



Lee–Yang–Fisher Zeros for the DHL and 2D Rational Dynamics, II. Global Pluripotential Interpretation

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Abstract

In a classical work of the 1950s, Lee and Yang proved that for fixed nonnegative temperature, the zeros of the partition functions of a ferromagnetic Ising model always lie on the unit circle in the complex magnetic field. Zeros of the partition function in the complex temperature were then considered by Fisher, when the magnetic field is set to zero. Limiting distributions of Lee–Yang and of Fisher zeros are physically important as they control phase transitions in the model. One can also consider the zeros of the partition function simultaneously in both complex magnetic field and complex temperature. They form an algebraic curve called the Lee–Yang–Fisher (LYF) zeros. In this paper, we continue studying their limiting distribution for the Diamond Hierarchical Lattice (DHL). In this case, it can be described in terms of the dynamics of an explicit rational function R in two variables (the Migdal–Kadanoff renormalization transformation). We study properties of the Fatou and Julia sets of this transformation and then we prove that the LYF zeros are equidistributed with respect to a dynamical $(1, 1)$ -current in the projective space. The free energy of the lattice gets interpreted as the pluripotential of this current. We also prove a more general equidistribution theorem which applies to rational mappings having indeterminate points, including the Migdal–Kadanoff renormalization transformation of various other hierarchical lattices.

Keywords Complex dynamics in higher dimensions · Invariant currents · Equidistribution problems · Lee–Yang–Fisher zeros

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1 Introduction

1.1 Lee–Yang–Fisher Zeros

We will begin with providing a brief background on the Lee–Yang–Fisher zeros that continues the discussion in Part I [4].

We consider the Ising model on a finite graph Γ and its partition function Z_Γ , which is a Laurent polynomial in two variables (z, t) , where $z = e^{-h/T}$ is a “field-like” variable and $t = e^{-2J/T}$ is “temperature-like” one. They are expressed in terms of the externally applied magnetic field h , the temperature T , and the coupling constant $J > 0$; see [4, Section 2.1] for more details.

For a fixed $t \in [0, 1]$, the complex zeros of $Z(z, t)$ in z are called the Lee–Yang zeros. The Lee–Yang Theorem [63, 67] asserts that for the ferromagnetic Ising model on any graph, *the zeros of the partition function lie on the unit circle \mathbb{T} in the complex plane*.

If we have a hierarchy of graphs Γ_n of increasing size, then under fairly general conditions, zeros of the partition functions $Z_n = Z_{\Gamma_n}$ will have a limiting distribution μ_t on the unit circle. This distribution captures phase transitions in the model.

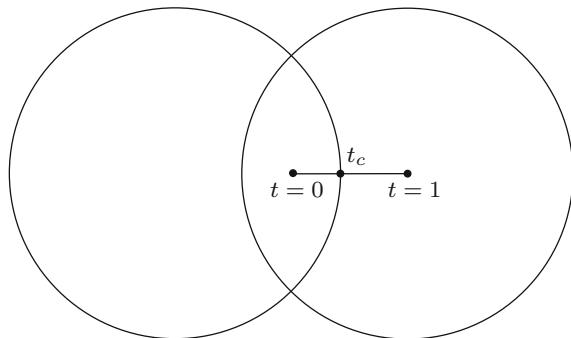
Instead of freezing temperature, one can freeze the external field, and study zeros of $Z(z, t)$ in the t -variable. They are called *Fisher zeros* as they were first studied by Fisher for the regular two-dimensional lattice, see [55, 58]. Similarly to the Lee–Yang zeros, asymptotic distribution of the Fisher zeros is supported on the singularities of the magnetic observables, and is thus related to phase transitions in the model. However, Fisher zeros do not lie on the unit circle any more. For instance, for the regular 2D lattice at zero field (corresponding to $z = 1$), the asymptotic distribution lies on the union of two *Fisher circles* depicted on Fig. 1.

We can also consider the zeros of $Z_n(z, t)$ as a single object in \mathbb{C}^2 . While $\{Z_n(z, t) = 0\}$ is an algebraic curve in \mathbb{C}^2 , we want to keep track of the multiplicities to which $Z_n(z, t)$ vanishes along each irreducible component of this curve. We will do this using the notion of *divisor*, which is a sum of finitely many irreducible algebraic curves, each with integer multiplicities (see [4, Appendix A.3]). Thus, the way $Z_n(z, t)$ vanishes in \mathbb{C}^2 defines a divisor¹ \mathcal{S}_n^c on \mathbb{C}^2 which we call the *Lee–Yang–Fisher (LYF) zeros*.

To study the limiting distribution of the LYF zeros \mathcal{S}_n^c , as n tends to infinity, we will use the theory of currents; see [38, 43]. A $(1, 1)$ -current ν on \mathbb{C}^2 is a linear functional on the space of $(1, 1)$ -forms that have compact support (see Appendix A.3). A basic example is the current $[X]$ of integration over an irreducible algebraic curve X . Meanwhile, the current of integration $[D]$ over a divisor D is the weighted sum of currents of integration over each of the irreducible components, weighted according to the multiplicities. A plurisubharmonic function G is called a *pluripotential* of ν if $\frac{i}{\pi} \partial \bar{\partial} G = \nu$, in

¹ We will see in Remark 2.1 that each of these multiplicities is one, and hence there is no harm in thinking in terms of the algebraic curve.

Fig. 1 The Fisher circles:
 $|t \pm 1| = \sqrt{2}$



the sense of distributions. (Informally, this means that $\frac{1}{2\pi} \Delta(G|L) = v|L$ for almost any complex line L , so $G|L$ is the electrostatic potential of the charge distribution $v|L$.)

Let d_n be the degree of divisor \mathcal{S}_n^c . It is natural to ask whether there exists a $(1, 1)$ -current \mathcal{S}^c so that

$$\frac{1}{d_n} [\mathcal{S}_n^c] \rightarrow \mathcal{S}^c. \quad (1.1)$$

It would describe the limiting distribution of Lee–Yang–Fisher zeros. Within almost any complex line L , the limiting distribution of zeros can be obtained as the restriction $\mathcal{S}^c|L$.

To justify existence of \mathcal{S}^c , one considers the sequence of “free energies”

$$F_n^\#(z, t) := \log |\check{Z}_n(z, t)|,$$

where $\check{Z}_n(z, t)$ is the polynomial obtained by clearing the denominators of Z_n . We will say that the sequence of graphs Γ_n has a *global thermodynamic limit* if

$$\frac{1}{d_n} F_n^\#(z, t) \rightarrow F^\#(z, t)$$

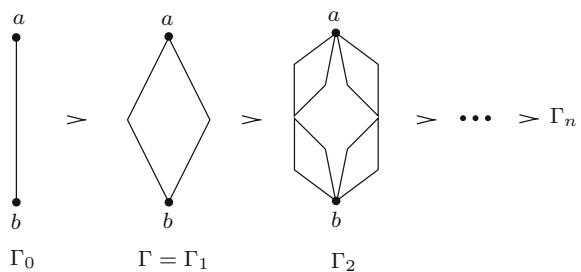
in $L^1_{\text{loc}}(\mathbb{C}^2)$. In Proposition 2.2 we will show that this is sufficient for the limiting current \mathcal{S}^c to exist and convergence (1.1) to hold.

The support of \mathcal{S}^c consists of the singularities of the magnetic observables of the model, thus describing “global phase transitions” in \mathbb{C}^2 . Connected components of $\mathbb{C}^2 \setminus \text{supp } \mathcal{S}^c$ describe the distinct “complex phases” of the system.

1.2 Diamond Hierarchical Model

The *diamond hierarchical lattice* (DHL) is a sequence of graphs Γ_n illustrated on Fig. 2. Part I [4] and much of the present paper are both devoted to study of this lattice.

Fig. 2 Diamond hierarchical lattice



The Migdal–Kadanoff renorm-group RG equations for the DHL have the form:

$$(z_{n+1}, t_{n+1}) = \left(\frac{z_n^2 + t_n^2}{z_n^{-2} + t_n^2}, \frac{z_n^2 + z_n^{-2} + 2}{z_n^2 + z_n^{-2} + t_n^2 + t_n^{-2}} \right) := \mathcal{R}(z_n, t_n), \quad (1.2)$$

where z_n and t_n are the renormalized field-like and temperature-like variables on Γ_n . The map \mathcal{R} that relates these quantities is also called the *renormalization transformation*.

To study the Fisher zeros, we consider the line $\mathcal{L}_{\text{inv}} = \{z = 1\}$ in \mathbb{C}^2 . This line is invariant under \mathcal{R} , and $\mathcal{R} : \mathcal{L}_{\text{inv}} \rightarrow \mathcal{L}_{\text{inv}}$ reduces to a fairly simple one-dimensional rational map

$$\mathcal{R} : t \mapsto \left(\frac{2t}{t^2 + 1} \right)^2.$$

The Fisher zeros at level n are obtained by pulling back the point $t = -1$ under \mathcal{R}^n . As shown in [51], the limiting distribution of the Fisher zeros in this case exists and it coincides with the measure of maximal entropy (see [6, 24, 39, 40]) of $\mathcal{R}|L$. The limiting support for this measure is the Julia set for $\mathcal{R}|L_{\text{inv}}$, which is shown in Fig. 3. It was studied by [51, 56, 57, 60] and others.

In this paper, we will use the Migdal–Kadanoff RG equations to study the global limiting distribution of Lee–Yang–Fisher zeros for the DHL in the complex projective plane \mathbb{CP}^2 . (The divisors \mathcal{S}_n^c are extended to \mathbb{CP}^2 in the natural way.) The first main result of this paper is:

Global Lee–Yang–Fisher Current Theorem *For the DHL, the currents $\frac{1}{2 \cdot 4^n} [\mathcal{S}_n^c]$ converge distributionally to some $(1, 1)$ -current \mathcal{S}^c on \mathbb{CP}^2 whose pluripotential coincides with the free energy $F^\#$ of the system.*

It would seem natural to prove this theorem by extending \mathcal{R} as a rational map $\mathcal{R} : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ and then considering the normalized pullbacks $\frac{1}{2 \cdot 4^n} (\mathcal{R}^n)^* \mathcal{S}_0^c$. However, an important subtlety arises because the degrees of \mathcal{R} do not behave properly under iteration:

$$4^n < \deg(\mathcal{R}^n) < (\deg(\mathcal{R}))^n = 6^n.$$

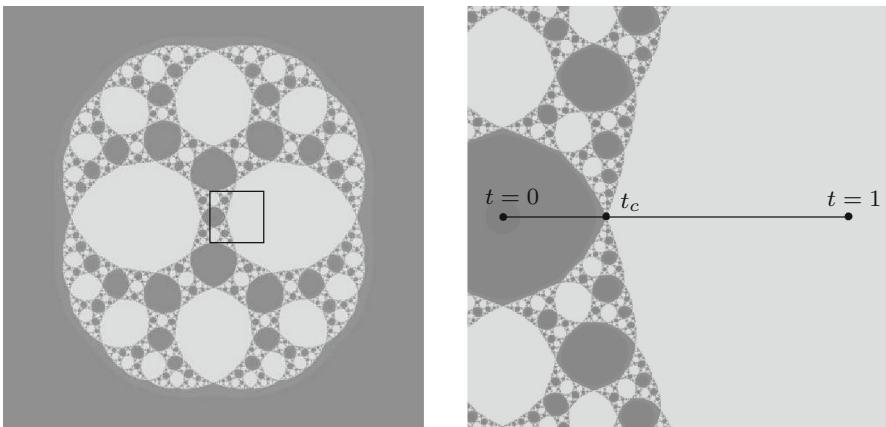


Fig. 3 On the left is the Julia set for $\mathcal{R}|_{\mathcal{L}_{\text{inv}}}$. On the right is a zoomed-in view of a boxed region around the critical point t_c . The invariant interval $[0, 1]$ corresponds to the states with real temperatures $T \in [0, \infty]$ and vanishing field $h = 0$

This *algebraic instability*² of \mathcal{R} has the consequence that

$$\mathcal{S}_n^c \neq (\mathcal{R}^n)^* \mathcal{S}_0^c.$$

The issue is resolved by working with another rational mapping $R : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ coming directly from the Migdal–Kadanoff RG Equations, without passing to the “physical” (z, t) -coordinates. This map is semi-conjugate to \mathcal{R} by a degree two rational map $\Psi : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$. Moreover, R is algebraically stable, satisfying $\deg(R^n) = (\deg(R))^n = 4^n$. For each $n \geq 0$, we have:

$$\mathcal{S}_n^c = \Psi^{-1}(R^{-n} \mathcal{S}_0^c),$$

where \mathcal{S}_0^c is an appropriate projective line.

Note that even though R is algebraically stable, it is still not well defined at two indeterminate points a_{\pm} which strongly influence the global dynamics.

1.3 Equidistribution of Curves to the Green Current

Associated to any (dominant, algebraically stable) rational mapping $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ is a canonically defined invariant current S , called the Green current³ of f . It satisfies $f^*S = d \cdot S$, where $d = \deg f$. Such invariant currents are a powerful tool of higher-dimensional holomorphic dynamics: see Bedford–Smillie [2], Fornaess–Sibony [22], Hubbard–Papadopol [33], and others (see [16, 47] for surveys of this subject).

² For the definition, see [47, §1.4] or [4, Appendix A.6].

³ It is common in the literature to denote the Green current by T , but we use S to avoid any confusion with the temperature.

Let $A \subset \mathbb{CP}^2$ be an algebraic curve of degree $\deg(A)$. Since the early 1990s there has been extensive research proving that

$$\frac{1}{d^n \deg(A)} (f^n)^*[A] \rightarrow S \quad (1.3)$$

under certain hypotheses on f and A . See [2, 10, 11, 14, 19–22, 27, 28, 42, 45, 48] for a sample of papers on the subject. Note also the recent survey [15].

If (1.3) holds for $f = R$ (the Migdal–Kadanoff RG mapping for the DHL) and $A = S_0^c$ (the principal LYF zeros), then we obtain the Global Lee–Yang–Fisher Theorem by pulling everything back under Ψ . In this way, the classical Lee–Yang–Fisher theory gets linked to the contemporary Dynamical Pluripotential Theory.

However, the majority of the papers studying (1.3) focus on the case that either

1. f is birational (f has a rational “inverse”), or
2. f is a holomorphic endomorphism (no indeterminate points),

to obtain the sharpest possible results. Otherwise, they either assume A is generic, or they work with a more “diffused” current in place of A . In any case, because $I(R) \neq \emptyset$ and $d_{\text{top}}(R) > 1$, there does not seem to be an existing result that applies to our setting:

Equidistribution Theorem for the DHL *Convergence (1.3) holds for the Migdal–Kadanoff Renormalization mapping $R : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ and any algebraic curve $A \subset \mathbb{CP}^2$.*

Remark 1.1 The dynamical approach to studying the limiting distribution of Lee–Yang–Fisher zeros for hierarchical lattices has independently been considered in [8] and studied numerically in [9].

The strategy of the proof of the Equidistribution Theorem for the DHL can be adapted to prove a more general Equidistribution Theorem, also suitable for rational maps whose indeterminacy locus satisfies certain properties.

Let $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ be a dominant algebraically stable rational mapping of algebraic degree d . Denote the indeterminacy set of f by $I(f)$. For any $Y \subset \mathbb{CP}^2$ we define $f(Y)$ and $f^{-1}(Y)$ using a resolution of $I(f)$; see Appendix A.1.

We say that an algebraic curve A is *backward invariant*⁴ if $f^{-1}(A) = A$ and we say that A is *collapsed* by f if $f(A \setminus I(f))$ is a single point. Let

$$I^-(f) := \{f(A \setminus I(f)) : A \text{ is a collapsed curve of } f\}.$$

Since each collapsed curve is critical for f , $I^-(f)$ is finite.

Choose a volume form on \mathbb{CP}^2 normalized so that $\text{vol}(\mathbb{CP}^2) = 1$. For any $z \in \mathbb{CP}^2$ we define the *volume exponent* $\sigma(z, f)$ to be the smallest positive number so that for any $\gamma > \sigma(z, f)$ there is a constant $K > 0$ and a neighborhood N of z such that for any measurable set $Y \subset \mathbb{CP}^2$ we have

$$\text{vol}(f^{-1}(Y) \cap N) \leq K (\text{vol } Y)^{1/\gamma}. \quad (1.4)$$

⁴ Note that such a curve is also forward invariant unless it contains an indeterminate point that blows-up to a different curve.

In Sect. 5 we will give two estimates on $\sigma(z, f)$ in terms of how the complex Jacobian $\text{Jac } f := \det Df$ vanishes at z . One of them is simply in terms of the order of vanishing $\mu(z, f)$ of $\text{Jac } f$ at z , while the second is stronger, but requires a more detailed assumption on $\text{Jac } f$ near z .

A sequence of points $\{z_n\} \subset \mathbb{CP}^2$ is an *orbit* of f if $z_{n+1} \in f(\{z_n\})$ for each $n \geq 0$. (If $z_n \in I(f)$, z_{n+1} can be any point on the algebraic curve $f(\{z_n\})$.) If $z_n \notin I(f)$ for every n , we will refer to the orbit as a *regular orbit*. Otherwise, we will refer to it as an *indeterminate orbit*.

If f^n is holomorphic in a neighborhood of $z \in \mathbb{CP}^2$ let $c(z, f^n)$ denote the order of vanishing of the power series expansion for f^n expressed in local coordinates centered at z and $f^n(z)$, respectively. If z_0, \dots, z_{k-1} is a regular periodic orbit of period k for f then

$$c_\infty(z_0, f) := \lim_{n \rightarrow \infty} c(z_0, f^{nk})^{1/nk}$$

exists and satisfies $c_\infty(z_0, f) \leq d$; see Sect. 7.2. We say that a regular periodic point z_0 is *superattracting* if $c_\infty(z_0, f) > 1$ and is *maximally superattracting* if $c_\infty(z_0, f) = d$. In these cases, the orbit of z_0 is attracting at superexponential rate in all directions.

Let \mathcal{E} be the finite set containing all

- (a) maximally superattracting periodic points, and
- (b) superattracting periodic points z_0 of period k for which there is an algebraic curve C that is backward invariant under f^k , collapsed to z_0 under some iterate of f^k , and for which z_0 is a singular point of C .

Denote the respective subsets of \mathcal{E} where (a) or (b) holds as $\mathcal{E}(a)$ and $\mathcal{E}(b)$. We will call \mathcal{E} the *exceptional set* for f .

Equidistribution Theorem *Let $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ be a dominant algebraically stable rational map of degree $d \geq 2$ and let S denote the Green Current of f . Assume that*

- (i) $I(f) \neq \emptyset$,
- (ii) $\sigma(z, f) < d$ for every $z \in I(f)$, and
- (iii) no periodic orbit passes through both the finite set
 $D_{>d} := \{z \in \mathbb{CP}^2 : \sigma(z, f) > d\}$ and $I(f) \cup I^-(f)$.

Then, for any algebraic curve A that does not pass through the exceptional set \mathcal{E} we have

$$\frac{1}{d^n \deg A} (f^n)^*[A] \rightarrow S.$$

Note that Hypotheses (i) and (ii) are verifiable algebraic conditions on the map f itself. The last Hypothesis (iii) is more problematic as it is dynamical; still it is amenable to verification under favorable circumstances as it requires that a certain finite set of points (specified algebraically) is aperiodic. We illustrate application of the Equidistribution Theorem to a few examples in Sect. 8.

Let us also compare⁵ our result to the cases of birational maps and endomorphisms:

Equidistribution for Birational Mappings [10,19,20] *Let $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ be an algebraically stable birational mapping of degree $d \geq 2$ and let S denote the Green Current of f . Let $A \subset \mathbb{CP}^2$ be an algebraic curve. Then, there is a exceptional set \mathcal{E} consisting of at most one point such that*

$$\frac{1}{d^n \deg A} (f^n)^*[A] \rightarrow S \quad \text{if and only if} \quad A \text{ does not pass through } \mathcal{E}.$$

The exceptional set \mathcal{E} consists of a maximally superattracting fixed point through which there passes a backward invariant curve. In the special case that f is a Hénon mapping,

$$f[x : y : z] = [x^2 + ayz : xz : z^2],$$

we have that $\mathcal{E} = [1 : 0 : 0]$ is the superattracting fixed point at infinity, with the totally invariant curve corresponding to the line at infinity $\{z = 0\}$ [2,23].

Equidistribution for Endomorphisms [14,21] *Let $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ be a holomorphic endomorphism of degree $d \geq 2$ and let S denote the Green Current of f . Then, there is a totally invariant algebraic set \mathcal{E}_1 consisting of at most three projective complex lines and a finite totally invariant set \mathcal{E}_2 with the following property: If A is an algebraic curve such that*

- (i) $A \not\subset \mathcal{E}_1$, and
- (ii) $A \cap \mathcal{E}_2 = \emptyset$,

then

$$\frac{1}{d^n \deg A} (f^n)^*[A] \rightarrow S.$$

The exceptional set \mathcal{E}_1 corresponds to curves on which the order of vanishing of the Jacobian grows at rate $\geq d^n$ under iteration, and hence the volume exponent $\sigma(z, f^n)$ growing at rate $\geq d^n$ as well. Meanwhile the set \mathcal{E}_2 consists of maximally superattracting periodic points.

