Covariance Steering of Discrete-Time Linear Systems with Mixed Multiplicative and Additive Noise

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Abstract— **In this paper, we study the covariance steering (CS) problem for discrete-time linear systems subject to multiplicative and additive noise. Specifically, we consider two variants of the CS problem. The goal of the first problem, which is called the exact CS problem, is to steer the mean and covariance of the state process to their desired values in finite time. In the second problem, which is called the "relaxed" CS problem, the covariance assignment constraint is relaxed into a positive semi-definite constraint. We show that the relaxed CS problem can be cast as an equivalent convex semi-definite program (SDP) after applying suitable variable transformations and constraint relaxations. Furthermore, we propose a two-step solution procedure for the exact CS problem based on the relaxed problem formulation which returns a feasible solution, if there exists one. Finally, results from numerical experiments are provided to show the efficacy of the proposed solution methods.**

I. Introduction

In this paper, we study the problem of characterizing causal feedback policies for discrete-time stochastic linear systems which steer the state mean and the state covariance to desired values. This class of problems is known as Covariance Steering (CS) problems in the relevant literature [1]–[3]. The CS methods can be used for robust trajectory optimization for uncertain systems [4] and density control of robotic swarms [5].

Typically, CS problems are addressed only for the case of additive noise. By contrast, in this work, we consider discrete-time linear systems which are excited by both additive and multiplicative noise. Throughout the paper, we will study two variations of the CS problem. In the first problem formulation, the main goal is to find a causal policy which will steer the mean and the covariance of the state process to their respective desired values in finite time. In the second problem formulation, we keep the hard constraint on the state mean but the constraint on the state covariance is "relaxed" into a positive semi-definite constraint. We will refer to the former variation of the CS problem as the "exact" CS problem and the latter as the "relaxed" CS problem.

Literature Review: Early attempts to address CS problems were focused on the infinite horizon case for linear time-invariant systems in which the set of assignable covariance matrices can be characterized in terms of linear matrix inequalities (LMI) [6], [7]. More recently, finite horizon CS problems have gained significant attention. Unconstrained CS problem formulations with continuous-time linear systems were first addressed in [2], [8] whereas the constrained CS problems for discrete-time linear systems are considered in [1], [9]. Soft constrained versions of the CS problems are studied in [10], [11]. Furthermore, CS problems for partially observable systems are studied in [12]. In all of the aforementioned papers, the system model is assumed to be linear and the noise process is assumed to be an additive white noise process. Besides finite horizon CS problems, density control problems are studied in [5], [13], [14] under simplifying assumptions.

The problem of finding stabilizing controllers for linear systems subject to multiplicative noise using LMIs has been studied in [15]. Model Predictive Control (MPC) algorithms for linear systems subject to state and control multiplicative noise have been developed in [16], [17]. Estimation and control design problems are studied in [18]. More recently, sampling-based methods for learning the optimal state feedback controllers for linear systems subject to multiplicative noise have been proposed in [19], [20].

The CS problem with continuous-time dynamics and multiplicative noise is studied in [21] where a solution based on coupled Riccati equations is obtained. However, the authors of [21] consider the case in which the system is only affected by the state multiplicative noise, and the state mean at the initial stage and its desired terminal value are both zero. In our work, we consider a more general problem with system dynamics having both state and control multiplicative noise, and nonzero initial and desired mean dynamics. To the best of our knowledge, this is the first paper that addresses the finite horizon CS problem for discrete-time linear systems excited by both state and control multiplicative noise.

Main Contributions: First, we present a formulation of the CS problem as a nonlinear program (NLP) assuming an affine state feedback policy parametrization and subsequently, we show that this NLP can be transformed into an equivalent semi-definite program by applying suitable variable transformations and semi-definite relaxations. Second, we show that SDP relaxations, which are tight in the relaxed problem, are loose in the exact CS problem. In view of these results, we propose a two-step procedure to solve the exact CS problem which is based on the solution to the relaxed problem. Third, we provide an instance of the exact CS problem in which the semi-definite relaxations used in the second step of the solution procedure are loose. Then, we show that the semi-definite relaxations in the second step of the solution procedure are tight if there is no control multiplicative noise acting on the system.

II. Problem Formulation

Notation: The space of n-dimensional real vectors and $n \times m$ matrices are denoted as $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively. The

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set of positive integers is denoted as $\mathbb{Z}_+$. The cone of $n \times n$ positive definite (semi-definite) matrices are denoted by $\mathbb{S}_n^{++}(\mathbb{S}_n^+)$. $\mathbf{0}$ denotes the zero matrix (or vector) with the appropriate dimension. We use I_n to denote the $n \times n$ identity matrix. For $A, B \in \mathbb{S}_n$, $A \succ B$ ($A \succeq B$) means $A - B \in \mathbb{S}_n^{++}$ ($A - B \in \mathbb{S}_n^+$). We use $\mathrm{tr}(\cdot)$ to denote the trace operator. $\mathrm{bdiag}(A_1, \ldots, A_N)$ denotes the block diagonal matrix whose diagonal blocks are the matrices $A_1, \ldots, A_N$. The expectation and the covariance of a random variable x are denoted as $\mathbb{E}[x]$ and $\mathrm{Cov}(x)$, respectively.

Problem Setup and Formulation: We consider discrete-time linear systems of the form:

$$x_{k+1} = \tilde{A}_k x_k + \tilde{B}_{k,\ell} u_k + w_k + d_k \tag{1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ are the state and the input processes, respectively. $\tilde{A}_k := A_k + \sum_{\ell=1}^{M} \delta_{k,\ell} \bar{A}_{k,\ell}$, $\tilde{B}_k := B_k + \sum_{\ell=1}^{M} \gamma_{k,\ell} \bar{B}_{k,\ell}$. We assume that $\mathbb{E}[x_0] = \mu_0 \in \mathbb{R}^n$ and $\mathrm{Cov}(x_0) = \Sigma_0 \in \mathbb{S}_n^{++}$ are given. $d_k \in \mathbb{R}^n$ is known for all $k \in \{0, \ldots, N-1\}$. The state and control multiplicative noise processes are represented by i.i.d. random variables $\delta_{k,\ell}, \gamma_{k,\ell}$ where $\mathbb{E}[\delta_{k,\ell}] = \mathbb{E}[\gamma_{k,\ell}] = 0$ and $\mathrm{Cov}(\delta_{k,\ell}) = \mathrm{Cov}(\gamma_{k,\ell}) = 1$. Note that this representation of the multiplicative noise process is not restrictive, i.e., any random matrix $S \in \mathbb{R}^{n \times m}$ whose entries have finite second moments can be represented in this form as shown in [22]. The additive noise $\{w_k\}_{k=0}^{N-1}$ is also an i.i.d. random process with $\mathbb{E}[w_k] = \mathbf{0}$ and $\mathrm{Cov}(w_k) = W_k \in \mathbb{S}_n^+$.

