

CONVERGENCE OF NORMALIZED BETTI NUMBERS IN NONPOSITIVE CURVATURE

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Abstract. We study the convergence of volume-normalized Betti numbers in Benjamini-Schramm convergent sequences of non-positively curved manifolds with finite volume. In particular, we show that if X is an irreducible symmetric space of noncompact type, $X = \mathbb{H}^3$, and (M_n) is any Benjamini-Schramm convergent sequence of finite volume X -manifolds, then the normalized Betti numbers $b_k(M_n)/\text{vol}(M_n)$ converge for all k .

As a corollary, if X has higher rank and (M_n) is any sequence of distinct, finite volume X -manifolds, the normalized Betti numbers of M_n converge to the L^2 Betti numbers of X . This extends our earlier work with Nikolov, Raimbault and Samet in [1], where we proved the same convergence result for uniformly thick sequences of compact X -manifolds. One of the novelties of the current work is that it applies to all quotients $M = \Gamma \backslash X$ where Γ is arithmetic; in particular, it applies when Γ is isotropic.

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1. Introduction

We begin with a fair amount of general motivation, mostly from Elek [16] and Bowen [10]. The well-versed reader can skip ahead to §1.1 for the statements of our results.

The normalized Betti numbers of a space X are the quotients

$$b_k(X)/\text{vol}(X), \text{ where } b_k(X) := \dim H_k(X, \mathbb{R}).$$

All spaces in this paper will be either Riemannian manifolds or simplicial complexes. In the latter case, volume should be interpreted as the number of vertices.

Fix $d > 0$. A simplicial complex K has degree at most d if every vertex in K is adjacent to at most d edges. In [16], Elek shows that the normalized Betti numbers of finite simplicial complex K with degree at most d are testable, meaning that there is a way to read off approximations of the normalized Betti numbers while only looking at bounded random samples of K . More precisely, given $\varrho > 0$, there is some $R(\varrho)$ as follows. Given K , select R vertices of K at random and look at the R -neighborhood of each in K . Testability means there is a way to guess from this data what the normalized Betti numbers of K are, that is correct up to an error of ϱ with probability $1 - \varrho$.

This is really a continuity result, in the following sense. Consider the topological space
 $K = \text{connected, pointed finite degree simplicial complexes } (K, p) / \boxtimes,$

where each $p \boxtimes K$ is a vertex, two pointed complexes are equivalent if they are isomorphic via a map that takes basepoint to basepoint, and where two complexes are close if for large R , the R -balls around their basepoints are isomorphic. Each finite (even possibly disconnected) complex K induces a finite measure μ_K on K , defined by pushing forward the counting measure on the vertex set $V(K)$ under the map

$$V(K) \rightarrow K, p \rightarrow [(K_p, p)],$$

where $K_p \boxtimes K$ is the connected component of p . One then says that a sequence (K_n) in K Benjamini-Schramm (BS) converges¹ if the probability measures $\mu_{K_n}/\text{vol}(K_n)$ weakly converge to some probability measure on K . One can then reformulate the testability of normalized Betti numbers above as saying:

Theorem 1.1 (Elek [16, Lemma 6.1]). If (K_n) is a BS-convergent sequence of finite, simplicial complexes, each with degree at most d , the normalized Betti numbers $b_k(K_n)/\text{vol}(K_n)$ converge for all k .

Informally, the relationship with testability is that if we fix $R > 0$ and take $n, m > > 0$, convergence says the measures associated to the two complexes K_n, K_m will be close. So by the definition of the topology on K , we will have that for large R , the distribution of randomly sampled R -balls in K_n will be almost the same as that in K_m , so having a way

¹Benjamini-Schramm convergence of graphs was first studied in their paper [8]. See also Aldous–Lyons [3] for a broader picture of BS-convergence in the case of graphs.

to accurately guess the normalized Betti numbers from these (nearly identical) data sets means that the normalized Betti numbers of K_n and K_m must be close.

Recently, a number of authors, see e.g. [1, 2, 9, 10, 26], have studied the analogous version of BS-convergence for Riemannian manifolds. Adopting the language of [2], set

$$M = \{\text{pointed Riemannian manifolds } (M, p)\}/\text{pointed isometry},$$

endowed with the topology of pointed smooth convergence. See §2.1. Here and below, Riemannian manifolds are always assumed to be connected and complete. Really, all of the results below hold for disconnected manifolds, just as Theorem 1.1 applies to disconnected complexes, but it seems unnecessarily confusing to continue working in this generality.

A finite volume (connected, complete) Riemannian manifold M induces a finite measure μ_M on M , by pushing forward the Riemannian measure on M via the map $p \rightarrow [(M, p)]$, and we say that a sequence (M_n) Benjamini-Schramm (BS) converges if the measures $\mu_M / \text{vol}(M_n)$ weakly converge to some probability measure. In full generality, the Riemannian analogue of Theorem 1.1 is not true, since if no geometric constraints are imposed, we can pack as much homology as desired into a part of a manifold with negligible volume. For example: connect sum a small volume genus $g(n)$ surface, say with volume 1, somewhere on a round radius- n sphere. The resulting surfaces will BS-converge to an atomic measure on the single point $[(R^2, p)] \in M$, where $p \in R^2$ is any basepoint. But by choosing $g(n)$ appropriately, we can make the first Betti numbers whatever we like.

In the above example, the real problem is injectivity radius. For a Riemannian manifold M and a point $x \in M$ we denote the injectivity radius of M at x by $\text{inj}_M(x)$. Given $\eta > 0$, the η -thick part and the η -thin part of M are

$$M_{\geq \eta} = \{x \in M : \text{inj}_M(x) \geq \eta/2\} \text{ and } M_{< \eta} = M \setminus M_{\geq \eta}.$$

One says that M is η -thick if $M = M_{\geq \eta}$. Now, under geometric constraints like curvature bounds, there is a standard way to model an η -thick manifold M by a simplicial complex $K(M)$ with comparable volume and bounded degree: one selects an η -net S in M , and lets $N(S)$ be the nerve of the covering of M by η -balls. One can then show:

Theorem 1.2 (Elek, Bowen + ABBG²). If (M_n) is a BS-convergent sequence of compact, η -thick Riemannian manifolds with upper and lower curvature bounds, then the normalized Betti numbers $b_k(M_n)/\text{vol}(M_n)$ converge.

A word is in order about the attributions: it was originally conceived by Elek, and then written up and published by Bowen [10, Theorem 4.1], but this writeup was not complete, and we (ABBG) provide a slightly different argument that avoids this gap in §2.3. Briefly, the idea is to superimpose a bunch of Poisson processes on M_n , discarding points that are too close together, until enough points are laid down so that the nerve complex N_n associated to a collection of balls around these points sees the Betti numbers of M_n up to a small error. One then proves that the constructed sequence of (random) nerve complexes

²By (ABBG) we refer to the current paper.

BS-converges. By (a slight generalization of) Theorem 1.1 above, the expected normalized Betti numbers $E[b_k(N_n)/\text{vol}(N_n)]$ will converge, from which one can deduce convergence of the normalized Betti numbers $b_k(M_n)/\text{vol}(M_n)$.

Theorem 1.2 is really a special case of a more general result, see §2.3. Indeed, the essence of the current work is that we deal with general manifolds with no assumptions on the injectivity radius. The thick part $M_{\geq 0}$ is then a proper submanifold with boundary and we rely on Gelander's techniques [18] in order to associate a random simplicial complex to the thick part. As shown in [5] the boundary of the thick part corresponds to a sub-simplicial complex. This allows us to consider the thick and the thin parts separately.

1.1. Main results. Our interest in this paper is whether for certain manifolds of nonpositive curvature, one can control the thin parts well enough so that BS-convergence implies convergence of normalized Betti numbers, without any assumption of thickness.

Although almost all of the real work in this paper is done more generally, we start as follows. Let X be an irreducible symmetric space of noncompact type. An x -manifold is a complete Riemannian manifold whose universal cover is isometric to x .

Theorem 1.3. Suppose that $\dim(X) = 3$ and (M_n) is a BS-convergent sequence of finite volume x -manifolds. Then for all k , the sequence $b_k(M_n)/\text{vol}(M_n)$ converges.

Here, the only three-dimensional irreducible symmetric spaces of noncompact type are scales of H^3 . In fact, the conclusion of Theorem 1.3 is false when $X = H^3$. As an example, let $K \sqsubset S^3$ be a knot such that the complement $M = S^3 \setminus K$ admits a hyperbolic metric, e.g. the figure-8 knot. Using meridian–longitude coordinates, let M_n be obtained by Dehn filling M with slope $(1, n)$; then each M_n is a homology 3-sphere. The manifolds $M_n \rightarrow M$ geometrically, see [7, Ch E.6], so the measures μ_{M_n} weakly converge to μ_M (c.f. [5, Lemma 6.4]) and the volumes $\text{vol}(M_n) \rightarrow \text{vol}(M)$. However, $b_1(M_n) = 0$ while $b_1(M) = 1$, so the normalized Betti numbers of the BS-convergent sequence M_1, M, M_2, M, \dots do not converge. See also Example 3.1 for a similar counterexample in which volume goes to infinity. In fact, there is a real sense in which the only counterexamples come from Dehn filling. See §3.

To illustrate a special case of Theorem 1.3, let's say that (M_n) BS-converges to X when the measures μ_{M_n} weakly converge to the atomic probability measure on the point

$$[(X, x)] \sqsubset M,$$

where $x \sqsubset X$ is any basepoint. Now any X as above admits a (compact, even) X -manifold M , by a theorem of Borel [27, Theorem 14.1]. A theorem of Mal'cev [25] says that $\pi_1 M$ is residually finite. So, we can take a tower of regular covers

$$\dots \rightarrow M_2 \rightarrow M_1 \rightarrow M$$

corresponding to a nested sequence of normal subgroups of $\pi_1 M$ with trivial intersection, and such a sequence (M_n) will BS-converge to x , see [1] for details. Moreover, if M is compact then DeGeorge–Wallach [14] showed that the normalized Betti numbers of (M_n)

converge to the L^2 -Betti numbers $b_k^{(2)}(X)$ of X . See [1, 24] for more information about L^2 -Betti numbers, and for a more general result.

In fact, any sequence of manifolds that BS-converges to X can be interleaved with a tower of covers of a compact x -manifold as in the example above, and the result still BS-converges. So, Theorem 1.3 and the result of DeGeorge-Wallach [14] above give:

Corollary 1.4. Suppose that (M_n) is a sequence of finite volume x -manifolds that BS-converges to X . Then for all $k \in \mathbb{N}$, we have $b_k(M_n)/\text{vol}(M_n) \rightarrow b_k^{(2)}(X)$.

With Nikolov, Raimbault and Samet, we proved this in [1] for sequences of compact, q -thick manifolds, using analytic methods. One could also prove it in the thick case by using Theorem 1.2 above (the Bowen–Elek simplicial approximation technique) and interleaving with a covering tower. In the thin case, we were able to push our analytic methods far enough to give a proof for $X = H^d$, see [1, Theorem 1.8]. Hence, there is no problem in allowing $X = H^3$ in Corollary 1.4, even though Theorem 1.3 does not apply.

While we were finishing this paper, Alessandro Carderi sent us an interesting preprint where, among other things, he proves the same result as Corollary 1.4 if either $k = 1$, or k is arbitrary and the symmetric space $X = G/K$ is of higher rank and M_n is non compact, or in most cases when X is of rank 1. His proof is quite different, he considers the ultraproduct of the sequence of actions of G on G/Γ_n . He then identifies the L^2 -Betti numbers of the resulting G -action with the L^2 -Betti numbers of the group G .

Corollary 1.4 is particularly powerful when X has real rank at least two. In this case, we proved with Nikolov, Raimbault and Samet that any sequence of distinct finite volume x -manifolds BS-converges to x , see [1, Theorem 4.4]. So, Corollary 1.4 implies:

Corollary 1.5. Suppose that $\text{rank}_R X \geq 2$ and (M_n) is any sequence of distinct finite volume x -manifolds. Then for all $k \in \mathbb{N}$, we have $b_k(M_n)/\text{vol}(M_n) \rightarrow b_k^{(2)}(X)$.

In the two corollaries above, we can identify the limit of the normalized Betti numbers when the BS-limit is X . In general, one can think of Theorem 1.3 as giving a definition of ‘ L^2 -Betti numbers’ for arbitrary limits of BS-convergent sequences. The measures on M that arise as such limits have a special property called unimodularity, see [2], and it would be interesting to find a good intrinsic definition of the ‘ L^2 -Betti numbers’ of a unimodular measure that is compatible with Theorem 1.3.

1.2. The proof, and generalities in nonpositive curvature. To prove Theorem 1.3, we split into cases depending on $\text{rank}_R X$. When the rank is one, we need to deal with general BS-convergent sequences, but the thin parts of rank one locally symmetric spaces are easy to understand. And when the rank is at least two, the only possible BS-limit we need to consider is x . We now give two theorems that handle these two cases. We state them very generally, without any assumption of symmetry.

Theorem 1.6 (Pinched negative curvature, arbitrary BS-limits). Let (M_n) be a BS-convergent sequence of finite volume Riemannian d -manifolds, with $d = 3$, and with sectional curvatures in the interval $[-1, \delta]$, for some $-1 \leq \delta < 0$. Then the normalized Betti numbers $b_k(M_n)/\text{vol}(M_n)$ converge for all k .

Theorem 1.7 (Nonpositive curvature, with a thick BS-limit). Let $q > 0$ and let (M_n) be a sequence of real analytic, finite volume Riemannian d -manifolds with sectional curvatures in the interval $[-1, 0]$, and assume the universal covers of the M_n do not have Euclidean de Rham-factors. If (M_n) BS-converges to a measure μ on M that is supported on q -thick manifolds, the normalized Betti numbers $b_k(M_n)/\text{vol}(M_n)$ converge for all k .

Let's see how to deduce Theorem 1.3 from these results. Suppose X is an irreducible symmetric space of noncompact type, $\dim(X) = 3$. When X has rank one, X has pinched negative curvature, so therefore Theorem 1.3 follows from Theorem 1.6. When X has higher rank, [1, Theorem 4.4] says that any BS-convergent sequence (M_n) of X -manifolds BS-converges to X , as mentioned above. Since X is actually q -thick for any q , Theorem 1.7 applies, and Theorem 1.3 follows.

The reader may wonder where we use $d = 3$ in the proof of Theorem 1.6. When $d = 2$, one can deduce the claim from Gauss–Bonnet. In general, the point is that the boundary of a Margulis tube is homeomorphic to an s^{n-2} -bundle over s^1 . When $d \geq 4$, this bundle is not aspherical, so it can be distinguished from a cusp cross section, which prevents one from doing Dehn filling as in our problematic 3-dimensional example. More to the point, one can show that when $d \geq 4$, Margulis tubes with very short cores have boundaries with large volume, see Proposition 3.1, which implies that the number of Margulis tubes with short cores one can see in a manifold is sublinear in volume. Hence, the contribution of the tubes to homology cannot affect the normalized Betti numbers much.

The key to Theorem 1.7 is a celebrated theorem of Gromov, see [6, Theorem 2], that bounds the Betti numbers of an analytic manifold with sectional curvatures in $[-1, 0]$ and no local Euclidean deRham factors linearly in terms of its volume. Delving into its proof, one can show that in the setting of Theorem 1.7, the Betti numbers of the thin parts of the M_n grow sublinearly with $\text{vol}(M_n)$. One can then combine the proof of Theorem 1.2 (the Bowen–Elek simplicial approximation argument), which handles the thick parts of the M_n , making use of the techniques from [18] and [5] to control the complexity of the boundary, where the thick and thin parts are glued, with Mayer–Vietoris sequence to get Theorem 1.7.

Remark 1.1. Recently, the work [1] has been extended by Gelander and Levit to analytic groups over non-archimedean local fields [19]. For non-archimedean local fields of characteristic 0 the uniform discreteness assumption holds automatically for the family of all lattices and more generally all discrete IRS. However this is not the case in positive characteristic. We conjecture that the analogue of the stronger results concerning Betti numbers obtained in the current work can be extended to general analytic groups over non-archimedean local fields.

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2. Spaces of spaces and simplicial approximation

In this section, we discuss the topology on M and a similar topology on the space M of all pointed metric measure spaces. We then state and prove a generalization of the Bowen–Elek theorem on the convergence of Betti numbers of thick spaces, which was stated in a weak form in the introduction as Theorem 1.2.

2.1. **The smooth topology.** In the introduction, we introduce the space

$M = \{\text{pointed, connected, complete Riemannian manifolds } (M, p)\}/\text{pointed isometry}$, endowed with the topology of pointed smooth convergence. Here, a sequence (M_n, p_n) converges smoothly to (M_∞, p_∞) if there is a sequence of smooth embeddings

$$(1) \quad \phi_n : B_{M_\infty}(p_\infty, R_n) \rightarrow M_n$$

with $R_n \rightarrow \infty$ and $\phi_n(p_\infty) = p_n$, such that $\phi_n^* g_n \rightarrow g_\infty$ in the C^∞ -topology, where g_n are the Riemannian metrics on M_n . We call (ϕ_n) a sequence of almost isometric maps coming from smooth convergence. Note that each metric $\phi_n^* g_n$ is only partially defined on M_∞ , but their domains of definition exhaust M_∞ , so it still makes sense to say that $\phi_n g_n \rightarrow g_\infty$ on all of M_∞ , even if the language is a bit abusive. Alvaréz López, Barral Lijó and Candel [4] have shown that M , with the smooth topology, is a Polish space. See also the appendix of Abert–Biringer [2] for a slightly simpler proof.

2.2. **Metric measure spaces.** A metric measure space (or mm-space) is a proper, separable metric space M equipped with a Radon measure vol . Let

$$M = \{\text{pointed mm-spaces } (M, \text{vol}, p)\}/\text{pointed measure preserving isometry}.$$

Following Bowen [10, Definitions 28 and 29], an (ϱ, R) -relation between pointed mm-spaces $M_1 = (M_1, \text{vol}_1, p_1)$ and $M_2 = (M_2, \text{vol}_2, p_2)$ is a pair of isometric embeddings

$$M_i \rightarrow Z, \quad i = 1, 2$$

into some common metric space Z having the following properties:

- (a) $d_Z(p_1, p_2) < \varrho$,
- (b) $B_{M_1}(p_1, R) \sqsupseteq (M_2)_\varrho$ and $B_{M_2}(p_2, R) \sqsupseteq (M_1)_\varrho$,
- (c) for all Borel subsets $F_i \subseteq B_{M_i}(p_i, R)$, we have

$$\text{vol}_1(F_1) < (1 + \varrho)\text{vol}_2((F_1)_\varrho) + \varrho, \quad \text{vol}_2(F_2) < (1 + \varrho)\text{vol}_1((F_2)_\varrho) + \varrho.$$

Here, if F is a subset of a metric space, the notation $(F)_\varrho$ refers to the ϱ -neighborhood of F . See also §3.2. The multiplicative factors of $(1 + \varrho)$ in (c) are not really necessary, and are not present in [10]. However, some of our statements, e.g. Lemma 2.1 below, are simpler because of them.

For each $M = (M, \text{vol}, p) \sqsupseteq M$ and $\varrho, R > 0$, define the (ϱ, R) -neighborhood of M to be the set $N_{\varrho, R}(M)$ of all $M' \sqsupseteq M$ that are (ϱ', R') -related to M for some $\varrho' < \varrho$ and $R' > R$. Note that if $M' \sqsupseteq N_{\varrho, R}(M)$, then for all sufficiently small $\delta > 0$ and large $r > 0$, we have

$$(2) \quad N_{\delta, r}(M') \sqsupseteq N_{\varrho, R}(M).$$

This follows from the fact that one can ‘concatenate’ a relation between M_1 and M_2 with one between M_2 and M_3 , by gluing the two metric spaces Z together along M_2 .

If we endow M with the topology generated by all (ϱ, R) -neighborhoods, then the neighborhood nesting property referenced in (2) implies that

$$M_i \rightarrow M_\infty \iff \varrho_i \rightarrow 0, R_i \rightarrow \infty \text{ such that } M_i \text{ is } (\varrho_i, R_i)\text{-related to } M_\infty.$$

The next lemma will help us relate smooth convergence of Riemannian manifolds to their convergence as metric measure spaces.

Lemma 2.1. Suppose that (M_i, p_i) , $i = 1, 2$, are pointed Riemannian d -manifolds and for some $R > 0$ there is an embedding $\phi : B_{M_1}(p_1, R) \rightarrow M_2$ with $\phi(p_1) = p_2$ and

$$(3) \quad (1 - \delta)|v| \leq |\phi'(v)| \leq (1 + \delta)|v|, \quad \forall v \in T B_{M_1}(p_1, R).$$

Then if $\delta = \delta(\varrho, d)$ is small, the triples (M_i, vol_i, p_i) are (ϱ, R) -related, where vol_i is the Riemannian measure on M_i .

Proof. Take $\delta < \varrho$ and let ϕ be as in the statement of the lemma. We want to produce an (ϱ, R) relation between M_1 and M_2 . Define the common space Z as the disjoint union

$$Z = M_1 \sqcup M_2,$$

endowed with a metric that restricts to the given metrics on M_1, M_2 , and where for $x \in M_1, y \in M_2$,

$$d(x, y) = \inf\{d(x, x') + \delta + d(\phi(x'), y) \mid x' \in B_{M_1}(p_1, R + 1)\}.$$

We now verify that Z gives an (ϱ, R) -relation. First, $d_Z(p_1, p_2) = \delta < \varrho$. Second, if $x \in M_1 \cap B_Z(p_1, R) = B_{M_1}(p_1, R)$, then $d(x, \phi(x)) = \delta < \varrho$, so $x \in (M_2)_\varrho$. Third, if $F_1 \sqsubset B_Z(p_1, R)$ is a Borel subset, then we have

$$\text{vol}_1(F_1) = \text{vol}_1(F_1 \cap M_1) \leq (1 + \delta)^d \text{vol}_2(\phi(F_1 \cap M_1)) \leq (1 + \delta)^d \text{vol}_2((F_1)_\delta),$$

where the first inequality follows from (3), and the second follows from the fact that $d(x, \phi(x)) = \delta$. So, as long as δ is small, the right side will be at most $(1 + \varrho)\text{vol}_2((F_1)_\varrho)$. The two remaining parts of properties (a) and (b) follow similarly.