Our general strategy is similar to that in the above-mentioned works: We prove the L^1_{loc} -convergence of the potentials of the currents under consideration, which requires estimates on the volume growth under the iterated pullbacks. (For the latter, we have especially profited from the techniques developed by Favre and Jonsson [21]). However, in our setting there is a possibility that the orbit of a point z recurs to $I(f)$, while also having bad growth of the volume exponent. In the case of birational maps, this is eliminated since the only critical points are on collapsed curves, whose orbits stay away from $I(f)$ (by the algebraic stability assumption). In our case, Hypotheses (i)–(iii) allow us to rule out the problematic scenario.

⁵ We will specialize the following two results to the case of pulling back a curve, instead of pulling back an arbitrary closed positive $(1, 1)$ current.

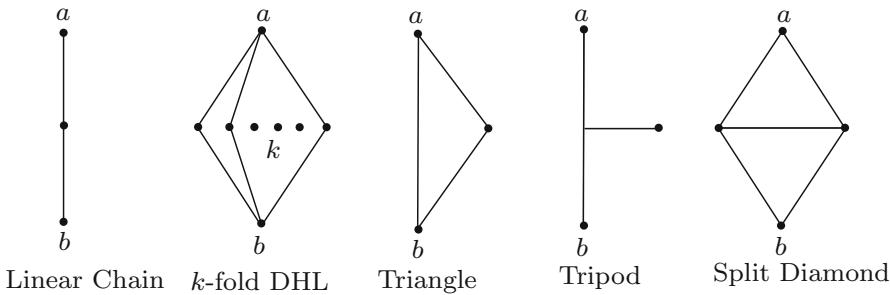


Fig. 4 Generating graphs for some other hierarchical lattices

The final punch line of our argument is an application of the Borel–Cantelli Lemma, which makes it quite elementary and general.

1.4 Other Hierarchical Lattices

The Diamond Hierarchical Lattice has the merit of being one of the simplest non-trivial hierarchical lattices. Instead of using the diamond to generate our sequence of graphs $\{\Gamma_n\}_{n=0}^\infty$ (as shown in Fig. 2) we can use any finite graph Γ with two marked vertices a and b that is symmetric under interchange of a and b . One obtains Γ_{n+1} by replacing each edge of Γ_n with a copy of Γ , using the marked vertices a and b as “endpoints”. We will call the sequence of graphs *the hierarchical lattice generated by Γ* .

Associated to each generating graph Γ is a Migdal–Kadanoff renormalization mapping, which is a rational map $R_\Gamma : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$. In Sect. 8 we discuss the Migdal Kadanoff renormalization mappings associated to the five different hierarchical lattices whose generating graphs are shown in Fig. 4. We will see that the Equidistribution Theorem applies to the Migdal Kadanoff renormalization mappings associated to the k -fold DHL ($k \geq 2$), the Triangle, and the Split Diamond thus proving the Global LYF Theorem for each of those lattices. We find that the Equidistribution Theorem does not apply to the Migdal Kadanoff renormalization mappings for the Linear Chain or the Tripod, however, an easy argument directly shows that the Global LYF Theorem holds for the Linear Chain. We do not know if it holds for the Tripod.

Problem 1.2 *Does the Global LYF Theorem hold for every hierarchical lattice?*

Several additional open problems are listed in Appendix C.

1.5 Structure of the Paper

We begin in Sect. 2 by recalling the definitions of free energy and the classical notion of thermodynamic limit for the Ising model. We then discuss the notion of global thermodynamic limit, which is sufficient to guarantee that some lattice have a $(1, 1)$ -current \mathcal{S}^c describing its limiting distribution of LYF zeros in \mathbb{C}^2 . We also give an alternative interpretation of the partition function as a section of (an appropriate tensor

power of) the hyperplane bundle over \mathbb{CP}^2 that will be central to the proof of the Global LYF Current Theorem. We conclude Sect. 2 by summarizing material on the Migdal-Kadanoff RG equations, including the details for reducing the proof of the Global LYF Theorem to proof of (1.3) for $A = S_n^c$ and $f = R$.

In Sect. 3 we summarize the global features of the mappings \mathcal{R} and R on the complex projective space \mathbb{CP}^2 that were studied in [4], including their critical and indeterminacy loci, superattracting fixed points and their separatrices.

In the next section, Sect. 4, we define the Fatou and Julia sets for \mathcal{R} and show that the Julia set coincides with the closure of preimages of the invariant complex line $\{z = 1\}$ (corresponding to the vanishing external field). It is based on M. Green's criteria for Kobayashi hyperbolicity of the complements of several algebraic curves in \mathbb{CP}^2 [25,26] that generalize the classical Montel Theorem. We then use this result to prove that points in the interior of the solid cylinder $\mathbb{D} \times I$ are attracted to a superattracting fixed point $\eta = (0, 1)$ of \mathcal{R} .

The proofs of the equidistribution theorems require estimates on the volume of a tubular neighborhood of an algebraic curve and estimates on how volume is transformed under a rational map. These estimates are presented in Sect. 5.

Section 5 is devoted to proving the Equidistribution Theorem for the DHL. In Sect. 7 we then show how to adapt its proof to prove the Equidistribution Theorem. In Sect. 8 we discuss applications of the Equidistribution Theorem to other hierarchical lattices.

Like Part I, this paper is written for readers from both complex dynamics and statistical physics, so we provide background material in two appendices. To minimize overlap, we will refer the reader to appendices of Part I when possible. In Appendix A, we collect needed background in complex geometry (line bundles over \mathbb{CP}^2 , currents and their pluri-potentials, Kobayashi hyperbolicity, and normal families). In Appendix B, we provide information on the Green current. In Appendix C, we collect several open problems.

1.6 Basic Notation and Terminology

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\mathbb{T} = \{|z| = 1\}$, $\mathbb{D}_r = \{|z| < r\}$, $\mathbb{D} \equiv \mathbb{D}_1$, and $\mathbb{N} = \{0, 1, 2, \dots\}$. Given two variables x and y , $x \asymp y$ means that $c \leq |x/y| \leq C$ for some constants $C > c > 0$.

2 Description of the Model

2.1 Free Energy and Thermodynamic Limit

The partition function of the Ising model on a graph Γ is a symmetric Laurent polynomial in (z, t) of the form

$$Z_\Gamma = \sum_{n=0}^d a_n(t)(z^n + z^{-n}) \quad (2.1)$$

of degree d equal to the the number of edges in Γ . (See [4, Section 2.1] for the definition.)

Remark 2.1 Setting $\check{Z}(z, t) := z^d t^{d/2} Z(z, t)$ clears the denominators of $Z(z, t)$ and results in a polynomial in z and t of degree $2d$ whose divisor of zeros is the same as that of $Z(z, t)$. It follows from the definition of $Z(z, t)$ that $\check{Z}(z, 0) = z^{2d} + 1$, each of whose zeros is simple. In particular the divisor of zeros assigns multiplicity one to each irreducible component of $\{Z(z, t) = 0\}$.

The *free energy* of the system is defined as

$$F_\Gamma = -T \log |Z_\Gamma|, \quad (2.2)$$

where T is the temperature (related to the temperature-like variable by $t = e^{-J/T}$, where J is the coupling constant of the model).

It will be more convenient to consider the following variant of the free energy:

$$F_\Gamma^\#(z, t) := -\frac{1}{T} F_\Gamma(z, t) + d \left(\log |z| + \frac{1}{2} \log |t| \right) = \log |\check{Z}(z, t)|.$$

The advantage of using $F_\Gamma^\#$, instead of F_Γ , is that it extends as a plurisubharmonic function on all of \mathbb{C}^2 . We will also refer to $F_\Gamma^\#$ as the “free energy”.

Assume that we have a lattice given by a hierarchy of graphs Γ_n with $d_n \rightarrow \infty$ edges. Let us consider its partition functions Z_n and free energies $F_n^\#$. To pass to the thermodynamic limit we normalize the free energy *per bond*. One says that the hierarchy of graphs has a (pointwise) *thermodynamic limit* if

$$\frac{1}{2d_n} F_n^\#(z, t) \rightarrow F^\#(z, t) \quad \text{for any } z \in \mathbb{R}_+, t \in (0, 1). \quad (2.3)$$

In this case, the function $F^\#$ is called the (modified) *free energy* of the lattice. For many⁶ lattices (e.g. \mathbb{Z}^d), existence of the thermodynamic limit can be justified by van Hove’s Theorem [59, 66]. If the classical thermodynamic limit exists, then one can justify existence of the limiting distribution of Lee–Yang zeros and relate it to the limiting free energy; see [4, Prop. 2.2].

To prove existence of a limiting distribution for the Fisher zeros, one needs to prove existence of the thermodynamic limit in the $L^1_{\text{loc}}(\mathbb{C})$ -sense:

$$\frac{1}{2d_n} F_n^\#(1, t) \rightarrow F^\#(1, t) \quad \text{in } L^1_{\text{loc}}(\mathbb{C}). \quad (2.4)$$

For the \mathbb{Z}^2 lattice this is achieved by the Onsager solution, which provides an explicit formula for the limiting free energy; see, for example, [50]. Similar techniques apply to the triangular, hexagonal, and various homopolygona lattices (see [64, 65] for suitable references and an investigation of the distribution of Fisher zeros for these lattices). For various hierarchical lattices, (2.4) can be proved by dynamical means.

The situation is similar for the Lee–Yang–Fisher zeros.

⁶ Note that the DHL is not in this class—instead, dynamical techniques are used to justify its classical thermodynamic limit.

Proposition 2.2 Let (Γ_n) be a lattice for which the thermodynamic limit exists in the $L^1_{\text{loc}}(\mathbb{C}^2)$ -sense:

$$\frac{1}{2d_n} F_n^\#(z, t) \rightarrow F^\#(z, t) \quad \text{in } L^1_{\text{loc}}(\mathbb{C}^2). \quad (2.5)$$

Then, there is a closed positive $(1, 1)$ -current \mathcal{S}^c on \mathbb{C}^2 describing the limiting distribution of Lee–Yang–Fisher zeros. Its pluripotential coincides with the free energy $F^\#(z, t)$.

For the DHL, we will prove existence of the limit (2.5) in the Global LYF Current Theorem.

Remark 2.3 It is an open question whether the limit (2.5) exists for any classical lattice, including the \mathbb{Z}^2 lattice. Moreover, it seems to also be an open question whether the limit (2.4) exists for the \mathbb{Z}^d lattice, when $d \geq 3$, and other classical three dimensional lattices. See Problem C.1.

Proof of Proposition 2.2 The locus of Lee–Yang–Fisher zeros \mathcal{S}_n^c are the zero set (counted with multiplicities) of the degree $2d_n$ polynomial $\check{Z}_n(z, t)$. The Poincaré–Lelong Formula describes its current of integration:

$$[\mathcal{S}_n^c] = \frac{i}{\pi} \partial \bar{\partial} \log |\check{Z}_n(z, t)| = \frac{i}{\pi} \partial \bar{\partial} F_n^\#(z, t).$$

Hypothesis (2.5) implies

$$\frac{1}{2d_n} [\mathcal{S}_n^c] = \frac{i}{\pi} \partial \bar{\partial} \frac{1}{2d_n} F_n^\#(z, t) \rightarrow \frac{i}{\pi} \partial \bar{\partial} F^\#(z, t) =: \mathcal{S}^c.$$

□

2.2 Global Consideration of Partition Functions and Free Energy on \mathbb{CP}^2

It will be convenient for us to extend the partition functions Z_n and their associated free energies $F_n^\#$ from \mathbb{C}^2 to \mathbb{CP}^2 . We will use the homogeneous coordinates $[Z : T : Y]$ on \mathbb{CP}^2 , with the copy of \mathbb{C}^2 given by the affine coordinates $(z, t) \mapsto [z : t : 1]$.

For each n , we clear the denominators of $Z_n(z, t)$, obtaining a polynomial $\check{Z}_n(z, t)$ of degree $d_n := 2|\mathcal{E}_n|$. It lifts to a unique homogeneous polynomial $\hat{Z}_n(Z, T, Y)$ of the same degree that satisfies $\check{Z}_n(z, t) = \hat{Z}_n(z, t, 1)$. The associated free energy becomes a plurisubharmonic function

$$\hat{F}_n^\#(Z, T, Y) := \log |\hat{Z}_n(Z, T, Y)|$$

on \mathbb{C}^3 . It is related by the Poincaré–Lelong Formula to the current of integration over the Lee–Yang–Fisher zeros: $\pi^*[\mathcal{S}_n^c] := \frac{i}{\pi} \partial \bar{\partial} \hat{F}_n^\#(Z, T, Y)$.

Both of these extensions are defined on \mathbb{C}^3 , rather than \mathbb{CP}^2 . In the proof of the Global LYF Current Theorem, it will be useful for us to interpret the partition function as an object defined on \mathbb{CP}^2 . Instead of being a function on \mathbb{CP}^2 , it gets interpreted as a section s_{Z_n} of an appropriate tensor power of the hyperplane bundle; See Appendix A.2. The Lee–Yang–Fisher zeros \mathcal{S}_n^c are described as the zero locus of this section.

2.3 Migdal–Kadanoff Renormalization for the DHL

The renormalized field-like and temperature-like variables z_n and t_n that appear in the Migdal–Kadanoff RG equations (1.2) are defined through certain “conditional partition functions of level n ” in the following way:

$$z_n^2 = W_n/U_n, \quad t_n^2 = \frac{V_n^2}{U_n W_n}. \quad (2.6)$$

In the (U, V, W) -coordinates the Migdal–Kadanoff RG equation assumes the homogeneous form

$$U_{n+1} = (U_n^2 + V_n^2)^2, \quad V_{n+1} = V_n^2(U_n + W_n)^2, \quad W_{n+1} = (V_n^2 + W_n^2)^2,$$

and the total partition function becomes a linear form

$$Z_n \equiv Z_{\Gamma_n} = U_n + 2V_n + W_n.$$

(See Part I [4] for the derivation of these equations.) This leads us to a homogeneous degree 4 polynomial map

$$\hat{R} : (U, V, W) \mapsto ((U^2 + V^2)^2, V^2(U + W)^2, (W^2 + V^2)^2), \quad (2.7)$$

called the *Migdal Kadanoff Renormalization*, such that $(U_n, V_n, W_n) = \hat{R}^n(U_0, V_0, W_0)$. (The corresponding map $R : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ will be referred to in the same way.) Moreover, letting $Y_0 := U + 2V + W$, we obtain:

$$\hat{Z}_n = Y_0 \circ \hat{R}^n, \quad (2.8)$$

so the partition functions \hat{Z}_n are obtained by pulling the linear form $Y_0 \equiv \hat{Z}_0$ back by \hat{R}^n .

We will often write R in the system of local coordinates $u = U/V$ and $w = W/V$, in which it has the form

$$R : (u, w) \mapsto \left(\frac{u^2 + 1}{u + w}, \frac{w^2 + 1}{u + w} \right)^2. \quad (2.9)$$

Notice that the form Y_0 is *not a function* on \mathbb{CP}^2 but rather a *section* s_{Y_0} of the hyperplane line bundle over \mathbb{CP}^2 , see Appendix A.2. Respectively, the partition functions \hat{Z}_n

are sections of the tensor powers of this line bundle. Accordingly, the Lee–Yang–Fisher loci S_n^c are the *zero divisors* of these sections.

The free energy is also no longer a function on \mathbb{CP}^2 , rather it is lifted to become a function on \mathbb{C}^3 , given by

$$\hat{F}_n^\# := \log |\hat{Z}_n|. \quad (2.10)$$

The above formulae express the partition functions and free energies in terms of the U, V, W coordinates. To re-express them in terms of the physical coordinates, we pull each of them back by

$$\Psi : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2, \quad (U : V : W) = \Psi(z, t) = (z^{-1}t^{-1/2} : t^{1/2} : zt^{-1/2}). \quad (2.11)$$

This change of variables also semi-conjugates the map

$$\mathcal{R} : (z, t) \mapsto \left(\frac{z^2 + t^2}{z^{-2} + t^2}, \frac{z^2 + z^{-2} + 2}{z^2 + z^{-2} + t^2 + t^{-2}} \right), \quad (2.12)$$

corresponding to RG equation (1.2), to R .

3 Global Properties of the RG Transformation in \mathbb{CP}^2

We will now summarize (typically without proofs) results from [4] about the global properties of the RG mappings.

3.1 Preliminaries

The renormalization mappings \mathcal{R} and R are semi-conjugate by the degree two rational mapping $\Psi : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ given by (2.6).

Both mappings have topological degree 8 (see Proposition 4.3 from Part I). However, as noted in the Introduction, their algebraic degrees behave differently: R is algebraically stable, while \mathcal{R} is not. Since $\deg(R^n) = 4^n$, for any algebraic curve D of degree d , the pullback $(R^n)^*D$ is a divisor of degree $d \cdot 4^n$. (For background on divisors, see [4, Appendix A.3].) For this reason, we will focus most of our attention on the dynamics of R .

The semiconjugacy Ψ sends the Lee–Yang cylinder $\mathcal{C} := \mathbb{T} \times [0, 1]$ to a Möbius band C that is invariant under R . It is obtained as the closure in \mathbb{CP}^2 of the topological annulus

$$C_0 = \{(u, w) \in \mathbb{C}^2 : w = \bar{u}, |u| \geq 1\}. \quad (3.1)$$

Let $T = \{(u, \bar{u}) : |u| = 1\}$ be the “top” circle of C , and let B be the slice of C at infinity. In fact, $\Psi : \mathcal{C} \rightarrow C$ is a conjugacy, except that it maps the bottom B of \mathcal{C} by a 2-to-1 mapping to B (see Proposition 3.1 from Part I).

3.2 Indeterminacy Points for R

In homogeneous coordinates on \mathbb{CP}^2 , the map R has the form:

$$R : [U : V : W] \mapsto [(U^2 + V^2)^2 : V^2(U + W)^2 : (V^2 + W^2)^2]. \quad (3.2)$$

One can see that R has precisely two points of indeterminacy $a_+ := [i : 1 : -i]$ and $a_- := [-i : 1 : i]$. Resolving all of the indeterminacies of R by blowing-up the two points a_{\pm} (see [4, Appendix A.2]), one obtains a holomorphic mapping $\tilde{R} : \widetilde{\mathbb{CP}}^2 \rightarrow \mathbb{CP}^2$.

In coordinates $\xi = u - i$ and $\chi = (w + i)/(u - i)$ near $a_+ = (i, -i)$, we obtain the following expression for the map $\tilde{R} : \widetilde{\mathbb{CP}}^2 \rightarrow \mathbb{CP}^2$ near $L_{\text{exc}}(a_+)$:

$$u = \left(\frac{\xi + 2i}{1 + \chi} \right)^2, \quad w = \left(\frac{\chi^2 \xi - 2i \chi}{1 + \chi} \right)^2. \quad (3.3)$$

(Similar formulas hold near $a_- = (-i, i)$.) The exceptional divisor $L_{\text{exc}}(a_+)$ is mapped by \tilde{R} to the conic

$$G := \{(u - w)^2 + 8(u + w) + 16 = 0\}.$$

3.3 Superattracting Fixed Points and Their Separatrices

We will often refer to $L_0 := \{V = 0\} \subset \mathbb{CP}^2$ as the *line at infinity*. It contains two symmetric superattracting fixed points, $e = (1 : 0 : 0)$ and $e' = (0 : 0 : 1)$. Let $\mathcal{W}^s(e)$ and $\mathcal{W}^s(e')$ stand for the attracting basins of these points. It will be useful to consider local coordinates $(\xi = W/U, \eta = V/U)$ near e .

The line at infinity $L_0 = \{\eta = 0\}$ is R -invariant, and the restriction $R|_{L_0}$ is the power map $\xi \mapsto \xi^4$. Thus, points in the disk $\{|\xi| < 1\}$ in L_0 are attracted to e , points in the disk $\{|\xi| > 1\}$ are attracted to e' , and these two basins are separated by the unit circle B . We will also call L_0 the *fast separatrix* of e and e' .

Let us also consider the conic

$$L_1 = \{\xi = \eta^2\} = \{V^2 = UW\} \quad (3.4)$$

passing through points e and e' . It is an embedded copy of \mathbb{CP}^1 that is invariant under R , with $R|_{L_1}(w) = w^2$, where $w = W/V = \xi/\eta$. Thus, points in the disk $\{|w| < 1\}$ in L_1 are attracted to e , points in the disk $\{|w| > 1\}$ are attracted to e' , and these two basins are separated by the unit circle T (see §3.1 from Part I). We will call L_1 the *slow separatrix* of e and e' .

If a point x near e (resp. e') does not belong to the fast separatrix L_0 , then its orbit is “pulled” towards the slow separatrix L_1 at rate ρ^{4^n} , with some $\rho < 1$, and converges to e (resp. e') along L_1 at rate r^{2^n} , with some $r < 1$.