Remark 1. The only assumption that we make on the distributions of the initial state x_0 and the noise processes $w_k, \delta_{k,\ell}, \gamma_{k,\ell}$ is that their first two moments are known. Thus, the distribution of the initial state x_0, the random variables $\delta_{k,\ell}, \gamma_{k,\ell}$ and w_k can have any distribution (not necessarily Gaussian) with given covariance values.

A state feedback control policy for the system in (1) is a sequence $\pi = \{\pi_k\}_{k=0}^{N-1}$ where each $\pi_k : \mathbb{R}^n \to \mathbb{R}^m$ is a function from the state x_k to control u_k. We denote the set of admissible policies by Π. Throughout the paper, we consider a performance measure with a quadratic running cost:

$$J(U_{0:N-1}, X_{0:N}) := \sum_{k=0}^{N-1} u_k^{\mathrm{T}} R_k u_k + x_k^{\mathrm{T}} Q_k x_k \tag{2}$$

where $U_{0:N-1} = \{u_0, \ldots, u_{N-1}\}$ (input process) and $X_{0:N} = \{x_0, \ldots, x_N\}$ (state process). Next, we provide the precise formulations of the two main problems of this work.

Problem 1 (Exact Covariance Steering Problem)**.** *Let $N \in \mathbb{Z}_+$, $\{A_k, B_k, d_k, W_k, \{\bar{A}_{k,\ell}\}_{\ell=1}^M, \{\bar{B}_{k,\ell}\}_{\ell=1}^M\}_{k=0}^{N-1}$, $\mu_0, \mu_\mathrm{d} \in \mathbb{R}^n, \Sigma_0, \Sigma_\mathrm{d} \in \mathbb{S}_n^{++}$, and $\{R_k, Q_k\}_{k=0}^{N-1}$ where $R_k, Q_k \in \mathbb{S}_m^{++}, W_k \in \mathbb{S}_n^+$ be given. Then, find an admissible control policy $\pi^\star \in \Pi$ that solves the following stochastic optimal control problem:*

$$\begin{aligned}
\min_{\pi \in \Pi} \quad & \mathbb{E}[J(U_{0:N-1}, X_{0:N})] && \text{(3a)}\\
\text{s.t.} \quad & (1) && \text{(3b)}\\
& \mathbb{E}[x_N] = \mu_\mathrm{d} && \text{(3c)}\\
& \mathrm{Cov}(x_N) = \Sigma_\mathrm{d} && \text{(3d)}\\
& u_k = \pi_k(x_k) && \text{(3e)}
\end{aligned}$$

Many practical applications of stochastic optimal control problems require the terminal covariance of the state to be upper bounded by some acceptable covariance matrix in the Löwner partial order sense. Thus, we consider the 'relaxed' variation of the Problem 1 in which the terminal covariance constraint in (3d) is relaxed to the constraint in (4c).

Problem 2 (Relaxed Covariance Steering Problem)**.** *Let $N \in \mathbb{Z}_+$, $\{A_k, B_k, d_k, W_k, \{\bar{A}_{k,\ell}\}_{\ell=1}^M, \{\bar{B}_{k,\ell}\}_{\ell=1}^M\}_{k=0}^{N-1}$, $\mu_0, \mu_\mathrm{d} \in \mathbb{R}^n, \Sigma_0, \Sigma_\mathrm{d} \in \mathbb{S}_n^{++}$, and $\{R_k, Q_k\}_{k=0}^{N-1}$ where $R_k \in \mathbb{S}_m^{++}, W_k, Q_k \in \mathbb{S}_n^+$ be given. Then, find an admissible control policy $\pi^\star \in \Pi$ that solves the following stochastic optimal control problem:*

$$\begin{aligned}
\min_{\pi \in \Pi} \quad & \mathbb{E}[J(U_{0:N-1}, X_{0:N})] && \text{(4a)}\\
\text{s.t.} \quad & (1), (3c), (3e) && \text{(4b)}\\
& \mathrm{Cov}(x_N) \preceq \Sigma_\mathrm{d} && \text{(4c)}
\end{aligned}$$

Remark 2. The solution to Problem 2, besides its own practical value, will be used in the proposed solution procedure for Problem 1 (exact CS), as explained in Section III.

III. Main Results

Since the proposed solution method for Problem 1 requires the solution obtained by solving Problem 2, we first present our results on the relaxed CS problem. Both Problem 1 and Problem 2 are stochastic optimal control problems over infinite dimensional policy spaces which make them computationally intractable for most cases. However, the optimal policy for the CS problem takes the form of an affine state feedback control policy [2], [3], [23], [24]. Thus, we restrict the set of policies that we optimize over to the set of affine state feedback policies which is denoted as Π^{sf}. In particular, a policy $\pi = \{\pi_k\}_{k=0}^{N-1} \in \Pi^{sf}$ is given as

$$\pi_k(x_k) = \bar{u}_k + K_k(x_k - \mu_k), \tag{5}$$

for every $\pi_k : \mathbb{R}^n \to \mathbb{R}^m$ where $\mu_k = \mathbb{E}[x_k]$. With this formulation, the policy space Π^{sf} is parametrized by a finite number of decision variables which are $\{\bar{u}_k, K_k\}_{k=0}^{N-1}$ where $\bar{u}_k \in \mathbb{R}^M, K_k \in \mathbb{R}^{m \times n}$.

