^{2\Phi} = c_2^2 \left(\frac{1}{\alpha_1^2 + k_1 r^2} \right) , \tag{A.48}$$

where c_2 is an unknown constant, as in the solution (A.9) with $\widetilde{J} = 0$. We also find the algebraic constraints

$$\alpha_x = \sqrt{\frac{k_x}{k_1}} \alpha_1 , \qquad \tilde{\alpha}_e = \left(\frac{k_1}{k_x}\right)^{1/4} \alpha_j , \qquad \alpha_5 = r_5 . \tag{A.49}$$

As in the discussion around equation (A.31), we can rescale x_5 to set $k_x = k_1$ which significantly simplifies the number of parameters. In the spinning case, the relationship between c_2 and the other parameters, imposed by the dilaton equation of motion, is modified. In this case we find

$$c_2^4 = \frac{r_5^4 \left(\alpha_1^4 + k_1 \left(\alpha_1^2 \alpha_e^2 + k_1 \alpha_j^4\right)\right)}{\hat{r}_1^4} . \tag{A.50}$$

Likewise, the equations of motion determine $f_r(r)$ to be

$$f_r(r) = \frac{r_5^2}{r^2 - \alpha_e^2 + \frac{\alpha_j^4}{r^2}} . \tag{A.51}$$

In determining the parameter map, we see that the relations,

$$\frac{r^2}{\tilde{r}^2} = \frac{\alpha_1^2 r_5^2}{R^2}\,, \qquad \ell^2 = \frac{\alpha_1^4}{c_2^4} r_5^2\,, \tag{A.52}$$

remain unchanged from the case without spin. With this identification, we can determine α_j in terms of $\widetilde{J}$ appearing in the metric (A.46). The final string-frame metric and dilaton take the form,

$$\begin{aligned} ds^2 &= -\frac{1-\frac{\alpha_e^2}{r^2}+\frac{\alpha_j^4}{r^4}}{k_1+\frac{\hat{r}_1^2}{r^2}}\,dt^2 + \frac{1}{k_1+\frac{\hat{r}_1^2}{r^2}}\left(dx_5 - \frac{\alpha_j^2}{r^2}dt\right)^2 + \frac{r_5^2}{r^2-\alpha_e^2+\frac{\alpha_j^4}{r^2}}\,dr^2 + r_5^2\,d\Omega_3^2 + ds_{T^4}^2\,, \\ e^{2\Phi} &= \frac{r_5^2\sqrt{1+8k_1\frac{\widetilde{M}r_5^2}{R^2}+16\,(k_1)^2\,\frac{\widetilde{J}^2 r_5^4}{R^4}}}{\hat{r}_1^2+k_1 r^2}\,, \qquad \alpha_e^2 = 8\widetilde{M}\frac{\hat{r}_1^2 r_5^2}{R^2}\,, \qquad \alpha_j^2 = 4\widetilde{J}\,\frac{\hat{r}_1^2 r_5^2}{R^2}\,. \end{aligned} \tag{A.53}$$

The form of the dilaton solution is eerily reminiscent of the solution of the $T\overline{T}$-deformed energy flow equation (1.3).

A.2 The asymptotically flat case where $k_5 > 0$

We will now attempt to recouple the asymptotic region by taking $k_5 > 0$ for the general class of metrics with mass and spin. This means that asymptotic theory will be 6-dimensional rather than 3-dimensional because the 3-sphere will decompactify as $r \to \infty$, while the T^4 remains compact.

Interpolating Ansatz

To simplify our analysis, we take the following ansatz for the ten-dimensional metric which is a slight generalization of the prior cases,

$$\begin{aligned} f_1(r) &= k_1 + \frac{\alpha_1^2}{r^2}, \qquad f_e(r) = 1 - \frac{\alpha_e^2}{r^2} + \frac{\tilde{\alpha}_e^4}{r^4}, \qquad f_5(r) = k_5 + \frac{\alpha_5^2}{r^2}, \\ f_x(r) &= k_x + \frac{\alpha_x^2}{r^2}\,, \qquad f_j(r) = -\frac{\alpha_j^2}{r^2}\,. \end{aligned} \tag{A.54}$$