The strong separatrix L_0 is transversally superattracting: all nearby points are pulled towards L_0 uniformly at rate r^{2^n} . It follows that these points either converge to one of the fixed points, e or e' , or converge to the circle B .

Given a neighborhood Ω of B , let

$$\mathcal{W}_{\mathbb{C},\text{loc}}^s(B) = \{x \in \mathbb{CP}^2 : R^n x \in \Omega \text{ (}n \in \mathbb{N}\text{)} \text{ and } \mathbb{R}^n x \rightarrow B \text{ as } n \rightarrow \infty\} \quad (3.5)$$

(where Ω is implicit in the notation, and an assertion involving $\mathcal{W}_{\mathbb{C},\text{loc}}^s(B)$ means that it holds for arbitrary small suitable neighborhoods of B). It is shown in Part I (§9.2) that $\mathcal{W}_{\mathbb{C},\text{loc}}^s(B)$ has the topology of a 3-manifold that is laminated by the union of holomorphic local stable manifolds $\mathcal{W}_{\mathbb{C},\text{loc}}^s(x)$ of points $x \in B$.

We conclude:

Lemma 3.1 $\mathcal{W}^s(e) \cup \mathcal{W}^s(e') \cup \mathcal{W}_{\mathbb{C},\text{loc}}^s(B)$ fills in some neighborhood of L_0 .

3.4 Regularity of $\mathcal{W}_{\mathbb{C},\text{loc}}^s(x)$

For a diffeomorphism, the existence and regularity of the local stable manifold for a hyperbolic invariant manifold N has been studied extensively in [32]. To guarantee a C^1 local stable manifold $\mathcal{W}_{\text{loc}}^s(N)$, a strong form of hyperbolicity known as *normal hyperbolicity* is assumed. Essentially, N is normally hyperbolic for f if the expansion of Df in the transverse unstable direction dominates the maximal tangent expansion of $D(f|_N)$ and the contraction of Df in the transverse stable direction dominates the maximal tangent contraction of $D(f|_N)$. See [32, Thm. 1.1]. If, furthermore, the expansion of Df in the transverse unstable direction dominates the r -th power of the maximal tangent expansion of $D(f|_N)$ and the contraction of Df in the transverse stable direction dominates the r -th power of the maximal tangent contraction of $D(f|_N)$, this guarantees that the stable manifold is of class C^r . The corresponding theory for endomorphisms is less developed, although note that some aspects of [32], related to persistence of normally hyperbolic invariant laminations, have been generalized to endomorphisms in [3].

In our situation, B is not normally hyperbolic because it lies within the invariant line L_0 and R is holomorphic. This forces the expansion rates tangent to B and transverse to B (within this line) to coincide. Therefore, the following result does not seem to be part of the standard hyperbolic theory.

Lemma 3.2 $\mathcal{W}_{\mathbb{C},\text{loc}}^s(B)$ is a C^∞ manifold and the stable foliation is a C^∞ foliation by complex analytic discs.

Proof In Proposition 9.11 from Part I, we showed that within the cylinder \mathcal{C} the stable foliation of \mathcal{B} has C^∞ regularity and that the stable curve of each point is real analytic. Mapping forward under Ψ , we obtain the same properties for the stable foliation of B within C .

Let us work in the local coordinates $\xi = W/U$ and $\eta = V/U$. In these coordinates, $B = \{\eta = 0, |\xi| = 1\}$. The stable curve $\mathcal{W}_{\mathbb{C},\text{loc}}^s(\xi_0)$ of any $\xi_0 \in B$ can be given by expressing ξ as a holomorphic function of η :

$$\xi = h(\eta, \xi_0) = \sum_{i=0}^{\infty} a_i(\xi_0) \eta^i. \quad (3.6)$$

The right hand side is a convergent power series with coefficients depending on ξ_0 , having a uniform radius of convergence over every $\xi_0 \in \mathcal{B}$. The series is uniquely determined by its values on the real slice C , in which the leaves depend with C^∞ regularity on ξ_0 . Therefore, each of the coefficients $a_i(\xi_0)$ is C^∞ in ξ_0 . This gives that each $W_{\mathbb{C}, \text{loc}}^s(\xi_0)$ depends with C^∞ regularity on ξ_0 , implying the stated result. \square

Remark 3.3 The technique from the proof of Lemma 3.2 applies to a more general situation: Suppose that M is a real analytic manifold and $f : M \rightarrow M$ is a real analytic map. Let $N \subset M$ be a compact real analytic invariant submanifold for f , with $f|N$ expanding and with N transversally attracting under f . Then N will have a stable foliation $\mathcal{W}_{\text{loc}}^s(N)$ of regularity C' , for some $r > 0$ (see the beginning of this subsection), with the stable manifold of each point being real-analytic. The stable manifold $\mathcal{W}_{\mathbb{C}, \text{loc}}^s(N)$ for the extension of f to the complexification $M_{\mathbb{C}}$ of M will then also have C' regularity.

Remark 3.4 It has been shown by Kaschner and the third author that $\mathcal{W}_{\mathbb{C}, \text{loc}}^s(\mathcal{B})$ is not real analytic at any point [34, Thm. B].

3.5 Critical Locus

The complex Jacobian of $\hat{R} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ (2.7) is equal to

$$\text{Jac } \hat{R} \equiv \det D\hat{R} = 32 V (UW - V^2) (U + W)^2 (U^2 + V^2) (W^2 + V^2), \quad (3.7)$$

and therefore, the critical locus of R consists of six complex lines and one conic:

$$\begin{aligned} L_0 &:= \{V = 0\} = \text{line at infinity}, \\ L_1 &:= \{UW = V^2\} = \text{conic } \{uw = 1\}, \\ L_2 &:= \{U = -W\} = \{u = -w\} = \text{the collapsing line}, \\ L_3^\pm &:= \{U = \pm iV\} = \{u = \pm i\}, \\ L_4^\pm &:= \{W = \pm iV\} = \{w = \pm i\}. \end{aligned}$$

(Here, the curves are written in the homogeneous coordinates $(U : V : W)$ and in the affine ones, $(u = U/V, w = W/V)$.) The critical locus is schematically depicted on Fig. 5, while its image, the critical value locus, is depicted on Fig. 6.

It will be helpful to also consider the critical locus for the lift $\tilde{R} : \tilde{\mathbb{CP}}^2 \rightarrow \mathbb{CP}^2$. Each of the critical curves L_i lifts by proper transform (see [4, Appendix A.2]) to a critical curve $\tilde{L}_i \subset \tilde{\mathbb{CP}}^2$ for \tilde{R} . Moreover, any critical point for \tilde{R} belongs to either one of these proper transforms or to one of the exceptional divisors $L_{\text{exc}}(a_\pm)$.

By symmetry, it is enough to consider the blow-up of a_+ . We saw in Part I that there are four critical points on the exceptional divisor $L_{\text{exc}}(a_+)$ occurring at $\chi = -1, 1, \infty$,

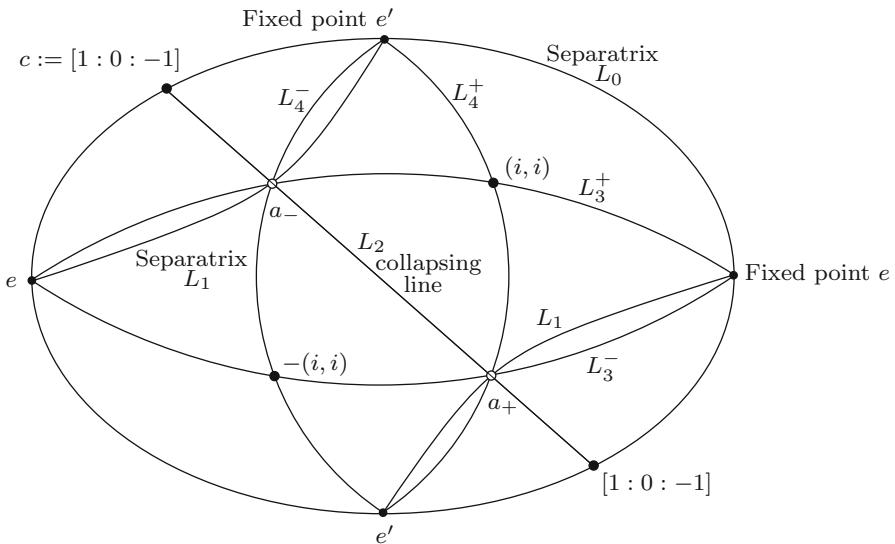
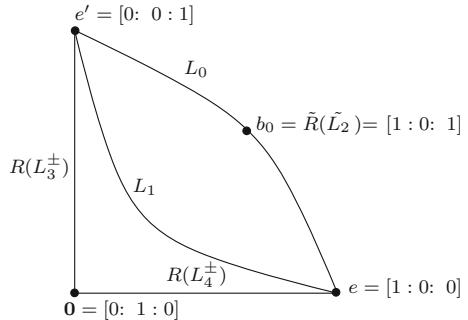


Fig. 5 Critical locus for R shown with the separatrix L_0 at infinity

Fig. 6 Critical values locus of R



and 0, where $\chi = (w+i)/(u-i)$. They correspond to intersections of $L_{\text{exc}}(a_+)$ with the collapsing line \tilde{L}_2 , the \tilde{L}_1 , and the critical lines \tilde{L}_3^+ and \tilde{L}_4^- , respectively.

Remark 3.5 In Part I we showed that all of the critical points of \tilde{R} except the fixed points e, e' , the collapsing line \tilde{L}_2 , and two points $\{\pm(i, i)\} = \tilde{L}_3^\pm \cap \tilde{L}_4^\pm$, are degree two *Whitney folds*, i.e. they can be brought into the normal form $(x, y) \mapsto (x, y^2)$ in holomorphic coordinates. See [4, Appendix D.2] and also Lemma A.4.

3.6 Local Study of the Critical Locus of R and \tilde{R}

In the proof of the Equidistribution Theorem for the DHL we will need details about how the Jacobian of R vanishes, when R is expressed in local coordinates; See Lemmas 5.2 and 5.5.

These details can be recovered from Formula (3.7) for $\text{Jac } \hat{R}$ as follows: Suppose $z \in N \subset \mathbb{CP}^2 \setminus I(R)$, with N an open set admitting a local section $s : N \rightarrow \mathbb{C}^3 \setminus \{0\}$ of the canonical projection $\pi : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathbb{CP}^2$. If we express R in local coordinates on N (in the domain) and on $R(N)$ (in the codomain), then the complex Jacobian of this local expression for R differs from $\text{Jac } \hat{R} \circ s$ by a non-vanishing analytic function.

For any $z \in \mathbb{CP}^2 \setminus I(R)$ define $\mu(z, R)$ to be the order of vanishing at z for the complex Jacobian of any local coordinate expression for R . By the chain rule, the result is independent of the choice of charts. Moreover, by the discussion in the previous paragraph, $\mu(z, R)$ equals the order of vanishing of $\text{Jac } \hat{R} \circ s$ at z .

Lemma 3.6 *Let c be the point of intersection between L_0 and L_2 . We have*

- (a) $\mu(z, R) \leq 2$ for any $z \in \mathbb{CP}^2 \setminus \{c, e, e', a_{\pm}\}$,
- (b) $\mu(c, R) = 3$, moreover, there are local coordinates (x, y) centered at c in which $\text{Jac } R \asymp xy^2$, and
- (c) $\mu(e, R) = \mu(e', R) = 4$.

Proof Any $z \in \mathbb{CP}^2 \setminus \{\pm(i, i), c, a_{\pm}, e, e'\}$ is either regular or is a smooth point of the critical locus; See Fig. 5. Since each of the irreducible factors of $\text{Jac } \hat{R}$ occurs to the first or second power, at any smooth point z of the critical locus we have $\mu(z, R) \leq 2$.

Since the lines L_3^{\pm} and L_4^{\pm} intersect transversally at $\pm(i, i)$, we can choose local coordinates (x, y) centered at $\pm(i, i)$ so that L_3^{\pm} is given by $\{x = 0\}$ and L_4^{\pm} is given by $\{y = 0\}$. As the irreducible factors of $\text{Jac } \hat{R}$ corresponding to L_3^{\pm} and L_4^{\pm} occur to the first power, near the origin in these coordinates we have $\text{Jac } R \asymp xy$, giving that $\mu(\pm(i, i), R) = 2$.

Similarly, we can choose local coordinates (x, y) centered at c with L_0 corresponding to $\{x = 0\}$ and L_2 corresponding to $\{y = 0\}$. As the irreducible factors of $\text{Jac } \hat{R}$ corresponding to these two lines have first and second powers, respectively, in the expression for $\text{Jac } \hat{R}$ we have $\text{Jac } R \asymp xy^2$, implying $\mu(c, R) = 3$.

The four separate critical lines L_0, L_1, L_3^+ and L_3^- meet at e and $\text{Jac } \hat{R}$ vanishes to order one along each of them, so that $\mu(e, R) = 4$. The same result holds at e' , by symmetry. \square

To deal with the indeterminate points a_{\pm} we need the following:

Lemma 3.7 *For every $\tilde{z} \in L_{\text{exc}}(a_+) \cup L_{\text{exc}}(a_-)$ we have $\mu(\tilde{z}, \tilde{R}) \leq 2$.*

Proof By symmetry, we can focus on a_+ . There is a neighborhood N of the exceptional divisor $L_{\text{exc}}(a_+)$ in which the critical locus of \tilde{R} consists of the proper transforms $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3^+$ and \tilde{L}_4^- and in which these curves are disjoint. As these four critical curves of \tilde{R} are smooth and disjoint in N , the order on vanishing of $\text{Jac } \tilde{R}$ is locally constant on each of them. Since each of the defining equations for L_1, L_3^+ , and L_4^- occur to the first power in $\text{Jac } \hat{R}$, we have $\mu(\tilde{z}, \tilde{R}) = 1$ for any $\tilde{z} \in N$ that is on $\tilde{L}_1, \tilde{L}_3^+$ or \tilde{L}_4^- . Meanwhile, the defining equation for L_2 occurs to the second power in $\text{Jac } \hat{R}$, so that $\mu(\tilde{z}, \tilde{R}) = 2$ for any $\tilde{z} \in N \cap \tilde{L}_2$. \square

4 Fatou and Julia Sets and the Measure of Maximal Entropy

4.1 Julia Set

For a rational map $R : \mathbb{CP}^n \rightarrow \mathbb{CP}^n$, the *Fatou set* F_R is defined to be the maximal open set on which the iterates $\{R^m\}$ are well-defined and form a normal family. The complement of the Fatou set is the *Julia set* J_R .

If R is dominant and has no collapsing varieties, Lemma A.1 from [4] gives that R is locally surjective (except at indeterminate points), so that the Fatou set is forward invariant and consequently, the Julia set is backward invariant.

If R has indeterminate points, then according to this definition they are in J_R . In this case, F_R and J_R are not typically totally invariant. One can see this by blowing up an indeterminate point and observing that the image of the resulting exceptional divisor typically intersects F_R . Note also that any algebraic curve A that is mapped by some iterate of R to an indeterminate point (such a curve exists iff R is not algebraically stable) is in J_R .

The Migdal–Kadinoff renormalization R is not locally surjective at any $x \in L_2 \setminus L_0$. More specifically, if N is a small neighborhood of x that avoids L_0 , then

$$R(N) \cap L_0 = b_0 = R(x),$$

where $b_0 = [1 : 0 : 1]$, since any point of $L_0 \setminus \{b_0\}$ has preimages only in L_0 . However, we still have the desired invariance.

Lemma 4.1 *The Migdal–Kadinoff renormalization R has forward invariant Fatou set and, consequently, backward invariant Julia set.*

Proof It suffices to show that $L_2 \subset J_R$, since R is locally surjective at any other point, by Lemma A.1 from [4]. By definition, $\{a_{\pm}\} \subset J_R$, so we consider $x \in L_2 \setminus \{a_{\pm}\}$. Let N be any small neighborhood of x . Note that $R(x) = b_0$ is a fixed point of saddle-type, with one-dimensional stable and unstable manifolds. Therefore, in order for the iterates to form a normal family on N , we must have $R(N) \subset \mathcal{W}^s(b_0)$. However, this is impossible, since there are plenty of regular points for R in N . \square

Lemmas 3.1 and 3.2 give a clear picture of J_R in a neighborhood of the line at infinity L_0 .

Proposition 4.2 *Within some neighborhood N of L_0 we have that $J_R \cap N = \mathcal{W}_{\mathbb{C}, \text{loc}}^s(B) \cap N$. Within this neighborhood, J_R is a C^∞ 3-dimensional manifold.*

Let us consider the locus $\{h = 0\}$ of vanishing magnetic field in \mathbb{CP}^2 for the DHL. In the affine coordinates, it is an R -invariant line $L_{\text{inv}} = \{u = w\}$; in the physical coordinates, it is an \mathcal{R} -invariant line $\mathcal{L}_{\text{inv}} = \{z = 1\}$. The two maps are conjugated by the Möbius transformation $\mathcal{L}_{\text{inv}} \rightarrow L_{\text{inv}}$, $u = 1/t$. Dynamics of $\mathcal{R} : t \mapsto \left(\frac{2t}{t^2+1}\right)^2$ on \mathcal{L}_{inv} was studied in [51]. In particular, it is shown in that paper that the Fatou set for $\mathcal{R}|_{\mathcal{L}_{\text{inv}}}$ consists entirely of the basins of attraction of the fixed points $\beta_0 := \{t = 0\}$ and $\beta_1 := \{t = 1\}$ which are superattracting within this line: see Fig. 3. Under the

conjugacy, the Fatou set for $R|L_{\text{inv}}$ consists of the basins of attraction for the two fixed points

$$b_0 := \{u = \infty\} = [1 : 0 : 1] \quad \text{and} \quad b_1 := \{u = 1\} = [1 : 1 : 1].$$

Proposition 4.3 $J_R = \overline{\bigcup_n R^{-n}(L_{\text{inv}})}$.

Proof Since every point in the Fatou set of $R|L_{\text{inv}}$ is in the basin of attraction of either b_0 or b_1 and since these points are of saddle-type in \mathbb{CP}^2 , the family of iterates R^n cannot be normal in an open neighborhood of any point on L_{inv} . Thus $L_{\text{inv}} \subset J_R$. It follows that $\overline{\bigcup_n R^{-n}(L_{\text{inv}})} \subset J_R$ since J_R is closed and backward invariant.

We will now show that $\bigcup_n R^{-n}(L_{\text{inv}})$ is dense in J_R . Consider a configuration of five algebraic curves

$$\begin{aligned} X_0 &:= \{V = 0\} = \text{the separatrix } L_0, \\ X_1 &:= \{U = W\} = \text{the invariant line } L_{\text{inv}}, \\ X_2 &:= \{U = -W\} = \text{the collapsed line } L_2 \subset R^{-1}(L_{\text{inv}}), \\ X_3 &:= \{U^2 + 2V^2 + W^2 = 0\} = \text{a component of } R^{-1}(L_{\text{inv}}), \\ X_4 &:= \{U^4 + 2U^2V^2 + 2V^4 + 2W^2V^2 + W^4 = 0\} = \text{a component of } R^{-1}(X_3). \end{aligned}$$

We will use the results of M. Green to check that the complement of these curves, $M := \mathbb{CP}^2 \setminus \bigcup_i X_i$, is a complete Kobayashi hyperbolic manifold hyperbolically embedded in \mathbb{CP}^2 (see Appendix A.4). We will first check that M is Brody hyperbolic, i.e., there are no non-constant holomorphic maps $f : \mathbb{C} \rightarrow M$. To this end, we will apply Green's Theorem A.7. It implies that the image of f must lie in a line L that is tangent to the conic X_3 at an intersection point with X_i , for one of the lines X_i , $i = 0, 1, 2$, and contains the intersection point $X_j \cap X_l$ of the other two lines. It is a highly degenerate situation which does not occur for a generic configuration. However, this is exactly what happens in our case, as the lines X_0, X_1, X_2 form a self-dual triangle with respect to the conic X_3 (see §A.5). However, one can check by direct calculation that the last curve, X_4 , must intersect each of these tangent lines L in at least one point away from X_0, \dots, X_3 . Since any holomorphic map from \mathbb{C} to $L \setminus \bigcup_i X_i$ must then omit 3 points in L , it must be constant.

Thus, M is Brody hyperbolic. Moreover, for each $i = 0, \dots, 4$ the remaining curves $\bigcup_{j \neq i} X_j$ intersect X_i in at least three points so that there is no non-constant holomorphic map from \mathbb{C} to $X_i \setminus \bigcup_{j \neq i} X_j$. Therefore, another of Green's results (Thm. A.6) applies showing that M is complete hyperbolic and hyperbolically embedded. It then follows from Proposition A.5 that the family $\{R^n\}$ is normal on any open set $N \subset \mathbb{CP}^2$ for which $R^n : N \rightarrow M$ for all n .