As an immediate corollary, we get the following:

Corollary 2.2. The natural inclusion $M \rightarrow M$ from the space of pointed Riemannian manifolds (with the smooth topology) to the space of pointed mm-spaces is continuous.

2.2.1. Extended mm-spaces. We will need a slight variant of M for our work below. Let

$$M^{\text{ext}} = \{(M, \text{vol}, p, E) \mid (M, \text{vol}, p) \sqsupseteq M, E \sqsupseteq M \text{ a super-metric space}\}/\sim,$$

where a super-metric space is just a proper, separable metric space that contains M as a submetric space. We call a quadruple (M, vol, p, E) an extended pointed mm-space; two quadruples are identified in M^{ext} if there is a pointed isometry between the super-metric spaces E that restricts to a measure preserving isometry from one mm-space M to the other. The topology on M^{ext} is similar to that on M : we say that $(M_i, \text{vol}_i, p_i, E_i)$, $i = 1, 2$, are (ϱ, R) -related if there are isometric embeddings

$$E_i \rightarrow Z, i = 1, 2$$

that restrict to give an (ϱ, R) -relation between the triples (M_i, vol_i, p_i) , and where also

$$(4) \quad B_{E_1}(p_1, R) \sqsupseteq (E_2)_\varrho, \quad B_{E_2}(p_2, R) \sqsupseteq (E_1)_\varrho.$$

One then defines (ϱ, R) -neighborhoods just as before and the topology on M^{ext} is that generated by these neighborhoods, in which $M_i \rightarrow M_\infty$ if and only if there are $\varrho_i \rightarrow 0$ and $R_i \rightarrow \infty$ such that M_i is (ϱ_i, R_i) -related to M_∞ for all i .

We then have the following variant of Lemma 2.1.

Lemma 2.3. Suppose that (M_i, p_i) , $i = 1, 2$, are pointed Riemannian d -manifolds with distinguished subsets $T_i \sqsupseteq M_i$ and that for some $R > 0$ there is an embedding

$$\phi : B_{M_1}(p_1, R) \rightarrow M_2$$

with $\phi(p_1) = p_2$ that satisfies the following three properties:

- (i) $(1 - \delta)|v| \leq |d\phi(v)| \leq (1 + \delta)|v|$, $\forall v \in T B_{M_1}(p_1, R)$.
- (ii) $\phi^{-1}(T_2) \sqsupseteq (T_1)_\delta$, and $\phi(T_1 \cap B_{M_1}(p_1, R)) \sqsupseteq (T_2)_\delta$,
- (iii) $\text{vol}_1(\phi^{-1}(T_2) \setminus T_1) < \delta$,

where \setminus is the symmetric difference. Then if $\delta = \delta(\varrho, d)$ is sufficiently small, the quadruples $(T_i, \text{vol}_i|_{T_i}, p_i, M_i)$ are (ϱ, R) -related, where here vol_i is the Riemannian measure on M_i .

Proof. The proof is similar to that of Lemma 2.1. With $Z = M_1 \sqsupseteq M_2$ and d the metric defined in Lemma 2.1, equation (4) above follows exactly as before as long as $\delta < \varrho$. So, we just need to verify that Z gives an (ϱ, R) -relation between the subsets T_1, T_2 . Property (a) is immediate from the definition of the metric on Z . For (b), if $x \in B_T(p_1, R_1)$ then $\phi(x) \in (T_2)_\delta$, so $d_Z(x, T_2) < 2\delta$. So, (b) holds if $\delta \leq \varrho/2$, as the proof of the other part is similar. For (c), suppose $F \sqsupseteq B_Z(p_1, R)$ is Borel. Then

$$\begin{aligned} \text{vol}_1|_{T_1}(F_1) &= \text{vol}_1(F_1 \cap T_1) \\ &\leq \text{vol}_1(F_1 \cap \phi^{-1}(T_2)) + \text{vol}_1(\phi^{-1}(T_2) \setminus T_1) \\ &< (1 + \delta)^d \text{vol}_2(\phi(F_1) \cap T_2) + \delta \\ &= (1 + \delta)^d \text{vol}_2|_{T_2}(\phi(F_1)) + \delta \end{aligned}$$

So since $\phi(F_1) \sqsupseteq (F_1)_\delta$, (c) holds if $(1 + \delta)^d \leq (1 + \varrho)$. The other part of (c) is similar.

2.2.2. Extended mm-spaces with multiple measures or distinguished subsets. Let

$$M^{n,ext} := \{(M, vol_1, vol_2, \dots, vol_n, p, E)\} / \mathbb{R},$$

where here (M, p) is a pointed metric space embedded in some super metric space $E \supseteq M$, the vol_i are Radon measures on M , and the equivalence relation is pointed isometry that preserves all the measures. The space $M^{n,ext}$ comes equipped with projection maps

$$\pi_i : M^{ext,n} \rightarrow M^{ext}, \quad (M, vol_1, vol_2, \dots, vol_n, p, E) \rightarrow (M, vol_i, p, E)$$

for each $i = 1, \dots, n$, and we say that two tuples

$$M = (M, vol_1, vol_2, \dots, vol_n, p, E), \quad M' = (M', vol'_1, vol'_2, \dots, vol'_n, p', E)$$

are (ϱ, R) -related if there are fixed embeddings $E \hookrightarrow Z, E' \hookrightarrow Z$ of the two super metric spaces into some common metric space Z that induce (ϱ, R) -relations between the projections $\pi_i(M), \pi_i(M')$ for all i . The (ϱ, R) -neighborhood $N_{\varrho, R}(M)$ of $M \in M^{ext,n}$ is again defined to be the set of all M' that are (ϱ', R') -related to M for some $\varrho' < \varrho$ and $R' > R$, and we endow $M^{ext,n}$ with the topology generated by these neighborhoods, in which $M_i \rightarrow M$ when there are $\varrho_i \rightarrow 0$ and $R_i \rightarrow \infty$ such that M_i, M are (ϱ_i, R_i) -related for large i .

We also consider the space

$$MS^{ext} := \{(M, vol, p, E, S)\} / \mathbb{R},$$

of pointed, extended mm-spaces equipped with locally finite subsets $S \subseteq M$. The topology is defined so that the natural map $MF^{ext} \rightarrow M^{2,ext}$ that interprets a locally finite set S as the atomic Radon measure 1_S is a homeomorphism onto its image. Finally, we let

$$MF^{ext} := \{(M, vol, p, E, S, f)\} / \mathbb{R},$$

be the space of pointed, extended mm-spaces equipped with locally finite subsets S that come weighted with functions $f : S \rightarrow [0, 1]$. We topologize MF^{ext} so that the natural map $MF^{ext} \rightarrow M^{3,ext}$ is a homeomorphism onto its image; here, the three measures on the image of (M, vol, p, E, S, f) are vol , the atomic Radon measure 1_S determined by S , and the atomic Radon measure 1_f where points $s \in S$ have mass $f(s)$ instead of unit weight. Note that the natural projection $MF^{ext} \rightarrow MS^{ext}$ is continuous, and that there is also an embedding $MS^{ext} \rightarrow MF^{ext}$ obtained by letting f be the constant function whose values are all 1. With this embedding in mind, we state most results below just for MF^{ext} , knowing that they also apply to the subspace MS^{ext} . Finally, an (ϱ, R) -relation between two elements of MS^{ext} , or between two elements of MF^{ext} , is just an (ϱ, R) -relation between their images in $M^{2,ext}$, or in $M^{3,ext}$. The topologies on MS^{ext} and MF^{ext} can then also be described via these relations, just as above.

One difficulty that arises when working with (ϱ, R) -relations is that you have a different pair of embeddings for each relation. In order to work with probability measures on sets of pointed mm-spaces, it is more convenient to have all our spaces be subsets of a fixed metric space. So, let Z be some proper separable metric space. A pointed, extended mm-space,

possibly with a distinguished discrete set and a function, is embedded in Z if the extended space E is a subset of Z , and we write the associated spaces of such spaces as

$$M^{ext}(Z), \quad MS^{ext}(Z), \quad MF^{ext}(Z).$$

We say that two spaces are (ϱ, R) -related within Z if their inclusions into Z induce an (ϱ, R) -relation, and we equip the spaces of spaces above with the topologies generated by (ϱ, R) -relations within Z . In particular, we say that $M_i \rightarrow M_\infty$ within Z if for any (ϱ, R) we have that for large i , M_i, M_∞ are (ϱ, R) -related within Z .

A sequence of Radon measures μ_i on Z weak* converges to μ_∞ if $\int_R f d\mu_i \rightarrow \int_R f d\mu_\infty$ for all continuous functions $f : Z \rightarrow R$ with compact support³. When a sequence of mm-spaces with weighted subsets is embedded in a single Z , convergence of the weighted subsets can be interpreted as weak* convergence.

Lemma 2.4 (c.f. Lemma A.2 of [10]). Suppose that $M_i = (M_i, \text{vol}_i, p_i, E_i, S_i, f_i) \sqsubseteq MF^{ext}(Z)$, where $i = 1, 2, \dots, \infty$. Then $M_i \rightarrow M_\infty$ within Z if and only if the embedded extended pointed mm-spaces $(M_i, \text{vol}_i, p_i, E_i) \rightarrow (M_\infty, \text{vol}_\infty, p_\infty, E_\infty)$ and the measures 1_{S_i} and 1_{f_i} converge in the weak* topology to 1_{S_∞} and 1_{f_∞} .

In fact, every convergent sequence in MF^{ext} can be embedded in some Z .

Lemma 2.5. Suppose that $M_i = (M_i, \text{vol}_i, p_i, E_i, S_i, f_i) \sqsubseteq MF^{ext}$, where $i = 1, 2, \dots, \infty$, and $M_i \rightarrow M_\infty$. Then there is a proper, separable metric on

$$Z = \bigcup_{i=1,2,\dots,\infty} E_i$$

such that $M_i \rightarrow M_\infty$ within Z . Furthermore, we can assume that for all i, j ,

$$d_Z(E_i, E_j) \geq 1/i + 1/j.$$

Note that this lemma also applies to sequences of extended mm-spaces without weighted subsets, just by taking $S_i = \emptyset$. The proof is a modification of Lemma B.2 in [10].

Proof. For each i , choose an (ϱ_i, R_i) -relation between M_i and M_∞ , where $\varrho_i \rightarrow 0$ and $R_i \rightarrow \infty$. Instead of writing this relation as a pair of embeddings of E_i, E_∞ into some third metric space, we can consider it as a pseudometric on the disjoint union $E_i \sqcup E_\infty$ that restricts to the original metrics on E_i and E_∞ . We can then change each such pseudometric into a metric d by adding $1/i$ to the distance between any point in E_i and any point in E_∞ , and combine all of them into a single partially defined metric d on the disjoint union

$$Z = \bigcup_{i \in \mathbb{N} \cup \{\infty\}} E_i,$$

³In this paper weak* convergence involves integrating against continuous functions with compact support, while weak convergence integrates against bounded continuous functions. Bowen uses weak* convergence in [10] when defining Benjamini-Schramm convergence on M , but it really should be weak convergence. Indeed, M is not locally compact at any point, so there are no nonzero continuous functions with compact support on M .

which we extend to a (fully defined) metric d by setting

$$d(x_i, x_j) := \inf_{x_\infty \in E_\infty} d(x_i, x_\infty) + d(x_j, x_\infty)$$

for all finite i, j and $x_i \in E_i, x_j \in E_j$. The reader can verify the desired properties.

In order to talk about convergence of measures on $MF^{\text{ext}}(Z)$ we will need an explicit basis of neighborhoods. Of course, one could just take the sets $N_{\varrho, R}(M)$ of all M' that are (ϱ', R') -related to M within Z , but then it is a little unclear exactly what condition this places on the weighted discrete subsets. The following system of neighborhoods is more convenient in that respect.

Lemma 2.6. Suppose that $M_1 := (M_1, \text{vol}_1, p_1, E_1, S_1, f_1) \in MF^{\text{ext}}(Z)$. For $\varrho, R > 0$, let $B_{\varrho, R} \subseteq MF^{\text{ext}}$ be the set of all $M_2 := (M_2, \text{vol}_2, p_2, E_2, S_2, f_2) \in MF^{\text{ext}}(Z)$ such that

- the pointed extended mm-spaces $(M_i, p_i, \text{vol}_i, E_i)$ are (ϱ', R') -related in Z , for some $\varrho' < \varrho$ and $R < R'$, and where
- there is a bijection

$$\phi : S_1 \cap B_{M_1}(p_1, R) \rightarrow S_2 \cap B_{M_2}^*(p_2, R)$$

such that $d_Z(s, \phi(s)) < \varrho$ and $|f_1(s) - f_2(\phi(s))| < \varrho$ for all $s \in S_1 \cap B_{M_1}(p_1, R)$.

Then there is a family of ‘admissible’ pairs (ϱ, R) such that the sets $B_{\varrho, R}$ form a basis of open neighborhoods of $M_1 \in MF^{\text{ext}}(Z)$. Moreover, for every R_0 , there is some $R > R_0$ such that (ϱ, R) is admissible for all sufficiently small ϱ .

Here, $B_{\varrho}^*(\varrho, \varrho)$ denotes the closed ball of the given center and radius, while $B_{\varrho}(\varrho, \varrho)$ is the open ball. A pair (ϱ, R) is admissible if the following conditions hold:

- (1) $d(s, t) > 3\varrho$ for all $s, t \in S_1 \cap B(p_1, R)$, and
- (2) there are no points $s \in S_1$ with $d(p_1, s) \in (R - 2\varrho, R + 2\varrho)$.

Since S_1 is locally finite, for any given R condition (1) holds whenever ϱ is sufficiently small, and if we perturb R so that there are no $s \in S_1$ with $d(p_1, s) = R$ and then shrink ϱ further, we can ensure that (2) holds as well. This justifies the last line of the lemma.

Note also that if we drop the condition on f_1, f_2 from ϕ , then the above gives a description of a neighborhood basis for a point in $Ms^{\text{ext}}(Z)$ rather than in $MF^{\text{ext}}(Z)$.

Proof. Suppose (ϱ, R) is an admissible pair as defined above. Below, one should consider all relations as taken within Z .

We first want to show that $B_{\varrho, R}$ is open. If $M_2 := (M_2, \text{vol}_2, p_2, E_2, S_2, f_2) \in B_{\varrho, R}$, it suffices to find some δ, T such that any M_3 that is (δ, T) -related to M_2 lies in $B_{\varrho, R}$. So, let $E_i \overset{\sim}{\rightarrow} Z$, $i = 1, 2$, ϱ', R' and ϕ be the data witnessing that $M_2 \in B_{\varrho, R}$, and let δ be very small and T be very large. Take some M_3 that is (δ, T) -related to M_2 .

Given $s_1 \in S_1 \cap B_{M_1}(p_1, R)$, as long as T is large we can apply the definition of a (δ, T) -relation to get that

$$1 = |\{\phi(s_1)\} \cap S_1| < (1 + \delta)|S_3 \cap (\{\phi(s_1)\})_\delta| + \delta.$$

As long as $\delta < 1$ this implies that there is at least one element $s_3 \in S_3$ that is within δ of $\phi(s_1)$. Since $d(s_1, \phi(s_1)) < \varrho$, we then have $d(s_1, s_3) < \varrho$ as well as long as δ is small. (The set of all s_1 is finite, so δ can be chosen small enough that this works for all s_1 simultaneously.) Now if we had two elements $s_3, s' \in S_3$ within ϱ of s_1 , we have $2 = |S_3 \cap \{s_3, s'\}| < (1 + \delta)|S_2 \cap \{s_3, s'\}| + \delta$, so there are at least two elements $s_2, s' \in S_2$ within $\delta + \varrho$ of s_1 . As long as δ is small, property (2) in the definition of admissibility implies that these two points lie in the image of ϕ , so $\phi^{-1}(s_2), \phi^{-1}(s')$ both lie within $\delta + 2\varrho$ of s_1 , contradicting property (1) of admissibility. So, if we let $\psi(s_1)$ be the unique element of S_3 with $d(s_1, \phi(s_3)) < \varrho$, we get a map

$$\psi : S_1 \cap B_{M_1}(p_1, R) \rightarrow S_3$$

such that $d_Z(s_1, \psi(s_1)) < \varrho$ for all $s \in S_1 \cap B_{M_1}(p_1, R)$. By property (2) of admissibility, the image of ψ lies in $S_3 \cap B_{M_3}(p_3, R)$. The argument above shows that ϕ is an injection. And if $s_3 \in B_{M_3}(p_3, R)$, we have that

$$(1 + \delta)|S_2 \cap \{s_3\}| + \delta \geq |S_3 \cap \{s_3\}| = 1,$$

implying there's some $s_2 \in S_2$ within a distance of δ of s_3 . If δ is very small relative to the minimum distance from an element of $S_2 \setminus B^+(p_2, R)$ to $B^+(p_2, R)$, we can assume that this $s_2 \in S_2 \cap B^+(p_2, R)$, so that $s_2 = \phi(s_1)$ for some s_1 . Taking δ small again, we have $d_Z(s_3, s_1) < \varrho$, so s_3 is in the image of ψ as desired. This proves ψ is a bijection. The fact that $|f_1(s_1) - f_2(\psi(s_1))| < \varrho$ if δ is small follows from similar techniques. This verifies that $M_3 \in B_{\varrho, R}$, so the set $B_{\varrho, R}$ is open. Note also that condition (2) in the definition of admissibility implies that $M_1 \in B_{\varrho, R}$, so $B_{\varrho, R}$ is an open neighborhood as required.

Next, we need to show that the sets $B_{\varrho, R}$ with (ϱ, R) admissible form a neighborhood basis for M_1 . For this, it suffices to fix (δ, T) and show that for sufficiently small ϱ and large R , any $M_2 \in B_{\varrho, R}$ is (δ, T) -related to M_1 . By choosing $\varrho < \delta$ and $T < R$, we get automatically that the embeddings $E_i : M_i \rightarrow Z$ that verify that $M_2 \in B_{\varrho, R}$ induce (δ, T) -relations of the corresponding pointed extended mm-spaces. If $F \in B_M(p_1, T)$ is Borel, then for any $s_1 \in F \cap S_1$ we have $\phi(s_1) \in (F)_\delta \cap S_2$ and hence

$$|S_1 \cap F| \leq |S_2 \cap (F)_\delta|.$$

Moreover, as long as $\varrho < \delta / |S_1 \cap B_{M_1}(p_1, T)|$, we have

$$\begin{array}{ccccccccc} X & & X & & X & & X & & \\ (f_1(s_1) - \varrho) < & & f_2(s_2) & \Rightarrow & f_1(s_1) < & & f_2(s_2) + \varrho. \\ s_1 \in S_1 \cap F & & s_2 \in S_2 \cap (F)_\delta & & s_1 \in S_1 \cap F & & s_2 \in S_2 \cap (F)_\delta & & \end{array}$$

The two inequalities associated to a subset $F \in B_{M_2}(p_2, T)$ are proved similarly, using ϕ^{-1} instead of ϕ , so we have a (δ, T) -relation between M_1 and M_2 .

2.2.3. Poisson processes on mm-spaces. The reason we introduce so many spaces of spaces above is that we need to make precise the notion that the Poisson process on a pointed mm-space varies continuously with the space.

Let (M, vol) be a mm-space and let S be the set of all locally finite subsets of M . Regarding a locally finite subset S as an atomic Radon measure μ_S on M , we endow S

with the weak* topology, where measures are tested against continuous functions with compact support, as discussed before Lemma 2.4. The Poisson process of M (of intensity 1) is the unique Borel probability measure ρ_M on S such that the following hold.

- When A_1, \dots, A_n are disjoint Borel subsets of M , the random variables that record the sizes of the intersections $S \cap A_i$ are independent.
- If $A \subseteq M$ is Borel, the size of $S \cap A$ is a random variable having a Poisson distribution with expectation $\text{vol}(A)$.

For a finite volume subset $A \subseteq M$ and $n \in \mathbb{N}$, we have

$$(5) \quad \text{Prob} \begin{array}{l} \text{for } (x_1, \dots, x_n) \in A^n, \text{ we have } D \cap A = \{x_1, \dots, x_n\}, \\ \text{given that } D \cap A \text{ has } n \text{ elements.} \end{array} = d\text{vol}^n(x_1, \dots, x_n).$$

In other words, if D is chosen randomly, the elements of $D \cap A$ are distributed within A independently according to vol . See [12, Example 7.1(a)] for details on Poisson processes in \mathbb{R}^n . The general case is similar. In fact, every mm-space is measure-isomorphic modulo null sets to the union of an interval in \mathbb{R} with a countable set of atoms, c.f. [28], so as the definition of the Poisson process is totally measure theoretic, most analyses of it can be performed on the latter space.

Suppose now that $M = (M, \text{vol}, p, E)$ is a pointed, extended mm-space. Push forward the Poisson process on M to a measure ρ_M on MS^{ext} , using the map

$$(6) \quad S \rightarrow MS^{\text{ext}}, \quad S \subseteq M \rightarrow (M, \text{vol}, p, E, S).$$

Note that the map in (6) is continuous: if S is weakly close to S' , the identity inclusions $E \rightarrow E'$ generate an (ϱ, R) -relation between (M, vol, p, E, S) and $(M, \text{vol}, p, E, S')$.

The following is the main result of this subsection. A variant of it is claimed, but not proved, in the proof of Claim 1 on pg 582 in Bowen [10].

Lemma 2.7 (Poisson processes vary continuously with the mm-space). The map

$$M^{\text{ext}} \rightarrow P(MS^{\text{ext}}), \quad M \rightarrow \rho_M$$

is continuous.

Proof. Suppose that we have $M_i = (M_i, \text{vol}_i, p_i, E_i) \subseteq M^{\text{ext}}$ and $M_i \rightarrow M_\infty$. By Lemma 2.5, we can assume that all E_i are embedded in some fixed Z , and that the convergence happens within Z . Let $S(Z)$ be the set of all $S \subseteq Z$ that are locally finite subsets of M_i for some $i = i(S)$; endow $S(Z)$ with the weak* topology. Then for each i , the Poisson process on M_i can be considered as a probability measure ρ_i on $S(Z)$. By Lemma 2.4, the map

$$(7) \quad S(Z) \rightarrow MS^{\text{ext}}, \quad S \rightarrow (M_{i(S)}, \text{vol}_{i(S)}, p_{i(S)}, E_{i(S)}, S_{i(S)}),$$

is continuous, and each ρ_i pushes forward under this map to the measure ρ_{M_i} on MS^{ext} . So, to prove the lemma it suffices to show that $\rho_i \rightarrow \rho_\infty$ weakly.