We also demand that $k_1 > 0, k_x > 0$ and $k_5 > 0$ for a sensible metric at large r. We can repeat the steps of the preceeding analysis and see what constraints emerge from trying to solve the spacetime equations of motion. Firstly, there are algebraic constraints generalizing what we found in the $k_5 = 0$ case:

$$\tilde{\alpha}_e = \left(\frac{k_1}{k_x}\right)^{1/4}\alpha_j\,, \qquad \alpha_x = \sqrt{\frac{k_x}{k_1}}\alpha_1\,, \qquad \alpha_e = \frac{\sqrt{k_x(r_5^4-\alpha_5^4)-k_1 k_5^2\alpha_j^4}}{\alpha_5\sqrt{k_5 k_x}}\,. \tag{A.55}$$

Once again it is natural to rescale the x_5 circle to set $k_x = k_1$ and replace both $\tilde{\alpha}_e$ by α_j and α_x by α_1. With these replacements the solution for $f_r(r)$ is given by,

$$f_r(r) = \frac{k_5\alpha_5^2 r^2(k_5 r^2+\alpha_5^2)}{r^2\left(\alpha_5^4+k_5\alpha_5^2 r^2-r_5^4\right)+k_5\alpha_j^4\left(k_5 r^2+\alpha_5^2\right)}\,, \tag{A.56}$$

while the dilaton takes the form:

$$e^{2\Phi} = \frac{k_5 r^2+\alpha_5^2}{\hat{r}_1^2\left(k_1 r^2+\alpha_1^2\right)}\cdot\sqrt{\alpha_1^4+k_1^2\alpha_j^4-\frac{k_1\alpha_1^2\left(\alpha_5^4+k_5\alpha_j^4-r_5^4\right)}{k_5\alpha_5^2}}\,. \tag{A.57}$$

It is more convenient to invert the relation between α_5 and α_e,

$$\alpha_5^2 = \frac{1}{2}\left(\sqrt{k_5^2\alpha_e^4+4\left(r_5^4-k_5^2\alpha_j^4\right)}-k_5\alpha_e^2\right) \tag{A.58}$$

As a check, notice that $\alpha_5 \to r_5$ when we take $k_5 \to 0$ which is the solution we found earlier in (A.8). In this more general case, α_5 is mass-dependent. In a small k_5 expansion, we see that

$$\alpha_5^2 = r_5^2 - \frac{k_5\alpha_e^2}{2} + O(k_5^2)\,. \tag{A.59}$$

Substituting (A.58) into (A.57) gives the following nicer form for the dilaton:

$$e^{2\Phi} = \frac{k_5 r^2+\alpha_5^2}{k_1 r^2+\alpha_1^2}\cdot\sqrt{\frac{\alpha_1^4+k_1\alpha_1^2\alpha_e^2+k_1^2\alpha_j^4}{\hat{r}_1^4}}\,. \tag{A.60}$$

Let us count parameters versus expectations. First α_e^2 and α_j^2 are determined by the mass and spin of the deep interior BTZ black hole via (A.53). Together with k_5, these parameters determine α_5^2 using (A.58). That leaves α_1, k_1 and k_5 to be identified with physical parameters of the asymptotic solution. Now the existence of a non-zero k_5 is the statement that we have not taken the partial decoupling limit. Therefore it should be possible to identify k_5 with the asymptotic value of the string coupling, called g_s in (2.6).

To determine the precise identifications, we can start by examining the asymptotic value of the dilaton. This cannot be mass or spin-dependent because it is part of the data defining the quantum gravity Hilbert space. Unlike the prior discussion, we now need to make a conformal transformation in a 6-dimensional gravity theory rather than a 3-dimensional theory. The Einstein frame metric is now given by,

$$ds^2_{\text{Einstein}} = e^{-(\Phi-\Phi_0)}ds^2_{\text{string}}\,, \tag{A.61}$$

where Φ_0 is the now finite $r \to \infty$ value of the dilaton. We will require that Φ_0 be independent of mass and spin since it determines the gravitational constant. To determine how to implement this condition, it is useful to examine the nicer form of the dilaton appearing in (A.60). It is not possible to make the square root independent of mass and spin by modifying α_1 in a way proportional to k_5 so we expect that the relation

$$\alpha_1 = \hat{r}_1 \,, \tag{A.62}$$

is unchanged. This means we must absorb the mass and spin-dependence in k_5. So we define

$$\hat{k}_5 = k_5 \cdot \gamma, \qquad \gamma = \sqrt{1 + \frac{k_1 \alpha_e^2}{\alpha_1^2} + \frac{k_1^2 \alpha_j^4}{\alpha_1^4}} \,. \tag{A.63}$$

