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Lebesgue measure of Feigenbaum Julia sets

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Abstract

We construct Feigenbaum quadratic-like maps with a Julia set of positive Lebesgue measure. Indeed, in the quadratic family $P_c : z \mapsto z^2 + c$ the corresponding set of parameters c is shown to have positive Hausdorff dimension. Our examples include renormalization fixed points, and the corresponding quadratic polynomials in their stable manifold are the first known rational maps for which the hyperbolic dimension is different from the Hausdorff dimension of the Julia set.

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1. Introduction

One of the major successes of the theory of one-dimensional dynamical systems was the conceptual explanation, in terms of the dynamics of a renormalization operator, of the striking universality phenomena discovered by Feigenbaum and Coullet-Tresser in 1970s. At the center of the picture lies the concept of a *Feigenbaum map*, which is a quadratic-like map that can be renormalized infinitely many times with bounded combinatorics and a priori bounds (a certain uniform control on the non-linearity). The successive renormalizations are then exponentially asymptotic to a *renormalization attractor*; see [Sul92], [McM96], [Lyu99]. In the simplest case of stationary combinatorics,

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the renormalization attractor consists of a single renormalization fixed point. As a consequence, the dynamics of such Feigenbaum maps display remarkable self-similarity reflected in the geometry of the corresponding Julia sets.

In fact, understanding the geometry of Feigenbaum Julia sets already played a key role in the first proof of exponential convergence of the renormalization [McM96]. However, for a long time the theory had been unable to tackle natural geometric problems: do Feigenbaum Julia sets have full Hausdorff dimension or even positive area? (See [McM96, p. 177, question 3]). In [AL08], a new approach to these problems was developed, which allowed us to show, in particular, that Feigenbaum Julia sets can have Hausdorff dimension strictly less than two, while leaving open the problem of whether they can ever have positive area. The goal of this work is to settle the latter question affirmatively. Namely, we will show that Julia sets of positive area appear already among Feigenbaum quadratic polynomials with stationary combinatorics. (Note that there are only countably many such polynomials.) At the same time, we construct a set of parameters c of positive Hausdorff dimension such that the quadratic polynomials $P_c : z \mapsto z^2 + c$ are Feigenbaum maps with Julia sets of positive area.

Note that our results (as well as the earlier results of [AL08]) go against intuition coming from hyperbolic geometry. Indeed, according to the philosophy known as *Sullivan's dictionary*,[1] there is a correspondence between certain objects and results in complex dynamics and hyperbolic geometry. As McMullen suggested in [McM96] (see especially the last paragraph on page 177), Feigenbaum maps are analogous to 3-manifolds with two ends, one of which is geometrically finite, while the other one is asymptotically fibered over the circle. The limit sets $\Lambda(\Gamma)$ of the corresponding Kleinian groups have zero area but full Hausdorff dimension; see Thurston [Thu82] and Sullivan[Sul81]. So, it may look like the dictionary completely breaks down at this point, though in fact there is a way to rehabilitate it; see Section 1.2.6 below.

1.1. *Feigenbaum maps.* Let us begin with reviewing briefly the main concepts of the complex renormalization theory. (See Section 2 for a precise brief account and [Lyu] for details.) A *quadratic-like map* is a holomorphic double covering $f : U \to V$, where U and V are quasidisks with U compactly contained in V. The *filled-in Julia set* of f is the set $K(f)$ of points z with $f^n(z) \in U$ for all $n \geq 0$; its boundary is the *Julia set* $J(f)$. The filled-in Julia set is always a full compact set (i.e., a compact set with connected complement) that is either connected or totally disconnected, according to whether or not it contains the critical point.

[1] Remarkably, such a dictionary was already anticipated by Fatou; see [Fat29, p. 22].

Simplest examples of quadratic-like maps are given by restrictions of quadratic maps $P_c : z \mapsto z^2 + c$ to suitable neighborhoods of $K(P_c)$. The precise choice of the restriction is dynamically inessential, which is expressed by saying that they all define the same *quadratic-like germ*.

The *Mandelbrot set* $\mathcal{M}$ is defined as the set of parameters $c \in \mathbb{C}$ for which $K(P_c)$ is connected.

The central role of the quadratic family is made clear by Douady-Hubbard's *Straightening Theorem* that states that each quadratic-like map with connected Julia set is *hybrid conjugate* to a unique quadratic map P_c; i.e., there exists a quasiconformal map $h : (\mathbb{C}, K(f)) \to (\mathbb{C}, K(P_c))$ satisfying $h \circ f = P_c \circ h$ near $K(f)$ and with $\bar{\partial} h | K(f) = 0$ a.e. (almost everywhere). We say that P_c is the *straightening* of f, and we write $c = \chi(f)$.

A quadratic-like map $f : U \to V$ is said to be *renormalizable* with period $\mathfrak{p} \geq 2$ if the $\mathfrak{p}$-th iterate of f can be restricted to a quadratic-like map $g : U' \to V'$ such that the *little Julia sets* $K_j := f^j(K(g))$, $0 \leq j \leq \mathfrak{p} - 1$, are connected and do not cross each other (meaning that $K_j \setminus K_i$ are connected for $i \neq j$). We can always choose g to have the same critical point as f, and such a g is called the *pre-renormalization of period* $\mathfrak{p}$ of f. The smallest possible value of $\mathfrak{p}$ is called the *renormalization period* of f, and the corresponding pre-renormalization, considered up to affine conjugacy, is called the *renormalization* of f and denoted by Rf. The *renormalization operator* $f \mapsto Rf$ is then well defined at the level of affine conjugacy classes of quadratic-like germs.

The set of parameter values corresponding to renormalizable quadratic maps is disconnected. Its connected components are called (maximal) *Mandelbrot copies*, which can be of two types, *primitive* or *satellite*, according to whether they are canonically homeomorphic (via the straightening map $c \mapsto \chi(R(P_c))$) to the full Mandelbrot set or to $\mathcal{M} \setminus \{1/4\}$. (Note that $1/4$ is the *cusp* of the *main cardioid* bounding the "largest" component of the interior of $\mathcal{M}$.) Alternatively, (maximal) satellite copies can be distinguished by the property that they are "attached" to the main cardioid at the "missing" cusp.[2] They can also be distinguished dynamically: For the satellite renormalization (with the minimal period), all little Julia sets have a common touching point, while for the primitive renormalization, they are pairwise disjoint.

The *renormalization combinatorics* of a renormalizable quadratic-like map f is the Mandelbrot copy $\mathcal{M}'$ containing $\chi(f)$. The renormalization period only depends on the renormalization combinatorics, but the converse is false (except for period two). There are however only finitely many combinatorics corresponding to each period.

[2]Note that our terminology is slightly different from the conventional one: usually the "missing cusp" is added to a satellite copy.

Remark 1.1. The renormalization combinatorics can be alternatively encoded by a finite graph, the *Hubbard tree*, which describes the positioning of the little Julia sets (of the first pre-renormalization) inside the full Julia set. It coincides with the Hubbard tree of the superattracting map $\mathbf{f}_{c'}$, $c' \in \mathcal{M}'$, whose period is equal to the renormalization period of f.

If a renormalization Rf is itself renormalizable, then f is called *twice renormalizable* and its second renormalization is denoted $R^2 f$. Similarly, we can define n times renormalizable maps and the corresponding n-th renormalizations $R^n f$ with some periods $\mathfrak{p}_n$. Note that $R^n f$ is the renormalization of $R^{n-1} f$ with *relative* renormalization period $\mathfrak{q}_n = \mathfrak{p}_n / \mathfrak{p}_{n-1}$.

Assume now that f is *infinitely renormalizable*, i.e., the renormalizations $R^n f$ are well defined for all $n \geq 0$. We say that f has *bounded combinatorics* if the relative renormalization periods of the successive renormalizations $R^n f$, $n \geq 0$, remain bounded. The combinatorics is *stationary* if it is the same for all $R^n f$.

The "analytic quality" of a quadratic-like map $f : U \to V$ is measured by the *modulus* of the *fundamental annulus* $V \setminus \overline{U}$, denoted by $\operatorname{mod} f$. (The quality is poor if $\operatorname{mod} f$ is small.) An infinitely renormalizable map is said to have *a priori bounds* if all of its renormalizations have definite quality; i.e., the corresponding moduli are bounded away from zero. (A priori bounds are equivalent to precompactness of the full renormalization orbit $\{R^n f\}_{n \geq 0}$ in a suitable topology.) While by no means all infinitely renormalizable maps have *a priori bounds*, many do and, in particular, it is conjectured that bounded combinatorics implies a priori bounds (which has indeed been proved whenever the renormalization combinatorics of all the $R^n f$ are primitive [Kah06]).

A *Feigenbaum map* is an infinitely renormalizable quadratic-like map with bounded combinatorics and a priori bounds.

THEOREM 1.1. *There exists a Feigenbaum quadratic polynomial P_c with primitive stationary combinatorics whose Julia set J_c has positive area.*

Our methods yield, in fact, an infinite family of primitive Mandelbrot copies that have the property that all infinitely renormalizable maps whose renormalization combinatorics (for all the renormalizations) belong to this family have Julia sets of positive area. We recall that any finite family $\mathcal{F}$ of primitive Mandelbrot copies with $\#\mathcal{F} \geq 2$ defines an associated *renormalization horseshoe* $\mathcal{A}$ consisting of all quadratic-like maps that belong to the ω-limit of the renormalization operator restricted to those combinatorics; see [AL11] (complemented with [Kah06]) for a recent account of this result . The dynamics of $R|\mathcal{A}$ is topologically semiconjugate to the shift on $\mathcal{F}^{\mathbb{Z}}$, and the corresponding quadratic parameters in $\chi(\mathcal{A})$ form a Cantor set naturally labeled by $\mathcal{F}^{\mathbb{N}}$. This Cantor set has bounded geometry by [Lyu99, Lemma 9.6 and §1.5], so we can conclude (see Section 6.9):

THEOREM 1.2. *The set of Feigenbaum quadratic maps with Julia sets of positive area has positive Hausdorff dimension in the parameter space.*

In fact, we will show that this Hausdorff dimension is at least $1/2$.

1.2. *What do we learn about Julia sets of positive area?*

1.2.1. *Preamble*: *Area problem.* The problem of whether all nowhere dense Julia sets have zero area goes back to the classical Fatou's memoirs who gave first examples of such Julia sets [Fat19].[3] In 1980-90s, broad classes of Julia sets with zero area were given in [Lyu83], [Lyu91], [Shi95], [Yar95] and [Urb94], [PR98], [GS09]. First examples of rational maps[4] (in fact, quadratic polynomials) with nowhere dense Julia sets with positive area have been recently constructed by Buff and Cheritat [BC12] in a remarkable development that successfully brought to completion Douady's program from the mid-1990s. (See also Yampolsky [Yam08] for an alternative point of view on the final piece of their argument.) An important technical input to this program was supplied by the recent breakthrough in the Parabolic Renormalization Theory by Inou and Shishikura [IS08].

The strategy carried by Buff and Cheritat depends on a *Liouvillian mechanism* of fast rational approximation. It produces three type of examples: *Cremer*, *Siegel*, and *infinitely renormalizable with unbounded satellite combinatorics*. (We recall that a quadratic map with a periodic orbit β with irrationally indifferent multiplier $e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, is classified as *Siegel* or *Cremer* according to whether it is locally linearizable near β or not.)

Feigenbaum Julia sets have quite a different nature, so our work brings new light on the realm of Julia sets of positive area.

1.2.2. *Parameter visibility.* Julia sets of positive area are supposed to be *visible* objects. However, sets of parameters produced by the Liouvillian mechanisms (such as in [BC12]) tend to be tiny: they probably have zero Hausdorff dimension. (This is definitely so in the Cremer case as the whole set of Cremer parameters has zero Hausdorff dimension.)

By our previous work [AL08], Feigenbaum Julia sets of positive area are more robust: the existence of a single Feigenbaum Julia set of positive area

[3]What Fatou showed is that if $|Df(z)| > \deg f$ for all $z \in J(f)$, then $J(f)$ is a Cantor set of zero length.

[4]For transcendental entire functions, a class of Julia sets of zero area was described in [EL84], which included some exponential maps (see also [McM87] for this particular case), while examples of Julia sets of positive area appeared in [EL87], [McM87].

inside some renormalization horseshoe implies that there is a whole "sub-horseshoe" of them, restricted to which the renormalization dynamics is topologically conjugate to a subshift of finite type. This creates a parameter set of positive Hausdorff dimension. The construction we use to prove Theorem 1.2 is even more precise, providing us with full renormalization horseshoes and allowing us to obtain an effective estimate: the set of parameters c such that P_c is a Feigenbaum map of positive area has Hausdorff dimension at least $1/2$.

We note that it is expected that Lebesgue almost every quadratic map is hyperbolic,[5] and hence has a Julia set of not only zero Lebesgue measure but even of Hausdorff dimension less than two. It is unclear whether the set of all complex Feigenbaum parameters has Hausdorff dimension strictly less than two.[6] At the moment, it is only known that the Hausdorff dimension of these parameters is at least 1 [Lyu98].

1.2.3. *Poincaré series and Hausdorff dimension.* The notion of *Poincaré series* was transferred from the theory of Kleinian groups to Holomorphic Dynamics by Sullivan [Sul83], and it became an efficient tool in the study of Hausdorff dimension of Julia sets. Previously to our work, in all known cases the Hausdorff dimension of rational Julia sets coincided with the critical exponent of the Poincaré series (see [Urb94], [PR98], [GS09] and [AL08]). On the other hand, it was shown in [AL08] that equality must break down in the case of a Feigenbaum map with periodic combinatorics and positive Lebesgue measure Julia set.[7]

The critical exponent does coincide with the *hyperbolic dimension* for all Feigenbaum Julia sets (and indeed for all known cases of rational maps), so our examples display a definite gap between the Hausdorff dimensions of the Julia set and of its hyperbolic subsets. It is conceivable, however, that *for Julia sets of zero area, the critical exponent, Hausdorff dimension and hyperbolic dimension, are all equal* (without any further assumptions on the rational map). Note that this is the case for Feigenbaum Julia sets of zero area [AL08].

1.2.4. *Positive measure vs non-local connectivity.* There was a general feeling that these two phenomena are tightly linked as the examples constructed by Buff and Cheritat are probably all non-locally connected. (Note, in particular, that Cremer Julia sets are never locally connected.) On the other hand, all Feigenbaum Julia sets have well-behaved geometry and, in particular, are locally connected; see [HJ93], [Jia00], [McM94]. Note that local connectivity

[5] It would follow from the property (somewhat supported by the computer evidence) that all little Mandelbrot copies may have a "uniformly bounded shape."

[6] The real analogue of this statement is known to be true [AM].

[7] More recently, such a phenomenon was also observed in transcendental dynamics [UZ07].

makes a Julia set *topologically tame*: it admits an explicit topological model (see [Dou93]). Thus, our examples show that positive area is compatible with topological tameness.

Related to this issue is the fact that all Feigenbaum Julia sets constructed here have primitive combinatorics, while the previously known infinitely renormalizable examples had satellite combinatorics. In fact all known examples of infinitely renormalizable maps with non-locally connected Julia set have satellite combinatorics.

Remark 1.2. A priori bounds have been recently proved for some Feigenbaum maps with satellite combinatorics [DL18], which led, by adapting our methods, to the first examples of satellite Feigenbaum maps whose Julia sets have positive area.

1.2.5. *Wild attractors and ergodicity.* The measure-theoretic dynamics on Feigenbaum Julia sets of positive area had been well understood long before first examples (presented in this paper) were constructed. In particular, it is *ergodic* with respect to the Lebesgue measure [Pra98], and there is a uniquely ergodic Cantor attractor $\mathcal{O} \subset J(f)$ (of Hausdorff dimension strictly less than two) such that $\omega(z) = \mathcal{O}$ for a.e. $\in J(f)$; see [Lyu83]. Moreover, almost all orbits are equidistributed with respect to the canonical measure μ on $\mathcal{O}$ such that $\operatorname{supp} \mu = \mathcal{O}$.[8]

The measure-theoretic picture for Buff-Cheritat examples is more delicate and exotic, and has been addressed more recently. In the Cremer and Siegel cases, it was proven by D. Cheraghi [Che13], [Che19] that there is a unique measure-theoretic attractor of zero area such that $\omega(z) = \mathcal{O}$ for a.e. $z \in J(f)$. However, this attractor has quite an intricate topology: it is a non-locally connected "hedgehog." Moreover, it was proven in [AC18] that the dynamics on $\mathcal{O}$ is uniquely ergodic, with the canonical invariant measure μ that is either the delta-mass at the Cremer point or the inner harmonic measure on the boundary of the Siegel disk. This measure governs distribution of almost all points on the Julia set, so typical points spend lion's share of time near $\operatorname{supp} \mu$, which is a proper subset of $\mathcal{O}$.[9] This can be viewed as a "second order wild phenomon." Note also that it remains unknown whether the Lebesgue measure on $J(f)$ is ergodic.

1.2.6. *Sullivan's Dictionary.* A parallel spectacular development in the problem of area and Hausdorff dimension has happened in the Theory of

[8] See [Lyu87, Ree86] and [BKNvS96] for related phenomena in transcendental and real dynamics.

[9] Such a phenomen had been earlier encountered in the real dynamics; see [HK90], [Zak78, BM10].

Kleinian groups. However, the outcome appeared to be quite different. In the mid 1990s, it was proved by Bishop and Jones [BJ97] that the limit set $\Lambda = \Lambda(\Gamma)$ of a (finitely generated) Kleinian group Γ has full Hausdorff dimension if and only if the group is geometrically infinite or $\Lambda = \mathbb{C}$. As geometrically finite groups correspond to hyperbolic or parabolic rational maps, we see that the answer for Kleinian groups is much simpler.

As the area is concerned, it had been the subject of the long-standing *Ahlfors Area Conjecture* asserting that any limit set $\Lambda(\Gamma)$ has zero area as long as it is different from the whole sphere. Through the work of Thurston [Thu82], Bonahon [Bon86] and Canary [Can93], this conjecture was reduced to *Marden's Tameness Conjecture*, and the latter was proved in the mid 2000s by Agol [Ago04] and Calegary-Gabai [CG06]. Thus, there are no non-trivial limit sets Λ of positive area; again, the situation for Kleinian groups is much more definite compared with rational maps.

It does not mean, however, that Sullivan's Dictionary between Kleinian groups and rational maps completely breaks down at this point. Kleinian groups belong to a special class of *reversible* dynamical systems: the corresponding geodesic flow on the hyperbolic 3-manifold M_Γ admits a nice involution that conjugates it to the inverse flow. The analogous flow for a rational map f lives on the hyperbolic 3-lamination $\mathcal{H}_f$ constructed in [LM97]. However, this flow is not reversible, which reflects the *unbalanced* property (see the next section) of the underlying maps and bears responsibility for richer geometric properties of Julia sets.

1.3. *Basic trichotomy.* To put our result into deeper perspective, let us briefly recall the basic trichotomy of [AL08]. Consider the following alternative for Feigenbaum maps:

Lean case: $\mathrm{HD}(J(f)) < 2$;
Balanced case: $\mathrm{HD}(J(f)) = 2$ but $\operatorname{area} J(f) = 0$;
Black Hole case: $\operatorname{area} J(f) > 0$.

In that paper, we showed that if a periodic point of renormalization is either of Lean or Black Hole type, then this can be verified "in finite time," by estimating some geometric quantities associated to some (not necessarily the first) renormalization of f. Namely, let us define two parameters:

- η_n gives the probability for an orbit starting in the domain of f to enter the domain of the n-th pre-renormalization (see Section 2.3);
- ξ_n gives the probability that an orbit starting in the domain of the n-th pre-renormalization will never come back to it.

We showed that in the Lean case $\eta_n/\xi_n \to 0$ exponentially, in the Black Hole case $\eta_n/\xi_n \to \infty$ exponentially, and that in the Balanced case η_n/ξ_n remains bounded away from zero and infinity. Moreover, there is an effective constant

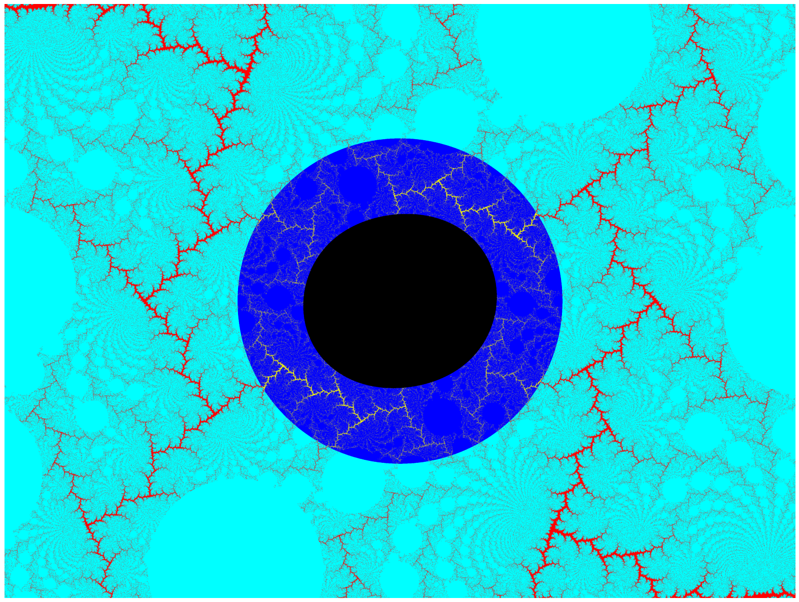

Figure 1.1. Black Hole. We see a renormalization of a quadratic-like map of type constructed in this paper. Light bubbles comprise the landing set. Rays emanating by the Black Hole correspond to escaping points. The probability of landing clearly dominates that of escape.

$C > 1$ (given in terms of some rough geometric parameters, like $\operatorname{mod} f$, but independent of n) such that if $R^n f = f$, then

- $\eta_n/\xi_n > C$ implies the Black Hole case;
- $\eta_n/\xi_n < C^{-1}$ implies the Lean case.

Remark 1.3. The latter condition has been recently used by A. Dudko and S. Sutherland [DS20] to give a computer assisted proof that area $J(f) = 0$ for the most classical Feigenbaum map corresponding to the doubling renormalizations.

Regarding the Balanced case, Theorem 8.2 of [AL08] asserts that the existence of both Lean and Black Hole Feigenbaum maps inside some renormalization horseshoe implies that *there exist some Balanced Feigenbaum maps* in this horseshoe, but the construction does not yield a renormalization periodic point. In fact, in seems unlikely that Balanced maps with periodic combinatorics exist. (The geometric parameters would be too fine tuned for this to happen "by chance" given that there are only countably many periodic points of renormalization.)

Remark 1.4. See also the discussion in [AL08] on a related problem for real maps of the form $x \mapsto |x|^\alpha + c$. Therein one can vary the degree α of the critical point continuously to fine-tune the parameters, so the corresponding Balanced case is believed to exist (and a conditional proof is given, subject to a Renormalization Conjecture), but it is unlikely that the fine tuned degrees would ever correspond to an integer (even) number (i.e., to a polynomial).

1.4. *Strategy.* As discussed above, [AL08] gives a probabilistic criterion for the Black Hole property to hold for a fixed point of renormalization; it suffices to check that η_n/ξ_n is sufficiently large for some n. Below we will use this only in the particular case $n = 1$. We will produce a sequence of fixed points of renormalization $f_m : U_m \to V_m$ with $\mathfrak{p}_m \to \infty$ such that $\inf \eta(m) > 0$ while $\lim \xi(m) = 0$, where $\eta(m) = \eta_1(f_m)$ and $\xi(m) = \xi_1(f_m)$. We will also verify that the rough initial geometry of the fundamental annuli $V_m \setminus \overline{U}_m$ remains under uniform control. Since the "constant to beat" in the criterion only depends on such a control, this will show that for m sufficiently large, the criterion is satisfied so that the Julia set of f_m has positive Lebesgue measure.

It is easy to see that if the sequence $\chi(f_m)$ converges to a parameter c for which $\operatorname{area} K(P_c) = 0$, and the rough initial geometry remains under control, then $\eta(m) \to 0$. Given this observation, it is natural to consider sequences of renormalization combinatorics that approach a parameter c with either a Siegel disk or a parabolic point. In our argument, we will take c to have a Siegel disk of bounded type. One still has to select the combinatorics very carefully, and a number of natural options we had initially tried had either displayed degeneration of the geometry (for instance, with growing modulus of the fundamental annulus, which would make the landing probability $\eta(m)$ go to 0), or could not be treated in a definitive way without computer assistance.

We now describe the idea more precisely. Let us consider a quadratic polynomial P_c that has a Siegel disc S with rotation number $\theta = [N, N, \dots]$, N being big enough. Let $p_m/q_m = [N, \dots, N]$ be the continued fraction approximants to θ, and let P_{c_m} be the corresponding quadratic maps with parabolic fixed points with rotation numbers p_m/q_m. We perturb c_m within the (p_m/q_m)-limb (the connected component of $\mathcal{M} \setminus \{c_m\}$ not containing 0) to a Misiurewicz map P_{a_m}, i.e., one for which the critical orbit is eventually periodic but not periodic. Then we further perturb a_m to a superattracting parameter b_m. This parameter is the center of some maximal primitive Mandelbrot copy $\mathcal{M}_m$.

Let $f_m : U_m \to V_m$ be the corresponding renormalization fixed points with stationary combinatorics $\mathcal{M}_m$. To control the dynamics of these maps in what follows, we need a good control of the postcritical set after all the perturbations. This has also been crucial in Buff and Cheritat's work [BC12], who proved using the Inou-Shishikura renormalization theory [IS08] (which currently is only available for large N, hence the choice above,[10] that the postcritical set of P_{c_m} stays in a small neighborhood of the Siegel disk S. Our further choice of a_m and b_m is in part designed to keep this property for the further perturbations. In particular, excursions of the critical orbit away from the Siegel disk must be prevented to avoid excessive expansion (which would again lead to growing fundamental annuli). Thus, the periodic orbit on which the critical point eventually lands must be taken quite close to the Siegel disk. The most natural choice would be the periodic orbit with combinatorial rotation number p_m/q_m that arises from the bifurcation of P_{c_m}, but for technical implementation reasons we actually use an orbit of rotation number $p_{m-\kappa}/q_{m-\kappa}$, for some big but bounded (as $m \to \infty$) κ (so that the critical point still only goes a bounded number of levels up in terms of the cylinder Siegel renormalization).

We then fine tune the superattracting parameter b_m to get a suitable control on the initial geometry of the first renormalization. While we want the moduli of fundamental annuli to remain bounded, we would like them to be sufficiently large to obtain control on the actual renormalization fixed point. Indeed, there is a "threshold" lower bound on the moduli of the fundamental annuli of the first renormalization of a Feigenbaum quadratic map with stationary primitive combinatorics, which, once surpassed, implies uniform control for the associated renormalization fixed point. Below this threshold, current techniques do not give such uniform bounds without further restrictions (which would, in particular, not apply when approaching Siegel parameters). Thus, we make the critical orbit (after perturbation) follow closely the periodic orbit for large but bounded number of turns around the Siegel disk, picking up the right amount of expansion from the periodic orbit before drifting apart and closing.

Once the geometry of the first renormalization is controlled, we construct a *safe trapping disk* D that stays away from the postcritical set, captures all orbits that escape from the Siegel disk S to infinity and has the property that a definite portion of D lands in the renormalization domain U. Then a direct Distortion Argument implies that the pullbacks of U occupy a definite proportion of S, which implies that the landing probability η_m stays bounded away from 0.

To control the escaping parameter ξ_m, we make use of the *Siegel Return Machinery* that ensures high probability of returns back to the trapping disk, and hence high probability of eventual landing in the renormalization

[10]Recent developments in the Pacman Renormalization Theory [DLS20], [DL18], [DL21] give a good chance to extend our construction to arbitrary N's.

domain U. (The Return Machinery makes use of the *hyperbolic expansion* outside the postcritical set [McM94], which was also used by Buff and Cheritat [BC12]).

In this construction, there is one free parameter that can be varied without significant impact on the geometry of the first renormalization, which is the time the critical point spends in the *parabolic gate* created when the parabolic map P_{c_m} is perturbed to the Misiurewicz map P_{a_m}. There is a uniform control of this perturbation governed by the limiting transit map (the *geometric limit*). Varying this time parameter produces a sequence of Black Hole combinatorics whose Mandelbrot copies decay quadratically. Alternating these combinatorics creates a Cantor set of Hausdorff dimension $> 1/2 - \epsilon$ consisting of Black Hole parameters.

To carry out the above strategy, we make use of four Renormalization Theories:

- Renormalization of *quadratic-like maps*, including the probabilistic criterion of [AL08], is discussed in Section 2.
- Renormalization of *quasicritical circle maps* is developed in Section 3. (Roughly speaking, "quasicritical" means that the map is allowed to lose analyticity at the critical point, but is assumed to be quasiregular there.)
- *Siegel* renormalization theory based upon renormalization of quasicritical circle maps is laid down in Section 4.
- Finally, in Section 5 we briefly discuss the *parabolic* renormalization, and particularly, the *Inou-Shishikura Theory*.

With these renormalization tools in hands, we proceed to the main construction (Section 6).

1.5. *Basic terminology and notation.*

- $\mathbb{N}_0 = \{0, 1, \dots\}$, $\mathbb{N} \equiv \mathbb{N}_1 = \{1, 2, \dots\}$, and in general, $\mathbb{N}_\kappa = \{n \in \mathbb{N} : \ n \geq \kappa\}$;
- $\bar{\mathbb{N}}_\kappa = \mathbb{N}_\kappa \cup \infty$ (with the natural topology);
- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$;
- $\mathbb{D}_R(a) = \{z : \ |z - a| < R\}$;
- $\mathbb{D}_R = \mathbb{D}_R(0)$, $\mathbb{D} = \mathbb{D}_1$;
- the notation $\mathbb{T}$ will be used for both the unit circle in $\mathbb{C}$ and its angular parametrization by $\mathbb{R}/\mathbb{Z}$;
- $\mathbb{H} = \mathbb{H}_+ = \{z : \ \operatorname{Im} z > 0\}$ is the upper half-plane;
- $\mathbb{H}_-$ is the lower half-plane;
- "area" refers to the Lebesgue measure;
- for a set $Z \subset \mathbb{C}$ and a point $z \in Z$, we let $\operatorname{Comp}_z(Z)$ be the component of Z containing z;
- for a topological annulus $A \Subset \mathbb{C}$, we let $\partial^o A$ and $\partial^i A$ be its *outer* and *inner* boundaries;

- a *topological triangle* is a Jordan disk with three points marked on the boundary;
- the dashed arrow notation $f : X \dashrightarrow Y$ is used for a partially defined map;
- $\operatorname{Dom} f$ is the domain of a map f;
- $\operatorname{orb} z = \operatorname{orb}_f z$ is the forward orbit of a point z;
- we use the notation c_0 for the critical point of various maps, $c_n := f^n c_0$;
- $\mathcal{O}_f$ is the postcritical set of a map f, i.e., the closure of $\operatorname{orb} c_1$;
- $\mathbf{f}_\theta : z \mapsto e^{2\pi i\theta} z + z^2$, $\theta \in \mathbb{C}/\mathbb{Z}$;
- $\boldsymbol{\mathcal{F}} = (\mathbf{f}_\theta)_{\theta \in \mathbb{C}}$ is the quadratic family — note that we are using a non-standard parametrization for the quadratic family, which is more suitable for our purposes;
- $\mathcal{M}$ is the Mandelbrot set.

By saying that some quantity, e.g., η, depending on parameters is *definite*, we mean that $\eta \geq \epsilon > 0$ where ϵ is independent of the parameters (or rather, it may depend only on some, explicitly specified, parameters). By saying that a set K is *well inside a domain* $D \Subset \mathbb{C}$ we mean that $K \Subset D$ with a definite $\operatorname{mod}(D \setminus K)$ (which is equivalent to saying that $\operatorname{dist}(K, \partial D) \geq \epsilon \operatorname{diam} K$). The meaning of the expressions *bounded*, *comparable*, etc. is similar. If we need to specify a constant, then we say "ϵ-definite," "C-comparable ($\asymp$)," etc.

Given a pointed domain (D, β), we say that β *lies in the middle of* D, or equivalently, that D *has a bounded shape around* β if

$$\max_{\zeta \in \partial D} |\beta - \zeta| \leq C \min_{\zeta \in \partial D} |\beta - \zeta|, \tag{1.1}$$

where C is a constant that may depend only on specified parameters.

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2. **Quadratic-like maps**

2.1. *Basic definitions.*

2.1.1. *Quadratic-like maps.* A *quadratic-like map* $f : U \to V$ [DH85b], which will also be abbreviated as a *q-l map*, is a holomorphic double branched covering between two Jordan disks $U \Subset V \subset \mathbb{C}$. It has a single critical point that we denote c_0. The annulus $A = U \setminus \overline{V}$ is called the *fundamental annulus* of f. We let $\operatorname{mod} f := \operatorname{mod} A$. The *filled Julia set* $K(f)$ is the set of non-escaping points

$$K(f) = \{z : f^n z \in U,\ n = 0, 1, 2, \dots\}.$$

Its boundary is called the *Julia set* $J(f)$. The (filled) Julia set is either connected or Cantor, depending on whether the critical point is non-escaping (i.e., $c_0 \in K(f)$) or otherwise.

Two quadratic-like maps $f : U \to V$ and $\tilde{f} : \tilde{U} \to \tilde{V}$ are called *hybrid conjugate* if they are conjugate by a quasiconformal map $h : (V, U) \to (\tilde{V}, \tilde{U})$ such that $\bar{\partial} h = 0$ a.e. on $K(f)$.

A simplest example of a quadratic-like map is provided by a quadratic polynomial $P_c : z \mapsto z^2 + c$ restricted to a disk $\mathbb{D}_R$ of sufficiently big radius. The Douady and Hubbard *Straightening Theorem* asserts that any quadratic-like map f is hybrid conjugate to some restricted quadratic polynomial P_c. Moreover, if $J(f)$ is connected, then the parameter $c \in \mathcal{M}$ is unique.

As for quadratic polynomials, the two fixed points of a quadratic-like maps with connected Julia set have a different dynamical meaning. One of them, called β, is the landing point of a proper arc $\gamma \subset U \setminus K(f)$ such that $f(\gamma) \supset \gamma$. It is either repelling or parabolic with multiplier one. The other fixed point, called α, is either non-repelling or a *cut-point* of the Julia set (can be both).