Let $T \subseteq S(Z)$ with $T \subseteq M_\infty$, let $\varrho, R > 0$ and let $B_{\varrho, R}(T)$ be the set of all $S \subseteq S(Z)$ such that there is a bijection

$$f : T \cap B_{M_\infty}(p_\infty, R) \rightarrow S \cap B_{M_{i(S)}}^*(p_{i(S)}, R)$$

such that $d(t, f(t)) < \varrho$ for all t . Lemmas 2.5 and 2.6 imply that for admissible pairs (ϱ, R) , the sets $B_{\varrho, R}(T)$ form a basis of neighborhoods for $T \in S(Z)$. So by the Portmanteau theorem and the fact that p_∞ is supported on subsets of M_∞ , it suffices to show that

$$(8) \quad \liminf_i p_i(B) \geq p_\infty(B)$$

for all $B := B_{\varrho, R}(T)$, where T, ϱ, R are as above.

Fixing some such B , let $t \in T \cap B_{M_\infty}(p_\infty, R)$ and define

$$V_i(t) := \text{vol}_i(B_Z(t, \varrho)).$$

By the definition of the Poisson process and the fact that the points of $T \cap B_{M_\infty}(p_\infty, R)$ are 3ϱ -separated when (ϱ, R) is an admissible pair, we have

$$(9) \quad \begin{aligned} p_i(B) &= \prod_{t \in T \cap B_{M_\infty}(p_\infty, R)} V_i(t) e^{-V_i(t)} \cdot e^{-\text{vol}_i(B_{M_i}(p_i, R)) - \sum_{t \in T \cap B_{M_\infty}(p_\infty, R)} V_i(t)} \\ &= \prod_{t \in T \cap B_{M_\infty}(p_\infty, R)} V_i(t) \cdot e^{-\text{vol}_i(B_{M_i}^\bullet(p_i, R))}, \end{aligned}$$

where $t \in T \cap B_{M_\infty}(p_\infty, R)$ and $V_i(t) = \text{vol}_i(B_Z(t, \varrho) \cap M_i)$. For a p_i -random S , the product in the first line of (9) is the probability that there is exactly one point of S within ϱ of each t , and the second factor is the probability that there are no points of $S \cap B_{M_i}^\bullet(p_i, R)$ other than those within ϱ of the various t .

Recall that the inclusions of E_i and E_∞ into Z form an (ϱ_i, R_i) relation where $\varrho_i \rightarrow 0$ and $R_i \rightarrow \infty$. Pick any $0 < \varrho' < \varrho$ and apply property (c) in the definition of an (ϱ_i, R_i) -relation to $B_{M_\infty}(t, \varrho')$. Then if i is large enough so that $\varrho' + \varrho_i < \varrho$, we have for all t that

$$\text{vol}_\infty(B_{M_\infty}(t, \varrho')) < (1 + \varrho_i) \text{vol}_i((B_{M_\infty}(t, \varrho'))_{\varrho_i}) + \varrho_i < (1 + \varrho_i) \text{vol}_i(B_{M_\infty}(t, \varrho)) + \varrho_i$$

By taking ϱ' close enough to ϱ , we can make $\text{vol}_\infty(B_{M_\infty}(t, \varrho'))$ arbitrarily close to $\text{vol}_\infty(B_{M_\infty}(t, \varrho))$. Combining this with the fact that $\varrho_i \rightarrow 0$, we get that

$$(10) \quad \text{vol}_\infty(B_Z(t, \varrho)) = \text{vol}_\infty(B_{M_\infty}(t, \varrho)) \leq \liminf_i \text{vol}_i(B_Z(t, \varrho)).$$

We now apply property (c) in the definition of an (ϱ_i, R_i) -relation to $B_{M_i}^\bullet(p_i, R)$, giving

$$\text{vol}_i(B_{M_i}^\bullet(p_i, R)) < (1 + \varrho_i) \text{vol}_\infty((B_{M_i}^\bullet(p_i, R))_{\varrho_i}) + \varrho_i.$$

But since $d(p_i, p_\infty) < \varrho_i$, we have $(B_{M_i}^\bullet(p_i, R))_{\varrho_i} \cap M_\infty \subseteq B_{M_\infty}(p_\infty, R + 2\varrho_i)$, which implies

$$\text{vol}_i(B_{M_i}^\bullet(p_i, R)) < (1 + \varrho_i) \text{vol}_\infty(B_{M_\infty}(p_\infty, R + 2\varrho_i)) + \varrho_i.$$

As $i \rightarrow \infty$, the right hand side converges to $\text{vol}_\infty(B_{M_\infty}^\bullet(p_\infty, R))$, so we get

$$(11) \quad \limsup_i \text{vol}_i(B_{M_i}^\bullet(p_i, R)) \leq \text{vol}_\infty(B_{M_\infty}^\bullet(p_\infty, R)).$$

Combining (10) and (11) proves (8), so we are done.

2.3. **Normalized Betti numbers of mm-spaces.** If (M, vol) is a finite volume mm-space, let $\mu_{(M, \text{vol})}$ be the measure on M obtained by pushing forward the vol under

$$M \rightarrow M, p \rightarrow (M, \text{vol}, p).$$

A sequence of finite volume mm-spaces (M_n, vol_n) Benjamini-Schramm (BS) converges if the associated sequence of probability measures $\mu_{(M_n, \text{vol}_n)} / \text{vol}_n(M_n)$ weakly converges to some limit probability measure on M .

An mm-space M is special if M has finitely many path components⁴, the measure vol is non-atomic and fully supported, and metric spheres have measure zero. In [10], Bowen claims the following result, and justifies it by fleshing out an argument of Elek.

Theorem 2.8 (Compare [10, Theorem 4.1]). Suppose (M_n, vol_n) is a BS-convergent sequence of finite volume special mm-spaces and that there are constants r, v_0, v_1 such that

- (1) all $r/2$ -balls in M_n have volume at least v_0 ,
- (2) all $20r$ -balls have volume at most v_1 ,
- (3) all p -balls in M_n with $p < 10r$ are strongly convex, meaning that for any two points x, y in a p -ball B , there is a unique point $z \in B$ with $d(x, z) = d(y, z) = 1/2d(x, y)$.

Then the normalized Betti numbers $b_k(M_n) / \text{vol}(M_n)$ converge for all k .

As mentioned in the introduction, Bowen's proof of the Theorem 2.8 is not quite complete. Briefly, the Elek/Bowen argument is to construct, for each n , a random q -net $S_n \subset M_n$, i.e. a set of points that are $q/2$ -separated in M_n and where every point in M_n is within q of a point of S_n . Letting N_n be the nerve complex associated to the cover of M_n by q -balls centered at the points of S_n , they then say that the random complexes N_n BS-converge, and then they use Elek's Theorem 1.1 to conclude that the expected normalized Betti numbers of the N_n converge. By the strong convexity in condition (3) above and the Nerve Lemma (c.f. [20, Corollary 4G.3]), each N_n is homotopy equivalent to M_n , so $b_k(N_n) = b_k(M_n)$. One can also relate the number of vertices of N_n to the volume of M_n , so this implies the convergence of the normalized Betti numbers of M_n .

Above, the random nets S_n are constructed as subsets of the union of infinitely many randomly chosen discrete subsets of M_n , each of which is chosen according to a Poisson process. In order to ensure separation of the net, Elek/Bowen enumerate all the discrete subsets and their points, and add them into S_n one by one, throwing out the points that are too close to the previously added points. The problem with this is that it is very hard to prove that such random nets vary continuously when the underlying space is changed, which is essential for BS-convergence of the associated nerve complexes. In [10], this issue is not really addressed. The construction of these subsets is the content of Lemma 4.2 of [10], and the last line of the proof (see the end of the first paragraph of pg 584) seems to indicate that continuity of the q -nets follows immediately from continuity of the 'almost nets' one would obtain by superimposing only a fixed number of Poisson processes, instead

⁴Bowen requires M to be path connected in his definition of special, but finitely many components suffices everywhere below.

of infinitely many of them. However, this is not true; it is like saying that the limit of continuous functions is always continuous. The question of whether the Elek/Bowen random nets do vary continuously with the underlying mm-space seems quite subtle in general, and while we do not have a counterexample, we think that a proof of this would be more difficult than the alternative approach we take in the current paper.

Below, we will prove a slightly more (and less) general result, Theorem 2.9. While it does not strictly imply Theorem 2.8, it can be used in all Bowen's applications. The proof essentially follows the Elek/Bowen argument, but we get around the continuity issue by only superimposing a fixed finite number of Poisson processes, creating an $q/2$ -separated 'almost net' $S_n \sqsubset M_n$. While S_n may not be a net, we show that it can be completed to a net using a small number of points, so the Betti numbers of the associated nerve complex still approximates that of M_n , allowing us to run the rest of the Elek/Bowen argument.

To motivate the statement of the more general result, look again at the statement of Theorem 2.8. Condition (3) is only used to say that the nerve complex is homotopy equivalent to M_n , so we should be able to state a version of Theorem 2.8 in which (3) is omitted, if we talk about the Betti numbers of the nerve complexes directly instead of the Betti numbers of the M_n . Next, to make a result that is compatible with the machinery of Gelander described in §3.2, it is also important for us to take nets in the M_n , but construct the corresponding nerves using balls in larger spaces E_n . In other words, we need to work with the extended mm-spaces of §2.2.

To that end, we say that an extended mm-space $M = (M, \text{vol}, E)$ is finite volume or special if the mm-space M is. When M has finite volume, we can construct a finite measure μ_M on M^{ext} by pushing forward vol under the map

$$p : M \rightarrow (M, \text{vol}, p, E).$$

If $M_n = (M_n, \text{vol}_n, E_n)$ is a sequence of extended mm-spaces, then we say that (M_n) BS-converges if the sequence of measures $\mu_{M_n}/\text{vol}_n(M_n)$ weakly converges.

We define an (r_0, r_1) -net in M to be a subset $S \sqsubset M$ such that

- (1) S is r_0 -separated, i.e. $d(x, y) > r_0$ for all $x = y \in S$,
- (2) S r_1 -covers M , i.e. for every $p \in M$, there is some $x \in S$ with $d(p, x) < r_1$,

and an $[r_2, r_3]$ -weighted (r_0, r_1) -net is a (r_0, r_1) -net S with a function

$$\rho : S \rightarrow [r_2, r_3],$$

where here $r_0 < r_1 \leq r_2 < r_3$. Given any weighted net (S, ρ) in M , we let $N_E(S, \rho)$ be the nerve complex associated to the collection of E -balls $B_E(x, \rho(x))$, where $x \in S$.

Theorem 2.9. Fix k , let $M_n = (M_n, \text{vol}_n, E_n)$ be a BS-convergent sequence of extended finite volume special mm-spaces and suppose we have constants $v_{\min} > 0$, $r_1 > r_0 > 0$, and $r_3 > r_2 \geq 2r_1$, a function $v_{\max} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

- (1) all $r_0/2$ -balls in every M_n have volume at least v_{\min} ,
- (2) for all $r \in \mathbb{R}_+$, every r -ball in M_n has volume at most $v_{\max}(r)$.

Now suppose that we have a sequence b_n of positive numbers such that

(3) for any sequence of $[r_2, r_3]$ -weighted $(r_0, 2r_1)$ -nets (S_n, ρ_n) in M_n ,

$$\frac{b_k(N_{E_n}(S_n, \rho_n)) - B_n}{\text{vol}_n(M_n)} \rightarrow 0.$$

Then the ratios $B_n/\text{vol}_n(M_n)$ converge.

In our two applications, Theorem 1.6 and Theorem 1.7, the numbers B_n will be the Betti numbers $b_k(M_n)$ and the Betti numbers $b_k(E_n)$, respectively. We state it as above to have a single unified statement that applies in both situations. Note that when applying Theorem 2.9, one has to show that the Betti numbers of the nerve complexes associated to all nets in (3) are approximated by a single sequence B_n . This usually requires an argument that goes through the Nerve Lemma at some point.

Given a sequence (M_n, vol_n) of finite volume special mm-spaces, we can apply Theorem 2.9 to the extended mm-spaces (M_n, vol_n, M_n) , with $B_n = b_k(M_n)$, to get a slightly weaker version of Theorem 2.8. The difference is that hypothesis (2) in Theorem 2.9 is formally stronger than it is in Theorem 2.8, but in basically all applications, upper bounds on ball volumes come from curvature lower bounds, which imply both versions of (2). Note, however, that by the Nerve Lemma the nerve of any covering of M_n by strongly convex balls is homotopy equivalent to M_n , so condition (3) in Theorem 2.8 implies condition (3) in Theorem 2.9 with $B_n = b_k(M_n)$, after adjusting the constants appropriately.

Before starting the proof, we also record two brief lemmas. First, as mentioned above Elek's Theorem 1.1 is crucial in the proof below. Here is a formal consequence of his result with a more general sounding statement.

Lemma 2.10. Suppose that for $n = 1, 2, \dots$, we have a probability measure η_n on the space of pointed complexes K that is of the form

$$\eta_n = \frac{\prod_{m=1}^{M_n} t_{n,m} \mu_{X_{n,m}}}{\prod_{m=1}^{M_n} t_{n,m} \text{vol}(X_{n,m})},$$

where $X_{n,m}$ are finite complexes with universally bounded degree, and $\mu_{X_{n,m}}$ is the measure on K obtained by pushing forward the counting measure on the vertex set of $X_{n,m}$, as in §1. Then if the measures η_n weakly converge, the ratios

$$\frac{\prod_{m=1}^{M_n} t_{n,m} b_k(X_{n,m})}{\prod_{m=1}^{M_n} t_{n,m} \text{vol}(X_{n,m})},$$

converge for all k .

The reader can compare this with Lemma 2.2 in Bowen [10], although that lemma is incorrectly stated⁵.

⁵In Lemma 2.2 of [10], Bowen sets $\eta_{i,j} = t_{i,j} \mu_{K_{i,j}}$, where in his paper the $\mu_{K_{i,j}}$ are the *normalized probability measures* associated to the $K_{i,j}$, but in order to get his conclusion you need to normalize to a probability measure *after* taking the convex combination, like we do in our lemma. Also, it is worth noting that in his proof of Lemma 2.2, Bowen wedges complexes together in order to create a connected complex,

Proof. As in Bowen's proof of [10, Lemma 2.2], it suffices to prove the lemma when the coefficients $t_{n,m}$ are rational, so we can assume we have integers D_n such that $D_n t_{n,m} \in \mathbb{N}$ for all n, m . Then we can create a complex Y_n by taking the disjoint union of $D_n t_{n,m}$ copies of each $X_{n,m}$. Since $\mu_Y / \text{vol}(Y_n) = \eta_n$, these Y_n BS-converge, so by Elek's Theorem 1.1 their normalized Betti numbers converge, and the conclusion of the lemma follows.

Note that in the argument above it is important that Elek's Theorem holds for disconnected complexes, which is why we wrote it that way in the introduction.

Finally, we record the following elementary measure theoretic lemma.

Lemma 2.11. Suppose that (M, vol) is a mm-space and that every ball in M with radius in the interval $[r_0/2, r_0]$ has volume between v_{\min} and v_{\max} . Set $c = v_{\min}^2 / (2v_{\max})$, $c' = v_{\min}^2 / (2v_{\max}^2)$. Then for every measurable subset $A \subseteq M$, we have that

$$v_{\text{ol}} \int_M \chi_A \text{vol}(B_{r_0}(x) \cap A) dx \geq \frac{c \cdot \text{vol}(A)}{\text{vol}(M)} \geq \frac{c' \cdot \text{vol}(A)}{\text{vol}(M)} v_{\text{ol}}(A).$$

Here, the ball $B_{r_0}(x)$ is the metric ball in M . Note that if $\text{vol}(A)/\text{vol}(M)$ is bounded away from zero, the lemma says that a definite proportion of A is taken up by points $x \in A$ such that A takes up a definite proportion of $B_{r_0}(x)$.

Proof. Note that

$$\begin{aligned} \int_M \text{vol}(B_{r_0}(x) \cap A) dx &= \int_M \text{vol}(\{(a, b) \in A^2 \mid d(a, b) < r_0\}) dx \\ &\geq \frac{1}{v_{\max}} \text{vol}(\{(x, a, b) \in M \times A^2 \mid d(a, x) < r_0/2 \text{ and } d(a, b) < r_0\}) \\ &\geq \frac{1}{v_{\max}} \text{vol}(\{(x, a, b) \in M \times A^2 \mid d(a, x) < r_0/2 \text{ and } d(x, b) < r_0/2\}) \\ &= \frac{1}{v_{\max}} \int_M \text{vol}(B_{r_0/2}(x) \cap A)^2 dx \\ &\geq \frac{1}{v_{\max} \text{vol}(M)} \int_M \text{vol}(B_{r_0/2}(x) \cap A)^2 dx = \\ &\quad \frac{1}{v_{\max} \text{vol}(M)} \int_M \text{vol}(B_{r_0/2}(x)) dx \\ &\geq \frac{v_{\min}^2 \text{vol}(A)^2}{v_{\max} \text{vol}(M)} \\ &= 2 \cdot c \cdot \frac{\text{vol}(A)}{\text{vol}(M)} \cdot \text{vol}(A). \end{aligned}$$

but as Elek's theorem actually applies to disconnected complexes, he could have just taken the disjoint union instead of the wedge, which makes his hypothesis on the sizes of the complexes unnecessary.

From this the lemma follows immediately, since if $f : A \rightarrow [0, \max]$ is a function then

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$$\int_A f \geq 2 \cdot \varrho \cdot \text{vol}(A) \implies \text{vol}\{x \in A \mid f(x) \geq \varrho\} \geq \frac{\varrho}{\max} \text{vol}(A).$$

2.4. The proof of Theorem 2.9. We now begin the proof. Recall from the previous section that the general strategy is to show how to construct a random ‘almost net’ in a given finite volume space such that the associated nerve complex N_n has Betti numbers close to those of the nerve complex associated to an actual net. Then one uses Elek’s Theorem (or rather, Lemma 2.10) to show that the normalized Betti numbers of these nerve complexes N_n converge, and finally one deduces from this and property (3) in the statement of Theorem 2.9 that the ratios $B_n/\text{vol}_n(M_n)$ converge.

2.4.1. Random ‘almost nets’. Fix a finite volume special mm-space (M, vol) and real numbers $0 < r_0 < r_1$. For each $j \in \mathbb{N}$, let P^j be a Poisson process on M with intensity 1, and let $f^j : P^j \rightarrow [0, 1]$ be a random function whose values are chosen independently according to Lebesgue measure. Each function f^j is almost surely injective, and when it is, it induces a linear order “ $<$ ” on P^j via $s < t \iff f^j(s) < f^j(t)$. Set

$$P^j(< s) = \{t \in P^j \mid t < s\}.$$

Pick some r with $r_0 < r < r_1$ and choose a continuous function

$$\phi : [0, \infty) \rightarrow [0, 1], \text{ where } \phi(t) = 0 \text{ if } t \leq r_0, \phi(t) = 1 \text{ if } t \geq r.$$

Let $P = \bigcup_j P^j$ be the disjoint union, and for each pair $s, t \in P$, let $x(s, t)$ be a Lebesgue-random element of $[0, 1]$, where the $x(s, t)$ are independent as s, t are varied. We now recursively define subsets

$$S^j \subseteq P^j, \quad S^{\leq j} := S^1 \cup \dots \cup S^j, \quad S^{< j} := S^1 \cup \dots \cup S^{j-1},$$

where given S^1, \dots, S^{j-1} , the rule is that for $s \in P^j$, we say that $s \in S^j$ if

$$\text{for all } t \in P^j (< s) \cap S^{< j}, \quad \phi(d(s, t)) \geq x(s, t).$$

In other words, go through all the elements s of a Poisson process P^1 one by one, in some random order. For each s , backtrack through all previously considered t , flip for each a $[0, 1]$ -valued coin, and add s to S^1 if for each t , the value $\phi(d(s, t))$ is bigger than the result of the coin flip. After finishing with all available s , switch over to a new Poisson process, and add points to S^2 using a similar rule, comparing them against previous points in P^2 and also against all points in S^1 . Then repeat this with a third Poisson process to define S^3 , and a fourth to define S^4 , etc...

For later use, we record:

Claim 2.12. There is some $c = c(r_0, r_1, v_{\min}, v_{\max}) > 0$ such that for all j , if M satisfies conditions (1) and (2) in the statement of the theorem, we have $E[|S^{\leq j}|] \geq c \cdot \text{vol}(M)$.

Proof. Certainly, it suffices to set $j = 1$. Let B_1, \dots, B_k be a maximal collection of disjoint r_0 -balls in M , and note that $k \geq c_1 \cdot \text{vol}(M)$ for some uniform c_1 , by (1) and (2). For each i , there is a probability bigger than some fixed constant that the Poisson process P_1 will intersect the r_1 -neighborhood of B_i in a single point x that lies in B_i . When this happens, this x will automatically be included in S^1 . So, $E[|S^1 \cap B_i|] \geq c_2$ for some uniform $c_2 > 0$. Hence, $E[|S^1|] \geq c_1 \cdot c_2 \cdot \text{vol}(M)$ by linearity of expectation.

