

Feedback control for fluid mixing via advection

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Abstract

This work is concerned with nonlinear feedback control design for the problem of fluid mixing via advection. The overall dynamics is governed by the transport and Stokes equations in an open bounded and connected domain $\Omega \subset \mathbb{R}^d$, with $d = 2$ or $d = 3$. The feedback laws are constructed based on the ideas of instantaneous control as well as a direct approximation of the optimality system derived from an optimal open-loop control problem. It can be shown that under appropriate numerical discretization schemes, two approaches generate the same sub-optimal feedback law. On the other hand, different discretization schemes may result in feedback laws of different regularity, which determine different mixing results. The Sobolev norm of the dual space $(H^1(\Omega))'$ of $H^1(\Omega)$ is used as the mix-norm to quantify mixing based on the known property of weak convergence. The major challenge is encountered in the analysis of the asymptotic behavior of the closed-loop systems due to the absence of diffusion in the transport equation together with its nonlinear coupling with the flow equations. To address these issues, we first establish the decay properties of the velocity, which in turn help obtain the estimates on scalar mixing and its long-time behavior. Finally, mixed continuous Galerkin (CG) and discontinuous Galerkin (DG) methods are employed to discretize the closed-loop system. Numerical experiments are conducted to demonstrate our ideas and compare the effectiveness of different feedback laws.

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1. Introduction

The processes of mass transport and mixing of fluid flows comprise fundamental, yet highly challenging problems in engineering and natural sciences. These topics have broad applications ranging from industrial and chemical mixing on small and large scales, to preventing the spreading of pollutants in geophysical flows. The question of what fluid flow maximizes mixing rate, slows it down, or even steers a quantity of interest towards a desired target distribution draws great attention from a broad range of scientists and engineers. The current work is aimed at achieving effective fluid mixing via an internal (distributed) control of the flow dynamics governed by the incompressible Stokes equations. The transport equation is used to describe the scalar field, where diffusion is negligible. In this case, transport and mixing are purely driven by flow advection. This naturally leads to a nonlinear control and optimization problem of a semi-dissipative system (cf. [8]), which presents new and significant challenges arising from both analysis and computation.

Our recent work in [28,29,31,34] has applied optimal control design for fluid mixing through Navier slip boundary controls. Numerical schemes and experiments have been formulated and conducted in [32] via mixed continuous Galerkin (CG) and discontinuous Galerkin (DG) methods together with gradient descent. These tools are specific to the treatment of the parabolic and hyperbolic features of the semi-dissipative systems. However, as it is well-known, to solve the resulting optimality system, one has to solve the governing system forward in time, coupled with the adjoint system backward in time together with a nonlinear optimality condition. Furthermore, the small-scale structures and large gradients of the scalar field will develop during the mixing process. The mesh size must be sufficiently refined to capture the smallest spatial scales of the thin filaments that arise in the evolutions of the scalar distribution. Straightforward use of this theory can result in extremely to prohibitively high computational costs. The objective of the current work is to establish nonlinear feedback controls and the corresponding feasible computational methods for scalar transport and mixing. Such feedback laws are aimed at achieving a balance between the efficiency in real-time implementation and the accuracy in steering the system behavior.

Constructing an optimal feedback law for a nonlinear system and a useful approximation thereof are highly challenging problems. One of the well-known approaches is to solve the related Hamilton-Jacobi-Bellman (HJB) equation, however, it suffers from the curse of dimensionality. In this work, we consider a feasible synthesis of nonlinear state feedback control via interpolation of the optimality system resulting from the optimal open-loop control problem. This idea is motivated by the method of instantaneous control design and essentially leads to a sub-optimal feedback law. Instantaneous control design is a powerful tool for dealing with the computational limitations of open-loop control and provides a feedback law for flow control problems at a sustainable control cost [14,27,24,11,13,54]. The idea behind it is that it successively determines approximations of the objective functional while marching forward in time. The uncontrolled dynamical system is first discretized in time. Then, at the selected time slice an instantaneous version of the cost functional is approximately minimized subject to a stationary system, whose structure depends on the chosen discretization method. The control so obtained is used to steer the system to the next time slice, where the procedure is repeated [27]. This method is closely tied to

receding horizon control (RHC) or model predictive control (MPC) with finite time horizon [21, 45, 47, 7, 38, 4, 22]. Essentially, instantaneous control is a discrete-in-time and suboptimal feedback control approach as shown in [12, 27, 24], and can be interpreted as the stable time discretization of a closed-loop control law [12, 40, 27, 24, 44, 25].

Alternatively, if the optimality system from an optimal open-loop control problem is known, one may derive the feedback laws by first discretizing the optimality system in time and solving it at the selected time slice, and then marching forward in time successively. In the following, we will call the former “discretize-then-optimize (DTO)” or direct approach and the latter “optimize-then-discretize (OTD)” or indirect approach. It is worth stressing that the OTD approach involves directly discretizing the optimality system over one step time sub-interval, and then carry the information to the next one. Then the state and the adjoint equations are solved forward and backward in time, respectively, but just for one time step, so that the adjoint state can be related to the state explicitly. Consequently, this procedure leads to a sub-optimal continuous feedback law. Establishing the well-posedness and asymptotic behavior of the nonlinear closed-loop is a second focus of this work.

On the other hand, it can be shown that different time discretization schemes may result in feedback laws of different regularity, and hence affect the regularity of flow velocity and the mixing decay rate in time. Understanding the relation between the regularity of the velocity and mixing decay rate in time is a prominent problem in the study of mathematical fluid mixing (cf. [2, 9, 16, 19, 23, 35, 42, 48, 58]). Loosely speaking, a less regular velocity field may lead to a faster mixing decay rate. Analyzing the well-posedness and asymptotic behavior of transport and mixing via feedback control of the flows of different regularity is at the core of this work. Finally, numerical experiments will be conducted with the aim of comparing the mixing decay rates in time using different feedback control designs.

The remainder of this paper is organized as follows. In section 2, the mathematical model and control designs are introduced for fluid mixing via Stokes flows. In section 3, the internal feedback control is constructed using both DTO and OTD approaches, which lead to the same feedback law under appropriate discretization schemes. The well-posedness and asymptotic behavior of the closed-loop system will be addressed in section 4. Numerical implementation of our control designs is presented in section 5, where the finite element formulation and nonlinear iterative solvers are used to construct our numerical schemes. Several numerical experiments are conducted to demonstrate and compare the performances of different feedback laws for fluid mixing.

2. Mathematical model and control designs

Consider a passive scalar field governed by a transport equation that is advected by a controlled incompressible Stokes flow in an open bounded and connected domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with a smooth boundary $\Gamma := \partial\Omega$. The passive scalar has no dynamical effect on the fluid motion itself. As a result of the one-way coupling between the scalar field and the velocity field, investigating the optimal control design for the flow-transport system is tied to the study of control problems for the Stokes equations. The current work mainly addresses an internal feedback control design for mixing.

2.1. Internal control for fluid mixing via Stokes flows

In the absence of diffusion, transport and mixing occur due to pure advection. Applying an internal control means steering the advection by supplying energy to the system in the interior of the flow domain. For example, stirring a fluid back and forth can generate fluctuating velocities with respect to the flow barriers leading to transport across them and achieving a better mixing [55,56]. In order to formulate an internal control problem, let $\omega \subset \Omega$ be a subdomain with a smooth boundary $\partial\omega$. The controlled system in this case is given by

$$\begin{cases} \partial_t \theta + v \cdot \nabla \theta = 0, \\ \partial_t v = \nu \Delta v - \nabla p + m_\omega u, \quad \nabla \cdot v = 0, \\ v \cdot n|_\Gamma = 0 \quad \text{and} \quad (2\nu n \cdot \mathbb{D}(v) \cdot \tau + \kappa v \cdot \tau)|_\Gamma = 0, \end{cases} \quad (\text{M})$$

with the initial condition $(v(0), \theta(0)) = (v_0, \theta_0)$, and where θ is the mass distribution or scalar concentration, v is the velocity of the flow, $\nu > 0$ is the viscosity, p is the pressure, m_ω is a smooth function with compact support at ω , and u represents the internal control input. Moreover, n and τ denote the outward unit normal and tangential vectors with respect to the domain boundary Γ , respectively, and $\mathbb{D}(v) = (1/2)(\nabla v + (\nabla v)^T)$ is the deformation tensor. The friction between the fluid and the wall is proportional to $-v$ with the positive coefficient κ .

Due to incompressibility and no-penetration boundary condition, i.e. $v \cdot n|_\Gamma = 0$, it can be easily verified that any L^p -norm of θ in Ω is conserved, i.e.,

$$\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p}, \quad t \geq 0, \quad p \in [1, \infty]. \quad (2.1)$$

To quantify mixing, a classical measure is the spatial variance of the scalar concentration [17], which is related to the L^2 -norm of the scalar field. However, this measurement fails in the case of zero diffusivity since it is unable to quantify pure stirring effects: From (2.1) we know that every L^p -norm of θ is invariant with respect to time. Recently, the mix-norm and negative Sobolev norms have been adopted to quantify this process based on ergodic theory, which are sensitive to both stirring and diffusion [43,52,51]. In [43] Mathew et al. first showed the equivalence of the mix-norm to the $H^{-1/2}$ -norm on a d -dimensional torus $\mathbb{T}^d$. In fact, any negative Sobolev norm H^{-s} , for $s > 0$, can be used as a mix-norm; see [51]. Since a general open and bounded domain will be considered in this paper, the negative Sobolev norm is replaced by the norm of the dual space $(H^s(\Omega))'$ of $H^s(\Omega)$ with $s > 0$, as in [28,29]. Without loss of generality, we use $(H^1(\Omega))'$, following [28,29,31,34], to quantify mixing in this work.

Throughout this paper, we use $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ for the L^2 -inner product in the domain Ω and on the boundary Γ , respectively. For the convenience of our discussion, define

$$\begin{aligned} V_n^s(\Omega) &= \{v \in H^s(\Omega) : \operatorname{div} v = 0, \ v \cdot n|_\Gamma = 0\}, \quad s \geq 0, \\ V_n^s(\Gamma) &= \{g \in H^s(\Gamma) : g \cdot n|_\Gamma = 0\}, \quad s \geq 0, \end{aligned}$$

where $H^0(\Omega) = L^2(\Omega)$, and $H^0(\Gamma) = L^2(\Gamma)$.

In the sequel, the symbol C denotes a generic positive constant, which is allowed to depend on the domain as well as on indicated parameters without ambiguity.