Under the policy parameterization defined in (5), the mean and the covariance dynamics of the state process x_k obey the following recursive equations:

$$\mu_{k+1} = A_k \mu_k + B_k \bar{u}_k + d_k \tag{6}$$

$$\Sigma_{k+1} = \tilde{A}_k \Sigma_k \tilde{A}_k^{\mathrm{T}} + \mathcal{F}_k(\Sigma_k, K_k, \bar{u}_k) + \mathcal{G}_k(\Sigma_k, \mu_k) \tag{7}$$

where $\Sigma_k := \mathrm{Cov}(x_k)$, $\mu_k := \mathbb{E}(x_k)$, $\tilde{A}_k := A_k + B_k K_k$, $\mathcal{F}_k(\Sigma_k, K_k, \bar{u}_k) := \sum_{\ell=0}^{M} \bar{B}_{k,\ell}(K_k \Sigma_k K_k^{\mathrm{T}} + \bar{u}_k \bar{u}_k^{\mathrm{T}}) \bar{B}_{k,\ell}^{\mathrm{T}} + W_k$, $\mathcal{G}(\Sigma_k, \mu_k) := \sum_{\ell=0}^{M} \bar{A}_{k,\ell}(\Sigma_k + \mu_k \mu_k^{\mathrm{T}}) \bar{A}_{k,\ell}^{\mathrm{T}}$. Also, we used the fact that random variables $\delta_{k,\ell}$ and $\gamma_{k,\ell}$ are i.i.d. random processes. Besides the system dynamics, we need to represent the objective function $\mathbb{E}[J(U_{0:N-1}, X_{0:N})]$ in terms of the policy parameters $\{\bar{u}_k, K_k\}_{k=0}^{N-1}$ where $J(U_{0:N-1}, X_{0:N})$ is defined in (2) to formulate both Problem 1 and Problem 2 as finite dimensional nonlinear programs. To this aim, we use the following identities:

$$\mathbb{E}\left[u_k^{\mathrm{T}} R_k u_k\right] = \mathrm{tr}(R_k \bar{u}_k \bar{u}_k^{\mathrm{T}}) + \mathrm{tr}(R_k K_k \Sigma_k K_k^{\mathrm{T}}), \tag{8a}$$

$$\mathbb{E}\left[x_k^{\mathrm{T}} Q_k x_k\right] = \mathrm{tr}(Q_k \mu_k \mu_k^{\mathrm{T}}) + \mathrm{tr}(Q_k \Sigma_k), \tag{8b}$$

whose derivation is based on the linearity of the expectation operator $\mathbb{E}[\cdot]$ and the cyclic permutation property of the trace operator $\mathrm{tr}(\cdot)$. By the summation of the equalities in (8) over all $k \in \{0, \dots, N-1\}$, we observe that

$$\begin{aligned}\mathbb{E}\left[J(U_{0:N-1}, X_{0:N})\right] = \sum_{k=0}^{N-1} \mathrm{tr}(R_k(\bar{u}_k\bar{u}_k^{\mathrm{T}} + K_k\Sigma_k K_k^{\mathrm{T}})) \\ + \mathrm{tr}(Q_k(\mu_k\mu_k^{\mathrm{T}} + \Sigma_k)) \\ =: \mathcal{J}(\{\bar{u}_k, K_k, \mu_k, \Sigma_k\}_{k=0}^{N-1}). \quad (9)\end{aligned}$$

Now that we have written both the dynamics of the mean μ_k and the covariance Σ_k of the state and the objective function $\mathbb{E}[J(U_{0:N-1}, X_{0:N})]$ in terms of the policy parameters $\{\bar{u}_k, K_k\}_{k=0}^{N-1}$, we are ready to formulate Problem 1 and 2 as finite dimensional optimization problems.

A. Relaxed Covariance Steering

A finite dimensional optimization problem over the variables $\{\mu_k, \Sigma_k, \bar{u}_k, K_k\}$ can be written as follows:

$$\begin{aligned}\min_{\substack{\bar{u}_k, K_k \\ \mu_k, \Sigma_k}} \quad & \mathcal{J}(\{\bar{u}_k, K_k, \mu_k, \Sigma_k\}_{k=0}^{N-1}) && (10a)\\ \text{s.t.} \quad & (6), (7), \mu_N = \mu_{\mathrm{d}}, \Sigma_{\mathrm{d}} \succeq \Sigma_N.\end{aligned}$$

The optimization problem in (10) is a general nonlinear program (NLP) for which there are no algorithms that can guarantee convergence or global optimality. The difficulty of the problem in (10) comes from the bilinear terms $K_k\Sigma_k$ and $K_k\Sigma_k K_k^{\mathrm{T}}$ that appear in the objective function $\mathcal{J}(\{\bar{u}_k, K_k, \mu_k, \Sigma_k\}_{k=0}^{N-1})$ and the state covariance dynamics given in (7). The mean dynamics in (6) are affine and the other terms in the objective function in (10a) are either affine or convex quadratic function of the decision variables.

To isolate the nonlinearities in the optimization problem in (10), we introduce the following decision variables:

$$\begin{aligned}L_k = K_k\Sigma_k, \quad & \mathbf{M}_k = L_k\Sigma_k^{-1}L_k^{\mathrm{T}}, \\ \mathbf{X}_k = \mu_k\mu_k^{\mathrm{T}}, \quad & \mathbf{U}_k = \bar{u}_k\bar{u}_k^{\mathrm{T}}.\end{aligned} \quad (11)$$

The objective function $\mathcal{J}(\cdot)$ in (9) can be equivalently represented in terms of the new decision variables which are defined in (11) as follows: $\hat{\mathcal{J}}(\{\mathbf{M}_k, \mathbf{U}_k, \Sigma_k, \mathbf{X}_k\}_{k=0}^{N-1}) := \sum_{k=0}^{N-1} \mathrm{tr}(R_k(\mathbf{U}_k + \mathbf{M}_k)) + \mathrm{tr}(Q_k(\Sigma_k + \mathbf{X}_k))$. By using the new decision variables, we can formulate a new optimization problem that is equivalent to (10) as follows:

$$\begin{aligned}\min_{\substack{\bar{u}_k, L_k, \Sigma_k, \\ \mathbf{M}_k, \mathbf{X}_k, \mathbf{U}_k}} \quad & \hat{\mathcal{J}}(\{\mathbf{M}_k, \mathbf{U}_k, \Sigma_k, \mathbf{X}_k\}_{k=0}^{N-1}) && (12a)\\ \text{s.t.} \quad & \mu_{k+1} = A_k\mu_k + B_k\bar{u}_k + d_k, && (12b)\\ & \Sigma_{k+1} = A_k\Sigma_k A_k^{\mathrm{T}} + A_k L_k^{\mathrm{T}} B_k^{\mathrm{T}} + B_k L_k A_k^{\mathrm{T}} \\ & \quad + B_k\mathbf{M}_k B_k^{\mathrm{T}} + W_k \\ & \quad + \tilde{\mathcal{F}}_k(\Sigma_k, \mathbf{X}_k) + \tilde{\mathcal{G}}_k(\mathbf{M}_k, \mathbf{X}_k), && (12c)\\ & \mathbf{M}_k = L_k\Sigma_k^{-1}L_k^{\mathrm{T}}, && (12d)\\ & \mathbf{X}_k = \mu_k\mu_k^{\mathrm{T}}, && (12e)\\ & \mathbf{U}_k = \bar{u}_k\bar{u}_k^{\mathrm{T}}, && (12f)\\ & \mu_N = \mu_{\mathrm{d}}, \quad \Sigma_{\mathrm{d}} \succeq \Sigma_N, && (12g)\end{aligned}$$

where $\tilde{\mathcal{F}}_k(\Sigma_k, \mathbf{X}_k) := \sum_{\ell=1}^{M} \bar{A}_{k,\ell}(\Sigma_k + \mathbf{X}_k)\bar{A}_{k,\ell}^{\mathrm{T}}$, and $\tilde{\mathcal{G}}_k(\mathbf{M}_k, \mathbf{U}_k) := \sum_{\ell=1}^{M} \bar{B}_{k,\ell}(\mathbf{M}_k + \mathbf{U}_k)\bar{B}_{k,\ell}^{\mathrm{T}}$. We replaced the bilinear terms $K_k\Sigma_k$ in the recursive equation for the propagation of the state covariance (7) with L_k. The term $K_k\Sigma_k^{-1}K_k^{\mathrm{T}}$ is rewritten as $K_k\Sigma_k\Sigma_k^{-1}\Sigma_k K_k$ then turned into $L_k\Sigma_k^{-1}L_k^{\mathrm{T}}$ which is subsequently replaced with $\mathbf{M}_k$. The terms $\mu_k\mu_k^{\mathrm{T}}$, $\bar{u}_k\bar{u}_k^{\mathrm{T}}$ are replaced with $\mathbf{X}_k$, $\mathbf{U}_k$, respectively. Note that the constraints that include the decision variables denoted with subscript k are imposed for all $k \in \{0, \dots, N-1\}$ in the rest of the optimization problems defined throughout the paper. Finally, to keep the equivalence of the problems in (10) and (12), we add the nonlinear equalities in (11) as constraints in (12d), (12e) and (12f).

After introducing the new decision variables, the problem in (10) takes the form in (12) where the objective function now is expressed as affine functions of the decision variables $\Sigma_k, \mathbf{X}_k, \mathbf{U}_k, \mathbf{M}_k$. Furthermore, the covariance dynamics constraint in (12c) is now represented as an affine constraint. To convefixy the problem, we relax the nonlinear equality constraints (12d)-(12f) as follows:

$$\mathbf{M}_k \succeq L_k\Sigma_k^{-1}L_k^{\mathrm{T}}, \quad \mathbf{X}_k \succeq \mu_k\mu_k^{\mathrm{T}}, \quad \mathbf{U}_k \succeq \bar{u}_k\bar{u}_k^{\mathrm{T}}. \quad (13)$$

In light of Schur's complement lemma [25], the relaxed nonlinear SDP constraints in (13) can be transformed into LMI constraints in (14a)-(14d). The resulting optimization problem after the SDP relaxations is given as follows:

$$\begin{aligned}\min_{\substack{\bar{u}_k, \mu_k, L_k, \Sigma_k, \\ \mathbf{M}_k, \mathbf{X}_k, \mathbf{U}_k}} \quad & \hat{\mathcal{J}}(\{\mathbf{M}_k, \mathbf{U}_k, \Sigma_k, \mathbf{X}_k\}_{k=0}^{N-1}) \\ \text{s.t.} \quad & (12b), (12c), (12g), \\ & \begin{bmatrix}\mathbf{M}_k & L_k \\ L_k & \Sigma_k\end{bmatrix} \succeq \mathbf{0}, && (14a)\\ & \begin{bmatrix}\mathbf{X}_k & \mu_k \\ \mu_k^{\mathrm{T}} & 1\end{bmatrix} \succeq \mathbf{0}, && (14b)\\ & \begin{bmatrix}\mathbf{U}_k & \bar{u}_k \\ \bar{u}_k^{\mathrm{T}} & 1\end{bmatrix} \succeq \mathbf{0}, && (14c)\\ & \Sigma_{\mathrm{d}} \succeq \Sigma_N. && (14d)\end{aligned}$$

To be able to recover the optimal state feedback policy parameters $\{\bar{u}_k, K_k\}_{k=0}^{N-1}$ from the solution of the SDP in (14) which is denoted as $(\{\bar{u}_k^\star, L_k^\star, \Sigma_k^\star, \mathbf{M}_k^\star, \mathbf{X}_k^\star, \mathbf{U}_k^\star\}_{k=0}^{N-1})$, we need the optimal parameters to satisfy the relaxed (non-strict) inequality constraints in (13) with equality. Next, we show that the optimal parameters of problem in (14) satisfy the nonlinear equality constraints (12d)-(12f).