2.1.2. *Quadratic-like families.* A *quadratic-like family* $\mathbb{F} = (f_\lambda : U_\lambda \to V_\lambda)$ over a parameter domain[11] $\Lambda \subset \mathbb{C}$ is a family of quadratic-like maps f_λ holomorphically depending on λ. The latter means more precisely that the set

$$\mathbf{U} = \bigcup_{\lambda \in \Lambda} U_\lambda$$

is a domain in $\mathbb{C}^2$ and the function $f_\lambda(z)$ is holomorphic on $\mathbf{U}$. Let us normalize it so that 0 is the critical point for all f_λ. The associated Mandelbrot set is defined as

$$\mathcal{M}_{\mathbb{F}} = \{\lambda \in \Lambda :\ J(f_\lambda) \text{ is connected}\}.$$

Let us select a base point $\lambda_\circ$ and let $U_\circ \equiv U_{\lambda_\circ}$ etc. We say that a quadratic-like family $\mathbb{F}$ is *equipped* if there is a holomorphic motion

$$h_\lambda : \overline{V}_\circ \setminus U_\circ \to \overline{V}_\lambda \setminus U_\lambda$$

of the (closed) fundamental annulus $\overline{V}_\lambda \setminus U_\lambda$ over the pointed domain $(\Lambda, \lambda_\circ)$ that is *equivariant* on the boundary of the annulus, i.e.,

$$h_\lambda(f_\circ(z)) = f_\lambda(h_\lambda(z)), \quad z \in \partial U_\circ.$$

[11] In what follows, Λ is assumed to be a Jordan disk.

An equipped quadratic-like family $\mathbb{F}$ is called *proper* if $f_\lambda(0) \in \partial V_\lambda$ for $\lambda \in \partial\Lambda$ (which assumes implicitly that the family f_λ is continuous up to $\partial\Lambda$).

A quadratic-like family $\mathbb{F}$ is called *unfolded* if the curve

$$\lambda \mapsto f_\lambda(0), \ \lambda \in \partial\Lambda,$$

has winding number 1 around 0.

THEOREM 2.1 ([DH85b]). *For any equipped proper unfolded quadratic-like family $\mathbb{F}$, the Mandelbrot set $\mathcal{M}_{\mathbb{F}}$ is canonically homeomorphic to the standard Mandelbrot set $\mathcal{M}$.*

The proof can be also found in [Lyu].

2.2. *Renormalization.* A quadratic-like map $f : U \to V$ is called *DH renormalizable* (after Douady and Hubbard) if there is a quadratic-like restriction

$$Rf \equiv R_{\mathrm{DH}} f = f^{\mathfrak{p}} : U' \to V'$$

with connected Julia set K' such that the sets $f^i(K')$, $k = 1, \dots, \mathfrak{p} - 1$, are either disjoint from K' or else touch it at its β-fixed point.[12] In the former case the renormalization is called *primitive*, while in the latter it is called *satellite*.

The map $Rf : U' \to V'$ is called the *pre-renormalization* of f. If it is considered up to rescaling (i.e., up to conjugacy by a linear map $z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$), it is called the *renormalization* of f.

The sets $f^i(K')$, $i = 0, \dots, \mathfrak{p} - 1$, are referred to as the *little (filled) Julia sets*. Their “positions”[13] in the big Julia set $K(f)$ determines the renormalization *combinatorics*. The set of parameters c for which the quadratic polynomial P_c is renormalizable with a given combinatorics forms a *little Mandelbrot copy* $\mathcal{M}' \subset \mathcal{M}$. In fact, the family of renormalizations $R(P_c)$, $c \in \mathcal{M}'$, with a given combinatorics can be included in a quadratic-like family $\mathbb{F} = (f^{\mathfrak{p}} : U_c \to V_c)$ over some domain $\Lambda \supset \mathcal{M}'$ so that $\mathcal{M}' = \mathcal{M}_{\mathbb{F}}$. A natural base point $c_\circ \in \mathcal{M}'$ in this family is the superattracting parameter with period $\mathfrak{p}$. It is called the *center* of $\mathcal{M}'$. Any superattracting parameter in $\mathcal{M}$ with period $\mathfrak{p} > 1$ is the center of some Mandelbrot copy $\mathcal{M}'$ like this. Moreover, in case of primitive combinatorics the quadratic-like family $\mathbb{F}$ is proper and unfolded. (See [DH85b], [Dou87a], [Lyu] for a discussion of all these facts.)

We can encode the renormalization combinatorics by the corresponding copy $\mathcal{M}'$ itself. Equivalently, it can be encoded by the center $c_\circ$ of $\mathcal{M}'$ or by the corresponding Hubbard tree H'.

A little Mandelbrot copy is called *primitive* or *satellite* depending on the type of the corresponding renormalization. They can be easily distinguished

[12] See [McM94] for a discussion of this condition.

[13] They can be defined precisely in terms of the combinatorial model for f (see [Lyu, §37.11.2]).

as any satellite copy is attached to some hyperbolic component of $\operatorname{int}\mathcal{M}$ and does not have the cusp at its root point.

In the introduction (Section 1.1), we have introduced n times renormalizable maps and their renormalizations $R^n f$. Moreover, for infinitely renormalizable maps, we have defined notions of *stationary / bounded combinatorics*, a priori bounds and *Feigenbaum maps*. We say that a Feigenbaum map is *primitive* if all its renormalizations are such.

One says that a family $\mathcal{F}$ of Feigenbaum maps (e.g., the family of maps with a given combinatorics) has *beau bounds* if there exists $\mu > 0$ such that for any $\nu > 0$, there exists $n_0 = n_0(\nu)$ such that for any $f \in \mathcal{F}$ with $\operatorname{mod} f \geq \nu$, we have

$$\operatorname{mod} R^n f \geq \mu \quad \text{for all } n \geq n_0.$$

It was proved by Kahn [Kah06] that *infinitely renormalizable maps of bounded primitive type have beau bounds*, with μ depending only on the combinatorial bound. In fact, μ can be made uniform over some class of bounded primitive combinatorics [KL08].[14]

The *renormalization fixed point* f_* is a quadratic-like map that is invariant under renormalization: $Rf_* = f_*$. In terms of the *pre*-renormalization, there exists a *scaling factor* $\lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}$ such that

$$Rf_*(z) = \lambda^{-1} f_*(\lambda z).$$

Theorem 2.2. *For any stationary combinatorics with a beau bound, there exists a unique renormalization fixed point f_* with this combinatorics. Moreover,* $\operatorname{mod} f_* \geq \mu$, *where $\mu > 0$ is the beau bound.*

This theorem was originally proved by Sullivan [Sul92]. Other proofs were given by McMullen [McM96], and recently, by the authors [AL11].

A priori bounds are called *unbranched* if the renormalizations

$$f_n \equiv R^n f : \mathbb{U}^n \to \mathbb{V}^n$$

with definite moduli can be selected so that $\mathbb{V}^n \cap \mathcal{O} = K^n \cap \mathcal{O}$ (where K^n is the filled Julia set of $R^n f$). For instance, it is sufficient that

$$\mathbb{V}^n \cap \mathcal{K}^n = K^n, \quad \text{where } \mathcal{K}^n = \bigcup_{j=0}^{q_n - 1} f^j(K^n)$$

and q_n is the period of K^n under f. In turn, it is sufficient to choose the renormalization domains so that the images $f^j(\mathbb{U}^n)$, $j = 1, \dots, q_n$, are pairwise disjoint. (Of course, such a choice is possible only for a primitive renormalization.)

[14] We will not use these results as the combinatorics we construct do not fall into the class [KL08]. On the other hand, *beau bounds* can be easily supplied for our class.

For maps with unbranched a priori bounds the renormalization domains $\mathbb{U}^n$ and $\mathbb{V}^n$ can be adjusted (replaced with domains U^n and V^n below) to assume nice topological and geometric properties:

(C1) $\mathbb{V}^n \cap \mathcal{O} \subset V^n \subset \mathbb{U}^n$;
(C2) $V^{n+1} \subset U^n$;
(C3) $f^k(\partial V^n) \cap V^n = \emptyset$, $k = 0, 1, \dots$ (the number of iterates depends on $z \in \partial V^n$ and continues for as long as $f^k_{n-2}(z)$ is well defined);
(G1) the fundamental annuli $V^n \setminus \overline{U^n}$ have bounded hyperbolic diameters in $V \setminus \mathcal{O}$;
(G2) $\operatorname{area}(V^n \setminus \overline{U^n}) \asymp \operatorname{area}(U^n) \asymp (\operatorname{diam} U^n)^2 \asymp (\operatorname{diam} V^n)^2$,

with constants depending only on the unbranched a priori bounds (see [AL08, §2.7 and Appendix A]). Under the above circumstances, bounds (G1)–(G2) together with unbranched a priori bounds are called *geometric bounds* for f. We will measure them by a single number $\mathfrak{g} = \mathfrak{g}(f) > 1$ such that $\mathfrak{g}$ or $\mathfrak{g}^{-1}$ gives an upper or a lower bound for the above geometric constants, e.g.,

$$\mathfrak{g}^{-1}(\operatorname{diam} V^n)^2 \le \operatorname{area}(V^n \setminus \overline{U}^n) \le \mathfrak{g}(\operatorname{diam} V^n)^2.$$

Remark 2.1. In the primitive case, the above domains can be selected so that the $\operatorname{mod}(V^n \setminus U^n)$ are definite. However, in the satellite case, the annuli $V^n \setminus U^n$ can degenerate.

2.3. *Probabilistic criterion for positive area.* Let us now introduce precisely probabilistic parameters η and ξ mentioned in the introduction. Let $f : U \to V$ be a Feigenbaum map with unbranched a priori bounds, and let $Rf : U' \to V'$ be its first pre-renormalization, $A' = U' \setminus \overline{V}'$ be the corresponding fundamental annulus.

The *landing parameter* η is the probability of landing in U'. Precisely, let $\mathcal{X} = \bigcup_{n \in \mathbb{N}} f^{-n} U'$ be the set of points in U that eventually land in U'. Then

$$\eta = \frac{\operatorname{area} \mathcal{X}}{\operatorname{area} U}. \tag{2.1}$$

The *escaping parameter* ξ is the probability of escaping from the fundamental annulus A'. Precisely, let $\mathcal{Y}$ be the set of points in A' that never return back to V':

$$\mathcal{Y} = \{z \in A' : \ f^n z \notin V' \text{ for } n \ge 1 \text{ (as long as } f^n z \in V)\}.$$

Then

$$\xi = \frac{\operatorname{area} \mathcal{Y}}{\operatorname{area} A'}. \tag{2.2}$$

The following result asserts that if the landing probability is much higher than the escaping one, then the Julia set has positive area.

THEOREM 2.3 (Black Hole Criterion [AL08]). *There exists $C = C(\mathfrak{g})$ (independent of combinatorial bounds) with the following property. Let f be a renormalization fixed point with a geometric bound $\mathfrak{g}$. If $\eta \geq C\xi$, then* area $J(f) > 0$.

3. **Quasicritical circle maps**

An (*analytic*) *critical circle map* is an analytic homeomorphism $f : \mathbb{T} \to \mathbb{T}$ of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with a single critical point c_0 of cubic type (that is, $f'''(c_0) \neq 0$). It is usually normalized so that $c_0 = 0$ in the angular coordinate.

To study Siegel disks of non-polynomial maps we need to enlarge this class allowing the map be only quasiregular at the critical point.

3.1. *Definition.* For this definition, it is convenient to use the complex model $\{|z| = 1\}$ for the circle $\mathbb{T}$ and the Blaschke maps

$$B_\alpha(z) = e^{2\pi i\alpha} z^2 \, \frac{z-3}{1-3z} \tag{3.1}$$

as the standard family of critical circle maps.

A *quasicritical circle map* is a homeomorphism $f : \mathbb{T} \to \mathbb{T}$ of the circle with the following properties:

(Q1) f is a real analytic diffeomorphism outside a single critical point c_0 normalized so that $c_0 = 1$; we let $c_n = f^n c_0$.

(Q2) f admits a quasiregular extension to a $\mathbb{T}$-symmetric annulus $\operatorname{Dom} f$ around $\mathbb{T}$ of the form $B_\alpha \circ h$ with some $\alpha \in \mathbb{R}/\mathbb{Z}$ and a global quasiconformal map h that is holomorphic near $z \in \operatorname{Dom} f$ whenever $f(z)$ lies on the same side of $\mathbb{T}$ as z.

It follows, in particular, that $f|\,\mathbb{T}$ *is quasisymmetric*. Moreover, it admits a quasiregular extension to a neighborhood of $\mathbb{T}$, symmetric with respect to $\mathbb{T}$, that is holomorphic in the domain

$$\operatorname{Dom}^h f = \{z \in \operatorname{Dom} f : \; z \text{ and } f(z) \text{ lie on the same side of } \mathbb{T}\ \} \cup \mathbb{T} \setminus \{c_0\}.$$

(Q3) $\operatorname{Dom}^h f$ is a topological disk whose upper part, $\operatorname{Dom}^h f \setminus \mathbb{D}$, is obtained from the outer annulus $\operatorname{Dom} f \setminus \mathbb{D}$ by removing a topological triangle

$$\mathcal{T} = \mathcal{T}_f \subset \operatorname{Dom} f \setminus \overline{\mathbb{D}}$$

with a vertex at c_0 and the opposite side on the outer boundary of $\operatorname{Dom} f$ (which are not included to the triangle). We let $\hat{\mathcal{T}}$ be the "double-triangle" that is the union of $\mathcal{T} \cup \{c_0\}$ and its mirror image, so $\operatorname{Dom} f = \operatorname{Dom}^h f \cup \hat{\mathcal{T}}$.

(Q4) $f : \operatorname{Dom}^h f \to \mathbb{C}$ is an immersion and $f : \mathcal{T} \to \mathbb{D}$ is an embedding.

Passing to the angular coordinate amounts to taking the universal covering[15] $f : (\mathrm{Dom}\, f, \mathbb{R}) \to (\mathbb{C}, \mathbb{R})$, where $\mathrm{Dom}\, f$ is a $\mathbb{Z}$-periodic $\mathbb{R}$-symmetric open strip around $\mathbb{R}$. Then $\mathrm{Dom}^h f$ is the union of disjoint $\mathbb{R}$-symmetric open disks D_k interescting $\mathbb{R}$ along the intervals $(k, k+1)$, respectively. The difference $(\mathrm{Dom}\, f \setminus \mathrm{Dom}^h f) \cap \mathbb{H}_+$ is the union of triangles $\mathcal{T}_k \subset \mathbb{H}_+$ with vertices at k. The difference $\mathrm{Dom}\, f \setminus \mathrm{Dom}^h f$ is the union of the corresponding double triangles $\hat{\mathcal{T}}_k$. Moreoover, each restriction $f : D_k \setminus \mathbb{H}_- \to \overline{\mathbb{H}}_+$ is an embedding intersecting the real line along an interval strictly containing the image of $[k, k+1]$. We will impose one more geometric property:

(Q5) The restriction $f|\, D_0$ admits a representation $\psi(h(z)^3)$, where h is as in (Q2) and ψ is a conformal map whose image contains $f[0,1]$.

Let Cir stand for the space of all quasicritical circle maps. The *geometry* of such a map is specified by the dilatation of the map h from (Q2), the size of $\mathrm{Dom}\, f$, and the space between the image of ψ and $f[0,1]$ in (Q5). We call f a (K, ϵ)-*quasicritical* if $\mathrm{Dil}\, h \le K$, $\mathrm{Dom}\, f$ contains the (2ϵ)-neighborhood of $\mathbb{T} \subset \mathbb{C}$, and the image of ψ in (Q5) contains the ϵ-neighborhood of $f[0,1]$. Let $\mathrm{Cir}(\overline{N}, K, \epsilon)$ denote the class of (K, ϵ)-quasicritical circle maps of type bounded by $\overline{N}$ (i.e., in the continued fraction expansion for the rotation number of f all the entries are bounded by $\overline{N}$), and let $\mathrm{Cir}(K, \epsilon) = \bigcup_{\overline{N}} \mathrm{Cir}(\overline{N}, K, \epsilon)$.

3.2. *Local properties near the critical point.*

3.2.1. *Scaling limits and John Property.* Let $\mathcal{N}(K)$ stand for the class of degree three normalized $\mathbb{R}$-symmetric K-quasiregular branch coverings $F : (\mathbb{C}, \mathbb{R}, 0, 1) \to (\mathbb{C}, \mathbb{R}, 0, 1)$ that are conformal in the topological sectors $S_\pm = F^{-1}(\mathbb{C} \setminus \mathbb{R}_\mp) \supset \mathbb{R}_\pm$, where F^{-1} is the branch of the inverse map preserving $\mathbb{R}_\pm$. Such a map can be represented in the form $F(z) = H(z)^3$, where $H : (\mathbb{C}, \mathbb{R}, 0, 1) \to (\mathbb{C}, \mathbb{R}, 0, 1)$ is a normalized $\mathbb{R}$-symmetric K-qc homeomorphism that conformally maps S_- onto the straight sector $\{|\arg z - \pi| < \pi/3\}$, and conformally maps S_+ onto $\{|\arg z| < \pi/3\}$. Let $\mathcal{N} = \bigcup \mathcal{N}(K)$.

Lemma 3.1. *For a map $F \in \mathcal{N}(K)$, we have*

$$S_- \supset \{|\arg z - \pi| \le \alpha\pi\}, \quad S_+ \supset \{|\arg z| \le \alpha\pi\},$$

where $\alpha > 0$ depends only on K.

Proof. We will deal with S_- only, as the argument for S_+ is the same. The inverse branch $F^{-1} : \mathbb{C} \setminus \mathbb{R}_+ \to S_-$ is the composition of $z \mapsto z^{1/3}$ with H^{-1}, so

$$S_- = H^{-1}(T_-), \quad \text{where } T_- = \{|\arg z - \pi| < \pi/3\}.$$

[15]We will not notationally distinguish a circle map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and its universal covering $f : \mathbb{R} \to \mathbb{R}$ (and neither the associated objects).

Since $H^{-1} : (\mathbb{C}, 0, 1) \to (\mathbb{C},0,1)$ is a normalized K-qc map, it is L_0-quasisymmetric on the whole plane with $L_0 = L_0(K)$.

For any $\zeta \in \mathbb{R}_-$, we have $\operatorname{dist}(\zeta, \partial T_-) = (\sqrt{3}/2)\,|\zeta|$. Take any $z \in \mathbb{R}_-$, and let $\zeta = H(z)$. By definition of L_0-quasisymmetry, we have

$$\frac{\operatorname{dist}(z, \partial S_-)}{|z|} \geq \frac{1}{L} \cdot \frac{\operatorname{dist}(\zeta, \partial T_-)}{|\zeta|} = \frac{\sqrt{3}}{2L},$$

with some L depending only on L_0. The conclusion follows. □

Any quasicritical circle map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ of class $\operatorname{Cir}(K, \epsilon)$, viewed as a map on a neighborhood of $\mathbb{R}$, can be non-dynamically normalized without changing its dilatation so that it fixes 0 and 1. Namely, for any $t \in (0, 1/2)$, let

$$F_t : (\mathbb{C}, \mathbb{R}, 0, 1) \to (\mathbb{C}, \mathbb{R}, 0, 1), \quad F_t(x) = \frac{f(tx) - c_1}{f(t) - c_1}. \tag{3.2}$$

LEMMA 3.2. *For $f \in \operatorname{Cir}(K, \epsilon)$, $t \in (0, 1/2)$, the family of rescalings F_t (3.2) is precompact in the topology of uniform convergence on compact subsets of $\hat{\mathbb{C}}$. All limit maps as $t \to 0$ belong to the class $\mathcal{N}(K)$.*

Proof. Given any radius $r > 1$, all the rescalings F_t are well defined on the disk $\mathbb{D}_r$ for t small enough. Moreover, by Property (Q2) they can be represented in the form $F_t(z) = H_t(z)^3$ on this disk, where H_t are normalized $\mathbb{R}$-symmetric K-qc maps. It follows that the F_t form a precompact family with limit maps of the form $H(z)^3$, where H are normalized $\mathbb{R}$-symmetric K-qc maps.

Moreover, the inverse branches F_t^{-1} are conformal in $(\mathbb{C} \setminus \mathbb{R}_\pm) \cap \mathbb{D}_{\delta/t^\gamma}$, with some $\delta > 0$ and $\gamma \in (0, 1)$ depending only on the geometry of f. (We use here that $|f(t) - c_1| = O(t^\gamma)$ due to the Hölder continuity of quasisymmetric maps.) Hence in the limit we obtain a map whose inverse branches are conformal in the whole slit planes $\mathbb{C} \setminus \mathbb{R}_\pm$. The conclusion follows. □

The above two lemmas imply

PROPOSITION 3.3. *For any quasicritical circle map $f \in \operatorname{Cir}(K, \epsilon)$, the domain $\operatorname{Dom}^h f$ contains local sectors*

$$T_-(f) = \{|\arg z - \pi| \leq \alpha\pi,\ |z| < \epsilon\} \quad \text{and} \quad T_+(f) = \{|\arg z| \leq \alpha\pi,\ |z| < \epsilon\}$$

with some $\alpha > 0$ depending only on (K, ϵ).

3.2.2. *Schwarzian derivative.* We will now show that quasicritical circle maps have negative Schwarzian derivative near the critical point. Let us begin with maps of class $\mathcal{N}$:

LEMMA 3.4. *Any map $F \in \mathcal{N}$ has negative Schwarzian derivative on the whole punctured line $\mathbb{R} \setminus \{0\}$.*

Proof. Let us consider an open interval $I = (a, d) \subset \mathbb{R} \setminus \{0\}$ as a Poincaré model of the hyperbolic line. Given a subinterval $J = (b, c) \Subset I$, let

$$|J : I| = \log \frac{(c-a)(d-b)}{(b-a)(d-c)} \tag{3.3}$$

stand for its hyperbolic length. The condition of negative Schwarzian derivative for F is equivalent to the property that F^{-1} is a hyperbolic contraction

$$|F^{-1}(J) : F^{-1}(I)| \le |J : I|$$

for any pair of intervals I and J as above.

Let us now consider the slit plane $\mathbb{C}(I) := \mathbb{C} \setminus (\mathbb{R} \setminus I)$ endowed with its hyperbolic metric. Then I is a hyperbolic geodesic in $\mathbb{C}(I)$. Let $\mathbb{D}(I)$ be the round disk based upon I as a diameter. It is the hyperbolic neighborhood of I in $\mathbb{C}(I)$ of certain radius r independent of I.

If $F \in \mathcal{N}$, then the inverse map $F^{-1} : I \to I'$ (where $I' = F^{-1}(I)$) extends to a holomorphic map $F^{-1} : \mathbb{C}(I) \to \mathbb{C}(I')$. By the Schwarz Lemma, it is a hyperbolic contraction. Since $F^{-1}(I) = I'$, we conclude that $F^{-1}(\mathbb{D}(I)) \subset \mathbb{D}(I')$. Applying the Schwarz Lemma again, we obtain that $F^{-1} : \mathbb{D}(I) \to \mathbb{D}(I')$ is contracting with respect to the hyperbolic metric in these disks. Since the hyperbolic metrics on I and I' are induced by the hyperbolic metrics in the corresponding disks, we are done. □

Remark 3.1. In fact, in the applications to the distortion bounds, the contracting property for the cross-ratios from (3.3), rather than the Schwarzian derivative, is directly used (see Theorem 3.7).

Proposition 3.5. *Any quasicritical circle map $f \in \mathrm{Cir}(K, \epsilon)$ has negative Schwarzian derivative in the δ-neighborhood of the critical point, where $\delta = \delta(K, \epsilon)$ depends only on the geometry of f.*

Proof. By Lemma 3.2, the rescalings F_t accumulate as $t \to 0$, uniformly over $f \in \mathrm{Cir}(K, \epsilon)$, on a compact set $\mathcal{K} \subset \mathcal{N}(K)$. By Lemma 3.4, the latter have negative Schwarzian derivative. By Proposition 3.3, the maps F_t are eventually (for $t < t_0(K, \epsilon)$) holomorphic in definite sectors $\{|\arg z| < \alpha\pi\} \cap \mathbb{D}$ and $\{|\arg z - \pi| < \alpha\pi\} \cap \mathbb{D}$. It follows that $SF_t \to SF$, $F \in \mathcal{K}$, uniformly on $\pm[1/2, 1]$, and hence the Schwarzian derivatives SF_t are eventually negative on these two intervals. By the scaling properties of the Schwarzian, we have $SF_t(x) = t^2 Sf(tx)$, and hence $Sf < 0$ on some punctured interval $[-\delta, \delta]$, with $\delta > 0$ depending only on the geometry of f. □

3.2.3. *Power expansion.* Let us consider a map F of class $\mathcal{N}$, and let $\mathrm{Dom}^h F = \{z : (\mathrm{Im}\, z) \cdot (\mathrm{Im}\, F(z)) > 0\}$. Recall from Lemma 3.1 that it consists of two disjoint topological sectors $S_\pm$ with the axes $\mathbb{R}_\pm$ mapped conformally onto $\mathbb{C} \setminus \mathbb{R}_\mp$ respectively. Let us slightly shrink these sectors; namely, for

$\beta \in (0,1)$, let

$$S_-(\beta) = \{z \in S_- : |\arg F(z)| > \beta\pi\},$$
$$S_+(\beta) = \{z \in S_+ : |\arg F(z)| < (1-\beta)\pi\}.$$

LEMMA 3.6. *Let us consider a map F of class $\mathcal{N}(K)$, and let $\beta \in (0,1)$. Then*

$$|F(z)| \geq C|z|^{1+\sigma} \quad \text{for } z \in S_\pm(\beta),\ |z| \geq 1,$$

where $\sigma > 0$ and $C > 0$ depend only on K and $\beta > 0$.

Proof. Since by Lemma 3.1 S_- contains the sector $\{|\arg z - \pi| < \alpha\pi\}$, we have

$$S_+ \subset \{|\arg z| < (1-\alpha)\,\pi\}.$$

Hence the inverse branch $F^{-1} : \mathbb{C} \setminus \mathbb{R}_- \to S_+$ can be decomposed as $\phi(z)^{1-\alpha}$, where $\phi : (\mathbb{C} \setminus \mathbb{R}_-, 0, 1) \to (\mathbb{C} \setminus \mathbb{R}_-, 0, 1)$ is a conformal embedding. For such a map, we have

$$|\phi(z)| \leq A|z| \quad \text{as long as } |z| \geq 1,\ |\arg z| < \pi(1-\beta), \tag{3.4}$$

where A depends only on $\beta > 0$. Indeed, the hyperbolic distance (in $\mathbb{C} \setminus \mathbb{R}_-$) from z as above to 1 is $\log|z| + O(1)$. (Note that by the scaling invariance, the hyperbolic distance from z to $|z|$ depends only on $\arg z$.) Since 1 is fixed under ϕ, the Schwarz Lemma implies (3.4). The conclusion for F on S_+ follows.

The argument for S_- is similar, except -1 is not the fixed point any more. But since F is quasiregular, $|\phi(-1)| \asymp 1$, and the Schwarz Lemma implies the assertion again. □

3.3. *Real geometry.* Due to the above local properties, quasicritical circle maps enjoy the same geometric virtues as usual analytic critical circle maps. The main results formulated below are proven in a standard way; see e.g., the monograph by de Melo and van Strien [dMvS93, Ch. IV, §1–5] for a reference.

3.3.1. *Koebe distortion bounds.* The following statement extends the usual Koebe distortion bounds to quasicritical circle maps:

THEOREM 3.7. *Let $f \in \mathrm{Cir}(K, \epsilon)$ be a quasicritical circle map. Let $J \subset I \subset \mathbb{R}/\mathbb{Z}$ be two nested intervals in $\mathbb{T}$, with I open. Assume that for some $n, m \in \mathbb{N}$, the intersection multiplicity of the intervals $f^{-k}I$, $k = 0, 1, \ldots, n$ is bounded by m and $|f^{-k}I| < \delta/2$ with δ from Proposition 3.5. Then*

$$|f^{-k}J : f^{-k}I| \leq C(K, \epsilon, m)\,|J : I|.$$

Proof. It is obtained by the standard cross-ratio distortion techniques; see [dMvS93]. To see the role of various properties of f, let us recall the main ingredients.

• *Denjoy Distortion control outside the $(\delta/2)$-neighborhood of c_0.* The distortion bound depends on C^2-norm of f on $\mathbb{T}$ and on $\sum_{k\in\mathcal{L}} |f^{-k}I|$, where $\mathcal{L}$ is the set of moments $k \le n$ for which $f^{-k}I \cap (-\delta/2, \delta/2) = \emptyset$. The C^2-norm of f depends only on (K, ϵ) by compactness of $\mathrm{Cir}(K, \epsilon)$ and the Cauchy control of the derivatives of holomorphic functions. The total length of the intervals $f^{-k}I$ is bounded m.

• *Contraction of the cross-ratio in the punctured δ-neighborhood of c_0.* This is concerned with the moments $k \le n$ when $f^{-k}I \subset (-\delta, \delta) \setminus \{0\}$. At these moments the hyperbolic length $|f^{-k}J : f^{-k}I|$ is contracted under f^{-1} by Proposition 3.5.

• *Quasisymmetric distortion control at the critical moments.* At the moments $k \le n$ when $f^{-k}I \ni c_0$, we have

$$|f^{-k-1}J : f^{-k-1}I| \le C(H, L) \cdot |f^{-k}J : f^{-k}I|,$$

where L is an upper bound for $|f^{-k}J : f^{-k}I|$ and $H = H(K, \epsilon)$ is the qs-dilatation of f near c_0. Due to the above contraction property, L is bounded in terms of H and the number s of critical moments. Since $s \le m$, the contribution of the critical moments to the total distortion is bounded. □

3.3.2. *No wandering intervals.* Recall that an interval $J \subset I$ is called *wandering* if $f^n J \cap J = \emptyset$ for any $n > 0$. The above Koebe distortion bounds lead to the following generalization of Yoccoz's No Wandering Intervals Theorem [Yoc84]:

Theorem 3.8. *A quasicritical circle map $f \in \mathrm{Cir}$ with an irrational rotation number $\theta \in \mathbb{R} \setminus \mathbb{Q} \pmod{\mathbb{Z}}$ does not have wandering intervals.*

It follows by the classical theory (Poincaré's thesis) that such a map f is topologically conjugate to the rigid rotation

$$T_\theta : x \mapsto x + \theta \mod 1.$$

When we want to specify the rotation number of circle maps under consideration, we will use notation Cir_θ, $\mathrm{Cir}_\theta(K, \epsilon)$, etc.

3.3.3. *Bounded geometry and dynamical scales.* The further theory largely depends on the Diophantine properties of θ encoded in its continued fraction expansion $[N_1, N_2, \dots]$. Let $p_m/q_m = [N_1, \dots, N_m]$ be the m-fold rational approximant to θ. The rotation number (and the map f itself) is called of *bounded type* if the entries of the expansion are bounded by some $\overline{N}$. The spaces of circle maps with rotation number bounded by $\overline{N}$ will be denoted $\mathrm{Cir}(\overline{N})$, $\mathrm{Cir}_\theta(\overline{N}, K, \epsilon)$, etc. (depending on how many parameters we need to specify).

The Koebe distortion bounds also imply a more general version of the Herman-Swiatek Theorem [Her86], [Świ98]:

THEOREM 3.9. *A quasicritical circle map $f \in \mathrm{Cir}(\overline{N}, K, \epsilon)$ of bounded type is H-quasisymmetrically conjugate to the rigid rotation T_θ, with $H = H(\overline{N}, K, \epsilon)$.*

The circle dynamics naturally encodes the continued fraction expansion of the rotation number, as the denominators q_n are the *moments of combinatorially closest approaches*[16] of the critical orbit $\{c_n\}$ back to the critical point c_0. Let us consider the corresponding intervals $I^n = [c_0, c_{q_n}]$ (i.e., the combinatorially shortest intervals bounded by c_0 and c_{q_n}). The orbits of two consecutive ones,

$$f^k(I^n),\ k = 1, \dots, q_{n+1} - 1 \quad \text{and} \quad f^k(I^{n+1}),\ k = 1, \dots, q_n - 1, \tag{3.5}$$

together with the *central* interval $I_0^n := I^n \cup I^{n+1}$ form a dynamical tiling $\mathcal{I}^n$ of $\mathbb{T}$. Moreover, these tilings are nested: $\mathcal{I}^{n+1}$ is a refinement of $\mathcal{I}^n$.

We label the intervals $I_k^n \in \mathcal{I}^n$, $k = 1, \dots, q_n + q_{n+1} - 2$, in an arbitrary way. Each of these intervals is homeomorphically mapped onto either $f^{q_{n+1}}(I^n)$ or $f^{q_n}(I^{n+1})$ by some iterate of f. We call it the *landing map* $L = L_n$ of level n. On the central interval I_0^n, we let $L_n = \mathrm{id}$.

In case of bounded type, Theorem 3.9 ensures that these tilings have bounded geometry,[17] i.e., the neighboring tiles are comparable, and hence the consecutive nested tiles are also comparable. This gives us a notion of *n-th dynamical scale* at any point $z \in \mathbb{T}$ (well defined up to a constant); it is the size of any tile $I^n(z) \in \mathcal{I}^n$ containing z.

More precisely, let $C_0 = C(\overline{N}, K, \epsilon) \geq 2$ be an upper bound for the ratios of any two neighboring and any two consecutive nested dynamical tiles. We say that a point $\zeta \in \mathbb{C}$ lies in n-th dynamical scale around $z \in \mathbb{T}$ if

$$C_0^{-1} |I_k^n| \leq |\zeta - z| \leq C_0 |I_k^n| \tag{3.6}$$

for the dynamical tile I_k^n of depth n containing z. Any point $\zeta \in \mathbb{D}_2$ lies in some dynamical scale around any $z \in \mathbb{T}$, and the number of such scales is bounded in terms of $(\overline{N}, K, \epsilon)$.

For $z \in \mathbb{T}$, we use notation $I^n(z)$ for the interval of the tiling $\mathcal{I}^n$ containing z. (If there are two such intervals, make an arbitrary choice.)

3.4. *Quasicritical circle pairs and their renormalizations.* A quasicritical circle map can be represented as a discontinuous map of the fundamental interval $[c_1 - 1, c_1]$, which motivates the following definition: A (*real*) *quasicritical*

[16] This means that these are the closest approaches for the corresponding circle rotation T_θ.

[17] This property is also referred to as *real a priori bounds*.

circle pair $F = (\phi_-, \phi_+)$ is a pair of real analytic homeomorphisms

$$\phi_- : [\beta_-, 0) \to [b, \beta_+), \quad \phi_+ : (0, \beta_+] \to (\beta_-, b] \tag{3.7}$$

with some $\beta_- \le 0 \le \beta_+$, $\beta_+ - \beta_- = 1$. Moreover, $c_0 = 0$ is the only critical point of the $\phi_\pm$ and this point is of quasicubic type, i.e., it has a local representation $h(x)^3 + c_1$ with a quasisymmetric h; compare with properties (Q1) and (Q2).

Renormalization R_{cp} of circle pairs is defined as follows. In the degenerate case $\beta_- = 0$ or $\beta_+ = 0$ (so that the critical point is fixed under ϕ_+ or ϕ_-) F is non-renormalizable. In the non-degenerate case, assume for definiteness that $b \in (\beta_-, 0]$. (Otherwise , one should change the roles of β_- and β_+.) If $\phi_-^N(\beta_-) \le 0$ for all $N \in \mathbb{N}$ (equivalently, there is a fixed point in $(\beta_-, 0)$), then F in still non-renormalizable.[18] Otherwise, let $N \ge 1$ be the biggest integer such that

$$\beta'_- := \phi_-^N(\beta_-) \le 0, \quad \beta'_+ := \beta_+,$$

and let

$$\phi'_- \,|\, [\beta'_-, 0] = \phi_-, \quad \phi'_+ \,|\, [0, \beta'_+] = \phi_-^N \circ \phi_+.$$

Rescaling the interval $[\beta'_-, \beta'_+]$ to the unit size by an orientation preserving[19] linear map, we obtain $R_{\mathrm{cp}}F$.