2.2. Preliminary

To proceed, we first rewrite the model (M) as an abstract Cauchy problem

$$\begin{cases} \partial_t \theta = -v \cdot \nabla \theta, \\ \partial_t v = Av + Bu, \end{cases} \quad (2.2)$$

with $(v(0), \theta(0)) = (v_0, \theta_0)$, where $A = \mathbb{P} \Delta$ with domain

$$\mathcal{D}(A) = \{v \in V_n^2(\Omega) : (2n \cdot \mathbb{D}(v) \cdot \tau + \kappa v \cdot \tau)|_\Gamma = 0\}$$

is the Stokes operator associated with the Navier slip boundary conditions, and $\mathbb{P} : L^2(\Omega) \rightarrow V_n^0(\Omega)$ is the Leray projector (cf. [49, p. 18, Remark 1.6], [20, p. 37–38]). Note that A is self-adjoint, strictly negative, and

$$(Av, \phi) = -2(\mathbb{D}(v), \mathbb{D}(\phi)) - \frac{\kappa}{v} \langle v, \phi \rangle, \quad \forall v, \phi \in V_n^1(\Omega).$$

Here B is the control input operator, which is defined by how control is introduced to the system. Stirring the fluids at the interior subdomain $\omega \subset \Omega$ leads to an internal (distributed) control. In this case, let $U_{ad} = L^2(0, T; L^2(\omega))$ and

$$B = \mathbb{P} m_\omega : U_{ad} \rightarrow L^2(0, T; V_n^0(\Omega)).$$

Moreover, the L^2 -adjoint operator of B is given by

$$B^* = m_\omega \mathbb{P} : L^2(0, T; V_n^0(\Omega)) \rightarrow U_{ad}. \quad (2.3)$$

As a first step to construct the feedback control law, we consider the following optimal control problem for the flow-transport system (2.2): For a given $T > 0$, find a control u minimizing the cost functional

$$J(u, \theta) = \frac{\alpha}{2} \|\theta(T)\|_{(H^1(\Omega))'}^2 + \frac{\beta}{2} \int_0^T \|\theta\|_{(H^1(\Omega))'}^2 dt + \frac{\gamma}{2} \|u\|_{U_{ad}}^2, \quad (2.4)$$

subject to (2.2), where $\alpha, \beta \geq 0$ are the state weight parameters that do not vanish simultaneously, $\gamma > 0$ is the control weight parameter, and U_{ad} is the set of admissible controls. The choice of U_{ad} is often determined based on the physical properties as well as the need to guarantee the existence of an optimal solution. To compute the dual norm $(H^1(\Omega))'$, we introduce a higher regularity counterpart of θ , denoted as η , and define it by

$$(-\Delta + I)\eta = \theta \quad \text{in } \Omega, \quad \frac{\partial \eta}{\partial n} = 0 \quad \text{on } \Gamma. \quad (2.5)$$

Then

$$\|\eta\|_{W^{2,p}} \leq C(\Omega, p) \|\theta\|_{L^p}, \quad (2.6)$$

for $1 < p < \infty$. Let $\mathcal{A} = -\Delta + I$ with domain $\mathcal{D}(\mathcal{A}) = \{\eta \in H^2(\Omega) : \frac{\partial \eta}{\partial n} = 0\}$, and $\Lambda = \mathcal{A}^{1/2}$. Then $\|\theta\|_{(H^1(\Omega))'} = \|\Lambda^{-1}\theta\|_{L^2} = \|\Lambda\eta\|_{L^2}$ and the cost functional (2.4) can be rewritten as

$$J(u, \theta) = \frac{\alpha}{2} \|\Lambda^{-1}\theta(T)\|_{L^2}^2 + \frac{\beta}{2} \int_0^T \|\Lambda^{-1}\theta\|_{L^2}^2 dt + \frac{\gamma}{2} \|u\|_{U_{\text{ad}}}^2. \quad (2.7)$$

Note that controlling of the velocity field gives rise to a bilinear control problem of the scalar equation due to the coupling via the advective term $v \cdot \nabla \theta$. As a result, the optimal control problem (2.7) subject to (2.2) becomes non-convex, and the uniqueness of the optimal solution may not hold in general. Given $\theta_0 \in L^\infty(\Omega)$, the existence of an optimal solution to (2.7) can be obtained following the same procedure as in [29, Theorem 3.2] for $v_0 \in V_n^0(\Omega)$ and $u \in L^2(0, T; L^2(\omega))$. However, the challenge arises in deriving the first order necessary conditions of optimality. As shown in [28, 29], establishing the Gâteaux differentiability of the control to state map $u \mapsto (\theta, v)$ leads to the major obstruction in the design of boundary control. In this case, one needs $\sup_{t \in [0, T]} \|\nabla \theta\|_{L^2} < \infty$, which in turn requires $\theta_0 \in H^1(\Omega)$ and the flow velocity to satisfy

$$\int_0^T \|\nabla v\|_{L^\infty} dt < \infty. \quad (2.8)$$

Consequently, the initial conditions v_0 and U_{ad} have to be chosen such that this *a priori* estimate is satisfied, which presents the major obstruction compared to the full dissipative system (with nonzero diffusivity in the scalar equation). To lower the regularity requirement on the velocity, an approximating control design was used in [28] by first introducing a small diffusivity $\varepsilon \Delta \theta$, with $\varepsilon > 0$, to the transport equation associated with an appropriate boundary condition. Then, we established a rigorous analysis of convergence of the approximating control problem to the original one as the diffusivity approaches zero. Employing the same approach, we can derive the first-order optimality system for solving our current problem (2.7) with the internal control by setting $\theta_0 \in L^\infty(\Omega) \cap H^1(\Omega)$, $v_0 \in V_n^{d/2-1+\epsilon}(\Omega)$, $d = 2, 3$, for any $\epsilon > 0$, and $u \in U_{\text{ad}} = L^2(0, T; L^2(\omega))$. Given γ sufficiently large, and following the same procedure as in [28, Theorem 5.1], we can also obtain the uniqueness of the optimal solution for $d = 2$.

With these results at our disposal, in the following lemma we provide a formal derivation of the first-order optimality conditions using the Euler-Lagrange method (the so-called “Pontryagin maximum principle”; cf. [26, p. 63, 1.6.4], [6, p. 242, 4.1.3] and [53, p. 84, 2.10]). Again, the rigorous proof via an approximating approach can be found in [28, Theorems 5.2-5.5]. In the lemma we make use of the following: for a given $u \in U_{\text{ad}}$, let (v, θ) be the solution of the state equations (2.2), then the associated adjoint state (ρ, w) is defined via the adjoint equations

$$\begin{cases} -\partial_t \rho = v \cdot \nabla \rho + \beta \Lambda^{-2} \theta, \\ -\partial_t w = A w + \mathbb{P}(\theta \nabla \rho), \end{cases} \quad (2.9)$$

with final time condition

$$(\rho(T), w(T)) = (\alpha \Lambda^{-2} \theta(T), 0). \quad (2.10)$$

Lemma 2.1. Assume $(\theta_0, v_0) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times V_n^1(\Omega)$. If $\hat{u}$ is an optimal control to our problem of interest, that is, $(\hat{u}, \hat{\theta}, \hat{v})$ solves $\min J(u, \theta)$ subject to (2.2), then it satisfies the optimality condition

$$\hat{u} = -\frac{1}{\gamma} B^* \hat{w}, \quad (2.11)$$

where B^* is given by (2.3) and $\hat{w}$ is determined by $(\hat{\rho}, \hat{w})$, the associated adjoint state to $(\hat{v}, \hat{\theta})$ satisfying (2.9)–(2.10). Moreover,

$$(\hat{\theta}, \hat{v}) \in L^\infty(0, T; L^\infty(\Omega) \cap H^1(\Omega)) \times C([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \quad (2.12)$$

$$(\hat{\rho}, \hat{w}) \in L^\infty(0, T; L^\infty(\Omega) \cap H^1(\Omega)) \times C([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \quad (2.13)$$

and $\hat{u} \in C([0, T]; H^1(\omega)) \cap L^2(0, T; H^2(\omega))$.

A sketch of the derivation of the optimality conditions is given below. The basic idea is to convert the constrained optimization problem (2.4) to an unconstrained one (cf. [26, p. 63, 1.6.4], [6, p. 242, 4.1.3]). Define the Lagrangian

$$\mathcal{L}(\theta, v, u; w, \rho) = J(u, \theta) - \int_0^T (\rho, \partial_t \theta + v \cdot \nabla \theta) dt - \int_0^T (w, \partial_t v - (Av + Bu)) dt, \quad (2.14)$$

where by Stokes formula we have

$$(\rho, v \cdot \nabla \theta) = (\rho, \nabla \cdot (v\theta)) = \int_\Gamma \rho((v\theta) \cdot n) dx - \int_\Omega \nabla \rho \cdot (v\theta) dx = -(v \cdot \nabla \rho, \theta). \quad (2.15)$$

Applying integration by parts to the last two terms of the right hand side of (2.14) together with (2.15) yields

$$\begin{aligned} \mathcal{L}(\theta, v, u; w, \rho) = & J(u, \theta) - [(\rho(T), \theta(T)) - (\rho_0, \theta_0) + \int_0^T (-\partial_t \rho - v \cdot \nabla \rho, \theta) dt] \\ & - [(w(T), v(T)) - (w_0, v_0) + \int_0^T (-\partial_t w - Aw, v) - (B^* w, u) dt]. \end{aligned}$$

Moreover, due to the divergence-free condition of velocity, we have

$$(v \cdot \nabla \rho, \theta) = (v, \theta \nabla \rho) = (v, \mathbb{P}(\theta \nabla \rho)). \quad (2.16)$$

The adjoint state (ρ, w) is chosen such that the first derivatives of $\mathcal{L}$ with respect to θ and v vanish, i.e. $\frac{\partial \mathcal{L}}{\partial \theta} = 0$ and $\frac{\partial \mathcal{L}}{\partial v} = 0$, which lead to (2.9). The final time condition (2.10) is derived by setting $\frac{\partial \mathcal{L}}{\partial \theta(T)} = 0$. Finally, the equation $\frac{\partial \mathcal{L}}{\partial u} = 0$ yields the desired optimality condition (2.11).

The detailed explanation on a Lagrangian-based view of the adjoint approach can be found in [26, p. 63, 1.6.4]. The optimality system (2.9)–(2.11) can be also obtained using a variational inequality as in [29, Theorem 4.1] and [28, Theorem 5.3].

Remark 2.2. The well-posedness and regularity of the solution $(\hat{\theta}, \hat{v})$ to (2.2) for $(\theta_0, v_0) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times V_n^1(\Omega)$ and any $B\hat{u} \in L^2(0, T; V_n^0(\Omega))$ can be established following similar approaches as in [36, Theorem 4.1] and [30, Theorem 1.1] for the Boussinesq system with zero diffusivity. In fact, since we consider Stokes flow in the current work, the analysis is less involved compared to the Navier-Stokes case addressed in the references. Similarly we could claim (2.13) for the adjoint problem. The regularity of the optimal control $\hat{u}$ holds immediately as $B^*: L^2(0, T; V_n^0(\Omega)) \rightarrow U_{ad}$ is a bounded operator given by (2.3).