Proposition 1. *Let* $\{\bar{u}_k^\star, L_k^\star, \Sigma_k^\star, \mathbf{M}_k^\star, \mathbf{X}_k^\star, \mathbf{U}_k^\star\}_{k=0}^{N-1}$ *be the optimal solution of Problem in* (14)*. Then, it satisfies the equalities in* (12d), (12e), (12f)*. Therefore, it is an optimal solution to Problem 2.*

Proof. Suppose for the sake of contradiction that the parameters corresponding to the optimal solution satisfy $\mathbf{M}_k - L_k\Sigma_k^{-1}L_k^{\mathrm{T}} = \mathbf{N}^m \neq \mathbf{0}$, $\mathbf{X}_k - \mu_k\mu_k^{\mathrm{T}} = \mathbf{N}^x \neq \mathbf{0}$, $\mathbf{U}_k - \bar{u}_k\bar{u}_k^{\mathrm{T}} = \mathbf{N}^u \neq \mathbf{0}$ for some $k \in \{0, \dots, N-1\}$. Now, let's define $\mathbf{M}_k' = L_k\Sigma_k^{-1}L_k^{\mathrm{T}}$, $\mathbf{U}_k' = \bar{u}_k\bar{u}_k$ and $\mathbf{X}_k' = \mu_k\mu_k^{\mathrm{T}}$. It follows readily that $\mathbf{M}_k \succeq \mathbf{M}_k'$, $\mathbf{U}_k \succeq \mathbf{U}_k'$ and $\mathbf{X}_k \succeq \mathbf{X}_k'$. Since $R_k \succ 0$, $Q_k \succ \mathbf{0}$; $\mathrm{tr}(R_k(\mathbf{M}_k + \mathbf{U}_k)) >$

$\mathrm{tr}(R_k(\mathbf{M}'_k + \mathbf{U}'_k))$. Thus, the value of the objective function is strictly lower with $\mathbf{M}'_k, \mathbf{U}'_k, \mathbf{X}'_k$. Furthermore, let Σ'_t be the value of the state covariance under $\mathbf{M}'_k, \mathbf{U}'_k, \mathbf{X}'_k$ for all $t \geq k+1$. Then, we have that $\Sigma_t \succeq \Sigma'_t$ for all $t \geq k+1$. Now, replace L_t with $L_t(\Sigma'_t)^{-1}\Sigma_t$ to ensure feasibility of constraint (14a) for all $t \geq k+1$. Since $\mathbf{M}_t \succeq L_t \Sigma_t^{-1} L_t^{\mathrm{T}} = L'_t(\Sigma'_t)^{-1}\Sigma_t\Sigma_t^{-1}\Sigma_t(\Sigma'_t)^{-1}L_t'^{\mathrm{T}}$. Since $\Sigma_t \succeq \Sigma'_t$ implies that $(\Sigma'_t)^{-1}\Sigma_t(\Sigma'_t)^{-1} \succeq (\Sigma'_t)^{-1}$; we have $\mathbf{M}_t \succeq L'_t(\Sigma'_t)^{-1}L'_t$; thus the constraint (14a) is satisfied. Combining both results, we conclude that if the inequalities in (13) are not strict, one can pick new values for $\mathbf{M}_k, L_k$ which decrease the value of the objective function without violating the constraints which contradicts the optimality assumption of $\mathbf{M}_k, L_k$. This completes the proof. ■

B. Exact Covariance Steering

For the upper bound constraint on the covariance given in (4c), the CS problem 1 can be relaxed into problem in (14) without changing the nature of the problem according to Proposition 1. However, the SDP relaxations in (13) for the constraints in (11) for $\mathbf{X}_k, \mathbf{U}_k$ do not hold with equality in the exact covariance steering problem (Problem 1).

In our numerical experiments, we observed that the loose constraints were the ones with $\mathbf{X}_k, \mathbf{U}_k$ in the optimal solution. Furthermore, one can show that if the feed-forward control inputs ($\{\bar{u}_k\}_{k=0}^{N-1}$) are fixed, then the state mean μ_k is also fixed through (6) thus $\mathbf{X}_k, \mathbf{U}_k$ can be set to their respected values for fixed $\bar{u}_k$.

Now, suppose that the relaxed CS problem is feasible and let $\{\bar{u}_k^\star, K_k^\star\}_{k=0}^{N-1}$, $\{\mu_k^\star, \Sigma_k^\star\}_{k=0}^{N}$ be the policy parameters and the state statistics that is found by solving problem in (12), respectively. Then, we have that $\mu_N = \mu_\mathrm{d}$, and the terminal covariance constraint $\Sigma_\mathrm{d} \succeq \Sigma_N$ is also satisfied.

After $\bar{u}_k, \mu_k$ are fixed based on the values obtained from solving (14), the decision variables $\{\bar{u}_k, \mu_k, \mathbf{X}_k, \mathbf{U}_k\}_{k=0}^{N-1}$ become problem parameters for the exact CS problem. Thus, we formulate another optimization problem with $L_k, \Sigma_k, \mathbf{M}_k$ as the decision variables as follows:

$$\min_{L_k, \Sigma_k, \mathbf{M}_k} \quad \tilde{\mathcal{J}}(\{\mathbf{M}_k, \Sigma_k\}_{k=0}^{N-1}) \tag{15a}$$

$$\begin{aligned}\text{s.t.} \quad \Sigma_{k+1} &= A_k \Sigma_k A_k^{\mathrm{T}} + A_k L_k^{\mathrm{T}} B_k^{\mathrm{T}} + B_k L_k A_k^{\mathrm{T}} \\ &\quad + B_k \mathbf{M}_k B_k^{\mathrm{T}} + \mathbf{H}_k + \mathcal{H}_k(\Sigma_k, \mathbf{M}_k)\end{aligned} \tag{15b}$$

$$\begin{bmatrix} \mathbf{M}_k & L_k \\ L_k^{\mathrm{T}} & \Sigma_k \end{bmatrix} \succeq \mathbf{0} \tag{15c}$$

$$\Sigma_N = \Sigma_\mathrm{d} \tag{15d}$$

where $\tilde{\mathcal{J}}(\{\mathbf{M}_k, \Sigma_k\}_{k=0}^{N-1}) := \sum_{k=0}^{N-1} \mathrm{tr}(R_k \mathbf{M}_k + Q_k \Sigma_k)$, $\mathbf{H}_k = W_k + \sum_{\ell=1}^{M}(\bar{A}_{k,\ell}\mathbf{X}_k^\star \bar{A}_{k,\ell}^{\mathrm{T}} + \bar{B}_{k,\ell}\mathbf{U}_{k,\ell}^\star \bar{B}_{k,\ell}^{\mathrm{T}})$, $\mathbf{U}_k^\star = \bar{u}_k^\star \bar{u}_k^{\star\mathrm{T}}$, $\mathbf{X}_k^\star = \mu_k^\star \mu_k^{\star\mathrm{T}}$, $\mathcal{H}_k(\Sigma_k, \mathbf{M}_k) := \sum_{\ell=1}^{M}(\bar{A}_{k,\ell}\Sigma_k \bar{A}_{k,\ell}^{\mathrm{T}} + B_{k,\ell}\mathbf{M}_k B_{k,\ell}^{\mathrm{T}})$.