To see how the renormalization acts on the rotation numbers, let us consider the linear case (corresponding to the pure rotation). In this case, a convenient normalization of F is to let $\max(|\beta_-|, \beta_+) = 1$ (instead of $\beta_+ - \beta_- = 1$) leaving only one parameter $\beta = \min(|\beta_-|, \beta_+) \in [0, 1]$, together with the sign $s \in \pm$ such that $\beta = |\beta_s|$ (which are related to the rotation number θ of f by $\theta = s\beta/(1+\beta)$). Then N is the biggest integer such that $N\beta \le 1$, so N is the integer part of $1/\beta$. Under the renormalization, we obtain

$$\beta' = \frac{1 - N\beta}{\beta} = \frac{1}{\beta} \mod \mathbb{Z},$$

which is the Gauss map G applied to β, while $s' = -s$. (As in this renormalization scheme, the cases $\beta_+ = 1$ and $\beta_- = -1$ alternate.) Moreover, the β-number is equivariant under the renormalization: $\beta(RF) = G(\beta(F))$. In this way, the continued fraction expansion of β (and hence θ) is directly related to the renormalization dynamics. See [dMvS93, Ch. I, §1] for a detailed discussion.

Let us now adapt properties (Q1)–(Q5) to the setting of circle pairs $F = (\phi_\pm)$. We will rely upon the universal covering description of f from the end of Section 3.1. Let $\phi_\pm$ be the lifts of f such that $\phi_\pm(0) = \beta_\mp$. Their restrictions to

[18] In other words, maps with zero rotation number are non-renormalizable.

[19] Under the usual convention, the rescaling is orientation reversing. However, in further applications to Siegel maps, this would lead to some inconvenience.

the disks $D_- := D_{-1}$ and $D_+ := D_0$ are conformal. We let $\mathcal{T} := \mathcal{T}_0$, $\hat{\mathcal{T}} := \hat{\mathcal{T}}_0$, and

$$\operatorname{Dom} F := D_- \cup D_+ \cup \hat{\mathcal{T}}, \quad \operatorname{Dom}^h F := D_- \cup D_+.$$

We also let $I_- := [\beta_-, 0]$, $I_+ := [0, \beta_+]$, and $J_\pm := \phi_\pm(I_\pm)$. For $\epsilon > 0$, let $J_\pm(\epsilon)$ be the interval $J_\pm$ scaled by factor $(1+\epsilon)$ centered at its mid-points.

We can now record the following properties of our maps:

(P1) Each branch $\phi_\pm : D_\pm \to \mathbb{C}$ admits a quasiregular extension of the form $\phi_\pm = \psi_\pm \circ G$, where $G : \mathbb{C} \to \mathbb{C}$ is a global quasiregular map of some class $\mathcal{N}(K)$ while each $\psi_\pm$ is a conformal map on a domain $\Upsilon_\pm \supset G(\operatorname{Dom}^h_\pm F)$ whose range $\psi_\pm(G(\operatorname{Dom}^h_\pm F)) \Supset J_\pm(\epsilon)$ is a topological disk slit along two real rays.

(P2) The maps $\phi_\pm : \operatorname{Dom}^h_\pm F \cap \mathbb{H}_+ \to \mathbb{H}_+$ and $F : \mathcal{T} \to \overline{\mathbb{H}}_-$ are embeddings.

We let $\operatorname{Cir}_{\mathrm{cp}}(\overline{N}, K, \epsilon)$ be the class of quasicritical circle pairs of type bounded by $\overline{N}$ such that $G \in \mathcal{N}(K)$, $\operatorname{Dom} F$ contains the (2ϵ)-neighborhood of $[\beta_-, \beta_+] \subset \mathbb{C}$, and ϵ satisfies (P1).

We say that a quasicritical circle pair $F \in \operatorname{Cir}_{\mathrm{cp}}(\overline{N}, K, \epsilon)$ belongs to *Epstein class* $\mathcal{E}(\overline{N}, K, \epsilon)$ if the range of each of the above univalent maps $\psi_\pm$ contains the whole slit plane $\mathbb{C} \setminus (\mathbb{R} \setminus J_\pm(\epsilon))$.

Examples of such maps are provided by Blaschke maps B_θ (3.1) lifted to $\mathbb{C}$ by the exponential map $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^{2\pi i z}$.

The renormalization R_{cp} acts on the class of quasicritical circle pairs of type bounded by $\overline{N}$, as well as on the corresponding Epstein class.

3.5. *Butterflies and complex bounds.*

3.5.1. *Butterflies.* We will now introduce a class of quasicritical circle pairs with a nice external structure (which is a quasicritical version of butterfies introduced by Edson de Faria in the early 1990s). A *butterfly map*

$$f = (\phi_-, \phi_+) : (\hat{X}_-, \hat{X}_+) \to \hat{Y} \tag{3.8}$$

is a quasicritical circle pair with the following properties:

- $\hat{X}_\pm \supset \operatorname{int} I_\pm$ are disjoint $\mathbb{R}$-symmetric Jordan disks whose closures touch only at 0; we let $X_\pm = \hat{X}_\pm \cap \mathbb{H}$.
- $\hat{Y}$ is an $\mathbb{R}$-symmetric topological disk compactly containing the $X_\pm$; we let
$$Y := \hat{Y} \cap \mathbb{H}.$$
- Each $\phi_\pm$ maps the corresponding $X_\pm$ univalently onto Y.
- The maps $\phi_\pm$ admit a quasiregular extension as descrbed in (P1) (with $\hat{Y}_\pm = D_\pm$).

The configuration of the domains $X := X_+ \cup X_-$ sitting inside Y is called a *butterfly* (see Figure 3.1). The *filled Julia set* $K(f)$ is the set of points that never escape $\operatorname{Dom} f = X_+ \cup X_-$. Let $\operatorname{mod} f = \min(\operatorname{mod}(\hat{Y} \setminus \hat{X}_\pm))$.

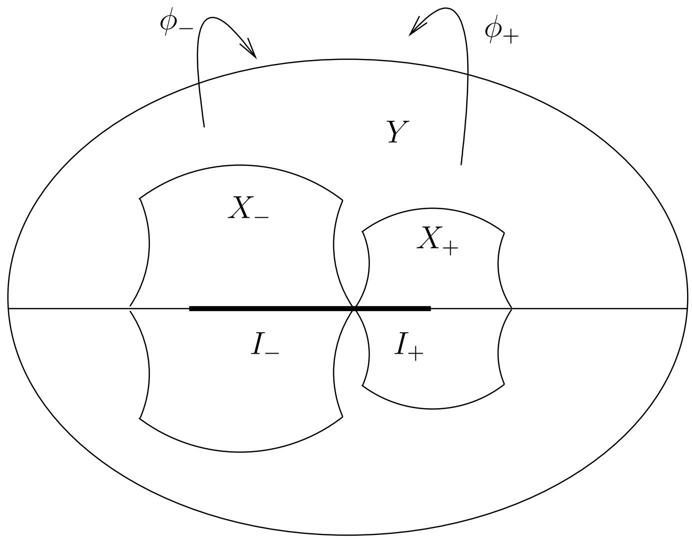

Figure 3.1. Butterfly.

Let us mark in $\hat{Y}$ the critical point $c_0 = 0$, and in $\hat{X}_\pm$ the critical value $c_\mp = \phi_\pm(0)$. We say that a butterfly has a *κ-bounded shape* if each of the marked domains can be mapped onto the marked unit disk $(\mathbb{D}, 0)$ by a global $\mathbb{R}$-symmetric κ-qc map.

The *geometry* of a butterfly is controlled by three parameters: μ (a lower bound on $\operatorname{mod} f$), κ (a bound on the shape of the butterfly), and B, a bound on the geometry of the intervals $\hat{X}_\pm \cap \mathbb{R}$ inside $\hat{Y} \cap \mathbb{R}$. The latter is defined as the best dilatation of a quasisymmetric map $(\hat{Y} \cap \mathbb{R}, 0) \to ([-1, 1], 0)$ that moves the boundary points of the intervals in question to some standard configuration. Let $\mathcal{B}(\overline{N}, \mu, \kappa, B)$ stand for the space of butterflies of type bounded by $\overline{N}$ whose geometry is controlled by the specified parameters. As usual, the class $\mathcal{B}(\mu, \kappa, B)$ is defined as the union of those.

3.5.2. *Complex bounds.* We are ready to state a quasicritical version of de Faria-Yampolsky complex bounds [dF99], [Yam99]:

THEOREM 3.10. *Let $f \in \mathrm{Cir}_{\mathrm{cp}}(\overline{N}, K, \epsilon)$ be a quasicritical circle pair. Then there exists an $\underline{l}$ depending only on $(\overline{N}, K, \epsilon)$ such that for all $m \geq \underline{l} - 1$, the pre-renormalizations $R^m_{\mathrm{cp}} f$ can be represented as butterflies $X^m_- \cup X^m_+ \to Y^m$ of class $\mathcal{B}(\overline{N}, \mu, \kappa, B)$ with $\hat{Y}^{\underline{l}-1} \Supset \hat{Y}^{\underline{l}} \Supset \dots$, and*

$$\operatorname{dist}(\partial \hat{Y}^m,\, \hat{Y}^{m+1}) \asymp \operatorname{dist}(\partial \hat{Y}^m,\, \hat{X}^m_\pm) \asymp \operatorname{diam} \hat{Y}^m.$$

All constants and bounds depend on $(\overline{N}, K, \epsilon)$ only.

Proof. The proof is the same as in the analytic case (at the last moment making use of Lemma 3.6). We recall the main steps, in the case of Epstein class, following the strategy of [LY97], [Yam99]. The general case can be carried following [dFdM00] by a more careful analysis at small scales, based on the real bounds (which are available in our setting by Theorem 3.9).

Let us consider the dynamical interval $I = I^n = [c_0, c_{q_n}]$ of some level $n \in \mathbb{N}$ attached to the critical point (see Section 3.3.3), and let $q = q_{n+1}$, $J = f^q(I)$. Then $f^q|I$ can be decomposed as $\psi \circ f$ where $\psi^{-1} : J \to f(I)$ admits a conformal extension to the slit plane $\mathbb{C} \setminus (\mathbb{R} \setminus J)$. Here is the Key Estimate: for any z outside $\mathbb{R}$, we have

$$\frac{\operatorname{dist}(\psi^{-1}(z)), |f(I)|}{|f(I)|} \leq A \left(\frac{\operatorname{dist}(z, J)}{|J|} \right) + B. \tag{3.9}$$

The proof uses only the real bounds and the Schwarz Lemma for holomorphic maps between slit planes. As both these ingredients are available for our class (as we always apply only holomorphic inverse branches of f), the Key Estimate is valid in this generality.

At the last moment we apply the inverse branch of the cubic quasiregular map f near its critical point. By Lemma 3.6, it is highly contracting in big (rel I) scales, beating a bounded expansion allowed by (3.9).

Now take a big $k \in \mathbb{N}$ and consider a disk $\mathbb{D}_\rho$ of size comparable with I^{n-k}. Let $\hat{Y}$ be $\mathbb{D}_\rho$ slit along two real rays corresponding to the range of the Epstein map $R^n f$. The contracting property discussed above implies that the pullbacks of $\hat{Y}$ by $R^n f$ are well trapped inside $\hat{Y}$. This produces a butterfly with a definite modulus $\mu > 0$.

Slightly shrinking $\hat{Y}^n$ (using the space in between $\hat{Y}^n$ and the $\hat{X}^n_\pm$) and taking its pullbacks under $R^n f$ once again, we obtain a butterfly with a bounded shape. ☐

Let us mention the following important special case that can be reduced directly to the Epstein class setting:

COROLLARY 3.11. *The above a priori bounds hold for a butterfly map* $f : X_- \cup X_+ \to Y$ *of class* $\mathcal{B}(\overline{N}, \mu_0, \kappa_0, B_0)$.

Proof. By uniformizing Y with the slit plane $\mathbb{C} \setminus (\mathbb{R} \setminus [-1, 1])$, we conformally conjugate our butterfly f to a map of Epstein class $\mathcal{E}(\overline{N}, K, \epsilon)$ with parameters depending only on $(\overline{N}, \mu_0, \kappa_0, B_0)$, which reduces the problem to this setting. ☐

Remark 3.2. For the same reason, all the statements formulated below for maps of Epstein class are also valid for butterfly maps.

3.5.3. *Expansion.* In this section we adapt some of McMullen's results (see [McM96, §6.2] and [McM98]) to our setting. For z in the upper half-plane, we will use notation $\operatorname{ang} z$ for $\min\{\arg z, \pi - \arg z\}$, where $\arg z \in (0, \pi)$. Together with the Schwarz Lemma, the complex bounds imply

LEMMA 3.12. *Under the conditions of Theorem 3.10, the renormalizations $f_m := R^m_{\mathrm{cp}} f$ are expanding in the hyperbolic metric of $\mathbb{H}$. Moreover,*

$$\|Df_m(z)\|_{\mathrm{hyp}} \geq \rho > 1, \quad m \geq \underline{l},$$

with ρ depending only on $(\overline{N}, K, \epsilon)$ and a lower bound on $\operatorname{ang} z$.

Proof. Assume for definiteness that $z \in X^m_+$ and $R^m f | X^m_+ = f^{q_m}$. Since each $X_\pm$ is univalently mapped onto $\mathbb{H}$ under f^{q_m}, there exists a disk $\mathbf{X}^m_+ \supset X^m_+$ that is univalently mapped onto $\mathbb{H}$ under f^{q_m}. The hyperbolic expanding factor of this map is equal to the inverse of $\|Di(z)\|_{\mathrm{hyp}}$, where $i : \mathbf{X}^m_+ \to \mathbb{H}$ is the natural embedding. This hyperbolic norm is bounded in terms of the upper bound on $\operatorname{dist}_{\mathrm{hyp}}(z, \partial \mathbf{X}^m_+)$ measured in $\mathbb{H}$. But by Lemma 3.1, if $\operatorname{ang} z > \omega > 0$, then z can be connected to ∂X^m_- by a circle arc γ whose Euclidean length divided by its Euclidean distance to $\mathbb{H}$ is bounded by some constant $C(\overline{N}, K, \epsilon; \omega)$. All the more, the same bound holds for the piece of this arc connecting z to $\partial \mathbf{X}_+$. The conclusion follows. ☐

THEOREM 3.13. *Let $f \in \mathcal{E}(\overline{N}, K, \epsilon)$ be a map of Epstein class. Then there exists $\rho > 1$ depending on $(\overline{N}, K, \epsilon)$ only such that if $z \in Y^m \cap \operatorname{Dom}^h f^n$ while $f^n z \in Y^{m-k} \setminus Y^{m-k+1}$ for some $n \in \mathbb{N}$, $0 < k < m$ (with $m - k > \underline{l}$), then*

$$\|Df^n(z)\|_{\mathrm{hyp}} \geq \rho^k,$$

where the norm is measured in the hyperbolic metric of the upper half-plane $\mathbb{H}$.

Proof. On its way from Y^m to $Y^{m-k} \setminus Y^{m-k+1}$, there exist $\asymp k$ levels $X^i := X^i_+ \cup X^i_-$ and corresponding moments n_i such that $z_i := f^{n_i} z \in X^i$ but $(R^i f)(z_i) \notin X^i$. Such a point z_i stays away from $\mathbb{R}$ (in the rescaled plane), unless either $\operatorname{ang} z_i$ or $\operatorname{ang}(R^i f)(z_i)$ is definite. Lemma 3.12 implies the desired assertion. ☐

3.5.4. *Compactness.* Let us normalize a complex pair $f : \hat{X}_+ \cup \hat{X}_- \to \hat{Y}$ so that $|\hat{Y} \cap \mathbb{R}| = 1$ and introduce the following topology on the space of normalized pairs. A sequence $f_n : \hat{X}^n_+ \cup \hat{X}^n_- \to \hat{Y}^n$ converges to a pair $f : \hat{X}_+ \cup \hat{X}_- \to \hat{Y}$ if the domains $\hat{Y}^n$ Carathéodory converge to $\hat{Y}$ and the inverse branches $(f_n)^{-1} : \hat{Y}^n \to \hat{X}^n_\pm$ converge to the corresponding branches of f^{-1} uniformly on compact subsets of $\hat{Y}_\pm$. (See [McM94, §5.1] or [Lyu, §7.7] for a discussion of the Carathéodory topology.) Standard compactness properties of the Carathéodory topology imply

PROPOSITION 3.14. *The butterfly space $\mathcal{B}(\mu, \kappa, B)$ is compact.*

3.6. *Periodic points α^l, collars A^l, and trapping disks D^l.*

3.6.1. *Periodic points α^l.* Let us start collecting consequences of the complex bounds.

Proposition 3.15. *Under the circumstances of* Theorem 3.10, *for any* $l \geq \underline{l} - 1$, *there exists a repelling periodic point* $\alpha^l \in X^l_- \cup X^l_+$ *of period* q_l. *Moreover*,

(i) $\operatorname{dist}(\alpha^l, \mathbb{T})$ *is comparable to* $I^l(c_0)$;

(ii) *the multiplier of* α^l *is bounded and bounded away from* 1 *in absolute value.*

Proof. Each restriction $R^l f : X^l_\pm \to Y^l$ is a conformal map from a smaller domain onto a bigger one. By the Wolff-Denjoy Theorem (applied to the inverse map) it has a fixed point in the closure $\overline{X}^l_\pm$. However, it does not have fixed points on the boundary since f does not have periodic points on $\mathbb{R}$, while the image of $\partial X^l_\pm \setminus \mathbb{R}$ under $R^l f$ (contained in ∂Y^l) is disjoint from itself. So, there is a fixed point $\alpha^l_\pm \in X^l_\pm$.

Assertions (i) and (ii) follow from compactness (Proposition 3.14).

Finally one of the points $\alpha^l_\pm$ has period q_l. □

3.6.2. *Collar Lemma and trapping disks D^l.* For all sufficiently big l, complex a priori bounds allow us to construct nice collars A^l around $\overline{\mathbb{D}}$ and nice trapping disks D^l that capture all orbits that escape beyond the corresponding collars.

We say that a point $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ *lies on depth* l, $d(z) = l$, if

$$C_0^{-1} |I^l(\zeta)| \leq \operatorname{dist}(z, \mathbb{T}) \leq C_0 |I^l(\zeta)|,$$

where ζ is the closest to z point of $\mathbb{T}$, and $C_0 = C_0(\overline{N}, K, \epsilon)$ is the constant from (3.6). Of course, any point can lie on several depths (so $d(z)$ is multivalued), but this number is bounded in term of $(\overline{N}, K, \epsilon)$.

Lemma 3.16. *Under the circumstances of* Theorem 3.10, *for any* $l \geq \underline{l} - 1$, *there exists a pair of smooth annuli* ("*collars*") $A^l_0 \Subset A^l$ *surrounding*[20] $\mathbb{D}$ *in* $\operatorname{Dom} f \setminus \overline{\mathbb{D}}$, *and a smooth quasidisk* $D^l \ni \alpha^l$ *in* Y^l *with the following properties*:

(A1) *Any boundary point* $z \in \partial^o A^l_0 \cup \partial^o A^l$ *of these collars lies on depth* $d(z)$ *with*

$$|d(z) - l| \leq \bar{\iota} = \bar{\iota}(\overline{N}, K, \epsilon).$$

Moreover, $\operatorname{dist}(z, \partial^o A^l) \asymp \operatorname{dist}(z, \mathbb{T})$ *for any* $z \in \partial^o A^l_0$, *and similarly for the inner boundaries* $\partial^i A^l_0$ *and* $\partial^i A^l$.

(A2) *It is impossible to* "*jump over the collar*":

$$\text{If } z \in \operatorname{Comp}_0(\mathbb{C} \setminus A^l_0) \setminus \overline{\mathbb{D}} \text{ while } f(z) \notin \operatorname{Comp}_0(\mathbb{C} \setminus A^l_0), \text{ then } f(z) \in A^l_0.$$

(D1) *The disk* D^l *has a bounded shape around* α^l; *it also has the hyperbolic diameter of order* 1 *in* $Y^l \setminus \overline{\mathbb{D}}$ *and in* $\mathbb{C} \setminus \overline{\mathbb{D}}$.

[20] We prepare a *pair* of collars for each l to make the statements robust under perturbations. By "surrounding" we mean that $\overline{\mathbb{D}} \subset \operatorname{Comp}_0(\mathbb{C} \setminus A^l)$.

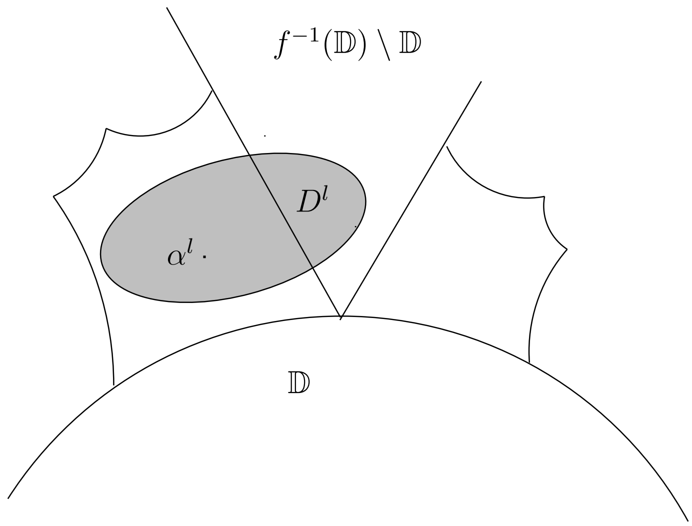

Figure 3.2. Trapping disk.

(D2) *A definite portion of* D^l (*with respect to the plane area*) *is contained in* $f^{-1}(\mathbb{D}) \setminus \mathbb{D}$. *Moreover,*

there is a point $\beta \in f^{-1}(\mathbb{T}) \setminus \overline{\mathbb{D}}$ *that lies in the middle of* D^l.

See (1.1) *and Figure* 3.2.

(D3) *If* $z \in A^l$, *then there exists a moment* $k < q_{l+1}$ *such that* $f^k z$ *lies in the middle of* D^l.

(D4) *There exists* $\underline{\iota} = \underline{\iota}(\overline{N}, K, \epsilon)$ *such that for any* $\iota > \underline{\iota}$ *and* $l > \underline{l} + 2\iota$, *under the circumstances of* (D3), *we have*

$$f^i z \notin D_1^{l-\iota}, \quad i = 0, 1, \ldots, k,$$

where $D_1^{l-\iota} \Subset Y^{l-\iota} \setminus \overline{\mathbb{D}}$ *is a disk containing* $D^{l-\iota}$ *with a definite* $\operatorname{mod}(D_1^{l-\iota} \setminus D^{l-\iota})$; *in particular,* $D^l \cap D_1^{l-\iota} = \emptyset$.

(D5) *Moreover, under the above circumstances,*

$$f^i z \in \mathrm{Comp}_0(\mathbb{C} \setminus A^{l-\iota}), \quad i = 0, 1, \ldots, k,$$

and $A^{l-\iota} \Subset \mathrm{Comp}_0(\mathbb{C} \setminus A^{l-2\iota})$.

All the bounds and constants depend only on $(\overline{N}, K, \epsilon)$.

Proof. Let us consider the butterfly renormalization $R^l_{\mathrm{cp}} f : X^l_- \cup X^l_+ \to Y^l$. For Y^l, we will also use the notation Y^l_0.

Any dynamical tile $I^l_k \in \mathcal{I}^l$ is compactly contained in the topological disk Y^l_k obtained by pulling Y^l back by the conformal landing map, the complex extension of the landing map $L_l : I^l_k \to I^l_0$. Complex a priori bounds imply that I^l_k is contained well inside Y^l_k (since $\operatorname{mod}(Y^l_k \setminus I^l_k) = \operatorname{mod}(Y^l \setminus I^l_0)$ is definite). Hence each Y^l_k contains a half-ellipse $\Delta_\epsilon(I^l_k)$ of bounded eccentricity based on the $(1+\epsilon)$-scaled interval I^l_k, where $\epsilon > 0$ and the bound on eccentricity depend only on real and complex a priori bounds. The union of these half-ellipses is an annulus whose inner boundary is $\mathbb{T}$ and the outer boundary lies on dynamical depth $l+O(1)$. Moreover, for $k \neq 0$, these disks lie well inside $\mathrm{Dom}^h f$, since $Y^l_k \Subset \mathrm{Dom}^h f$.

Now existence of collars $A^l_0 \Subset A^l$ satisfying (A1) is obvious. Moreover, one can easily secure the following property:

(M) Every point $z \in A^l_1$ lies in the middle of some Y^l_k.

Furthermore, since f is quasiregular, there is $\bar\iota = \bar\iota(\overline{N}, K, \epsilon)$ such that

$$d(f(z) \geq d(z) - \bar\iota, \ \ z \in \mathrm{Dom}^h f.$$

It follows that if the collar A^l_0 is selected sufficiently thick (i.e., contains a round annulus going over more than $\bar i$ depth levels), then points cannot jump over it, securing (A2).

Let us view the topological half-disk $Y^l \setminus \overline{\mathbb{D}}$ as the hyperbolic plane, and let $D^l = D^l(R)$ be the hyperbolic disk of radius R in Y^l centered at α^l. By the Koebe Distortion Theorem, these disks satisfy property (D1) with constants depending on R (or better to say, on an upper bound for R) .

For R big enough (depending only on ($\overline{N}$, K, ϵ)), they also satisfy (D2). Indeed, since f is quasiregular, any sufficiently small disk $\mathbb{D}(c_0, r)$ contains a comparable disk $\mathbb{D}(\zeta, ar) \subset f^{-1}(\mathbb{D}) \setminus \mathbb{D}$. Since the domains Y^l have a bounded shape around c_0, while the disks $D^l(R)$ closely approximate $Y^l \setminus \overline{\mathbb{D}}$ (uniformly in l), we conclude that for R big enough,

$$D^l(R) \supset \mathbb{D}(\zeta, ar/2) \text{ and } \operatorname{area} D^l(R) \asymp \operatorname{area} \mathbb{D}(\zeta, ar/2),$$

which yields the first part of (D2).

The second part of (D2) follows from Proposition 3.3, which implies that there is a point $\zeta \in f^{-1}(\mathbb{T})$ lying in the middle of Y^l. For R big enough, it lies in the middle of $D^l(R)$ as well.

If $z \in A^l$, then by Property (M), z lies in the middle of some domain Y^l_k. By the Koebe Distortion Theorem, under the landing map $L_k : Y^l_k \to Y^l$, it lands in the middle of Y^l. Hence for R big enough, $L_k(z)$ lies in the middle of $D^l(R)$ as well, which establishes property (D3).

Since the whole orbit $\{f^i z\}_{i=0}^k$ lies on depth $\geq l - O(1)$, it is separated from $D^{l-\iota} + O(1)$ and from $A^{l-\iota}$, as long as ι is sufficiently big. Similarly, since $A^{l-\iota}$ lies on depth $l - \iota$, it is separated from $A^{l-2\iota}$ for ι big enough. These remarks prove (D4) and (D5). □

We say that the trapping disk $D = D^l$ is *centered* at α^l, or that depth $D = l$.

3.7. *Cylinder circle renormalization.*

3.7.1. *Real definition.* There is a different approach to the circle renormalization that avoids using circle pairs. For any non-critical point $\theta \in \mathbb{R}/\mathbb{Z}$, consider the oriented interval $I = [\theta, f(\theta)] \subset \mathbb{R}/\mathbb{Z}$. Identifying its endpoints by means of f, we obtain an oriented real analytic circle $\mathbb{T}'$. The first return map to I descends to a quasicritical circle map of $\mathbb{T}'$ (defined up to an orientation preserving analytic conjugacy) that is called the *cylinder renormalization* $R_{\mathrm{cyl}}f$ of f. The rotation number of $R_{\mathrm{cyl}}f$ is equal to $-1/\theta \mod \mathbb{Z}$.

This leads to the modified Gauss map $G_* : \theta \mapsto -1/\theta \mod \mathbb{Z}$ accompanied by the modified continued fraction expansion

$$\theta = \frac{1}{N_1 - \frac{1}{N2 - \cdots}} \equiv [N_1,\, N_2, \dots]_*, \quad N_i \geq 2.$$

We will use the same notation for the rational approximands in this expansion, $p_m/q_m = [N_1, \dots, N_m]_*$. Of course, the notion of "bounded type" is independent of which expansion we use.

The rotation numbers $\theta_N = [N,\, N,\, N, \dots]_*$ with equal entries[21] $N \geq 3$ are called of *stationary type* (with respect to the modified expansion). The most familiar of these is the golden mean $\theta_3 = (3 - \sqrt{5})/2$.

3.7.2. *Complexification.* Let us start with a topological lemma:

LEMMA 3.17. *For any butterfly map $f \in \mathcal{B}(\mu, \kappa, B)$ (3.8), there exists an arc γ connecting the fixed point $\alpha \in X_+$ to β_+ (3.7) in such a way that α is the only common point of γ and $f(\gamma)$. Moreover, the triangle bounded by γ, $f(\gamma)$ and the arc of $J := [\phi_+(\beta_+), \beta_+] \in \mathbb{R}$ is $L(\kappa, \mu, B)$-qc equivalent (by a global map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$) to the half-strip*

$$\{z : \ \mathrm{Im}\, z \geq 0,\ 0 \leq \mathrm{Re}\, z \leq 1\} \cup \{\infty\} \subset \hat{\mathbb{C}}. \tag{3.10}$$

Proof. Let $\phi := \phi_+$, $X := X_+$, $\beta_+ := \beta$. Notice that there is a subarc $\sigma \subset \partial X$ that touches J at β and is mapped homeomorphically onto the subinterval of $\partial Y \cap \mathbb{R}$ that begins at $\phi(\beta)$ (covering J). Hence the pullback $J' := \phi^{-1}(J)$ is a subarc of σ touching J at β. Since X is κ-qc equivalent to the unit semi-disk, the concatenation $J \cup J'$ is a quasiarc. Pulling it further, we obtain a sequence of quasiarcs $J^n := \phi^{-n}(J) \subset \overline{X}$, $n = 0, 1, \dots$, one attached to the previous one, such that the $J^n \cup J^{n+1}$ are quasiarcs with uniform dilatation (depending only on (κ, μ, B)) shrinking to α at a geometric rate. Then

$$\Gamma := \{\alpha\} \cup \bigcup_{n=1}^{\infty} J^n$$

is a quasiarc (with dilatation depending only on κ, μ, B) connecting α to β whose image $f(\Gamma)$ is a longer quasiarc connecting α to $\phi(\beta)$. To see that Γ is a quasiarc, notice that it is so away from α since Γ is composed from overlapping quasiarcs $T^k := \phi^{-k}(J \cup J')$. Moreover, since ϕ near α acts as a linear expansion by some $\rho > 1$, both length and diameter of the arc $\bigcup_{m \leq k \leq n} T^k$ are comparable with ρ^n, implying that Γ is a quasiarc near α as well.

[21]Note that $\theta_2 = 1$.

The map ϕ on X can be globally linearized by a Q-qc homeomorphism

$$\psi : (\mathbb{C}, X) \to (\mathbb{C}, \psi(X))$$

that is conformal on X, $\psi(\phi(z)) = \lambda\psi(z)$, $z \in X$, with Q depending only on (κ, μ, B). It can be further conjugate to the doubling map $T : z \mapsto 2z$ by a qc homeomorphism $h : \mathbb{C} \to \mathbb{C}$ that straightens the quasiarc Γ to the unit interval $[0, 1]$. In this model, we can let $\tilde{\gamma} \equiv h((\psi(\gamma))$ be a segment of a circle passing through 0 and 1 sufficiently close to $\mathbb{R}$ so that it fits to the domain $h(\psi(X))$. Moreover, the triangle bounded by $\tilde{\gamma}$, $2 \cdot \tilde{\gamma}$ and $[1, 2]$ is qc equivalent to the half-strip (3.10), implying the conclusion. □

For m sufficiently big, the cylinder renormalizations $R^m_{\mathrm{cyl}} f$ we have described above can be complexified as follows; see Yampolsky [Yam02]. Let us consider a periodic point α^m, $m \geq \underline{l}$, from Corollary 3.15. Then there is a $\mathbb{T}$-symmetric arc γ_m connecting α^m to the symmetric point[22] $1/\bar{\alpha}^m$ in such a way that $f^{q_m}(\gamma_m)$ does not intersect γ_m. Let us consider the fundamental region $\Upsilon^m = \Upsilon^m(f)$ bounded by these two arcs.

Lemma 3.18. *Let $f \in \mathrm{Cir}(\overline{N}, K, \epsilon)$. Then the regions Υ^m are κ-qc equivalent to the strip $\{0 \leq \operatorname{Re} z \leq 1\}$, with κ depending only on $(\bar{N}, K, \epsilon)$.*

Let us now identify the boundary components of Υ^m by means of f^{q_m}. We obtain a cylinder Cyl^m that is conformally equivalent to the standard bi-infinite cylinder $\mathbb{C}/\mathbb{Z}$ (the symmetrization of the half-cylinder corresponding to (3.10)). The first return map to Υ^m descends to a holomorphic map on Cyl^m near the circle, and then can be transferred to $\exp(\mathbb{C}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = (\mathbb{C}^*, \mathbb{T})$. This is the *cylinder renormalization* of a holomorphic circle map (well defined up to affine conjugacy).

3.8. *Quasiconformal conjugacy.*

Theorem 3.19 (compare [dFdM00]). *Two quasicritical circle maps,*

$$f : \mathrm{Dom}^h f \to Y \text{ and } \tilde{f} : \mathrm{Dom}^h \tilde{f} \to \tilde{Y}, \text{ of class } \mathrm{Cir}_{\mathrm{cp}}(\overline{N}, K, \epsilon),$$

with the same rotation number are L-qc conjugate in a δ-neighborhood of $\mathbb{T}$, with L and $\delta > 0$ depending only on $(\overline{N}, K, \epsilon)$.

Proof. The proof is an application of *Sullivan's Pullback Argument*; see [dMvS93].

Without loss of generality, we can assume that f is a butterfly renormalization of the Blaschke product B_θ (3.1) provided by complex bounds (Theorem 3.10). It is easy to see, using the general description of the dynamics on

[22] Here we describe it in terms of the unit circle $\mathbb{T}$ in $\mathbb{C}$.

the Fatou set, that the filled Julia set $K(B_\theta)$, and hence $K(f)$, are nowhere dense.

By Theorem 3.9, there is a quasiconformal map $h_0 : \mathbb{C} \to \mathbb{C}$ conjugating f and $\tilde{f}$ on the unit circle (with dilatation depending only on $\overline{N}$). Using the complex bounds (Theorem 3.10) this map can be adjusted, after passing to some renormalization, so that it is equivariant on the boundary of the butterfly, with dilatation depending only on $(\overline{N}, K, \epsilon)$.