3. Feedback control design

In this section, we discuss the feedback control design for system (2.2). We first utilize the idea of instantaneous control, which is a DTO (discretize-then-optimize) approach, and then compare it with the OTD (optimize-then-discretize) approach. The former, as mentioned earlier, is to first discretize the uncontrolled state equations in time and conduct the optimization procedure over discrete time steps, and then progress recursively in time (cf. [24, 27]). In contrast, the OTD approach is to directly discretize the optimality system (2.9)–(2.11) on one step time sub-interval, and then carry the information for the next time sub-interval. Finally, we observe that under appropriate time discretization schemes, these two approaches lead to the same nonlinear continuous feedback law. In the following discussion, we set $\alpha = 0$ and $\beta = 1$.

3.1. Discretize-then-optimize approach

With the understanding of the original optimal control of system (2.2), we now describe the moving horizon approach, which is at the core of the instantaneous control design. There are various ways to discretize the control system (2.2) in time. To focus on our discussion, we employ two Euler's methods and compare the resulting feedback laws.

3.1.1. Semi-implicit Euler's method I

Consider a uniform partition of $[0, T]$ and let $h = \frac{T}{N}$ for $N \in \mathbb{N}$ and $t_i = ih$ for $i = 0, 1, \dots, N-1$. Applying the semi-implicit Euler's method to system (2.2) in time t gives

$$\begin{cases} \theta^{i+1} - \theta^i = -h v^{i+1} \cdot \nabla \theta^i, \\ v^{i+1} - v^i = h A v^{i+1} + B u^{i+1}, \end{cases} \quad \text{that is,} \quad \begin{cases} \theta^{i+1} = \theta^i - h v^{i+1} \cdot \nabla \theta^i, \\ (I - h A) v^{i+1} = v^i + B u^{i+1}, \end{cases} \quad (3.1)$$

where $\theta^i = \theta(\cdot, t_i)$, $v^i = v(\cdot, t_i)$, and $u^i = u(\cdot, t_i)$. The semi-implicit scheme is applied to the advection term $v \cdot \nabla \theta$ and the implicit scheme is applied to Av . The instantaneous version of the minimization problem over the subinterval $I_i = [t_i, t_{i+1}]$ becomes

$$\min J^i(u^{i+1}, \theta^{i+1}) = \frac{1}{2} \|\Lambda^{-1} \theta^{i+1}\|_{L^2}^2 + \frac{\gamma}{2} \|u^{i+1}\|_{U_i}^2, \quad i = 0, 1, \dots, N-1,$$

subject to (3.1), where $U_i = L^2(\omega)$.

Let (ρ^{i+1}, w^{i+1}) be the adjoint state of (θ^{i+1}, v^{i+1}) . Define the Lagrangian

$$\begin{aligned} L^i(\theta^{i+1}, v^{i+1}, u^{i+1}; \rho^{i+1}, w^{i+1}) &= \frac{1}{2}(\Lambda^{-2}\theta^{i+1}, \theta^{i+1}) + \frac{\gamma}{2}(u^{i+1}, u^{i+1}) \\ &\quad - (\rho^{i+1}, \theta^{i+1} - (\theta^i - hv^{i+1} \cdot \nabla \theta^i)) \\ &\quad - (w^{i+1}, (I - hA)v^{i+1} - (v^i + Bu^{i+1})). \end{aligned} \quad (3.2)$$

Rewriting the right hand side of (3.2) with the help of (2.16), we get

$$\begin{aligned} L^i(\theta^{i+1}, v^{i+1}, u^{i+1}; \rho^{i+1}, w^{i+1}) &= \frac{1}{2}(\Lambda^{-2}\theta^{i+1}, \theta^{i+1}) + \frac{\gamma}{2}(u^{i+1}, u^{i+1}) \\ &\quad - [(\rho^{i+1}, \theta^{i+1}) - (\rho^{i+1}, \theta^i) - h(v^{i+1}, \mathbb{P}(\theta^i \nabla \rho^{i+1}))] \\ &\quad - [((I - hA)w^{i+1}, v^{i+1}) - (w^{i+1}, v^i) - (B^*w^{i+1}, u^{i+1})]. \end{aligned}$$

Setting $\frac{\partial L^i}{\partial \theta^{i+1}} = 0$ and $\frac{\partial L^i}{\partial v^{i+1}} = 0$, we obtain the adjoint system

$$\begin{cases} \rho^{i+1} = \Lambda^{-2}\theta^{i+1} = \eta^{i+1}, \\ (I - hA)w^{i+1} = h\mathbb{P}(\theta^i \nabla \rho^{i+1}), \end{cases} \quad (3.3)$$

where $\eta^{i+1} = \eta(\cdot, t_{i+1})$. Setting $\frac{\partial L}{\partial u^{i+1}} = 0$ yields the optimality condition

$$u^{i+1} = -\frac{1}{\gamma}B^*w^{i+1}. \quad (3.4)$$

The optimality system (3.1), (3.3), and (3.4) admits a unique solution due to the quadratic cost functional and the linearity and uniqueness of solutions to (3.1). Since there are several layers of couplings in the optimality conditions, if one follows the recursive procedures as in (cf. [24,27]) to solve u^{i+1} explicitly in terms of (θ^{i+1}, v^{i+1}) and η^{i+1} , one ends up with a highly nonlinear feedback law. In what follows, we adopt a different approach to overcome this obstacle.

First, from the adjoint system (3.3) we observe that

$$w^{i+1} = h(I - hA)^{-1}\mathbb{P}(\theta^i \nabla \rho^{i+1}) = h(I - hA)^{-1}\mathbb{P}(\theta^i \nabla \eta^{i+1}).$$

Thus, by the optimality condition (3.4), we obtain

$$u^{i+1} = -\frac{1}{\gamma}B^*h(I - hA)^{-1}\mathbb{P}(\theta^i \nabla \eta^{i+1}). \quad (3.5)$$

As a result, the governing system (3.1) together with (3.5) is the semi-implicit time discretization of the closed-loop dynamical system

$$\begin{cases} \partial_t \theta = -v \cdot \nabla \theta, \\ \partial_t v = Av - \frac{h}{\gamma}BB^*(I - hA)^{-1}\mathbb{P}(\theta \nabla \eta), \end{cases} \quad (3.6)$$

with initial condition $(\theta(0), v(0)) = (\theta_0, v_0)$, where η is given by (2.5). Since the Stokes operator A is strictly negative, $I - hA$ is strictly positive, and hence invertible for any $h > 0$.

3.1.2. Semi-explicit Euler's method II

If using an explicit scheme to discretize Av in the velocity equation in time, we have

$$\begin{cases} \theta^{i+1} - \theta^i = -h v^{i+1} \cdot \nabla \theta^i, \\ v^{i+1} - v^i = h A v^i + B u^{i+1}, \end{cases} \quad \text{that is,} \quad \begin{cases} \theta^{i+1} = \theta^i - h v^{i+1} \cdot \nabla \theta^i, \\ v^{i+1} = (I + h A) v^i + B u^{i+1}. \end{cases} \quad (3.7)$$

Again by the Euler-Lagrange method, we can easily derive the adjoint system and the optimality condition as follows

$$\begin{cases} \rho^{i+1} = \Lambda^{-2} \theta^{i+1} = \eta^{i+1}, \\ w^{i+1} = h \mathbb{P}(\theta^i \nabla \rho^{i+1}), \end{cases} \quad (3.8)$$

and

$$u^{i+1} = -\frac{1}{\gamma} B^* w^{i+1}. \quad (3.9)$$

It is clear that (3.8)–(3.9) are the semi-implicit time discretization of the closed-loop dynamical system given by

$$\begin{cases} \dot{\theta} = -v \cdot \nabla \theta, \\ \dot{v} = Av - \frac{h}{\gamma} B B^* \mathbb{P}(\theta \nabla \eta), \end{cases} \quad (3.10)$$

with initial condition $(\theta(0), v(0)) = (\theta_0, v_0)$.

Compared to the closed-loop (3.6), the control input becomes less regular and hence the velocity of (3.10) has a lower regularity, which however, provides a better mixing effect as shown in our numerical experiments in Section 5.

3.2. Optimize-then-discretize approach

Since the optimality conditions are at our disposal, it is natural to consider a direct approximation to obtain the feedback law. To be more specific, we first discretize the optimality system (2.2) and (2.9)–(2.11) restricted to one time subinterval $I_i = [t_i, t_{i+1}]$, $i = 0, 1, \dots, N-1$, and then carry the information for the next time subinterval, where the state and the adjoint equations will be formulated forward and backward in time, respectively. This is done only for one step, so that the adjoint state can be related to the state explicitly.

Recall that (θ, v) and (ρ, w) are continuous in time according to (2.12)–(2.13). We now let $(\theta^i, v^i) = (\theta(\cdot, t_i)|_{I_i}, v(\cdot, t_i)|_{I_i})$, $(\rho^i, w^i) = (\rho(\cdot, t_i)|_{I_i}, w(\cdot, t_i)|_{I_i})$, and $u^i = u(\cdot, t_i)|_{I_i}$, denote the state, adjoint state, and optimal control on each I_i , respectively. We apply a semi-implicit Euler's method with the same step size h in time as in section 3.1 to discretize the optimality conditions (2.2) and (2.9)–(2.11) on I_i . This procedure leads to

$$\begin{aligned} \text{State Equations} & \begin{cases} \theta^{i+1} = \theta^i - h v^{i+1} \cdot \nabla \theta^i, \\ v^{i+1} = (I + h A) v^i + B u^{i+1}, \end{cases} \\ \text{Adjoint Equations} & \begin{cases} -(\rho^{i+1} - \rho^i) = h v^{i+1} \cdot \nabla \rho^{i+1} + h \eta^{i+1} \\ -(w^{i+1} - w^i) = h A w^i + h \mathbb{P}(\theta^i \nabla \rho^i), \end{cases} \end{aligned} \quad (3.11)$$

where

$$(\rho^{i+1}, w^{i+1}) = (0, 0) \quad (3.12)$$

and $\eta^{i+1} = \Lambda^{-2}\theta^{i+1}$. The optimality condition is given by

$$u^{i+1} = \frac{1}{\gamma} B^* w^i, \quad i = 1, 2, \dots, N. \quad (3.13)$$

Applying the final time condition (3.12) to (3.11) follows

$$\begin{cases} \rho^i = h\eta^{i+1}, \\ w^i = h(I - hA)^{-1} \mathbb{P}(\theta^i \nabla \rho^i). \end{cases} \quad (3.14)$$

With the help of (3.14), (3.13) becomes

$$u^{i+1} = \frac{h}{\gamma} B^* (I - hA)^{-1} \mathbb{P}(\theta^i \nabla \rho^i) = \frac{h^2}{\gamma} B^* (I - hA)^{-1} \mathbb{P}(\theta^i \nabla \eta^{i+1}).$$

Consequently, this procedure gives rise to a system which is the semi-implicit time discretization of the closed-loop dynamical system (3.6).