To recover the optimal state feedback control parameters $\{K_k\}_{k=0}^{N-1}$ from the identity $K_k = L_k \Sigma_k^{-1}$, we need the optimal solution of the problem in (15) to satisfy the equality $\mathbf{M}_k = L_k \Sigma_k^{-1} L_k^{\mathrm{T}}$, otherwise the recovered policy will not satisfy the terminal covariance constraint. Although we observed that the LMI constraint in (15c) holds with equality in our numerical experiments in Section IV, this may not always be the case. The next problem instance is one example of such cases where the LMI constraint in (15c) is loose.

Example 1. Let parameters of the example problem instance be given as: $N = 1$, $A_0 = \left[\begin{smallmatrix} 1.04 & -0.22 \\ -0.07 & 1.341 \end{smallmatrix}\right]$, $B_k = \left[\begin{smallmatrix} -0.5 \\ -0.38 \end{smallmatrix}\right]$, $A_{0,1} = \left[\begin{smallmatrix} -0.16 & -0.2 \\ -0.14 & 0.24 \end{smallmatrix}\right]$, $\bar{B}_{0,1} = \left[\begin{smallmatrix} 0.26 \\ -0.16 \end{smallmatrix}\right]$, $d_k = \mathbf{0}$, $W_0 = \mathbf{0}$, $\mu_0 = \mu_\mathrm{d} = \mathbf{0}$, $R_0 = 10.0$, $Q_0 = 0.1 I_2$, $\Sigma_0 = I_2$, $\Sigma_d = \left[\begin{smallmatrix} 1.26 & -0.36 \\ -0.36 & 1.91 \end{smallmatrix}\right]$. The terminal mean constraint in (3c) dictates that $A_0\mu_0 + B_0\bar{u}_0 + d_0 = \mu_1 = \mu_\mathrm{d} = \mathbf{0}$. Since $\mu_0 = \mu_\mathrm{d} = d_0 = \mathbf{0}$ then it follows that $B_0\bar{u}_0 = \mathbf{0}$ which implies that $\bar{u}_0 = 0$ assuming that B_0 is full-rank. Now that $\bar{u}$ is fixed to 0 the optimal solution for Problem 2 for this given instance can be obtained by solving the SDP in (15). By solving the aforementioned SDP using MOSEK [26], we obtain the following optimal values for decision variables: $\mathbf{M}_0 = 0.149$, $L_0 = [-0.0181, -0.008]$, which yields, $\mathbf{M}_0 - L_0\Sigma_0^{-1}L_0^{\mathrm{T}} = 0.148 \neq 0$ which shows that for the given problem instance, the constraint in (15c) is loose.

Although the solution of the problem instance in Example 1 does not correspond to an affine state feedback policy since $\mathbf{M}_k = L_k \Sigma_k^{-1} L_k^{\mathrm{T}}$ is not satisfied, the mean and the covariance of the state and the control processes which can be found by solving (15) can still be realized by considering randomized affine state feedback policies as in [11].

The set of randomized affine state feedback policies is denoted by Π^{rsf}. Every $\pi \in \Pi^{rsf}$ is a sequence $\pi = \{\pi_k\}_{k=0}^{N-1}$ where each π_k is given by:

$$\pi_k(x_k) = \bar{u}_k + K_k(x_k - \mu_k) + v_k \tag{17}$$

where $v_k \in \mathbb{R}^m$ is a random variable with $\mathbb{E}[v_k] = \mathbf{0}$, $\mathrm{Cov}(v_k) = P_k \in \mathbb{S}_n^+$ and each v_k satisfies that $\mathbb{E}\left[v_k x_\ell^{\mathrm{T}}\right] = \mathbf{0}$ for all $\ell \leq k$, $\mathbb{E}[v_k \delta_{n,\ell}] = \mathbb{E}[v_k \gamma_{n,\ell}] = \mathbf{0}$, $\mathbb{E}\left[v_k w_\ell^{\mathrm{T}}\right] = \mathbf{0}$ for all n, ℓ. Thus, the randomized affine state feedback policies are parametrized by the decision variables $\{\bar{u}_k, K_k, P_k\}_{k=0}^{N-1}$. Now, setting the parameters of the randomized policy to $K_k = L_k \Sigma_k^{-1}$ and $P_k = \mathbf{M}_k - L_k \Sigma_k^{-1} L_k^{\mathrm{T}}$, the randomized affine state feedback policy induces a state process and a control process whose first and second moments are equal to the ones found by solving (15).

Despite the fact that deterministic affine state feedback policies are sufficient for CS problems for systems excited by additive noise [2], [3], [11], Example 1 shows that the optimal policy for Problem 1 may require randomized policies for systems excited by multiplicative noise. If we consider the special case of Problem 1 where the multiplicative noise only acts through the state, which means that $\bar{B}_{k,\ell} = \mathbf{0}$ for all k, ℓ, then, we can show that the LMI constraint in (15c) holds with equality. Proposition 2 formally states that claim.

Proposition 2. *Assuming that the problem in* (15) *is feasible,* $\bar{B}_{k,\ell} = \mathbf{0}$ *and* A_k^{-1} *exists for all* k, ℓ*, then the optimal values of decision variables* $\{L_k^\star, \Sigma_k^\star, \mathbf{M}_k^\star\}_{k=0}^{N-1}$ *satisfy* $\mathbf{M}_k^\star = L_k^\star \Sigma_k^{\star -1} L_k^{\star\mathrm{T}}$.