Given any $\zeta \in J_R$ and any neighborhood N of ζ , we will show that some preimage $R^{-n}(L_{\text{inv}})$ intersects N . Since $\zeta \in J_R$, the family of iterates R^n are not normal on N , hence $R^n(N)$ must intersect $\bigcup_i X_i$ for some n . If the intersection is with X_i for $i > 0$ then $R^{n+2}(N)$ intersects L_{inv} .

Thus, some iterate $R^n(N)$ must intersect $X_0 = L_0$. Suppose first that $\zeta \in L_0$. Then, by Lemma 3.1, $\zeta \subset \mathcal{W}^s(e) \cup \mathcal{W}^s(e') \cup \mathbb{T}$. Since the first two basins are contained in the Fatou set, $\zeta \in \mathbb{T}$, where preimages of the fixed point $b_0 \in L_{\text{inv}}$ are dense.

Finally, assume $\zeta \notin L_0$. By shrinking N if needed, we can make it disjoint from L_0 . Hence, there is $n > 0$ such that $R^n(N)$ intersects L_0 , while $R^{n-1}(N)$ is disjoint from L_0 . But since $R^{-1}(L_0) = L_0 \cup L_2$, we conclude that $R^{n-1}(N)$ must intersect L_2 . But L_2 collapses under R to the fixed point $b_0 \in L_{\text{inv}}$. Hence $R^n(N)$ intersects L_{inv} . \square

Remark 4.4 We thank the referee for pointing out that the above proof is similar to the one that Bonifant and Dabija use to show that if an endomorphism $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ of positive degree has an invariant elliptic curve Q then any point of Q has backward orbit under f that is dense in the Julia set J_f ; see [5, Thm. 5.4].

We will now relate J_R to the Green current S . (See Appendix B for the definition and basic properties of S .)

Proposition 4.5 $J_R = \text{supp } S$.

Proof The inclusion $\text{supp } S \subset J_R$ follows immediately from Theorem B.3. We will use Proposition 4.3 to show that $J_R \subset \text{supp } S$. Since $\text{supp } S$ is a backward invariant closed set, it is sufficient for us to show that $L_{\text{inv}} \subset \text{supp } S$.

Note that $L_{\text{inv}} = \mathcal{W}^s(b_0) \cup \mathcal{W}^s(b_1) \cup J_{R|L_{\text{inv}}}$. The basin $\mathcal{W}^s(b_0)$ is open and contained within the normal set for R (see Appendix B for the definition of normal set). Therefore, $\mathcal{W}^s(b_0) \subset J_R \cap N \subset \text{supp } S$, by Theorem B.3. Since $\text{supp } S$ is closed, we also have that $J_{R|L_{\text{inv}}} \subset \text{supp } S$.

Consider the “top” unit circle $T = \{(u, \bar{u}) : |u| = 1\}$ and note that $R|T$ is the squaring map $u \mapsto u^2$. Since the fixed point b_1 and indeterminate points a_{\pm} are on T , we therefore have

$$b_1 \in \overline{\bigcup_{n \geq 0} R^{-n}\{a_{\pm}\}},$$

implying that none of the points of $\mathcal{W}^s(b_1)$ are normal. Hence, we cannot directly use Theorem B.3 to conclude that $\mathcal{W}^s(b_1) \subset \text{supp } S$.

Notice that the points of $L_2 \setminus \{a_{\pm}\}$ are normal, since they are in $\mathcal{W}^s(L_0)$. Theorem B.3 gives that $L_2 \setminus \{a_{\pm}\} \subset \text{supp } S$, since $L_2 \subset J_R$. Because $\text{supp } S$ is closed, $L_2 \subset \text{supp } S$. Let $D_2 \subset L_2$ be a small disc centered around a_+ . Preimages of D_2 under appropriate branches of R^n will give discs intersecting L_1 transversally at a sequence of points converging to b_1 . By the Dynamical λ -Lemma (see [41, pp. 80–84]), this sequence of discs will converge to $\mathcal{W}_0^s(b_1) \subset L_{\text{inv}}$, where $\mathcal{W}_0^s(b_1)$ is the immediate basin of b_1 . Since each of the discs is in $\text{supp } S$, and the latter is closed, we find that $\mathcal{W}_0^s(b_1) \subset \text{supp } S$. Further preimages show that all of $\mathcal{W}^s(b_1) \subset \text{supp } S$. \square

4.2 Fatou Set

Because $J_R = \text{supp } S$, we immediately have:

Corollary 4.6 *The Fatou set of R is pseudoconvex.*

For the definition of pseudoconvexity, see [36].

Proof It is well-known that in the complement in \mathbb{CP}^2 of the support of a closed positive $(1, 1)$ -current is pseudoconvex. See [7, Thm. 6.2] or [49, Lem. 2.4]. \square

Remark 4.7 We thank the referee for pointing out that Corollary 4.6 can also be obtained directly from Proposition 4.3. Suppose that $L_{\text{inv}} = \{l_{\text{inv}} = 0\}$ and $L_0 = \{l_0 = 0\}$. For any $N \geq 0$ one can define a holomorphic function which does not extend to $\bigcup_{n=0}^N R^{-n} L_{\text{inv}}$ by

$$z \mapsto \frac{(l_0(\hat{z}))^k}{\left(\prod_{n=0}^N l_{\text{inv}} \circ \hat{R}^n(\hat{z})\right)}, \text{ where } \pi(\hat{z}) = z \text{ and } k = \sum_{n=0}^N 4^n.$$

Therefore, the Fatou set of R is the interior of the intersection of domains of holomorphy, so it is also a domain of holomorphy. Hence, it is pseudoconvex.

Computer experiments indicate that the Fatou set of R may be precisely the union of the basins of attraction $\mathcal{W}^s(e)$ and $\mathcal{W}^s(e')$ for the two superattracting fixed points e and e' . See Problem C.4.

We can prove the following more modest statement. Consider the solid cylinders⁷

$$\begin{aligned} SC &:= \left\{ [U : V : W] : \frac{V^2}{UW} \in [0, 1] \text{ and } \left| \frac{W}{U} \right| < 1 \right\} \text{ and} \\ SC' &:= \left\{ [U : V : W] : \frac{V^2}{UW} \in [0, 1] \text{ and } \left| \frac{W}{U} \right| > 1 \right\}. \end{aligned}$$

Theorem 4.8 *For the mapping R we have $SC \subset \mathcal{W}^s(e)$ and $SC' \subset \mathcal{W}^s(e')$.*

In the proof, we will need to use an important property of $R : C \rightarrow C$ that was proved in Part I. Recall from Sect. 3.1 that $C = \Psi(\mathcal{C})$ is the invariant real Möbius band and that $C_0 = C \setminus B$ is the topological annulus obtained by removing the “core curve” B .

The key property is:

(P9') Every proper path γ in C_0 lifts under R to at least 4 proper paths in C_0 . If γ crosses G at a single point, then $R^{-1}\gamma = \cup \delta_i$.

Proof of Theorem 4.8 It suffices to prove the proposition for SC , since the statement for SC' follows from the symmetry ρ .

We will decompose SC as a union of complex discs and show that each disc is in $\mathcal{W}^s(e)$. Let

$$P_c := \left\{ [U : V : W] : \frac{V^2}{UW} = c \in [0, 1] \right\},$$

⁷ They correspond to actual solid cylinders in the (z, t) coords; see Corollary 4.9.

and

$$P_c^* := \left\{ [U : V : W] : \frac{V^2}{UW} = c \in [0, 1] \text{ and } \left| \frac{W}{U} \right| < 1 \right\},$$

so that $SC = \bigcup_{c \in [0, 1]} P_c^*$.

The discs P_0^* and P_1^* are in $\mathcal{W}^s(e)$ because they are each within the forward invariant critical curves L_0 and L_1 , respectively, on which the dynamics is given by $(W/U) \rightarrow (W/U)^4$ and $(W/U) \rightarrow (W/U)^2$, respectively.

We now show that for any $c \in (0, 1)$ we also have $P_c^* \subset \mathcal{W}^s(e)$. In fact $e \in P_c^*$, so it suffices to show that R^n forms a normal family on P_c^* . Consider any $x \in P_c^*$. If $x = e$, then $x \in \mathcal{W}^s(e)$ so that R^n is normal on some neighborhood of x in P_c^* .

Now consider any $x \in P_c^* \setminus \{e\}$. There is a neighborhood of $N \subset P_c^*$ of x with $e \notin N$, on which we will show that R^n forms a normal family. Recall the family of curves X_0, \dots, X_4 from the proof of Proposition 4.3, where we showed that $\mathbb{CP}^2 \setminus \bigcup_i X_i$ is complete hyperbolic and hyperbolically embedded. We will show for every n that $R^n(N)$ is in $\mathbb{CP}^2 \setminus \bigcup_i X_i$, so that R^n is normal on N .

Since $P_c^* \cap X_0 = \{e\}$, and $e \notin N$, we have that $N \cap X_0 = \emptyset$. Therefore, by reasoning identical to that in the proof of Proposition 4.3, if $R^n(N)$ intersects X_i for any $i = 0, \dots, 4$ we must have that some iterate $R^m(N)$ intersects $X_1 = L_{\text{inv}}$.

We will check that forward iterates of $R^n(P_c^*)$ are disjoint from L_{inv} , which is sufficient since $N \subset P_c^*$. The line L_{inv} intersects the invariant annulus C_0 in two properly embedded radial curves, so Property (P9') gives that $(R^n)^* L_{\text{inv}}$ intersects C in at least $2 \cdot 4^n$ properly embedded radial curves.

One can check that P_c intersects the invariant annulus C in the horizontal curve

$$\left\{ [U : V : W] : \frac{V^2}{UW} = c \in [0, 1] \text{ and } \left| \frac{W}{U} \right| = 1 \right\},$$

which corresponds to $|u| = 1/\sqrt{c} > 1$ in the u coordinate for C . Therefore, the $2 \cdot 4^n$ radial curves in C from $(R^n)^* L_{\text{inv}}$ intersect P_c in at least $2 \cdot 4^n$ distinct points within C .

We will now show that these are the only intersection points between $(R^n)^* L_{\text{inv}}$ and P_c in all of \mathbb{CP}^2 . Since R is algebraically stable, Bezout's Theorem gives $\deg(P_c) \cdot \deg((R^n)^* L_{\text{inv}}) = 2 \cdot 4^n$ intersection points, counted with multiplicities, in all of \mathbb{CP}^2 . Therefore $P_c \cap (R^n)^* L_{\text{inv}} \subset C$.

Since $P_c^* \subset P_c$ with $P_c^* \cap C = \emptyset$, we conclude that $P_c^* \cap (R^n)^* L_{\text{inv}} = \emptyset$ for every n . In other words, $R^n(P_c^*) \cap L_{\text{inv}} = \emptyset$ for every n . Thus, the same holds for $N \subset P_c^*$, implying that R^n is a normal family on N . \square

Theorem 4.8 has an interesting consequence for \mathcal{R} . The fixed point e' for R has a single preimage $\eta' = \Psi^{-1}(e')$, which is a superattracting fixed point for \mathcal{R} . However, e has the entire collapsing line $Z = 0$ as preimage under Ψ . Within this line is another superattracting fixed point $\eta = [0 : 1 : 1]$ for \mathcal{R} and every point in $\{Z = 0\} \setminus \{\mathbf{0}, \gamma\}$ is collapsed by \mathcal{R} to η .

We obtain:

Corollary 4.9 *For the mapping \mathcal{R} , the solid cylinder $\{(z, t) : |z| < 1, t \in (0, 1]\}$ is in $\mathcal{W}^s(\eta)$ and, symmetrically, the solid cylinder $\{(z, t) : |z| > 1, t \in [0, 1]\}$ is in $\mathcal{W}^s(\eta')$.*

Notice that we had to omit the “bottom”, $t = 0$, of the solid cylinder in $\mathcal{W}^s(\eta)$ because points on it are forward asymptotic to the indeterminate point $\mathbf{0}$.

4.3 Measure of Maximal Entropy

There is a conjecture specifying the expected ergodic properties of a dominant rational map of a projective manifold⁸ in terms of the relationship between various *dynamical degrees* of the map; see [30].

Since the Migdal–Kadanoff renormalization R is an algebraically stable map of \mathbb{CP}^2 , there are only two relevant dynamical degrees, the topological degree $\deg_{top} R$ and the algebraic degree $\deg R$, which satisfy

$$\deg_{top} R = 8 > 4 = \deg R.$$

This case of *high topological degree* was studied by Guedj [29], who made use of a bound on topological entropy obtained by Dinh and Sibony [13]. In our situation, his results give

Proposition 4.10 *R has a unique measure ν of maximal entropy $\log 8$ with the following properties*

- (i) ν is mixing;
- (ii) The Lyapunov exponents of ν are bounded below by $\log \sqrt{2}$;
- (iii) If θ is any probability measure that does not charge the postcritical set⁹ of R , then $8^{-n}(R^n)^*\theta \rightarrow \nu$;
- (iv) If P_n is the set of repelling periodic points of R of period n then

$$8^{-n} \sum_{a \in P_n} \delta_a \rightarrow \nu.$$

(In fact, it suffices to take just the repelling periodic points in $\text{supp } \nu$.)

The measure ν satisfies the backwards invariance $R^*\nu = 8\nu$, hence its support is totally invariant. In our situation, $\text{supp } \nu \subsetneq J_R$ because (for example) the points in $\mathcal{W}^s(\mathcal{B})$ are not in $\text{supp } \nu$. It can be thought of as the “little Julia set” within J_R on which the “most chaotic” dynamics occurs.

Remark 4.11 The statement of (iv) in [29, Thm. 3.1] does not emphasize that one can restrict his or her attention to the periodic points within $\text{supp } \nu$, but it follows from the proof in [29] and the fact that $\text{supp } \nu$ is totally invariant. See [16, Thm. 1.4.13] for the analogous argument for holomorphic $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$.

Remark 4.12 We know very little about the support of ν . See Problem C.3.

⁸ It is stated more generally in [30], for meromorphic maps of compact Kähler manifolds.

⁹ Here, the postcritical set is defined without taking the closure.

5 Volume Estimates

This section is devoted to estimating the volume exponent $\sigma(z, f)$ in terms of the Jacobian Jf and to studying the sets

$$D_{\geq a} := \{z \in \mathbb{CP}^2 : \sigma(z, f) \geq a\} \quad \text{and} \quad D_{>a} := \{z \in \mathbb{CP}^2 : \sigma(z, f) > a\} \quad (5.1)$$

for various values of $a > 0$.

Lemma 5.1 *Let $h(x, y)$ be a non-constant holomorphic function vanishing with multiplicity ℓ at $(0, 0)$. Then, there is a neighborhood N of $(0, 0)$ in \mathbb{C}^2 and a constant $K > 0$ such that for any $s > 0$ we have*

$$\text{vol}(\{|h(x, y)| < s\} \cap N) \leq K s^{2/\ell}.$$

Proof We can suppose that the coordinates (x, y) satisfy $h(0, y) \not\equiv 0$. The Weierstrass Preparation Theorem then gives that there is a sufficiently small bidisc \mathbb{D}_ϵ^2 centered at $(0, 0)$ in which

$$h(x, y) = \alpha(x, y) \left(y^\ell + \beta_{\ell-1}(x) y^{\ell-1} + \cdots + \beta_0(x) \right),$$

where $\alpha(x, y)$ is a non-vanishing holomorphic function and the coefficients $\beta_j(x)$ are holomorphic in \mathbb{D}_ϵ .

Up to a multiplicative constant, we can suppose $\alpha(x, y) = 1$. Then, in each vertical slice we have

$$\{x = x_0\} \cap \{|h(x_0, y)| < s\} \subset \{x = x_0\} \times \bigcup_{i=1}^{\ell} \mathbb{D}_{s^{1/\ell}}(r_i)$$

where r_1, \dots, r_ℓ are the roots of $h(x_0, y)$, listed with multiplicities. In particular,

$$\text{area}(\{x = x_0\} \cap \{|h(x_0, y)| < s\}) \leq K_1 s^{2/\ell}$$

for some constant K_1 . The result then follows by Fubini's Theorem, after integrating over all $x_0 \in \mathbb{D}_\epsilon$. \square

We now estimate the volume exponent $\sigma(z, f)$ in terms of the order of vanishing $\mu(z, f)$ of the complex Jacobian $\text{Jac } f$ at z (see p. 6).

Lemma 5.2 (Favre–Jonsson [21, Prop. 6.3]) *Let $U, V \subset \mathbb{C}^2$ and $f : U \rightarrow V$ be holomorphic and at most $d_{\text{top-to-one}}$ off of a measure zero subset of U . Then for any $z \in U$ we have*

$$\sigma(z, f) \leq \mu(z, f) + 1. \quad (5.2)$$

Proof Lemma 5.1 gives a neighborhood N of z and constant $K_0 > 0$ for which

$$\text{vol}(\{|\text{Jac } f(x, y)|^2 < s\} \cap N) \leq K_0 s^{1/\mu},$$

where $\mu \equiv \mu(z, f)$. Let X be a measurable subset of N and choose s so that $K_0 s^{1/\mu} = \frac{1}{2} \text{vol}(X)$. The Chebyshev Inequality gives

$$\begin{aligned} \text{vol } f(X) &\geq \frac{1}{d_{\text{top}}} \int_X |\text{Jac } f|^2 d \text{vol} \geq \frac{s}{d_{\text{top}}} (\text{vol } X - \text{vol}(\{|\text{Jac } f|^2 < s\})) \\ &\geq \frac{s}{2d_{\text{top}}} \text{vol } X \geq K (\text{vol } X)^{1+\mu}. \end{aligned}$$

for an appropriate constant $K > 0$. \square

Lemma 5.3 Suppose $f : U \rightarrow V$ satisfies the hypotheses of Lemma 5.2. If z is a smooth point of the critical locus of f , then $\sigma(z, f) = \mu(z, f) + 1$, i.e. the estimate from Lemma 5.2 is sharp at such points.

Proof It suffices to prove that $\sigma(z, f) \geq \mu(z, f) + 1$. We will do this by showing that in any neighborhood N of z there is a constant $C > 0$ such that there are measurable sets $X \subset N$ of arbitrarily small measure for which $\text{vol } f(X) \leq C(\text{vol } X)^{\mu+1}$.

Since z is a smooth point of the critical locus, one can choose local coordinates (x, y) centered at z so that $\text{Jac } f(x, y) \asymp y^\mu$. Given any neighborhood N of z we can apply a linear rescaling to our coordinates in order to assume that the unit bidisc \mathbb{D}^2 is contained in N . For any $0 < \epsilon < 1$ let $X := \mathbb{D} \times \mathbb{D}_\epsilon \subset N$. We have

$$\text{vol } f(X) \leq \int_X |y|^{2\mu} d \text{vol} = \pi \int_0^{2\pi} \int_0^\epsilon r^{2\mu} r dr d\theta = 2\pi^2 \frac{\epsilon^{2\mu+2}}{2\mu+2} \asymp (\text{vol } X)^{\mu+1}.$$

\square

Remark 5.4 Unlike in one-dimensional dynamics, one can have points with $\sigma(z, f) > d$. Consider $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by $f(x, y) = (xy^2, y^3)$. One has $\text{Jac} = 3y^4$ so that $\sigma(\mathbf{0}, f) = 5$, by Lemma 5.3.

If the zero set of $\text{Jac } f$ has a normal crossing singularity at z , then one has the following stronger estimate

Lemma 5.5 Let $U, V \subset \mathbb{C}^2$ and $f : U \rightarrow V$ be holomorphic and at most d_{top} -to-one off of a measure zero subset of U . If $\text{Jac } f \asymp x^a y^b$ in suitable local coordinates (x, y) centered at $z \in U$. Then

$$\sigma(z, f) \leq \max(a, b) + 1. \quad (5.3)$$

The proof of Lemma 5.5 will use:

Lemma 5.6 Let $Q : w \mapsto w^d$. For any measurable set $Y \subset \mathbb{C}$,

$$\text{area}(Q^{-1}Y) \leq (\text{area } Y)^{1/d}.$$

Proof We can assume that area $Y > 0$. Let us take the radius $r > 0$ such that $\pi r^2 = \text{area } Y$. Let $Y_- = Y \cap \mathbb{D}_r$, $Y_+ = Y \setminus Y_-$, $Y_c = \mathbb{D}_r \setminus Y_-$. Then,

$$\begin{aligned} (\text{area } Y)^{1/d} &= (\text{area } \mathbb{D}_r)^{1/d} = \text{area}(Q^{-1}\mathbb{D}_r) = \text{area}(Q^{-1}Y_-) + \text{area}(Q^{-1}Y_c) \\ &\geq \text{area}(Q^{-1}Y_-) + d \text{Jac } Q^{-1}(r) \text{ area } Y_c \geq \text{area}(Q^{-1}Y_-) + \text{area}(Q^{-1}Y_+) = \text{area}(Q^{-1}Y). \end{aligned}$$

Proof of Lemma 5.5 Let $\gamma > \max(a, b) + 1$ and let $N = \{|x| < \epsilon\} \times \{|y| < \epsilon\}$. For any measurable set $X \subset N$ we have:

$$\text{vol } f(X) \geq \frac{1}{d_{\text{top}}} \int_X |\text{Jac } f|^2 d \text{vol} \asymp \int_{X^v} |y|^{2b} \int_{X_y^h} |x|^{2a} d \text{area}(x) d \text{area}(y),$$

where X^v is the projection of X onto the y -axis and X_y^h are the slices of X by horizontal lines. The inner integral above is exactly $\text{area}(Q_{a+1}(X_y^h))/(a+1)^2$, where $Q_{a+1}(x) = x^{a+1}$. By Lemma 5.6, it is bounded below by $(\text{area } X_y^h)^{a+1}/(a+1)^2$. Using the Hölder inequality, with $p = \gamma/(\gamma-1)$ and $q = \gamma$, we find

$$\begin{aligned} &\left(\int_{X^v} 1/|y|^{2b/(\gamma-1)} d \text{area}(y) \right)^{\gamma-1} \int_{X^v} |y|^{2b} (\text{area } X_y^h)^{a+1} d \text{area}(y) \\ &\geq \left(\int_{X^v} (\text{area } X_y^h)^{(a+1)/\gamma} d \text{area}(y) \right)^\gamma \geq (\text{vol } X)^\gamma. \end{aligned}$$

The conclusion follows since $b < \gamma-1$ implies that $1/|y|^{2b/(\gamma-1)}$ is locally integrable. \square

Remark 5.7 Favre and Jonsson prove a similar volume estimate within their study of the the exceptional set \mathcal{E}_1 ; see [21, Prop. 7.1 and Lem. 7.2].