On the other hand, if we apply an implicit scheme for discretizing $Aw(\cdot, t)|_{I_i}$ in the adjoint velocity equation in (2.9), then we obtain

$$\text{Adjoint Equations } \begin{cases} -(\rho^{i+1} - \rho^i) = hv^{i+1} \cdot \nabla \rho^{i+1} + h\eta^{i+1}, \\ -(w^{i+1} - w^i) = hAw^{i+1} + h\mathbb{P}(\theta^i \nabla \rho^i), \end{cases}$$

and hence by (3.12) we get

$$\begin{cases} \rho^i = h\eta^{i+1}, \\ w^i = h\mathbb{P}(\theta^i \nabla \rho^i). \end{cases}$$

Thus,

$$u^{i+1} = \frac{h^2}{\gamma} B^* \mathbb{P}(\theta^i \nabla \eta^{i+1}).$$

This results in the semi-implicit time discretization of the same closed-loop dynamical system (3.10).

Remark 3.1. Both DTO and OTD designs allow for an interpretation as a nonlinear discrete-in-time sub-optimal closed-loop control method. Under appropriate discretization schemes, they turn out to be the time discretization of certain continuous closed-loop controller. We are particularly interested in the long-time behavior of the corresponding closed-loop system. For such a semi-dissipative system, however, we can not simply set $T = \infty$ in the cost functional (2.7) in the first place, since $\|\theta\|_{(H^1(\Omega))'}$ may not even converge to zero as $T \rightarrow \infty$. A discount factor $e^{-\xi t}$ for some $\xi > 0$ may need to be taken into account in the second term of (P) if setting $T = \infty$.

4. Well-posedness and asymptotic behavior of the nonlinear closed-loop systems

In this section, we address the well-posedness and asymptotic behavior of the nonlinear closed-loop systems (3.6) and (3.10) incorporated with feedback laws of different regularity. In the current work, we mainly investigate the internal control design that is applied to the entire domain, that is, $\omega = \Omega$ and $\chi_\omega = I$, and hence $B = \mathbb{P}$. The main results are stated in the following theorems.

Theorem 4.1. *For $(\theta_0, v_0) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times \mathcal{D}(A)$, where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, there exists a unique solution (θ, v) to the closed-loop system (3.6) satisfying*

$$(\theta, v) \in L^\infty(0, T; L^\infty(\Omega) \cap H^1(\Omega)) \times L^\infty(0, T; V_n^2(\Omega)) \cap L^2(0, T; V_n^3(\Omega))$$

for any $T > 0$. Moreover, (θ, v) obeys the following asymptotic behavior

$$\begin{aligned} \|v\|_{L^2} &\rightarrow 0, \quad \|\nabla v\|_{L^2} \rightarrow 0, \quad \|\Delta v\|_{L^2} \rightarrow 0, \quad \|\partial_t v\|_{L^2} \rightarrow 0, \\ \|\theta\|_{(H^1(\Omega))'} &\rightarrow c_0 \quad \text{as } t \rightarrow \infty, \quad \text{where} \\ c_0 &< \sqrt{\|\Lambda \eta_0\|_{L^2}^2 + \frac{\gamma}{h} \|v_0\|_{L^2}^2 + \gamma(2\|\mathbb{D}v_0\|_{L^2}^2 + k\|v_0 \cdot \tau\|_{L^2(\Gamma)}^2)}, \end{aligned}$$

and

$$\|u\|_{L^2} = \left\| -\frac{h}{\gamma}(I - hA)^{-1} \mathbb{P}(\theta \nabla \eta) \right\|_{L^2} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (4.1)$$

which indicates

$$\|\theta \nabla \eta - \nabla q\|_{(H^2(\Omega))'} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (4.2)$$

for some $q \in L^2(\Omega)$.

Since the feedback law for the closed-loop system (3.10) has a much lower regularity compared to (3.6), our discussion for (3.6) only focuses on $d = 2$.

Theorem 4.2. *For $(\theta_0, v_0) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times \mathcal{D}(A)$, where $\Omega \subset \mathbb{R}^2$, there exists a unique solution (θ, v) to the closed-loop system (3.10) satisfying*

$$(\theta, v) \in L^\infty(0, T; L^\infty(\Omega) \cap H^1(\Omega)) \times L^\infty(0, T; V_n^2(\Omega)) \cap L^2(0, T; V_n^3(\Omega)), \quad (4.3)$$

for any $T > 0$. Moreover,

$$\|v\|_{L^2} \rightarrow 0, \quad \|\nabla v\|_{L^2} \rightarrow 0, \quad \|\partial_t v\|_{L^2} \rightarrow 0, \quad (4.4)$$

$$\|\theta\|_{(H^1(\Omega))'} \rightarrow C_0 \quad \text{as } t \rightarrow \infty, \quad \text{where } C_0 < \sqrt{\|\Lambda \eta_0\|_{L^2}^2 + \frac{\gamma}{h} \|v_0\|_{L^2}^2}, \quad (4.5)$$

and the feedback law satisfies

$$\|u\|_{(V_n^1(\Omega))'} = \left\| -\frac{h}{\gamma} \mathbb{P}(\theta \nabla \eta) \right\|_{(V_n^1(\Omega))'} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (4.6)$$

which indicates

$$\|\theta \nabla \eta - \nabla q\|_{(H^1(\Omega))'} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (4.7)$$

for some $q \in L^2(\Omega)$.

The critical step in the proofs of Theorems 4.1–4.2 lies in establishing the global-in-time *a priori* bound for $(v, \Lambda \eta)$. The main difficulties are due to the lack of dissipation in the transport equation and the nonlinear couplings involved. To tackle these challenges, we first establish the decay properties of the velocity, which in turn help obtain the estimates on $\Lambda \eta$ and its long-time behavior. The uniqueness of solution to the closed-loop system (3.6) for $d = 2, 3$, is straightforward to prove once velocity satisfies $v \in L^2(0, T; V_n^3(\Omega))$ because the estimate (2.8) needs to hold. However, for the closed-loop system (3.6), the lower regularity of the feedback law demands a more delicate analysis in order to establish (4.3) even for $d = 2$.

To start with, we introduce the following lemma which will be often used in our discussion. Its proof is elementary and can be found in [39, Lemma 2.3] (also see [18, Lemma 3.1]).

Lemma 4.3. *Let $f = f(t)$ with $t \in [0, \infty)$ be a nonnegative continuous function, and suppose that f is integrable on $(0, \infty)$, i.e.,*

$$\int_0^\infty f(t) dt < \infty.$$

Assume further that for any $\epsilon > 0$, there is $\delta > 0$ such that for any $0 \leq t_1 < t_2$ with $t_2 - t_1 < \delta$,

$$\text{either } f(t_2) \leq f(t_1) \quad \text{or} \quad f(t_2) \geq f(t_1) \quad \text{and} \quad f(t_2) - f(t_1) \leq \epsilon.$$

Then

$$f(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Proof of Theorem 4.1. The critical ingredient in the proof is to establish the global-in-time *a priori* bound for (θ, v) in the Bochner-Sobolev space

$$\theta \in L^\infty(0, T; L^\infty(\Omega) \cap H^1(\Omega)), \quad v \in L^\infty(0, T; V_n^2(\Omega)) \cap L^2(0, T; V_n^3(\Omega)).$$

Once the global bound is established, the existence of solutions then follows from standard procedures such as the Galerkin approximation scheme. We shall present our arguments in the following steps.

Step 1. We show that $\|v\|_{L^2} \rightarrow 0$ as $t \rightarrow \infty$. To this end, we first establish the necessary *a priori* bounds. Applying $\frac{\gamma}{h}(I - hA)$ to the velocity equation in (3.6) we get

$$\frac{\gamma}{h} \frac{d(I - hA)v}{dt} = \frac{\gamma}{h} (I - hA)Av - \mathbb{P}(\theta \nabla \eta). \quad (4.8)$$

Now taking the inner product of (4.8) with v yields

$$\begin{aligned} \frac{\gamma}{2h} \frac{d\|v\|_{L^2}^2}{dt} + \frac{\gamma}{2} \frac{d}{dt} (2\|\mathbb{D}v\|_{L^2}^2 + k\|v \cdot \tau\|_{L^2(\Gamma)}^2) \\ + \frac{\gamma}{h} (2\|\mathbb{D}v\|_{L^2}^2 + k\|v \cdot \tau\|_{L^2(\Gamma)}^2) + \gamma \|Av\|_{L^2}^2 = -(\mathbb{P}(\theta \nabla \eta), v). \end{aligned} \quad (4.9)$$

Next by taking the inner product of the transport equation in (3.6) with η it follows that

$$\frac{1}{2} \frac{d\|\Lambda \eta\|_{L^2}^2}{dt} = - \int_{\Omega} (v \cdot \nabla \theta) \eta \, dx = \int_{\Omega} v \cdot (\theta \nabla \eta) \, dx = (v, \mathbb{P}(\theta \nabla \eta)). \quad (4.10)$$

Adding (4.9) to (4.10) we have

$$\begin{aligned} \frac{1}{2} \frac{d\|\Lambda \eta\|_{L^2}^2}{dt} + \frac{\gamma}{2h} \frac{d\|v\|_{L^2}^2}{dt} + \frac{\gamma}{2} \frac{d}{dt} (2\|\mathbb{D}v\|_{L^2}^2 + k\|v \cdot \tau\|_{L^2(\Gamma)}^2) \\ + \frac{\gamma}{h} (2\|\mathbb{D}v\|_{L^2}^2 + k\|v \cdot \tau\|_{L^2(\Gamma)}^2) + \gamma \|Av\|_{L^2}^2 = 0, \end{aligned} \quad (4.11)$$

which gives

$$\begin{aligned} \|\Lambda \eta\|_{L^2}^2 + \frac{\gamma}{h} \|v\|_{L^2}^2 + \gamma (2\|\mathbb{D}v\|_{L^2}^2 + k\|v \cdot \tau\|_{L^2(\Gamma)}^2) \\ + 2\frac{\gamma}{h} \int_s^t (2\|\mathbb{D}v\|_{L^2}^2 + k\|v \cdot \tau\|_{L^2(\Gamma)}^2) \, d\sigma + 2\gamma \int_s^t \|Av\|_{L^2}^2 \, d\sigma \\ = \|\Lambda \eta(s)\|_{L^2}^2 + \frac{\gamma}{h} \|v(s)\|_{L^2}^2 + \gamma (2\|\mathbb{D}v(s)\|_{L^2}^2 + k\|v(s) \cdot \tau\|_{L^2(\Gamma)}^2). \end{aligned} \quad (4.12)$$