We can now start lifting the map h_0 under the dynamics to make it equivariant on bigger and bigger parts of Ω^h_f. Since f is conformal on Ω^h_f, these lifts h_n have the same dilatation as h. By compactness of the space of normalized L-qc maps, we can pass to a subsequential limit, $h_{n_k} \to h$.

Moreover, outside the filled Julia set $K(f)$, h is independent of the subsequence (n_k) since the lifts h_n stabilize pointwise on $\mathbb{C} \setminus K(f)$. Since $K(f)$ is nowhere dense, by continuity h is independent of the subsequence on the whole plane. Hence $h_n \to h$ on the whole plane, implying that h conjugates f to $\tilde{f}$.

Let us finally spread the conjugacy around the circle. To this end let us consider an arc Γ connecting two boundary points of ∂Y and composed of two external rays of B_θ through the point $b = \phi_\pm(\beta_\mp)$. (Such rays exist since the Julia set of B_θ is locally connected [Pet96].) Let $\Gamma_\pm$ be similar arcs obtained as the pullbacks of Γ through $\beta_\pm$, respectively. Since the external rays form a foliation, $\Gamma_\pm$ are disjoint from Γ. Let $\Pi_\pm$ be the topological rectangles each bounded by $\Gamma_\pm$ and Γ, respectively, and by a pair of arcs of ∂Y. Pulling these rectangles back by the first landing map to Y (compare with the proof of Lemma 3.16), we obtain a tiling of a neighborhood of $\mathbb{T}$ by topological rectangles.

Transfer the arcs Γ and $\Gamma_\pm$ by h to the butterfly range $\tilde{Y}$ and use them to construct a similar tiling near $\mathbb{T}$ for $\tilde{f}$. Then the conjugacy h between the butterflies can be lifted via these tilings to a desired qc conjugacy between f and $\tilde{f}$ near $\mathbb{T}$. ∎

4. **Siegel maps and their perturbations**

4.1. *Douady-Ghys surgery.*

4.1.1. *Blaschke model for Siegel polynomials.* Let us consider a quadratic polynomial

$$\mathbf{f}_\theta : z \mapsto e^{2\pi i\theta} z + z^2, \quad \theta \in \mathbb{R}/\mathbb{Z}. \tag{4.1}$$

When the rotation number θ has bounded type, it is linearizable near the origin, and thus has a Siegel disk $\boldsymbol{\mathcal{B}} \equiv \boldsymbol{\mathcal{B}}_{P_\theta} \equiv \boldsymbol{\mathcal{B}}_\theta$. Here we will briefly describe the Blaschke model for this quadratic map due to Douady and Ghys (see [Dou87b]). It is based on a surgery that turns an appropriate Blaschke product into $\mathbf{f}_\theta$.

Consider a family of Blaschke products (3.1). It induces a family of critical circle maps on the unit circle $\mathbb{T}$. Adjusting the parameter α one can make the rotation number of B_α assume an arbitrary value, so it can be made equal to the rotation number θ from (4.1).

Assume θ is of bounded type. Then by Theorem 3.9, $B_\alpha : \mathbb{T} \to \mathbb{T}$ is quasisymmetrically conjugate to the pure rotation T_θ. We can use this conjugacy to glue the Blaschke product on $\mathbb{C} \setminus \mathbb{D}$ to the rotation of $\mathbb{D}$. This produces a degree two quasiregular map F of a quasiconformal sphere. Moreover, F preserves the conformal structure obtained by spreading around the standard structure on the disk $\mathbb{D}$. By the Measurable Riemann Mapping Theorem, F is quasiconformally conjugate to some quadratic polynomial $z \mapsto \lambda z + z^2$. Since this quadratic polynomial has an invariant Siegel disk with rotation number θ, it coincides with $\mathbf{f}_\theta$.

4.2. *Expansion.* Let us endow the complement $\mathbb{C}\setminus\overline{\boldsymbol{\mathcal{B}}}$ of a Siegel disk $\boldsymbol{\mathcal{B}}=\boldsymbol{\mathcal{B}}_\theta$ of bounded type with the hyperbolic metric $\|\cdot\|_{\text{hyp}}$. A standard application of the Schwarz Lemma shows that the map $\mathbf{f} = \mathbf{f}_\theta$ is expanding in this metric,

$$\|D\mathbf{f}(z)\|_{\text{hyp}} > 1 \quad \text{if } z, \mathbf{f}(z) \in \mathbb{C} \setminus \overline{\boldsymbol{\mathcal{B}}}.$$

Indeed, the map $\mathbf{f} : \mathbb{C} \setminus f^{-1}(\overline{\boldsymbol{\mathcal{B}}}) \to \mathbb{C} \setminus \overline{\boldsymbol{\mathcal{B}}}$ is a covering and hence a hyperbolic isometry. By the Schwarz Lemma, the embedding

$$i : \mathbb{C} \setminus \mathbf{f}^{-1}(\overline{\boldsymbol{\mathcal{B}}}) \to \mathbb{C} \setminus \overline{\boldsymbol{\mathcal{B}}} \tag{4.2}$$

is a hyperbolic contraction. Hence $\mathbf{f} \circ i^{-1} : \mathbb{C} \setminus \overline{\boldsymbol{\mathcal{B}}} \dashrightarrow \mathbb{C} \setminus \overline{\boldsymbol{\mathcal{B}}}$ is expanding on its domain of definition (i.e., on $\mathbb{C} \setminus \mathbf{f}^{-1}(\overline{\boldsymbol{\mathcal{B}}})$).

Using the Blaschke model, McMullen showed that the expansion is uniform near the critical point:

Lemma 4.1 ([McM94]). *Let $\mathbf{f} = \mathbf{f}_\theta$ be a Siegel quadratic polynomial of type bounded by $\overline{N}$, and let $C > 0$. Then there exists $\rho = \rho(\overline{N}, C) > 1$ such that*

$$\|D\mathbf{f}(z)\|_{\text{hyp}} > \rho \quad \text{if } z, \mathbf{f}(z) \in \mathbb{C} \setminus \overline{\boldsymbol{\mathcal{B}}}, \text{ and } |z - c_0| \le C \operatorname{dist}(z, \boldsymbol{\mathcal{B}}),$$

where the dist *stands for the Euclidean one.*

Proof. From the above argument we see that $\|D\mathbf{f}(z)\|_{\text{hyp}} = \|Di^{-1}\|_{\text{hyp}}$, where i is embedding (4.2). The latter is bounded away from 1 in terms of the hyperbolic distance Δ from z to $\mathbf{f}^{-1}\overline{\boldsymbol{\mathcal{B}}}$ (in $\mathbb{C} \setminus \overline{\boldsymbol{\mathcal{B}}}$). For the Blaschke model, the corresponding hyperbolic distance is bounded in terms of the constant corresponding to C (i.e., with the Sielge disk $\boldsymbol{\mathcal{B}}$ replaced by the unit disk $\mathbb{D}$). The Blaschke model is K-qc equivalent to $\mathbf{f}$ where K is bounded in terms of $\overline{N}$. Since global qc maps are quasisymmetries and hyperbolic quasi-isometries (quantitatively), the hyperbolic distance Δ is bounded in terms of C. The conclusion follows. $\square$

Let us now consider a perturbation $\tilde{\mathbf{f}} = \mathbf{f}_{\tilde{\theta}}$ (not necessarily with real $\tilde{\theta}$) of the Siegel polynomial $\mathbf{f} = \mathbf{f}_\theta$. Let $\tilde{\mathbf{O}}$ be the postcritical set of $\tilde{\mathbf{f}}$. Endow its complement $\mathbb{C} \setminus \tilde{\mathbf{O}}$ with the hyperbolic metric $\|\cdot\|_{\widetilde{\text{hyp}}}$. Then the map $\tilde{\mathbf{f}}$ is expanding with respect to this metric (for the same reason as the Siegel map $\mathbf{f}$). In fact, under certain circumstances it is also uniformly expanding near the critical point:

LEMMA 4.2. *Let the type of θ be bounded by $\overline{N}$, and let $C > 0$. Then there exists $\rho = \rho(\overline{N}, C) > 1$ such that for any compact set $K \Subset \mathbb{C} \setminus \overline{\boldsymbol{\mathcal{B}}}$, there exists $\delta > 0$ with the following property. Let $|\tilde{\theta} - \theta| < \delta$, and assume $\tilde{\mathbf{O}}$ is contained in the δ-neighborhood of the Siegel disk $\boldsymbol{\mathcal{B}}$. Then for any point $z \in K \setminus \tilde{\mathbf{f}}^{-1}(\overline{\mathbf{O}})$ such that*

$$|z - c_0| \le C \operatorname{dist}(z, \overline{\boldsymbol{\mathcal{B}}}), \tag{4.3}$$

we have

$$\|D\tilde{\mathbf{f}}(z)\|_{\widetilde{\text{hyp}}} \ge \rho.$$

Proof. As the proof of Lemma 4.1 shows, the expansion factor ρ is bounded from below in terms of the hyperbolic distance from z to $\tilde{\mathbf{f}}^{-1}(\tilde{\mathbf{O}})$ in $\mathbb{C} \setminus \tilde{\mathbf{O}}$.

Let $U \equiv U_\delta$ be the δ-neighborhood of $\overline{\boldsymbol{\mathcal{B}}}$. For δ small enough, $\overline{U}$ is disjoint from K. Then the hyperbolic metrics on $\mathbb{C} \setminus \overline{\boldsymbol{\mathcal{B}}}$ and on $\mathbb{C} \setminus \overline{U}$ restricted to K are comparable (and in fact, close for δ small).

By assumption, the postcritical set $\tilde{\mathbf{O}}$ is contained in U. By the Schwarz Lemma, the hyperbolic metric $\|\cdot\|_{\widetilde{\text{hyp}}}$ on $\mathbb{C} \setminus \tilde{\mathbf{O}}$ restricted to K is bounded from above by the hyperbolic metric on $\mathbb{C} \setminus \overline{U}$. Altogether, for δ sufficiently small, we conclude

$$\|\cdot\|_{\widetilde{\text{hyp}}} \le C_1 \|\cdot\|_{\text{hyp}} \quad \text{on } K,$$

with the constant C_1 depending only on $\overline{N}$. (In fact, C_1 can be taken arbitrary close to 1 for δ small.)

Since the dynamics of $\mathbf{f}$ on $\partial\boldsymbol{\mathcal{B}}$ is minimal, the set $\tilde{\mathbf{O}}$ makes an ϵ-net for $\partial\boldsymbol{\mathcal{B}}$ provided δ is small enough. Hence $\tilde{\mathbf{f}}^{-1}(\tilde{\mathbf{O}})$ makes an $O(\epsilon)$-net for $\mathbf{f}^{-1}(\partial\boldsymbol{\mathcal{B}})$. As we know (see the proof of Lemma 4.1), condition (4.3) implies that the hyperbolic distance from z to $\mathbf{f}^{-1}(\boldsymbol{\mathcal{B}})$ in $\mathbb{C} \setminus \overline{\boldsymbol{\mathcal{B}}}$ is bounded. It follows that the hyperbolic distance from z to $\tilde{\mathbf{f}}^{-1}(\tilde{\mathbf{O}})$ in $\mathbb{C} \setminus \tilde{\mathbf{O}}$ is bounded as well. □

4.3. *Siegel maps.*

4.3.1. *Definition.* A *Siegel map* $f : (\Omega, 0) \to (\mathbb{C}, 0)$ is a holomorphic map on a Jordan disk $\Omega \equiv \Omega_f = \operatorname{Dom} f$ with the following properties (see Figure 4.1):

(S1) f has a Siegel disk $S \equiv S_f$ (centered at 0) that is a *quasidisk compactly contained in* Ω;

(S2) f has a non-degenerate critical point $c_0 \in \partial S$;

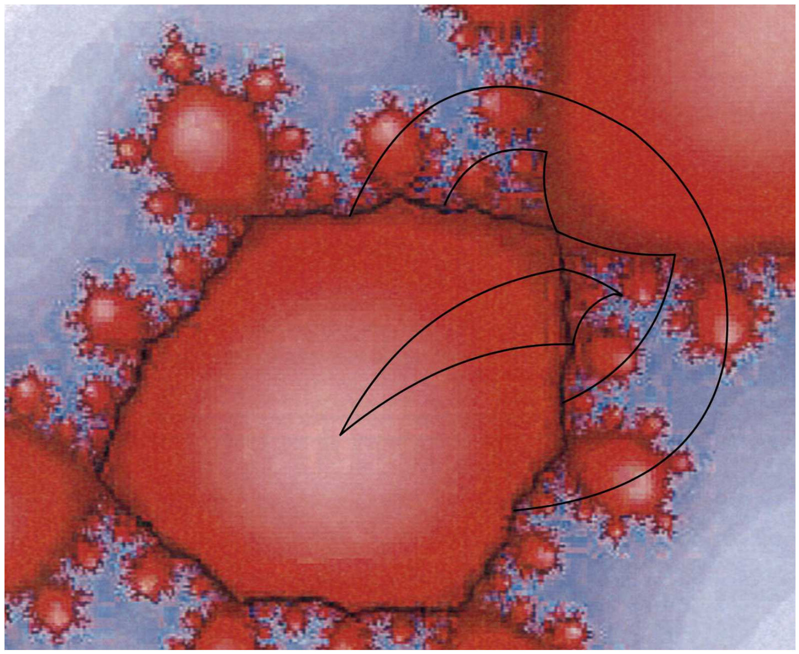

Figure 4.1. A Siegel map supplied with a renormalization butterfly and a fundamental domain for the corresponding cylinder renormalization. *Courtesy of D. Dudko.*

(S3) the domain $\Omega^h \equiv \Omega^h_f := \{z \in \Omega \setminus \overline{S} :\ fz \in \Omega \setminus \overline{S}\}$ is obtained from the annulus $\Omega \setminus \overline{S}$ by removing a topological triangle

$$\mathcal{T} \equiv \mathcal{T}_f \subset (\Omega \setminus \overline{S}) \cup \{c_0\}$$

with a vertex at c_0 and the opposite side on the boundary of Ω;

(S4) $f : \Omega^h_f \to \mathbb{C} \setminus \overline{S}$ is an immersion, and $f : \mathcal{T} \to S \cup \{c_1\}$ is an embedding.

We let $\mathrm{Dom}^h f = \Omega^h_f \cup \overline{S}$.

Remark 4.1. Note that Siegel maps are holomorphic by definition, so in this case superscript "h" is taken only by analogy with the circle case.

(S5) A lift of f to the universal covering $\mathbb{H}$ of $S \setminus \overline{\mathbb{D}}$ admits a representation as in (Q5).

Given $\overline{N} \in \mathbb{N}$ and $\mu > 0$, let $\mathrm{Sieg}(\overline{N}, \mu, K)$ stand for the space of Siegel maps $f : \Omega \to \mathbb{C}$ of type bounded by $\overline{N}$ and such that $\mathrm{mod}(\Omega \setminus S) \geq \mu$ and ∂S is a K-quasicircle. (If irrelevant, some of these parameters can be skipped in the notation.)

We will later use the notation $\mathrm{Sieg}_\theta(\mu, K) \equiv \mathrm{Sieg}_N(\mu, K)$ for the class of Siegel maps $f \in \mathrm{Sieg}(\mu, K)$ with stationary rotation number $\theta = \theta_N$.

4.3.2. *Circle model for Siegel maps.* By performing the Douady-Ghys surgery on an arbitrary *analytic* critical circle map g of bounded type (not only on the Blaschke map), we can produce plenty of Siegel maps. However, to obtain all of them, we need to allow *quasicritical* circle maps.

PROPOSITION 4.3. *Any Siegel map $f : (\Omega, 0) \to (\mathbb{C}, 0)$ of class $S(\overline{N}, \mu, K)$ can be obtained by performing a Douady-Ghys surgery on a quasicritical circle map. Moreover, the dilatation of the surgery depends on K only.*

Proof. Let $\psi_+ : \mathbb{C} \setminus S \to \mathbb{C} \setminus \mathbb{D}$ be the uniformization of the complement of S normalized so that $\psi_+(c_0) = 1$. Since S is a quasidisk, it extends to a global quasiconformal map $\psi_+ : (\mathbb{C}, S) \to (\mathbb{C}, \mathbb{D})$ (with the dilatation depending on K only). Then

$$g_0 := \psi_+ \circ f \circ \psi_+^{-1} : (\psi_+(\Omega), \mathbb{D}) \to (\mathbb{C}, \mathbb{D})$$

is a quasiregular map in a neighborhood of $\overline{\mathbb{D}}$ that is a holomorphic immersion on $\psi_+(\Omega^h)$. Applying to $g_0 | \psi_+(\Omega) \setminus \mathbb{D}$ the Schwarz Reflection Principle, we obtain a quasiregular map g near $\mathbb{T}$ that restricts to a homeomorphism $\mathbb{T} \to \mathbb{T}$. Moreover, it is a holomorphic immersion on $\operatorname{Dom}^h g$, and hence is real analytic on $\mathbb{T} \setminus \{1\}$. At the critical point $c_0 = 1$, it has local degree 3. Moreover, properties (S3) and (S4) of f readily translate to properties (Q3)–(Q5) of g. Thus, g is a quasicritical circle map.

On the other hand, the uniformization $\psi_- : \overline{S} \to \overline{\mathbb{D}}$ conjugates f to the rotation T_θ (and extends to a global qc map). Hence f is the quasiconformal welding between g and T_θ. □

Remark 4.2. Notice that the above construction is softer than the Douady-Ghys surgery (as it does not involve an infinite procedure of spreading around a Beltrami diffirential and does not use the Measurable Riemann Mapping Theorem).[23] The price for this simplification is that the outcome is quasicritical rather than holomorphic.

4.4. *Circle $\rightsquigarrow$ Siegel transfer.* By means of the Douady-Ghys surgery, we can transfer the objects defined above for quasicritical circle maps to their Siegel counterparts. Somewhat abusing notation, we will usually keep the same notation for the transferred objects.

4.4.1. *Dynamical scales.* For any $f \in \operatorname{Sieg}(\overline{N}, \mu, K)$, we can transfer the circle dynamical tilings (3.5) to the boundary of the Siegel disk S. Since the surgery is quasisymmetric, these *Siegel dynamical tilings* $\mathcal{I}^m$ have bounded geometry as well (depending only on $(\overline{N}, \mu, K)$), which gives us for any $z \in \partial S$ a

[23]It is more similar to the external circle map construction for quadratic-like maps.

notion of the *dynamical scales* near z (with the constant C_0 from (3.6) replaced with an analogous constant $C_0 = C_0(\overline{N}, \mu, K)$ controlling the geometry of the tilings for Siegel maps).

4.4.2. *Siegel butterfly renormalization.* Since any Siegel map f of bounded type is conjugate on the boundary of S to a quasicritical circle map, we can immediately define the *Siegel pairs* renormalizations $R_{\mathrm{Sp}} f$ on ∂S. The complexification of this notion, a *Siegel butterfly*

$$R^m_{\mathrm{Sp}} : X^m_+ \cup X^m_- \to Y^m, \tag{4.4}$$

corresponds, via the surgery, to the external part of the circle butterfly (see Figure 4.1). Theorem 3.10 implies

THEOREM 4.4. *Let $f \in \mathrm{Sieg}(\overline{N}, \mu, K)$ be a Siegel map of bounded type. Then there exists an $\underline{l}$ depending only on $(\overline{N}, \mu, K)$ such that for all $m \geq \underline{l} - 1$, the renormalizations $R^m_{\mathrm{Sp}} f$ on ∂S can be extended to Siegel butterflies*

$$R^m_{\mathrm{Sp}} f : X^m_- \cup X^m_+ \to Y^m$$

with $Y^{\underline{l}-1} \supset Y^{\underline{l}} \supset \cdots$ such that the Y^m are quasidisks of bounded shape and

$$\mathrm{dist}(\partial Y^m \setminus \partial S,\, Y^{m+1}) \asymp \mathrm{dist}(\partial Y^m \setminus \partial S,\, X^m_\pm) \asymp \mathrm{diam}\, Y^m.$$

All constants and bounds depend on $(\overline{N}, \mu, K)$ only.

As in the circle case, these a priori bounds lead to external expansion:

COROLLARY 4.5. *Under the circumstance of Theorem 4.4, the renormalizations $f_m := R^m_{\mathrm{Sp}} f$ are expanding in the hyperbolic metric of Y^0. Moreover,*

$$\|Df_m(z)\|_{\mathrm{hyp}} \geq \rho > 1$$

with ρ depending only on $(\overline{N}, \mu, K)$ and a lower bound on $\mathrm{dist}(z, \overline{S}) / \mathrm{dist}(z, c_0)$.

Proof. To deduce it directly from the statement of Lemma 3.12 just map Y^0 conformally onto $\mathbb{H}$. ☐

This leads to a direct analogue of Theorem 3.13 (with an additional constant factor a due to the comparison of the hyperbolic metrics in Y^0 and $\mathbb{C} \setminus \overline{S}$).

COROLLARY 4.6. *Let $f \in \mathrm{Sieg}(\overline{N}, \mu, K)$ be a Siegel map. Then there exist $a > 0$ and $\rho > 1$ depending only on $(\overline{N}, \mu, K)$ such that if $z \in Y^m \cap \mathrm{Dom}^h f^n$ and $f^n z \in Y^{m-k} \setminus Y^{m-k+1}$ for some $n \in \mathbb{N}$, $0 < k < m$ (with $m - k > \underline{l}$), then*

$$\|Df^n(z)\|_{\mathrm{hyp}} \geq a\rho^k,$$

where the norm is measured in the hyperbolic metric of $\mathbb{C} \setminus \overline{S}$.

Here $\mathrm{Dom}\, f^n$ denotes (as in the circle case) the set of points whose orbits $(f^k z)^n_{k=0}$ stay outside $\mathbb{C} \setminus \overline{S}$.

4.4.3. *Periodic points α^l.* Proposition 3.15 implies

COROLLARY 4.7. *For any Siegel map $f \in \operatorname{Sieg}(\overline{N}, \mu, K)$, there exists $\underline{l} = \underline{l}(\overline{N}, \mu, K)$ such that for any $l \geq \underline{l} - 1$, f has a repelling periodic point α_l of period q_l in the l-th dynamical scale near the critical point c_0.*

Remark 4.3. If $\mathbf{f} = \mathbf{f}_\theta$ is a Siegel quadratic polynomial with rotation number of bounded type, then the periodic point α^l was born in the *parabolic explosion* from the parabolic approximand $\mathbf{f}_{p_\kappa/\ q_\kappa}$. It can be characterized as the landing point of a ray with rotation number p_κ/q_κ.

4.4.4. *External collars of A^l and trapping disks D^l.* Let us now transfer, by means of the surgery, the collars and trapping disks from the circle plane to the Siegel plane. It is a direct consequence of Lemma 3.16 and quasisymmetry of quasiconformal maps.

PROPOSITION 4.8. *For any Siegel map $f \in \operatorname{Sieg}(\overline{N}, \mu, K)$ and any $l > \underline{l} + 2\iota$, there exist a pair of smooth annuli (collars) $A^l_0 \Subset A^l$ surrounding the Siegel disk $S = S_f$ in $\operatorname{Dom} f \setminus \overline{S}$ and a smooth quasidisk $D^l \Subset \operatorname{Dom} f \setminus \overline{S}$ containing α^l with the following properties:*

(A1) *For any $z \in \partial^o A^l_0$, $\operatorname{dist}(z, \partial^o A^l) \asymp \operatorname{dist}(z, \partial S)$, and similarly for the inner boundaries $\partial^i A^l_0$ and $\partial^i A^l$.*

(A2) *It is impossible to "jump over the collar":*

If $z \in \operatorname{Comp}_0(\mathbb{C} \setminus A^l_0)$ while $f(z) \notin \operatorname{Comp}_0(\mathbb{C} \setminus A^l_0)$ then $f(z) \in A^l_0$.

(D1) *The disk D^l has a bounded shape around α^l and it has the hyperbolic diameter of order 1 in $\mathbb{C} \setminus \overline{S}$.*

(D2) *A definite portion of D^l is contained in $f^{-1}(S) \setminus S$; moreover,*

there is a point $\beta \in f^{-1}(\partial S) \setminus \overline{S}$ that lies in the middle of D^l;

see Figure 3.2.

(D3) *If $z \in A^l$ then there exists a moment $k < q_{l+1}$ such that $f^k z$ lies in the middle of D^l.*

(D4) *There exists $\underline{\iota} = \underline{\iota}(\overline{N}, \mu, K)$ such that for any $\iota > \underline{\iota}$ and $l > \underline{l} + 2\iota$, under the circumstances of* (D3), *we have*

$$f^i z \notin D_1^{l-\iota}, \quad i = 0, 1, \dots, k,$$

where $D_1^{l-\iota} \Subset Y^{l-\iota} \setminus \overline{S}$ is a topological disk containing $D^{l-\iota}$ with a definite $\operatorname{mod}(D_1^{l-\iota} \setminus D^{l-\iota})$; in particular, $D^l \cap D_1^{l-\iota} = \emptyset$.

(D5) *Moreover, under the above circumstances,*

$$f^i z \in \operatorname{Comp}_0(\mathbb{C} \setminus A^{l-\iota}), \quad i = 0, 1, \dots, k,$$

and $A^{l-\iota} \Subset \operatorname{Comp}_0 \mathbb{C} \setminus (A^{l-2\iota})$.

All bounds and constants depend only on $(\overline{N}, \mu, K)$.

4.5. *Siegel cylinder renormalization.*

4.5.1. *Definition.* Using the circle model, we can extend Yampolsky's construction of the *cylinder renormalization* R_S [Yam08] to all Siegel maps $f \in \mathrm{Sieg}_\theta$ of bounded type. Let g be the quasicritical circle map corresponding to f through the surgery. Let us transfer the arc used for the m-th cylinder renormalization of g (see Section 3.7.2) to an arc δ_m connecting the periodic point α^m of f from Corollary 4.7 to the boundary of S_f. By continuing along the internal ray of S_f, extend δ_m to an arc γ_m connecting α^m to the Siegel fixed point 0. Then $f^{q_m}(\gamma_m)$ does not intersect γ_m, and these two arcs bound a *fundamental crescent* $\mathcal{C}^m$ for f^{q_m}. Now we can proceed with the construction as in the circle case: identifying the boundary arcs of $\mathcal{C}^m$, we produce a map of the standard cylinder $\mathbb{C}/\mathbb{Z}$ whose upper end corresponds to the Siegel fixed point. To recover this point back, let us map $\mathbb{C}/\mathbb{Z}$ onto $\mathbb{C}^*$ by means of $e^{2\pi i z}$. We obtain a Siegel map with rotation number $-1/\theta \mod 1$ (see [Yam08]).

The following statement is a Siegel counterpart of Lemma 3.18 that follows from the latter by surgery.

LEMMA 4.9. *Let f be a Siegel map of class $\mathrm{Sieg}(\overline{N}, \mu, K)$. Then for any $m \geq \underline{l} - 1$, the fundamental crescent $\mathcal{C}^m$ is κ-qc equivalent to the quadrilateral composed by attaching the half-strip* (3.10) (*corresponding to $\mathcal{C}^m \setminus S$*) *to a triangle with angle $2\pi/q$ at* 0 (*corresponding to $\mathcal{C}^m \cap \overline{S}$*) (*see Figure* 4.1). *The dilatation κ depends only on* $(\overline{N}, \mu, K)$.

Let $\pi_m = \pi_m^f$ stand for the change of variable projecting the original dynamical plane to the renormalized one. It starts in the fundamental crescent $\mathcal{C}^m$ and then is spread around by means of pullbacks.

4.5.2. *Hybrid classes.* Two Siegel maps, f and $\tilde{f}$, are said to be *L-hybrid conjugate* if there exists an equivariant L-qc map $\mathrm{Dom}_f^h \to \mathrm{Dom}_{\tilde{f}}^h$ that is conformal on the Siegel disk S_f.

By means of the Douady-Ghys surgery, Theorem 3.19 can be immediately transferred to the Siegel setting:

THEOREM 4.10. *Two Siegel maps $f, \tilde{f} \in \mathrm{Sieg}(\overline{N}, \mu, K)$ with the same rotation number are hybrid $L-$conjugate in a δ-neighborhood of S, with L and $\delta > 0$ depending only on $(\overline{N}, \mu, K)$.*

5. **Inou-Shishikura class**

5.1. *Parabolic renormalization.* Here we will briefly outline the Parabolic Renormalization Theory that provides us with a good control of bifurcations of parabolic maps. It was laid down in the work by Douady and Sentenac (see [DH85a], [Dou94]), Lavaurs [Lav89], and Shishikura [Shi98], which can be consulted for details.

5.1.1. *Parabolic Puiseux germs and their transit maps.* For $q \in \mathbb{N}$ and a small convex neighborhood U of 0, let $\mathcal{G}_0(U)$ be the space of parabolic maps on U given by Puiseux series

$$f : z \mapsto z + z^2 + \sum_{k\in\mathbb{N}} a_k z^{2+k/q} \tag{5.1}$$

(continuous up to the boundary). By definition, it is isomorphic (as a Banach space) to the space of holomorphic germs

$$\hat{f} : \zeta \mapsto \zeta^q + \zeta^{2q} + \sum_{k\in\mathbb{N}} a_k \zeta^{2q+k},$$

continuous up to the boundary, on the neighborhood $\hat{U}$, the full preimage of U under the power change of variable $z = \zeta^q$. The latter space is endowed with uniform topology, which is inherited by $\mathcal{G}_0(U)$.

Let us consider the principal branch of f (for which $z^{1/q}$ is positive on $\mathbb{R}_+$) in the slit plane $U \setminus e^{-\pi i/4}\,\mathbb{R}_+$. It is endowed with the following structure:

(C1) An *attracting petal* $\mathcal{P}^a \equiv \mathcal{P}^a(f)$, which is an open piecewise smooth Jordan disk with the following properties:

- $\mathcal{P}^a$ is $\mathbb{R}$-symmetric and $\mathcal{P}^a \cap \mathbb{R} = (-\delta, 0)$ for some $\delta > 0$;
- $\mathcal{P}^a$ touches the origin at the angle $\pi/2$ with $\mathbb{R}$;
- f univalently maps $\mathcal{P}^a$ into itself, $f(\partial\mathcal{P}^a) \cap \partial\mathcal{P}^a = \{0\}$, and $f^n(z) \to 0$ as $n \to +\infty$ uniformly on $\mathcal{P}^a$.

Along with the attracting petal, there is a *repelling petal* $\mathcal{P}^r \equiv \mathcal{P}^r(f)$ containing an interval $(0, \delta)$ with some $\delta > 0$ that can be defined as the attracting petal for f^{-1}.

(C2) *The horn map* $H \equiv H_f : \mathcal{P}^r \dashrightarrow \mathcal{P}^a$. For any angle $\theta > 0$, there exist $\epsilon > 0$ with the following property: for any $\epsilon' \in (0, \epsilon)$, there exists $n \in \mathbb{N}$ such that for any point $z \in \mathcal{P}^r$ with $\epsilon' < |z| < \epsilon$ and $\theta < \arg z < \pi/2$ (where $\arg z$ is the principal value of the argument), we have $f^n z \in \mathcal{P}^a$. Moreover, ϵ can be selected the same for all maps $\tilde{f} \in \mathcal{G}_0(U)$ near f, and then n can be selected depending on (ϵ, ϵ') only. This transit map is called the horn map H_f.

(C3) The attracting and repelling *Fatou coordinates*

$$\phi^a \equiv \phi^a_f : \mathcal{P}^a \to \mathbb{C}, \quad \phi^r \equiv \phi^r_f : \mathcal{P}^r \to \mathbb{C}, \quad \phi^{a/r}(z) \sim -1/z + \text{const as } z \to 0$$

conformally conjugate f and f^{-1} to the translations $z \mapsto z + 1$ and $z \mapsto z - 1$ respectively. The Fatou coordinates are defined up to translation, so they are uniquely determined by normalization that specifies which points $c^{a/r} \equiv c^{a/r}_f \in \mathcal{P}^{a/r}(f)$ correspond to 1 and $-A-1$, respectively (with some $A \in \mathbb{N}$ to be chosen below). Moreover, if the base points $c^{a/r}_f$ depend holomorphically on f, then so do the normalized Fatou coordinates.

(C4) An *attracting fundamental crescent* $\mathcal{C}^a \equiv \mathcal{C}^a(f)$ is a strip properly embedded into the attracting petal $\mathcal{P}^a$ such that $\partial\mathcal{C}^a \cap \partial\mathcal{P}^a = \{0\}$ and $f(\mathcal{C}^a) \cap \mathcal{C}^a$ is a boundary component of $\mathcal{C}^a$. To be definite, we will use the following choice:

$$\mathcal{C}^a \equiv \mathcal{C}^a(f) = \{z \in \mathcal{P}^a : \ 3/4 \le \operatorname{Re} \phi^a(z) \le 7/4\}.$$

Since the Fatou coordinate depends holomorphically on f, the crescent $\mathcal{C}^a(f)$ moves holomorphically with f.

Similarly, one can define the *repelling fundamental crescent*

$$\mathcal{C}^r \equiv \mathcal{C}^r(f) = \{z \in \mathcal{P}^r : \ -A - 5/4 \le \operatorname{Re} \phi^r(z) \le -A - 1/4\}.$$

(C5) The *Écalle-Voronin cylinders* $\mathrm{Cyl}^{a/r} \equiv \mathrm{Cyl}^{a/r}(f)$ are the quotients of the petals $\mathcal{P}^{a/r}$ by the dynamics. They can be obtained by identifying the boundary components of the corresponding fundamental crescents $\mathcal{C}^{a/r}$ by means of $z \sim f(z)$. The normalized Fatou coordinates induce isomorphisms of the pointed cylinders $(\mathrm{Cyl}^{a/r}, c^{a/r})$ to the standard cylinder $(\mathbb{C}/\mathbb{Z}, 0)$, and in what follows, we will freely identify the cylinders with the standard model.

(C6) A complex one-parameter family of *transit* isomorphisms

$$I_\lambda : \mathrm{Cyl}^a \approx \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z} \approx \mathrm{Cyl}^r, \quad z \mapsto z + \lambda, \quad \lambda \in \mathbb{C}/\mathbb{Z}. \tag{5.2}$$

Let

$$\mathbf{\Lambda}^b := \{-1/4 \le \operatorname{Re}\lambda < 3/4, \ |\operatorname{Im}\lambda| < b\}. \tag{5.3}$$

Then for any $\lambda \in \mathbf{\Lambda}^1$, the isomorphism I_λ lifts to the translation

$$\{3/4 \le \operatorname{Re} z \le 7/4\} \to \{-A - 5/4 \le \operatorname{Re} z \le -A - 1/4\}, \quad z \mapsto z - A - 2 + \lambda,$$

which induces, by means of the Fatou coordinates $\phi_f^{a/r}$, a conformal embedding

$$I_{f,\lambda} : \mathcal{C}^a(f) \to \mathcal{P}^r(f). \tag{5.4}$$

Holomorphic dependence of the Fatou coordinates on f implies that these embeddings depend nicely on the parameters:

LEMMA 5.1. *Assume the base points $c_f^{a/r} \in \mathcal{P}^{a/r}(f)$ are selected holomorphically in f over some neighborhood $\mathcal{U}_0 \subset \mathcal{G}_0(U)$. Then the family of transit maps (5.4) depends holomorphically on $(f, \lambda) \in \mathcal{U}_0 \times \mathbf{\Lambda}^1$.*

The horn map $H \equiv H_f$ from (C2) also descends to the cylinders, and we will keep the same notation, $H : \mathrm{Cyl}^r \dashrightarrow \mathrm{Cyl}^a$, for the quotient.