Let us return to the case that $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ is a rational map. For any $z \in I(f)$ we can estimate $\sigma(z, f)$ by applying either Lemmas 5.2 or 5.5 at each point of the the exceptional divisor $\pi^{-1}(z)$:

Proposition 5.8 *Consider the resolution of indeterminacy (A.1) for f . For any $z \in \mathbb{CP}^2$ the exponent $\sigma(z, f)$ exists and satisfies*

$$\sigma(z, f) \leq \max_{\tilde{z} \in \pi^{-1}(\{z\})} \sigma(\tilde{z}, \tilde{f}). \quad (5.4)$$

Proof Note that $\pi : \widetilde{\mathbb{CP}^2} \rightarrow \mathbb{CP}^2$ decreases volumes. Therefore, for any $z \in \mathbb{CP}^2$ and any $\gamma > \max_{\tilde{z} \in \pi^{-1}(\{z\})} \sigma(\tilde{z}, \tilde{f})$ it suffices for us to find a neighborhood \tilde{N} of $\pi^{-1}(\{z\})$ and a constant $K > 0$ so that for any measurable set $Y \subset \mathbb{CP}^2$ we have

$$\text{vol}_{\widetilde{\mathbb{CP}^2}} \left(\tilde{N} \cap \tilde{f}^{-1}(Y) \right) \leq K (\text{vol } Y)^{1/\gamma}. \quad (5.5)$$

This follows by applying the definition of $\sigma(\tilde{z}, \tilde{f})$ at each point of $\pi^{-1}(\{z\})$ and using that $\pi^{-1}(\{z\})$ is compact. \square

Lemma 5.9 *The volume exponent σ satisfies:*

- (i) $\sigma(z, f)$ is an integer away from $I(f)$ and singular points of $\text{Crit}(f)$,
- (ii) $\sigma(z, f)$ assumes finitely many values as z varies over \mathbb{CP}^2 , and
- (iii) for any $a > 1$ the set $D_{\geq a}$ and $D_{>a}$ (see Equation 5.1) is algebraic, consisting of finitely many algebraic curves together with finitely many isolated points.

Proof Property (i) is a consequence of Lemma 5.3 and Property (ii) follows because $I(f)$ and the set of singular points of $\text{Crit}(f)$ are finite. Property (iii) follows from Lemma 5.3 and upper semicontinuity of σ :

$$\limsup_{z \rightarrow z_0} \sigma(z, f) \leq \sigma(z_0, f),$$

which is a consequence of the definition of σ . \square

Proposition 5.10 *Suppose that C is an irreducible algebraic curve contained in $D_{>d}$. Then, C is collapsed by f .*

Proof Suppose for contradiction that C is not collapsed by f . Then Lemma A.4 gives that C is a Whitney Fold of f , i.e. there exists $r \in \mathbb{Z}_+$ and a finite $S \subset C$ such that for any $p \in C \setminus S$ there are systems of holomorphic coordinates (x, y) centered at p and (z, w) centered at $f(p)$ in which

$$(z, w) = f(x, y) = (x, y^r). \quad (5.6)$$

Moreover, for all $p \in C \setminus S$ we have $\sigma(p, f) = r > d$, since we suppose $C \subset D_{>d}$.

In these coordinates, for any $w_0 \neq 0$ we have that $f^{-1}(\{w = w_0\})$ intersects the y -axis transversally in $r > d$ points. Let $L \subset \mathbb{CP}^2$ be a projective line through p that is tangent to the y -axis in these local coordinates. If we let Λ be a complex projective line in \mathbb{CP}^2 that is tangent to $\{w = w_0\}$ at $(0, w_0)$ and take $|w_0|$ sufficiently small, then $f^{-1}(\Lambda)$ will be an algebraic curve of degree d that intersects L transversally in $r > d$ points, by the stability of transverse intersections between analytic curves under small perturbations. This contradicts the Bezout Theorem. \square

6 Proof of the Equidistribution Theorem for the DHL

6.1 Transformation of Volume by R

Proposition 6.1 *There exist constants $K > 0$ and $1 < \tau < \deg(R^2) = 4^2$ such that*

$$\text{vol} \left(R^{-2}(Y) \right) \leq K (\text{vol } Y)^{1/\tau}$$

for any measurable $Y \subset \mathbb{CP}^2$.

Proof Let Ω be any forward invariant neighborhood of the two superattracting fixed points e and e' . We will first prove that there exists $K' > 0$ such that

$$\text{vol} \left(R^{-1}(Y) \right) \leq K' (\text{vol } Y)^{1/3} \quad (6.1)$$

for any measurable $Y \subset \mathbb{CP}^2 \setminus \Omega$. By compactness, it suffices to prove that every $z \in \mathbb{CP}^2 \setminus \Omega$ has volume exponent $\sigma(z, R) \leq 3$.

Lemma 3.6(a) gives that for any $z \in \mathbb{CP}^2 \setminus \{c, e, e', a_{\pm}\}$ we have $\mu(z, R) \leq 2$ and hence $\sigma(z, R) \leq 3$, by Lemma 5.2. Meanwhile, Lemma 3.6(b) gives local coordinates (x, y) centered at $\{c\} = L_0 \cap L_2$ in which $\text{Jac } R \asymp xy^2$, so that Lemma 5.5 gives $\sigma(c, R) \leq 3$.

We now use Proposition 5.8 to check that the indeterminate points a_{\pm} satisfy $\sigma(a_{\pm}, R) \leq 3$. By Lemmas 3.7 and 5.5 we have $\sigma(\tilde{z}, \tilde{R}) \leq 3$ for every $\tilde{z} \in L_{\text{exc}}(a_+) \cup L_{\text{exc}}(a_-)$. This completes the proof of (6.1).

A calculation¹⁰ shows that $\mu(e, DR^2) = 14$ and hence Lemma 5.2 gives $\sigma(e, R^2) \leq 15$. By symmetry, the same holds at e' . Combined with (6.1), this completes the proof of Proposition 6.1. \square

Remark 6.2 In Lemma 3.6(c) we saw that $\mu(e, R) = 4$, so that Lemma 5.2 gives $\sigma(e, R) \leq 5$, which is insufficient for our purposes. Meanwhile, the four separate critical curves L_0, L_1, L_3^+ and L_3^- meeting at e imply that $\det DR$ does not have the form needed to apply Lemma 5.5. (The same holds at the symmetric fixed point e' .) This is why we needed to pass to the second iterate of R in the proof of Proposition 6.1.

6.2 Completing the Proof of the Equidistribution Theorem for the DHL

Let $g := R^2$ and $d := 16 = \deg(g)$. It suffices to prove that

$$\frac{1}{d^n \deg(A)} (g^n)^*[A] \rightarrow S \quad (6.2)$$

for any algebraic curve $A \subset \mathbb{CP}^2$ because the Green current S for R is also the Green current for $g = R^2$ and because the normalized pullback $\frac{1}{4}R^*$ acts continuously on the space of closed positive $(1, 1)$ currents and has S as a fixed point.

Let $\pi : \mathbb{C}^3 \setminus \{0\} \rightarrow \mathbb{CP}^2$ denote the canonical projection. For any $z \in \mathbb{CP}^2$ we will denote by $\hat{z} \in \mathbb{C}^3 \setminus \{0\}$ any choice of a point of $\pi^{-1}(z)$. Let

$$A = \{z \in \mathbb{CP}^2 : P(\hat{z}) = 0\},$$

with P a homogeneous polynomial of degree $a = \deg(A)$. We must show that the limit

$$\lim_{n \rightarrow \infty} \frac{1}{ad^n} \log |P \circ \hat{g}^n(\hat{z})| \quad (6.3)$$

¹⁰ We omit the calculation, but the reader can readily check it using Maple.

exists in $L^1_{\text{loc}}(\mathbb{C}^3)$ and is equal to the Green Potential

$$G(\hat{z}) := \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \|\hat{g}^n \hat{z}\| \quad (6.4)$$

of g .

The homogeneous polynomial P determines a section s_P of the a -th tensor power of the hyperplane bundle; see Appendix A.2. For each $n \geq 0$ we will consider the function

$$\phi_n : \mathbb{CP}^2 \rightarrow [-\infty, \infty) \quad \text{where} \quad \phi_n(z) := \frac{1}{ad^n} \log \|s_P(g^n(z))\|,$$

with $\|\cdot\|$ denoting the Hermitian norm on this bundle. By definition of the norm,

$$\phi_n(z) = \frac{1}{ad^n} \log \frac{|P \circ \hat{g}^n(\hat{z})|}{\|\hat{g}^n \hat{z}\|^a} = \frac{1}{ad^n} \log |P \circ \hat{g}^n(\hat{z})| - \frac{1}{d^n} \log \|\hat{g}^n \hat{z}\|.$$

The limit in (6.4) exists by the hypothesis that R (and hence g) is algebraically stable; see Theorem B.1. Therefore, the desired convergence of currents (6.2) will follow from:

Theorem 6.3 $\phi_n \rightarrow 0$ in $L^1_{\text{loc}}(\mathbb{CP}^2)$ as $n \rightarrow \infty$.

Proof We will use the following general convergence criterion:

Lemma 6.4 *Let ϕ_n be a sequence of L^2 functions on a finite measure space (X, m) with bounded L^2 -norms. If $\phi_n \rightarrow 0$ a.e. then $\phi_n \rightarrow 0$ in L^1 .*

Proof Take any $\epsilon > 0$ and $\delta > 0$. By Egorov's Lemma, there exists a set $X' \subset X$ with $m(X \setminus X') < \epsilon$ such that $\phi_n \rightarrow 0$ uniformly on X' . So, eventually the sup-norms of the ϕ_n on X' are bounded by δ . Hence

$$\int |\phi_n| dm = \int_{X'} |\phi_n| dm + \int_{X \setminus X'} |\phi_n| dm \leq \delta \cdot m(X) + B\sqrt{\epsilon},$$

where the last estimate follows from the Cauchy-Schwarz Inequality (with B the L^2 -bound on the ϕ_n). The conclusion follows. \square

Lemma 6.5 *There exists $K > 0$ such that for any $s < 0$ we have*

$$\text{vol}(\{\phi_0 < s\}) \leq K e^{2s}.$$

Proof Since A is compact, it suffices to work in a neighborhood of any point $z \in A$. Without loss of generality, we can suppose $z = [0 : 0 : 1]$ so that it is the origin in the affine coordinates $(x, y) \mapsto [x : y : 1]$. In these coordinates,

$$\frac{|P(x, y, 1)|}{\|(x, y, 1)\|^a} \asymp |P(x, y, 1)|.$$

The local multiplicity of $P(x, y, 1)$ at $(0, 0)$ is less than or equal to $a = \deg(P)$. The estimate then follows in a neighborhood of z from Lemma 5.1. \square

Lemma 6.6 *For any measurable set $Y \subset \mathbb{CP}^2$ there exists $1 < \tau < d$ and $K > 0$ such that for any $n \geq 0$ we have*

$$\text{vol } g^{-n} Y \leq K (\text{vol } Y)^{1/\tau^n}. \quad (6.5)$$

Proof The estimate for a single iterate of $g = R^2$ is Proposition 6.1. The result then follows inductively if we let $K := K_0^s$, where $s = \sum_{n=0}^{\infty} \frac{1}{\tau^n}$ and K_0 is the constant given by Proposition 6.1. \square

We will estimate the distribution of the tails of the random variables ϕ_n :

Lemma 6.7 *Let $M = \sup \phi_0(z)$. Then, there exists $K > 0$ so that*

$$\text{vol}\{|\phi_n| > r\} \leq K \exp\left(-2r \left(\frac{d}{\tau}\right)^n\right) \quad \text{for any } r > Md^{-n}.$$

Proof We have:

$$\begin{aligned} X_n(r) &:= \{|\phi_n| > r\} = \{\phi_0 \circ g^n > rd^n\} \cup \{\phi_0 \circ g^n < -rd^n\} \\ &= \{\phi_0 \circ g^n < -rd^n\} = g^{-n}\{\phi_0 < -rd^n\}. \end{aligned}$$

We have used that $\phi_0 < rd^n$ to see that the first term in the union is empty. According to Lemma 6.5 there is a constant $K_1 > 0$ such that

$$\text{vol}\{\phi_0 < -rd^n\} \leq K_1 \exp(-2rd^n).$$

The result then follows from Lemma 6.6. \square

We can now show that the functions ϕ_n satisfy the conditions of Lemma 6.4.

Lemma 6.8 *Assuming (6.5) holds for some $1 \leq \tau \leq d$, the sequence ϕ_n is L^2 -bounded.*

Proof We have:

$$\|\phi_n\|^2 \leq \sum_{\ell=0}^{\infty} (\ell + 1)^2 \text{vol}\{|\phi_n| \geq \ell\}.$$

By Lemma 6.7, this sum is bounded by

$$\begin{aligned} & \sum_{\ell=0}^{M+1} (\ell+1)^2 + K \sum_{\ell>M} (\ell+1)^2 \exp\left(-2\ell\left(\frac{d}{\tau}\right)^n\right) \\ & \leq K_0 + K \sum_{\ell=0}^{\infty} (\ell+1)^2 \exp(-2\ell) < \infty. \end{aligned}$$

□

Lemma 6.9 *The sequence ϕ_n exponentially converges to 0 almost everywhere.*

Proof Fix any $\lambda \in (1, d/\tau)$. For sufficiently large n , we have $\lambda^{-n} > Md^{-n}$, hence Lemma 6.7 gives

$$\text{vol}\{|\phi_n| > \lambda^{-n}\} \leq K \exp\left(-2\left(\frac{d}{\tau\lambda}\right)^n\right).$$

Since the sum of these volumes converges, the Borel–Cantelli Lemma gives that for a.e. $x \in \mathbb{CP}^2$, we eventually have $|\phi_n(x)| \leq \lambda^{-n}$. □

This completes the proof of Theorem 6.3 and hence of the Equidistribution Theorem for the DHL. □

7 Proof of the Equidistribution Theorem

This section is devoted to proving:

Proposition 7.1 (Finding a good iterate) *Suppose that $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ is a dominant algebraically stable rational map with $\deg f = d$ that satisfies the hypotheses the Equidistribution Theorem (see p. 6). For any forward invariant neighborhood Ω of exceptional set \mathcal{E} there exists an iterate n_0 and constants $K > 0$ and $0 < \tau < d^{n_0}$ such that*

$$\text{vol}(f^{-n_0}(Y)) \leq K (\text{vol } Y)^{1/\tau}. \quad (7.1)$$

for any measurable $Y \subset \mathbb{CP}^2 \setminus \Omega$.

Once this proposition is proved, the remainder of the proof of the Equidistribution Theorem follows in exactly the same way as the proof of the Equidistribution Theorem for the DHL (Sect. 6.2).

Throughout this section, we will suppose that $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ is a dominant algebraically stable rational map with $\deg f = d$. However, we will keep track of which specific Hypotheses (i)–(iii) of the Equidistribution Theorem are used.

7.1 No Curves of Maximal Degeneracy

Proposition 7.2 *Suppose f satisfies Hypothesis (ii). Then $D_{>d}$ is a finite set.*

Proof By Corollary 5.9, $D_{>d}$ is algebraic, so if it were an infinite set it would contain some irreducible algebraic curve C . Then Proposition 5.10 implies that C is collapsed by f , which results in C containing an indeterminate point for f , by Lemma A.1. This violates Hypothesis (ii). \square

Proposition 7.3 *Suppose f satisfies Hypotheses (i) and (ii) of the Equidistribution Theorem. Then there exists $N > 0$ and a finite forward invariant “non-escaping set” $\mathcal{N}_{\geq d} \subset D_{\geq d}$ such that if $z \in D_{\geq d} \setminus \mathcal{N}_{\geq d}$ then there exists $0 < n < N$ such that $f^n(z) \notin D_{\geq d}$.*

The proof of Proposition 7.3 relies on several basic lemmas about the structure of one iterate of a rational map that are presented in Appendix A.1. We will also need the following definition. Suppose $p \in \mathbb{CP}^2 \setminus I(f)$ is not on a collapsed curve. Then, f induces a germ of an open mapping at p and the *local topological degree* $e(p, f)$ is the topological degree of that germ. It satisfies the chain rule

$$e(p, f^2) = e(p, f) \cdot e(f(p), f), \quad (7.2)$$

so long as $e(p, f)$ and $e(f(p), f)$ are defined.

The key step in the proof of Proposition 7.3 is:

Lemma 7.4 *Suppose that f satisfies Hypothesis (i). Then, there is no irreducible algebraic curve C such that for some $k \geq 1$ we have $f^k(C) \subset C$, $C \cap I(f^k) = \emptyset$, and $\sigma(z, f^k) \geq d^k$ for all $z \in C$.*

Proof Suppose for contradiction that such a curve C exists. Then, Lemma A.1 implies that C is not collapsed by f^k . Lemma A.4 then gives that C is a Whitney Fold and that there is a finite $S \subset C$ such that for every $p \in C \setminus S$ we have

$$e(p, f^k) = \sigma(p, f^k) \geq d^k. \quad (7.3)$$

Let us now consider the irreducible component C as a divisor (C) , assigning it multiplicity one. (See [4, Appendix A] for basic background on divisors.) Since C is disjoint from $I(f^k)$, the pushforward of (C) by f^k is defined by

$$(f^k)_*(C) := \deg_{\text{top}}(f^k|_C : C \rightarrow f^k(C))(f^k(C)) = \deg_{\text{top}}(f^k|_C : C \rightarrow C)(C).$$

Meanwhile, we have that

$$\deg((f^k)_*(C)) = d^k \deg((C));$$

see [4, Lem. A.5]. Therefore, we conclude that

$$\deg_{\text{top}}(f^k|_C : C \rightarrow C) = d^k. \quad (7.4)$$

Equation (7.4) implies that a generic $z \in C$ will have d^k preimages under $f^k|_C$ and Equation (7.3) then implies that a generic point z' near z , but not on C , will have d^{2k}

preimages under f^k . (There are d^k preimages of z' near each preimage of z under $f^k|_{C'}$.) By Hypothesis (i) $I(f) \neq \emptyset$, so this violates the fact that $\deg_{\text{top}}(f) < d^{2k}$; see Lemma A.2. \square

Proof of Proposition 7.3 By Lemma 5.9, the set $D_{\geq d}$ is algebraic. Moreover, none of the irreducible components of $D_{\geq d}$ is collapsed by f , using Hypothesis (ii) that $D_{\geq d}$ is disjoint from $I(f)$ and Lemma A.1.

It suffices to show that for each irreducible component $C \subset D_{\geq d}$ there is an iterate n such that $f^n(C) \not\subset D_{\geq d}$. In this case, the Bezout Theorem implies $f^n(C) \cap D_{\geq d}$ is finite. This implies that all but finitely many points of C are mapped out of $D_{\geq d}$ by f^n , since f does not collapse any irreducible component of $D_{\geq d}$.

Suppose for the purpose of obtaining a contradiction that $C \subset D_{\geq d}$ is some irreducible component with $f^n(C) \subset D_{\geq d}$ for every $n \geq 1$. As $D_{\geq d}$ contains finitely many irreducible components, we conclude that some iterate $f^{n_0}(C)$ is periodic under f . Therefore, without loss of generality, we can suppose that C itself is periodic under f with some period $k \geq 1$.