(4.12) implies that $Y(t) = \|\Lambda \eta\|_{L^2}^2 + \frac{\gamma}{h} \|v\|_{L^2}^2 + \gamma (2\|\mathbb{D}v\|_{L^2}^2 + k\|v \cdot \tau\|_{L^2(\Gamma)}^2)$ is a decreasing function of $t \in [0, \infty)$ and

$$0 < Y(t) \leq Y_0 = \|\Lambda \eta_0\|_{L^2}^2 + \frac{\gamma}{h} \|v_0\|_{L^2}^2 + \gamma (2\|\mathbb{D}v_0\|_{L^2}^2 + k\|v_0 \cdot \tau\|_{L^2(\Gamma)}^2). \quad (4.13)$$

As a result, there exists $c_0 > 0$ such that

$$Y(t) \rightarrow c_0 < Y_0. \quad (4.14)$$

Note that (4.11) also implies

$$\int_0^\infty \frac{\gamma}{h} (2\|\mathbb{D}v\|_{L^2}^2 + k\|v \cdot \tau\|_{L^2(\Gamma)}^2) + \gamma \|Av\|_{L^2}^2 \, dt \leq Y_0. \quad (4.15)$$

Using the Poincaré inequality stating that $\|v\|_{L^2} \leq C\|Av\|_{L^2}$ for $v \in \mathcal{D}(A)$ and some constant $C > 0$, we immediately obtain

$$\|v\|_{L^2}^2 \in L^1(0, \infty). \quad (4.16)$$

On the other hand, a direct L^2 -estimate of the velocity equation in (3.6) gives

$$\begin{aligned} \frac{1}{2} \frac{d\|v\|_{L^2}^2}{dt} + 2\|\mathbb{D}v\|_{L^2}^2 + k\|v \cdot \tau\|_{L^2(\Gamma)}^2 &\leq \frac{h}{\gamma} \|(I - hA)^{-1} \mathbb{P}(\theta \nabla \eta)\|_{L^2} \|v\|_{L^2} \\ &\leq C \frac{h}{\gamma} \|\theta_0\|_{L^\infty} \|\nabla \eta\|_{L^2} \|v\|_{L^2} \leq C \frac{h^2}{\gamma^2} \|\theta_0\|_{L^\infty}^2 \|\theta_0\|_{L^2}^2 + \|\mathbb{D}v\|_{L^2}^2, \end{aligned}$$

from which it follows that

$$\frac{d\|v\|_{L^2}^2}{dt} + 2\|\mathbb{D}v\|_{L^2}^2 + 2k\|v \cdot \tau\|_{L^2(\Gamma)}^2 \leq C \frac{h^2}{\gamma^2} \|\theta_0\|_{L^\infty}^2 \|\theta_0\|_{L^2}^2.$$

Thus for any $0 \leq s \leq t < \infty$, integrating in time yields

$$\|v(t)\|_{L^2}^2 - \|v(s)\|_{L^2}^2 \leq C \frac{h^2}{\gamma^2} \|\theta_0\|_{L^\infty}^2 \|\theta_0\|_{L^2}^2 |t - s|. \quad (4.17)$$

Combining (4.17) with (4.16) and Lemma 4.3, we obtain

$$\|v\|_{L^2} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (4.18)$$

Step 2. We show that $\|\mathbb{D}v\|_{L^2} \rightarrow 0$ as $t \rightarrow \infty$. Applying an H^1 -estimate of the velocity equation in (3.6), we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (2\|\mathbb{D}v\|_{L^2}^2 + k\|v \cdot \tau\|_{L^2(\Gamma)}^2) + \|Av\|_{L^2}^2 &\leq \frac{h}{\gamma} \|(I - hA)^{-1} \mathbb{P}(\theta \nabla \eta)\|_{L^2} \|Av\|_{L^2} \\ &\leq C \frac{h}{\gamma} \|\theta_0\|_{L^\infty} \|\nabla \eta\|_{L^2} \|Av\|_{L^2} \leq C \frac{h^2}{\gamma^2} \|\theta_0\|_{L^\infty}^2 \|\theta_0\|_{L^2}^2 + \frac{1}{2} \|Av\|_{L^2}^2, \end{aligned}$$

from which we observe

$$\frac{d}{dt} (2\|\mathbb{D}v\|_{L^2}^2 + k\|v \cdot \tau\|_{L^2(\Gamma)}^2) + \|Av\|_{L^2}^2 \leq C \frac{h^2}{\gamma^2} \|\theta_0\|_{L^\infty}^2 \|\theta_0\|_{L^2}^2.$$

Again, integrating in time yields

$$\begin{aligned} 2\|\mathbb{D}v(t)\|_{L^2}^2 + k\|v(t) \cdot \tau\|_{L^2(\Gamma)}^2 - (2\|\mathbb{D}v(s)\|_{L^2}^2 + k\|v(s) \cdot \tau\|_{L^2(\Gamma)}^2) \\ \leq C \frac{h^2}{\gamma^2} \|\theta_0\|_{L^\infty}^2 \|\theta_0\|_{L^2}^2 |t - s|, \end{aligned} \quad (4.19)$$

where $0 \leq s \leq t < \infty$. With the help of (4.15) and Lemma 4.3 we obtain that

$$\|\mathbb{D}v\|_{L^2} \rightarrow 0 \quad \text{and} \quad \|v \cdot \tau\|_{L^2(\Gamma)} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty. \quad (4.20)$$

As a result of (4.18), (4.20) and (4.14), we can conclude that

$$\|\theta\|_{(H^1(\Omega))'} = \|\Lambda\eta\|_{L^2} \rightarrow c_0 \quad \text{as} \quad t \rightarrow \infty.$$

Step 3. We further prove that $\|Av\|_{L^2} \rightarrow 0$ as $t \rightarrow \infty$. Taking the inner product of (4.8) with $-Av$ leads to

$$\frac{\gamma}{h} \left(\frac{d}{dt} (I - hA)v, -Av \right) = \frac{\gamma}{h} ((I - hA)Av, -Av) - (\mathbb{P}(\theta \nabla \eta), -Av),$$

which implies that

$$\begin{aligned} & \frac{\gamma}{h} \frac{1}{2} \frac{d}{dt} (2\|\mathbb{D}v\|_{L^2}^2 + k\|v \cdot \tau\|_{L^2(\Gamma)}^2) + \frac{\gamma}{2} \frac{d}{dt} \|Av\|_{L^2}^2 + \frac{\gamma}{h} \|Av\|_{L^2}^2 \\ & \quad + \gamma (2\|\mathbb{D}(Av)\|_{L^2}^2 + k\|(Av) \cdot \tau\|_{L^2(\Gamma)}^2) \\ & = (\mathbb{P}(\theta \nabla \eta), Av) \leq C \|\theta_0\|_{L^\infty}^2 \|\theta_0\|_{L^2}^2 + \frac{\gamma}{2h} \|Av\|_{L^2}^2, \end{aligned}$$

and thus

$$\begin{aligned} & \frac{\gamma}{h} \frac{d}{dt} (2\|\mathbb{D}v\|_{L^2}^2 + k\|v \cdot \tau\|_{L^2(\Gamma)}^2) + \gamma \frac{d}{dt} \|Av\|_{L^2}^2 + \frac{\gamma}{h} \|Av\|_{L^2}^2 \\ & \quad + 2\gamma (2\|\mathbb{D}(Av)\|_{L^2}^2 + k\|(Av) \cdot \tau\|_{L^2(\Gamma)}^2) \leq C \|\theta_0\|_{L^\infty}^2 \|\theta_0\|_{L^2}^2. \end{aligned} \quad (4.21)$$

This indicates that

$$\begin{aligned} & \frac{\gamma}{h} (2\|\mathbb{D}v(t)\|_{L^2}^2 + k\|v(t) \cdot \tau\|_{L^2(\Gamma)}^2) + \gamma \|Av(t)\|_{L^2}^2 \\ & \quad - \left[\frac{\gamma}{h} (2\|\mathbb{D}v(s)\|_{L^2}^2 + k\|v(s) \cdot \tau\|_{L^2(\Gamma)}^2) + \gamma \|Av(s)\|_{L^2}^2 \right] \\ & \leq C \|\theta_0\|_{L^\infty}^2 \|\theta_0\|_{L^2}^2 |t - s|. \end{aligned}$$

Again by (4.15) and Lemma 4.3 we have

$$\frac{\gamma}{h} (2\|\mathbb{D}v(t)\|_{L^2}^2 + k\|v(t) \cdot \tau\|_{L^2(\Gamma)}^2) + \gamma \|Av(t)\|_{L^2}^2 \rightarrow 0, \quad \text{as} \quad t \rightarrow \infty,$$

so that $\|Av\|_{L^2} \rightarrow 0$ as $t \rightarrow \infty$. Furthermore, (4.21) also implies that

$$\int_0^T \|(-A)^{3/2}v\|_{L^2}^2 dt = \int_0^T 2\|\mathbb{D}(Av)\|_{L^2}^2 + k\|(Av) \cdot \tau\|_{L^2(\Gamma)}^2 dt \leq C(h, \gamma, \theta_0, v_0, T) \quad (4.22)$$

for any $0 < T < \infty$.