(C7) *Parabolic renormalization $R_{\mathrm{par}} f$.* Composing the transit maps with the horn map, we obtain a one-parameter family of return maps

$$I_\lambda \circ H_f : \mathbb{C}/\mathbb{Z} \dashrightarrow \mathbb{C}/\mathbb{Z} \tag{5.5}$$

defined near the upper end of the repelling cylinder $\mathrm{Cyl}^r \approx \mathbb{C}/\mathbb{Z}$. By means of[24]

$$\mathrm{Exp} : \mathbb{C}/\mathbb{Z} \to \mathbb{C}^*, \quad \mathrm{Exp}(z) = -(4/27)e^{2\pi i z},$$

we can identify the cylinder $\mathbb{C}/\mathbb{Z}$ with $\mathbb{C}^*$ so that its upper end corresponds to 0 and the boundary of the fundamental crescents $\mathcal{C}^{a/r}$ corresponds to the ray $i\mathbb{R}_-$. Then family of return maps (5.5) becomes a one-parameter family $g_{f,\lambda}$ of conformal germs near 0.

Moreover, there is a unique choice of the transit parameter λ that makes the map $g_{f,\lambda}$ parabolic, with multiplier 1 at 0. This map $g_{f,\lambda}$ is called the *parabolic renormalization* $R_{\mathrm{par}} f$ of f.

5.1.2. *Transit maps for perturbations and their geometric limits.* Let us now consider the space $\mathcal{G}(U)$ of Puiseux germs (continuous up to the boundary)

$$f : z \mapsto e^{2\pi i\gamma}(z + z^2) + \sum_{k\in\mathbb{N}} a_k z^{2+k/q} \tag{5.6}$$

on U. We will refer to $\gamma \in \mathbb{C}/\mathbb{Z}$ as the *complex rotation number* of 0.

Let $\mathcal{U}_0 \subset \mathcal{G}_0(U)$ be a neighborhood of a parabolic map f_0. Let us consider a neighborhood $\mathcal{U}$ in $\mathcal{G}(U)$ consisting of maps $f = e^{2\pi i\gamma}\tilde{f}$, where $\tilde{f} \in \mathcal{U}_0$ and $|\arg\gamma| < \pi/4$.

If $\mathcal{U}$ is sufficiently small, then any map $f \in \mathcal{U}\setminus\mathcal{U}_0$ has a second fixed point $\beta \equiv \beta_f$ near 0. These points can be connected by two disjoint (closed) arcs, $\omega^{a/r} \equiv \omega_f^{a/r}$, with the following properties:

- Together with the interval $[0,\beta]$, each arc $\omega^{a/r}$ bounds an (open) Jordan domain $\mathcal{P}^{a/r} \equiv \mathcal{P}^{a/r}(f)$, called a *perturbed attracting/repelling petal*, whose shape is close to a round disk. Moreover, $\mathcal{P}^a \cap \mathcal{P}^r = \emptyset$.
- The image arc $f^{\pm 1}(\omega^{a/r})$ is contained in $\mathcal{P}^{a/r}\cup\{0\}$, is disjoint from $\omega^{a/r}$, and together with $\omega^{a/r}$ bounds a crescent-shaped region $\mathcal{C}^{a/r} \equiv \mathcal{C}^{a/r}(f)$ called the *attracting/repelling fundamental crescent* (respectively).

The domain $\mathcal{P} = \mathcal{P}(f)$ bounded by the arcs ω_f^a and ω_f^r will be referred to as the *petal* for f. (What happens is that the attracting and repelling petals of a parabolic map “merge” under the perturbation, to form $\mathcal{P}$; see [Dou94], [Shi98].)

As in the parabolic case, the perturbed map can be linearized on its petal. The linearizing coordinate

$$\phi \equiv \phi_f : \mathcal{P} \to \mathbb{C}, \quad \phi(fz) = \phi(z) + 1,\ z \in \mathcal{P} \cap f^{-1}(\mathcal{P})$$

[24]This special normalization of the exponential map is chosen to make it consistent with the one used by Inou and Shishikura; see below.

is called the *Fatou-Douady coordinate* (or *the perturbed Fatou coordinate*). It is defined uniquely up to translation, so it can be normalized by prescribing a point $c_f \in \mathcal{C}^a$ corresponding to 1. If this point is selected so that it depends holomorphically on $f \in \mathcal{U}$ (including parabolic maps $f \in \mathcal{U}_0$), then the linearizing coordinate ϕ_f depends holomorphically, and hence continuously, on $f \in \mathcal{U}$. Thus, if $f_n \to f$, then for any compact set $K \subset \mathcal{P}^{a/r}(f)$, the ϕ_{f_n} are eventually well defined on K, and $\phi_{f_n} \to \phi_f^{a/r}$ uniformly on K.

Furthermore, all the above choices can be adjusted so that the attracting/repelling fundamental petals and crescents are given as follows:

$$\begin{aligned} \mathcal{P}^a &= \{3/4 < \operatorname{Re}\phi(z) < 3/4 + [N_1/2]\}, \\ \mathcal{C}^a \equiv \mathcal{C}^a(f) &= \{3/4 \le \operatorname{Re}\phi(z) \le 7/4\}, \\ \mathcal{P}^r &= \{3/4 + [N_1/2] - N_1 < \operatorname{Re}\phi(z) < -A - 1/4\}, \\ \mathcal{C}^r \equiv \mathcal{C}^r(f) &= \{-A - 5/4 \le \operatorname{Re}\phi(z) \le -A - 1/4\}. \end{aligned} \tag{5.7}$$

Remark 5.1. Here the petals $\mathcal{P}^{a/r}$ can be viewed as domains on the cylinder $\mathbb{C}/N_1\mathbb{Z}$ merging along the vertical line

$$\{\operatorname{Re}\phi(z) = 3/4 + [N_1/2]\} = \{\operatorname{Re}\phi(z) = 3/4 + [N_1/2] - N_1\} \mod N_1.$$

into a single petal $\mathcal{P}$

The quotients of the petals $\mathcal{P}^{a/r}$ by the dynamics provide us with a pair of *Douady cylinders* $\mathrm{Cyl}^{a/r} = \mathrm{Cyl}^{a/r}(f)$. They can be obtained by identifying the boundary arcs of the crescents $\mathcal{C}^{a/r}$ by means of $z \sim f(z)$. As in the purely parabolic case, the Fatou-Douady coordinate ϕ induces an isomorphism between the cylinders $\mathrm{Cyl}^{a/r}$ and the standard cylinder $\mathbb{C}/\mathbb{Z}$, and we will freely identify the cylinders with the standard model.

Let us consider the transit map $T \equiv T_f : \mathcal{C}^a \to \mathcal{C}^r$, i.e., $Tz = f^j z$ where $f^k z \in \mathcal{P}$, $k = 0, 1, \dots, j$, and $f^j z \in \mathcal{C}^r$. It is usually discontinuous, but it induces a conformal isomorphism between the cylinders:

$$I_f : \mathrm{Cyl}^a \approx \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z} \approx \mathrm{Cyl}^r, \quad z \mapsto z + \lambda, \quad \lambda = \lambda(f) \in \mathbb{C}/\mathbb{Z}. \tag{5.8}$$

Remark 5.2. Notice an essential difference with the parabolic case. In that case, there is a one-parameter family of isomorphisms between the cylinders, all on equal footing, while in the perturbed case, (5.8) is a *preferred* isomorphism induced by the dynamics.

THEOREM 5.2. *Assume that the base points $c_f^{a/r} \in \mathcal{P}^{a/r}(f)$ are selected holomorphically in f over some neighborhood $\mathcal{U}_0 \subset \mathcal{G}_0(U)$. Let (Λ_f^b, c_f^r) be the lift of $(\mathbf{\Lambda}^b, 0) \subset (\mathbb{C}/\mathbb{Z}, 0)$ (defined in (5.3)) to $\mathcal{P}^r(f)$ (by means of the Fatou-Douady coordinate). Then for sufficiently small $\mathcal{U}_0$, b and for every sufficiently* *big j, there exists a holomorphic embedding*

$$\Phi_j : \mathcal{U}_0 \times \mathbf{\Lambda}^b \to \mathcal{U}, \quad (\tilde{f}, \lambda) \mapsto e^{2\pi i \gamma_j} \tilde{f},$$

where $\gamma_j \equiv \gamma_{j,\tilde{f}} : \mathbf{\Lambda}^b \to \mathbb{C}$ is a conformal embedding such that

- *Letting $f = \Phi_j(\tilde{f}, \lambda)$, we have*
$$f^k(c^a) \in \mathcal{P}(f),\ k = 0, 1, \dots, j,\ f^j(c^a) \in \Lambda^b_f,\ \textit{and } \lambda(f) = \lambda.$$
- *Letting $\mathcal{C}^a_\epsilon(f) = \{z : 3/4 - \epsilon < \operatorname{Re} \phi^a_f(z) < 7/4 + \epsilon\}$ for $\epsilon > 0$, the transit maps $f^j : \mathcal{C}^a_\epsilon(f) \to \mathcal{P}(f)$ converge as $j \to \infty$ to the parabolic transit map $I_{\tilde{f},\lambda} : \mathcal{C}^a(\tilde{f}) \to \mathcal{P}^r(\tilde{f})$ uniformly on compact subsets of $\mathcal{C}^a(\tilde{f})$, and uniformly over the tube $\mathcal{U}_0 \times \mathbf{\Lambda}^b$.*[25]
- $\operatorname{diam}(\operatorname{Image} \gamma_{j,\tilde{f}}) \asymp j^{-2}$.

The images Q^j of the maps Φ_j will be called *parabolic tubes*. They are endowed with the *horizontal foliation* whose leaves $\mathcal{L}^j(\lambda) \approx \mathcal{U}_0$, $\lambda \in \mathbf{\Lambda}^b$, correspond to the same transit parameter $\lambda \in \mathbf{\Lambda}^b$.

The horn map from (C2) is robust under a perturbation $f = e^{2\pi i \gamma} \tilde{f}$ (5.6). The perturbed horn map $H \equiv H_f : \mathcal{P} \to \mathcal{P}$ is defined for $z \in \mathcal{P}$ with $|z| < \epsilon$ and $0 < \theta < \arg z < \pi/2$. It induces the cylinder horn map $\mathrm{Cyl}^r \to \mathrm{Cyl}^a$ near the upper[26] end of the Douady cylinders. We will use the same notation $H \equiv H_f$ for this map.

Composing it with the transit map $I_f : \mathrm{Cyl}^a \to \mathrm{Cyl}^r$, we obtain the return map $I_f \circ H_f : \mathrm{Cyl}^r(f) \dashrightarrow \mathrm{Cyl}^r(f)$ near the upper end of the cylinders. Viewed in the Exp-coordinate, it becomes a germ $g_f : (\mathbb{C}, 0) \dashrightarrow (\mathbb{C}, 0)$. Its rotation number is given by the (modified) complex Gauss map $G_*(\gamma) = -1/\gamma \mod \mathbb{Z}$. If $G_*(\gamma)$ is small, then this return map is close to the parabolic renormalization of $\tilde{f}$. It is called the *near parabolic renormalization* of f. We will keep the same notation R_{par} for this operator.

5.1.3. *Case of rotation number p/q.* Let us now consider a holomorphic parabolic germ

$$f : \zeta \mapsto e^{2\pi i p/q} \zeta + \zeta^2 + \cdots \ . \tag{5.9}$$

with rotation number p/q. Assume it is non-degenerate, i.e., it has q petals (rather than a multiple of q petals). Then the q-th iterate f^q has a form

$$f^q : \zeta \mapsto \zeta + a_{q+1} \zeta^{q+1} + \cdots, \quad \text{with } a_{q+1} \neq 0.$$

Performing a power change of variable $z = c\zeta^q$, we bring f^q to Puiseux form (5.1).

[25]Under these circumstances, the pair $(\tilde{f}, I_\lambda)$ is called the *geometric limit* of the sequence $\{f_j\}$.

[26]The assumption that $\arg \gamma > \theta$ breaks the symmetry between the ends as it ensures that the points within a compact set of $\mathcal{C}^a(f)$ escape through the upper end of $\mathcal{C}^r(f)$.

Let us now perturb the parabolic map f to

$$f_\epsilon : \zeta \mapsto e^{2\pi i(p/q+\epsilon)}\zeta + \zeta^2 + \cdots . \tag{5.10}$$

The q-th iterate f_ϵ^q has non-vanishing terms $a_k z^k$ with $1 < k < q+1$, but these terms can be killed by a conformal change of variable. Performing further a power change of variable $z = c\zeta^q$, we bring f_ϵ to Puiseux form (5.6). As all the above coordinate changes depend holomorphically on f, this allows us to apply the above theory to the space of germs (5.9).

5.2. *Inou-Shishikura class.* Inou and Shishikura [IS08] have constructed a class $\mathcal{IS}_0$ of maps with the following properties:

(P1) Any map $f \in \mathcal{IS}_0$ is holomorphic on some quasidisk Ω_f containing 0 and has a form $P_0 \circ \phi^{-1}$ where P_0 is the restriction of $z \mapsto z(1+z)^2$ to some domain Ω_0, and $\phi : \Omega_0 \to \mathbb{C}$ is an appropriately normalized univalent map that admits a global qc extension to $\mathbb{C}$.
(P2) 0 is the parabolic fixed point of any $f \in \mathcal{IS}_0$.
(P3) Any $f \in \mathcal{IS}_0$ has a single quadratic critical point $c_0 = c_0(f)$; moreover, the orbit of c_0 does not escape Ω_f, and $f^n(c_0) \to 0$ as $n \to \infty$.
(P4) The class is endowed with the *Bers-Teichmüller topology* and complex structure inherited from the space of Schwarzian derivatives $S\phi$ (see [Ahl06, Ch. VI]); they make it isomorphic to the Universal Teichmüller Space.
(P5) The class is also endowed with *weak* topology induced by the compact-open topology on the space of univalent functions $\phi : \Omega_0 \to \mathbb{C}$; the weak completion $\overline{\mathcal{IS}}_0$ is compact.
(P6) The parabolic renormalization R acts from $\overline{\mathcal{IS}}_0$ to $\mathcal{IS}_0$; its restriction to $\mathcal{IS}_0$ is a compact holomorphic operator.
(P7) The parabolic renormalization of the quadratic map $z \mapsto z + z^2$ has a restriction in $\mathcal{IS}_0$.

For $\theta \in \mathbb{R}/\mathbb{Z}$, define the class $\mathcal{IS}_\theta$ as $e^{2\pi i\theta} \cdot \mathcal{IS}_0$, and let $\mathcal{IS} := \bigcup_\theta \mathcal{IS}_\theta$. (The notation $\overline{\mathcal{IS}}_\theta$ and $\overline{\mathcal{IS}}$ has a similar meaning.) Property (P6) is robust under perturbation:

Theorem 5.3 ([IS08]). *If θ is sufficiently small, then the near parabolic renormalization R_{par} induces an operator $R_{\mathcal{IS}} : \overline{\mathcal{IS}}_\theta \to \mathcal{IS}_{-1/\theta}$ that restricts to a compact holomorphic operator $R_{\mathcal{IS}} : \mathcal{IS}_\theta \to \mathcal{IS}_{-1/\theta}$.*

We will call this operator $R_{\mathcal{IS}}$. (In this section we will often abbreviate it, without saying, to R.) Recall also modified continued fraction expansions from Section 3.7.

COROLLARY 5.4. *There exists* $\underline{N}$ *such that if* $\theta = [N_1, N_2, \dots N_m, \dots]_*$ *with* $N_i > \underline{N}$, $i = 1\dots, m$, *then any map* $\bar{f} \in \overline{\mathcal{IS}}_\theta$ *is* m *times renormalizable under* $R_{\mathcal{IS}}$. *Hence it is infinitely renormalizable if* $m = \infty$.

We say that a rotation number $\theta \in \mathbb{R}/\mathbb{Z}$ (rational or irrational) has *high type* if all $N_i > \underline{N}$ with $\underline{N}$ as above. Let $\mathcal{IS}(\underline{N})$ stand for the union of the spaces $\mathcal{IS}_\theta$ over all θ of high type. For $\theta = [N, N, \dots]_*$ of high stationary type ($N > \underline{N}$), we will also use the notation $\mathcal{IS}_N \equiv \mathcal{IS}_\theta$. Similar notation will be used for the weak completion $\overline{\mathcal{IS}}$.

5.3. *Postcritical set.* Inou and Shishikura have deduced from the above results

PROPOSITION 5.5 ([IS08]). *For any map* $f \in \overline{\mathcal{IS}}(\underline{N})$, *the critical point is non-escaping* (*i.e.*, $f^n(c_0) \in \Omega_f$, $n = 0, 1, \dots$) *and stays away from the boundary of* $\operatorname{Dom} f$. *Thus, the postcritical set* $\mathcal{O}_f$ *is compactly contained in* Ω_f (*uniformly over* $\overline{IS}$). *In the parabolic case we have* $f^n(c_0) \to 0$ *as* $n \to \infty$. *In general,* $\operatorname{orb} c_0$ *is non-periodic.*

Sketch of proof. The mere fact that the IS renormalization Rf is well defined implies that the first N_1 iterates of the critical point stay in Ω_f (where N_1 is the first entry of the rotation number). Existence of all the renormalizations imply that the whole critical orbit stays in Ω_f. Uniform bounds on the postcritical set follow from compactness of $\overline{\mathcal{IS}}$.

In the parabolic case, the map is finitely renormalizable and its last renormalization falls to the class $\mathcal{IS}_0$. Property (P3) implies that $f^n(c_0) \to 0$ as $n \to \infty$. In the irrational case, f is infinitely renormalizable and by increasing $\underline{N}$ if needed, we can make all the renormalizations $R^m f$ to be small perturbations of parabolic maps of class $\mathcal{IS}_0$. Hence $R^m f(c_0) \neq c_0$. On the other hand, if c_0 was periodic, then it would be the fixed point for some renormalization. ☐

5.4. *Renormalization telescope.* In this section we will collect some technical results, essentially contained in the work of Buff-Cheritat [BC12] and Cheraghi [Che13].

Given a map $f \in \overline{\mathcal{IS}}_\theta$ and a (open) topological sector $\mathfrak{S}$ centered at 0, a *principal branch of the first return map to* $\mathfrak{S}$ is an iterate $f^l : V \to \mathfrak{S}$, where V is a relatively open subset of $\mathfrak{S}$ with $0 \in \partial V$ such that for any $z \in V$, $f^l(z)$ is the first return of $\operatorname{orb} z$ to $\mathfrak{S}$.

The following statement provides us with a convenient domain of definition for the renormalization change of variable (see Figure 5.1):

LEMMA 5.6 ([Che13, §2]). *For any map* $f \in \overline{\mathcal{IS}}_\theta$ *with* $\theta = [N_1, N_2, \dots]_*$ *of sufficiently high type, there exists a piecewise smooth sector* $\mathfrak{S} = \mathfrak{S}_f$ *attached to the fixed point* 0 *with the following properties*:

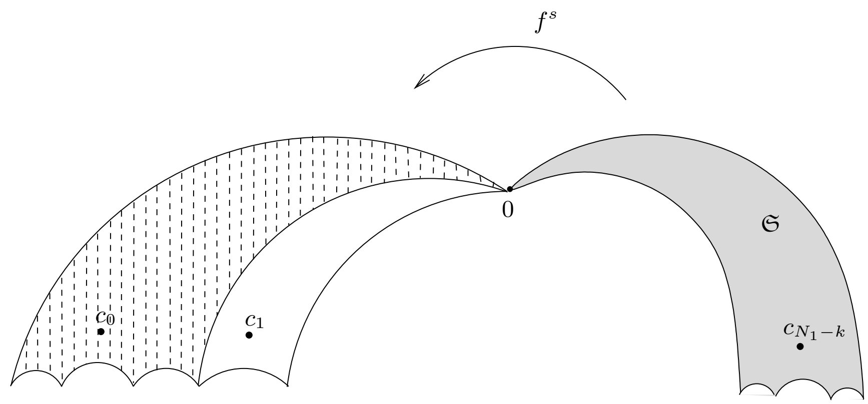

Figure 5.1. Renormalization sectors for an Inou-Shishikura map

(0) *It has angle θ at 0 and is contained in the repelling crescent.*

(i) *There exists a bounded $s = s_f$ such that $f^s(\mathfrak{S})$ is a sector containing the critical value c_1 of f. In an appropriate Fatou coordinate,*[27] *the latter sector becomes the half-strip*

$$\{3/4 \le \operatorname{Re} z \le 7/4,\ \operatorname{Im} z \ge -2\}. \tag{5.11}$$

(ii) *There exists a well-defined change of variable $\pi = \pi_f : \mathfrak{S} \to \mathbb{C}$ that is univalent on $\mathfrak{S}$ and $\sim z^{1/\theta}$ as $z \to 0$ (uniformly over the class $\overline{\mathcal{IS}}_\theta$). Moreover, the image $\pi(\mathfrak{S})$ is a slit topological disk containing $\mathfrak{S}_{Rf}$, and the slit touches the boundary of $\mathfrak{S}_{Rf}$ at a single point, the fixed point 0.*

(iii) *The change of variable is equivariant: it conjugates two principal branches of the first return map to $\mathfrak{S}$ and the renormalization Rf on its full domain.*

(iv) *For some k independent of f, $\mathfrak{S} \ni c_{N_1-k}$ and the union*[28]

$$\Omega^1_f = \bigcup_{n=0}^{N_1+s-k} f^n(\mathfrak{S})$$

is a neighborhood of 0 uniformly compactly containing $\{c_n\}_{n=0}^{N_1+s-k}$.

(v) *The sectors $\mathfrak{S}_f$ depend continuously on $f \in \overline{\mathcal{IS}}(\underline{N})$.*

For $t \ge 2$, let $\Delta = \Delta_f(t)$ be the subset of the sector $\mathfrak{S}_f$ corresponding to the box

$$\{3/4 \le \operatorname{Re} z \le 7/4,\ -2 \le \operatorname{Im} z \le t\} \tag{5.12}$$

in the Fatou coordinate (compare (5.11)).

[27]This coordinate is normalized so that the critical value is placed at 1.

[28]There is a precise formula for the return times in terms of the arithmetic of θ; see Lemma 2.2 in [Che13].

LEMMA 5.7. *Under the circumstances of* Lemma 5.6, *for t sufficiently big, the image $\pi_f(\Delta_f(t))$ compactly contains $\Delta_{Rf}(t)$, with a definite space in between. Moreover, the domain $\Delta_f(t)$ depends continuously on f.*

Proof. The last statement follows from item (v) of Lemma 5.6 and continuous dependence of the Fatou coordinate on f. Together with the weak compactness of the Inou-Shishikura class $\overline{\mathcal{IS}}$ and item (ii) of the lemma, this implies that the change of variable π_f on $\mathfrak{S}_f$ is uniformly comparable with $z \mapsto z^{1/\theta}$ near 0. This map is attracting near 0, so the "bottom" of Δ_f (corresponding to $\{\operatorname{Im} z = t\}$ in the Fatou coordinate) goes even closer to 0. Together with item (ii) of the lemma, this implies that $\pi_f(\Delta_f(t))$ compactly contains $\Delta_{Rf}(t)$. Using weak compactness of $\overline{\mathcal{IS}}$ once again, we conclude that there is a definite space in between. $\square$

From now on, t will be fixed, and it will not appear in the notation.

If f is m times IS-renormalizable, then we can compose the above changes of variable to obtain a map

$$\pi_f^m = \pi_{R^{m-1}f} \circ \cdots \circ \pi_f,$$

well-defined and univalent on a sector $\mathfrak{S}_f^m$ attached to 0. Spreading these sectors around by the iterates of f, we obtain a neighborhood of 0,

$$\Omega_f^m = \bigcup_{n=0}^{r_m} f^n(\mathfrak{S}_f^m), \tag{5.13}$$

where r_m is an appropriate time expressed in terms of the arithmetic of θ (see [Che13, §2]), and $f^n | \mathfrak{S}_f^m$ is at most 2-to-1 for $n \le r_m$. (Note that these maps are not branched coverings over their images.) Moreover, the iterate $f^{s_m - 1} | \mathfrak{S}_f^m$ whose image $\mathfrak{S}^m(c_0) \equiv \mathfrak{S}_f^m(c_0)$ contains the critical point c_0 is univalent.

We let

$$\Pi_m \equiv \Pi_f^m = \pi_m \circ f^{-(s_m-1)} : \mathfrak{S}^m(c_0) \to \mathbb{C}, \tag{5.14}$$

where $f^{-(s_m-1)} | \mathfrak{S}^m(c_0)$ is the branch of the inverse map with image $\mathfrak{S}^m$. (Note also that $\operatorname{Image} \Pi_m = \operatorname{Image} \pi_m$ is a slit topological disk around 0.)

LEMMA 5.8. *Let f be an m times IS-renormalizable map such that $R^m f$ is a parabolic map with multiplier 1. Then the postcritical set $\mathcal{O}_f$ is trapped inside Ω_f^m.*

Let us also consider the lifts Δ_f^m of the domains $\Delta_{R^m f}$ under π_f^m. We let

$$\mathcal{N}_f^m = \bigcup_{n=0}^{r_m} f^n(\Delta_f^m), \tag{5.15}$$

where the times r_m are the same as in (5.13). Moreover, f^{s_m-1} maps Δ_f^m univalently and with bounded distortion onto its image $\Delta^m(c_0) \equiv \Delta_f^m(c_0)$ containing the critical point c_0. Thus, the change of variable Π_m (5.14) restricted to $\Delta^m(c_0)$,

$$\Pi_m : \Delta^m(c_0) \to \mathbb{C}, \tag{5.16}$$

is a univalent map with bounded distortion (over the class $\overline{\mathcal{IS}}(\underline{N})$). Notice also that by compactness of $\overline{\mathcal{IS}}(\underline{N})$ and continuous dependence of Δ^m_g on $g = R^m f$, the image of the restricted map Π_m contains a definite neighborhood of the critical point. Also, the inverse branch $f^{-(r_m - s_m)}$ on Δ^m sending c_{r_m} to c_1 admits an extension to a bigger domain $\tilde{\Delta}^m$ with a definite modulus (by applying the same construction to slightly bigger boxes than defined by (5.12)). It follows that

$$\operatorname{diam}(f^{-(r_m - s_m)}(\Delta^m)) \to 0 \quad \text{as } m \to \infty. \tag{5.17}$$

We call the sets $\mathcal{N}^m_f$ *necklaces*. Lemma 5.7 implies

COROLLARY 5.9. *Under the circumstances of Lemma 5.6, $\pi^m_f(\Delta^{m-1}_f)$ compactly contains $\Delta_{R^m f}$, with a definite space in between. There exist $\rho = \rho(\underline{N}) > 1$ such that $\operatorname{diam} \Delta^m_f = O(\rho^{-m})$. Moreover, for each m, the domain Δ^m_f depends continuously on f.*

COROLLARY 5.10. *Let $f \in \overline{\mathcal{IS}}_\theta$ be a map of IS class with irrational rotation number. Then the critical point is recurrent (but non-periodic).*

Proof. The critical point returns (infinitley many times) to each of the domains $\Delta^m_f \ni c_{r_m - s_m + 1}$. Take such a moment $n_m > r_m - s_m + 1$ and apply to $c_{r_m - s_m + 1}$ and c_{n_m} the inverse branch $f^{-(r_m - s_m)}$ that sends $c_{r_m - s_m + 1}$ to c_1. It follows from (5.17) that it brings the point $c_{n_m - r_m + s_m}$ close to c_1, implying the assertion. □

5.5. *Siegel disks.* The next statement shows that maps $f \in \mathcal{IS}_\theta$ with θ of high bounded type are Siegel maps:

PROPOSITION 5.11 ([Yam08]). *Let $f \in \overline{\mathcal{IS}}_\theta$, where θ is a rotation number of high type bounded by some $\overline{N}$. Then f is a Siegel map; its Siegel disk S_f is a quasidisk compactly contained in Ω_f, and $\partial S_f \ni c_0$. Moreover, $f | \partial S_f$ is quasisymmetrically conjugate to $\mathbf{f}_\theta | \partial \boldsymbol{\mathcal{B}}_\theta$.*

Proof. By replacing f with its IS renormalization $Rf \in \mathcal{IS}$, we can assume that $f \in \mathcal{IS}$ (see Property (P6)).

By Section 4.1.1, we know that the assertion is valid for the quadratic map $\mathbf{f}_\theta$ and hence for its renormalization $\mathbf{g} := R\mathbf{f}_\theta \in \mathcal{IS}_{\theta'}$, $\theta' = -1/\theta$. Let $g := Rf \in \mathcal{IS}_{\theta'}$. Since $\mathcal{IS}_{\theta'}$ is isomorphic to the Universal Teichmüller Space, the map $\mathbf{g}$ can be connected to g by a holomorphic Beltrami path g_λ, $\lambda \in \mathbb{D}$.

Let $c_0(\lambda)$ be the critical point of g_λ, and let $c_n(\lambda) = g^n_\lambda(c_0(\lambda))$, $n \in \mathbb{N}$. By Proposition 5.5, all points $c_n(\lambda)$ are well defined, and then, they depend holomorphically on on λ. Moreover, they do not collide: $c_n(\lambda) \neq c_m(\lambda)$ for $n \neq m$ (by Proposition 5.5 and Corollary 5.10). Hence, they form a holomorphic motion over $\mathbb{D}$.

By the λ-lemma, this motion extends to the postcritical set $\mathbf{O}$ of $\mathbf{g}$, and provides us with a family of quasisymmetric homeomorphisms $h_\lambda : \mathbf{O} \to \mathcal{O}_\lambda$, $\lambda \in \mathbb{D}$, where $\mathcal{O}_\lambda$ is the postcritical set for g_λ. It follows that $\mathcal{O}_\lambda$ is a quasicircle for any $\lambda \in \mathbb{D}$, in particular, for the map g.

Let D be a quasidisk bounded by $\mathcal{O}_g$. Then the family of iterates g^n is normal on D, so $D \subset S_g$. On the other hand, as the Siegel disk S_g does not contain preimages of c_0, which are dense in $\partial D = \mathcal{O}_g$, S_g is contained in D. The conclusions follow. $\square$

5.6. *IS Renormalization fixed point.* Now the whole theory of Siegel maps developed in Section 4 (external tilings, periodic points, trapping disks, renormalization fixed points, etc.) is applicable to any class $\mathcal{IS}_N$, $N > \underline{N}$.

Theorem 5.12 ([IS08]). *Let $\theta = \theta_N$ be a stationary rotation number of high type. Then the IS renormalization R has a unique hyperbolic fixed point $f_\infty \in \mathcal{IS}_N$. The unstable manifold $\mathcal{W}^u(f_\infty)$ is a complex curve that can be parametrized by the complex rotation number ranging over a neighborhood of $[0, \theta]$. Moreover, $R^n f \to f_\infty$ exponentially fast for any Siegel map $f \in \mathcal{IS}_N$.*

Corollary 5.13. *Under the circumstances of the above lemma, let us consider a holomorphic family $\mathcal{F}$ passing through a Siegel map $f_\circ \in \mathcal{IS}_N$ transversally to $\mathcal{IS}_N$. Then there is a sequence of topological disks $\mathcal{F}_n \subset \mathcal{F}$ around $f_\circ$ such that the IS renormalizations $R^n(\mathcal{F}_n)$, $n = 0, 1, \dots$, converge to the unstable manifold $\mathcal{W}^u(f_\infty)$.*

5.7. *Perturbations of Siegel maps.* The above control of one renormalization, together with existence of the hyperbolic renormalization fixed point, provides us with a good control of perturbations of Siegel disks of stationary type (compare [BC12, §1.5]).

Lemma 5.14. *Let $f_\circ$ be a Siegel map of Inou-Shishikura class with stationary rotation number $\theta_\circ = [N, N, \dots]_*$ of high type, and let $\mathcal{F} = \{f_\lambda\}$ be a holomorphic family through $f_\circ = f_{\lambda_0}$ transverse to $\mathcal{IS}_N$. Then for any $m \in \mathbb{N}_0$ and any rotation number θ in a neighborhood of $[0, \theta_\circ]$, there exists a map $f_\lambda \in \mathcal{F}$ such that the Siegel renormalization $R^m f_\lambda$ with the same combinatorics as $R^m f_\circ$ is well defined and has rotation number θ. Moreover, the domain $\Omega^m_{f_\lambda}$ (5.13) is contained in the $O(\rho^{-m})$-neighborhood of $S_\circ$, where $\rho = \rho(N) > 1$.*

Proof. Existence of $f = f_\lambda$ follows from the Renormalization Theorem 5.12. Moreover, the renormalizations of f_λ shadow those of $f_\circ$:

$$\operatorname{dist}(R^n f, R^n f_\circ) \le C|\theta - \theta_0|\, \rho_0^{-(m-n)},\ n = 0, 1, \dots, m, \tag{5.18}$$

where $\rho_0 = \rho_0(N) > 1$.

Let us now apply the lifting and spreading procedure to control the necklaces (5.15), and hence the Ω^m-domains. Assume we have already constructed a necklace $\mathcal{N}^{m-n}_{R^n f}$ that is confined to a δ-neighborhood of $S_{R^n f_\circ}$ By Corollary 5.9, under sufficiently many (k) further lifts, it will shrink by a big factor. Spreading this pullback around by a bounded number of iterates of $R^{m-n-k} f$, the necklace can be pulled father away from $S_\circ$ only by an exponentially small (in $m-n$)) distance; see (5.18). These two mechanisms imply the desired. $\square$

Together with Lemma 5.8, this leads us to the following important conclusion:

Corollary 5.15 ([BC12]). *Under the circumstances of Lemma 5.14, assume the map $R^m f_\lambda$ is parabolic with multiplier 1. Then the postcritical set $\mathcal{O}_\lambda$ of f_λ is contained in the $O(\rho^{-m})$-neighborhood of the Siegel disk $S_\circ$.*

6. Main construction

6.1. *Outline.* Let us start with a quick guide to our construction (outlined in the third paragraph of Section 1.4). Take a big $l \in \mathbb{N}$, a bigger $\kappa \in \mathbb{N}$, and an even much bigger $m \in \mathbb{N}$. Begin with a *Siegel* quadratic polynomial

$$\mathbf{f} = \mathbf{f}_\theta : z \mapsto e^{2\pi i\theta} z + z^2$$

with a stationary rotation number of high type, and consider its cylinder renormalization $f = R_S^{m-\kappa}\mathbf{f}$. It is a Siegel map of Inou-Shishikura class.