Proof. Let $\mathbf{M}_t^\star - L_t^\star \Sigma_t^{\star -1} L_t^{\mathrm{T}} \neq \mathbf{0}$ for some $t \in \{0, \ldots, N-1\}$ for the sake of contradiction and consider the SDP:

$$\min_{\substack{L \in \mathbb{R}^{m \times n} \\ \mathbf{M} \in \mathbb{S}_m^+}} \quad \mathrm{tr}(R_t \mathbf{M}) \tag{18a}$$

$$\text{s.t.} \quad \Sigma_{t+1}^\star = \mathcal{R}(L) + B_t \mathbf{M} B_t^{\mathrm{T}} + \mathbf{H}_t \tag{18b}$$

$$\begin{bmatrix} \mathbf{M} & L \\ L^{\mathrm{T}} & \Sigma_t^\star \end{bmatrix} \succeq \mathbf{0} \tag{18c}$$

where $\mathbf{H}_t = W_t + A_t\Sigma_t^\star A_t^{\mathrm{T}} + \sum_{\ell=1}^{M} \bar{A}_{t,\ell}(\Sigma_t^\star + \mathbf{X}_t)\bar{A}_{t,\ell}^{\mathrm{T}}$, $\mathcal{R}(L) := A_t L^{\mathrm{T}} B_t^{\mathrm{T}} + B_t L A_t^{\mathrm{T}}$. The SDP in (18) represents the covariance evolution from time step t to $t+1$ but covariances are fixed. So, the objective is to find policy parameters $\mathbf{M}, L$ to steer the covariance from $\Sigma_t^\star$ to $\Sigma_{t+1}^\star$. We establish the contradiction by showing that the values of $\mathbf{M}, L$ that optimize problem in (18) have to satisfy $\mathbf{M} - L\Sigma_t^{-1}L^{\mathrm{T}} = 0$. Multiplying both sides of (18b) by A_t^{-1} from left and $A_t^{-\mathrm{T}}$ from the right, we obtain:

$$\min_{\substack{L\in\mathbb{R}^{m\times n}\\ \mathbf{M}\in\mathbb{S}_m^+}} \quad \mathrm{tr}(R_n\mathbf{M}) \tag{19a}$$

$$\text{s.t.} \quad Z = L^{\mathrm{T}}Y^{\mathrm{T}} + YL + Y\mathbf{M}Y^{\mathrm{T}} \tag{19b}$$

(18b)

where $Z = A_t^{-1}(\Sigma_{t+1}^\star - \mathbf{H}_t)A_t^{-\mathrm{T}}$, $Y = A_t^{-1}B_t$. It is shown in [11, Theorem 3] that the SDP in (19) admits a solution that satisfies $M - L\Sigma_t^{-1}L^{\mathrm{T}} = \mathbf{0}$ if $R_t \succ \mathbf{0}$ which contradicts with our initial assumption. This completes the proof. ∎

Remark 3. Note that, the assumption that A_k is non-singular is not restrictive. This is because in practice, A_k is computed as the state transition matrix between discrete time steps of a continuous-time linear dynamical system [27].

Remark 4. The results that we obtained in Proposition 2 coincide with the result in [21] where the authors show that the optimal policy corresponds to a deterministic state feedback policy under the state multiplicative noise for a continuous-time linear system.

Note that the condition in Proposition 2 is not a necessary condition for the optimality of deterministic policies. It is a sufficient condition, thus even if the condition in Proposition 2 is not satisfied i.e. $\bar{B}_{k,\ell} \neq \mathbf{0}$ for some k, ℓ, the SDP constraint $\mathbf{M}_k \succeq L_k\Sigma_k^{-1}L_k^{\mathrm{T}}$ can be tight for all k in our numerical experiments which is presented in Section IV. Establishing a necessary condition for the tightness of the SDP relaxations will be left for future work.

IV. Numerical Experiments

All numerical experiments in this section run on a Mac M1 with 8GB of RAM. We used the CVXPY [28] package to parse the SDPs and used MOSEK [26] as the SDP solver. Specifically, we consider a UAV path planning problem. The UAV is modeled as a point mass with double integrator dynamics (which is a standard assumption in the relevant literature [4], [29]). The state and the control input are defined as $x_k = [p_k^x, p_k^y, v_k^x, v_k^y]^{\mathrm{T}} \in \mathbb{R}^4$ and $u_k = [a_k^x, a_k^y]^{\mathrm{T}}$ respectively, where p, v, a denote the position, velocity, and acceleration of the UAV, respectively.

The dynamics matrices of the UAV are given as: $A_k = \left[\begin{smallmatrix} I_2 & \Delta t I_2 \\ \mathbf{0} & I_2 \end{smallmatrix}\right]$, $B_k = \left[\begin{smallmatrix} \Delta t^2/2 \\ \Delta t \end{smallmatrix}\right]$, $W_k = \left[\begin{smallmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0.01 I_2 \end{smallmatrix}\right]$, $\sqrt{\Delta t} = 0.1$. The number of multiplicative noise processes is given as $M = 2$ for all k. Therefore, $\bar{A}_{k,1} = \mathrm{bdiag}(\mathbf{0}, A_{b,1})$, $\bar{A}_{k,2} = \mathrm{bdiag}(\mathbf{0}, A_{b,2})$, $\bar{B}_{k,1} = [\mathbf{0}^{\mathrm{T}}, B_{b,1}^{\mathrm{T}}]^{\mathrm{T}}$, $\bar{B}_{k,2} = [\mathbf{0}^{\mathrm{T}}, B_{b,2}^{\mathrm{T}}]^{\mathrm{T}}$ where $A_{k,1} = \beta_1\sqrt{\Delta t}\left[\begin{smallmatrix} 1.0 & 0 \\ 0.5 & 0 \end{smallmatrix}\right]$, $A_{k,2} = \beta_2\sqrt{\Delta t}\left[\begin{smallmatrix} 0 & 0.5 \\ 0 & 1.0 \end{smallmatrix}\right]$, $B_{k,1} = \theta_1\sqrt{\Delta t}\left[\begin{smallmatrix} 1.0 & 0 \\ 0.5 & 0 \end{smallmatrix}\right]$, $B_{k,2} = \theta_2\sqrt{\Delta t}\left[\begin{smallmatrix} 0 & 0.5 \\ 0 & 1 \end{smallmatrix}\right]$. $[\beta_1, \beta_2, \theta_1, \theta_2] = [0.1, 0.3, 0.1, 0.6]$ are the noise intensity parameters.