Let $C_n := f^n(C)$ for each $0 \leq n < k$. Lemma A.4 gives that there is a finite set $S_n \subset C_n$ and an exponent r_n so that for each $p \in C_n \setminus S_n$ we have

$$\sigma(p, f) = e(p, f) = r_n \geq d,$$

with the last inequality coming from the fact that $C_n \subset D_{\geq d}$. Meanwhile, applying Lemma A.4 to f^k and C , together with the chain rule (7.2), we see that there is a finite $S \subset C$ so that for all $p \in C \setminus S$ we have

$$\sigma(p, f^k) = e(p, f^k) = e(p, f) \cdot e(f(p), f) \cdots e(f^{n-1}(p), f) \geq d^k. \quad (7.5)$$

Using Hypothesis (i), we can now apply Lemma 7.4 to conclude that C contains a point of $I(f^k)$. However, this contradicts that for each $0 \leq n < k$ the curve $C_n \subset D_{\geq d}$ and is, therefore, disjoint from $I(f)$ by Hypothesis (ii). \square

7.2 Superattracting Periodic Points

An important aspect of the work of Favre and Jonsson [21] is to consider asymptotic versions of the multiplicities μ and c (defined in Sects. 3.6 and 1.3, respectively) along the orbit of any $z \in \mathbb{CP}^2$. In our paper, $I(f)$ is not assumed to be empty, so that a point for which an iterate lands on $I(f)$ will have uncountably many different forward orbits, causing such an asymptotic multiplicity not to exist. However, we can apply the results of [21] at regular periodic points, as we will now summarize.

Let $g : (\mathbb{C}^2, \mathbf{0}) \rightarrow (\mathbb{C}^2, \mathbf{0})$ be a dominant holomorphic germ. Then $\hat{\mu}(z, g^n) := 3 + 2\mu(z, g^n)$ is submultiplicative:

$$\hat{\mu}(z, g^{n+m}) \leq \hat{\mu}(z, g^n)\hat{\mu}(g^n(z), g^m).$$

In particular,

$$\mu_\infty(z, g) := \lim \hat{\mu}(z, g^n)^{1/n}$$

exists.

For any $n \in \mathbb{N}$, let $c(z, g^n)$ be the order of the lowest order term in the power series for g^n centered at the origin. It follows directly from the definition that $c(z, g^{n+m}) \geq c(z, g^n)c(g^n(z), g^m)$. Moreover, Favre and Jonsson proved that $c(z, g^n) \leq \frac{1}{2}\mu(z, g^n) + 1$. Therefore, the limit

$$c_\infty(z, g) := \lim c(z, g^n)^{1/n}$$

exists and satisfies

$$c_\infty(z, g) \leq \mu_\infty(z, g). \quad (7.6)$$

Suppose z is a regular periodic point of period k for the dominant rational map $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$. Then, in suitable local coordinates, f^k defines a dominant holomorphic germ $f^k : (\mathbb{C}^2, \mathbf{0}) \rightarrow (\mathbb{C}^2, \mathbf{0})$. By the discussion of the previous two paragraphs, we can define

$$\mu_\infty(z, f) := \lim_{n \rightarrow \infty} \hat{\mu}(z, f^{nk})^{1/nk} \quad \text{and} \quad c_\infty(z, f) := \lim_{n \rightarrow \infty} c(z, f^{nk})^{1/nk}.$$

The total degree of the Jacobian divisor $\text{Jac } f^k$ on \mathbb{CP}^2 is $3(d^k - 1)$, so $\text{Jac } f^k$ cannot vanish to more than this order at $\mathbf{0}$; see Remark A.3. This implies that

$$c_\infty(z, f) \leq \mu_\infty(z, f) \leq d = \deg(f). \quad (7.7)$$

In particular, this justifies the definition of *maximally superattracting regular periodic point* from the Introduction.

Theorem 7.5 (Favre–Jonsson [21, Thm. 4.2]) *Let $g : (\mathbb{C}^2, \mathbf{0}) \rightarrow (\mathbb{C}^2, \mathbf{0})$ be a holomorphic germ. Let V_1, \dots, V_k be the irreducible components of the critical set of g . Assume that $c_\infty(\mathbf{0}, g) < \mu_\infty(\mathbf{0}, g)$. Then, there exists $a_1, \dots, a_k \geq 0$ (not all zero) such that*

$$g^* \left(\sum_i a_i [V_i] \right) \geq \mu_\infty(\mathbf{0}, g) \left(\sum_i a_i [V_i] \right). \quad (7.8)$$

Inequality (7.8) means that if one subtracts the current on the right from the current on the left, then the result is a weighted sum of currents of integration over finitely many analytic curves, each assigned a non-negative weight.

Corollary 7.6 Suppose that $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ is a dominant algebraically stable rational map of degree d . If z is a regular periodic point of period k for f such that

$$c_\infty(z, f) < \mu_\infty(z, f) = d, \quad (7.9)$$

then f^k has a backward invariant curve C through z . Each of the irreducible components of C passes through z .

Proof Taking the k -th iterate, we can suppose z is a regular fixed point for f . Theorem 7.5 implies that there exist weights $a_1, \dots, a_k \geq 0$ (not all zero) such that (7.8) holds with $\mu_\infty(\mathbf{0}, f) = d$. Suppose two irreducible branches V_i and V_j of the critical locus of the germ $f : (\mathbb{C}^2, \mathbf{0}) \rightarrow (\mathbb{C}^2, \mathbf{0})$ are obtained as the restriction of the same algebraic curve from the critical locus of $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$. In this case, it is straightforward to check the proof of Theorem 7.5 from [21] the weights are equal: $a_i = a_j$. Therefore, the local current $\sum_i a_i[V_i]$ extends to a global closed positive $(1, 1)$ current

$$T = \sum \alpha_k[C_k]$$

on \mathbb{CP}^2 which also satisfies $f^*T \geq d T$. Since the pullback under f multiplies the total mass of a closed positive $(1, 1)$ current on \mathbb{CP}^2 by exactly d , we conclude that $f^*T = d T$ (see Appendix A.3). In particular, the support of T is a backward invariant curve for f passing through z . \square

7.3 Backward Invariant Curves

Because of Corollary 7.6 we need a better understanding of algebraic curves that are backward invariant under some iterate of f . This subsection is devoted to proving the following:

Proposition 7.7 Suppose that f satisfies Hypotheses (i)–(iii) of the Equidistribution Theorem and that z_0 is a regular periodic point of period k for f . If C is a (possibly reducible) algebraic curve that is backward invariant under f^k , each of whose irreducible components passes through z_0 , then $z_0 \in \mathcal{E}(b)$ (see p. 6 for the definition of $\mathcal{E}(b)$).

Proof We will show that z_0 is superattracting and that there is some algebraic curve $C' \subset C$ (possibly also reducible) satisfying the conditions necessary for z_0 to be in $\mathcal{E}(b)$.

Let C_1, \dots, C_m denote the irreducible components of C and suppose for contradiction that none of them is collapsed under f^k . In this case, we claim that f^{-k} induces a permutation on $\{C_1, \dots, C_m\}$. Consider an arbitrary $0 \leq i \leq m$ and notice that $f^{-k}(C_i)$ is not reduced to an indeterminate point. Indeed, by considering C_i as a divisor (C_i) of multiplicity one, the fact that $(f^k)^*(C_i)$ is a divisor of degree $d^k \deg(C_i) > 0$ (see [4, Lem. A.5]) implies that $f^{-k}(C_i)$ is a non-trivial algebraic curve. Since C is backward invariant under f^k , this implies that there is at least one

$1 \leq j \leq m$ such that $C_j \subset f^{-k}(C_i)$. Meanwhile, for each $1 \leq n \leq m$ the irreducible component C_n is not collapsed by f^k so that it can occur as a component of $f^{-k}(C_p)$ for at most one value of $1 \leq p \leq m$. We conclude that, for each $1 \leq i \leq m$ there exists a unique j such that $f^{-k}(C_i) = C_j$.

In particular, there exists $\ell \geq 1$ such that $B := C_1$ is backward invariant under $f^{\ell k}$. If we consider B as a divisor (B) of multiplicity one, then this implies that

$$(f^{\ell k})^*(B) = d^{\ell k}(B) \quad (7.10)$$

(see again [4, Lem. A.5]).

Backward invariance under $f^{\ell k}$ also implies $f^{\ell k}(B \setminus I(f^{\ell k})) \subset B$. Moreover, since we are supposing that none of the irreducible components of C is collapsed by f^k , we can use (7.10) and the Whitney Fold normal form given by Lemma A.4 to find that $\sigma(z, f^{\ell k}) \geq d^{\ell k}$ for all $z \in B$. Using Hypothesis (i), it follows from Lemma 7.4 that that B contains a point of $I(f^{\ell k})$.

For each $0 \leq j < \ell k$ we let $B_j := \overline{f^j(B \setminus I(f^j))}$, which is a non-trivial irreducible algebraic curve that is not collapsed by f . Proposition 5.10 gives that at most finitely many points of B_j are in $D_{>d}$. Thus, in order to have $\sigma(z, f^{\ell k}) = d^{\ell k}$ for generic points of B , we must have that $B_j \subset D_{\geq d}$ for each $1 \leq j \leq \ell k$. Hypothesis (ii) implies that B_j is disjoint from $I(f)$ for each $0 \leq j < \ell k$, contradicting that B contains a point of $I(f^{\ell k})$.

We can, therefore, let $C' \subset C$ denote the (non-trivial) union of all irreducible components of C that are collapsed by some iterate of f^k . Since each of these components passes through the regular periodic point z_0 , each of them collapses to z_0 . Since C is backward invariant under f^k , C' is also backward invariant under f^k .

Suppose for contradiction that z_0 is a smooth point of C' . Since every irreducible component of C passes through z_0 , this implies that C' is itself irreducible. In particular, backward invariance of C' under f^k implies that

$$(f^k)^*(C') = d^k(C'). \quad (7.11)$$

Then, we can choose local coordinates (x, y) centered at z_0 such that $C' = \{x = 0\}$. Write f^k in these local coordinates as

$$f^k(x, y) = (f_1^k(x, y), f_2^k(x, y)). \quad (7.12)$$

Equation (7.11) implies that $f_1^k(x, y) = x^{d^k} g(x, y)$ for some non-vanishing holomorphic function g . From this, it is immediate that f^k contracts the volume of a small bidisc centered along C' with exponent d^k , i.e. that for ever $z \in C'$ we have $\sigma(z, f^k) \geq d^k$.

Let z_0, \dots, z_{k-1} denote the periodic orbit of z_0 . Moreover, let $1 \leq j_0 \leq k$ be the smallest iterate for which C' is collapsed by f^j . For each $0 \leq j < j_0$ we let

$$C'_j := \overline{f^j(C' \setminus I(f^j))}.$$

Then C'_{j_0-1} is collapsed by f and $z_{j_0} \in I^-(f)$, by definition. By Hypothesis (iii), we have $\sigma(z_j, f) \leq d$ for each $0 \leq j \leq k-1$. Meanwhile, for each $0 \leq j \leq j_0-1$ the curve $C'_j \not\subset D_{>d}$, by Proposition 7.2. Therefore, to have $\sigma(z, f^k) \geq d^k$ for every $z \in C'$ we must have $C'_{j_0-1} \subset D_{\geq d}$. This contradicts Hypothesis (ii) since C'_{j_0-1} contains a point of $I(f)$, by Lemma A.1.

Therefore, z_0 is a singular point of C' . We now use this to show that z_0 is super-attracting. Let (x, y) be local holomorphic coordinates centered at z_0 and suppose that C' is given in these coordinates as $C' = \{q(x, y) = 0\}$ for some holomorphic function q (more specifically, we choose q so that it defines the divisor (C') with multiplicity one). Since z_0 is a singular point of C' we have

$$q(0, 0) = \frac{\partial q}{\partial x}(0, 0) = \frac{\partial q}{\partial y}(0, 0) = 0. \quad (7.13)$$

Let $\ell \geq 1$ be chosen so that $f^{\ell k}(C' \setminus I(f^{\ell k})) = z_0$. Writing $f^{\ell k}(x, y) = (f_1^{\ell k}(x, y), f_2^{\ell k}(x, y))$ in these local coordinates we have that

$$f_1^{\ell k}(x, y) = q(x, y)g_1(x, y) \quad \text{and} \quad f_2^{\ell k}(x, y) = q(x, y)g_2(x, y), \quad (7.14)$$

for some holomorphic functions $g_1(x, y)$ and $g_2(x, y)$. Equations (7.13) and (7.14) imply that $Df^{\ell k}(0, 0) = \mathbf{0}$, i.e. that z_0 is a superattracting periodic point. We conclude that $z_0 \in \mathcal{E}(b)$. \square

7.4 Behavior of the Volume Exponent σ Under Iteration

The exponents $\sigma(z, f)$ do not transform very well under iteration: if an iterate of z lands on $I(f)$, this leads to many different orbits of z , all of which we need to control. Let $Y \subset \mathbb{CP}^2$. Given a sequence of open sets $N_0, \dots, N_{n-1} \subset \mathbb{CP}^2$ let

$$f_{N_0, N_1, \dots, N_{n-1}}^{-n}(Y) = \{z_0 \in N_0 : \text{there exists orbit } z = (z_0, z_1, \dots, z_n) \text{ with } z_i \in N_i \text{ for } 1 \leq i \leq n-1 \text{ and } z_n \in Y\}.$$

If $z = (z_0, z_1, \dots, z_{n-1})$ is an orbit of f we let $\sigma(z, f^n) \equiv \sigma(z_0, z_1, \dots, z_{n-1}, f^n)$ be the smallest positive number such that for any $\gamma > \sigma(z, f^n)$ there is an $K > 0$ and neighborhoods N_0, \dots, N_{n-1} of z_0, \dots, z_{n-1} such that

$$\text{vol}(f_{N_0, \dots, N_{n-1}}^{-n} Y) \leq K(\text{vol } Y)^{1/\gamma} \quad (7.15)$$

for any measurable $Y \subset \mathbb{CP}^2$.

It is clear from the definition that for any orbit z_0, \dots, z_{n-1} and any $1 \leq k \leq n-1$

$$\begin{aligned} \sigma(z_0, z_1, \dots, z_{n-1}, f^n) &\leq \sigma(z_0, \dots, z_{k-1}, f^k) \sigma(z_k, \dots, z_{n-1}, f^{n-k}) \quad \text{and} \\ \sigma(z_0, z_1, \dots, z_{n-1}, f^n) &\leq \sigma(z_0, f^n), \end{aligned}$$

where the exponent on the right-hand side of the second inequality is from (1.4).

Lemma 7.8 *For any $z_0 \in \mathbb{CP}^2$ we have*

$$\sigma(z_0, f^n) \leq \sup_z \sigma(z, f^n), \quad (7.16)$$

where the supremum is taken over all orbits $z = (z_0, \dots, z_{n-1})$.

Proof Since the indeterminate points of f^i for each $1 \leq i \leq n$ are isolated, we can find a neighborhood V of z_0 containing no indeterminate points for each f^i other than (potentially) z_0 . We do a finite sequence of blow-ups over z_0 forming $\pi : \tilde{V} \rightarrow \mathbb{CP}^2$ so that for each $1 \leq i \leq n$ the iterate f^i lifts to a regular map \tilde{f}^i making the diagram

$$\begin{array}{ccc} \tilde{V} & & \\ \downarrow \pi & \searrow \tilde{f}^i & \\ V & \xrightarrow{f^i} & \mathbb{CP}^2, \end{array}$$

commute wherever $f^i \circ \pi$ is defined.

Let γ be any exponent greater than $\sup_z \sigma(z, f^n)$. Any point $\tilde{z} \in \pi^{-1}(z)$ determines the following orbit of length n of z :

$$z \equiv (z_0, \dots, z_{n-1}) := \left(z, \tilde{f}^1(\tilde{z}), \dots, \tilde{f}^{n-1}(\tilde{z}) \right). \quad (7.17)$$

By choice of γ , there is a sequence of neighborhoods $U_0(z), \dots, U_{n-1}(z)$ of z_0, \dots, z_{n-1} , respectively, so that (7.15) holds. Associated to this sequence is a neighborhood $\tilde{U}(\tilde{z})$ of \tilde{z} such that for any $Y \subset \mathbb{CP}^2$ we have

$$f_{U_0(z), \dots, U_{n-1}(z)}^{-n} Y = \pi \left((\tilde{f}^n)^{-1} Y \cap \tilde{U}(\tilde{z}) \right). \quad (7.18)$$

Since $\pi^{-1}\{z_0\}$ is compact, there exist finitely many length n orbits z^1, \dots, z^ℓ of z_0 whose neighborhoods $\tilde{U}(z^1), \dots, \tilde{U}(z^\ell)$ cover $\pi^{-1}(z_0)$. If we let $U = \bigcup_{i=1}^{\ell} \pi(\tilde{U}(z^i))$ then (7.18) implies that for any $Y \subset \mathbb{CP}^2$ we have

$$f^{-n}(Y) \cap U \subset \bigcup_{i=1}^{\ell} f_{U_0(z^i), \dots, U_{n-1}(z^i)}^{-n} Y.$$

The result follows since each $f_{U_0(z^i), \dots, U_{n-1}(z^i)}^{-n} Y$ satisfies (7.15) with some suitable multiplicative constant. \square

We will also use a modified notation $\sigma(z, f, W)$ for the volume exponent in the case that we require that (1.4) only holds for measurable $Y \subset W$. A straightforward adaptation of the proof of Lemma 7.8 yields:

Lemma 7.9 *Let $\Omega \subset \mathbb{CP}^2$ be a forward invariant open set. For any $z_0 \in \mathbb{CP}^2 \setminus \Omega$ we have*

$$\sigma(z_0, f^n, \mathbb{CP}^2 \setminus \Omega) \leq \sup_z \sigma(z, f^n), \quad (7.19)$$

where the supremum is taken over all orbits $z = (z_0, \dots, z_{n-1}) \subset \mathbb{CP}^2 \setminus \Omega$.

7.5 Proof of Proposition 7.1.

Let Ω be any forward invariant neighborhood of the exceptional set \mathcal{E} . We will prove that there exist $B > 0$ and $1 < \alpha < d$ such that for any $n \geq 0$ and any finite orbit z_0, \dots, z_{n-1} remaining in $\mathbb{CP}^2 \setminus \Omega$ we have

$$\sigma(z_0, \dots, z_{n-1}, f^n) \leq B\alpha^n. \quad (7.20)$$

We can then let n_0 be sufficiently large that $\tau := B\alpha^{n_0} < d^{n_0}$. Then Lemma 7.9 will imply that $\sigma(z_0, f^{n_0}, \mathbb{CP}^2 \setminus \Omega) < \tau$ for any $z_0 \in \mathbb{CP}^2 \setminus \Omega$. Since $\mathbb{CP}^2 \setminus \Omega$ is compact, it is then covered by finitely many neighborhoods such that on each neighborhood N we have

$$\text{vol}(f^{-n_0}Y \cap N) \leq K_N(\text{vol } Y)^{1/\tau},$$

which will, therefore, prove the proposition.

To prove (7.20), we consider three types of finite orbits.

Type I Periodic orbits that pass through $D_{>d} \setminus \mathcal{E}$.

Let p be any point from such an orbit. We could¹¹ have $\sigma(f^n(p), f) \geq d$ for every $n \geq 0$, but the hypothesis that $p \notin \mathcal{E}$ and the results of Sects. 7.2 and 7.3 will imply that $\sigma(p, f^n)$ decays sufficiently as we iterate and indeed grows at exponential rate slower than d .

By Hypothesis (iii), p is a regular periodic point, i.e. the orbit of p is disjoint from $I(f)$. Since $p \notin \mathcal{E}(a)$ it is not maximally superattracting, $c_\infty(p, f) < d$. Therefore, since $p \notin \mathcal{E}(b)$, Corollary 7.6 and Proposition 7.7 imply that $\mu_\infty(p, f) < d$. Using the volume estimate from Lemma 5.2, this implies that $\limsup(\sigma(p, f^n))^{1/n} < d$. Hence, (7.20) holds for such an orbit.

Type II Periodic orbits contained in $\mathcal{N}_{\geq d} \setminus \mathcal{E}$.

Recall that $\mathcal{N}_{\geq d} \subset D_{\geq d}$ is finite and forward invariant by Proposition 7.3. It is disjoint from $I(f)$ by Hypothesis (ii). Inequality (7.20) holds for these orbits using exactly the same reasoning as for orbits of Type I.

Type III Finite orbits $z = (z_0, \dots, z_{n-1}) \subset \mathbb{CP}^2 \setminus \Omega$ that are disjoint from orbits of Type I and Type II.

Let k denote the number of elements of $\mathcal{N}_{\geq d}$, which is finite and forward invariant by Proposition 7.3. Since $\mathcal{N}_{\geq d}$ is forward invariant and orbit z is disjoint from periodic

¹¹ For example, p could be a fixed point in $D_{>d}$.

orbits of Type II, we have $z_0, \dots, z_{n-k-1} \notin \mathcal{N}_{\geq d}$. Moreover, Proposition 7.3 gives $N \in \mathbb{N}$ such that at least one out of every N points of z_0, \dots, z_{n-k-1} is “good”, landing in $\mathbb{CP}^2 \setminus D_{\geq d}$. Moreover, Lemma 5.9 gives a uniform $1 \leq \sigma_0 < d$ such that $\sigma(p, f) \leq \sigma_0$ for these “good” points.