Moreover, f has a distinguished repelling periodic point $\alpha = \alpha^l$ of period q_l (that approximates the dynamics on ∂S_f in scale l). Perturb f to a *parabolic* approximant $\tilde f$ with rotation number p_κ/q_κ. Then α gets perturbed to a periodic point $\tilde\alpha$ with the same period; see Section 6.2

Furthermore, using the theory of parabolic bifurcation, one can perturb $\tilde f$ to a *Misiurewicz* map f_{Mis} for which $\tilde\alpha$ becomes a postcritical point α_{Mis}. Since α_{Mis} can be approximated with precritical points, f_{Mis} can be further perturbed to a *superattracting* map $f_\circ$; see Sections 6.3 and 6.4.1.

The last map can be anti-renormalized to obtain a superattracting quadratic polynomial $\mathbf{f}_\circ$ such that $f_\circ = R_S^{m-\kappa}\mathbf{f}_\circ$. This quadratic polynomial determines a renormalization combinatorics. The unique infinitely renormalizable quadratic polynomial $\mathbf{f}_*$ with this combinatorics is desired; see Section 6.4.

Our construction depends on six large integer parameters N, l, κ, t, and m, j, selected consecutively as listed, where the last two play a somewhat different role than the first four. Once we select one of the first four parameters, we assume, sometimes without saying, that all that follows depends on this choice. A statement *For any consecutively selected* $(N, l, \kappa) > (\underline{N}, \underline{l} + 2\iota, \underline{\kappa}) \cdots$ (or *For any consecutively selected sufficiently big* $(N, l, \kappa) \cdots$) will mean

$$\exists \underline{N} \quad \forall N > \underline{N}\ \exists \underline{l} = \underline{l}(N) \quad \forall\, l > \underline{l} + 2\iota\ \exists\, \underline{\kappa} = \underline{\kappa}(N, l) \quad \forall\, \kappa > \underline{\kappa} \cdots .$$

We will also assume that the choice $\underline{l}(N)$ is made *monotonically increasing* in N, the choice of $\underline{\kappa}(N, l)$ is monotonically increasing in each variable, and similarly for any other parameter in question.

Let us now supply the details.

6.2. *Perturbed periodic points and trapping disks.*

6.2.1. *General perturbations.* Recall that $\mathrm{Sieg}(\bar{N}, \mu, K)$ stands for the space of Siegel maps $f : (\Omega, 0) \to (\mathbb{C}, 0)$ introduced in Section 4.3.1.

Remark 6.1. In what follows,

- when we perturb f, we will use the uniform metric on Ω;
- objects associated with $\tilde{f}$ are usually marked with "tilde", but it can be skipped if the object is independent of $\tilde{f}$, e.g., $\tilde{A}^l \equiv A^l$, $\tilde{D}^l \equiv D^l$, etc.

LEMMA 6.1. *There exist natural numbers*[29] $\underline{l}$ *and* ι *depending on* $(\bar{N}, \mu, K)$ *such that for any* $l \geq \underline{l} + 2\iota$, *there exists a* $\delta_0 = \delta_0(\bar{N}, \mu, K, l) > 0$ *with the following property. For any* $\delta < \delta_0$, *if a holomorphic map* $\tilde{f} : \Omega \to \mathbb{C}$ *is* δ*-close to a Siegel map* $f : \Omega \to \mathbb{C}$ *of class* $\mathrm{Sieg}(\bar{N}, \mu, K)$, *then*

(i) *There exists a periodic point* $\tilde{\alpha}^l$ *of period* q_l *that is a perturbation*[30] *of the* α^l.

(ii) *There exists a collar* A^l *in* $\Omega \setminus \bar{S}_f$ *such that it is impossible to jump over it under* $\tilde{f}$:

$$\text{if } z \in \mathrm{Comp}_0(\mathbb{C} \setminus A^l),\ \tilde{f}(z) \notin \mathrm{Comp}_0(\mathbb{C} \setminus A^l),\ \text{then } \tilde{f}(z) \in A^l.$$

(iii) *There exists a trapping quasidisk* $D^l \Subset \Omega \setminus \bar{S}_f$ *with bounded shape around* $\tilde{\alpha}^l$ *whose hyperbolic diameter in* $\Omega \setminus \bar{S}_f$ *is of order* 1. *Moreover,*

$$D^l \cap D^{l+\iota} = \emptyset.$$

(iv) *A definite part of the disk* D^l *is contained in* $\tilde{f}^{-1}(S_f) \setminus \bar{S}_f$. *Moreover, there is a point* $\tilde{\beta} \in \tilde{f}^{-1}(\partial S_f) \setminus \bar{S}_f$ *that lies in the middle of* D^l *(in the sense of* (1.1)*).*

(v) *If* $z \in A^l$, *then at some moment* $k < q_{l+1}$, $f^k z$ *lands in the middle of* D^l, *while*

$$f^i z \in \mathrm{Comp}_0(\mathbb{C} \setminus A^{l-\iota}) \setminus D^{l-\iota}, \quad i = 0, 1, \ldots, k.$$

All geometric bounds depend only on N, μ, *and* K.

Proof. The properties of Proposition 4.8 are manifestly robust under perturbations, keeping the same collars A^l and trapping disks D^l. (The auxiliary

[29] In the polynomial case, we can let $\underline{l} = 1$.

[30] This means that $\tilde{\alpha}^l$ is $\epsilon(\delta)$-near α^l where $\epsilon(\delta) \to 0$ as $\delta \to 0$.

collars A^l_0 and disks D^l_1, as well as the collars $A^{l-2\iota}$ in the last statement (D5), were designed to secure robustness.) $\square$

As before, we say that the trapping disk $D = D^l$ is centered at α^l, or that depth $D = l$.

6.2.2. *Expansion.* For a perturbation $\tilde{f}$ of a Siegel map f, we will use the notation $R^l_{\mathrm{Sp}}\tilde{f} : X^l_+ \cup X^l_- \to \mathbb{C}$ for the corresponding perturbation of the butterfly renormalization $R^l_{\mathrm{Sp}} f$. (That is, if $R^l_{\mathrm{Sp}} f \,|\, X^l_\pm = f^{q^\pm_l}$ then we let $R^l_{\mathrm{Sp}}\tilde{f} \,|\, X^l_\pm = \tilde{f}^{q^\pm_l}$.)[31]

Away from the Siegel disk, Corollary 4.5 is robust under perturbations:

LEMMA 6.2. *Let $f : \Omega \to \mathbb{C}$ be a Siegel map of class* $\mathrm{Sieg}(\bar{N}, \mu, K)$. *For any $\epsilon > 0$, there exists $\delta = \delta(\bar{N}, \mu, K; \epsilon) > 0$ with the following property. Let $\tilde{f} : \Omega \to \mathbb{C}$ be a holomorphic map that is δ-close to f, and let $z \in X^l_+ \cup X^l_-$ be a point with the property that $R^l_{\mathrm{Sp}}\tilde{f}(z) \in Y^l$ and* $\mathrm{dist}(R^l_{\mathrm{Sp}}\tilde{f}(z), \bar{S}) \geq \epsilon$. *Then*

$$\|D(R^l_{\mathrm{Sp}}\tilde{f})(z)\|_{\mathrm{hyp}} \geq \rho > 1$$

with ρ depending only on $(\bar{N}, \mu, K)$ and $\mathrm{dist}(z, \bar{S})/\mathrm{dist}(z, c_0)$ *and the norm being measured in the hyperbolic metric of Y^0.*

Remark 6.2. Note that in the unperturbed case, the above geometric assumption is stronger than the one imposed in Corollary 4.5.

In turn, the last lemma implies a perturbed version of Corollary 4.6:

COROLLARY 6.3. *Let $f : \Omega \to \mathbb{C}$ be a Siegel map of class* $\mathrm{Sieg}(\bar{N}, \mu, K)$. *There exist $a > 0$ and $\rho > 1$ such that for every $\epsilon > 0$, there exists $\delta > 0$ with the following property. For any holomorphic map $\tilde{f} : \Omega \to \mathbb{C}$ that is δ-close to f, if $z \in Y^m$, $\tilde{f}^n z \in Y^{m-k} \setminus Y^{m-k+1}$ for some $n \in \mathbb{N}$, $0 < k < m$ (with $m - k > \underline{l}$), while*

$$\mathrm{dist}(\tilde{f}^i z, \bar{S}) \geq \epsilon, \quad i = 0, 1, \ldots, n,$$

then

$$\|D\tilde{f}^n(z)\|_{\mathrm{hyp}} \geq a\rho^k,$$

where the norm is measured in the hyperbolic metric of $\mathbb{C} \setminus \bar{S}$.

6.2.3. *Cylinder renormalization of polynomial maps.* To make the exposition more transparent, we will focus on the stationary case when $\theta = \theta_N$ is a stationary rotation number with $N > \underline{N}$. Let $\mathbf{f} = \mathbf{f}_\theta : z \mapsto e^{2\pi i\theta} z + z^2$ be the corresponding Siegel quadratic polynomial, and let $\tilde{\mathbf{f}} = \mathbf{f}_{\tilde{\theta}}$ be its polynomial perturbation (where $\tilde{\theta}$ is not necessarily real). By the Inou-Shishikura theory,

[31] Note that we keep the same domain $X^l_+ \cup X^l_-$ for the perturbed map.

all cylinder renormalizations of $\mathbf{f}$ are well defined and belong to the IS class:

$$f_i = R_S^i(\mathbf{f}) \in \mathcal{IS}_\theta, \quad i = 1, 2, \dots . \tag{6.1}$$

Moreover, for any n, if $\tilde{\theta}$ is sufficiently close to θ, then the same is true for the first n cylinder renormalizations of $\tilde{\mathbf{f}}$. In this case, we let

$$\tilde{f}_i = R_S^i(\tilde{\mathbf{f}}) \in \mathcal{IS}_{G_*^i \tilde{\theta}}, \quad i = 1, 2, \dots, n, \tag{6.2}$$

where $G_* : \gamma \mapsto -1/\gamma \mod \mathbb{Z}$ is the modified and complexified Gauss map.

Theorem 5.12 and its Corollary 5.13 provide us with a good control of the maps $\tilde{f}_i$:

LEMMA 6.4. *There exist positive μ, K, ϵ_0, C, and $\rho > 1$ depending only on N such that*

- $f_i \in \mathrm{Sieg}(N, \mu, K), \quad i = 0, 1, \dots ;$
- *for any $\gamma \in \mathbb{C}$ that is ϵ_0-close to θ and any $n \in \mathbb{N}$, there exists a unique $\tilde{\theta}_n$ such that the cylinder renormalizations $\tilde{f}_{n,i} = R_S^i(\mathbf{f}_{\tilde{\theta}_n})$, $i = 0, 1, \dots, n$, are well defined, and $\tilde{f}_{n,n}$ has complex rotation number γ;*
- $\mathrm{dist}(f_i, \tilde{f}_{n,i}) \le C \,\mathrm{dist}(f_n, \tilde{f}_{n,n})\, \rho^{-(n-i)}, \quad i = 0, 1, \dots, n;$
- *the Siegel maps f_i converge to the Siegel renormalization fixed point f_∞, while the nearby maps $\tilde{f}_{n,n}$ converge to a map $\tilde{f}_\infty$ in the unstable manifold $\mathcal{W}^u(f_\infty)$.*

6.2.4. *Parabolic approximant $\tilde{\mathbf{f}}$.* We will now choose a specific perturbation $\tilde{\mathbf{f}} = \mathbf{f}_{\tilde{\theta}}$ of the Siegel polynomial $\mathbf{f} = \mathbf{f}_\theta$ with $\theta = \theta_N$. Take two natural numbers $\kappa < m$. Let $\tilde{\theta} = p_m/q_m$ be the (modified) continued fraction approximant to θ, so that $\tilde{\mathbf{f}}$ is the parabolic quadratic polynomial with rotation number p_m/q_m at 0. It is m times cylinder renormalizable with all the renormalizations $\tilde{f}_i = R_S^j \tilde{\mathbf{f}}$, $i \ge 1$, in the IS class. Moreover, $\tilde{f}_i$ is parabolic with rotation number p_{m-i}/q_{m-i} at 0. We will consider the maps

$$f_{m-\kappa} = R_S^{m-\kappa}(\mathbf{f}) \in \mathcal{IS}_\theta, \quad \tilde{f}_{m,m-\kappa} = R_S^{m-\kappa}(\tilde{\mathbf{f}}) \in \mathcal{IS}_{p_\kappa/q_\kappa}, \tag{6.3}$$

and their limits f_∞ and $\tilde{f}_\infty$ (as $m \to \infty$ with κ being fixed). To simplify notation, we will often skip the subscript $m - \kappa \in \overline{\mathbb{N}}$ letting

$$f \equiv f_{m-\kappa}, \quad \tilde{f} \equiv \tilde{f}_{m,m-\kappa}, \ m \in \overline{\mathbb{N}}_\kappa.$$

By Lemma 6.4, $\tilde{f}$ is δ-close to $f : \Omega \to \mathbb{C}$ for κ big enough, so Lemma 6.1 is applicable, providing us with the trapping discs D^l and the collars A^l.

6.2.5. *Transit from $\tilde{\mathcal{C}}^r$ to the trapping disk D^l.* For the parabolic map $\tilde{f} = \tilde{f}_{m,m-\kappa}$, we let

- $\tilde{\mathcal{C}}^r$ be its repelling crescent (specified as (5.7) in any repelling petal);
- $\tilde{\Delta}^\kappa$ be the domain of the renormalization change of variable $\tilde{\pi}_\kappa$ (see Section 5.4).

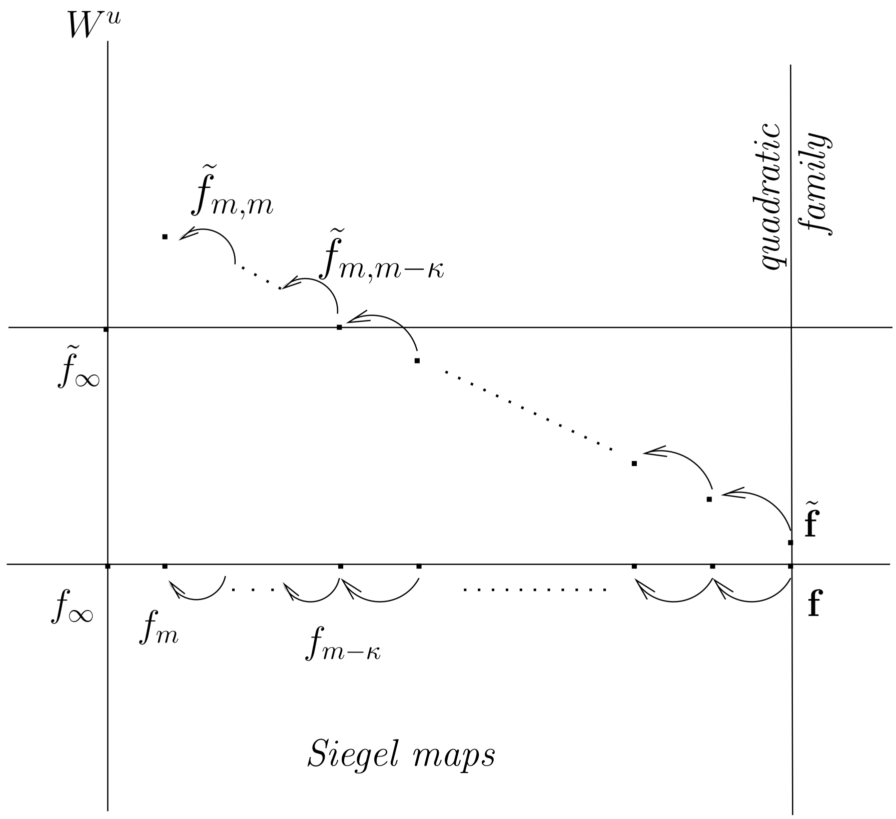

Figure 6.1. Parabolic approximants.

LEMMA 6.5. *For any consecutively selected sufficiently big N and l, there exists $\underline{\kappa}$ such that for any natural $m > \kappa > \underline{\kappa}$, the parabolic map $\tilde{f} = \tilde{f}_{m,m-\kappa}$ has the following property. There exist $\bar{s} = \bar{s}(N, l, \kappa)$ and a point $\tilde{a} \in \tilde{\mathcal{C}}^r \cap \tilde{\Delta}^\kappa$ such that $\tilde{f}^s(\tilde{a}) \in D^l$ for some $s \leq \bar{s}$, and this happens before the orbit of $\tilde{a}$ passes through the collar $A^{l-\iota}$, where $\iota = \iota(N)$. Moreover, the projection $\tilde{\pi}_\kappa(\tilde{a})$ lies in the middle of the repelling crescent $\mathcal{C}^r(\tilde{f}_{m,m})$, with a constant depending on κ but independent of N and l.*

Proof. The range $\tilde{\pi}_\kappa(\tilde{\Delta}^\kappa)$ contains an annulus $\{\epsilon < |z| < r\}$ with a definite r and a small[32] ϵ, slit along the straight ray $i\mathbb{R}_-$. Let us also consider a truncated repelling crescent

$$\mathcal{C}^r_{\mathrm{tr}}(\tilde{f}_{m,m}) := \mathcal{C}^r(\tilde{f}_{m,m}) \cap \{|\arg z| < \pi/2 - \epsilon\}).$$

[32]How small it is depends on the truncation level t defining the domains Δ^κ; see Section 5.4.

Then

- The ray $i\mathbb{R}_-$ does not intersect the truncated repelling crescent $\mathcal{C}^r_{\rm tr}(\tilde{f}_{m,m})$, since the latter is contained in the $\mathbb{R}_+$-symmetric wedge of size $\pi - \epsilon$. (Recall that according to our normalization (5.1) of parabolic maps, $\mathbb{R}_+$ is the repelling direction for $\tilde{f}_{m,m}$.)
- If ϵ is sufficiently small, then the end of $\mathcal{C}^r(\tilde{f}_{m,m})$ is contained in a (slightly enlarged) attracting petal of $\tilde{f}_{m,m}$ (by property (C2) of Section 5.1.1 and compactness of $\overline{\mathcal{IS}}_0$).

The truncated repelling crescent lifts under $\tilde{\pi}_\kappa$ to a truncated crescent $\tilde{\mathcal{C}}^r_{\rm tr}$ for $\tilde{f}$. The latter contains a point $\tilde{a}$ that escapes the domain Ω. (For otherwise, the union of the repelling and attracting petals would form a neighborhood of 0 on which the family of iterates, $\{\tilde{f}^n\}_{n=0}^\infty$, would be well defined and normal.) By Lemma 6.1, this forces $\operatorname{orb}\tilde{a}$ to pass through the trapping disk D^l at some moment s before it passes through the collar $A^{l-\iota}$ with $\iota = \iota(N)$.

If we fix κ, then we obtain a compact family of maps $\tilde{f} \in \overline{\mathcal{IS}}_{p_\kappa/q_\kappa}$, and the fundamental crescent $\tilde{\mathcal{C}}^r$ can be selected in a locally continuous way. This allows us to make a locally continuous choice of $\tilde{a}_{\tilde{f}}$, which, by compactness, makes the escaping time s bounded and puts $\tilde{a}$ in the middle of $\tilde{\mathcal{C}}^r_{\rm tr}$. Since $\tilde{\pi}_\kappa$ has a bounded distortion on $\tilde{\Delta}^\kappa$, this puts $\tilde{\pi}_\kappa(\tilde{a})$ in the middle of $\mathcal{C}^r_{\rm tr}(\tilde{f}_{m,m})$. ☐

6.2.6. *Pullback of D.*

LEMMA 6.6. *For any consecutively selected N and l, there exists $\underline{\kappa}$ such that for any natural $m > \kappa > \underline{\kappa}$, the parabolic map $\tilde{f}_{m,m-\kappa}$ has the properties of Lemma 6.5, and the trapping disc $D = D^l$ can be univalently and with bounded distortion pulled back to $\tilde{a}$ along the orbit $\{\tilde{f}^i\tilde{a}\}_{i=0}^s$. Moreover, the whole pullback $\{\tilde{D}_{-k}\}_{k=0}^s$ is contained in $\operatorname{Comp}_0(\mathbb{C} \setminus A^{l-\iota})$ for some $\iota = \iota(N)$, while the last domain $\tilde{D}_{-s}$ is contained in the repelling crescent $\tilde{\mathcal{C}}^r$.*

Proof. By Proposition 5.15, for κ big enough, the postcritical set $\tilde{\mathcal{O}}$ of $\tilde{f}$ stays close to $S = S_f$. Since D is contained well inside $\Omega \setminus \bar{S}$, it is also contained well inside $\Omega \setminus \tilde{\mathcal{O}}$. So it has a bounded hyperbolic diameter in $\Omega \setminus \tilde{\mathcal{O}}$.

Let us consider the parabolic map $\tilde{f}_{m,m} = R_S^m \tilde{\mathbf{f}} = R_S^\kappa(\tilde{f}_{m,m-\kappa})$ with multiplier 1 at the origin. By Lemma 6.5, there is an escaping point $\tilde{a}$ in $\tilde{\Delta}^\kappa$ such that $\tilde{\pi}_\kappa(\tilde{a})$ lies in the middle of the repelling crescent $\mathcal{C}^r(\tilde{f}_{m,m})$, while $\tilde{a}_s \equiv \tilde{f}^s(\tilde{a})$ lands in $D = D^l$.

Corollary 4.6 implies that for any $\epsilon > 0$, if κ is sufficiently big, there is $k \le s$ such that

(i) $\tilde{a}_{s-k}$ is ϵ-close to the critical point $\tilde{c}_0$;
(ii) D can be univalently pulled back along the orbit $\{\tilde{a}_{s-n}\}_{n=0}^k$ — let $\tilde{D}_{-n}$ denote the corresponding disks;
(iii) the hyperbolic diameter of $\tilde{D}_{-k}$ in $\Omega \setminus \tilde{\mathcal{O}}$ is less than ϵ;

(iv) the orbit $\{\tilde{a}_{s-n}\}_{k\le n\le s}$ is contained in the ϵ-neighborhood of S.

Property (iii) allows us to enlarge $\tilde{D}_{-k}$ to a disk $\tilde{D}'_{-k} \Subset \Omega \setminus \tilde{\mathcal{O}}$ such that

$$\operatorname{mod}(\tilde{D}'_{-k} \setminus \tilde{D}_{-k}) > \mu, \quad \operatorname{diam}_{\text{hyp}} \tilde{D}'_{-k} < \epsilon, \tag{6.4}$$

where $\mu = \mu(\epsilon) \to \infty$ as $\epsilon \to 0$.

Property (iv) ensures that the orbit $\{\tilde{a}_{s-n}\}_{k\le n\le s}$ lies in the range of the renormalization change of variable $\tilde{\pi}_{m-\kappa}$, so it can be lifted to a return orbit $\{\tilde{\mathbf{a}}_{s-n}\}_{k\le n\le s}$ in the domain $\tilde{\boldsymbol{\Delta}}^{m-\kappa}$ of $\tilde{\pi}_{m-\kappa}$.

Moreover, by (iii) the disks $\tilde{D}'_{-k} \supset \tilde{D}_{-k} \ni \tilde{a}_{s-k}$ also lie in the range of $\tilde{\pi}_{m-\kappa}$, so they lift to disks $\tilde{\boldsymbol{D}}'_{\mathbf{-k}} \supset \tilde{\boldsymbol{D}}_{\mathbf{-k}} \ni \tilde{\mathbf{a}}_{\mathbf{s-k}}$ in $\mathbb{C} \setminus \tilde{\mathbf{O}}$. Since $\tilde{\mathbf{f}}$ is a global polynomial map, the disks $\tilde{\boldsymbol{D}}'_{\mathbf{-k}} \supset \tilde{\boldsymbol{D}}_{\mathbf{-k}}$ can be further pulled back to disks $\tilde{\boldsymbol{D}}'_{\mathbf{-s}} \supset \tilde{\boldsymbol{D}}_{\mathbf{-s}} \ni \tilde{\mathbf{a}}$ in $\mathbb{C} \setminus \tilde{\mathbf{O}}$ (where $\tilde{\mathbf{a}}$ is the lift $\tilde{a}$).

As we know (see Section 4.2), this pullback contracts the hyperbolic diameter in $\mathbb{C} \setminus \tilde{\mathbf{O}}$. Since $\boldsymbol{D}'_{\mathbf{-k}}$ has a small hyperbolic diameter (see (6.4)), so does $\tilde{\boldsymbol{D}}'_{\mathbf{-s}}$. Hence it has a small Euclidean diameter compared with $\operatorname{dist}(\tilde{\mathbf{c}}_{\mathbf{-s}}, \tilde{\mathbf{c}}_{q_m\mathbf{-s}})$, where $\tilde{\mathbf{c}}_{\mathbf{-s}}$ is the center of $\tilde{\boldsymbol{\Delta}}^m$. On the other hand, $\operatorname{diam} \tilde{\boldsymbol{\Delta}}^m$ is comparable with the latter distance, and we conclude that $\tilde{\boldsymbol{D}}'_{\mathbf{-s}} \subset \tilde{\boldsymbol{\Delta}}^m$.

We can now apply to $\tilde{\boldsymbol{D}}'_{\mathbf{-s}} \supset \tilde{\boldsymbol{D}}_{\mathbf{-s}}$ the renormalization change of variable $\tilde{\pi}_{m-\kappa}$ to obtain disks $\tilde{D}'_{-s} \supset \tilde{D}_{-s} \ni \tilde{a}$ in $\tilde{\Delta}^\kappa \setminus \tilde{\mathcal{O}}$ that are univalent pullbacks of the disks $\tilde{D}'_{-k} \Supset \tilde{D}_{-k}$. Moreover, the change of variable $\tilde{\pi}_\kappa$ is well defined on these disks, and

$$\operatorname{mod}(\tilde{\pi}_\kappa(\tilde{D}'_{-s}) \setminus \tilde{\pi}_\kappa(\tilde{D}_{-s})) = \operatorname{mod}(\tilde{D}'_{-s} \setminus \tilde{D}_{-s}) = \operatorname{mod}(\tilde{D}'_{-k} \setminus \tilde{D}_{-k}) > \mu,$$

with a big μ; see (6.4). Hence the hyperbolic diameter of $\tilde{\pi}_\kappa(\tilde{D}_{-s})$ inside $\Omega \setminus \tilde{\mathcal{O}}(\tilde{f}_{m,m})$ is small. Since $\tilde{\pi}_\kappa(\tilde{a})$ lies in the middle of the repelling crescent of $\tilde{f}_m$, the disk $\tilde{\pi}_\kappa(\tilde{D}_{-s})$ lies inside the crescent. □

Passing to the limit as $m \to \infty$ (using Lemma 6.4), we conclude

LEMMA 6.7. *There exists $\underline{\kappa}$ such that for any natural $\kappa > \underline{\kappa}$, the map*

$$\tilde{f}_\infty = \lim_{m\to\infty} \tilde{f}_{m,m-\kappa} \in \mathcal{W}^u(f_\infty)$$

has the properties listed in Lemma 6.6.

6.3. *Various connections.* By a *connection* between two points, z and ζ, we mean a trajectory passing from a small neighborhood of z to a small neighborhood of ζ.

6.3.1. *Connection between $\tilde{c}_0$ and* 0. Property (P3) of the Inou-Shishikura class (Section 5.2) and compactness of the space $\operatorname{Sieg}(N, \mu, K)$ (with $\mu = \mu(N)$ and $K = K(N)$ as in Lemma 6.4) imply

LEMMA 6.8. *There exists an $\bar{n} = \bar{n}(N, \kappa)$ such that for any parabolic map $\tilde{f} = \tilde{f}_{m,m-\kappa}$, $m \in \overline{\mathbb{N}}_\kappa$, we have $\tilde{f}^n(\tilde{c}_0) \in \tilde{\mathcal{C}}^a$ for some $n \leq \bar{n}$.*

6.3.2. *Connection between $\tilde{\alpha}^l$ and $\tilde{c}_0$.* Let us now make a connection between the periodic point α^l and the critical point c_0:

LEMMA 6.9. *For any (N, μ, K), there exists $\underline{l}$ with the following property. For any natural $l > \underline{l}$ and any $\rho > 0$, there exists $\underline{t}$ such that for any $t > \underline{t}$ congruent to $\underline{t}$ mod q_l, any Siegel map $f \in \mathrm{Sieg}(N, \mu, K)$ has a t-precritical point c_{-t} in the ρ-neighborhood of the periodic point α^l. Moreover, the orbit $\{c_n\}_{n=-t}^0$ is contained well inside $\Omega^{l-\nu}$ with ν depending only on (N, μ, K). In particular, all these properties are valid uniformly for the maps $f_{m-\kappa}$, $m \in \overline{\mathbb{N}}_\kappa$.*

Proof. Let $\epsilon = \sigma \cdot \mathrm{dist}(\alpha^l, c_0)$ with a small $\sigma \in (0, 1)$, and let W be the ϵ^2-neighborhood of the critical value c_1. Any point $z \in W \cap \partial S$, except c_1 itself, has a preimage $z_{-1} \notin \bar{S}$. Let k be the first moment when the backward orbit $\{c_{-n}\}$ of c_0 (along ∂S) lands in W. Then $k = k(N, \mu, K; l)$ and $\mathrm{dist}(c_1, c_{-k}) \asymp \epsilon^2$ (with a constant depending on (N, μ, K) only).

The point c_{-k} has a preimage $c_{-k-1} \notin \bar{S}$ such that

$$\mathrm{dist}(c_{-k-1}, c_0) \asymp \mathrm{dist}(c_{-k-1}, \bar{S}) \asymp \epsilon.$$

It follows that if σ is sufficiently small, then $c_{-k-1} \in Y^l$ and the hyperbolic distance $d := \mathrm{dist}_{\mathrm{hyp}}(c_{-k-1}, \alpha^l)$ in Y^l is bounded. (Here Y^l corresponds through the surgery to the range of the holomorphic circle pair from Theorem 3.10).

Let $D \ni c_{-k-1}$ be the hyperbolic disk in Y^l of radius $2d$ centered at α^l. By the Schwarz Lemma, $f^{-q_l}(D)$ is a subset of D of bounded hyperbolic diameter (where f^{-q_l} is the inverse branch fixing α^l). A few more (of order $-\log \rho$) pullbacks of c_{-k-1} by f^{-q_l} will bring our point to the ρ-neighborhood of α^l.

Since this backward orbit stays in D, it is trapped inside $\mathrm{Comp}_0(\mathbb{C} \setminus A^{l-\iota})$ with $\iota = \iota(N)$. Since points of $\partial A^{l-\iota}$ lie on depth $l - \iota$, while those of $\partial \Omega^{l-\nu}$ lie on depth $l - \nu$, we see that $A^{l-\iota}$ is contained well inside $\Omega^{l-\nu}$ for ν big enough (depending on (N, μ, K) only). The conclusion follows. □

The above connection is robust:

COROLLARY 6.10. *For any (N, μ, K), there exists $\underline{l}$ with the following property. For any natural $l > \underline{l}$ and any $\rho > 0$, there exist $\underline{t}$ and $\delta_0 > 0$ such that for any $\delta < \delta_0$ and any natural $t > \underline{t}$ congruent to $\underline{t}$ mod q_l, the following holds. If a map $\tilde{f}$ is δ-close to a Siegel map $f \in \mathrm{Sieg}(N, \mu, K)$, then it has a t-precritical point $\tilde{c}_{-t}$ in the ρ-neighborhood of the periodic point $\tilde{\alpha}^l$. Moreover, the orbit $\{\tilde{c}_n\}_{n=-t}^0$ is contained in $\tilde{\Omega}^{l-\nu}$ with $\nu = \nu(N, \mu, K)$. In particular, these properties are valid for any parabolic map $\tilde{f}_{m,m-\kappa}$, $m \in \overline{\mathbb{N}}_\kappa$.*

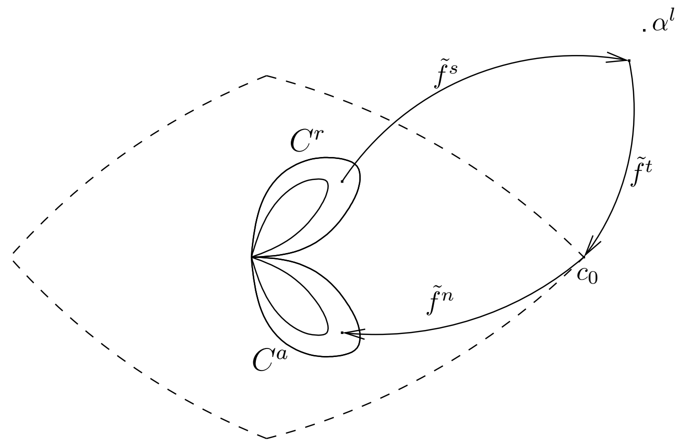

Figure 6.2. Various connections.

6.3.3. *Connection between* 0 *and* $\tilde\alpha^l$.

Lemma 6.11. *For any consecutively selected sufficiently big* N, l, κ *and any* $\rho > 0$, *there exist* $\underline{t}$ *and* $\bar s$ *such that for any natural* $t > \underline{t}$ *congruent to* $\underline{t}$ *mod* q_l, *and some* $s \le \bar s$, *the following holds. For any parabolic map* $\tilde f = \tilde f_{m,m-\kappa}$, $m \in \overline{\mathbb{N}}_\kappa$, *there exists a precritical point* $\tilde c_{-s-t}$ *lying in the middle of the repelling crescent* $\tilde{\mathcal{C}}^r$ *such that* $\tilde f^s(\tilde c_{-s-t}) = \tilde c_{-t}$, *where* $\tilde c_{-t}$ *is* ρ*-close to the periodic point* $\tilde\alpha^l$, *and the orbit* $\{\tilde f^i(c_{-s-t})\}_{i=0}^s$ *is contained in* $\tilde\Omega^{l-\nu}$ *with some* $\nu = \nu(N)$.

Proof. By Lemma 6.4, for κ sufficiently big, $\tilde f_{m,m-\kappa}$ is close to $f_{m-\kappa}$, uniformly in $m \in \overline{\mathbb{N}}_\kappa$. Hence we can apply

- Lemma 6.1 to find a trapping disk $D \equiv D^l$ around $\tilde\alpha^l$;
- Lemma 6.5 to find $\bar s$ and a point $\tilde a \in \tilde{\mathcal{C}}^r$ such that $\tilde f^s \tilde a \in D$ for some $s \le \bar s$;
- Corollary 6.10 to find, for any $t > \underline{t}$ congruent to $\underline{t}$ mod q_l, a precritical point $\tilde c_{-t} \in D$ that is ρ-close to $\tilde\alpha^l$.