By the definition of ϕ , we will almost never add s to S^j if $d(s, t) \leq r_0$ for some previously considered t . So, for each j , the subset $S^{\leq j}$ is almost surely r_0 -separated. On the other hand, we cannot ensure that any particular $S^{\leq j}$ is an (r_0, r_1) -net, since it may not r_1 -cover M . (You do get a net if you take $j = \infty$, as in Bowen's proof.) However, set

$$n^j = \min \{ |T \setminus S^{\leq j}| \mid T \text{ is a } (r_0, 2r_1)\text{-net in } M \text{ with } T \supseteq S^{\leq j} \},$$

random integer associated to each choice of M and j . We prove:

Proposition 2.13. Given $\phi > 0$, there is some $j = j(\phi, r_0, r_1, v_{\min}, v_{\max})$ such that for any M satisfying (1) and (2) in the statement of the theorem, we have

$$\frac{E[n^j]}{\text{vol}(M)} < \phi.$$

Proof. For each j , write R^j for the complement of the r_1 -neighborhood of $S^{\leq j} \setminus M$, and let R_2^j be the complement of the $2r_1$ -neighborhood. If X is any maximal r_0 -separated set in the space R_2^j , then the M -balls $B_{r_0}(x)$, $x \in X$ are disjoint and contained in R^j , so

$$v_{\min} \cdot |X| \leq \text{vol}(R^j).$$

Since the union $X \setminus S^j$ is a $(r_0, 2r_1)$ -net in M , this means $n^j \leq \text{vol}(R^j)/v_{\min}$, so to prove the proposition it suffices (after adjusting ϕ) to find j such that

$$(12) \quad \frac{E[\text{vol}(R^j)]}{\text{vol}(M)} < \phi.$$

Claim 2.14. There is some $\delta = \delta(\phi, r_0, r_1, v_{\min}, v_{\max}) < 1$ as follows. Suppose a fixed $S^{\leq j}$, and hence R^j , is given, and that $\text{vol}(R^j)/\text{vol}(M) \geq \phi/2$. Then

$$E[\text{vol}(R^{j+1}) \mid R^j] \leq \delta \cdot \text{vol}(R^j).$$

Here, we write $E[\text{vol}(R^{j+1}) \mid R^j]$ to indicate that this is the expected volume of R^{j+1} , conditioned on our particular choice of a fixed R^j . This is to remove some ambiguity when we apply the claim later. In the proof of Claim 2.14, though, we will always consider R^j as fixed and just write $E[\cdot]$, omitting any reference to R^j . Also, to avoid a proliferation of constants in the following proof, we will use the notation $x \asymp y$ to mean that $x \leq Cy$ for some constant $C > 0$ depending only on $\phi, r_0, r_1, v_{\min}, v_{\max}$.

Proof. Fix c, c' as in Lemma 2.11. Let R_0^j be the subset of R^j consisting of all points $x \in R^j$ such that $\text{vol}(B_{r_1}(x) \cap R^j) \geq c \cdot \phi/2$. Then Lemma 2.11 says that

$$(13) \quad \text{vol}(R_0^j) \asymp \text{vol}(R^j).$$

Let $R_{\circ\circ}^j$ be the further subset consisting of all points $x \in R^j$ such that $\text{vol}(B_{r_1}(x) \cap R^j) \geq \frac{\eta}{2}$. Applying Lemma 2.11 and using (13), we also have

$$\text{vol}(R_{\circ\circ}^j) \geq \text{vol}(R^j) \cdot \text{vol}(R^j).$$

Fix a maximal r_0 -separated (say) subset $Z \subseteq R^j$. By assumption, there is an upper bound for the volumes of all the r_0 -balls around points $z \in Z$, so since $R_{\circ\circ}^j$ is contained in the union of all such balls, our lower bound on the volume of $R_{\circ\circ}^j$ implies that

$$(14) \quad |Z| \geq \text{vol}(R^j).$$

For each $z \in Z \subseteq R^j$, the volume of $B_{r_1}(z) \cap R^j$ is bounded below and the volume of $B_{2r_1}(z)$ is bounded above and below. So if P^{j+1} is the Poisson process used in defining S^{j+1} , we have

$$(15) \quad P^{j+1} \cap B_{r_1}(z) \cap R^j = \emptyset \text{ and } |P^{j+1} \cap B_{2r_1}(z)| = 1$$

with probability bigger than some fixed constant. So by linearity of expectation and (14),

$$E^h \{z \in Z \mid (15) \text{ holds for } z\} \geq \text{vol}(R^j),$$

where the expectation is taken over the Poisson process P^{j+1} . But for every z such that (15) holds, the single point of P^{j+1} that is in $B_{r_1}(z) \cap R^j$ is at least r_1 -away from every other point of P^{j+1} , and is also at least r_1 -away from S^{j+1} , since $z \in R^j$. Hence, this single point lies not just in P^{j+1} , but in S^{j+1} . It follows that

$$(16) \quad E^h \{x \in S^{j+1} \cap R^j \mid B_{r_1}(x) \cap S^{j+1} = \{x\}\} \geq \text{vol}(R^j).$$

Note that $R^{j+1} = R^j \setminus \bigcup_{x \in S^{j+1}} B_{r_1}(x)$. By definition of $R_{\circ\circ}^j$, the r_1 -ball around each $x \in S^{j+1} \cap R^j$ intersects R^j in a set with volume bounded below, and if we only look at those x where $B_{r_1}(x) \cap S^{j+1} = \{x\}$, all the balls $B_{r_1}(x)$ are disjoint. So by (16),

$$E^h \text{vol} \left[\bigcap_{x \in S^{j+1}} B_{r_1}(x) \right] \geq \text{vol}(R^j),$$

and the claim follows.

We now complete the proof of the proposition. Let σ^j be the law of S^{j+1} . Then conditioning on whether $\text{vol}(R^j)/\text{vol}(M) \geq \frac{\eta}{2}$ or not, we have

$$(17) \quad \frac{E[\text{vol}(R^{j+1})]}{\text{vol}(M)} \leq \frac{\eta}{2} + \int_{\text{vol}(R^j)/\text{vol}(M) \geq \frac{\eta}{2}} \frac{E[\text{vol}(R^{j+1}) \mid R^j]}{\text{vol}(M)} d\sigma^j.$$

Let's call the second term on the right in (17) X^{j+1} . Then by Claim 2.14, we have

$$X^{j+1} \leq \delta \cdot \int_{\text{vol}(R^j)/\text{vol}(M) \geq \frac{\eta}{2}} \frac{\text{vol}(R^j)}{\text{vol}(M)} d\sigma^j \leq \delta \cdot \int_{\text{vol}(R^{j-1})/\text{vol}(M) \geq \frac{\eta}{2}} \frac{E[\text{vol}(R^j)]}{\text{vol}(M)} d\sigma^{j-1} \leq \delta X^j$$

for all j , where the middle inequality uses the inclusion $R^j \supseteq R^{j-1}$ to say that the condition on $\text{vol}(R^j)$ is at least as restrictive as the condition on $\text{vol}(R^{j-1})$. Since $\delta < 1$ is fixed, there is some uniform $j = j(\varrho, r_0, r_1, v_{\min}, v_{\max})$ such that $x^j < \varrho/2$, and then

$$\frac{E[\text{vol}(R^j)]}{\text{vol}(M)} < \varrho/2 + \varrho/2 = \varrho$$

as desired in (12).

2.4.2. Continuity of the random almost nets $S^{\leq j}$. If $M = (M, \text{vol}, p, E)$ is an extended special, pointed mm-space and j is fixed, we can choose a random element of MS^{ext} by choosing a random $S^{\leq j} \in M$ as above. The law of this random element is a measure $\sigma^{\leq j} = \sigma^{\leq j}(M)$ on MS^{ext} depending on M , and this defines a function

$$(18) \quad M^{\text{ext}} \rightarrow P(MS^{\text{ext}}), \quad M \rightarrow \sigma^{\leq j}(M)$$

where as usual $P(\cdot)$ denotes the space of probability measures.

Lemma 2.15. The function in (18) is continuous for all j .

This proof is suggested by Claim 2 on pg 583 of [10], but Bowen only does the $j = 1$ case, so for completeness we will give the argument here. Note that Bowen also does not prove continuity of the Poisson process, which we did above.

Proof. In Lemma 2.7, we showed that the map

$$(19) \quad M^{\text{ext}} \rightarrow P(MS^{\text{ext}}), \quad M \rightarrow \rho_M$$

that associates to an extended mm-space its Poisson process is continuous. Consider the map

$$(20) \quad MS^{\text{ext}} \rightarrow P(MF^{\text{ext}})$$

that takes a tuple $M = (M, \text{vol}, p, E, P)$ to the measure v_M whose random element is of the form $(M, \text{vol}, p, E, P, f)$, where the values of $f : P \rightarrow \mathbb{R}$ are chosen independently and uniformly from $[0, 1]$. We claim that the map (20) is continuous. For suppose $(M_i, \text{vol}_i, p_i, E_i, P_i) \rightarrow (M, \text{vol}, p, E, P)$ in MS^{ext} . Then Lemma 2.5 allows us to realize the convergence within $MS^{\text{ext}}(Z)$ for some Z . Fixing some $f : P \rightarrow [0, 1]$ and an admissible pair (ϱ, R) for $(M, \text{vol}, p, E, P, f)$, consider the neighborhood $B_{\varrho, R}$ given by Lemma 2.6. For large i , the hypothesized convergence gives bijections

$$\psi_i : P \cap B_M(p, R) \rightarrow P_i \cap B_{M_i}^{\bullet}(p_i, R)$$

such that $d_Z(P, \psi_i(s)) < \varrho$ for all s , and a tuple $(M_i, \text{vol}_i, p_i, E_i, P_i, f_i)$ is in $B_{\varrho, R}$ exactly when $|f_i(s) - f_i(\psi_i(s))| < \varrho$ for all s . This event has v_M -measure $\varrho^{|P \cap B_M(p, R)|}$ for $i = 1, 2, \dots, \infty$, so by the Portmanteau theorem we have $v_M \rightarrow v_M$ as desired.

It follows from the above that the composition

$$M^{\text{ext}} \rightarrow P(MS^{\text{ext}}) \rightarrow P(P(MF^{\text{ext}})) \rightarrow P(MF^{\text{ext}})$$

is continuous, where the first map is (19), the second is the weakly continuous map induced by (20), and the third is the expectation map. Given $M \in M^{ext}$, a random element of the associated measure on $P(MF^{ext})$ is exactly a $[0, 1]$ -weighted Poisson process (P, f) .

Let $MF^{j,ext}$ be the set of all pointed, extended mm-spaces (M, p, vol, E) that come equipped with j weighted locally finite subsets (P^i, f^i) , $i = 1, \dots, j$, which we topologize by regarding it as a subset of $M^{2j+1,ext}$. Consider the map

$$(21) \quad M^{ext} \rightarrow P(MF^{j,ext})$$

that takes (M, vol, p, E) to the measure whose random element is given by selecting j weighted Poisson processes randomly on M , as above. Then the same arguments as above show that (21) is continuous. Let $MF_{inj}^{j,ext} \subset MF^{j,ext}$ be the subset consisting of tuples where all the f^i are injective and consider the map

$$(22) \quad MF_{inj}^{j,ext} \rightarrow P(MS^{ext})$$

that takes $(M, vol, p, E, P^1, f^1, \dots, P^j, f^j)$ to the measure whose random element is the tuple $(M, vol, p, E, S^{\leq j})$ constructed from the given data as in the previous section. (This last element of randomness comes from the need to pick a random value $X(s, t) \in [0, 1]$ for every pair of elements $s, t \in P := \bigcup P^i$.) Since (21) is continuous and any measure in its image gives $MF_{inj}^{j,ext}$ full mass, it suffices now to show that (22) is continuous.

So, suppose that $M_i = (M_i, vol_i, p_i, E_i, P_i^1, f_i^1, \dots, P_i^j, f_i^j) \in MF_{inj}^{j,ext}$, where $i = 1, \dots, \infty$, and that $M_i \rightarrow M_\infty$. Applying the appropriate analogue of Lemma 2.5, we can assume that this convergence happens within $MF_{inj}^{j,ext}(Z)$ for some fixed Z . Then the construction in (22) gives a sequence of probability measures v_i on $MS^{ext}(Z)$, where a v_i -random element is $(M_i, vol_i, p_i, E_i, S_i^{\leq j})$, with $S_i^{\leq j}$ constructed from (P_i^j, f_i^j) as in the previous section. We want to show that the measures v_i weakly converge.

Set $P_i := \bigoplus_{k=1}^j P_i^k$. After discarding finitely many i , Lemma 2.6 says that there is a sequence (ϱ_i, R_i) of pairs, where $\varrho_i \rightarrow 0$ and $R_i \rightarrow \infty$ and where each pair is admissible with respect to $(M_\infty, vol_\infty, p_\infty, E_\infty, P_\infty)$, and bijections

$$\phi_i : P_\infty \cap B_{M_\infty}(p_\infty, R_i) \rightarrow P_i \cap B_{M_i}(p_i, R_i)$$

such that $d_Z(s, \phi_i(s)) < \varrho_i$ and $|f_\infty(s) - f_i(\phi_i(s))| < \varrho_i$ for all s . The following claim is in some sense the heart of the proof of this lemma ⁶.

Claim 2.16. For any fixed $s_\infty \in P_\infty$, the v_i -probability that $\phi_i(s_\infty) \in S_i^{\leq j}$ converges to the v_∞ -probability that $s_\infty \in S_\infty^{\leq j}$.

Proof. Let us recall the definition of S^j . Fix some function $\rho : [0, \infty] \rightarrow [0, 1]$ with $\rho(t) = 0$ if $t \leq r_0$, $\rho(t) = 1$ if $t \geq r_1$; we called this function ϕ before. Select random

⁶The proof of Claim 2.16 is what fails if you let $j = \infty$ as in Bowen's paper [10]. For in that case, the decision to include an element $s \in S_i^{\leq \infty}$ does not depend just on the part of P_i that lies in a neighborhood of fixed radius around s , but potentially on the entire P_i .

elements $x(s, t) \in [0, 1]$ for all $s, t \in P_i$. Then $S_i^j \subseteq P_i^j$ and $S_i^{\leq j} = S_i \cup \dots \cup S^j$ are defined recursively with respect to j , where an element $s \in P_i^j$ is added to S_i^j exactly when

$$\rho(d(s, t)) \geq X(s, t)$$

for all $t \in P_i^j$ with $f_i(t) < f_i(s)$, and for all $t \in S_i^{\leq j-1}$. Note that since $\rho(t) = 1$ if $t \geq r_1$, the decision to include $s \in S_i^j$ only depends on points in the r_1 -neighborhood of s . So, the decision to include a point s in the set $S_i^{\leq j}$ depends only on the $j \cdot r_1$ -neighborhood of s .

Fix some $R > d_Z(s_\infty, p_\infty) + j \cdot r_1$. Since f_∞ is injective and $P_\infty^R := P_\infty \cap B_{M_\infty}(p_\infty, R)$ is finite, for all large i we have

$$f_\infty(s) < f_\infty(t) \iff f_i(\phi_i(s)) < f_i(\phi_i(t))$$

for all $s, t \in P_\infty^R$. Then the probability that s_∞ in $S_\infty^{\leq j}$ can be calculated using the same computation from the order on and the distance between elements of P_∞^R . For large i the probability that $\phi_i(s_\infty)$ in $S_i^{\leq j}$ is calculated in the same way from $\phi_i(P_\infty^R)$, which contains all points of P_i within a distance of $j \cdot r_1$ from $\phi_i(s_\infty)$. The order on P_∞^R agrees with that on its ϕ_i -image, and for large i the distances between points of P_∞^R almost agree with the distances between their ϕ_i -images. From this, the claim follows.

Returning to the proof of the lemma, the measure v_∞ is supported on points of the form

$$(23) \quad (M_\infty, \text{vol}_\infty, p_\infty, E_\infty, S_\infty) \in \text{MS}^{\text{ext}}(Z), \quad S_\infty \subseteq P_\infty.$$

If (ϱ, R) is an admissible pair for such a point, let $B_{\varrho, R}(S_\infty)$ be the neighborhood that is constructed in Lemma 2.6, i.e. the set of all $(M, \text{vol}, p, E, S) \in \text{MS}^{\text{ext}}(Z)$ such that there is a bijection from $S_\infty \cap B_{M_\infty}(p_\infty, R)$ to $S \cap B_{M}(p, R)$ that matches points that are within ϱ of each other in Z . As (ϱ, R) varies, these sets form a neighborhood basis for the point in (23). So by the Portmanteau Theorem, to prove $v_i \rightarrow v_\infty$ it suffices to show that

$$(24) \quad v_i(B_{\varrho, R}(S_\infty)) \rightarrow v_\infty(B_{\varrho, R}(S_\infty)),$$

But for large i , we have that

$$(M_i, \text{vol}_i, p_i, E_i, S_i) \in B_{\varrho, R}(S_\infty) \iff S_i \cap B_{M_i}^*(p_i, R) = \phi_i(S_\infty \cap B_{M_\infty}(p_\infty, R)),$$

since for such tuples and large i the bijection in the definition of $B_{\varrho, R}(S_\infty)$ must be the restriction of ϕ_i . This is equivalent to saying that for all $s_\infty \in P_\infty \cap B_{M_\infty}(p_\infty, R)$, we have $s_\infty \in S_\infty$ if and only if $\phi_i(s_\infty) \in S_i$. But then (24) follows from Claim 2.16 via inclusion-exclusion.

2.4.3. Random nerve complexes. Above, we defined the space MS^{ext} of pointed, extended special mm-spaces with distinguished discrete subsets. Here, we explain how to construct a random nerve complex from certain elements of MS^{ext} , in a way that depends continuously on the input. Consider the subset

$$(\text{MS}^{\text{ext}})^\circ \subseteq \text{MS}^{\text{ext}}$$

consisting of all tuples (M, vol, p, E, s) such that there is a unique element of S that is closest to p . Given some such tuple, construct a simplicial complex by choosing independently and Lebesgue-randomly a number $\rho(x) \in [r_2, r_3]$ for each $x \in S$ and taking the nerve $N_E(S, \rho)$ of the collection of balls $B_E(x, \rho(x))$, where $x \in S$. Note that the balls are in E , not in M . The unique element of S closest to p is a natural base point for the nerve, and if K is the connected component of $N_E(S, \rho)$ containing p then we have a map

$$(25) \quad (MS^{\text{ext}})' \rightarrow P(K),$$

where K is the space of all pointed, finite degree simplicial complexes, $P(\cdot)$ denotes the space of probability measures, and the map sends (M, vol, p, E, s) to the (law of the) random pointed complex (K, p) described above. Note that properties (1) and (2) in the statement of the theorem imply that there is a universal degree bound for all constructed K , that is independent of n .

Claim 2.17. The map in (25) is continuous.

Proof. In his Claim 1 at the beginning of the proof of Theorem 4.1, Bowen shows that a variant of (25) is continuous. Here are the discrepancies with our version. First, Bowen works only with $[r_2, r_3] = [5r_0, 6r_0]$, but his argument obviously generalizes. He also does not use extended mm-spaces, and so the map he constructs is from the subset $MS' \subset MS$ of all (M, p, vol, S) where there is a unique closest point in s to p , and the balls he uses in constructing the nerve are just in M , not in some larger space. However, the quick proof of continuity works verbatim in our extended setting: basically all that is used is that the topology comes from pointwise Hausdorff convergence of the mm-spaces M , so since we are putting the same topology on the super-sets E , the argument extends.

2.4.4. Convergence of normalized Betti numbers. We now begin on the main argument for the proof of Theorem 2.9. Most of the ideas below are from [10], altered so that we use our $S^{\leq j}$ instead of the Elek/Bowen random nets.

Let $M_n = (M_n, \text{vol}_n, E_n)$ be as in the theorem statement, and fix $\varrho > 0$. We will show

$$(26) \quad \limsup_{n \rightarrow \infty} \frac{B_n}{\text{vol}_n(M_n)} - \liminf_{n \rightarrow \infty} \frac{B_n}{\text{vol}_n(M_n)} \leq C \cdot \varrho$$

for some C depending only on the constants in the statement of the theorem. Since ϱ is arbitrary, this will suffice to prove convergence of $B_n/\text{vol}_n(M_n)$.

Pick $j = j(\varrho, r_0, r_1, v_{\min}, v_{\max})$ as in Proposition 2.13, and as in §2.4.2 let σ_n be the law of the random almost-net $S^{\leq j} \subset M_n$ constructed in §2.4.1. For simplicity in notation, we'll drop the superscript j below and just write S_n for a σ_n -random subset of M_n . Note that by our choice of j , the almost-net S_n can always be extended to a $(r_0, 2r_1)$ -net $T_n \subset M_n$ in such a way that for every n , we have

$$(27) \quad E |T_n \setminus S_n| \leq \varrho \cdot \text{vol}_n(M_n).$$

Fix some almost-net S_n and an extension T_n as above. If ρ_n is a function that assigns some $\rho_n(x) \in [r_2, r_3]$ to each $x \in S_n$, let $N_{E_n}(S_n, \rho_n)$ be the nerve complex of the collection

of balls $B_{E_n}(x, \rho_n(x))$, where for each $x \in S_n$ the radius is chosen Lebesgue-randomly, just as in §2.4.3. Extending ρ_n to T_n arbitrarily, define $N_E(T_n, \rho_n)$ similarly. Then $N_E(S_n, \rho_n)$ is a full subcomplex of $N_E(T_n, \rho_n)$. By conditions (1) and (2) in the statement of the theorem, the degrees of these complexes are bounded by some universal constant D . (Indeed, any $x \in T_n$ represents a vertex of the nerve that is connected only to vertices $y \in T_n$ such that $d(x, y) \leq 2r_3$. Since points in T_n are r_0 -separated, the degree of T_n is then bounded by the number of $r_0/2$ -balls one can pack into a $(2r_3 + r_0/2)$ -ball in M_n . The volume bounds in (1) and (2) imply that this packing constant is universally bounded.) Since S_n is a full subcomplex of T_n , and $\deg(T_n)$ is universally bounded, it follows that the total number of simplices in $T_n \setminus S_n$ is at most $C \cdot |T_n \setminus S_n|$, for some fixed constant C depending only on the constants in the statement of the theorem. Hence, we also have

$$(28) \quad b_k(N_{E_n}(T_n, \rho_n)) - b_k(N_{E_n}(S_n, \rho_n)) \leq C \cdot |T_n \setminus S_n|$$

for such a constant C , by Mayer-Vietoris.