Let ℓ denote the number of elements of $D_{>d}$, which is finite by Proposition 7.2. Since orbit z is disjoint from periodic orbits of Type I, it can only meet $D_{>d}$ at most ℓ times. In other words, $\sigma(z_i, f) > d$ for at most ℓ values of $0 \leq i \leq n-1$. If we absorb the excess $\sigma(z_i, f)/d > 1$ from these at most k points into the multiplicative constant B , then the previous paragraph implies that (7.20) holds for an orbit of Type III.

Let us now consider an arbitrary finite orbit $z = (z_0, \dots, z_{n-1}) \subset \mathbb{CP}^2 \setminus \Omega$. If it is not of Type I, II, or III, then there is some $0 < m < n-1$ such that (z_0, \dots, z_{m-1}) is of Type III and (z_m, \dots, z_{n-1}) is of either Type I or II. Inequality (7.20) holds for such an orbit using the submultiplicativity

$$\sigma(z_0, \dots, z_{n-1}, f^n) \leq \sigma(z_0, \dots, z_{m-1}, f^m) \sigma(z_m, \dots, z_{n-1}, f^{n-m}).$$

This completes the proof of Proposition 7.1 and hence of the Equidistribution Theorem. \square

7.6 A Useful Proposition for Determining \mathcal{E}

It is easy to detect collapsed curves by checking each irreducible component of the critical locus, so it is usually easy to determine $\mathcal{E}(b)$. The following proposition makes it easier to determine $\mathcal{E}(a)$.

Proposition 7.10 *Let $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ satisfy the hypotheses of the Equidistribution Theorem. Then, any maximally superattracting periodic orbit (i.e. any point of $\mathcal{E}(a)$) has orbit passing through $D_{\geq d}$. Moreover, it either*

- (i) *passes through the finite set $D_{>d}$, or*
- (ii) *is contained entirely in the finite non-escaping set $\mathcal{N}_{\geq d} \subset D_{\geq d}$.*

Proof Suppose z_0 is a maximally superattracting periodic point so that $d = c_\infty(z_0, f)$, by definition. Iterating a round ball of small radius centered at z_0 , this implies that $\liminf(\sigma(p, f^n))^{1/n} = d$. Using the submultiplicativity of the volume exponent σ , the only way this can happen is if (i) or (ii) holds. \square

8 Applications of the Equidistribution Theorem to Other Hierarchical Lattices

We now consider the Migdal–Kadanoff renormalization mappings associated to the five hierarchical lattices whose generating graphs are shown in Fig. 4 (in the introduction).

8.1 Linear Chain

The hierarchical lattice generated by the double edge shown in Fig. 4 is the same as the classical \mathbb{Z}^1 lattice. The corresponding renormalization mapping is

$$R[U : V : W] = [U^2 + V^2 : V(U + W) : W^2 + V^2].$$

Its critical locus consists of the curves $C_1 := \{U + W = 0\}$ and $C_2 := \{UW - V^2 = 0\}$. Generic points z on each of these curves have volume exponent $\sigma(z, R) = 2 = \deg(R)$. Since these two curves meet at the indeterminate points $a_{\pm} := [\pm i : 1 : \mp i]$, this implies that $\sigma(a_{\pm}, R) = \deg(R)$, violating Hypothesis (ii) of the Equidistribution Theorem.

Despite this, the Global LYF Theorem holds for this hierarchical lattice. Let \mathcal{F} denote the family of lines passing through the fixed point $[1 : 0 : 1]$. Each line $L \in \mathcal{F}$ is invariant under R and all but finitely many lines from \mathcal{F} intersect C_2 at two distinct points, neither of which is the intersection of the Principal LYF Locus $S_0 = \{U + 2V + W = 0\}$ with C_2 . For such generic lines L we have $R|_L(z) = z^2$ in a suitable choice of local coordinate z . The exceptional points $z = 0$ and $z = \infty$ correspond to the intersections of L with the conic C_2 and $S_0 \cap L$ corresponds to a non-exceptional point. Therefore, for all but finitely many $L \in \mathcal{F}$ we have that the (normalized) iterated preimages of $S_0 \cap L$ equidistribute to the measure of maximal entropy for $R|_L$, see [24, 39, 40], which is equal to the slice $S|_L$, where S is the Green current for R . This is sufficient to prove that $\frac{1}{2^n}(R^n)^*[S_0] \rightarrow S$. Pulling everything back under Ψ one finds that the Global LYF Theorem holds for this hierarchical lattice.

8.2 k -Fold DHL

One can consider a generalization of the DHL with the generating graph having valence $k \geq 2$ at the marked vertices a and b ; see Fig. 4. The renormalization mapping is

$$R[U : V : W] = [(U^2 + V^2)^k : V^k(U + W)^k : (W^2 + V^2)^k],$$

whose lift $\hat{R} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ has Jacobian

$$\det D\hat{R} = 2^2 k^3 V^{k-1} (UW - V^2) (U + W)^k (U^2 + V^2)^{k-1} (W^2 + V^2)^{k-1}.$$

In particular, the critical locus of R is the same as for the DHL (see Sect. 3.5). However, when considered as a divisor, the curves have higher multiplicities, depending on k .

This mapping has the same indeterminate points as for the DHL; $I(R) = \{a_{\pm}\}$. Since $I(R) \neq \emptyset$, Hypothesis (i) is satisfied. The mapping R also has the same superattracting fixed points $\{e, e'\}$. Just as in Sect. 6.1, one can check that for any $z \in \mathbb{CP}^2 \setminus \{e, e'\}$ one has $\sigma(z, R) < 2k = \deg(R)$. Therefore, $\sigma(a_{\pm}) < \deg(R)$, giving that Hypothesis (ii) is satisfied. Meanwhile, the only curve collapsed by R is $\{U + W = 0\}$, which is collapsed to the fixed point $b_0 := [1 : 0 : 1] \notin \{e, e'\}$. Since

$D_{>d} \subset \{e, e'\}$ it is invariant under R and disjoint from $I(R) \cup I^-(R) = \{a_\pm, b_0\}$, Hypothesis (iii) is also satisfied.

We now determine the exceptional set \mathcal{E} . Since $D_{\geq d} \subseteq \{e, e'\}$ and since $\{e, e'\}$ is invariant, Proposition 7.10 gives that any point of $\mathcal{E}(a)$ is from $\{e, e'\}$. As for the DHL, the curve $\{UW - V^2 = 0\}$ is invariant and on it R is conjugate to $z \mapsto z^k$, with $z = 0$ and $z = \infty$ corresponding to the superattracting fixed points e and e' , respectively. Therefore, $c_\infty(e, R) = c_\infty(e', R) = k < 2k = \deg R$, giving that these points are not maximally superattracting. Therefore, $\mathcal{E}(a) = \emptyset$. Meanwhile, the fixed point $b_0 \in I^-(R)$ is not superattracting, so $\mathcal{E}(b) = \emptyset$.

Therefore, the Equidistribution Theorem gives that the (normalized) iterated preimages of any algebraic curve $A \subset \mathbb{CP}^2$ equidistribute to the Green Current for R . In particular, the Global Lee–Yang–Fisher Current Theorem holds for the k -fold DHL.

8.3 Triangles

The renormalization mapping for the hierarchical lattice generated by the triangle shown in Fig. 4 is:

$$R[U : V : W] = [U^3 + UV^2 : UV^2 + V^2W : V^2W + W^3].$$

The lift $\hat{R} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ has Jacobian

$$\det D\hat{R} = 6V(U + W)(U^2V^2 + 3U^2W^2 - V^2WU + V^2W^2),$$

so that the critical locus is the union of two lines and a quartic:

$$\begin{aligned} L_1 &:= \{V = 0\}, \\ L_2 &:= \{U + W = 0\}, \text{ and} \\ Q &:= \{U^2V^2 + 3U^2W^2 - UV^2W + V^2W^2\}, \end{aligned}$$

each with multiplicity one.

Hypothesis (i) is satisfied because the indeterminacy locus $I(R)$ is non-empty, consisting of three points: $p := [0 : 1 : 0]$ and $q_\pm := [\pm i : 1 : \mp i]$. Point p is resolved in one blow-up. In a neighborhood of $L_{\text{exc}}(p)$ the lifted map \hat{R} has critical locus consisting of $L_{\text{exc}}(p)$ and the proper transforms \tilde{L}_2 and \tilde{Q} , each with multiplicity one. Moreover, \tilde{L}_2 and \tilde{Q} intersect $L_{\text{exc}}(p)$ at three distinct points, each of which is a normal crossing singularity. Therefore, Lemma 5.5 gives that any point $z \in L_{\text{exc}}(p)$ has $\sigma(z, \tilde{R}) = 2$. Proposition 5.8 then gives $\sigma(p, R) = 2 < \deg(R)$.

Both of the indeterminate points q_\pm are also resolved with one blow-up. The lift \tilde{R} does not have $L_{\text{exc}}(q_\pm)$ as a critical curve. The proper transforms \tilde{L}_2 and \tilde{Q} become disjoint and smooth in a neighborhood of the exceptional divisor $L_{\text{exc}}(q_\pm)$, so that $\det D\tilde{R}$ has multiplicity 0 or 1 at every point of $L_{\text{exc}}(q_\pm)$. Lemma 5.2 and Proposition 5.8 then imply that $\sigma(q_\pm, R) = 2 < \deg(R)$. We conclude that Hypothesis (ii) is satisfied.

We now check Hypothesis (iii). Since each of the critical curves has multiplicity one, Lemma 5.2 gives that $\sigma(z, R) \leq 2$ at every smooth point of the critical locus. Let $r := [1 : 0 : -1] = L_1 \cap L_2$. Then the singular points of the critical locus are $\{r, e, e'\} \cup I(R)$. Moreover, one can choose local coordinates (x, y) in a neighborhood of r such that $\det DR \asymp xy$. Hence Lemma 5.5 gives $\sigma(r, R) = 2$, so that $r \notin D_{\geq d}$. Thus, $D_{\geq d} \subset \{e, e'\}$ is, therefore, invariant under R .

Both L_1 and Q go through two superattracting fixed points $e := [1 : 0 : 0]$ and $e' := [0 : 0 : 1]$, so they are not collapsed by R . Meanwhile, a direct calculation shows that L_2 is collapsed by R to $s := [1 : 0 : -1] \in I^-(R)$, which is a fixed point of R . Since $D_{\geq d}$ is invariant under R and disjoint from $I(R) \cup I^-(R) = \{p, q_{\pm}, s\}$, Hypothesis (iii) is satisfied.

By Proposition 7.10, any point of $\mathcal{E}(a)$ passes through $D_{\geq d} \subset \{e, e'\}$, which is invariant under R . Moreover, the fixed points e and e' are not maximally superattracting since the curve $\{UW - V^2 = 0\}$ is invariant for this renormalization mapping as well, and on it R is conjugate to $z \mapsto z^2$. Therefore, $\mathcal{E}(a) = \emptyset$. The only point of $I^-(R)$ is the fixed point s , which is not superattracting. Therefore, $\mathcal{E}(b) = \emptyset$.

Thus, the Equidistribution Theorem gives that the (normalized) iterated preimages of any algebraic curve A equidistribute to the Green current for R . In particular, the Global Lee–Yang–Fisher Current Theorem also holds for the hierarchical lattice generated by the triangle.

8.4 Tripod

The renormalization mapping for the hierarchical lattice generated by the tripod shown in Fig. 4 is:

$$\begin{aligned} R[U : V : W] \\ = [U^3 + U^2V + V^3 + V^2W : V(U^2 + UV + VW + W^2) : UV^2 + V^3 \\ + VW^2 + W^3], \end{aligned}$$

which has indeterminacy set $I(R) = \{[\omega : 1 : \bar{\omega}] : \omega^3 = -1\}$.

One can resolve the indeterminacy point $p = [-1 : 1 : -1]$ by doing three blow-ups, each one done on the previously created exceptional divisor. When one applies Lemmas 5.2, 5.5, and Proposition 5.8 one finds $\sigma(p, R) \geq 3 = \deg(R)$. Therefore, Hypothesis (ii) of the Equidistribution Theorem fails for this mapping. In particular, we do not know if the Global Lee–Yang–Fisher Current Theorem holds for the hierarchical lattice generated by the tripod.

8.5 Split Diamond

The renormalization mapping for the hierarchical lattice generated by the split diamond shown in Fig. 4 is:

$$R[U : V : W] = [U^5 + 2U^2V^3 + V^4W : V^2(U^3 + 2UVW + W^3) : UV^4 + 2V^3W^2 + W^5].$$

The lift $\hat{R} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ has Jacobian

$$\begin{aligned} \det D\hat{R} = & 10V(UW - V^2) \\ & (4V^3U^6 + 5W^3U^6 + 5V^2W^2U^5 + 5V^4WU^4 + 15VW^4U^4 \\ & + V^6U^3 + 5V^3W^3U^3 + 5W^6U^3 + 5V^5W^2U^2 + 5V^2W^5U^2 \\ & - V^7WU + 5V^4W^4U + V^6W^3 + 4V^3W^6). \end{aligned} \quad (8.1)$$

One can check that $I(R) = \{[0 : 1 : 0], [\omega : 1 : \bar{\omega}] : \omega^3 = -1\}$. Each of these indeterminate points requires three blow-ups to resolve. One can work through the details in Maple and check that the lifted mappings have Jacobian of the form $\det D\tilde{R} \asymp x^a y^b$ with $\max(a, b) \leq 3$ at each of the points on each of the exceptional divisors. Thus, Lemma 5.5 and Proposition 5.8 give $\sigma(p, R) = 4 < \deg(R)$ for each $p \in I(R)$. Therefore, Hypotheses (i) and (ii) are satisfied.

Let us check Hypothesis (iii). Since each of the critical curves has multiplicity 1, we have $\sigma(p, R) \leq 2$ for any point outside the singular locus of the critical set. Meanwhile, each of the intersections of any two of the three critical curves and each of the singular points of C_3 lie in $I(R) \cup \{e, e'\}$. We conclude that $D_{\geq d} \subset \{e, e'\}$ and is therefore invariant under R and disjoint from $I(R)$.

Each of the irreducible components of the critical locus (corresponding to the three factors of (8.1)) passes through the two superattracting fixed points $e = [1 : 0 : 0]$ and $e' = [0 : 1 : 0]$ of R and, therefore, there are no curves collapsed by R . Therefore, $I^-(R) = \emptyset$. Since $D_{\geq d}$ is invariant under R and disjoint from $I(R) \cup I^-(R)$, Hypothesis (iii) is satisfied.

Since $I^-(R) = \emptyset$ we have that $\mathcal{E}(b) = \emptyset$. Meanwhile, the critical curve $\{UW - V^2 = 0\}$ is invariant under R and on it R is conjugate to $z \mapsto z^2$, with $z = 0$ and $z = \infty$ corresponding to the superattracting fixed points e and e' , respectively. Therefore, $c_\infty(e, R) = c_\infty(e', R) = 2 < 5 = \deg R$, giving that these points are not maximally superattracting. Therefore, we also have that $\mathcal{E}(a) = \emptyset$ and the Equidistribution Theorem gives that the (normalized) iterated preimages of any algebraic curve $A \subset \mathbb{CP}^2$ equidistributes to the Green Current for R . In particular, the Global Lee–Yang–Fisher Current Theorem also holds for the hierarchical lattice generated by the split diamond.

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Appendix A: Elements of Complex Geometry

This appendix presents background on the material from complex geometry that we need beyond what is described in [4, Appendix A]. We refer the reader there for background on rational maps, blow-ups, and divisors.

A.1: Structure of Rational Maps $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$

Throughout this subsection, we suppose $f : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ is a dominant rational map of degree $d \geq 2$.

There is a finite sequence of blow-ups $\pi : \widetilde{\mathbb{CP}^2} \rightarrow \mathbb{CP}^2$ done over the points of the indeterminacy set $I(f)$ so that f lifts to a regular map $\tilde{f} : \widetilde{\mathbb{CP}^2} \rightarrow \mathbb{CP}^2$, making the following diagram commute

$$\begin{array}{ccc} \widetilde{\mathbb{CP}^2} & & \\ \downarrow \pi & \searrow \tilde{f} & \\ \mathbb{CP}^2 & \xrightarrow{f} & \mathbb{CP}^2, \end{array} \quad (\text{A.1})$$

wherever $f \circ \pi$ is defined (see [46, Ch. IV, §3.3]). We call such a commutative diagram a *resolution of the indeterminacy* of f .

For any $Y \subset \mathbb{CP}^2$ one can use the resolution of indeterminacy (A.1) to define

$$f(Y) := \tilde{f}(\pi^{-1}(Y)) \quad \text{and} \quad f^{-1}(Y) := \pi\left(\tilde{f}^{-1}(Y)\right). \quad (\text{A.2})$$

These definitions are independent of the choice of resolution.

The following two lemmas are well-known; see for example [22]:

Lemma A.1 *Any curve that is collapsed by f passes through $I(f)$.*

Let $\text{Crit}(f)$ denote the critical locus of $f|_{\mathbb{CP}^2 \setminus I(f)}$ and let $V := f(I(f) \cup \text{Crit}(f))$. Then the mapping

$$f|_{\mathbb{CP}^2 \setminus f^{-1}(V)} : \mathbb{CP}^2 \setminus f^{-1}(V) \rightarrow \mathbb{CP}^2 \setminus V$$

is a finite degree covering map. The *topological degree* $d_{\text{top}}(f)$ is the degree of this cover.

Lemma A.2 *If $I(f) \neq \emptyset$, then $\deg_{\text{top}}(f) < d^2$.*

Remark A.3 Let $\hat{f} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ denote a homogeneous lift of f . Then, the complex Jacobian $\text{Jac}(\hat{f}) := \det D\hat{f}$ is a homogeneous polynomial of degree $3(d-1)$. It induces a divisor $\text{Jac}(f)$ on \mathbb{CP}^2 of the same degree.

Lemma A.4 (Whitney Fold Normal Form¹²) *Suppose C is an irreducible component of the critical locus that is not collapsed by f . Then, there is a finite set $S \subset C$ and $r \equiv r_C \in \mathbb{Z}_+$ such that for any $p \in C \setminus S$ there exist local holomorphic coordinates (x, y) in a neighborhood of p and (z, w) in a neighborhood of $f(p)$ in which*

¹² Here we are using a more general notion of “Whitney Fold” that was used in [4, Appendix D.2], where only exponent $r_C = 2$ was considered.

$$(z, w) = f(x, y) = (x, y^r).$$

In particular, at every $p \in C \setminus S$ we have $\sigma(p, f) = e(p, f) = r$ and for every $p \in C$ we have $\sigma(p, f) \geq r$.

(The volume exponent $\sigma(p, f)$ is defined in Sect. 1.3 and the local topological degree $e(p, f)$ in Sect. 7.1.)

Proof We include the following points from C in S :

- (i) singular points of the critical locus of f ,
- (ii) points $p \in C$ for which $T_p C \subset \text{Ker } Df(p)$, and
- (iii) points of $I(f)$.

Since C is not collapsed by f , the set S is finite and, since C is irreducible, $C \setminus S$ is connected. Essentially, the same proof as the characterization of Whitney Folds presented in Lemma D.2 of [4] gives that for each point p_0 satisfying (i) - (iii) we can find local coordinates in some neighborhoods of p_0 and $f(p_0)$ in which the map has the normal form $(z, w) = f(x, y) = (x, y^r)$ for some integer $r \geq 2$. Moreover, as $C \setminus S$ is connected, we see that the exponent r is constant on $C \setminus S$. From the normal form and Lemma 5.2, we see that at every point of $C \setminus S$ we have $\sigma(p, f) = \mu(p, f) + 1 = e(p, f) = r$. The fact that $\sigma(p, f) \geq r$ for every $p \in C$ is because $D_{\geq r}$ is algebraic, by Lemma 5.9. \square

A.2: Hyperplane Bundle

The hyperplane bundle and its tensor powers provide a convenient way to work with divisors on \mathbb{CP}^k .

The fibers of $\pi : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{CP}^k$ are punctured complex lines \mathbb{C}^* . Compactifying each of these lines at infinity, we add to $\mathbb{C}^{k+1} \setminus \{0\}$ the line at infinity $L_\infty \approx \mathbb{CP}^k$ obtaining the total space

$$(\mathbb{C}^{k+1})^* \cup L_\infty \approx (\mathbb{CP}^{k+1})^* := \mathbb{CP}^{k+1} \setminus \{0\}.$$

The projection naturally extends to $\pi : (\mathbb{CP}^{k+1})^* \rightarrow \mathbb{CP}^k$, whose fibers are complex lines \mathbb{C} . The resulting line bundle is called the *hyperplane bundle* over \mathbb{CP}^k .