By Lemma 6.6, the trapping disk D can be univalently pulled back to the point $\tilde a$. Moreover, this pullback is contained in $\tilde\Omega^{l-\nu}$ for some $\nu = \nu(N)$, and the last domain $\tilde D_{-s}$ is compactly contained in the repelling crescent $\tilde{\mathcal{C}}^r$. The corresponding pullback of the precritical point $\tilde c_{-t} \in D$ gives us the desired point $\tilde c_{-s-t}$. □

6.3.4. *Transit from the repelling crescent to the attracting one.* Combining the last lemma with Lemma 6.8 and Corollary 6.10, we obtain

Corollary 6.12. *For any consecutively selected sufficiently big* N, l, κ, *and for any* $\rho > 0$, *there exist* $\underline{t}$, $\bar n$ *and* $\bar s$ *with the following properties. For any* $m \in \overline{\mathbb{N}}_\kappa$ *and any* $t > \underline{t}$ *congruent to* $\underline{t}$ *mod* q_l, *there exist* $n \le \bar n$ *and* $s \le \bar s$ *such that the parabolic map* $\tilde f = \tilde f_{m,m-\kappa}$ *has a precritical point* $\tilde c_{-s-t} \in \tilde{\mathcal{C}}^r$ *and* *a postcritical point* $\tilde{c}_n \in \tilde{\mathcal{C}}^a$ *such that the whole orbit* $\{\tilde{c}_k\}_{k=-s-t}^n$ *is trapped in* $\tilde{\Omega}^{l-\nu}$ *with some* $\nu = \nu(N)$.

Recall that $\tilde{\mathbf{f}} = \mathbf{f}_{p_m/q_m}$ is the parabolic quadratic polynomial with rotation number p_m/q_m, and that $\tilde{\mathbf{C}}^{a/r}$ stand for the attracting and repelling crescents for $\tilde{\mathbf{f}}$. As $\tilde{f}_{m,m-\kappa} = R_S^{m-\kappa}\mathbf{f}$, we obtain

COROLLARY 6.13. *The points* $\tilde{c}_{-s-t}$ *and* $\tilde{c}_n$ *from* Corollary 6.12 *lift to a precritical point* $\tilde{\mathbf{c}}_{\mathbf{-s-t}} \in \tilde{\mathbf{C}}^r$ *and a postcritical point* $\tilde{\mathbf{c}}_{\mathbf{n}} \in \tilde{\mathbf{C}}^a$ *for the parabolic polynomial* $\tilde{\mathbf{f}}$ *such that the whole orbit* $\{\tilde{\mathbf{c}}_k\}_{k=\mathbf{-s-t}}^{\mathbf{n}}$ *is trapped in* $\tilde{\Omega}^{m-\kappa+l-\nu}$ *with* $\nu = \nu(N)$.

6.4. *Quadratic-like renormalization.*

6.4.1. *Superattracting parameter.* Let us now perturb the parabolic map $\tilde{f} \equiv \tilde{f}_{m,m-\kappa}$, $m \in \overline{\mathbb{N}}_\kappa$, to a superattracting map $f_\circ \equiv f_{m,m-\kappa,j;\circ}$, $j \in \mathbb{N}$, that will determine the desired renormalization combinatorics. Its superattracting cycle[33] $\{c_k^\circ\}_{k=0}^{p-1}$ follows the following route:

- first, it passes from the critical point c_0° to a postcritical point c_n° in the attracting crescent $\mathcal{C}_\circ^a$ (where n comes from Lemma 6.8);
- then it goes through the parabolic gate to a precritical point c_{-t-s}° in the repelling crescent $\mathcal{C}_\circ^r$ (where s and t come from Lemmas 6.5 and 6.9);
- then it penetrates trough the boundary of the virtual Siegel disk S_f, approaches a periodic point $\alpha_\circ$ just missing it to land at c_{-t}°;
- and finally, it returns to c_0°.

Here is a formal statement:

LEMMA 6.14. *Let* $\theta = \theta_N$ *be a stationary rotation number of high type* $N > \underline{N}$, *and let* $l > \underline{l}$ *be a level selected in* Lemma 6.1. *For any* $\delta > 0$, *there exists* $\underline{\kappa} = \underline{\kappa}(N, l; \delta)$ *such that for any* $\kappa > \underline{\kappa}$, *some* $n < \bar{n}(\kappa)$, $s < \bar{s}(\kappa)$, *and any* $t > \underline{t}(\underline{\kappa})$ *congruent to* $\underline{t}$ *mod* q_l, $j \geq \underline{j}$ *and* $m \geq \kappa$ *the following holds. There exists a superattracting map*

$$f_\circ \equiv f_{m,m-\kappa,j;\circ} = R_S^{m-\kappa}(\mathbf{f}_\circ) : \Omega \to \mathbb{C}$$

δ*-close to the parabolic map* $\tilde{f}$ (6.3), *with a superattracting cycle of period* $p = n + j + s + t$, *such that near the critical point* c_0°, *we have*

$$f_\circ^p = f_\circ^{s+t} \circ I_\circ \circ f_\circ^n,$$

where $I_\circ : \mathcal{C}_\circ^a \to \mathcal{C}_\circ^r$ *is a transit map between the crescents of* $f_\circ$. *Moreover, the whole cycle of* c_0° *is contained in* $\mathrm{Comp}_0(\mathbb{C} \setminus A_\circ^{l-\iota})$ *with some* $\iota = \iota(N)$.

[33] We will mark the objects related to $f_\circ$ with a subscript or superscript "$\circ$." On the other hand, for the (pre-/post-) critical points c_k, we skip (here and below) subscripts indicating their dependence on various parameters: m, j, etc.

The same properties hold for the limit map

$$f_{\infty,j;\circ} = \lim_{m\to\infty} f_{m,m-\kappa,j;\circ}$$

in the unstable manifold of the renormalization fixed point (cf. Lemma 6.7).

Proof. Let us consider the postcritical point $\tilde{\mathbf{c}}_{\mathbf{n}} \in \tilde{\mathbf{C}}^a$ and the precritical point $\tilde{\mathbf{c}}_{-\mathbf{s}-\mathbf{t}} \in \tilde{\mathbf{C}}^r$ from Corollary 6.13. Let $\mathbf{I} : \mathrm{Cyl}^a \to \mathrm{Cyl}^r$ be the isomorphism between the cylinders such that $\mathbf{I}(\mathbf{c}_{\mathbf{n}}) = \mathbf{c}_{-\mathbf{s}-\mathbf{t}}$. By the Parabolic Bifurcation Theory (Theorem 5.2) for any sufficiently big j, $\tilde{\mathbf{f}}$ can be perturbed to a superattracting polynomial map $\mathbf{f}_\circ \equiv \mathbf{f}_{j;\circ}$ for which

$$\mathbf{f}_\circ^j(\mathbf{c}_{\mathbf{n}}^\circ) = \mathbf{c}_{-\mathbf{s}-\mathbf{t}}^\circ.$$

Let $f_\circ = R_S^{m-\kappa}(\mathbf{f}_\circ)$ for $m \in \mathbb{N}_\kappa$. The desired properties for these maps, and their limit as $m \to \infty$, are evident. □

6.4.2. *Quadratic-like families for parabolic maps.* Similarly, we can construct the whole quadratic-like family with the desired renormalization combinatorics:

LEMMA 6.15. *For any consecutively selected natural* $(N, l, \kappa, t) > (\underline{N}, \underline{l}, \underline{\kappa}, \underline{t})$, *and any* $m \in \overline{\mathbb{N}}_\kappa$, *any parabolic map* $\tilde{f} \equiv \tilde{f}_{m,m-\kappa}$ *admits a family of transit maps*

$$I_\lambda : \tilde{\mathrm{Cyl}}^a \to \tilde{\mathrm{Cyl}}^r, \quad \lambda \in \Lambda,$$

with the following properties. There is a family of disks $\tilde{U}_\lambda \subset \tilde{V}$ *around* $\tilde{c}_0$ *and moments* $(n, s) \le (\bar{n}, \bar{s})$ (from Lemmas 6.8 and 6.5) *such that*

(0) V *is a quasidisk with bounded dilatation and definite size depending only on* (N, l, κ);[34]

(i) *the maps*

$$\tilde{F}_\lambda = \tilde{f}^{s+t} \circ I_\lambda \circ \tilde{f}^n : \tilde{U}_\lambda \to \tilde{V} \tag{6.5}$$

form a proper unfolded quadratic-like family over Λ;

(ii) *the closures of all intermediate disks*,

$$\tilde{f}^k(\tilde{U}_\lambda),\ k = 0, 1, \ldots, n, \quad \text{and} \quad \tilde{f}^k \circ I_\lambda \circ \tilde{f}^n(\tilde{U}_\lambda),\ k = 0, 1, \ldots s + t - 1,$$

that appear in composition (6.5) *are pairwise disjoint*;

(iii) $\bar{\mu}(N, l, \kappa, t) \ge \mathrm{mod}(V \setminus \tilde{U}_\lambda) \ge \underline{\mu}(N, l, \kappa, t) \to \infty$ *as* $t \to \infty$ *with* N, l, κ *fixed*;

(iv) *in case of connected Julia set* $J(\tilde{F}_\lambda)$ (*i.e., when* λ *belongs to the Mandelbrot set* $\mathcal{M}'_{N,l,\kappa,t,m}$ *of the q-l family* (6.5)), *the disk* $\tilde{U}_\lambda$ *is an L-quasidisk with*

$$\mathrm{area}\, \tilde{U}_\lambda \ge c(N, l, \kappa, t) > 0.$$

All constants and bounds are independent of m.

[34] In fact, for given (N, l, κ), the disk itself can be selected independently of t; for m sufficiently big, it can be selected independently of m either.

Proof. In the $\tilde{f}$-plane, select a disk $\tilde{V} \ni \tilde{c}_0$, and let $\tilde{V}_{-i}$, $i = 0, 1, \dots, t$, be its pullback to $\tilde{c}_{-t}$. Let us show that if $\tilde{V}$ is small enough, depending on N, l, and κ but independently of t, then the closures of these disks are pairwise disjoint. Consider a linearization domain W around the periodic point $\tilde{\alpha}_l$ (so, $\tilde{f}^{q_l}$ maps W univalently onto $\tilde{f}^{q_l}(W) \Supset W$). Note that its size depends on N and l only. It takes a bounded number of iterates ($\le t_0 = t_0(N, l, \kappa)$) for the pullback in question to get trapped in W. By adjusting W and selecting $\tilde{V}$ sufficiently small, we ensure that the first t_0 pullbacks $\tilde{V}_{-i}$, $i < t_0$, are pairwise disjoint and disjoint from W, while $\tilde{V}_{-t_0} \Subset W$. Then the further pullbacks $\tilde{V}_{-i} \Subset W$, $t_0 \le t$, will stay pairwise disjoint and disjoint form the first ones. So, independently of t, the whole pullback $\tilde{V}_{-i}$, $i = 0, 1, \dots t$, will consists of pairwise disjoint domains. Moreover,

$$\operatorname{diam} \tilde{V}_{-t} \to 0 \text{ as } t \to \infty \text{ with } (N, l, \kappa) \text{ fixed.} \tag{6.6}$$

Let us now pull $\tilde{V}_{-t}$ further to c_{-t-s}. The number s of iterates is bounded by $\bar{s}(N, l, \kappa)$ from Lemma 6.5, so for t sufficiently big, (6.6) ensures that these pullbacks stay small and pairwise disjoint. Since c_{-t-s} lies in the middle of the repelling crescent $\tilde{\mathcal{C}}^r$, the final domain $\tilde{V}_{-t}$ is trapped well inside $\tilde{\mathcal{C}}^r$. Hence it projects to a disk compactly contained in the repelling cylinder $\mathrm{Cyl}^r \approx \mathbb{C}/\mathbb{Z}$. (We will keep the same notation for it.)

Consider a parameter domain $\Lambda \subset \mathbb{C}/\mathbb{Z}$ such that $I_\lambda(\tilde{c}_n) \in \tilde{V}_{-t-s}$ for any transit parameter λ in Λ. (In fact, under our normalizations and notational conventions, $\Lambda = \tilde{V}_{-t-s}$.) Pull $\tilde{V}_{-t-s}$ further back by this transit map, and then further back to $\tilde{c}_0$ by the iterates of $\tilde{f}$ (by means of Lemma 6.8). Call the corresponding pullbacks $\tilde{V}_{\lambda,-t-s-I-i}$, $i \le n \le \bar{n}(N, \kappa)$. As the number of these pullbacks is bounded, all of them have a small diameter and remain pairwise disjoint and disjoint from the initial pullbacks, which proves assertion (ii). It also follows that the disc $\tilde{U}_\lambda := \tilde{V}_{\lambda,-t-s-I-n}$ is trapped well inside $\tilde{V}$, which implies that the maps F_λ defined by (6.5) are quadratic-like.

Since the transit map $I_\lambda : \mathcal{C}^a_\lambda \to \mathcal{C}^r$ depends holomorphically on $\lambda \in \Lambda$, these q-l maps form a quadratic-like family. For the same reason, the domains $\tilde{V}_{\lambda,-t-s-I}$, and hence their further pullbacks $\tilde{V}_{-s-t-I-i}$, move holomorphically with λ, so our family is equipped (see Section 2.1.2). For $\lambda \in \partial\Lambda$, we have $I_\lambda(c_n) \in \partial\tilde{V}_{-t-s}$, and hence $F_\lambda(c_0) \in \partial\tilde{V}$. Thus, our q-l family is proper. Finally, as λ goes once around $\partial\Lambda$, then $I_\lambda(c_n)$ goes once around $\tilde{V}_{-t-s}$. (Recall that with our normalizations, $\Lambda = \tilde{V}_{-t-s}$.) So, our q-l family is unfolded. This completes the proof of (i).

The upper estimate in items (iii) and (iv) follow from the property that the total number of $\tilde{f}$-iterates involved in our construction is bounded in terms of (N, l, κ) and t, while the transit maps I_λ, $\lambda \in \bar{\Lambda}$, form a compact family. Hence the size of U_λ is definite in terms of (N, l, κ) and t.

On the other hand, as $t \to \infty$ with (N, l, κ) fixed, (6.6) implies that $\operatorname{diam} \tilde{U}_\lambda \to 0$. This yields the lower bound in item (iii). □

6.4.3. *Quadratic-like families for parabolic perturbations.* For notational convenience, let us shift the m-parameter:

$$\mathfrak{m} = m - \kappa \in \overline{\mathbb{N}} = \{1, 2, \dots, \infty\}.$$

Let $\mathcal{F}_\mathfrak{m} = R_S^\mathfrak{m} \mathcal{F}$, where $\mathcal{F}$ is the quadratic family $(\mathbf{f}_\gamma)$. By Theorem 5.12, these are holomorphic curves converging to the unstable manifold $\mathcal{F}_\infty = \mathcal{W}^u(f_\infty)$ for the Siegel renormalization. Perturbing our parabolic maps $\tilde{f}_\mathfrak{m}$ within the families $\mathcal{F}_\mathfrak{m}$, we can construct genuinely renormalizable maps:

Lemma 6.16. *Under the circumstances of* Lemma 6.15, *for any* $\mathfrak{m} \in \overline{\mathbb{N}}_0$ *and* $j > \underline{j}(N, l, \kappa, t)$, *there exists a holomorphic subfamily* $\mathcal{F}_{\mathfrak{m},j} = (f_{\mathfrak{m},j;\lambda})$ *of* $\mathcal{F}_\mathfrak{m}$ *parametrized by some domain* $\Lambda_{\mathfrak{m},,j}$ *with the following properties*:

(i) *Each family* $\mathcal{F}_{\mathfrak{m},j}$ *gives rise to a proper unfolded q-l family* (see Section 2.1.2)

$$F_{\mathfrak{m},j;\lambda} = f^{\mathfrak{p}}_{\mathfrak{m},j;\lambda} : U_{\mathfrak{m},j;\lambda} \to V_\mathfrak{m}, \quad \lambda \in \Lambda_{\mathfrak{m},j},$$

with period $\mathfrak{p} = n + j + s + t$ *and pairwise disjoint disks* $f^i_{\mathfrak{m},j;\lambda}(\bar{U}_{\mathfrak{m},j;\lambda})$, $i = 1, \dots \mathfrak{p}$.

(ii) *As* $\mathfrak{m} \to \infty$, *the families* $\mathcal{F}_{\mathfrak{m},j}$ *converge, uniformly in* j, *to the families* $\mathcal{F}_{\infty,j}$ *in* $\mathcal{F}_\infty = \mathcal{W}^u(f_\infty)$.

(iii) $\bar{\mu}(N, l, \kappa, t) \geq \operatorname{mod}(V_\mathfrak{m} \setminus U_{\mathfrak{m},j;\lambda}) \geq \underline{\mu}(N, l, \kappa, t) \to \infty$ *as* $t \to \infty$ *with* N, l, κ *fixed.*

(iv) *In case of connected Julia set* $J(F_\lambda)$ (*i.e., when* λ *belongs to the corresponding little Mandelbrot set* $\mathcal{M}'_{N,l,\kappa,t,m,j}$), *the disks* $U_{\mathfrak{m},j;\lambda}$ *are L-quasi-disks with*

$$\operatorname{area} U_{\mathfrak{m},j;\lambda} \geq c(N, l, \kappa, t) > 0.$$

All geometric constants and bounds are independent of $\mathfrak{m}$ *and* j.

Proof. Throughout this argument, (N, l, κ, t) will be fixed, and dependences on them will not be mentioned. Parameters m and j will be free.

By Corollary 5.13, the families $\mathcal{F}_\mathfrak{m}$ stay within a compact collection of families crossing the Siegel class $\{f \in \mathcal{IS} : f'(0) = e^{2\pi i\theta}\}$ transversally at points $f_\mathfrak{m} = R_S^\mathfrak{m} \mathbf{f}_\theta$. In fact, they converge, as $\mathfrak{m} \to \infty$, to the unstable manifold $\mathcal{W}^u(f_\infty) \equiv \mathcal{F}_\infty$ of the Siegel fixed point. Moreover, the parabolic maps

$$\tilde{f}_\mathfrak{m} = R_S^\mathfrak{m}(\mathbf{f}_{p_m/q_m}) = f_{m;\, p_\kappa/q_\kappa} \in \mathcal{F}_\mathfrak{m}$$

converge to $\tilde{f}_\infty \in \mathcal{F}_\infty$. This allows us to apply the Parabolic Bifurcation Theory in a uniform way to the families $\mathcal{F}_\mathfrak{m}$ near the maps $\tilde{f}_\mathfrak{m}$.

Let us start with the limiting parabolic map $\tilde{f} = \tilde{f}_\infty$. Let $\tilde{V} \ni \tilde{c}_0$ be the disk selected for this map in Lemma 6.15, and let $\tilde{V}_{-s-t} \ni \tilde{c}_{-s-t}$ be its pullback constructed in that lemma. It is compactly contained in the repelling crescent $\tilde{\mathcal{C}}^r$, and hence it is compactly contained in some smooth disk $\Lambda \Subset \tilde{\mathcal{C}}^r$.

There is a neighborhood $\Upsilon \subset \mathcal{F}_\infty$ of $\tilde{f}$ such that for any map $f_\gamma \equiv f_{\infty,\gamma} \in \Upsilon$, the pullback $V^\gamma_{-s-t} \ni c_{-s-t}$ of $V \equiv \tilde{V}$ under f_γ^{s+t} is compactly contained in Λ as well (uniformly over $f_\gamma \in \Upsilon$). Moreover, since the disks V^γ_{-s-t} are univalent pullbacks of a fixed disk V by a holomorphic family of maps f_γ^{s+t}, they move holomorphically in γ; let

$$h_\gamma : \tilde{V}_{-s-t} \to V^\gamma_{-s-t}$$

be this holomorphic motion (based at $\tilde{f}$).

By Theorem 5.2, for any sufficiently big j, there exists a holomorphic function $\gamma = \gamma_j(\lambda)$ on Λ such that the transit maps $I^j_\gamma : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ induced by f^j_γ, have the following properties:

- $I^j_\gamma(0) = \lambda$. (Recall that the uniformizations of the Douady cylinders $\mathrm{Cyl}^{a/r}$ by $\mathbb{C}/\mathbb{Z}$ are selected so that $c_n \in \mathrm{Cyl}^a$ and $c_{-s-t} \in \mathrm{Cyl}^r$ correspond to $0 \in \mathbb{C}/\mathbb{Z}$.)
- As $j \to \infty$, the transit maps $I^j_{\gamma(\lambda)}$ converge uniformly on compact sets of $\mathbb{C}/\mathbb{Z}$ and uniformly in $\lambda \in \Lambda$ to the transit map $I_\lambda : z \mapsto z + \lambda$ between the Ecallé-Voronin cylinders for the parabolic map $\tilde{f}$.

By the Argument Principle, for any $z \in \partial\tilde{V}_{-s-t}$, there exists a unique $\lambda \in \Lambda$ such that

$$h_\gamma(z) = I^j_\gamma(0), \quad \text{with } \gamma = \gamma_j(\lambda),$$

and these λ's go around a Jordan curve $\Gamma^j \Subset \Lambda$. This implies that each quadratic-like family

$$F_{j;\lambda} = f^{s+t}_\gamma \circ I^j_\gamma \circ f^n_\gamma : U_{j;\gamma} \to V \tag{6.7}$$

is proper and unfolded over the disk $\Delta_j \Subset \Lambda$ bounded by Γ^j, where $\gamma = \gamma_j(\lambda)$ and $U_{j;\gamma} \ni c_0$ is the pullback of V^γ_{-t-s} by $I^j_\gamma \circ f^n_\gamma$. We obtain assertions (i) and (ii) for $\mathfrak{m} = \infty$.

Assertions (iii) and (iv) for $\mathfrak{m} = \infty$ follow from the corresponding assertions of Lemma 6.15 since the quadratic-like families (6.7) are small perturbations (for big j) of the family $\tilde{F}_\lambda$ (6.5).

For each finite $\mathfrak{m}$, we can apply the same argument to the family $\mathcal{F}_\mathfrak{m}$, which provides us with quadratic-like families $F_{\mathfrak{m},j;\lambda}$ with desired properties, except that the geometric constants and bounds may depend on $\mathfrak{m}$. To make them uniform, we can apply a perturbative argument near $\mathcal{F}_\infty$. Namely, let us start with the same disk $V \ni c_0$ as for $\tilde{f} \equiv \tilde{f}_\infty$, and pull it back by $f^{s+t}_{\mathfrak{m};\gamma}$. We obtain a holomorphically moving family of disks $V^{\mathfrak{m};\gamma}_{-s-t} \Subset \mathcal{C}^r(f_{\mathfrak{m};\gamma})$ which is a small perturbations of the above family (V^γ_{-s-t}) for the f_γ. In particular, for $\mathfrak{m}$ big enough, all these disks are uniformly compactly contained in the domain Λ used for $\mathfrak{m} = \infty$.

Moreover, by Theorem 5.2, as $m, j \to \infty$, the transit maps $I_\gamma^{\mathfrak{m},j}$, with $\gamma = \gamma_{\mathfrak{m},j}(\lambda)$, associated with $f_{\mathfrak{m};\gamma}$, converge to I_λ. It follows that for $\mathfrak{m}$ and j sufficiently big, the quadratic-like families $(F_{\mathfrak{m},j;\lambda})$ are small perturbations of the family $(\tilde{F}_\lambda)$ from Lemma 6.15. The uniformity of the geometric bounds follows. □

6.4.4. *Renormalizations in the quadratic family.* Lifting the above renormalization to the quadratic family by means of change of variable $\Pi_{m-\kappa}$ (5.16) we obtain

Corollary 6.17. *Let $\underline{N}, \underline{l}, \underline{\kappa}$, and $\underline{t}$ be as above. Then for any natural $(N, l, \kappa, t) > (\underline{N}, \underline{l}, \underline{\kappa}, \underline{t})$, there exist $\underline{m}$ and $\underline{j}$ with the following properties. For each natural $(m, j) > (\underline{m}, \underline{j})$, consider the holomorphic family $\boldsymbol{\mathcal{F}}_{m,j} = (\mathbf{f}_{m,j;\lambda})$ of quadratic polynomials such that*

$$f_{m-\kappa,j;\lambda} = R_S^{\mathfrak{m}}(\mathbf{f}_{m-\kappa,j;\lambda}),$$

where $(f_{m-\kappa,j;\lambda})$ is the family from Lemma 6.16. Then

(i) *each family $\boldsymbol{\mathcal{F}}_{m,j}$ admits a primitive proper unfolded q-l renormalization*
$$\boldsymbol{F}_{m,j;\lambda} = \mathbf{f}^{\mathfrak{p}}_{m,j;\lambda} : \mathbf{U}_{m,j;\lambda} \to \mathbf{V}_m, \quad \lambda \in \Lambda_{m,j}$$
with pairwise disjoint disks $\mathbf{f}^k_{\mathfrak{m},j;\lambda}(\bar{\mathbf{U}}_{\mathfrak{m},j;\lambda})$, $k = 1, \dots \mathfrak{p}$;

(ii) $\bar{\mu}(N, l, \kappa, t) \geq \operatorname{mod}(\mathbf{V}_{\mathfrak{m}} \setminus \mathbf{U}_{\mathfrak{m},j;\lambda}) \geq \underline{\mu}(N, l, \kappa, t) \to \infty$ *as $t \to \infty$ with N, l, κ fixed;*

(iii) *in case of connected Julia set $\mathbf{J} \equiv J(\boldsymbol{F}_\lambda)$ (i.e., when λ belongs to the corresponding little Mandelbrot set $\mathcal{M}'_{N,l,\kappa,t,m,j}$), the disks $\mathbf{U}_{m,j;\lambda}$ are $L(N, l, \kappa, t)$-quasidisks with*
$$\operatorname{area} \mathbf{U}_{\mathfrak{m},j;\lambda} \geq c \operatorname{area} \mathbf{V}_m, \text{ where } c = c(N, l, \kappa, t) > 0.$$

All geometric constants and bounds are independent of $\mathfrak{m}$ and j.

The little Mandelbrot copies $\mathcal{M}' = \mathcal{M}'_{N,l,\kappa,t,m,j} \subset \mathcal{M}$ generated by these quadratic-like families determine the desired renormalization combinatorics. Below, a map $\mathbf{f}_\lambda$ will be called *renormalizable* if it is DH renormalizable with these combinatorics (and similarly for f_λ).

6.5. *Geometric bounds.* Along with lower thresholds $(\underline{N}, \underline{l}, \underline{\kappa})$ let us select some upper bounds $(\bar{N}, \bar{l}, \bar{\kappa}, \bar{t}) > (\underline{N}, \underline{l}, \underline{\kappa}, \underline{t})$ satisfying the following requirements:

$$\bar{N} > \underline{N}, \quad \bar{l} > \underline{l} = \underline{l}(\bar{N}), \quad \bar{\kappa} > \underline{\kappa} = \underline{\kappa}(\bar{N}, \bar{l}), \quad \bar{t} > \underline{t} = \underline{t}(\bar{N}, \bar{l}, \bar{\kappa}).$$

Let $\mathbf{f}_* : \mathbf{U} \to \mathbf{V}$ be an infinitely renormalizable quadratic polynomial with bounded combinatorics $(\mathcal{M}^i)_{i=0}^\infty$, where $\mathcal{M}^i = M'_{N_i,l_i,\kappa_i,t_i,m_i,j_i}$ are the little Mandelbrot copies constructed above with

$$(\underline{N}, \underline{l}, \underline{\kappa}) < (N_i, l_i, \kappa_i) \leq (\bar{N}, \bar{l}, \bar{\kappa}) \tag{6.8}$$

(while the bounds on t_i, m_i and j_i are not yet specified[35]).

[35] In fact, in this section one can consider maps $\mathbf{f}_*$ with unbounded t_i, m_i, j_i

PROPOSITION 6.18. *For any sequence* (N_i, l_i, κ_i) *satisfying* (6.8), *there exists* $\underline{t}$ *such that if*

$$t_i > \underline{t}, \quad i = 0, 1, \dots,$$

then the quadratic polynomial $\mathbf{f}_*$ *has unbranched a priori bounds* $\nu(\bar{N}, \bar{l}, \bar{\kappa}) > 0$ *independent of* (t_i, m_i, j_i). *If additionally* $t < \bar{t}$, *then* $\mathbf{f}_*$ *has a geometric bound* $\mathfrak{g} = \mathfrak{g}(\bar{N}, \bar{l}, \bar{\kappa}, \bar{t})$ *independent of* (m_i, j_i).

Proof. If g is a quadratic-like map with $\operatorname{mod} g > \mu$, then it is K-qc conjugate to a quadratic polynomial $\mathbf{f}_\theta$, where $K = K(\mu) \searrow 1$ as $\mu \to \infty$. Hence, if g is DH renormalizable with any combinatorics $\mathcal{M}' = \mathcal{M}'_{N,l,\kappa,t,m,j}$ under consideration, then its renormalization Rg has modulus at least $K^{-1}\underline{\mu}$, where $\underline{\mu} = \underline{\mu}(N, l, \kappa, t)$ is from Corollary 6.17, and $K = K(\underline{\mu})$.

Let us select ν so that $K(\nu) < 2$ and then $\underline{t}$ so that $\underline{\mu}(N, l, \kappa, t) > 2\nu$ for any $t > \underline{t}$ congruent to $\underline{t}$ mod q_l (which is possible by Corollary 6.17). Then for any quadratic-like map g with $\operatorname{mod} g > \nu$ that is renormalizable with combinatorics $\mathcal{M}'$, we have $\operatorname{mod} Rg > \nu$ as well.

It follows that ν gives *a priori bound* for any quadratic-like map g with $\operatorname{mod} g > \nu$ that is infinitely renormalizable with combinatorics $(\mathcal{M}')$. These bounds are unbranched by Corollary 6.17(i). Hence the renormalization domains can be selected with a geometric bound $\mathfrak{g} = \mathfrak{g}(\bar{N}, \bar{l}, \bar{\kappa}, \bar{t})$. ☐

6.6. *Landing probability.* Let $f_* = R_S^{m-\kappa}\mathbf{f}_*$, and let $Rf_* : U_* \to V_*$ be its DH pre-renormalization (with the combinatorics constructed in Section 6.4.1).

The next lemma shows that there is a definite probability of landing in the renormalization domain U_* of the map f_*.

LEMMA 6.19. *Let* $\underline{l}$ *and* ι *be as in* Lemma 6.1. *Let* $l > \underline{l} + \iota$, *and let* $D_* = D_*^{l-\iota}$ *be the trapping disk for* f_* *constructed in that lemma. Then* D_* *contains domains* $U' \subset V'$ *of comparable* (*with* D_*) *size* (*with constants depending on* N, l, κ, *and* t) *that are mapped respectively to* U_* *and* V_* *under some iterate of* f_*. *Moreover,* D_* *is contained well inside* $\operatorname{Dom} f_* \setminus \mathcal{O}_*$ (*with a lower bound depending on* N *only*), *where* $\mathcal{O}_* = \mathcal{O}_{f_*}$ *is the postcritical set for* f_*.

Proof. Recall that f_* is a small perturbation of the Siegel map f whose Siegel disk is called $S = S_f$. Let S' be the component of $f^{-1}(S)$ that is different from S. The trapping disk $D^{l-\iota}$ for f contains in the middle some point of $\partial S'$. If f_* is sufficiently close to f, then $D_* \equiv D_*^{l-\iota}$ contains in the middle some point of $f_*^{-1}(\partial S)$. Hence $f_*(D_*)$ contains in the middle some point of ∂S.

The renormalization range V_* can be selected at a much deeper (but still depending only on N, l, κ, and t) dynamical scale than $f_*(D_*)$. Then $f_*(D_*)$ contains many (in fact, we need only one) univalent and bounded distortion pullbacks of V_* under the Siegel map f. Moreover, these pullbacks have size comparable with $\operatorname{diam} f_*(D_*)$. Selecting f_* sufficiently close to f, we ensure the same property for f_*. Then D_* also contains a comparable pullback of V_*. The corresponding pullback of U_* has a comparable size as well (all in terms of N, l, κ, and t) .

The last assertion follows from the property that the postcritical set $\mathcal{O}_*$ lies well inside A_*^{l-1} while D_* lies outside A_*^{l-1}. □

We call the disk $D = D_*^{l-\iota}$ (and similar disks that appear below) a *safe trapping disk* since it can be "safely" pulled back, with a bounded distortion (depending on N only), along any orbit landing in it. As before, we say that D is *centered* at $\alpha^{l-\iota}$, or that depth $D = l - \iota$.

Lifting this disk by the renormalization change of variable $\Pi_{m-\kappa}$ (5.16), we obtain

Corollary 6.20. *The quadratic polynomial* $\mathbf{f}_*$ *has a safe trapping disk* $\boldsymbol{D} := \boldsymbol{D}_*^{m-\kappa+l-\iota}$ *that contains domains* $\mathbf{U}' \subset \mathbf{V}'$ *of comparable* (*with* $\boldsymbol{D}$) *size that are mapped respectively to* $\mathbf{U}_*$ *and* $\mathbf{V}_*$ *under some iterate of* $\mathbf{f}_*$. *The constant depends on* N, l, κ, *and* t *but is independent of* m.

We will refer to the above disk $\boldsymbol{D}$ as the *base* safe trapping disk. Spreading the disks $\mathbf{U}' \subset \mathbf{V}'$ around by the landing map, we obtain

Corollary 6.21. *For any point* z *whose orbit passes through the safe trapping disk* $\boldsymbol{D}$ *under the iterates of* $\mathbf{f}_*$, *there exist quasidisks* $\mathbf{U}(z) \subset \mathbf{V}(z)$ *with bounded dilatation whose size is comparable with the Hausdorff* $\operatorname{dist}(z, \mathbf{V}(z))$, *and such that*

$$\mathbf{f}_*^n(\mathbf{U}(z)) = \mathbf{U}_*, \quad \mathbf{f}_*^n(\mathbf{V}(z)) = \mathbf{V}_* \quad \text{for some } n = n(z).$$

All constants and bounds depend on N, l, κ *and* t, *but not on* m.

We are now ready to show the map $\mathbf{f}_*$ has a definite landing probability η.

Proposition 6.22. *For the polynomial* $\mathbf{f}_*$, *the landing probability* η *is bounded from below in terms of* N, l, κ, *and* t, *uniformly in* m.

Proof. It is known that almost all point of the Julia set $\mathbf{J}_* = J(\mathbf{f}_*)$ land in $\mathbf{U}_*$ [Lyu83], so it is sufficient to deal with the Fatou set. Since the Siegel disk $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{B}}_{\mathbf{f}}$ occupies certain area, it is sufficient to check that a definite portion of points $z \in \boldsymbol{\mathcal{B}} \setminus \mathbf{J}_*$ land in $\mathbf{U}_*$. But any point $z \in \boldsymbol{\mathcal{B}} \setminus \mathbf{J}_*$ on its way from $\boldsymbol{\mathcal{B}}$ to ∞ must pass through the base safe trapping disk $\boldsymbol{D}$. Then Lemma 6.21 provides us with a domain $\mathbf{U}(z)$ of points landing in $\mathbf{U}_*$ that occupies a definite portion of some neighborhood of z. The conclusion follows from the Besikovich Covering Lemma (see [Mat95]). □

6.7. *Escaping probability* ξ.

6.7.1. *Porosity*. Let us start with a general measure-theoretic lemma asserting that if a set X has density less than $1 - \epsilon$ in many scales, then it has small area.

By a *gap* in X of radius r we mean a round disk of radius r disjoint from X.