Now condition (3) in the statement of the theorem implies there are $\delta_n \rightarrow 0$ such that

$$(29) \quad b_k(N_{E_n}(T_n, \rho_n)) - B_n \leq \delta_n \text{vol}_n(M_n)$$

for every sequence of nets T_n as above, and any radii ρ_n . (A priori, maybe this looks like a stronger statement, but if it were not true, we could apply (3) to a sequence of weighted nets with Betti numbers maximally different from B_n to get a contradiction.) It follows from (28) and (29) that for any S_n and any choice of radii ρ_n , we have

$$(30) \quad |b_k(N_{E_n}(S_n, \rho_n)) - B_n| \leq C \cdot |T_n \setminus S_n| + \delta_n \text{vol}_n(M_n).$$

So, if we σ_n -randomly chooses the almost-nets S_n and randomly choose the radii $\rho_n(x) \in [r_2, r_3]$ independently and uniformly for each $x \in S_n$, then we have

$$\begin{aligned} E \ b_k(N_{E_n}(S_n, \rho_n)) - B_n &\leq E \ b_k(N_{E_n}(S_n, \rho_n)) - B_n \\ &\leq C \cdot E[|T_n \setminus S_n|] + \delta_n \text{vol}_n(M_n) \\ &\leq C \cdot q \cdot \text{vol}_n(M_n) + \delta_n \text{vol}_n(M_n). \end{aligned}$$

where the second inequality is (30) and the third is (27). So, to prove that the liminf and limsup of $B_n/\text{vol}_n(M_n)$ are within Cq of each other, which is the goal we set in (26), it suffices to prove the following claim.

Claim 2.18. The ratio $E b_k(N_{E_n}(S_n, \rho_n))/\text{vol}_n(M_n)$ converges as $n \rightarrow \infty$.

We now show how to use Elek's Theorem 1.1 to reduce Claim 2.18 to two other convergence claims. For simplicity, let τ_n be the law of a σ_n -random $S_n \in M_n$ equipped with uniformly and independently chosen radii $\rho_n(x) \in [r_2, r_3]$ at each $x \in S_n$. So, the expectation in (30) is taken with respect to τ_n . As in the introduction, let K be the space of pointed complexes and consider the probability measure

$$(31) \quad \eta_n := \frac{\int_{\mathbb{R}} \mu_{N_{E_n}(S_n, \rho_n)} d\tau_n}{\text{vol}_n(S_n)} \in P(K).$$

Note that an η_n -random pointed complex is not produced by τ_n -randomly choosing (S_n, ρ_n) and then choosing a base point for the corresponding nerve complex uniformly randomly. For the cardinalities $|S_n|$ can vary depending on the particular subset, and the nerve complex of a particular S_n appears more often with respect to η_n if $|S_n|$ is larger. Intuitively, one should think that η_n assigns ‘equal weights’ to all the vertices in all the different complexes, rather than weighting the (unpointed) complexes themselves ‘equally’.

Now, the randomly produced nerve complexes $N_{E_n}(S_n, \rho_n)$ all have volume at most some universal constant times $\text{vol}_n(M_n)$. (Indeed, the points of S_n are r_0 -separated and by condition (2) in the statement of the theorem, we have a lower bound on the volume of every $r_0/2$ -ball in M_n .) So, for a fixed n the nerve complex takes on only finitely many isomorphism types. In other words, the measure η is really a finite linear combination of the measures μ_K associated to certain unpointed complexes K , as in Lemma 2.10, which was a slightly more general version of Elek’s Theorem 1.1.

Claim 2.19. The probability measures η_n weakly converge in $P(K)$.

Assuming this for a moment, we conclude from Lemma 2.10 that the ratios

$$\frac{E[b_k(N_{E_n}(S_n, \rho_n))]}{E[|S_n|]}$$

converges for all k . We will also show the following.

Claim 2.20. The ratio $E[|S_n|]/\text{vol}_n(M_n)$ converges.

Assuming both Claims 2.19 and 2.20, we then have that the ratio

$$\frac{E[b_k(N_{E_n}(S_n, \rho_n))]}{\text{vol}_n(M_n)} = \frac{E[b_k(N_{E_n}(S_n, \rho_n))]}{E[|S_n|]} \cdot \frac{E[|S_n|]}{\text{vol}_n(M_n)}$$

converges, as desired, proving Claim 2.18 and hence Theorem 2.9.

Claims 2.19 and 2.20 will be proved simultaneously. For Claim 2.19, the point is to use that the extended mm-spaces M_n BS-converge, and to translate that into convergence of the measures η_n . Since BS-convergence relies on randomly picking base points, one would like to relate randomly chosen basepoints in M_n to randomly chosen points of S_n , which can be used as base points of the associated nerve complexes. Intuitively, the idea is just to associate to a point $p \in M_n$ the point of S_n closest to p . However, there might not be a unique such closest point, and even if there is, the set of points in M_n closest to some $p \in S_n$ may have much different volume from the set of points closest to some other $q \in S_n$, so a Lebesgue-random basepoint in M_n may not correspond to a uniformly random point in S_n . The idea, then, is to show that we still obtain a weakly convergent sequence of measures if instead of choosing Lebesgue-random basepoints from M_n , we only choose them from small fixed-volume balls around the points of a random S_n . This convergence will translate directly into convergence of the measures η_n above.

To formalize this idea above, we work measure theoretically instead of probabilistically. For each n , let σ_n be the law of S_n and let λ_n be the measure on MS^{ext} (see §2.4.2) obtained

by pushing forward the product measure $(\text{vol}_n/\text{vol}_n(M_n)) \times \sigma_n$ under the map

$$(32) \quad M_n \times \{ \text{discrete } S \subset M_n \} \rightarrow MS^{\text{ext}}, \quad (p, S) \mapsto (M_n, \text{vol}_n, p, E_n, S).$$

This λ_n can also be obtained by pushing forward $\mu_{M_n}/\text{vol}_n(M_n)$ via the continuous map

$$M_{\text{sp}}^{\text{ext}} \rightarrow P(MS^{\text{ext}})$$

from (18) then taking the expected value. In symbols,

$$\mu_{M_n} \# P(M_{\text{sp}}^{\text{ext}}) \xrightarrow{(18)} P(P(MS^{\text{ext}})) \xrightarrow{E[\cdot]} P(MS^{\text{ext}}) \# \lambda_n$$

Since both the maps in this composition are continuous and (μ_{M_n}) is weakly convergent, it follows that (λ_n) converges weakly to some probability measure λ_∞ on MS^{ext} .

Fix some v with $0 < v < v_{\min}$, where v_{\min} is a lower bound for the volume of any $r_0/2$ -ball in any M_n , as in condition (1) of the theorem. If $S \subset M$ is a discrete set, let $S(v)$ be the union of all volume- v closed balls in M that are centered at points of S . Let

$$MS^{\text{ext}}(v) \# MS^{\text{ext}}$$

be the closed subset consisting of all tuples (M, vol, p, S, E) such that $p \# S(v)$.

Since $v < v_{\min}$, the volume- v balls around the points of any r_0 -separated set like $S_n \subset M_n$ are all disjoint, so we have $\text{vol}_n(S_n(v)) = v|S_n|$. Given S_n , the probability then that a randomly chosen $p \# M_n$ ends up in $S_n(v)$ is

$$v|S_n|/\text{vol}_n(M_n).$$

Integrating over S_n , we have for $n = 1, 2, \dots$ that

$$(33) \quad \lambda_n(MS^{\text{ext}}(v)) = v \cdot \frac{E[|S_n|]}{\text{vol}_n(M_n)},$$

where S_n is chosen σ_n -randomly.

Claim 2.21. We have $\lambda_n(MS^{\text{ext}}(v)) \rightarrow \lambda_\infty(MS^{\text{ext}}(v))$.

Proof. Fix some very small $\eta > 0$. As mentioned above, the sets S_n are r_0 -separated, so the ratios $|S_n|/\text{vol}_n(M_n)$ are bounded above by some C depending on the constants in properties (1) and (2) in the theorem. So if we set $a = v + \eta/C$, it follows from (33) that

$$\begin{aligned} \lambda_n(MS^{\text{ext}}(a)) &\leq a \frac{E[|S_n|]}{\text{vol}_n(M_n)} \\ &\leq \frac{v_{\min}}{2} \frac{E[|S_n|]}{\text{vol}_n(M_n)} + \eta \\ &= \lambda_n(MS^{\text{ext}}(v)) + \eta. \end{aligned}$$

Since $MS^{\text{ext}}(v)$ is closed and the interior of $MS^{\text{ext}}(a)$ contains $MS^{\text{ext}}(v)$, it then follows from the Portmanteau theorem that

$$\begin{aligned} \limsup \lambda_n(MS^{\text{ext}}(v)) &\leq \lambda_\infty(MS^{\text{ext}}(v)) \\ &\leq \liminf \lambda_n(MS^{\text{ext}}(a)) \end{aligned}$$

$$\leq \liminf \lambda_n(MS^{\text{ext}}(v)) + \eta.$$

And then the claim follows, since η was arbitrary.

Now let λ'_n be the probability measure on $MS^{\text{ext}}(v)$ that we get by normalizing the restriction of λ_n . If $C \subseteq MS^{\text{ext}}(v)$ is closed, then C is also closed in MS^{ext} . So by Claim 2.21 and the Portmanteau Theorem, we have that

$$\limsup \lambda'_n(C) = \limsup \frac{\lambda_n(C)}{\lambda_n(MS^{\text{ext}}(v))} \leq \frac{\lambda_\infty(C)}{\lambda_\infty(MS^{\text{ext}}(v))} = \lambda'_\infty(C).$$

Applying the Portmanteau Theorem again, we see that $\lambda'_n \rightarrow \lambda'_\infty$ weakly.

Since all the S_n are r_0 -separated, if v is sufficiently small relative to the constants in properties (1) and (2) in the theorem, the basepoint p of λ'_n -almost every 5-tuple

$$(M_n, \text{vol}_n, p, S_n, E_n) \in MS^{\text{ext}}(v)$$

is closest in M_n to the element $q \in S_n$ that is the center of the volume- v ball in which p lies. So, if K is the space of pointed complexes, the random nerve complex map

$$MS^{\text{ext}}(v) \rightarrow P(K)$$

one gets by restricting the map in (25) takes the 5-tuple above to a measure whose random element is obtained by picking a random p_n and then taking the connected component of $N_{E_n}(S_n, p_n)$ that is rooted at the vertex $q \in S_n$ in whose volume- v ball p lies.

Now, the restriction of λ_n to $MS^{\text{ext}}(v)$ is obtained by integrating the Lebesgue measure on $S_n(v)$ against σ_n . So, the image of λ'_n under the composition

$$(34) \quad P(MS^{\text{ext}}(v)) \rightarrow P(P(K)) \xrightarrow{E[\cdot]} P(K)$$

is a probability measure on K obtained by integrating the counting measure on the vertices of $N_{E_n}(S_n, p_n)$ against the measure τ_n that is the law of the random weighted nets (S_n, p_n) . In other words, λ'_n pushes forward to the measure η_n from (31). By Claim 2.17, the composition (34) is continuous, so the fact that the λ'_n weakly converge means that the η_n also weakly converge. This proves Claim 2.19. Claim 2.20 follows immediately from (33) and Claim 2.21, so our proof of Theorem 2.9 is done.

2.5. A variation of Theorem 2.9. The following variant of Theorem 2.9 will serve us in the sequel:

Corollary 2.22. Fix k , let $M_n = (M_n, \text{vol}_n, E_n)$ be a sequence of extended finite volume special mm-spaces and assume that for some sequence of constants V_n , the measures μ_{M_n}/V_n weakly converge to some finite measure μ on M^{ext} . Pick constants $v_{\min} > 0$, $r_1 > r_0 > 0$, and $r_3 > r_2 \geq 2r_1$, and a function $v_{\max} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

- (1) all $r_0/2$ -balls in every M_n have volume at least v_{\min} ,
- (2) for all $r \in \mathbb{R}_+$, every r -ball in M_n has volume at most $v_{\max}(r)$,

(3) for every sequence of $[r_2, r_3]$ -weighted $(r_0, 2r_1)$ -nets (S_n, ρ_n) in M_n ,

$$\frac{b_k(N_{E_n}(S_n, \rho_n)) - B_n}{V_n} \xrightarrow{n} 0.$$

Then the ratios B_n/V_n converge.

Note that here, the measures μ_{M_n}/V_n and their limit may not be probability measures.

Proof of Corollary 2.22 given Theorem 2.9. Let μ be the weak limit of μ_{M_n}/V_n . Then

$$\lim_{n \rightarrow \infty} \frac{\text{vol}_n(M_n)}{V_n} = \lim_{n \rightarrow \infty} \int M_n d(\mu_{M_n}/V_n) = \mu(M^{\text{ext}}) \in [0, \infty).$$

Suppose first that $\mu(M^{\text{ext}}) = 0$. By (1) and (2), the number of points in any $(r_0, 2r_1)$ -net in M_n is comparable to $\text{vol}_n(M_n)$, so the Betti numbers in (3) are $O(\text{vol}_n(M_n))$. Combining (3) and the triangle inequality, we have $B_n/V_n \rightarrow 0$.

If $\mu(M^{\text{ext}}) > 0$, then $V_n/\text{vol}_n(M_n)$ has a finite limit, so the probability measures

$$\frac{\mu_{M_n}}{\text{vol}_n(M_n)} = \frac{V_n}{\text{vol}_n(M_n)} \cdot \frac{\mu_{M_n}}{V_n} \xrightarrow{n} \frac{\mu}{\mu(M^{\text{ext}})}.$$

In other words, the extended mm-spaces M_n BS-converge. Also, $\text{vol}_n(M_n)$ and V_n are of bounded ratio, so (3) holds with $\text{vol}_n(M_n)$ instead of V_n . Theorem 2.9 then says that $B_n/\text{vol}_n(M_n)$ converges, from which it follows that B_n/V_n converges too.

3. Pinched negative curvature and Theorem 1.6

In this section, we consider only d -manifolds M with sectional curvature

$$-1 \leq K \leq -a^2 < 0,$$

and we let $q(d)$ be the corresponding d -dimensional Margulis constant. For any $q \leq q(d)$, each component of the q -thin part $(M_n)_{\leq q}$ is either:

- a Margulis tube, which is (topologically) a tubular neighborhood of a closed geodesic, and so is homeomorphic to a ball bundle over the circle, or
- a cusp neighborhood, which is homeomorphic to $S \times [0, \infty)$ for some compact aspherical $(d-1)$ -manifold S with virtually nilpotent fundamental group.

See for instance [6, §8] for a proof.

In the introduction, we explained how to produce BS-convergent sequences (M_n) of hyperbolic 3-manifolds where the normalized Betti numbers do not converge, using Dehn filling. In the example we gave, the volumes $\text{vol}(M_n)$ were bounded, but one can construct similar examples with unbounded volumes by filling the complements of links with unboundedly many components, instead of a fixed knot complement. Instead of doing the details of this approach, though, we'll briefly describe a similar example in which the BS-limit is easier to understand.

Example 3.1. Let M be the mapping torus of a homeomorphism $\phi : S \rightarrow S$, where ϕ is a pseudo-Anosov homeomorphism of some closed surface S with genus at least 2. So, M comes with a fibration $M \rightarrow S^1$. Identify S with a fiber of this fibration, and let γ be a simple closed curve on S . By Thurston's Hyperbolization Theorem [22], the manifold

$$M(\infty) := M \setminus \gamma$$

admits a finite volume hyperbolic metric.

Let $M(k)$ be the closed 3-manifold obtained from $M(\infty)$ by $(1, k)$ -Dehn filling⁷. For large k , Thurston's Dehn Filling Theorem [7] implies that $M(k)$ admits a hyperbolic metric; moreover, as $k \rightarrow \infty$ the manifolds $M(k) \rightarrow M(\infty)$ geometrically. Note that since we are doing $(1, k)$ filling, each $M(k)$ is also a genus g mapping torus. Indeed, if T_γ is a Dehn twist around γ , the monodromy map of $M(k)$ is $T_\gamma^k \circ \phi$.

For $k \in \mathbb{N} \setminus \{\infty\}$, let $M_n(k)$ be the degree n cyclic cover of $M(k)$ corresponding to the subgroup of $\pi_1 M(k)$ that is the preimage of $n\mathbb{Z} \leq \mathbb{Z} \oplus \pi_1(S^1)$ under the map induced by

$$M(k) \xrightarrow{\sim} M \rightarrow S^1.$$

Then for every n and $k < \infty$, the manifold $M_n(k)$ is a mapping torus over a genus g surface, and hence

$$b_1(M_n(k)) \leq 2g + 1.$$

On the other hand, setting $k = \infty$ the manifold $M_n(\infty)$ has n cusps, so we have

$$b_1(M_n(\infty)) \geq n.$$

Now set $k = n$. As $n \rightarrow \infty$, the sequences $M_n(n)$ and $M_n(\infty)$ both BS-converge to the same limit measure μ on M . This μ is supported on pointed manifolds isometric to the infinite cyclic cover $M_\infty(\infty)$ of $M(\infty)$ corresponding to the kernel of the map on fundamental groups induced by $M(\infty) \rightarrow S^1$; more carefully, μ is the push forward of the normalized Riemannian measure on $M(\infty)$ under the map

$$M(\infty) \rightarrow M, \quad p \rightarrow [(M_\infty(\infty), p_\infty)],$$

where p_∞ is any point that projects to p under the covering map $M_\infty(\infty) \rightarrow M(\infty)$. (This is a special case of the construction in Example 2.4 in [2].) However,

$$b_1(M_n(n))/\text{vol}(M_n(n)) \rightarrow 0, \quad b_1(M_n(\infty))/\text{vol}(M_n(\infty)) \rightarrow 0.$$

Essentially, the reason why Dehn filling is problematic is that from the perspective of most points in a manifold, a Margulis tube with very small core length can look nearly identical to a rank two cusp. (One can only see the difference if one is close enough to be able to distinguish the core geodesic of the tube, and when the core length is small, the set of points a bounded distance from the core has very small volume.) This coincidence is particularly three-dimensional, though. For instance, note that the boundary of a

⁷Here, we use meridian-longitude coordinates to parametrize the boundary of a cusp neighborhood, where the meridian is the curve that was homotopically trivial before we drilled out γ .

d -dimensional Margulis tube is a S^{d-2} -bundle over S^1 , while the boundary of a cusp neighborhood is a Euclidean $(d-1)$ -manifold. If $d=3$, the torus T^2 satisfies both descriptions, but when $d \geq 4$, S^{d-2} -bundles over S^1 are not aspherical, so cannot be Euclidean.

The plan for the rest of §3 is as follows. In §3.1 we show that Margulis tubes with short cores have large volume in dimension at least 4. In §3.2 we adapt some of Gelander's work in [18], showing that one can approximate (shrinkings of) the q -thick parts of manifolds with pinched negative curvature with certain nerve complexes. And then finally, in §3.3 we prove Theorem 1.6.

3.1. Lower volume bounds for Margulis tubes. As mentioned above, the basic idea in Theorem 1.6 is to show that the number of Margulis tubes with very short cores that appear in a manifold with pinched negative curvature is a very small fraction of its volume. To verify this, we will use the following proposition.

Proposition 3.1 (Short geodesics imply large volume). Let $d \geq 4$ and let M be a complete Riemannian d -manifold with sectional curvatures in the interval $[-1, -a^2]$, where $a > 0$. Suppose that $T \sqsupseteq M_{\leq q}$ is a component of the q -thin part of M whose core geodesic has length ℓ . Then $\text{vol}(T) \geq C := C(d, a, q, \ell)$, where $C \rightarrow \infty$ as $\ell \rightarrow 0$.

By Wang's finiteness theorem [29], for $d \geq 4$ a finite volume hyperbolic d -manifold M can only have a very short geodesic if its volume is very large. So for hyperbolic manifolds, one can think of the above as a strengthening of this statement that says that the large volume has to come from the Margulis tube around the short geodesic.

One could probably prove (at least a version of) the proposition using a geometric limit argument informed by the above discussion on Dehn filling. We assume there is a sequence of manifolds and Margulis tubes $T_n \sqsupseteq M_n$ where the core length $\ell_n \rightarrow 0$, but where $\sup \text{vol}(T_n) < \infty$. Take base points $p_n \in \partial T_n$ and extract pointed Gromov-Hausdorff limits of everything, giving $T_\infty \sqsupseteq M_\infty$ and $p_\infty \in \partial T_\infty$. Since $\ell_n \rightarrow 0$, this T_∞ is a cusp neighborhood, rather than a Margulis tube. And since $\sup \text{vol}(T_n) < \infty$, one can argue that the diameter of ∂T_n is bounded, which means that ∂T_∞ should actually be homeomorphic to ∂T_n . But as mentioned above, this is impossible since the boundary of a cusp neighborhood is always aspherical, but the boundary of a Margulis tube is not if $d \geq 4$.

We chose not to use the geometric limit approach because pushing through the limiting arguments requires control over higher order derivatives of the metric tensors, which we do not necessarily want to include in the statement of Theorem 1.6. Also, the proof we give below is attractive in that one could use it to write down an explicit formula for C .

Before starting the proof of Proposition 3.1, we establish the following simple lemma.

Lemma 3.2. Suppose that $n \geq 3$ and $A \leq O(n)$ is an abelian subgroup, which we consider as acting on the unit sphere S^{n-1} by isometries. Then $\text{diam}(G \backslash S^{n-1}) \geq \pi/2$.

Here, the distance between two points in the quotient is the minimal distance in S^{n-1} between points in their preimages. Note that $G \backslash S^{n-1}$ is a path metric space.

Proof. The subgroup A is contained in a subgroup $T \leq O(n)$ of the form

$$T = \begin{bmatrix} \mathbb{O}(2) & & & \\ & \ddots & & \\ & & \mathbb{O}(2) & \\ & & & \ddots \\ & & & & \mathbb{O}(2) \end{bmatrix} \text{ or } T = \begin{bmatrix} \mathbb{O}(2) & & & \\ & \ddots & & \\ & & \mathbb{O}(2) & \\ & & & \ddots \\ & & & & \mathbb{O}(2) \end{bmatrix},$$

$$\pm 1$$

written in suitable orthonormal coordinates (x_1, \dots, x_n) for \mathbb{R}^n , depending on whether n is even or odd. But in these coordinates, the action of A preserves the intersection I of S^{n-1} with the x_1x_2 -coordinate plane, and it also preserves the intersection J of S^{n-1} with either the x_3x_4 -coordinate plane or the x_3 -axis, depending on whether $n \geq 4$ or $n = 3$. The distance in S^{n-1} between I and J is $\pi/2$, so the lemma follows.