In homogeneous coordinates $(z_0 : \dots : z_k : t)$ on \mathbb{CP}^{k+1} , this projection is just

$$\pi : (z_0 : \dots : z_k : t) \mapsto (z_0 : \dots : z_k), \quad (\text{A.3})$$

with $L_\infty = \{t = 0\}$, $(\mathbb{C}^{k+1})^* = \{t = 1\}$, and the map $(z : t) \mapsto t/\|z\|$ parameterizing the fibers (here $\|z\|$ stands for the Euclidean norm of $z \in \mathbb{C}^{k+1} \setminus \{0\}$). This line bundle is endowed with the natural *Hermitian structure*: $\|(z : t)\| = |t|/\|z\|$.

Any non-vanishing linear form Y on \mathbb{C}^{k+1} determines a section of the hyperplane bundle:

$$s_Y : z \mapsto (z : Y(z)), \quad z \in \mathbb{C}^{k+1}. \quad (\text{A.4})$$

The divisor D_Y (a projective line counted with multiplicity 1) is precisely the zero divisor of s_Y .

The d th tensor power of the hyperplane bundle can be described as follows. Its total space X^d is the quotient of $(\mathbb{C}^{k+2})^*$ by the \mathbb{C}^* -action

$$(z_0, \dots, z_k, t) \mapsto (\lambda z_0, \dots, \lambda z_k, \lambda^d t), \quad \lambda \in \mathbb{C}^*.$$

We denote the equivalence class of (\hat{z}, t) using the “homogeneous” coordinates $(\hat{z} : t)$. The projection $X^d \rightarrow \mathbb{CP}^k$ is natural, as above (A.3). A non-vanishing homogeneous polynomial P on \mathbb{C}^{k+1} of degree d defines a holomorphic section s_P of this bundle given by $s_P(z) = (\hat{z} : P(\hat{z}))$. This bundle is endowed with the Hermitian structure:

$$\|(z : t)\| = |t|/\|z\|^d. \quad (\text{A.5})$$

More generally, any divisor $D = D_P - D_Q$ defines a section s_D of the $\deg(D)$ -th tensor power of the hyperplane bundle, defined by $s_D(z) = (\hat{z} : P(\hat{z})/Q(\hat{z}))$. One can recover D from s_D by taking its zero divisor.

A.3: Currents

We will now give a brief background on currents; for more details see [38, 43] and the appendices from [16, 47]. Currents are naturally defined on general complex (or even smooth) manifolds, however, to continue our discussion of rational maps, divisors, etc, we restrict our attention to projective manifolds.

A $(1, 1)$ -current T on V is a continuous linear functional on $(k-1, k-1)$ -forms with compact support. It can be also defined as a generalized differential $(1, 1)$ -form $\sum T_{ij} dz_i d\bar{z}_j$ with distributional coefficients.

A basic example is the current $[A]$ of integration over the regular points A_{reg} of an algebraic hypersurface A :

$$\omega \mapsto \int_{A_{\text{reg}}} \omega,$$

where ω is a test $(k-1, k-1)$ -form. The current of integration over a divisor D is defined by extending linearly.

The space of currents is given the distributional topology: $T_n \rightarrow T$ if $T_n(\omega) \rightarrow T(\omega)$ for every test form ω .

A differential $(k-1, k-1)$ -form ω is called positive if its integral over any complex subvariety is non-negative. A $(1, 1)$ -current T is called *positive* if $T(\omega) \geq 0$ for any positive $(1, 1)$ -form. A current T is called *closed* if $dT = 0$, where the differential d is understood in the distributional sense.

In this paper, we focus on closed, positive $(1, 1)$ -currents. They have a simple description in terms of local potentials, rather analogous to the definition of divisors.

Recall that ∂ and $\bar{\partial}$ stand for the holomorphic and anti-holomorphic parts of the external differential $d = \partial + \bar{\partial}$. Their composition $\frac{i}{\pi} \partial \bar{\partial}$ can be¹³ thought of as a kind of “pluri-Laplacian” because, given a C^2 -function h , the restriction of $\frac{i}{\pi} \partial \bar{\partial} h$ to any non-singular complex curve X is equal to the form $\frac{1}{2\pi} \Delta(h|X) dz \wedge d\bar{z}$, where z is a local coordinate on X and Δ is the usual Laplacian in this coordinate.

If U is an open subset of \mathbb{C}^k and $h : U \rightarrow [-\infty, \infty)$ is a plurisubharmonic (PSH) function, then $\frac{i}{\pi} \partial \bar{\partial} h$ is a closed $(1,1)$ -current on U . Conversely, the $\partial \bar{\partial}$ -Poincaré Lemma asserts that every closed, positive $(1,1)$ -current on U is obtained this way.

Therefore, any closed positive $(1,1)$ -current T on a manifold V can be described using an open cover $\{U_i\}$ of V together with PSH functions $h_i : U_i \rightarrow [-\infty, \infty)$ that are chosen so that $T = \frac{i}{\pi} \partial \bar{\partial} h_i$ in each U_i . The functions h_i are called *local potentials* for T and they are required to satisfy the compatibility condition that $h_i - h_j$ is pluriharmonic (PH) on any non-empty intersection $U_i \cap U_j \neq \emptyset$. The *support* of T is defined by:

$$\text{supp } T := \{z \in V : \text{if } z \in U_j \text{ then } h_j \text{ is not pluriharmonic at } z\}.$$

The compatibility condition assures that that above set is well-defined.

The Poincaré–Lelong formula describes the current of integration over a divisor $D = \{U_i, g_i\}$ by the system of local potentials $h_i := \log |g_i|$. I.e., on each U_i we have $[D] = \frac{i}{\pi} \partial \bar{\partial} \log |g_i|$. The result is a closed $(1,1)$ -current, which is positive iff D is effective (i.e. the multiplicities k_1, \dots, k_r are non-negative).

Suppose $R : V \rightarrow W$ is a dominant rational map and T is a closed-positive $(1,1)$ -current on W . The pullback R^*T is closed positive $(1,1)$ -current on V , defined as follows. First, one obtains a closed positive $(1,1)$ -current R^*T defined on $V \setminus I(R)$ by pulling-back the system of local potentials defining T under $R : V \setminus I(R) \rightarrow W$. One then extends R^*T trivially through $I(R)$, to obtain a closed, positive $(1,1)$ -current defined on all of V . (By a result of Harvey and Polking [31], this extension is closed.) See [47, Appendix A.7] for further details. Pullback is continuous with respect to the distributional topology.

Similarly to divisors, there is a particularly convenient description of closed, positive $(1,1)$ -currents on \mathbb{CP}^k . Associated to any PSH function $H : \mathbb{CP}^{k+1} \rightarrow [-\infty, \infty)$, having the homogeneity

$$H(\lambda \hat{z}) = m \log |\lambda| + H(\hat{z}), \quad (\text{A.6})$$

for some $m > 0$, is a closed, positive $(1,1)$ -current, denoted by $T_H \equiv \pi_* \left(\frac{i}{\pi} \partial \bar{\partial} H \right)$ that is defined as follows. For any open covering $\{U_i\}$ of \mathbb{CP}^k admitting local sections $s_i : U_i \rightarrow \mathbb{CP}^{k+1} \setminus \{0\}$ of the canonical projection π , T_H is defined by the system of local potentials $\{U_i, H \circ s_i\}$. i.e., in each U_i , T_H is defined by $T_H = \frac{i}{\pi} \partial \bar{\partial} H \circ s_i$. Moreover, every closed positive $(1,1)$ -current on \mathbb{CP}^k is described in this way; See [47, Thm. A.5.1]. The function H is called the *pluripotential* of T_H .

¹³ Many authors introduce real operators $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{2\pi} (\bar{\partial} - \partial)$ and write dd^c instead of $\frac{i}{\pi} \partial \bar{\partial}$. We use $\frac{i}{\pi} \partial \bar{\partial}$ to avoid confusion between the operator d and the algebraic degree of a map.

The *mass* of the current T is defined to be

$$\|T\| = \int_{\mathbb{CP}^k} T \wedge \omega_{\text{FS}}^{k-1},$$

where ω_{FS} is the Fubini–Study $(1, 1)$ form on \mathbb{CP}^k and $T \wedge \omega_{\text{FS}}^{k-1}$ is the positive measure defined by $\langle T \wedge \omega_{\text{FS}}^{k-1}, f \rangle := \langle T, f \omega_{\text{FS}}^{k-1} \rangle$. A calculation shows that the mass of T_H can be computed from the potential H as $\|T_H\| = m$, where m is the constant from (A.6).

If $R : \mathbb{CP}^k \rightarrow \mathbb{CP}^k$ is a rational map, the action of pull-back is described by

$$R^* T_H = T_{H \circ \hat{R}}. \quad (\text{A.7})$$

If R has algebraic degree d , then the lift \hat{R} is a $k+1$ -tuple of homogeneous polynomials of degree d . It follows that the mass satisfies

$$\|R^* T_H\| = d \|T_H\|.$$

A.4: Kobayashi Hyperbolicity and Normal Families

In Sect. 4 we use the Kobayashi metric to prove that the iterates R^n form a normal family on certain subspaces of \mathbb{CP}^2 . Here, we recall the relevant definitions and some important results that we use. The reader can consult the books [35, 37] and the original papers by Green [25, 26] for more details. For more dynamical applications, see e.g. [16, 47].

The Kobayashi pseudometric is a natural generalization of the Poincaré metric on Riemann surfaces. Let $\|\cdot\|$ stand for the Poincaré metric on the unit disk \mathbb{D} . Let M by a complex manifold. Pick a tangent vector $\xi \in TM$, and let $\mathcal{H}(\xi)$ be the family of holomorphic curves $\gamma : \mathbb{D} \rightarrow M$ tangent to the line $\mathbb{C} \cdot \xi$ at $\gamma(0)$. Then $Df(v) = \xi$ for some $v \equiv v_\gamma \in T_0 \mathbb{D}$, and the Kobayashi pseudometric is defined to be:

$$ds_M(\xi) = \inf_{\gamma \in \mathcal{H}(\xi)} \|v_\gamma\|. \quad (\text{A.8})$$

The Kobayashi pseudometric is designed so that holomorphic maps are distance decreasing: if $f : U \rightarrow M$ is holomorphic then $ds_M(Df(\xi)) \leq ds_U(\xi)$.

The reason for “pseudo-” is that for certain complex manifolds M , $ds(\xi)$ can vanish for some non-vanishing tangent vectors $\xi \neq 0$. For example, ds identically vanishes on \mathbb{C}^k or \mathbb{CP}^k . A complex manifold M is called *Kobayashi hyperbolic* if ds is non-degenerate: $ds(\xi) > 0$ for any non-vanishing $\xi \in TM$. Then, it induces a (Finsler) metric on M .

Let N be a compact complex manifold. Endow it with some Hermitian metric $|\cdot|_N$. A complex submanifold $M \subset N$ is called *hyperbolically embedded* in N if the Kobayashi pseudometric on M dominates the Hermitian metric on N , i.e., there exists $c > 0$ such that $ds_M(\xi) \geq c |\xi|_N$ for all $\xi \in TM$. Obviously, M is Kobayashi hyperbolic in this case.

A complex manifold M is called *Brody hyperbolic* if there are no non-constant holomorphic mappings $f : \mathbb{C} \rightarrow M$. If M is Kobayashi hyperbolic, it is also Brody hyperbolic, but the converse is generally not true unless M is compact.

An open subset of \mathbb{CP}^2 that is Kobayashi hyperbolic, but not hyperbolically embedded in \mathbb{CP}^2 is described in [35, Example 3.3.11] and an open subset of \mathbb{C}^2 that is Brody hyperbolic but not Kobayashi hyperbolic is described in [35, Example 3.6.6].

A family \mathcal{F} of holomorphic mappings from a complex manifold U to a complex manifold M is called *normal* if every sequence in \mathcal{F} either has a subsequence converging locally uniformly or a subsequence that diverges locally uniformly to infinity in M . In the case that M is embedded into some compact manifold Z , a stronger condition is that \mathcal{F} is *precompact* in $\text{Hol}(U, Z)$ (where $\text{Hol}(U, Z)$ is the space of holomorphic mappings $U \rightarrow Z$ endowed with topology of uniform convergence on compact subsets of U).

Proposition A.5 *Let M be a hyperbolically embedded complex submanifold of a compact complex manifold N . Then, for any complex manifold U , the family $\text{Hol}(U, M)$ is precompact in $\text{Hol}(U, N)$.*

See Theorem 5.1.11 from [35].

The classical Montel's Theorem asserts that the family of holomorphic maps $\mathbb{D} \rightarrow \mathbb{C} \setminus \{0, 1\}$ is normal (as $\mathbb{C} \setminus \{0, 1\}$ is a hyperbolic Riemann surface). It is a foundation for the whole Fatou-Julia iteration theory. Several higher dimensional versions of Montel's Theorem, due to M. Green [25, 26], are now available. Though their role in dynamics is not yet so prominent, they have found a number of interesting applications. Below we will formulate two particular results used in this paper (see Sect. 4). The following is Theorem 2 from [25]:

Theorem A.6 *Let X be a union of (possibly singular) hypersurfaces X_1, \dots, X_m in a compact complex manifold N . Assume $N \setminus X$ is Brody hyperbolic and*

$$X_{i_1} \cap \dots \cap X_{i_k} \setminus (X_{j_1} \cup \dots \cup X_{j_l}) \text{ is Brody hyperbolic}$$

for any choice of distinct multi-indices $\{i_1, \dots, i_k, j_1, \dots, j_l\} = \{1, \dots, m\}$. Then $N \setminus X$ is a complete hyperbolically embedded submanifold of N .

In the last section of [25], the following result is proved:

Theorem A.7 *Let $M = \mathbb{CP}^2 \setminus (Q \cup X_1 \cup X_2 \cup X_3)$, where Q is a non-singular conic and X_1, X_2, X_3 are lines. Then any non-constant holomorphic curve $f : \mathbb{C} \rightarrow M$ must lie in a line L that is tangent to Q at an intersection point with one of the lines, X_i , and that contains the intersection point $X_j \cap X_l$ of the other two lines.*

The configurations that appear in this theorem are related to amusing projective triangles:

A.5: Self-Dual Triangles

Let $Q(z) = \sum q_{ij}z_i z_j$ be a non-degenerate quadratic form in $E \approx \mathbb{C}^3$, and $X = \{Q = 0\}$ be the corresponding conic in \mathbb{CP}^2 . The form Q makes the space E Euclidean,

inducing duality between points and lines in \mathbb{CP}^2 . Namely, to a point $z = (z_0 : z_1 : z_2)$ corresponds the line $L_z = \{\zeta : Q(z, \zeta) = 0\}$ called the *polar* of z with respect to X (here we use the same notation for the quadratic form and the corresponding inner product). Geometrically, this duality looks as follows. Given a point $z \in \mathbb{CP}^2$, there are two tangent lines from z to X . Then L_z is the line passing through the corresponding tangency points. (In case $z \in X$, the polar is tangent to X at z).

Three points z_i in \mathbb{CP}^2 in general position are called a “triangle” Δ with vertices z_i . Equivalently, a triangle can be given by three lines L_i in general position, its “sides”. Let us say that Δ is *self-dual* (with respect to the conic X) if its vertices are dual to the opposite sides.

Lemma A.8 *A triangle Δ with vertices z_i is self-dual if and only if the corresponding vectors $\hat{z}_i \in E$ form an orthogonal basis with respect to the inner product Q .*

All three sides of a self-dual triangle satisfy the condition of Theorem A.7, so they can give us exceptional holomorphic curves $\mathbb{C} \rightarrow \mathbb{CP}^2 \setminus (Q \cup X_1 \cup X_2 \cup X_3)$.

Appendix B: Green Current

B.1: Green Potential

A rational mapping $R : \mathbb{CP}^m \rightarrow \mathbb{CP}^m$ is called *algebraically stable* if there is no integer n and no collapsing hypersurface $V \subset \mathbb{CP}^m$ so that $R^n(V)$ is contained within the indeterminacy set of R , [47, p. 109]. (See also [4, Appendix A4 and A5].)

Theorem B.1 (see [47], Thm. 1.6.1). *Let $R : \mathbb{CP}^m \rightarrow \mathbb{CP}^m$ be an algebraically stable rational map of degree d . Then the limit*

$$G = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \|\hat{R}^n\|$$

exists in $L^1_{\text{loc}}(\mathbb{C}^3)$ and determines a plurisubharmonic function. This function satisfies the following equivariance properties:

$$\begin{aligned} G(\lambda z) &= G(z) + \log |\lambda|, \quad \lambda \in \mathbb{C}^*, \\ G \circ \hat{R} &= dG. \end{aligned} \tag{B.1}$$

It is called the *Green potential* of R .

B.2: Green Current

Applying $\frac{i}{\pi} \partial \bar{\partial}$ to the Green potential, we obtain:

Theorem B.2 (see [47], Thm. 1.6.1). *Let $R : \mathbb{CP}^m \rightarrow \mathbb{CP}^m$ be an algebraically stable rational map of degree d . Then $S = \pi_*(\frac{i}{\pi} \partial \bar{\partial} G)$ is a closed positive $(1,1)$ -current on \mathbb{CP}^m satisfying the equivariance relation: $R^*S = d \cdot S$.*

The current S is called the *Green current* of R .

The set of *normal*¹⁴ points for an algebraically stable rational map $R : \mathbb{CP}^m \rightarrow \mathbb{CP}^m$ is:

$$N := \left\{ x \in \mathbb{CP}^m : \begin{array}{l} \text{there exists neighborhoods } U \text{ of } x \text{ and } V \text{ of } I(R) \\ \text{so that } f^n(U) \cap V = \emptyset \text{ for every } n \in \mathbb{N}. \end{array} \right\}$$

The normal points form an open subset of \mathbb{CP}^m .

One primary interest in the Green current S is the following connection between its support $\text{supp } S$ and the Julia set J_R . (See Sect. 4 for the definitions of the Fatou and Julia sets.) Note that $\text{supp } S$ is closed and backwards invariant, $R^{-1} \text{supp } S \subset \text{supp } S$, since $R^* S = d \cdot S$.

Theorem B.3 (See [47], Thm. 1.6.5) *Let $f : \mathbb{CP}^m \rightarrow \mathbb{CP}^m$ be an algebraically stable rational map. Then:*

$$J_R \cap N \subset \text{supp } S \subset J_R.$$

An algebraically stable rational map $R : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ for which $\text{supp } S \subsetneq J_R$ is given in [22, Example 2.1].

Appendix C: Open Problems

Problem C.1 (Existence of Fisher and Lee–Yang–Fisher distributions) *For which classical lattices (\mathbb{Z}^d for $d \geq 2$, etc) does the limit (2.5) exist? As explained in Proposition 2.2, this would justify existence of the limiting distributions of Lee–Yang–Fisher zeros for these lattices.*

Similarly, for which classical three and higher-dimensional lattices (\mathbb{Z}^d for $d \geq 3$, etc) does the limit (2.5) exist? It would justify the existence of a limiting distribution of Fisher zeros for these lattices.

Problem C.2 (Geometric properties of the Lee–Yang–Fisher current) *The theory of geometric currents has become increasingly useful in complex dynamics, see [1, 11, 12, 17, 18, 44] as a sample.*

The Green current S is strongly laminar in a neighborhood of B . The structure is given by the stable lamination of B (see Sect. 3.3) together with transverse measure obtained under holomorphy from the Lebesgue measure on B . However, S is not strongly laminar in a neighborhood of the topless Lee–Yang “cylinder” C_1 .

One can see this also follows: a disc within the invariant line L_{inv} centered at $L_{\text{inv}} \cap B$ is within the stable lamination of B . Therefore, an open neighborhood within L_{inv} of $L_{\text{inv}} \cap C_1$ would have to be a leaf of the lamination. However, S restricts to L_{inv} in a highly non-trivial way, coinciding with the measure of maximal entropy for $R|L_{\text{inv}}$. (It is supported on the Julia set shown in Fig. 3).

Does S have a weaker geometric structure? For example, is it non-uniformly laminar [1, 18] or woven [11, 12, 17]?

¹⁴ Not to be confused with the notion of normal families.

Problem C.3 (Support for the measure of maximal entropy) *What can be said about the support of the measure of maximal entropy ν that was discussed in Sect. 4.3? Is the critical fixed point $b_c \in L_{\text{inv}}$ within $\text{supp } \nu$? A positive answer to this question is actually equivalent to $C \cap \text{supp } \nu \neq \emptyset$ and also to $\text{supp } \nu \cap L_{\text{inv}} = J_{R|L_{\text{inv}}}$.*

Problem C.4 (Fatou Set) *In Theorem 4.9 we showed that certain “solid cylinders” are in $\mathcal{W}^s(e)$ and $\mathcal{W}^s(e')$. Computer experiments suggest a much stronger result:*

Conjecture $\mathcal{W}^s(e) \cup \mathcal{W}^s(e')$ *is the entire Fatou set for R .*

Problem C.5 (Julia Set) *Proposition 4.2 gives that in a neighborhood of B , J_R is a C^∞ 3-manifold. What can be said about the global topology of J_R ?*

Remark C.6 Note that each of the above Problems C.2–C.5 has a natural counterpart for \mathcal{R} .

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