In terms of $\hat{k}_5$, the dilaton now takes the form

$$e^{2\Phi} = \frac{\hat{k}_5 r^2 + \gamma \alpha_5^2}{k_1 r^2 + \alpha_1^2} \,. \tag{A.64}$$

As $r \to \infty$ the Einstein frame metric now contains the terms,

$$e^{-(\Phi - \Phi_0)} ds^2_{\text{string}} = \frac{\hat{k}_5}{\gamma} \left(dr^2 + r^2 d\Omega_3^2 \right) + \dots \,. \tag{A.65}$$

The natural radial coordinate at large distances is now

$$\rho^2 = \frac{\hat{k}_5}{\gamma} r^2 \,, \tag{A.66}$$

and the dilaton takes the form

$$\begin{aligned} e^{2\Phi} &= \frac{\hat{k}_5}{k_1} + \frac{k_1 \alpha_5^2 \gamma - \hat{k}_5 \alpha_1^2}{k_1^2 r^2} + O\left(\frac{1}{r^4}\right) \,, \\ &= \frac{\hat{k}_5}{k_1} + \frac{\hat{k}_5}{\gamma \rho^2} \frac{k_1 \alpha_5^2 \gamma - \hat{k}_5 \alpha_1^2}{k_1^2} + O\left(\frac{1}{\rho^4}\right) \,. \end{aligned} \tag{A.67}$$

The mass-dependence is now encoded in the subleading terms of the ρ expansion. Now we can identify $g_s^2 = \frac{\hat{k}_5}{k_1}$, or equivalently $\hat{k}_5 = k_1 g_s^2$. The periodicity of the x_5 circle is $\frac{2\pi R}{\sqrt{k_1}}$ at $r = \infty$.

The final string-frame metric and dilaton take the form,

$$
\begin{aligned}
ds^2 &= -\frac{1-\frac{\alpha_e^2}{r^2}+\frac{\alpha_j^4}{r^4}}{k_1+\frac{\hat{r}_1^2}{r^2}}\,dt^2+\frac{1}{k_1+\frac{\hat{r}_1^2}{r^2}}\left(dx_5-\frac{\alpha_j^2}{r^2}dt\right)^2 \\
&\quad+\frac{k_5 r^2-\frac{k_5\alpha_e^2}{2}+\sqrt{r_5^4-k_5^2\alpha_j^4+\frac{1}{4}k_5^2\alpha_e^4}}{r^4-r^2\alpha_e^2+\alpha_j^4}\,r^2\,dr^2+r^2\left(k_5+\frac{\alpha_5^2}{r^2}\right)d\Omega_3^2+ds_{T^4}^2\,, \\
e^{2\Phi} &= \frac{g_s^2+\frac{\gamma\alpha_5^2}{k_1 r^2}}{1+\frac{\hat{r}_1^2}{k_1 r^2}}\,, \qquad \alpha_e^2=8\widetilde{M}\frac{\hat{r}_1^2 r_5^2}{R^2}\,, \qquad \alpha_j^2=4\widetilde{J}\,\frac{\hat{r}_1^2 r_5^2}{R^2}\,, \qquad k_5=\frac{k_1 g_s^2}{\gamma}\,, \\
\alpha_5^2 &= \frac{1}{2}\left(\sqrt{k_5^2\alpha_e^4+4\left(r_5^4-k_5^2\alpha_j^4\right)}-k_5\alpha_e^2\right)\,, \qquad \gamma=\sqrt{1+\frac{k_1\alpha_e^2}{\hat{r}_1^2}+\frac{k_1^2\alpha_j^4}{\hat{r}_1^4}}\,.
\end{aligned}
\tag{A.68}
$$

This final form for the metric depends on $(g_s, k_1, \hat{r}_1, r_5)$ along with the mass and spin $(\widetilde{M}, \widetilde{J})$ of the interior BTZ black hole.

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