As we show next, with the help of (4.22), it is direct to obtain that

$$\theta \in L^\infty(0, T; H^1(\Omega)).$$

In fact, applying ∇ to the transport equation in (3.6) and taking the inner product with $\nabla \theta$ yield

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \theta\|_{L^2}^2 &= - \int_{\Omega} \partial_j (v_i \partial_i \theta) \partial_j \theta \, dx = - \int_{\Omega} \partial_j v_i \partial_i \theta \partial_j \theta \, dx - \frac{1}{2} \int_{\Omega} v_i \partial_i (\partial_j \theta \partial_j \theta) \, dx \\ &= - \int_{\Omega} \partial_j v_i \partial_i \theta \partial_j \theta \, dx + \frac{1}{2} \int_{\Omega} \partial_i v_i \partial_j \theta \partial_j \theta \, dx - \frac{1}{2} \int_{\Gamma} v_i n_i \partial_j \theta \partial_j \theta \, dx \\ &= - \int_{\Omega} \partial_j v_i \partial_i \theta \partial_j \theta \, dx \leq \|\nabla v\|_{L^\infty} \|\nabla \theta\|_{L^2}^2. \end{aligned}$$

By virtue of (4.22), we have

$$\|\nabla \theta\|_{L^2}^2 \leq \|\nabla \theta_0\|_{L^2}^2 e^{\int_0^T \|\nabla v\|_{L^\infty} \, dt} \leq \|\nabla \theta_0\|_{L^2}^2 e^{c \int_0^T \|v\|_{H^3} \, dt} \leq C(h, \gamma, \theta_0, v_0, T), \quad (4.23)$$

where in the second inequality we have used Agmon's inequality (cf. [50, (2.21), p. 11]) that

$$\|\nabla v\|_{L^\infty} \leq c \|v\|_{H^{1+d/2+\epsilon}}, \quad d = 2, 3,$$

for some $\epsilon > 0$,

Step 4. We establish that $\|\partial_t v\| \rightarrow 0$ as $t \rightarrow \infty$. Applying ∂_t to the velocity equation in (3.6) and making use of the transport equation yield

$$\begin{aligned} \partial_{tt} v &= A \partial_t v - \frac{h}{\gamma} (I - hA)^{-1} \mathbb{P} \partial_t (\theta \nabla \eta) \\ &= A \partial_t v - \frac{h}{\gamma} (I - hA)^{-1} \mathbb{P} (\partial_t \theta \nabla \eta + \theta \partial_t \nabla \eta) \\ &= A \partial_t v - \frac{h}{\gamma} (I - hA)^{-1} \mathbb{P} ((-v \cdot \nabla \theta) \nabla \eta + \theta \partial_t \nabla \eta). \end{aligned} \quad (4.24)$$

Taking the L^2 -inner product of (4.24) with $\partial_t v$ gives

$$\begin{aligned} \frac{1}{2} \frac{d \|\partial_t v\|_{L^2}^2}{dt} &+ 2 \|\mathbb{D}(\partial_t v)\|_{L^2}^2 + k \|(\partial_t v) \cdot \tau\|_{L^2(\Gamma)}^2 \\ &= - \frac{h}{\gamma} \left((-v \cdot \nabla \theta) \nabla \eta + \theta \partial_t \nabla \eta, \mathbb{P} (I - hA)^{-1} \partial_t v \right) \\ &= - \frac{h}{\gamma} \left(\int_{\Omega} (-\nabla \cdot (v \theta)) (\nabla \eta \cdot (I - hA)^{-1} \partial_t v) \, dx + \int_{\Omega} \theta \partial_t \nabla \eta \cdot (I - hA)^{-1} \partial_t v \, dx \right) \end{aligned}$$

$$\begin{aligned}
 &= -\frac{h}{\gamma} \int_{\Omega} (v\theta) \cdot \nabla(\nabla\eta \cdot (I - hA)^{-1} \partial_t v) dx - \frac{h}{\gamma} \int_{\Omega} \theta \nabla \partial_t \eta \cdot (I - hA)^{-1} \partial_t v dx \\
 &= I_1 + I_2,
 \end{aligned} \tag{4.25}$$

where

$$\begin{aligned}
 I_1 &\leq \frac{h}{\gamma} \|v\|_{L^2} \|\theta\|_{L^\infty} \|\nabla(\nabla\eta \cdot (I - hA)^{-1} \partial_t v)\|_{L^2} \\
 &\leq \frac{h}{\gamma} \|v\|_{L^2} \|\theta\|_{L^\infty} (\|\nabla \nabla \eta\|_{L^4} \|(I - hA)^{-1} \partial_t v\|_{L^4} \\
 &\quad + \|\nabla \eta\|_{L^\infty} \|\nabla(I - hA)^{-1} \partial_t v\|_{L^2})
 \end{aligned} \tag{4.26}$$

$$\begin{aligned}
 &\leq C \frac{h}{\gamma} \|v\|_{L^2} \|\theta\|_{L^\infty} (\|\theta\|_{L^4} \|\partial_t v\|_{L^2} + \|\nabla \nabla \eta\|_{L^r} \|\nabla(I - hA)^{-1} \partial_t v\|_{L^2}) \\
 &\leq C \frac{h}{\gamma} \|v\|_{L^2} \|\theta_0\|_{L^\infty} (\|\theta_0\|_{L^4} \|\partial_t v\|_{L^2} + \|\theta_0\|_{L^r} \|\partial_t v\|_{L^2})
 \end{aligned} \tag{4.27}$$

$$\begin{aligned}
 &\leq C \frac{h}{\gamma} \|v\|_{L^2} \|\theta_0\|_{L^\infty} \|\theta_0\|_{L^4} \|\partial_t v\|_{L^2}, \\
 &\leq C \frac{h^2}{\gamma^2} \|v\|_{L^2}^2 \|\theta_0\|_{L^\infty}^2 \|\theta_0\|_{L^4}^2 + \frac{1}{4} \|\mathbb{D} \partial_t v\|_{L^2}^2.
 \end{aligned} \tag{4.28}$$

From (4.26) to (4.27) we used Sobolev embedding, (2.1) and (2.6), that is, $\|\nabla \eta\|_{L^\infty} \leq C \|\nabla \nabla \eta\|_{L^r} \leq C \|\theta\|_{L^r} = C \|\theta_0\|_{L^r}$ for $d < r < \infty$.

To estimate I_2 , we first recall that $\eta = \mathcal{A}^{-1}\theta$, and hence

$$\partial_t \eta = \partial_t \mathcal{A}^{-1} \theta = -\mathcal{A}^{-1}(v \cdot \nabla \theta).$$

Therefore,

$$\begin{aligned}
 I_2 &= -\frac{h}{\gamma} \int_{\Omega} \theta \nabla(-\mathcal{A}^{-1}(v \cdot \nabla \theta)) \cdot (I - hA)^{-1} \partial_t v dx \\
 &\leq C \frac{h}{\gamma} \|\theta\|_{L^\infty} \|\mathcal{A}^{-1}(v \cdot \nabla \theta)\|_{L^2} \|\partial_t v\|_{L^2} \\
 &\leq C \frac{h}{\gamma} \|\theta_0\|_{L^\infty} \|v \cdot \nabla \theta\|_{(H^1(\Omega))'} \|\partial_t v\|_{L^2},
 \end{aligned} \tag{4.29}$$

where

$$\begin{aligned}
 \|v \cdot \nabla \theta\|_{(H^1(\Omega))'} &= \sup_{0 \neq \phi \in H^1(\Omega)} \frac{|\int_{\Omega} v \cdot \nabla \theta \phi dx|}{\|\phi\|_{H^1}} = \sup_{0 \neq \phi \in H^1(\Omega)} \frac{|\int_{\Omega} \theta v \cdot \nabla \phi dx|}{\|\phi\|_{H^1}} \\
 &\leq \sup_{0 \neq \phi \in H^1(\Omega)} \frac{C \|\theta\|_{L^\infty} \|v\|_{L^2} \|\phi\|_{H^1}}{\|\phi\|_{H^1}} \leq C \|\theta_0\|_{L^\infty} \|v\|_{L^2}.
 \end{aligned}$$

Finally, we get

$$I_2 \leq C \frac{h}{\gamma} \|\theta_0\|_{L^\infty}^2 \|v\|_{L^2} \|\partial_t v\|_{L^2} \leq C \frac{h^2}{\gamma^2} \|\theta_0\|_{L^\infty}^4 \|v\|_{L^2}^2 + \frac{1}{4} \|\mathbb{D} \partial_t v\|_{L^2}^2. \quad (4.30)$$

Combining (4.25) with (4.28)–(4.30) follows

$$\begin{aligned} \frac{d\|\partial_t v\|_{L^2}^2}{dt} + 2\|\mathbb{D}(\partial_t v)\|_{L^2}^2 + k\|(\partial_t v) \cdot \tau\|_{L^2(\Gamma)}^2 \\ \leq C \frac{h^2}{\gamma^2} \|\theta_0\|_{L^\infty}^2 (\|\theta_0\|_{L^\infty}^2 + \|\theta_0\|_{L^4}^2) \|v\|_{L^2}^2. \end{aligned} \quad (4.31)$$

With the help of (4.15)–(4.16), we obtain

$$\|\partial_t v\|_{L^2}^2 + \int_0^t 2\|\mathbb{D}(\partial_t v)\|_{L^2}^2 + k\|(\partial_t v) \cdot \tau\|_{L^2(\Gamma)}^2 d\tau \leq C(h, \gamma, \theta_0, v_0),$$

which by Poincaré and Korn's inequalities yields

$$\int_0^\infty \|\partial_t v\|_{L^2}^2 d\tau \leq C \int_0^\infty \|\mathbb{D} \partial_t v\|_{L^2}^2 d\tau \leq C(h, \gamma, \theta_0, v_0).$$

Moreover, by (4.31) and the uniform boundedness of $\|v\|_{L^2}^2$ in time shown in (4.13), we have for any $0 \leq s \leq t < \infty$ that

$$\begin{aligned} \|\partial_t v(t)\|_{L^2}^2 - \|\partial_t v(s)\|_{L^2}^2 &\leq C \frac{h^2}{\gamma^2} \|\theta_0\|_{L^\infty}^2 (\|\theta_0\|_{L^\infty}^2 + \|\theta_0\|_{L^4}^2) \int_s^t \|v\|_{L^2}^2 d\tau \\ &\leq C(h, \gamma, \theta_0, v_0) |t - s|. \end{aligned}$$

In light of Lemma 4.3, we conclude that $\|\partial_t v(t)\|_{L^2}^2 \rightarrow 0$ as $t \rightarrow \infty$. Consequently, it follows from the velocity equation in (3.6) that

$$\left\| \frac{h}{\gamma} (I - hA)^{-1} \mathbb{P}(\theta \nabla \eta) \right\|_{L^2} \leq \left\| \frac{\partial v}{\partial t} \right\|_{L^2} + \|Av\|_{L^2} \rightarrow 0,$$

and hence

$$\begin{aligned} \|\mathbb{P}(\theta \nabla \eta)\|_{(V_n^2(\Omega))'} &\rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad i.e., \\ \|\theta \nabla \eta - \nabla q\|_{(H^2(\Omega))'} &\rightarrow 0 \quad \text{as } t \rightarrow \infty, \end{aligned}$$

for some $q \in L^2(\Omega)$, which establishes (4.1)–(4.2).