Proof of Proposition 3.1. Pick a universal covering map $\tilde{M} \rightarrow M$ and lift the core geodesic $\tilde{\gamma} \subset T$ to a complete geodesic $\tilde{\gamma} \subset \tilde{M}$. Let \tilde{T} be the component of the preimage of T that contains $\tilde{\gamma}$, and let $g : \tilde{M} \rightarrow M$ be a nontrivial deck transformation stabilizing $\tilde{\gamma}$ that is primitive in the deck group. So, g is determined up to inversion, the cyclic group hgi is the stabilizer of \tilde{T} , and any deck transformation not in hgi moves \tilde{T} completely off itself.

Pick a point $\tilde{p} \in \tilde{\gamma}$ and isometrically identify the fiber $N^1(\tilde{\gamma})_{\tilde{p}}$ of the unit normal bundle of $\tilde{\gamma}$ with S^{n-2} . Parallel transport then determines a global trivialization

$$\tilde{\gamma} \times S^{n-2} \rightarrow N^1(\tilde{\gamma}),$$

and we can then write the action of g on $N^1(\tilde{\gamma})$ in these coordinates as g

$$= \tau \times r,$$

where τ is a translation by ℓ along $\tilde{\gamma}$ and $r \in O(d-1)$.

Since $O(d-1)$ is a compact manifold, there is some $c > 0$ such that if $S \subset O(d-1)$ is any set of $m^{\dim O(d-1)}$ points, there are $s, t \in S$ such that

$$d(s(\xi), t(\xi)) \leq c/m, \quad \forall \xi \in S^{d-2}.$$

Setting $m = \ell^{\frac{1}{\dim O(d-1)+1}}$, we get that there is some power r^k , $k \leq m^{\dim O(d-1)}$ with

$$(35) \quad d(r^k(\xi), \xi) \leq c/m \leq c \cdot \ell^{\frac{1}{\dim O(d-1)+1}}, \quad \forall \xi \in S^{d-2}.$$

Note that we also have

$$(36) \quad d(\tau^k(\tilde{x}), \tilde{x}) \leq \ell \cdot m^{\dim O(d-1)} \leq \ell \cdot \ell^{\frac{\dim O(d-1)}{\dim O(d-1)+1}} = \ell^{\frac{1}{\dim O(d-1)+1}}, \quad \forall \tilde{x} \in \tilde{\gamma}.$$

Since \tilde{M} has sectional curvatures in $[-1, -a^2]$, it follows from the triangle comparison theorems that if two unit speed geodesic segments α, β in M share an endpoint $\alpha(0) = \beta(0)$ at which they intersect with angle θ , then we have that

$$(37) \quad \theta t \leq d(\alpha(t), \beta(t)) \leq \theta \sinh(t), \quad \forall t > 0.$$

Similarly, by an application of Berger's extension of Rauch's comparison theorem [11, Theorem 1.34], if α, β start out with $\alpha(0), \beta(0) \in \tilde{\gamma}$, and both are perpendicular to $\tilde{\gamma}$, then

$$(38) \quad d(\alpha(t), \beta(t)) \leq d(\alpha(0), \beta(0)) \cdot \cosh(t), \quad \forall t > 0.$$

Combining (35) and (36) with the upper bounds in (37) and (38), and using the decomposition $g = \tau \times r$, we get that for a point $\tilde{x} \in \tilde{M}$ that lies at distance t from $\tilde{\gamma}$,

$$d(g^k(\tilde{x}), \tilde{x}) \leq \ell^{\dim O(d-1)+1} c \cdot \sinh(t) + \cosh(t) \leq 2\ell^{\dim O(d-1)+1} c \cdot \cosh(t).$$

So if ℓ is small, we can set $L = \cosh^{-1}(\varrho/(6c\ell^{\dim O(d-1)+1}))$, and then the L -neighborhood of $\tilde{\gamma}$ will be contained in the subset $\tilde{T}_{\varrho/3} \subset \tilde{T}$ that is the lift of $T \cap M_{\leq \varrho/3}$.

Since $d \geq 4$, Lemma 3.2 implies that

$$\text{diam}(\text{hri} \setminus S^{n-2}) \geq \pi/2.$$

Since the quotient is a path metric space, we can then choose for any $\theta > 0$ a set S of $\lceil \varrho/(2\theta) \rceil$ points in S^{n-2} , with the property that the hri-orbits of any two distinct points in S are at least a distance of θ from each other in S^{n-2} . Identify S^{n-2} with the fiber $N^1(\tilde{\gamma})_{\tilde{\gamma}}$ of the unit normal bundle as above, let $\exp_{\tilde{\gamma}}$ be the Riemannian exponential map, and let

$$S = \{\exp_{\tilde{\gamma}}(L \cdot \xi) \mid \xi \in S\}.$$

By the lower bound in (37), we get that the distance between the hgi-orbits of any two distinct points in S is at least θL . So taking $\theta = \varrho/2L$, the hgi-orbits of points in S are at least $\varrho/3$ apart in M .

Now $\tilde{T}_{\varrho/3}$ is star-shaped with respect to the geodesic $\tilde{\gamma}$, so we can now project $S \subset \tilde{T}_{\varrho/3}$ radially from $\tilde{\gamma}$ to a subset $S' \subset \partial T_{\varrho/3}$. Since \tilde{M} has negative curvature, this radial projection cannot decrease the distance between any two points in $\tilde{T}_{\varrho/3}$ that are the same distance from $\tilde{\gamma}$. Hence, the hgi-orbits of points in S' are still at least $\varrho/3$ apart in M . It follows that the covering map $\tilde{M} \rightarrow M$ restricts to an embedding on the union of the $\varrho/3$ -balls in M around the points of S' . (It is an embedding on each individual ball by definition of $T_{\varrho/3}$.) Each of these $\varrho/3$ -balls is contained in T , so volume of T is bounded below by the sum of the volumes of these balls. Each ball has volume at least some $V = V(\varrho, d, a)$, by the usual comparison arguments, and there are $\lceil \varrho/(2\theta) \rceil$ balls in total. By our definitions of θ and L , the number of balls goes to infinity with ℓ , and the proposition follows.

3.2. Simplicial approximation of the thick part. Suppose that M is a metric space and $A \subset M$. Following [18], we denote the metric ξ -neighborhood of A by $(A)_\xi$, and we define the ξ -shrinking of A to be the subset

$$A(\xi) := M \setminus (M \setminus A)_\xi \setminus A.$$

Fix now $\varrho, \xi > 0$, with ϱ less than the Margulis constant $\varrho(d)$, and let M be a Riemannian d -manifold with curvatures in $[-1, -a^2]$. The main result of this section is the following, which is an application of techniques of Gelander [18]. Informally, it says that the shrinking $M_{\geq \varrho}(\xi)$ of the ϱ -thick part of M can be simplicially modeled (up to homotopy) by the nerve complex associated to a certain open cover.

Proposition 3.3 ([18]). For any sufficiently small $q > q' > 0$ and any $c' \geq c \geq 1$, there is a constant $b = b(d, a, q) > 0$ and some small $\delta_0 = \delta_0(d, a, c, q') > 0$ such that the following holds for every d -manifold M with curvatures in $[-1, -a^2]$, and all $\delta < \delta_0$.

Set $\xi = q/2 + \delta$ and let S be a $(\delta, c\delta)$ -net in $\mathcal{M}_{\geq q}(\xi)$. Let

$$\rho : S \rightarrow [(b + c)\delta, (b + c')\delta]$$

be any function and let $N(S, \rho)$ be the nerve of the collection of balls $B_M(x, \rho(x))$, where $x \in S$. Then $N(S, \rho)$ is homotopy equivalent to $\mathcal{M}_{\geq q}$.

Here, recall from §2.3 that S is a $(\delta, c\delta)$ -net if it is δ -separated and $c\delta$ -covers. The key to the above is the following restatement of a result from [18].

Lemma 3.4 (essentially Lemma 4.1 in [18]). Let M be a complete Riemannian d -manifold with sectional curvatures in the interval $[-1, 0]$, let $M' \subset M$ be a connected submanifold with boundary and let $q, b, c, c' > 0$, with $c < c'$, be fixed. Suppose that

- (1) M' is contained in the q -thick part of M ,
- (2) M' is homotopy equivalent to $\mathcal{M}'_{(q/2)}$,
- (3) the preimage \tilde{X} of $X = M \setminus M'$ under a universal covering map $\tilde{M} \rightarrow M$ is a locally finite union $\tilde{X} = \bigcup_{\gamma} \tilde{X}_{\gamma}$ of convex open sets with smooth boundary,
- (4) for any point $x \in \tilde{M} \setminus \tilde{X}$ with $d(x, \tilde{X}) \leq q$, there is a unit tangent vector $n(x) \in T_x(\tilde{M})$ such that for each γ with $d(x, \tilde{X}_{\gamma}) = d(x, \tilde{X}_{\gamma})$, we have

$$n(x) \cdot d(\cdot, \tilde{X}_{\gamma})|_x \geq 1/b.$$

Then there is some small $\delta_0 = \delta_0(q, b, d) > 0$ such that the following holds for all $\delta < \delta_0$. Let S be any δ -separated subset of $\mathcal{M}'_{(q/2+\delta)}$ that $c\delta$ -covers $\mathcal{M}'_{(q/2+\delta)}$, and let

$$\rho : S \rightarrow [(b + c)\delta, (b + c')\delta]$$

be a function. Then the nerve of the collection of balls

$$C = \{B_M(x, \rho(x)) \mid x \in S\}$$

is homotopy equivalent to M' .

We should say that Lemma 4.1 in [18] is not quite stated as above. The biggest difference is that maximal δ -separated subsets of $\mathcal{M}'_{(q/2+\delta)}$ are used in [18] instead of subsets that $c\delta$ -cover, and the radii of the balls in the collection C are all chosen to be $(b+1)\delta$. However, the proof works just as well if all the 1 's are replaced by numbers between c and c' . (So for instance, one should use c' instead of 1 in Proposition 4.7 of [18], and allow the radius to vary in Proposition 4.8.) A purely cosmetic difference is that Lemma 4.1 in [18] is stated for locally symmetric spaces, but local symmetry is not used in its proof. Finally, the conclusion of Lemma 4.1 in [18] is that M' is homotopy equivalent to an unnamed simplicial complex, but if one looks at [18, Proposition 4.8], one will see that this unnamed complex is just the nerve mentioned above. (The statement of Proposition 4.8 references the cover of $\mathcal{M}'_{(q/2)}$ given by the collection of intersections $C \cap \mathcal{M}'_{(q/2)}$, rather than the cover by $C \subset C$, and a priori the difference matters when constructing the nerve complex.)

However, the proof of Proposition 4.8 shows that when a finite subset of C has a nonempty intersection, this intersection intersects $\partial M'_{(q/2)}$, so one gets the same nerve whether one considers the collection C referenced in our statement of Lemma 3.4, or the collection consisting of the intersections of its elements with $\partial M'_{(q/2)}$, as in [18].)

Proof of Proposition 3.3. Set $M' = M_{\geq q}$. First, note that M' is homotopy equivalent to its $q/2$ -shrinking, since its complement components are star-shaped neighborhoods of either a closed geodesic or a point at infinity, so we can deformation retract M' to its shrinking by flowing outwards. See the proof of Claim 8.5 of [18] for more details. So, by the Nerve Lemma it suffices to show that M' satisfies the conditions of the lemma above.

Conditions (1) and (2) are immediate from the definition of M' , where

$$\tilde{X}_\gamma = \{x \in X' \mid d(x, \gamma(x)) < q\}, \quad \gamma \in \pi_1 M.$$

For condition (3), we define the vector $n(x)$ in two cases. As long as q is small, we can assume that any x in condition (3) is contained in the preimage of the $q(d)$ -thin part of M , where $q(d)$ is the Margulis constant. If x lies in a component of this preimage that covers a Margulis tube, we define $n(x)$ exactly as in the proof of Lemma 7.4 of [18], i.e. by using Lemma 7.3 with $b = b(d)$ and a unit vector $n(x)$ whose inner products with the gradients $d(\cdot, \tilde{X}_\gamma)$ are all at least $1/b$. If x lies in a component that covers a cusp neighborhood, we let $n(x)$ point away from the point at infinity to which the lifted cusp neighborhood accumulates, just as in Section 6 of [18]. In [18] the control on the associated constant b makes use of the fact that only locally symmetric manifolds are considered. Instead, here we use the following:

Claim 3.5 (Moving away from the cusp). Suppose \tilde{M} is a simply connected Riemannian manifold with curvatures in $[-1, -a^2]$ and let $\xi \in \partial_\infty \tilde{M}$. Let γ be a parabolic isometry of \tilde{M} with $\gamma(\xi) = \xi$, let $x \in M$ be a point with $d(x, \gamma(x)) \geq q$ and let $c : \mathbb{R} \rightarrow M$ be a unit speed geodesic with $c(-\infty) = \xi$ and $c(0) = x$. Then we have

$$\frac{d}{dt} d(c(t), \tilde{X}_\gamma) \Big|_{t=0} \geq q \cdot a/2.$$

Proof. Since the geodesics $c(t)$ and $\gamma \circ c(t)$ are asymptotic to ξ as $t \rightarrow -\infty$ and always lie on the same horospheres, [21, Proposition 4.1] says that for any fixed s ,

$$d(c(t), \gamma \circ c(t)) \leq d(c(s), \gamma \circ c(s)) \cdot e^{a(t-s)}, \quad \forall t \leq s.$$

Since the two sides are equal at $t = s$ and we are saying that $t \leq s$, it follows that

$$\frac{d}{dt} d(c(t), \gamma \circ c(t)) \Big|_{t=s} \geq \frac{d}{dt} d(c(s), \gamma \circ c(s)) \cdot e^{a(t-s)} \Big|_{t=s}.$$

Apply this to the (unique) value $s \leq 0$ such that $c(s) \in \partial \tilde{X}_\gamma$. Then

$$(39) \quad \frac{d}{dt} d(c(t), \gamma \circ c(t)) \Big|_{t=s} \geq q \cdot a.$$

On $\partial\tilde{X}_y$, the gradients of d_y and $d(\cdot, \tilde{X}_y)$ are parallel. As

$$d_y(y) \leq d_y(x) + 2d(x, y) \quad \forall x, y,$$

we have $|\nabla d_y| \leq 2$, while $\nabla d(\cdot, \tilde{X}_y)$ is a unit vector. So, this and (39) imply:

$$\frac{d}{dt} d(c(t), \tilde{X}_y) \Big|_{t=s} = \nabla d(\cdot, \tilde{X}_y) \cdot c'(0) \geq \frac{1}{2} \nabla d_y \cdot c'(0) \geq \frac{1}{2} \varrho \cdot a.$$

Finally, as curvature is nonpositive and X_y is a convex set, $d(c(t), \tilde{X}_y)$ is a convex function and hence has increasing derivative. As $s \leq 0$, the claim follows.

So, to finish the proof of Proposition 3.3, we just take b to be at least the constant $b = b(d)$ from the Margulis tube case, and at least $\varrho \cdot a/2$. With this b and n , the conditions of Lemma 3.4 are satisfied, so the proposition follows.

Finally, we prove the following estimate on the volumes of balls in the shrunk thick parts $M_{\geq \varrho}(\delta)$, which is necessary if we want to invoke Theorem 2.9.

Lemma 3.6. Suppose that M is a complete Riemannian d -manifold with sectional curvatures in $[-1, -a^2]$. Fix $\varrho < \varrho(d)$, let $\delta, r < \min\{\varrho, \varrho(d) - \varrho\}/4$ and set $N := M_{\geq \varrho}(\delta)$. Then there is some $c = c(d, \varrho, a) > 0$ such that

$$\text{vol}(B_N(p, r)) \geq cr^d, \quad \forall p \in N.$$

Note that since M is non-positively curved, the volume of any embedded metric ball $B \cap M$ is at least the volume of a ball with the same radius in \mathbb{R}^d , see e.g. [17, Theorem 3.101]. So, as long as we choose $p < \varrho$, the lemma is trivial for balls $B_N(p, r)$ that do not intersect ∂N . The point of the lemma, then, is that the boundary of N is moderate enough that balls centered near ∂N still have a definite amount of volume that is contained in N .

Proof. As described in the paragraph above, it suffices to consider only points $p \in N$ that are within r of ∂N . The fact that $\delta < (\varrho(d) - \varrho)/4$ ensures that the radius r ball $B_M(p, r)$ in M around p will be an embedded ball contained in the $\varrho(d)$ -thin part $M_{<\varrho(d)}$. Choose a universal cover

$$\pi : \tilde{M} \rightarrow M,$$

components $\tilde{T}_{<\varrho} \cap \tilde{T}_{<\varrho(d)} \cap \tilde{M}$ of the preimages of $M_{<\varrho}$, $M_{<\varrho(d)}$, and a point \tilde{p}

$$\in \tilde{T}_{<\varrho(d)} \setminus (\tilde{T}_{<\varrho})_\delta, \quad \pi(\tilde{p}) = p.$$

Then we can write $\tilde{T}_{<\varrho}$ as the union

$$\tilde{T}_{<\varrho} = \bigsqcup_y \tilde{X}_y,$$

where y ranges over the nontrivial elements in the group of deck transformations stabilizing $\tilde{T}_{<\varrho}$, and

$$\tilde{X}_y := \{\tilde{x} \in \tilde{M} \mid d(y(\tilde{x}), \tilde{x}) < \varrho\}.$$

We claim that there is a unit vector $n \in T_{\tilde{p}} \tilde{M}$ and some $b = b(d, a, \varrho)$ such that

$$(40) \quad n \cdot \nabla d(\cdot, \tilde{X}_y) \Big|_{\tilde{p}} \geq 1/b > 0, \quad \forall y.$$

Now if $\tilde{T}_{<\varrho}$ covers a Margulis tube, any two γ, γ' as above commute, so we have

$$\partial d(\cdot, X_\gamma) \cdot \partial d(\cdot, X_{\gamma'}) \geq 0$$

by the argument of [18, Lemmas 7.1 and 7.2] (see also [5, Lemma 3.5]), and then one can construct n as in [18, Lemma 7.3] (or [5, Lemma 3.12]). If $\tilde{T}_{<\varrho}$ covers a cusp neighborhood, we can just let n be the unit vector that points away from the point $\xi \in \partial_\infty M$ to which $T_{<\varrho}$ accumulates (i.e. let $n = c'(0)$ where c is a unit speed geodesic with $c(-\infty) = \xi$ and $c(0) = \tilde{p}$) and then the claim follows from Claim 3.5 above, after setting $b = \varrho a/2$.

It follows from (40) that for every $v \in TM_{\tilde{p}}$ with $|v - n| < 1/b$ we have

$$\begin{aligned} v \cdot \partial d(\cdot, X_\gamma) &= n \cdot \partial d(\cdot, X_\gamma) + (v - n) \cdot \partial d(\cdot, X_\gamma) = \\ &= 1/b - |v - n| \\ &> 0, \end{aligned}$$

so v points out of the convex subset $(\tilde{X}_\gamma)_{d(\tilde{p}, \tilde{X}_\gamma)} \subset \tilde{M}$ on whose boundary \tilde{p} lies. And since

$$\tilde{p} \in (\tilde{T}_{<\varrho})_\delta \implies (\tilde{T}_{<\varrho})_\delta \subset \partial_\gamma (\tilde{X}_\gamma)_{d(\tilde{p}, \tilde{X}_\gamma)},$$

we then have that for all $v \in TM_{\tilde{p}}$ with $|v|/|v - n| < 1/b$, the Riemannian exponential

$$\exp_{\tilde{p}}(v) \in (\tilde{T}_{<\varrho})_\delta.$$

Now as explained in the beginning of the proof, r is small enough so that $B_M(p, r)$ is an embedded ball in $M_{<\varrho(d)}$. So, if we let

$$V = \{v \in TM_{\tilde{p}} \mid |v| < r, |v|/|v - n| < 1/b\},$$

the composition $\pi \circ \exp_{\tilde{p}}$ of the universal covering map and the Riemannian exponential map embeds V as a subset of N , where $N = M_{\geq \varrho(\delta)}$. The ratio of the Euclidean volume of V to r^d is certainly bounded below by some constant depending only on $b = b(d, a, \varrho)$, so nonpositive curvature implies that the same is true of $B_N(p, r)$, [23, Corollary 11.4].

3.3. The proof of Theorem 1.6. We will assume everywhere below that $d \geq 4$, since the theorem follows trivially from the Gauss–Bonnet theorem when $d = 2$ and we have assumed that $d = 3$.

Let $M_a^d \subset M$ be the subset consisting of all pointed Riemannian d -manifolds with sectional curvatures in $[-1, -a^2]$. Fix a sequence of finite volume d -manifolds (M_n) with curvatures in $[-1, -a^2]$ and for each n , let μ_n be the measure on M_a^d obtained by pushing forward the Riemannian measure on M_n under $p \rightarrow (M_n, p)$. By assumption, the sequence $(\mu_n/\text{vol}(M_n))$ converges weakly to some probability measure μ on M_a^d .

Claim 3.7. For some $\varrho_{\max} > 0$, we have that $\mu(E_\varrho) = 0$ for all but countably many $\varrho \in (0, \varrho_{\max})$, where here E_ϱ is the set of all $(M, p) \in M_a^d$ such that M has a primitive closed geodesic with length exactly 2ϱ .

Proof. Take ϱ_{\max} less than the Margulis constant $\varrho(d)$. Using Proposition 3.1, we may assume that ϱ_{\max} is small enough so that if $\varrho \in (0, \varrho_{\max})$, then any $\varrho(d)$ -Margulis tube with core length 2ϱ has volume at least 1. For each $\varrho \in (0, \varrho_{\max})$ and $R > 0$, consider the set $E_{\varrho, R}$ of all $(M, p) \in M^d$ such that there is an $\varrho(d)$ -Margulis tube with core length 2ϱ that is completely contained in the radius R ball around p . In any manifold M with sectional curvatures at least -1 , the radius R ball around any point has volume at most some constant $V(d, R)$, see [17, Theorem 3.101]. So, it follows that for fixed R , any $(M, p) \in M^d$ can be contained in $E_{\varrho, R}$ for at most $V(d, R)$ -many choices of ϱ . Hence, we have

$$\limsup_{\varrho \rightarrow 0} \mu(E_{\varrho, R}) \leq V(d, R),$$

implying that $\mu(E_{\varrho, R}) = 0$ for at most countably many ϱ . But letting $R \in \mathbb{N}$, there are then only countably many pairs (ϱ, R) such that $\mu(E_{\varrho, R}) = 0$, and hence only countably many ϱ such that $\mu(E_{\varrho, R}) = 0$ for some $R \in \mathbb{N}$. Since $E_\varrho = \bigcup_{R \in \mathbb{N}} E_{\varrho, R}$, the claim follows.