It is worth mentioning that the assumption of state and control multiplicative noise for UAV path planning tasks is more relevant than the assumption of additive noise since it is harder to follow the reference trajectory that describes an aggressive, jerky maneuver for the low-level controllers.

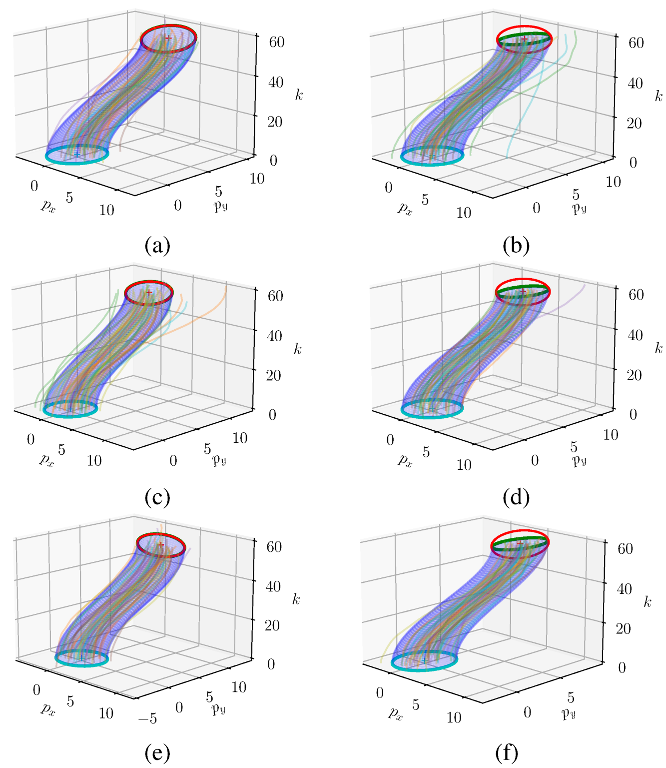

Fig. 1: Evolution of the state statistics and sample trajectories under different noise distributions. The figures on the left (Figure 1a, 1c 1e) show the results of the exact CS problem whereas the ones on the right (Figures 1b, 1d, 1f) correspond to the relaxed CS problem. Cyan, red and green ellipsoids correspond to the 2-σ confidence ellipsoids of initial, desired, and terminal covariances, respectively. The multiplicative noise terms $\delta_{k,\ell}, \gamma_{k,\ell}$ have uniform distribution over $[-\sqrt{3}, \sqrt{3}]$ in Figures 1a, 1b, unit normal distribution in Figures 1c, 1d and uniform distribution over $\{-\sqrt{1.5}, 0, \sqrt{1.5}\}$ in Figures 1e, 1f.

The initial state distribution of the UAV is a multivariate Gaussian with zero mean and covariance matrix $\Sigma_0 = \mathrm{bdiag}(2.0 I_2, 0.01 I_2)$ whereas the desired mean and covariance matrix are given as $\mu_{\mathrm{d}} = [7.0, 5.0, 0.0, 0.0]^{\mathrm{T}}$, $\Sigma_{\mathrm{d}} = \mathrm{bdiag}(\left[\begin{smallmatrix} 4.5 & -3.0 \\ -3.0 & 4.5 \end{smallmatrix}\right], 0.1 I_2)$. Finally, the problem horizon is given as $N = 60$. It can be seen that the number of decision variables are linearly proportional with N and the computational complexity of solving SDPs are $\mathcal{O}(N^3)$ therefore the computational complexity of solving both Problem 1 and Problem 2 are $\mathcal{O}(N^3)$.

In Figure 1, the evolution of the state mean and covariance together with sample trajectories of the UAV dynamics under the control policies obtained by solving both the exact and the relaxed CS problems are presented. In Figure 2, we illustrate the initial and terminal covariance matrices along with samples from the initial and terminal distributions.

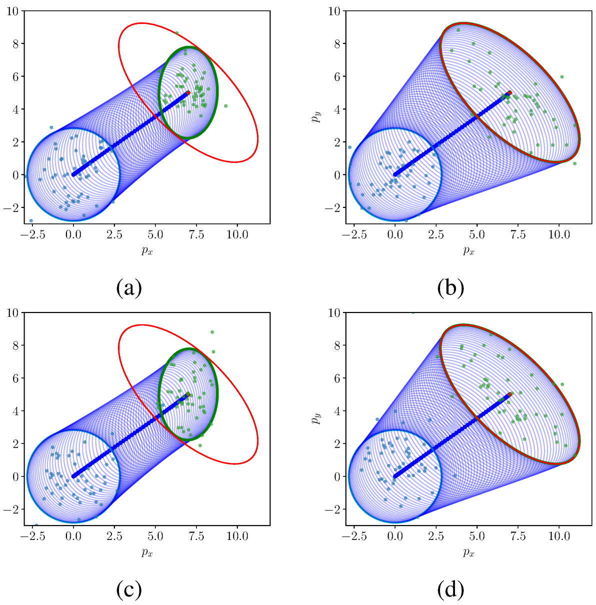

Fig. 2: Evolution of state statistics along with samples from initial and terminal distributions. Samples from the initial and the terminal distributions are illustrated by blue and green dots, respectively.

Specifically, Figure 2a and Figure 2b illustrate the evolution of the state covariance for the relaxed and exact CS problem where the multiplicative noise terms have discrete distribution. Figure 2c and Figure 2d illustrate the same for uniformly distributed multiplicative noise terms. Note that the state of the UAV is modeled as a 2-dimensional (vector) double integrator (4 states). However, we only show the distribution of the position in the $x - y$ plane. In both Figure 1 and 2, we sample 80 trajectories. It can be seen that terminal covariance constraints are satisfied for different multiplicative noise distributions.

V. Conclusion

In this paper, we have addressed the exact and relaxed versions of the CS problem for discrete-time linear systems subject to mixed additive and multiplicative noise. We first recast the relaxed CS problem as a convex SDP. Then, we proposed a two-step solution method which leverages the solution to the relaxed CS problem to solve the exact CS problem. Finally, we gave an example that shows the necessity of randomized policies for the exact CS problem and provided a condition that guarantees the set of deterministic policies is sufficiently rich to address the latter problem. We also demonstrated in our numerical simulations, however, that the optimal policy may turn out to be deterministic, even when the provided condition is violated.

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