Step 5. Uniqueness: Finally, to show the uniqueness of the solution, we assume that there are two pairs of solutions to system (3.6), denoted by (θ_i, v_i) , $i = 1, 2$. Let $\vartheta = \theta_1 - \theta_2$, $V = v_1 - v_2$, and $H = \eta_1 - \eta_2$. Then (ϑ, V) satisfies

$$\frac{\partial \vartheta}{\partial t} + V \cdot \nabla \theta_1 + v_2 \cdot \nabla \vartheta = 0, \quad \vartheta(0) = 0, \quad (4.32)$$

$$\frac{\partial V}{\partial t} = AV - \frac{\gamma}{h}(I - hA)^{-1}(\mathbb{P}(\vartheta \nabla \eta_1 + \theta_2 \nabla H)) \quad (4.33)$$

with $V(x, 0) = 0$ and $H(x, 0) = 0$. Multiplying (4.33) by $I - hA$ follows

$$\frac{\partial(I - hA)V}{\partial t} = (I - hA)AV - \frac{\gamma}{h}(\mathbb{P}(\vartheta \nabla \eta_1 + \theta_2 \nabla H)). \quad (4.34)$$

Taking the inner produce of (4.32) with ϑ , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d \|\vartheta\|_{L^2}^2}{dt} &= \int_{\Omega} (V \cdot \nabla \theta_1) \vartheta \, dx \leq \|V\|_{L^\infty} \|\nabla \theta_1\|_{L^2} \|\vartheta\|_{L^2} \\ &\leq \frac{h}{2} \|AV\|_{L^2}^2 + CM^2 \|\vartheta\|_{L^2}^2, \end{aligned} \quad (4.35)$$

where $M = \sup_{t \in [0, T]} \|\nabla \theta_1\|_{L^2}$. Taking the inner product of (4.34) with V yields

$$\begin{aligned} \frac{1}{2} \frac{d \|V\|_{L^2}^2}{dt} + \frac{h}{2} \frac{d}{dt} (2 \|\mathbb{D}V\|_{L^2}^2 + k \|V \cdot \tau\|_{L^2(\Gamma)}^2) &+ 2 \|\mathbb{D}V\|_{L^2}^2 + k \|V \cdot \tau\|_{L^2(\Gamma)}^2 + h \|AV\|_{L^2}^2 \\ &\leq \frac{\gamma}{h} (\|\vartheta\|_{L^2} \|\nabla \eta_1\|_{L^\infty} \|V\|_{L^2} + \|\theta_2\|_{L^\infty} \|\nabla H\|_{L^2} \|V\|_{L^2}) \\ &\leq C \frac{\gamma^2}{h^2} \|\vartheta\|_{L^2}^2 \|\theta_0\|_{L^r}^2 + \frac{1}{2} \|\mathbb{D}V\|_{L^2}^2 + C \frac{\gamma^2}{h^2} \|\theta_0\|_{L^\infty}^2 \|\vartheta\|_{L^2}^2 + \frac{1}{2} \|\mathbb{D}V\|_{L^2}^2, \end{aligned}$$

for $d < r < \infty$. Therefore,

$$\begin{aligned} \frac{1}{2} \frac{d \|V\|_{L^2}^2}{dt} + \frac{h}{2} \frac{d}{dt} (2 \|\mathbb{D}V\|_{L^2}^2 + k \|V \cdot \tau\|_{L^2(\Gamma)}^2) &+ \|\mathbb{D}V\|_{L^2}^2 + k \|V \cdot \tau\|_{L^2(\Gamma)}^2 + h \|AV\|_{L^2}^2 \\ &\leq C \frac{\gamma^2}{h^2} \|\vartheta\|_{L^2}^2 (\|\theta_0\|_{L^r}^2 + \|\theta_0\|_{L^\infty}^2). \end{aligned} \quad (4.36)$$

Combining (4.36) with (4.35) yields

$$\begin{aligned} \frac{1}{2} \frac{d \|V\|_{L^2}^2}{dt} + \frac{h}{2} \frac{d}{dt} (2 \|\mathbb{D}V\|_{L^2}^2 + k \|V \cdot \tau\|_{L^2(\Gamma)}^2) &+ \frac{1}{2} \frac{d \|\vartheta\|_{L^2}^2}{dt} \\ &+ \|\mathbb{D}V\|_{L^2}^2 + k \|V \cdot \tau\|_{L^2(\Gamma)}^2 + \frac{h}{2} \|AV\|_{L^2}^2 \leq C(\gamma, h, M, \theta_0) \|\vartheta\|_{L^2}^2, \end{aligned}$$

which implies

$$\begin{aligned} \frac{d}{dt}(\|V\|_{L^2} + h(2\|\mathbb{D}V\|_{L^2}^2 + k\|V \cdot \tau\|_{L^2(\Gamma)}^2) + \|\vartheta\|_{L^2}^2) \\ \leq C(\gamma, h, M, \theta_0)(\|V\|_{L^2}^2 + h(2\|\mathbb{D}V\|_{L^2}^2 + k\|V \cdot \tau\|_{L^2(\Gamma)}^2) + \|\vartheta\|_{L^2}^2). \end{aligned}$$

By the Gronwall inequality, it follows $\|V\|_{L^2} + h(2\|\mathbb{D}V\|_{L^2}^2 + k\|V \cdot \tau\|_{L^2(\Gamma)}^2) + \|\vartheta\|_{L^2}^2 = 0$, which indicates $V = 0$ and $\vartheta = 0$. The uniqueness of the solution is thus obtained which completes the proof. $\square$

Proof of Theorem 4.2. Due to the low regularity of the feedback law $u = -\frac{h}{\gamma}\mathbb{P}(\theta\nabla\eta)$ for the velocity equation in the closed-loop system (3.10), it is not straightforward to obtain

$$\int_0^T \|v\|_{H^3} < \infty,$$

as in the proof of Theorem 4.1. In fact, it is technically challenging to establish such an *a priori* estimate and the global-in-time boundedness of $\nabla\theta$. As shown in our previous work [33, Theorem 2.1] (see also [18, Theorem 1.1]) and [36, Section 3]), it requires a delicate analysis by combining the following Brezis-Gallouet inequality (cf. [10] and [33, (2.41)]) together with a new nonlinear Gronwall inequality [33, Lemma 2.3] to get $\|\nabla\theta\|_{L^2}$ bounded in a 2D domain:

$$\|\nabla v\|_{L^\infty} \leq C\|Av\|_{L^2} \left(1 + \log \frac{\|(-A)^{3/2}v\|_{L^2}^2}{\lambda_1\|Av\|_{L^2}^2} \right)^{1/2},$$

where $\lambda_1 > 0$ is the lowest eigenvalue of $-A$. For the detailed proof of (θ, v) satisfying (4.3), the reader is referred to the aforementioned references. Further discussion on the growth rate of $\|\nabla\theta\|_{L^2}$ in time can be found in [36, Theorem 3.2] and [37, Theorem 4.1].

The statements (4.4)–(4.7) can be obtained following the similar fashion as in Theorem 4.1. To establish the global-in-time *a priori* bound for $(v, \Lambda\eta)$ and show that

$$\|v\|_{L^2} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

we take the inner product of the velocity equation in (3.10) with $\frac{\gamma}{h}v$ and add the resulting equation to (4.10). The results follow easily by using the same approaches as in Step 1 of Theorem 4.1. The arguments for showing

$$\|\nabla v\|_{L^2} \rightarrow 0 \quad \text{and} \quad \|\partial_t v\| \rightarrow 0 \quad \text{as } t \rightarrow \infty, \tag{4.37}$$

also utilize the similar procedures as in Step 2 and Step 3 of Theorem 4.1. The details are omitted here. The uniform boundedness of $\|Av\|_{L^2}$ for $t \in [0, \infty)$ follows immediately from

$$\|Av\|_{L^2} \leq \|\partial_t v\|_{L^2} + \left\| \frac{h}{\gamma}\mathbb{P}(\theta\nabla\eta) \right\|_{L^2} \leq C(h, \gamma, \theta_0, v_0).$$

However, because of the low regularity of the feedback law, one can not claim that $\|Av\|_{L^2} \rightarrow 0$ as $t \rightarrow \infty$, as in Theorem 4.1.

Consequently, combining the velocity equation in (3.10) with (4.37) follows

$$\begin{aligned} \left\| \frac{h}{\gamma} (-A)^{-1/2} \mathbb{P}(\theta \nabla \eta) \right\|_{L^2} &\leq \| (-A)^{-1/2} \partial_t v \|_{L^2} + \| (-A)^{1/2} v \|_{L^2} \\ &\leq C (\| \partial_t v \|_{L^2} + \| \nabla v \|_{L^2}) \rightarrow 0, \end{aligned}$$

which verifies (4.6), and hence (4.7) holds for some $q \in L^2(\Omega)$.

Lastly, the proof of uniqueness can be obtained by using the same techniques as in Step 5 of Theorem 4.1. This completes the proof. $\square$

In the next section, numerical experiments are conducted to demonstrate and validate our ideas and designs for closed-loop systems (3.6) and (3.10).

5. Numerical implementation

One of the major obstacles encountered in numerical schemes is due to mass conservation of the scalar transport in incompressible flows. Specifically, the approximate velocity being divergence free is critical to preserving the accuracy, stability, and global conservation properties of the scalar. In this work, we employ mixed continuous Galerkin (CG) and discontinuous Galerkin (DG) methods to solve the closed-loop systems (3.6) and (3.10) as in our recent work [32], and compare the effectiveness of the feedback laws of different regularity.

Without loss of generality, let viscosity be $\nu = 1$. With internal control $B = \mathbb{P}$, we can rewrite the closed-loop system (3.6) resulting from the semi-implicit Euler's Method I succinctly as

$$\theta_t = -v \cdot \nabla \theta, \tag{5.1}$$

$$v_t = \Delta v - \nabla p + u, \quad \nabla \cdot v = 0. \tag{5.2}$$

$$(I - \Delta)\eta = \theta, \quad \frac{\partial \eta}{\partial n}|_{\Gamma} = 0, \tag{5.3}$$

$$-h\Delta u + \nabla q + u = -\frac{h}{\gamma} \theta \nabla \eta, \quad \nabla \cdot u = 0 \tag{5.4}$$

where $q \in L^2(\Omega)$ stands for the pressure and h and γ are fixed parameters. The algorithm for solving (5.1)–(5.4) is described below.

1. Assign the initial value of θ^0 and v^0 at $t = 0$. Set the time step Δt .
2. At time step $t_n = n\Delta t$, suppose θ^n and v^n are known.
 - (a) Solve η^n from (5.3);
 - (b) Solve u^n from (5.4) with θ^n and η^n ;
 - (c) Solve v^n from (5.2) with u^n ;
 - (d) Solve θ^{n+1} from (5.1);
 - (e) Let $n = n + 1$ and go back to (a).

To solve the above saddle point problems for two pairs, (u, q) in (5.4) and (v, p) in (5.2), an iterative projection BDF2/Taylor-Hood Finite element method introduced in [32] is used. A standard

Runge-Kutta Discontinuous Galerkin (RKDG) scheme [15], 3rd order accurate in time and piece-wise quadratic in space, is used to solve the scalar θ governed by the transport equation (5.1). The detailed discussion regarding the numerical schemes can be found in [32].

Since the only difference between the closed-loop systems (3.6) and (3.10) is that the control u of (3.10) satisfies the following equations

$$u + \nabla q = -\frac{h}{\gamma} \theta \nabla \eta, \quad \nabla \cdot u = 0, \quad (5.5)$$

to solve (3.10) it suffices to replace 2(b) by the solution of u from (5.5). To this end, we first apply divergence to (5.5) and then solve the resulting elliptic problem for q . After q is obtained, we solve u from (5.5) directly.