Fix now some small $\varrho, \xi > 0$, to be determined later, such that $\mu(E_\varrho) = 0$. Using the notation and terminology of §2.2, consider the extended mm-space

$$M_n := (\) (M_n)_{\geq \varrho} (\xi, M_n),$$

and let μ_{M_n} be the associated measure on M^{ext} . Then if

$$T = \{(M, p) \in M_a^d \mid d(p, M_{<\varrho}) > \xi\},$$

the measure μ_{M_n} is just the push forward of the restriction $\mu_n|_T$ under the map

$$T \rightarrow M^{ext}, \quad (M, p) \rightarrow (\) M_{\geq \varrho} (\xi, p, M).$$

Lemma 3.8. The measures $\mu_{M_n}/\text{vol}(M_n)$ weakly converge.

Note that these are not probability measures.

Proof. Let $f : M^{ext} \rightarrow \mathbb{R}$ be a bounded, continuous function and define

$$F : M_a^d \rightarrow \mathbb{R}, \quad F(M, p) = \begin{cases} f(\) M_{\geq \varrho} (\xi, p, M) & (M, p) \in T \\ 0 & \text{otherwise.} \end{cases}$$

We have $\int_M f d\mu_{M_n} = \int_T F d\mu_n$, so it suffices to show that the limit

$$\lim_{n \rightarrow \infty} \frac{1}{\text{vol}(M_n)} \int_T F d\mu_n$$

exists. Recall that the measures $\mu_n/\text{vol}(M_n) \rightarrow \mu$ weakly. So by the Portmanteau theorem, it suffices to show that F is continuous on a subset of M_a^d that has full μ -measure.

Claim 3.9. The map F is continuous on the difference $M_a^d \setminus (E_\varrho \setminus D)$, where

$$D := \{(M, p) \in M_a^d \mid d(p, M_{<\varrho}) = \xi\}$$

and E_ϱ is as in Claim 3.7.

Proof of Claim 3.9. Suppose that we have a convergent sequence

$$(N_n, p_n) \rightarrow (N, p) \in M_a \setminus (E_\varrho \setminus D).$$

Assume first that $d(p, N_{<\varrho}) < \xi$. By a result of Ehrlich [15], injectivity radius is continuous under smooth convergence, so it follows that $d(p_n, (N_n)_{<\varrho}) < \xi$ as well for large n . In this case, the continuity of F along our sequence is obvious, since for large n ,

$$0 = F(N_n, p_n) \rightarrow F(N, p) = 0.$$

So, assume that $d(p, N_{<\varrho}) > \xi$, i.e. that $(N, p) \notin T$. First, we claim that $(N_n, p_n) \notin T$ for large n . If not, then after passing to a subsequence there would be points $q_n \in N_n$ with $d(p_n, q_n) \leq \xi$ and $\text{inj}_{N_n}(q_n) \leq \varrho$. Again by continuity of injectivity radius, we can take a subsequential limit of the q_n to produce some $q \in N$ with $d(p, q) \leq \xi$ and $\text{inj}_N(q) \leq \varrho$. If $\text{inj}_N(q)$ is less than ϱ , then this contradicts that $d(p, N_{<\varrho}) > \xi$. So assume $\text{inj}_N(q) = \varrho$. Since $(N, p) \notin E_\varrho$, the point q cannot lie on a closed geodesic of length exactly 2ϱ , so q can be perturbed to a point q' with $\text{inj}_N(q') < \varrho$. Taking the perturbation small enough so that $d(p, q') < d(p, N_{<\varrho})$, we have a contradiction.

In order to avoid a debauch of parentheses, set $T_n = (N_n)_{\geq \varrho} \setminus \{\xi\}$ and define $T \setminus N$ similarly. To prove that F is continuous along $(N_n, p_n) \rightarrow (N, p)$, it suffices to show that

$$(41) \quad (T_n, p_n, N_n) \rightarrow (T, p, N) \in M^{\text{ext}}.$$

Fixing some large $R > 0$, choose a sequence of embeddings

$$\phi_n : B_N(p, R) \rightarrow N_n, \quad \phi_n(p) = p_n,$$

such that the pullback metrics $\phi_n^*(g_n) \rightarrow g$ in the smooth topology, as described in the appendix of [2]. To prove (41), we would like to apply Lemma 2.3 to say that for a given $\alpha > 0$, the triples in (41) are (α, R) -related for large n . This requires proving that for an arbitrary $\delta > 0$, conditions (1)–(3) in Lemma 2.3 hold for large n .

Condition (1) in Lemma 2.3 is immediate, since the maps ϕ_n are nearly isometries when n is large. The proof of condition (2) in Lemma 2.3 is similar to the first two paragraphs of the current claim. Namely, suppose that the first part of condition (2) fails for infinitely many n . Then for infinitely many n , there are points

$$q_n \in T_n \cap \phi(B_N(p, R)), \quad d(\phi_n^{-1}(q_n), T) > \delta.$$

Passing to a subsequence, we can assume that $\phi_n^{-1}(q_n) \rightarrow q \in B_N(p, R)$, and by continuity of injectivity radius we have $q \notin T$, a contradiction. The second part of condition (2) is similar, although as we did above one has to use that there are no closed geodesics of length exactly 2ϱ in N . So, it remains to prove condition (3) of Lemma 2.3, i.e. that

$$\text{vol}(\phi_n^{-1}(T_n) \setminus T) < \delta$$

for large n . Pick a neighborhood $U \subset \partial T \cap B_N(p, R)$ with volume less than δ . If n is large, then the same arguments as above show that $\phi_n^{-1}(T_n) \setminus T \subset U$, so we are done.

By our choice of ϱ , we have $\mu(E_0) = 0$. So, to prove Lemma 3.8 it suffices to show that $\mu(D) = 0$. Essentially, the point is that $d(p, M_{<\varrho}) = \xi$ is a measure zero condition within each fixed M , and as a weak limit of measures constructed using Riemannian measures on finite volume manifolds, μ is distributed on each ‘leaf’

$$L_M = \{(M, p) \mid p \in M\} \subset M_a^d$$

according to the Riemannian measure of M . (This is not quite precise, the leaves may be highly singular, but one can make this argument work in the foliated ‘desingularization’ of M constructed in [2, Theorem 1.6]). However, an easier approach is to use that μ satisfies the mass transport principle, see [2, (1)]. Namely, define a Borel function

$$\varphi : (M_a^d)_2 \rightarrow \{0, 1\}, \quad \varphi(M, p, q) = \begin{cases} 1 & d(p, M_{<\varrho}) = \xi \text{ and } d(p, q) \leq \varrho \\ 0 & \text{otherwise} \end{cases},$$

where $(M_a^d)_2$ is the space of doubly pointed d -manifolds with curvature in $[-1, -a^2]$, endowed with the natural version of smooth convergence, see [2]. Note that

$$d(p, M_{<\varrho}) = \xi \implies \int_M \varphi(M, p, q) d\text{vol} \geq \text{vol}_{B_{R^d}}(0, \varrho),$$

since embedded ϱ -balls in a d -manifold of nonpositive curvature have volume at least that of an ϱ -ball in R^d , c.f. [17, Theorem 3.101]. Then

$$\begin{aligned} \mu(D) &\leq 1/B_{R^d}(0, \varrho) \cdot \int_{Z^{(M, p) \times M_a^d}} \int_{Z^{q \times M}} \varphi(M, p, q) d\text{vol}_M d\mu \\ &= 1/B_{R^d}(0, \varrho) \cdot \int_{(M, p) \times M_a^d} \int_{q \times M} \varphi(M, q, p) d\text{vol}_M d\mu \\ &= 0, \end{aligned}$$

where the first equality is the mass transport principle [2, (1)], and the last equality is because for small ϱ , the set of points exactly at distance ξ from the ϱ -thin part has measure zero in any manifold with negative curvature.

We now know that the sequence of measures $\mu_{M_n}/\text{vol}(M_n)$ weakly converges, and we would like to apply Theorem 2.9, or really Corollary 2.22. Lemma 3.6 will give the lower bound on ball volumes needed in Theorem 2.9 (1), and the upper bound needed in (2) comes from the uniform lower sectional curvature bound, see e.g. [17, Theorem 3.101 on p. 169]. The key, though, is to use our work in §2 to define the appropriate r_0, r_1, r_2, r_3 . Namely, take $\varrho, \delta > 0$ small enough so that they work in Proposition 3.3, set $\xi = \varrho/2 + \delta$, and let b be as given in Proposition 3.3 for $c = 3, c' = 4$, say. If

$$r_0 = \delta, r_1 = 3\delta, r_2 = (b + 6)\delta, r_3 = (b + 7)\delta,$$

then Proposition 3.3 says that the nerve $N_{M_n}(s_n, p_n)$ in M_n associated to any $[r_2, r_3]$ -weighted (r_0, r_1) -net (s_n, p_n) in $(M_n)_{\geq \varrho}$ (ξ is homotopy equivalent to $(M_n)_{\geq \varrho}$). So, applying

Corollary 2.22 to the sequence of extended mm-spaces M_n , with $b_n = b_k((M_n)_{\geq 0})$, $v_n = \text{vol}(M_n)$ and r_0, r_1, r_2, r_3 as above, we get that the limit

$$(42) \quad \lim_{n \rightarrow \infty} \frac{b_k((M_n)_{\geq 0})}{\text{vol}(M_n)} = L \in [0, \infty).$$

But Proposition 3.1 says that the number of components of the q -thin part of M_n is at most $\text{vol}(M_n)/C$, where $C = C(q, d, a) \rightarrow \infty$ as $q \rightarrow 0$. Removing a cusp neighborhood from M_n does not change the homotopy type, and by Mayer–Vietoris removing a Margulis tube can only change Betti numbers by 1. So, we get that for each n and k ,

$$|b_k((M_n)_{\geq 0}) - b_k(M_n)| \leq \text{vol}(M_n)/C.$$

Combining this with (42), we get that

$$L - 1/C \leq \liminf_{n \rightarrow \infty} \frac{b_k(M_n)}{\text{vol}M_n} \leq \limsup_{n \rightarrow \infty} \frac{b_k(M_n)}{\text{vol}M_n} \leq L + 1/C,$$

so sending $q \rightarrow 0$, and hence $C \rightarrow \infty$, proves the theorem.

4. Manifolds of nonpositive curvature and Theorem 1.7

In this section we prove Theorem 1.7, i.e. the convergence of normalized Betti numbers for BS-convergent sequences of analytic d -manifolds of nonpositive curvature without Euclidean factors, when the limit is thick. For that purpose it will be more convenient to work not with the standard thick thin decomposition but a close variant of it, introduced in [6], which we call a ‘stable’ thick thin decomposition:

4.1. A stable thick thin decomposition. Suppose that M is a finite volume, real analytic d -manifold with sectional curvatures in the interval $[-1, 0]$ and that the universal cover X of M has no Euclidean deRham factors. Write $M = \Gamma \backslash X$. Then Γ operates freely and the displacement functions d_γ ($\gamma \in \Gamma$) are analytic. In particular the convex sets

$$\text{Min}(\gamma) = \{x \in X \mid d_\gamma(x) = \min(d_\gamma)\}$$

are complete submanifolds. An element $\gamma \in \Gamma$ is called J -stable if we have

$$\text{Min}(\gamma^i) = \text{Min}(\gamma), \quad i = 1, \dots, J.$$

Let q be less than the Margulis constant, and I the index constant in the Margulis lemma, and fix also the constants δ, I_δ , and $J = I_\delta \cdot I$ defined at the beginning of [6, §13.4], but using q instead of the actual Margulis constant. The interested reader can refer to [6] if necessary, but it is not necessary to know what these constants are to read our proof below. (As at the top of pg 141 of [6], though, we note that $0 < \delta < q/I_\delta$.) As in [6], let

$$\begin{aligned} \Delta_0 &:= \{\gamma \in \Gamma \setminus \{1\} \mid \gamma \text{ is } J\text{-stable and } \inf_{x \in X} d_\gamma(x) \leq \delta\}, \text{ and} \\ \Delta &:= \{\gamma^1, \dots, \gamma^{I_\delta} \mid \gamma \in \Delta_0\}. \end{aligned}$$

We now define two subsets $X_{\pm} \subset X$ with $X_{-} = \overline{X \setminus X_{+}}$ by

$$X_{+} := \{x \in X \mid d_{\gamma}(x) \geq \varrho \text{ for some } \gamma \in \Delta\}, \quad X_{-} := \{x \in X \mid d_{\gamma}(x) \leq \varrho \text{ for some } \gamma \in \Delta\},$$

and we define the stable ϱ -thick part M_{+} and the stable ϱ -thin part M_{-} by

$$M_{+} := \Gamma \setminus X_{+}, \quad M_{-} := \Gamma \setminus X_{-}$$

Lemma 4.1. M_{+}, M_{-} are topological submanifolds of M and their common boundary $\partial M_{-} = \partial M_{+}$ is compact.

Proof. We work mostly with X_{\pm} , and address M_{\pm} at the end. By definition, X_{-} is the union over all $\gamma \in \Delta$ of the sets

$$U_{\gamma, \varrho} := \{x \in X \mid d_{\gamma}(x) \leq \varrho\}.$$

Since X is a Hadamard manifold, the second variational formula implies that the distance function $d : X \times X \rightarrow \mathbb{R}$ is convex. Moreover, each d_{γ} is a submersion except along $\text{Min}(\gamma)$, see [6, Lemma, pg 96]. This implies that $U_{\gamma, \varrho}$ is a smooth, convex codimension zero submanifold of X .

Let N be the frontier of X_{+} and X_{-} in X . We claim that N is a topological submanifold of X . So, pick some $p \in N$. By discreteness of Γ , there is a small open neighborhood $W \subset X$ of p and a finite subset $F \subset \Delta$ such that $W \cap U_{\gamma, \varrho} = \emptyset$ only when $\gamma \in F$. By the Margulis lemma, F generates a subgroup of Γ that has a nilpotent subgroup with index at most l . Now $p \in \text{Min}(\gamma)$ for any $\gamma \in F$, since

$$\inf d_{\gamma}(x) \leq \delta \cdot l_{\delta} < \varrho/l_{\delta} \cdot l_{\delta} = \varrho$$

while the fact that $p \in F$ implies that $d_{\gamma}(p) \geq \varrho$ for all $\gamma \in F$. As every $\gamma \in \Delta$ is l -stable, this means that $p \in \text{Min}(\gamma^i)$ for any $i \leq l$, so [6, Lemma, part (2), pg 96] says that there is some $v \in T_{X_p}$ such that

$$(43) \quad h \in d_{\gamma}, \quad v_i > 0, \quad \forall \gamma \in F.$$

Note that at p , the gradient ∇d_{γ} is just the outward normal vector to the set $U_{\gamma, \varrho}$.

Shrinking W if necessary, pick a chart $\phi : W \rightarrow \mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$ with $\phi(p) = (0, 0)$ and such that $d\phi(v) = (0, 1)$. After shrinking W further, the implicit function theorem and (3.7) imply that for each $\gamma \in F$, we have

$$\phi(W \cap U_{\gamma, \varrho}) = \{(x, t) \in \phi(W) \subset \mathbb{R}^{d-1} \times \mathbb{R} \mid t \leq f_{\gamma}(x)\}, \text{ where}$$

f_{γ} is a smooth function defined on a neighborhood of $0 \in \mathbb{R}^{d-1}$. Hence,

$$\phi(W \cap N) = \{(x, t) \in \phi(W) \subset \mathbb{R}^{d-1} \times \mathbb{R} \mid t = \max_{\gamma \in F} f_{\gamma}(x)\},$$

which is the graph of a continuous function. Hence, N is a submanifold of X .

The frontier $\partial M_{-} = \partial M_{+}$ is the projection of N to M , and hence is a topological submanifold of M . It follows that M_{\pm} are topological submanifolds with boundary. Finally, frontiers are always closed, and since M has finite volume $M_{+} \setminus M_{\geq \varrho}$ is compact, so ∂M_{\pm} is compact.

By [6, Corollary 12.5], there is some integer $m = m(J)$ such that for any $\gamma \in \Gamma \setminus \{1\}$, there is some $j \leq m$ such that γ^j is J -stable. So, if $q' = q/m$ we have

$$(44) \quad M_{<q'} \sqsubseteq M_- \sqsubseteq M_{<q}, \quad M_{\geq q} \sqsubseteq M_+ \sqsubseteq M_{\geq q'}.$$

The following proposition is a modification of [6, Theorem 13.1].

Proposition 4.2. Suppose that M is a finite volume, real analytic d -manifold with sectional curvatures in the interval $[-1, 0]$ and that the universal cover X of M has no Euclidean de Rham factors. Then there is some $C = C(d, q)$ such that for all $k \in \mathbb{N}$, both $b_k(M_-)$ and $b_k(\partial M_-)$ are less than or equal to $C \text{vol}(M_{<2q})$.

It is necessary to assume here that M is analytic and that X has no Euclidean de Rham factors. If $M = N \times S^1$ for some $(d-1)$ -manifold N , we can scale the S^1 -factor so that $M = M_{\leq q} = M_-$ and $\text{vol}(M) \approx 0$. And unless we assume analyticity (or some weaker alternative, see [6, §A2]) there are finite volume manifolds with sectional curvatures in $[-1, 0]$ where the thin parts have infinite Betti numbers, see [6, §11.1].

Before starting the proof, we record a brief algebraic topology lemma.

Lemma 4.3. Suppose that N is a (possibly noncompact) topological manifold with compact boundary. If $b_k(N, R) \leq C$ for all k , then $b_k(\partial N, R) \leq 2C$ for all k .

Proof. By Poincaré-Lefschetz duality, using cohomology with compact support,

$$(45) \quad C \geq b_k(N) = \dim H_k(N; R) = \dim H_c^{d-k}(N, \partial N; R).$$

But by the long exact sequence of a pair, for every k we have

$$\begin{aligned} \dim H^k(\partial N; R) &\leq \dim H^k(N; R) + \dim \text{Image } H^k(\partial N; R) \rightarrow H^{k+1}(N, \partial N; R) \\ &\leq C + \dim \text{Image } H^k(\partial N; R) \rightarrow H^{k+1}(N, \partial N; R). \end{aligned}$$

However, since ∂N is compact, the map whose image we are interested in factors as

$$H^k(\partial N; R) \rightarrow H_c^k(\partial N; R) \rightarrow H_c^{k+1}(N, \partial N; R) \rightarrow H^{k+1}(N, \partial N; R).$$

So, the dimension of the image is at most C , by (45). Hence,

$$b_k(\partial N) = \dim H^k(\partial N; R) \leq 2C.$$

So for instance, to prove Proposition 4.2 it would suffice to just estimate the Betti numbers $b_k(M_-)$, and the estimates for $b_k(\partial M_-)$ would follow. It turns out, however, that this is not logically how the proof will go, since one needs an a priori estimate for $b_k(\partial M_-)$ in order to calculate $b_k(M_-)$. We will still apply Lemma 4.3 to estimate $b_k(\partial M_-)$, though, but the manifold N in the lemma will not be M_- .

We are now ready for the proof.

Proof of Proposition 4.2. We first estimate the Betti numbers of ∂M_- . By (44),

$$M_{<q'} \sqsubseteq M_- \sqsubseteq M_{<q},$$

where $q' < q$ depends only on q, d . Let $\{B_\alpha\}$ be a collection of open $q'/2$ -balls in M with centers on ∂M_- , such that the same centers determine a maximal collection of pairwise disjoint $q'/4$ -balls centered on ∂M_- . Since $\text{inj}(p) \geq q'$ on ∂M_- , each B_α is embedded and convex, and contained in $M_{<2q}$. In particular, we have

$$(46) \quad \#\{B_\alpha\} \leq C \cdot \text{vol}(M_{<2q}), \text{ where } C = c(q, d).$$

Choose arbitrary lifts $\tilde{B}_\alpha \sqsubset X$ of each ball. If $X_- \sqsubset X$ is the preimage of M_- , then as discussed in the proof of Lemma 4.1, we have

$$\text{int}(X_-) = \bigcup_{\gamma \in \Delta} \text{int}(U_{\gamma, q}), \text{ where } \text{int}(U_{\gamma, q}) := \{x \in X \mid d_\gamma(x) < q\}.$$

Since $\text{int}(U_{\gamma, q})$ is convex, each intersection $\tilde{B}_\alpha \cap \text{int}(U_{\gamma, q})$ projects to a convex open subset $B_{\alpha, \gamma} \sqsubset \text{int}(M_-)$. Note that for a given α , only some $N = N(q, d)$ of the sets $B_{\alpha, \gamma}$ are nonempty, for instance by Corollary 3.4 of [5], so (46) implies that the number of nonempty $B_{\alpha, \gamma}$ is bounded by $C \cdot \text{vol}(M_{<2q})$, after adjusting $C = c(q, d)$. It then follows from the Nerve Lemma (see the proof of Claim 4.7 below) that the Betti numbers of the union

$$U = \bigcup_{\alpha, \gamma} B_{\alpha, \gamma}$$

are bounded above by $C \cdot \text{vol}(M_{<2q})$ as well. But $U \sqsubset \partial M_-$ is a manifold with compact boundary by Lemma 4.1, so it follows from Lemma 4.3 that

$$(47) \quad b_k(\partial M_-) \leq C \cdot \text{vol}(M_{<2q})$$

for some $C = c(q, d)$ as well.