LEMMA 6.23. *For any $\rho \in (0,1)$, $C > 0$ and $\epsilon > 0$, there exist $\sigma \in (0,1)$ and $C_1 > 0$ with the following property. Assume that a measurable set $X \subset \mathbb{D}_r$ has the property that for any $z \in X$, there are n disks $\mathbb{D}(z, r_k)$ with radii*

$$C^{-1}\rho^{l_k} \le r_k/r \le C\rho^{l_k} \quad \textit{with some } l_k = l_k(z) \in \mathbb{N},\ l_1 < l_2 < \cdots < l_n,$$

containing gaps in X of radii $\epsilon\, r_k$. Then $\operatorname{area} X \le C_1 \sigma^n r^2$.

Proof. Since the assertion is scaling invariant, we can assume without loss of generality that $r = 1$. We can also assume that X is compact, and we can work with squares instead of disks. Using the first scale l_1 for points of X, we can subdivide the unit square $\mathbb{Q}$ into dyadic squares Q_i^1 (of varying scales) such that each Q_i^1 contains a comparable dyadic square B_i^1 (of relative scale depending on ϵ) disjoint from X. Let $\mathbb{Q}^1 \supset X$ be the union of $Q_i^1 \setminus B_i^1$. Then

$$\operatorname{area} \mathbb{Q}_1 \le \sigma_0 \operatorname{area} \mathbb{Q},$$

where $\sigma_0 \in (0,1)$ is roughly equal to $1 - \epsilon^2$.

Then we can subdivide each Q_i^1 into squares of size B_i^1 and repeat the construction with all non-empty squares of this subdivision (using a deeper scale l_k with a sufficiently big but bounded k, i.e., $k \le \bar{k}$ with some $\bar{k}$ independent of the square in question). It will produce a set $\mathbb{Q}_2 \supset X$ such that

$$\operatorname{area} \mathbb{Q}_2 \le \sigma_0 \operatorname{area} \mathbb{Q}_1.$$

We can repeat this procedure roughly $n/\bar{k}$ times, which implies the desired. ☐

6.7.2. *Landing branches.* Let us consider a safe trapping disk $\boldsymbol{D} = \boldsymbol{D}^{\mathbf{l}}$ for $\mathbf{f}_*$ centered at the periodic point $\boldsymbol{\alpha}_{\mathbf{l}}$. By definition, it has hyperbolic[36] diameter of order 1 in $\mathbb{C} \setminus \mathbf{O}_*$:

$$d^{-1} \le \operatorname{diam}_{\text{hyp}} \boldsymbol{D} \le d \quad \text{with } d = d(N). \tag{6.9}$$

For instance, $\boldsymbol{D}$ can be the base trapping disk of depth $\mathbf{l} = m - \kappa + l - \iota$ from Corollary 6.20, but we will also consider much more shallow disks.

For any point z, let

$$0 \le r_1(z) < \cdots < r_n(z) < \cdots$$

be all *landing times* of $\operatorname{orb} z$ at $\boldsymbol{D}$, i.e., the moments for which $\mathbf{f}_*^{r_n}(z) \in \boldsymbol{D}$ listed consecutively. (This list can be infinite, finite, or empty.) Let $T^n : \operatorname{Dom} T^n \to \boldsymbol{D}$ be the corresponding *landing maps*; i.e., for a point $z \in \operatorname{Dom} T^n$,

[36] Below, "hyperbolic" will always refer to the hyperbolic metric in $\mathbb{C} \setminus \mathbf{O}_*$.

the landing moment $r_n(z)$ is well defined and $T^n(z) = \mathbf{f}_*^{r_n}(z)$. Let $P^n(z) \ni z$ be the pullback of $\boldsymbol{D}$ along the orbit $\{\mathbf{f}_*^i(z)\}_{i=0}^{r_n}$. Since $\boldsymbol{D} \Subset \mathbb{C} \setminus \mathbf{O}_*$, the maps

$$\mathbf{f}_*^{r_n} : P^n(z) \to \boldsymbol{D} \tag{6.10}$$

are univalent. We will refer to these maps as the *landing branches*.

For a domain $P \equiv P^n(z)$, we will also use the notation r_P for the landing time $r_n(z)$ (which is independent of $z \in P$, though the associated level n may depend on z), and we will will use the notation $T_P = \mathbf{f}_*^{r_P}$ for the corresponding landing branch $P \to \boldsymbol{D}$.

Let $\mathcal{P}(\boldsymbol{D})$ be the family of all domains $P = P^n(z)$.

LEMMA 6.24.

- *The landing branches $T_P : P \to \boldsymbol{D}$, $P \in \mathcal{P}(\boldsymbol{D})$, have uniformly bounded distortion*; *the domains $P \in \mathcal{P}(\boldsymbol{D})$ have a bounded shape and are well inside $\mathbb{C} \setminus \mathbf{O}_*$ (with bounds and constants depending only on $\overline{N}$).*
- *Each domain $P \in \mathcal{P}(\boldsymbol{D})$ contains a pullback of $\mathbf{V}_*$ of comparable size (with the constant depending only on the parameters $\overline{N}, \underline{l}, \underline{\kappa}, \underline{t}$).*

Proof. The first assertion follows from the property that $\boldsymbol{D}$ is well inside $\mathbb{C} \setminus \mathbf{O}_*$ and the Koebe Distortion Theorem. Together with Corollary 6.20, it implies the second assertion. □

Along with $\boldsymbol{D}$, let us consider another trapping disk $\boldsymbol{D'}$ (which is allowed to coincide with $\boldsymbol{D}$). Let $\mathcal{P}_{\boldsymbol{D'}}(\boldsymbol{D})$ be the family of all the domains $P = P^n(z) \in \mathcal{P}(\boldsymbol{D})$ intersecting $\boldsymbol{D'}$.

LEMMA 6.25. *For any domain $P \in \mathcal{P}_{\boldsymbol{D'}}(\boldsymbol{D})$,*

$$\operatorname{diam} P \le C_0 \operatorname{diam} \boldsymbol{D'} \quad \text{with } C_0 = C_0(\overline{N}),$$

where $\operatorname{diam} \equiv \operatorname{diam}_{\mathrm{Euc}}$ *stands for the Euclidean diameter.*

Proof. By Lemma 4.2, the inverse branch $T_P^{-1} : \boldsymbol{D} \to P$ is a hyperbolic contraction. Hence $\operatorname{diam}_{\mathrm{hyp}} P \le \operatorname{diam}_{\mathrm{hyp}} \boldsymbol{D} \le d$. Since $P \cap \boldsymbol{D'} \neq \emptyset$ and $\operatorname{diam}_{\mathrm{hyp}} \boldsymbol{D} \le d$ as well, we have

$$\operatorname{diam}_{\mathrm{hyp}}(\boldsymbol{D} \cup P) \le 2d. \tag{6.11}$$

It follows that the conformal factor $\rho(z)$ between the hyperbolic and Euclidean metrics has a bounded oscillation on $\boldsymbol{D'} \cup P$:

$$\sup_{z \in \boldsymbol{D'} \cup P} \rho(z) \le C \inf_{z \in \boldsymbol{D'} \cup P} \rho(z), \quad C = C(N).$$

Hence

$$\frac{\operatorname{diam}_{\mathrm{Euc}} P}{\operatorname{diam}_{\mathrm{Euc}} \boldsymbol{D'}} \le C \frac{\operatorname{diam}_{\mathrm{hyp}} P}{\operatorname{diam}_{\mathrm{hyp}} \boldsymbol{D'}} \le C d^2. \tag{6.12}$$

□

The following lemma shows that pullbacks of trapping disks to some point z lie in different scales:

LEMMA 6.26. *For any $\sigma \in (0,1)$, there exists $\nu = \nu(N,\sigma) \in \mathbb{N}$ with the following property. Let $\boldsymbol{D}_i$, $i = 1,\dots,\nu$, be safe trapping disks, not necessarily distinct. Consider a point z landing at the $\boldsymbol{D}_i$ at moments r_i, where $0 \le r_1 < \cdots < r_\nu$, and let $P^i \ni z$ be the corresponding pullback of the $\boldsymbol{D}_i$. Then*

$$\operatorname{diam} P^\nu < \sigma \operatorname{diam} P^1.$$

Proof. Let $P \equiv P^\nu$, and let $P_i := \mathbf{f}_*^{r_i}(P)$, $i = 1,\dots,\nu$. Then $P_i \cap \boldsymbol{D}_i \neq \emptyset$. By property (6.11),

$$\operatorname{diam}_{\mathrm{hyp}} \boldsymbol{D}_i \cup P_i \le 2d, \tag{6.13}$$

which implies (4.3) for all $z \in P_i$. It allows us to apply Lemma 4.2 and to conclude that all the maps $\mathbf{f}_*^{r_{i+1}-r_i} : P_i \to P_{i+1}$ are hyperbolic expansions by some factor $\lambda = \lambda(N) > 1$. Hence the map $\mathbf{f}_*^{r_\nu - r_1} : P_1 \to P_\nu$ (which is the same as $\mathbf{f}_*^{r_1}(P) \to \boldsymbol{D}_\nu$) is a hyperbolic expansion by $\lambda^{\nu-1}$. Thus

$$\operatorname{diam}_{\mathrm{hyp}}(\mathbf{f}_*^{r_1}(P)) \le \lambda^{-\nu+1} \operatorname{diam}_{\mathrm{hyp}} \boldsymbol{D}_\nu \le d\,\lambda^{-\nu+1}.$$

On the other hand, $\operatorname{diam}_{\mathrm{hyp}}(\mathbf{f}_*^{r_1}(P^1)) \equiv \operatorname{diam}_{\mathrm{hyp}} \boldsymbol{D}_1 \ge d^{-1}$, so

$$\operatorname{diam}_{\mathrm{hyp}}(\mathbf{f}_*^{r_1}(P)) \le d^2 \lambda^{-\nu+1} \operatorname{diam}_{\mathrm{hyp}}(\mathbf{f}_*^{r_1}(P^1)).$$

Property (6.13) with $i = 1$ allows us to switch in the last estimate from the hyperbolic diameters to the Euclidean ones (as in (6.12)) and then to apply the Koebe Distortion Theorem to the map $\mathbf{f}_*^{r_1}$ on $P \cup P^1$. The conclusion follows. □

6.7.3. *Truncated Poincaré series.* Let us now fix a safe trapping disk $\boldsymbol{D}$ (in applications, it will be the base trapping disk), and let $\mathcal{P} := \mathcal{P}_{\boldsymbol{D}}(\boldsymbol{D})$. Of course, a domain $P \in \mathcal{P}$ can admit several representations as $P^n(z)$. Let

$$\chi(P) = \max\{n : \exists\, z \in P \text{ such that } P = P^n(z)\}.$$

Let $\mathcal{P}^n$ be the family of domains $P \in \mathcal{P}$ with $\chi(P) \le n$. We also let

$$\mathbb{P} = \bigcup_{\mathcal{P}} P, \quad \mathbb{P}^n = \bigcup_{\mathcal{P}^n} P.$$

LEMMA 6.27. *There exists $C = C(\overline{N})$ such that*

$$\sum_{\mathcal{P}^n} \operatorname{area} P \le Cn \operatorname{area} \boldsymbol{D}.$$

Proof. Note that the family $\mathcal{P}^n$ has the intersection multiplicity at most n. Indeed, if some point z is contained in k sets P_i of this family, then $P_i = P^{n_i}(z)$ with $n_i = n_i(z) \le n$. But since the n_i are pairwise distinct, $\max n_i \ge k$.

Hence

$$\sum_{\mathcal{P}^n} \operatorname{area} P \le n \operatorname{area} \mathbb{P}^n \le n \operatorname{area} \mathbb{P}. \tag{6.14}$$

By Lemma 6.25, $\mathbb{P}$ is contained in a Euclidean neighborhood of $\boldsymbol{D}$ of size $\le C_0 \operatorname{diam} \boldsymbol{D}$. Since $\boldsymbol{D}$ has a bounded shape, $\operatorname{area} \mathbb{P} \le C \operatorname{area} \boldsymbol{D}$, with $C = C(N)$. Together with (6.14), this implies the desired. ☐

Let us consider the following *truncated Poincaré series*: for $\zeta \in \boldsymbol{D}$, let

$$\phi_n(\zeta) = \sum_{P \in \mathcal{P}^n} \frac{1}{|DT_P(\zeta_P)|^2}, \quad \text{where } \zeta_P \in P \text{ and } T_P(\zeta_P) = \zeta.$$

LEMMA 6.28. *We have $\phi_n(\zeta) \le Cn$, where $C = C(\overline{N})$.*

Proof. We have

$$\int_{\boldsymbol{D}} \phi_n(\zeta)\, d \operatorname{area}(\zeta) = \sum_{\mathcal{P}^n} \operatorname{area} P \le Cn \operatorname{area} \boldsymbol{D},$$

where the last estimate is the content of Lemma 6.27. But since the branches $T_P : P \to \boldsymbol{D}$ have a bounded distortion, $\phi_n(\zeta) \asymp \phi_n(\zeta')$ for any $\zeta, \zeta' \in \boldsymbol{D}$ (with constants depending only on $\overline{N}$). The conclusion follows. ☐

6.7.4. *Probability of few returns to the base.* Let us start with an observation that for m big enough, our quadratic polynomial $\mathbf{f}_*$ has plenty of safe trapping disks:

LEMMA 6.29. *For any natural $\tau \in \mathbb{N}$, there exists $\underline{m} = \underline{m}(N, l.\kappa, t, \tau)$ such that for any $m > \underline{m}$, the polynomial $\mathbf{f}_*$ has at least τ safe trapping disks $\boldsymbol{D}_i$ satisfying the properties of Lemma 6.1. Moreover, these trapping disks are pairwise disjoint and disjoint from the base safe trapping disk $\boldsymbol{D} = \boldsymbol{D}^{m-\kappa+l-\iota}$.*

Proof. By Lemma 6.4, our polynomial $\mathbf{f}_*$ is ϵ_m-close to the Siegel polynomial $\mathbf{f}$, where $\epsilon_m \to 0$ as $m \to \infty$ (keeping the other parameters, N, l, κ and t, frozen). Hence for m big enough, Lemma 6.1 (applied directly to $\mathbf{f}_*$) supplies us with arbitrary many safe trapping disks $\boldsymbol{D}_i$. ☐

From now on, $\boldsymbol{D}$ will stand for the base trapping disk. Recall that $\mathbf{J}_*$ is the Julia set of $\mathbf{f}_*$. Let Z be the set of points $z \in \boldsymbol{D} \setminus \mathbf{J}_*$ that under the iterates of $\mathbf{f}_*$ never return back to $\boldsymbol{D}$. The following lemma shows that for m sufficiently big, it is difficult to escape from $\boldsymbol{D}$:

LEMMA 6.30. *For any natural $\tau \in \mathbb{N}$, there exists $\underline{m} = \underline{m}(N, l.\kappa, t, \tau)$ such that for any $m > \underline{m}$,*

$$\operatorname{area} Z \le C\sigma^\tau \operatorname{area} \boldsymbol{D},$$

with $\sigma \in (0,1)$ and $C > 0$ depending only on $\overline{N}$.

Proof. Let $z \in Z$. If m is sufficiently big, then on its way from $\boldsymbol{D}$ to ∞, the orbit of z must visit τ safe trapping disks $\boldsymbol{D}_i$ from Lemma 6.29 at some moments $r_1 < r_2 < \cdots < r_\tau$. By Lemma 6.1, definite parts W_i of these trapping disks are contained in $\mathbf{f}_*^{-1}(\boldsymbol{\mathcal{B}})$. Since $\operatorname{orb} z$ never returns back to $\boldsymbol{D}$, it cannot visit the Siegel disk $\boldsymbol{\mathcal{B}} = S_{\mathbf{f}}$, and hence it cannot land in the domains W_i either.

Since each disk $\boldsymbol{D}_i$ is safe, it can be univalently and with bounded distortion pulled back to z. The corresponding pullback of W_i creates a gap of definite size in Z near z. By Lemma 6.26, these gaps lie in $\asymp \tau$ different scales. Lemma 6.23 completes the proof. ∎

Let
$$Z_n = \bigcup_{P \in \mathcal{P}^n} T_P^{-1}(Z).$$
Notice that points of Z_n escape $\boldsymbol{D}$ forever after at most n returns.

Lemma 6.31. *For any natural $\tau \in \mathbb{N}$, there exists $\underline{m} = \underline{m}(N, l.\kappa, t, \tau)$ such that for any $m > \underline{m}$,*
$$\operatorname{area} Z_n \le C\, n\, \sigma^\tau \operatorname{area} \boldsymbol{D},$$
where $\sigma \in (0,1)$ and $C > 0$ depend only on $\overline{N}$.

Proof. Since
$$\operatorname{area} Z_n = \int_Z \phi_n(\zeta)\, d\operatorname{area}(\zeta),$$
the conclusion follows from Lemmas 6.28 and 6.30. ∎

6.7.5. *Many returns to the base.* Let
$$\mathbb{S}^n = \bigcup_{\chi(P) > n} P = \bigcup_{\mathcal{P} \setminus \mathcal{P}^n} P.$$

Lemma 6.32. *There exist $C > 0$ and $\sigma \in (0,1)$ depending on $N, \underline{l}, \underline{\kappa}$, and $\underline{t}$ such that for any $n \in \mathbb{N}$, the area of the set of points of $\mathbb{S}^n$ that never land in $\mathbf{V}_*$ is at most $C\sigma^n \operatorname{area} \boldsymbol{D}$.*

Proof. Take a point $\zeta \in \mathbb{S}^n$. It belongs to some domain $P \in \mathcal{P}$ with $\chi(P) > n$. Then P contains a point z that lands in $\boldsymbol{D}$ at least n times, and $P^n(z) = P$. By Lemma 6.26, the nest
$$P^1(z) \supset P^2(z) \supset \cdots \supset P^n(z) = P$$
represents $\asymp n$ different scales. By Lemma 6.24, each of these domains contains a pullback of $\mathbf{V}$ of comparable size. Now the desired follows from Lemma 6.23. ∎

6.7.6. *Escaping probability.* We are finally ready to show that the escaping probability ξ for $\mathbf{f}_*$ can be made arbitrary small by selecting m sufficiently big (while keeping the previously selected parameters, N, l, κ, and t, unchanged).

PROPOSITION 6.33. *For any $\epsilon > 0$, there exists $\underline{m}$ such that $\xi < \epsilon$ for any $m > \underline{m}$.*

Proof. Let Y be the set of points in $\boldsymbol{D}$ that never land in $\mathbf{V}_*$. We will show first that for m sufficiently big,

$$\operatorname{area} Y < \epsilon \operatorname{area} \boldsymbol{D}. \tag{6.15}$$

For any $n \in \mathbb{N}$, let us cover Y by three sets:

$$Y_0 = Y \cap J(\mathbf{f}_*), \quad Y_1^n = Y \cap \mathbb{S}^n, \quad Y_2^n = Y \setminus (Y_0 \cup Y_1^n).$$

It is known that almost all point of $J(\mathbf{f}_*)$ land in $\mathbf{V}_*$ [Lyu83], so area $Y_0 = 0$.

By Lemma 6.32,

$$\operatorname{area} Y_1^n \leq C\sigma^n \operatorname{area} \boldsymbol{D} < (\epsilon/2) \operatorname{area} \boldsymbol{D}$$

as long as n is sufficiently big.

Now let us take any point $z \in Y_2^n$. Then

$$\chi(z) := \max\{\chi(P) : \ P \in \mathcal{P}, \ P \ni z\} \leq n,$$

and orb z returns back to $\boldsymbol{D}$ at most n times. Let $k \leq n$ be the number of returns, and let $P := P^k(z)$. Since $P \ni z$, we have $P \in \mathcal{P}^{\chi(z)} \subset \mathcal{P}^n$. Moreover, under the return map $T_P : P \to \boldsymbol{D}$, the point z must land in Z since it will never come back to $\boldsymbol{D}$ again. Hence $z \in Z_n$. Thus $Y_2^n \subset Z_n$. Applying Lemma 6.31, we see that area $Y_2^n < (\epsilon/2) \operatorname{area} \boldsymbol{D}$ for m sufficiently big, and estimate (6.15) follows.

To pass from (6.15) to an estimate of ξ, we need to transfer the density estimate for Y to the fundamental annulus $\mathbf{V}_* \setminus \mathbf{U}_*$. Let $\mathcal{Y}$ be the set of points in $\mathbf{V}_* \setminus \mathbf{U}_*$ that never return back to $\mathbf{V}_*$. Again, since almost all points of $J(\mathbf{f}_*)$ land in $\mathbf{V}_*$, it is sufficient to deal with the Fatou set $\mathcal{Y} \setminus \mathbf{J}_*$. Any point $z \in \mathcal{Y} \setminus \mathbf{J}_*$ eventually lands in the "middle" of the base trapping disk $\boldsymbol{D}$. Pulling $\boldsymbol{D}$ back to z, we obtain a domain $Q(z)$ of bounded shape in which the set $\mathcal{Y} \cap Q(z)$ (the pullback of Y) has density $\leq C\epsilon$. Applying the Besikovich Covering Lemma, we conclude that $\mathcal{Y}$ has density $\leq C'\epsilon$ in $\mathbf{V}_* \setminus \mathbf{U}_*$. □

6.8. *Positive area: stationary case.*

THEOREM 6.34. *For any consecutively selected*

$$(N, l, \kappa, t, m, j) > (\underline{N}, \underline{l}, \underline{\kappa}, \underline{t}, \underline{m}, \underline{j})$$

(with t being congruent to $\underline{t}$ mod q_l), the Feigenbaum polynomial $\mathbf{f}_*$ with stationary combinatorics $\mathcal{M}'_{N,l,\kappa,t,m,j}$ has the Julia set of positive area.

Proof. By Proposition 6.18, the map $\mathbf{f}_*$ has *a priori bounds* depending only on N, l, κ, and t.

By Proposition 6.22, it has a definite *landing parameter* η depending on the same four parameters only.

By Proposition 6.33, it has an arbitrary small *escaping parameter* ξ as long as m, j are sufficiently big (with frozen N, l, κ, and t).

Now the Black Hole Criterion (Theorem 2.3) implies the desired. □

6.9. *Parameter visibility.* To prove Theorem 1.2, we need the following generalization of Theorem 6.34:

THEOREM 6.35. *Let $\mathcal{F}$ be a finite family of renormalization combinatorics as in* Theorem 6.34*. Then any map $f : U \to V$ in the corresponding renormalization horseshoe $\mathcal{A}$ has the Julia set of positive area.*

This result follows easily from the machinery developed in [AL08].

Let $f_m = f^{p_m} : U_m \to V_m$, $m = 0, 1, \dots$, be the consecutive quadratic-like renormalizations of f that are selected in a nice geometric way (specifying geometric bounds of f; see [AL08, §2.7]).

For any level $m \geq 1$, we consider the scaling factor

$$\rho_m = \frac{\operatorname{area} U_m}{\operatorname{area} U_{m-1}},$$

along with the escaping and landing parameters for f_{m-1}:

$$\xi_m = \xi(f_{m-1}), \quad \eta_m = \eta(f_{m-1}).$$

Recall that the *Poincaré series* with exponent 2 for a map f is defined as

$$\Theta(z; f) = \sum_{k=0}^{\infty} \sum_{f^k \zeta = z} \frac{1}{|Df^n(\zeta)|^2}, \quad z \in \mathbb{C} \setminus \mathcal{O},$$

where $\mathcal{O}$ is the postcritical set of f. The *truncated* at level j Poincaré series $\Theta^{[j]}(z; f)$, $j = 0, 1 \dots$ is defined by taking in this formula only preimages of z of order $k \leq j$. It is also convenient to let $\Theta^{[-1]}(z; f) = 0$.

Let us average the Poincaré series for the f_{m-1} over the renormalization fundamental annuli $A_m = V_m \setminus U_m$:

$$\omega_m = \frac{1}{\operatorname{area} A_m} \int_{A_m} \Theta(z; f_{m-1})\, dxdy.$$

The truncated version $\omega_m^{[j]}$ is defined similarly by averaging $\Theta^{[j]}(z; f_{m-1})$.

Finally, let us also consider the rescaled Poincaré series:

$$\sigma_m = \rho_m \omega_m, \quad \sigma_m^{[j]} = \rho_m \omega_m^{[j]}.$$

Lemma 6.36. *Let f be a Feigenbaum map in the renormalization horseshoe $\mathcal{A}$ with $\xi_m < \bar{\xi}$ and $\eta_m \geq \underline{\eta}$. There is $C > 1$ depending only on the geometric bounds for f such that*

$$\sigma_m^{[j+1]} \leq C + (1 - C^{-1}\underline{\eta})\,\sigma_m^{[j]} + C\bar{\xi}\,\sigma_{m+1}^{[j]} + C\bar{\xi}\,\sigma_m^{[j]}\sigma_{m+1}^{[j]}, \quad j = -1, 0, 1, \dots .$$

Proof. This follows from the Recursive Estimate of Lemma 4.2 [AL08] (applied to three consecutive levels $l = m-1, m, n = m+1$):

$$\begin{aligned}\omega_m^{[j+1]} \leq C\frac{\eta_m}{\rho_m} &+ (1 - C^{-1}(\eta_{m+1} + \xi_m))\omega_m^{[j]} \\ &+ C\,\frac{\rho_{m+1}}{\rho_m}\xi_{m+1}\eta_m\omega_{m+1}^{[j]} + C\rho_{m+1}\xi_{m+1}\,\omega_m^{[j]}\omega_{m+1}^{[j]},\end{aligned}$$

with $C > 0$ depending only on the geometric bounds for f. □

Lemma 6.37. *Under the circumstances of* Lemma 6.36, *there exist $\bar{\sigma}$ depending only on the geometric bounds for f such that $\sigma_m \leq \bar{\sigma}$, $m = 1, 2, \dots$, as long as $\bar{\xi}$ is sufficiently small, while $\underline{\eta}$ is bounded away from* 0.

Proof. Let us consider a quadratic polynomial

$$P(\sigma) = C + (1 - (2C)^{-1}\underline{\eta})\,\sigma + C\bar{\xi}\,\sigma^2.$$

For $\bar{\xi}$ sufficiently small, it has two positive fixed points, and the smallest one is bounded by some $\bar{\sigma}$.

Let us show inductively in j that $\sigma_m^{[j]} \leq \bar{\sigma}$ for all natural m. The base is obvious as $\sigma_m^{[-1]} = 0$. Furthermore, if the assertion is true for some j, then Lemma 6.36 (taking into account positivity of the coefficients in the recursive expression) implies

$$\sigma_m^{[j+1]} \leq P(\bar{\sigma}) = \bar{\sigma},$$

completing the induction step.

Hence

$$\sigma_m = \lim_{j\to\infty} \sigma_m^{[j]} \leq \bar{\sigma}.$$

□

On the other hand, by Lemma 5.6[37] from [AL08], we have

Lemma 6.38. *Let f be a Feigenbaum map in the renormalization horseshoe $\mathcal{A}$ with* area $J(f) = 0$. *Then*

$$\omega_m \asymp \frac{\eta_m}{\xi_m\rho_m}, \quad m = 1, 2, \dots$$

with a constant depending only on the geometric bounds for f.

[37] In [AL08] the lemma is stated under the assumption that f is a renormalization periodic point, but it was not used in the proof.

Proof of Theorem 6.35. If area $J(f) = 0$, then by the last lemma,

$$\sigma_m \geq \frac{c}{\bar{\xi}} \to \infty \text{ as } \bar{\xi} \to 0,$$

contradicting Lemma 6.37.

Proof of Theorem 1.2. For any finite family of combinatorics $\{\mathcal{M}'_{N,l,\kappa,t,m,j}\}$, consider the set of infinitely renormalizable parameters with these combinatorics (arbitrarily alternating). By [Lyu99], this is a Cantor set with bounded geometry, implying that it has positive Hausdorff dimension.

Remark 6.3. To see that the Hausdorff dimension of the parameter set in question is at least 1/2, freeze all the parameters except j and let $\underline{j} \leq j \leq \bar{j}$ with a big $\bar{j}$. The Hausdorff dimension of the corresponding Cantor set of infinitely renormalizable parameters is close to the exponent δ for which

$$\sum_j \frac{1}{(\operatorname{diam} \mathcal{M}'_{N,l,\kappa,t,m,j})^\delta} = 1.$$

The parabolic bifurcation theory [DBDS00] implies

$$\operatorname{diam} \mathcal{M}'_{N,l,\kappa,t,m,j} \asymp \frac{1}{j^2},$$

and the conclusion follows.

7. Appendix: Further comments and open problems

7.1. *Probabilistically balanced maps.* There is an interesting approach to creating balanced (in some stronger sense) maps by variation of a continuous parameter. (We thank Jean-Christophe Yoccoz for this suggestion.) Consider a renormalization horseshoe associated to a pair of renormalization combinatorics, such that one of the fixed points is lean and the other is a black hole. For each $0 \leq p \leq 1$, let μ_p be the Bernoulli measure on the horseshoe giving probability p to the "Lean" combinatorics and $1 - p$ to the "Black hole" one. Then conjecturally for each p, the limit

$$c_p = \lim \frac{1}{n} \log \frac{\eta_n}{\xi_n}$$

should exist μ_p-a.e. and be independent of a particular μ_p-typical combinatorics. Moreover, the dependence $p \mapsto c_p$ is conceivably continuous, and since $c_0 < 0 < c_1$, we must have $c_{p_*} = 0$ for some $0 < p_* < 1$. (Justification of all those facts would depend on a suitable extension of the analysis of [AL08].) Let us call a μ_{p_*}-typical Feigenbaum map *probabilistically balanced.* (They are "better balanced" than generic *topologically balanced* examples constructed in [AL08].) The geometry of the probabilistically balanced Julia sets would be a good approximation to the geometry of (perhaps, non-existing) balanced Julia sets with periodic combinatorics.

Remark 7.1. Note however that the μ_p-a.e. Feigenbaum Julia set has full hyperbolic dimension for every $0 < p < 1$ (see Lemma 7.2 and Theorem 8.1 of [AL08]), and while $c_p > 0$ should imply positive area, $c_p < 0$ would not imply Hausdorff dimension less than 2.

7.2. *Computer experiments.* After identifying theoretically the main dynamical phenomena that should lead to the Black Hole behavior, we have attempted an informal numeric investigation of a particularly simple sequence of renormalization combinatorics displaying them. Consider the quadratic map p_c with a golden mean Siegel disk, with rotation number $[1, 1, 1, \dots]$, and let p_m/q_m be the sequence of rational approximants ($p_m = q_{m-1}$ being the Fibonacci sequence). Visual inspection of the (p_m/q_m)-limb reveals a pair of largest primitive Mandelbrot copies with period $q_m + q_{m-2}$. Choosing one of them, we explore in detail the parameter z_m in this copy for which the first renormalization has a golden mean Siegel disk. This parameter is very close to the actual Feigenbaum parameter with this stationary combinatorics, and considerably easier to determine numerically.

In parameter space, one sees that $\frac{z_{2m-1}-c}{z_{2m+1}-c} \to \beta = \frac{7+3\sqrt{5}}{2}$. Moreover, centering the Mandelbrot copies at the superattracting parameter and rescaling by β^m shows manifest convergence of the copies in the Hausdorff topology.

In the dynamical plane, one sees that $p_{z_{2m+1}}^{q_{2m+1}+q_{2m-1}}$ restricts to a quadratic-like map $g_{2m+1} : U_{2m+1} \to V_{2m+1}$, where V_{2m+1} is a disk of radius $\sqrt{38}|w_{2m+1}|$ and w_{2m+1} is the center of the Siegel disk for g_{2m+1}. Moreover, $\frac{w_{2m-1}}{w_{2m+1}}$ converges to some real constant greater than 1, and up to rescaling by $|w_{2m+1}|^{-1}$, g_{2m+1} is seen to converge. The proportion of $p_{z_{2m+1}}$-orbits starting in the original Siegel disk of p_c that eventually land in V_{2m+1} is clearly seen to approach 1 (so that $\eta(2m + 1)$ is bounded from below), while $\xi(2m + 1)$ appears to decay exponentially. Julia sets of positive area might already emerge then for period 2207 ($\xi \approx 0.0622$), see Figure 1.1, and more likely for period 15127 ($\xi \approx 0.0215$).

Remark 7.2. Those estimates are valid for the quadratic map and not for the renormalization fixed point, so there is still some extra distortion to consider. Heuristically (ignoring distortion), ξ should be small compared to the relative area of the filled Julia set with a Siegel disk, which near the fixed point is around 0.06.

To justify all those observations one needs the existence of a hyperbolic Siegel renormalization fixed point with the golden mean rotation number and a one-dimensional unstable manifold containing (up to straightening) the Mandelbrot copies in question. As we know, the existence of a Siegel renormalization fixed point was established by McMullen [McM98]. Its hyperbolicity was proven in [GY20] (computer assisted) and [DLS20]. However, one still needs to show that the unstable manifold is large enough to contain those particular Mandelbrot copies that we want, which looks like a hard problem. See [DL18] for an approach to it through the analysis of external rays for an associated Transcendental Dynamics leading to new Julia sets of positive area.

7.3. *More Julia sets of positive area?* Recall that the renormalization in our examples is of primitive type. Recently, a priori bounds have been proven for some Feigenbaum maps of satellite type that made it possible to apply our machinery to those cases, providing Feigenbaum Julia sets of *satellite* type with positive area [DL18].

It remains an open problem whether Julia sets of positive area may exist for real quadratic maps. Any such example would have to be infinitely renormalizable, and would imply their existence already in the class of real Feigenbaum quadratic maps with periodic combinatorics. As we have already mentioned, A. Dudko and S. Sutherland have recently proven (with a computer assistance) that the "oringinal" Feigenbaum map corresponding to the period doubling bifurcation has the Julia set of zero area [DS20]. *It makes plausable that all real quadratic Feigenbaum Julia sets are Lean.*

In the higher degree case, the situation is even less conclusive. In this case, there is even a chance of existance of a *non-renormalizable* unicritical polynomial with positive area Julia set (and even real); see an attempt to prove it by Nowicki and van Strien for the Fibonacci map of high degree [Buf97] (stemming from computer experiments designed by the second author with Scott Sutherland in the early 1990s; see [Lyu95, §7.2]).

7.4. *Physical attractors for Hénon maps.* The complex Hénon family

$$F_{c,b} : (z, w) \mapsto (z^2 + c - bw, z)$$

with a small Jacobian b can be viewed as a perturbation of the one-dimensional quadratic family $f_c : z \mapsto z^2 + c$. The real renormalization theory developed in [DCLM05] can be adapted to the complex case to show that complex Feigenbaum maps admit infinitely renormalizable Hénon perturbations. Such a Feigenbaum-Hénon map has an invariant Cantor set $\mathcal{O}_F$ on which it acts as the adding machine. In the real case, this set is a global physical attractor; i.e., it attracts almost all orbits in the phase space (which is an invariant real bidisk).

In the general complex case, the random walk scheme associated to a Feigenbaum map f_c is robust under a perturbation, implying that the *forward Julia set $J^+(F)$ (see [HOV94]) has positive Lebesgue measure. In fact, a positive measure subset of orbits in $J^+(F)$ converges to $\mathcal{O}_F$, making it a* physical attractor for the complex Feigenbaum-Hénon map F.

We will supply details of this discussion elsewhere (manuscript in preparation).

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