5.1. Numerical results

In our numerical experiments, we let the boundary friction $\kappa = 0.5$. Here we choose a relatively small control parameter $\gamma = 1e-6$, as large control weight turns out to have weak mixing effects as shown in [32]. The initial distribution of the scalar field is given by $\theta(x, y) = \tanh(y/0.1)$, which is shown in Fig. 1[a]. For each feedback control design, we choose three values of h . Specifically, for the closed-loop system (3.6), we set $h = 0.01, 0.1, 1$. For the closed-loop system (3.10), we set $h = 0.01, 0.05, 0.1$. The spacial and temporal step sizes for the simulations are taken as 0.0125 and 0.01, respectively.

The evolution of the system state of different norms is plotted in Fig. 1. It is observed that all the norms decay in power rates, approximately. Especially, the velocity norms $\|v\|_{L^2}$ and $\|\nabla v\|_{L^2}$ of both systems, shown in Fig. 1 [b]-[c], decay to zero at the rates of $t^{-1.5}$ and t^{-1} , respectively. The mixing-norm $\|\theta\|_{(H^1(\Omega))^Y}$ of both systems, shown in Fig. 1 [d], converges at the rates of $t^{-0.4}$ and $t^{-0.65}$, respectively. To compare the feedback laws, we note that the second control law for the closed-loops system (3.10) only weakly decays to 0. Numerically, as we can see in Fig. 1 [e]-[f], both $\|\mathbb{P}(\theta \nabla \eta)\|_{L^2}$ and $\|u\|_{L^2}$ converge. These numerical experiments confirm the convergence results of Theorem 4.1 and 4.2 and further suggest that the rates of convergence obey power rules.

The time snapshots of scalar evolutions are shown in Fig. 2 and Fig. 3 for closed-loop systems (3.6) and (3.10), respectively. It is clear that the latter demonstrates much better mixing results. This is also reflected in the comparison of the decay rates of the mix-norm $\|\theta\|_{(H^1(\Omega))^Y}$ in time (see Fig. 1[d]). From Fig. 3, we can see that the feedback control of system (3.10) produces faster and thinner foldings than those of system (3.6). This is expected because the flow induced by the second feedback law is much more irregular over the entire domain, which enhances chaotic mixing (cf. [3,46,23,41,42,57,1,5]), and the control input does not necessarily vanish. However, the spatial scale of the thin filaments generated in mixing tends to approach infinitely small over time. The mesh size must be sufficiently small to capture the smallest spatial scales of the thin filaments for longer time simulations, which poses a significant difficulty to numerical methods. In addition, from Figs. 2 and 3 we also notice that a better homogenization of the mixing process is achieved when increasing the parameter h for both closed-loop systems.

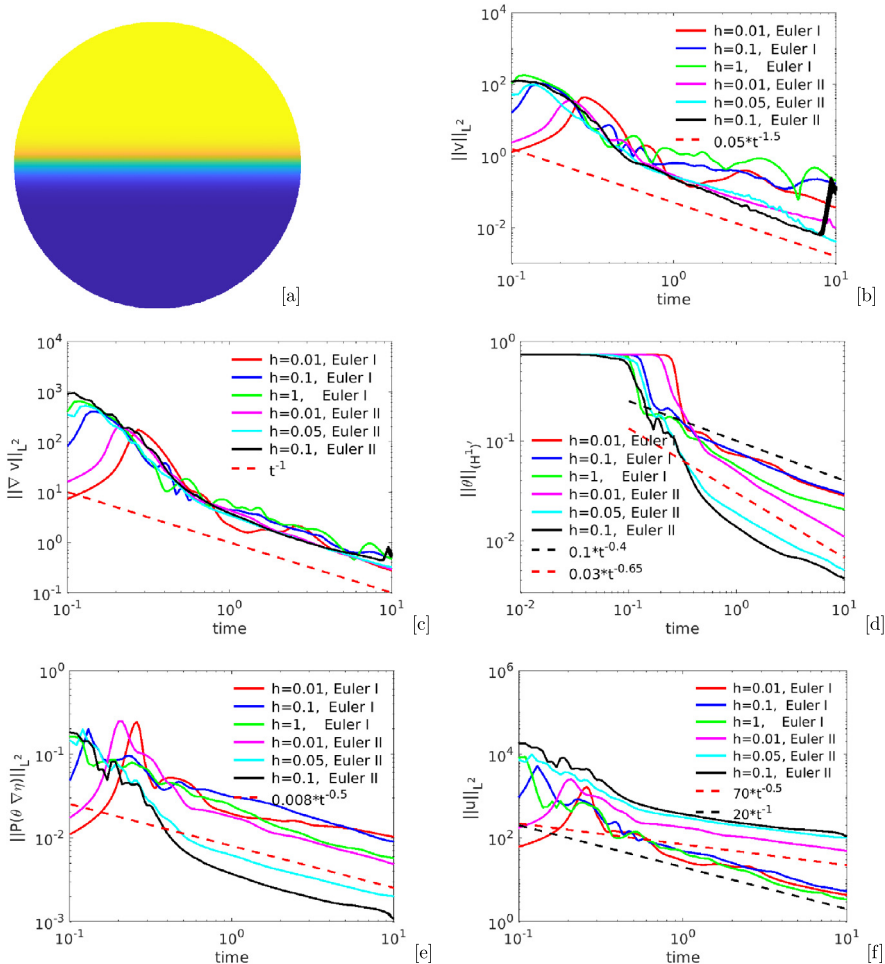


Fig. 1. [a]: Initial density distribution. [b]-[f]: Evolution of norms in time.

6. Conclusions

The current work considers a sub-optimal feedback control design for a fluid mixing problem governed by a flow-transport system. This essentially leads to a problem of nonlinear feedback control of a semi-dissipative system. The feedback law is established utilizing the ideas of instantaneous control method and a direct approximation of the optimality system derived from an optimal control problem. Rigorous analysis is presented to show the asymptotic behavior of the nonlinear closed-loop system subject to the feedback laws of different regularity. Numerical experiments show that when the flow has less regularity, the better mixing can be possibly achieved. Moreover, it is observed that the mixing decay rates of the closed-loop systems in terms of the $(H^1(\Omega))'$ -norm obey power rule in time.

However, a rigorous proof of optimality of such a nonlinear feedback law is still an open problem. Understanding how exactly the mechanism of the nonlinear feedback law plays in the enhancement of homogenization of a general scalar field, especially, its relation to the parameter

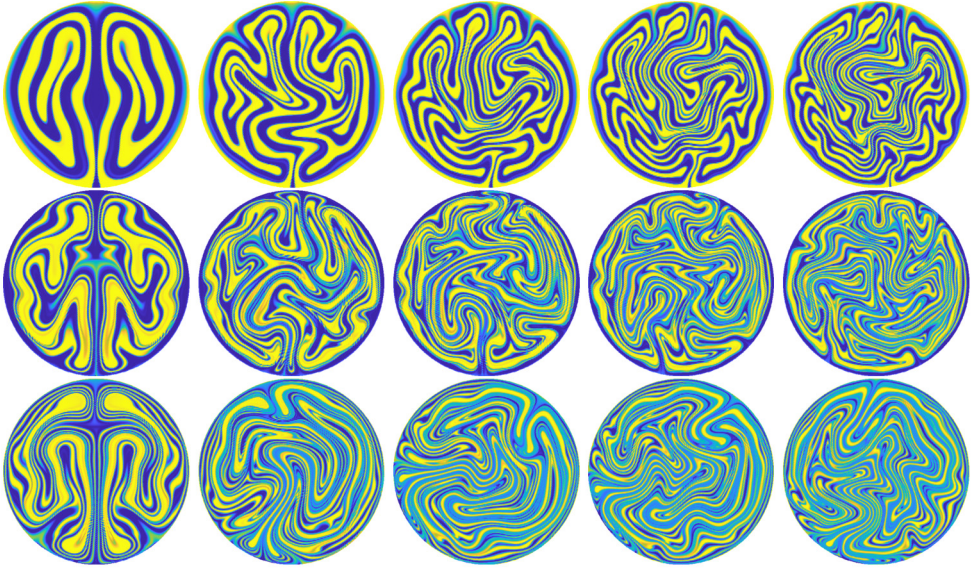


Fig. 2. Density snapshots of semi-implicit Euler method I. First row: $h = 0.01$. Second row: $h = 0.1$. Third row: $h = 1$. All the time frames are at $t = 1, 3, 5, 7, 10$.

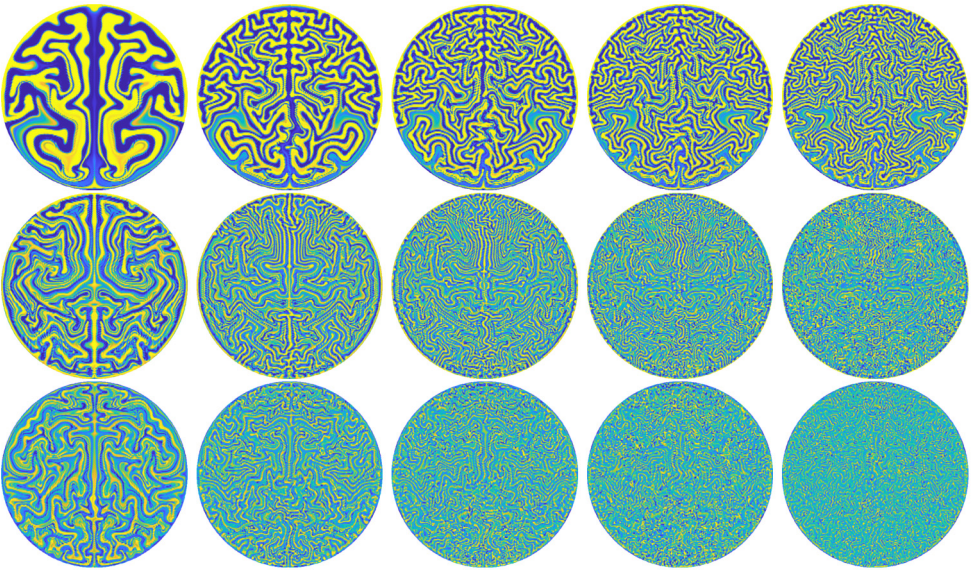


Fig. 3. Density snapshots of semi-implicit Euler method II. First row: $h = 0.01$. Second row: $h = 0.05$. Third row: $h = 0.1$. All the time frames are at $t = 1, 3, 5, 7, 10$.

h and the control weight γ requires a more in-depth analysis. Also, whether it is possible to establish the global-in-time *a priori* bound for $(v, \Lambda\eta)$ by using the internal control only exerted on a subdomain or boundary control will need a further investigation. These topics present new

and significant challenges and call for innovative analytical techniques from control theory and nonlinear analysis of semi-dissipative systems.

Data availability

No data was used for the research described in the article.

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