The estimate for $b_k(M_-)$ closely follows the proof of [6, Theorem 13.1]. Let g

$$: (0, \infty) \rightarrow [0, \infty)$$

be a C^∞ function with

- $g(t) > 0, g'(t) < 0$ for $t \in (0, q)$,
- $g(t) = 0$ for $t \geq q$,
- $g(t) \rightarrow \infty$ as $t \rightarrow 0$,
- $g(\delta) = 1$.

and with Δ as in the beginning of §4.1, consider the smooth function

$$F : X \rightarrow [0, \infty), F(x) = \bigcup_{\gamma \in \Delta} g \circ d_\gamma(x).$$

Since Δ is conjugation invariant in Γ , this F descends to a smooth function $f : M \rightarrow \mathbb{R}$. On [6, pg 145], it is shown that f and F have finitely many critical values

$$0 = r_0 < r_1 < \dots < r_s.$$

Note that the 0-critical set $f^{-1}(0)$ is exactly M_+ , and that $f^{-1}(0, \infty) = \text{int}(M_-)$.

In [6], Ballmann–Gromov–Schroeder use this function f to give a linear upper bound

$$(48) \quad b_k(M) \leq C \cdot \text{vol}(M), C = c(d).$$

We will describe this argument, then indicate how to modify it to prove that $b_k(M_-) \leq C\text{vol}(M_{2\varrho})$. First, at the top of pg 148 in [6], the authors prove:

$$(49) \quad b_k(M) \leq \sum_{i,j}^X b_k \{f_{x_{ij}} < r_i + \rho\}.$$

In the summation, each index i corresponds to a critical value r_i of f , and the indices j correspond to different pieces of the critical set $f^{-1}(r_i)$. More precisely, there is a collection of complete immersed submanifolds $V_{x_{ij}} \# M$ as follows⁸. For each j , let $f_{x_{ij}} := f|_{V_{x_{ij}}}$ be the restriction. Then the minimum value of $f_{x_{ij}}$ is r_i , this minimum is achieved on the set $\{f_{x_{ij}} = r_i\}$, which has nonempty interior in $V_{x_{ij}}$, and $f^{-1}(r_i)$ decomposes as:

$$f^{-1}(r_i) = \bigsqcup_{ij} \{f_{x_{ij}} = r_i\}.$$

So, in (49) the set $\{f_{x_{ij}} < r_i + \rho\}$ is just a small neighborhood of $\{f_{x_{ij}} = r_i\}$ in $V_{x_{ij}}$, since $\rho > 0$ is small. The proof of (49) is essentially via Morse theory, applied to the function f : one considers the homology of the sublevel set $\{f < r\}$, starting with $r < 0$ where $\{f < r\} = \emptyset$, and one shows that passing through the critical point $r = r_i$ contributes at most the corresponding index- i terms of the summation in (49) to the Betti numbers.

To derive (48), the authors show in [6, pg 148, (16)] that each term in (49) is bounded above by a constant times the essential volume⁹ of the immersed submanifold $V_{x_{ij}}$:

$$(50) \quad b_k(\{f_{x_{ij}} < r_i + \rho\}) \leq C \cdot \text{ess-vol}(V_{x_{ij}}), \quad C = C(d).$$

Here, the $\text{ess-vol}(V)$ is an integer that estimates volume up to some fixed multiplicative constant, but in a way that ignores small volume Euclidean factors, see [6, §12.8]. Finally, in [6, Theorem 12.11] they show¹⁰ that

$$(51) \quad \sum_{ij}^X \text{ess-vol}(V_{x_{ij}}) \leq C \cdot \text{vol}(M), \quad C = C(d),$$

so it follows from (49), (50) and (51) that $b_k(M) \leq C \cdot \text{vol}(M)$, where $C = C(d)$. We

now adapt this argument to M_- . It suffices to estimate the Betti numbers of

$$\text{int}(M_-) = f^{-1}(0, \infty),$$

since M_- is a manifold, by Lemma 4.1, and so is homotopy equivalent to $\text{int}(M_-)$. The idea is to run the Morse theory argument proving (49), but only on the interval $(0, \infty)$.

There are two main differences in the argument. First, we are no longer starting the Morse theory argument with an empty sublevel set, so we need to estimate independently the Betti numbers of $f^{-1}(0, r)$ when $r > 0$ is small. As long as $r < r_1$, Morse theory implies that $f^{-1}(0, r)$ is homeomorphic to a product $Z \times (0, r)$. The union $f^{-1}(0, r) \sqcup \partial M_-$ is a

⁸In [6], they set $V_x := Y_x/\Gamma_x$, but mostly use the latter notation in proofs.

⁹We are suppressing some constants in our notation. Really, essential volume depends on a choice of ϱ and $a > 0$, and is written $\varrho\text{-ess}^a\text{-vol}$ in [6].

¹⁰The conclusion of [6, Theorem 12.11] is about essential volume, but note that for M itself, essential volume agrees with volume up to a dimensional constant, since M has no Euclidean factors.

manifold with boundary by Lemma 4.1, so there is a collar neighborhood $\partial M_- \times [0, 1] \hookrightarrow f^{-1}(0, r)$. Since this collar gives an end neighborhood of $f^{-1}(0, r) \sqsupseteq Z \times (0, r)$, there is some $t \approx 0$ such that $f^{-1}(t) \sqsupseteq \partial M_- \times [0, 1]$. But as the composition

$$Z \xrightarrow{f^{-1}(t)} \partial M_- \times [0, 1] \xrightarrow{f^{-1}(0, r)} Z \times (0, r)$$

is a homotopy equivalence, it follows that the homology of Z injects into the homology of the collar $\partial M_- \times [0, 1]$, and therefore by (47) we have

$$(52) \quad b_k(f^{-1}(0, r)) = b_k(Z) \leq b_k(\partial M_-) \leq C \cdot \text{vol}(M_{<\varrho}).$$

Above, we are avoiding saying that Z is homeomorphic to ∂M_- , which is what you would expect in the current situation. This is probably true, but it is less obvious than the estimate in (52), which is all we need.

Second, we claim that for some $C = C(d, \varrho)$, we have

$$(53) \quad \sum_{i,j, i=0}^X \text{ess-vol}(v_{x_{ij}}) \leq C \cdot \text{vol}(M_{<\varrho}).$$

This follows from the arguments in [6].¹¹ Namely, their proof of (51) in [6, Theorem 12.11] is stronger than the statement: if $N := \sum_{i,j} \text{ess-vol}(v_{x_{ij}})$, the authors construct a collection of N injectively embedded r -balls centered at points of $M_{<\varrho/2}$ ¹¹ that overlap with uniformly bounded multiplicity, where here $r > 0$ depends only on d . Hence, this shows N is at most a dimensional constant times $\text{vol}(M_{<\varrho})$ as desired.

The Proposition now follows from (52) and (53). Namely,

$$\begin{aligned} b_k(M_-) &\leq b_k(f^{-1}(0, r)) + \sum_{i,j, i=0}^X b_k \{ f_{x_{ij}} < r_i + \varrho \} \\ &\leq b_k(f^{-1}(0, r)) + \sum_{i,j, i=0}^X \text{ess-vol}(v_{x_{ij}}) \\ &\leq C \cdot \text{vol}(M_{<\varrho}), \end{aligned}$$

where the first inequality is the Morse theory argument from [6], the second inequality is (50), and the third is (52) and (53).

4.2. Proof of Theorem 1.7. Let $\varrho_0 > 0$ and let (M_n) be a sequence of real analytic, finite volume Riemannian d -manifolds with sectional curvatures in the interval $[-1, 0]$, and assume the universal covers of the M_n do not have Euclidean de Rham-factors. Assume (M_n) BS-converges to a measure μ on M that is supported on ϱ_0 -thick manifolds. Here, recall that BS-convergence means that if μ_n are the associated measures on M^d , then

$$\mu_n / \text{vol}(M_n) \rightarrow \mu$$

¹¹See [6, (3) pp. 132–133]. In their construction of the centers z of these balls there exists an element $\beta_z \in \Gamma$ such that $d_{\beta_z}(z) = \varrho/2$, see top of page 132.

weakly. We want to show that the following limit exists for all k :

$$\lim_{n \rightarrow \infty} b_k(M_n)/\text{vol}(M_n).$$

First, here is the reason we assume that μ is supported on q_0 -thick manifolds.

Claim 4.4. For all $R > 0$ and $0 < q < q_0$, we have

$$\text{vol}(\{x \in M_n \mid d(x, M_{\leq q}) \leq R\})/\text{vol}(M_n) \rightarrow 0.$$

Proof. By the continuity of injectivity radius with respect to smooth convergence [15],

$$D := \{(M, p) \in M^d \mid d(p, M_{\leq q}) \leq R\} \subset M^d$$

is closed, so by the Portmanteau theorem,

$$\limsup_n \frac{\text{vol}(\{x \in M_n \mid d(x, M_{\leq q}) \leq R\})}{\text{vol}(M_n)} \leq \limsup_n \mu_n(D) \leq \mu(D) = 0,$$

so the limit of the sequence on the left is zero.

Pick some $q > 0$ that is less than q_0 and also less than the Margulis constant, and fix some $\xi < q/20$. With the notation $(\cdot)_+$ of the last section, with input q , let

$$N_n := \overline{((M_n)_+)_\xi} = \{x \in M_n \mid d(x, (M_n)_+) \leq \xi\},$$

and using the notation and terminology of §2.2, consider the extended mm-spaces

$$M_n := (M_n, M_n), \quad N_n := (N_n, M_n)$$

and their associated measures μ_{M_n}, μ_{N_n} on M^{ext} . (Note that the space N_n may be disconnected, but since it has finitely many components, it is special and hence our work in §2.3 still applies to N_n .) Here, if

$$\iota : M^d \rightarrow M^{\text{ext}}, \quad (M, p) \mapsto (M, p, M),$$

is the natural continuous map (see Corollary 2.2), then $\mu_{M_n} = \iota_\#(\mu_n)$, so

$$\mu_{M_n}/\text{vol}(M_n) = \iota_\# \mu_n/\text{vol}(M_n) \rightarrow \iota_\#(\mu).$$

Claim 4.5. We have $\mu_{N_n}/\text{vol}(M_n) \rightarrow \iota_\#(\mu)$ as well.

Here, note that by Claim 4.4, we have that $\text{vol}(N_n)/\text{vol}(M_n) \rightarrow 1$, so one could replace the normalizing factor by $\text{vol}(N_n)$ if desired.

Proof. Let $f : M^{\text{ext}} \rightarrow [0, m]$ be a continuous function, and fix $\alpha > 0$. Given δ, R , let

$$C_{\delta, R} = \{M \in M^{\text{ext}} \mid M \text{ is } (\delta, R)\text{-related to } N \in M^{\text{ext}} \implies |f(M) - f(N)| < \alpha\}.$$

Since the sets $C_{\delta, R}$ are open, are nested when δ is decreased and R is increased, and union to all of M^{ext} , we can choose δ, R such that

$$\iota_\#(\mu)(C_{\delta, R}) > 1 - \alpha.$$

By the Portmanteau theorem, $\liminf_n \mu_{M_n}(C_{\delta, R}) > 1 - \alpha$, so there is some N such that

$$\mu_{M_n}(C_{\delta, R}) > 1 - \alpha, \quad \forall n \geq N.$$

Furthermore, in light of Claim 4.4 and (44), we can also assume that

$$\frac{\text{vol}(\{x \in M_n \mid d(x, (M_n)_-) \leq R\})}{\text{vol}(M_n)} < \alpha, \quad \forall n \geq N.$$

Combining the above two estimates, we see that the $\text{vol}/\text{vol}(M_n)$ -measure of the set of points $p \in M_n$ such that both $d(x, (M_n)_-) > R$ and $(M, p, M) \in C_{\delta, R}$ is at least $(1 - 2\alpha)$. Now at any such point p , we have $p \in N_n$ as well, and the pointed extended mm-spaces (N_n, p, M_n) and (M_n, p, M_n) are obviously (δ, R) -related. Hence, at any such p , we have

$$(54) \quad |f(N_n, p, M_n) - f(M_n, p, M_n)| < \alpha.$$

Breaking the domains of the following integrals in two, and using the upper bound $m \geq f$ on the piece where (54) is not helpful, we see that

$$\int f \, d\mu_{N_n} - \int f \, d\mu_{M_n} \leq (1 - 2\alpha) \cdot \alpha + \alpha \cdot 2m, \quad \forall n \geq N.$$

So, since $\alpha > 0$ was arbitrary and $\int_R^R f \, d\mu_{M_n} \rightarrow \int_R^R f \, d\mu_{\mathbb{B}}(\mu)$, we have that $\int_R^R f \, d\mu_{N_n} \rightarrow \int_R^R f \, d\mu_{\mathbb{B}}(\mu)$ as well, and the claim follows.

We now want to apply Corollary 2.22 to the sequence $\mu_{N_n}/\text{vol}(M_n)$, in order to say something about normalized Betti numbers. We'll apply it with $r_0 = 4\xi, r_1 = 5\xi, r_2 = 10\xi$ and $r_3 = 11\xi$, with $B_n = b_k(M_n)$ and $V_n = \text{vol}(M_n)$. So, let's verify its hypotheses.

For condition (1) of Corollary 2.22, just note that any point $p \in N_n$ is within ξ of a point q in $(M_n)_+$, so $B_{N_n}(p, 2\xi)$ contains an embedded ξ -ball around q , which by nonpositive curvature has volume at least that of a ξ -ball in \mathbb{R}^d , see e.g. [17, Theorem 3.101]. Similarly, for condition (2) the lower curvature bound implies that any r -ball in N_n has volume at most that of an r -ball in H^d , again see [17, Theorem 3.101].

For condition (3), we need to prove the following.

Lemma 4.6. If (S_n, ρ_n) is a sequence of $[10\xi, 11\xi]$ -weighted $(4\xi, 10\xi)$ -nets in N_n , then

$$\frac{b_k(N_{M_n}(S_n, \rho_n)) - b_k(M_n)}{\text{vol}(M_n)} \rightarrow 0.$$

Assuming the lemma, the hypotheses of Corollary 2.22 are satisfied, so

$$B_n/V_n = b_k(M_n)/\text{vol}(M_n)$$

converges, proving Theorem 1.7. So, it remains to prove the lemma.

Proof of Lemma 4.6. Since $\text{inj} : M_n \rightarrow \mathbb{R}$ is 2-lipschitz, we have

$$\exists x \in S_n \in N_n := ((M_n)_+)_\xi, \text{inj}(x) \geq q - 2\xi > \frac{9q}{10} \xrightarrow{q \rightarrow \infty} \frac{11q}{20} \geq 11\xi \geq \rho_n(x).$$

Nonpositive curvature then implies that the balls $B_{\rho_n}(x)$ are convex, so the Nerve Lemma (c.f. [20, Corollary 4G.3]) says that $N_{S_n} := N_{M_n}(S_n, \rho_n)$ is homotopy equivalent to

$$U_n := \bigcup_{x \in S_n} B_{\rho_n}(x).$$

So to prove the claim, it suffices to show the following:

- (a) If $D_{k,n}$ is the dimension of the image of the map $H_k(U_n, R) \rightarrow H_k(M_n, R)$ induced by inclusion, then $b_k(M_n) = D_{k,n} + o(\text{vol}(M_n))$.
- (b) If $K_{k,n}$ is the dimension of the kernel of the map $H_k(U_n, R) \rightarrow H_k(M_n, R)$ induced by inclusion, then $K_{k,n} = o(\text{vol}(M_n))$.

For (a), apply Mayer–Vietoris to $M_n = \overline{(M_n)_-} \sqcup (M_n)_+$, giving the long exact sequence

$$\cdots \rightarrow H_k(\overline{(M_n)_-}; R) \sqcup H_k((M_n)_+; R) \rightarrow H_k(M_n; R) \rightarrow H_{k-1}(\partial(M_n)_-, R) \rightarrow \cdots.$$

By Proposition 4.2 and Claim 4.4, $b_k(\overline{(M_n)_-})$ and $b_{k-1}(\partial(M_n)_-)$ are $o(\text{vol}(M_n))$, so

$$b_k(M_n) = \dim \text{Im } H_k((M_n)_+; R) \rightarrow H_k(M_n; R) + o(\text{vol}(M_n)).$$

But the inclusion map $(M_n)_+ \rightarrow M_n$ factors through $U \rightarrow M_n$, so we have

$$b_k(M_n) \geq D_{k,n} \geq b_k(M_n) - o(\text{vol}(M_n))$$

as well, proving (a).

For (b), let $T_n \sqcup M_n \setminus N_n$ be a maximal collection of points such that

$$d(s, t) \geq \frac{1}{3} \min\{\text{inj}(s), \text{inj}(t)\}, \quad \forall s, t \in T_n.$$

Since inj is continuous, T_n is locally finite. Moreover, suppose $x \in M_n \setminus N_n$ and $x \in T_n$. By maximality, there must be some $t \in T_n$ with

$$d(x, t) \leq \frac{1}{3} \min\{\text{inj}(x), \text{inj}(t)\} \leq \frac{1}{3} \text{inj}(t),$$

so the open balls of radius $\rho_n(t) := \frac{1}{2} \text{inj}(t)$ around all $t \in T_n$ cover $M_n \setminus N_n$. Let $N_{S_n \sqcup T_n}$ be the nerve complex associated to the cover of M_n by the collection of all such balls $B_{\rho_n(t)}(t)$, $t \in T_n$, together with the balls $B_{\rho_n(x)}(x)$, $x \in S_n$. As all these balls are convex, $N_{S_n \sqcup T_n}$ is homotopy equivalent to M_n . In fact, more is true:

Claim 4.7. There is a diagram of maps

$$\begin{array}{ccc} U_n & \xrightarrow{\Phi} & N_{S_n} \\ \downarrow & & \downarrow \\ M_n & \xleftarrow{F} & N_{S_n \sqcup T_n} \end{array}$$

that is commutative up to homotopy, where the vertical maps are the natural inclusions and the horizontal maps Φ, F are homotopy equivalences.

The claim does not assert that the pairs (M_n, U_n) and $(N_{S_n \sqcup T_n}, N_{S_n})$ are homotopy equivalent, although it is certainly a result along those lines. We should note that there is at least one ‘Relative Nerve Lemma’ for pairs in the literature, see e.g. [5, Lemma 2.9], but this does not apply in our situation since $U_n \hookrightarrow M_n$ is not a cofibration. One can get

around this, but the fix is not particularly pretty, and it is much more direct just to prove the claim above without referencing any citations.

Before proving the claim, let us quickly indicate how to finish the proof of (b). Any point $x \in S_n$ such that $B_{p_{(y)}}(x)$ intersects a ball $B_{p_{(y)}}(y)$, where $y \in T_n$, must lie close to the q -thin part of M_n . By Claim 4.4 the volume of any fixed R -neighborhood of $(M_n)_-$ is $o(\text{vol}(M_n))$, so this means that there are only $o(\text{vol}(M_n))$ -many vertices of N_{S_n} that are adjacent to vertices of $N_{S_n \cap T_n} \setminus N_{S_n}$. So, Mayer–Vietoris implies that the kernel of the map

$$H_k(N_{S_n}; R) \rightarrow H_k(N_{S_n \cap T_n}; R)$$

induced by inclusion has rank $o(\text{vol}(M_n))$. Therefore, Claim 4.7 implies that the same is true for the kernel of the map on homology induced by $U_n \rightarrow M_n$.

Proof of Claim 4.7. Let's review the proof of the Nerve Lemma. For a much more general proof that essentially specializes to the one below, see Hatcher [20, 4G].

We start with a Riemannian manifold X and an open cover O by small convex balls. If N is the nerve complex of the cover O , we can define homotopy inverses

$$\alpha : X \rightarrow N, \quad \beta : N \rightarrow X$$

as follows. Pick a partition of unity $\{\phi_O \mid O \in O\}$ subordinate to O , and define

$$\alpha : X \rightarrow N, \quad \alpha(p) = \sum_{O \in O, p \in O} \phi_O(p) \cdot O.$$

Here, the values $\phi_O(p)$ are the barycentric coordinates of $\alpha(p)$, within the simplex of N spanned by those O containing p . The map β is defined inductively on the i -skeleta $B N^i$ of the first barycentric subdivision $B N$ of N . Before starting the construction, note that every vertex v of $B N$ is the barycenter of a simplex of N , which corresponds to some finite

$$F_v \cap O, \quad \cap_{O \in F_v} O = \emptyset,$$

and if Δ is a simplex of $B N$, there is one vertex $v(\Delta)$ of Δ such that $F_{v(\Delta)}$ is contained in F_w for every other vertex w of Δ . (This $v(\Delta)$ is just the vertex that is the barycenter of the simplex of N with minimal dimension.) For $i = 0, 1, 2, \dots$, we now construct the map β on $B N^i$ in such a way that for any i -simplex Δ ,

$$(55) \quad \beta(\Delta) \cap \cap_{O \in F_{v(\Delta)}} O.$$

If v is a vertex of $B N$, just pick $\beta(v) \in \cap_{O \in F_v} O$ arbitrarily. In general, assuming β has been defined on $\partial \Delta$, it follows from the definition of $v(\Delta)$ and (55) that

$$\beta(\partial \Delta) \cap \cap_{O \in F_{v(\Delta)}} O.$$

This intersection is contractible, so there is some extension of β to Δ satisfying (55). The homotopy $\alpha \circ \beta \circ 1$ is constructed inductively on the skeleta of $B N$, using the homotopy extension principle at each step. To see that $\beta \circ \alpha \circ 1$, one just notes that if $p \in X$, then $\alpha(p)$ is in some simplex Δ of $B N$ that has as a vertex some $O \ni p$, so by (55), $\beta \circ \alpha(p) \in O$. In other words, p and $\beta \circ \alpha(p)$ are both contained in one of the small convex balls in our cover, so we can just take a straight line homotopy from $\beta \circ \alpha$ to 1.

With the above presentation of the proof of the Nerve Lemma (which we could not find a reference for) the claim becomes trivial. Namely, let $\Phi : U_n \rightarrow N_{S_n}$ be the map called α above, where the manifold is U_n and the cover is by the $p_n(x)$ -balls around $x \in S_n$. Let $F : N_{S_n \sqcup T_n} \rightarrow M_n$ be the map called β above, where the manifold is M_n and the cover is by the $p_n(x)$ -balls around $x \in S_n \sqcup T_n$. These are both homotopy equivalences, and just as above the straight-line homotopy connects $F \circ \Phi$ to the inclusion $U_n \rightarrow M_n$.

Now that we have proved the claim, the lemma follows.

And so does the theorem.

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