



Unchaining surgery and topology of symplectic 4-manifolds

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Abstract

We study a symplectic surgery operation we call *unchaining*, which effectively reduces the second Betti number and the symplectic Kodaira dimension at the same time. Using unchaining, we give novel constructions of symplectic Calabi–Yau surfaces from complex surfaces of general type and completely resolve a conjecture of Stipsicz on the existence of exceptional sections in Lefschetz fibrations. Combining the unchaining surgery with others, which all correspond to certain monodromy substitutions for Lefschetz pencils, we provide further applications, such as new constructions of exotic symplectic 4-manifolds, and inequivalent pencils of the same genera and the same number of base points on families of symplectic 4-manifolds. Meanwhile, we present a handy criterion for determining from the monodromy of a pencil whether its total space is spin or not.

Keywords Lefschetz pencil · Monodromy substitution · Symplectic Calabi–Yau surface · Chain relation

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1 Introduction

Over the past couple of decades, many new surgeries, such as rational blowdown, generalized fiber sum, knot surgery, and Luttinger surgery, have been introduced to and successfully employed in the study of symplectic 4-manifolds. These symplectic surgeries have typically emerged as more flexible versions of complex algebraic operations or more rigid versions of topological ones. A diverse family of symplectic surgeries, many of which seem to have very little history in the complex algebraic or topological worlds, come from excising a compact Stein subdomain and replacing it with a new one, which too induces the same contact structure on its convex boundary. In this article we undertake an extensive study of a symplectic surgery of this kind, which we call *unchaining surgery*, and demonstrate several interesting features and applications of this surgery.

Since Lefschetz pencils and allowable Lefschetz fibrations are topological counterparts of closed symplectic 4-manifolds and compact Stein domains [1, 15, 28, 40], these operations have natural interpretations in the framework of positive factorizations of Dehn twists in the mapping class group of fibers. Swapping Stein subdomains locally correspond to swapping positive factorizations of the monodromy of a supporting open book of the boundary contact 3-manifold. Moreover, whenever the Stein subdomain, as a positive allowable Lefschetz fibration, embeds into a Lefschetz pencil on the closed symplectic 4-manifold, swapping the Stein subdomains has a global interpretation: it corresponds to a monodromy substitution in the global monodromy of the pencil. The unchaining surgery takes its name in this context from an important relation in the mapping class group of a surface: the *chain relation*, which exchanges a product of Dehn twists along a chain of an odd number of curves on a compact (sub)surface, with a pair of Dehn twists along the two boundary components of the tubular neighborhood of this chain; see Lemma 2.2. (And the analogous surgery corresponding to the chain relation for an even number of chains shares very similar features; see Remark 3.6.)

In all the examples we produce in this article, we will perform unchaining from the vantage point of substitutions in positive factorizations, and aim to preserve the global Lefschetz fibration structures, so we can determine the *Kodaira dimension* of the resulting symplectic 4-manifolds—which is a notion that measures the positivity of the symplectic canonical class, extended from the case of compact complex surfaces, and is a diffeomorphism invariant [37]. We will read off the symplectic Kodaira dimension using the additional data we derive on their exceptional multisections (symplectic (-1) -spheres which intersect all the fibers positively); see Theorem 2.3 and [11, 49]. Precise definitions and background results for all of the above are given in Sects. 2 and 3.

Our first application concerns the topology of *symplectic Calabi–Yau* surfaces, which constitute symplectic 4-manifolds of Kodaira dimension zero, up to finite covers. Recall that a symplectic Calabi–Yau surface (SCY) is a symplectic 4-manifold with a trivial canonical class, similar to a complex Calabi–Yau surface. Up to date, the only known examples of symplectic Calabi–Yau surfaces are torus bundles over tori and complex K3 surfaces [41], and any symplectic Calabi–Yau surface is known to necessarily have the rational homology type of these manifolds [6, 38]. While Simon Donaldson proposes in [16] that an analysis of monodromies of pencils on SCYs may shed light on their geometry and topology, much less is known about Lefschetz pencils on SCYs. In Sect. 4 we construct, for each $g \geq 3$, a variety of new positive factorizations yielding genus- g Lefschetz pencils on symplectic 4-manifolds with any Kodaira dimension, via careful applications of unchaining, starting with holomorphic pencils on complex surfaces of general type; see Theorems 4.6 and 4.8. In particular we obtain:

Theorem 1.1 *For each $g \geq 3$, there is a symplectic genus- g Lefschetz pencil (K_g, k_g) , where K_g is a symplectic Calabi–Yau homotopy K3 surface, and k_g has the monodromy factorization:*

$$t_{\delta_{g-1}} \cdots t_{\delta_2} t_{\delta_1} t_{\delta'_{g-1}} \cdots t_{\delta'_2} t_{\delta'_1} \\ = \begin{cases} t_{x_g} \cdots t_{x_1} t_{x'_g} \cdots t_{x'_1} (t_{c_1} t_{c_2} t_{c_3})^8 t_{d_4} \cdots t_{d_{2g+1}} t_{e_{2g+1}} \cdots t_{e_4} & (g : \text{odd}) \\ t_{x_g} \cdots t_{x_1} t_{x'_g} \cdots t_{x'_1} (t_{c_1} t_{c_2} t_{c_3})^6 (t_{c_3} t_{c_2} t_{c_1})^2 t_{d_4} \cdots t_{d_{2g+1}} t_{e_{2g+1}} \cdots t_{e_4} & (g : \text{even}) \end{cases}$$

in Γ_g^{2g-2} , where the curves $\delta_j, \delta'_j, c_j, d_j, e_j, x_j, x'_j$ are as in Figs. 4, 5, and 6.

These complement the explicit monodromies of pencils on symplectic Calabi–Yau surfaces with $b_1 > 0$ given in the works of Ivan Smith, the first author, and Noriyuki Hamada and the second author in [10, 30, 50].

It is an alluring open question whether this fairly short list of symplectic Calabi–Yau surfaces is complete up to diffeomorphisms, and in particular if there is no symplectic Calabi–Yau surfaces with $b_1 = 0$, that is not diffeomorphic to a K3 surface [17, 25, 37]. As Tian-Jun Li points out in [39], the poor state of this problem seems to also stem from the lack of any new constructions of symplectic Calabi–Yau surfaces; many widely used symplectic surgeries mentioned in the beginning of our article are seen not to yield any new SCYs, except for trivial cases [14, 32, 39, 55]. These constructions demonstrate how unchaining surgery can produce symplectic Calabi–Yau surfaces (Corollary 4.10) from complex surfaces of general type, or from rational or ruled surfaces via the natural inverse of the operation. The symplectic 4-manifolds K_g have the same fundamental group and Seiberg–Witten invariants as complex K3 surfaces, and we do not know at this point whether they are all diffeomorphic to them.

Our second application concerns a riveting conjecture of Andras Stipsicz on the existence of exceptional sections in *fiber-sum indecomposable* Lefschetz fibrations. In [51], having proved the converse, Stipsicz conjectured that any Lefschetz fibration, which cannot be expressed as a fiber-sum of any two non-trivial fibrations, always admits an exceptional section—an affirmative answer to which would mean that any Lefschetz fibration is a fiber-sum of blown-up pencils. This conjecture was shown to fail in genus-2 by just a handful of examples: a holomorphic fibration on a blown-up K3 surface by Auroux (which was the first counter-example, as observed by Yoshihisa Sato in [48]), a symplectic fibration on a homotopy Enriques surface by the first two authors of this article [11], a holomorphic fibration with 6 irreducible and 7 reducible fibers by Xiao [57] (as observed to be a counter-example by the first author), and one more (if different than Xiao’s), by an implicit argument in [3]. Only two more counter-examples, a pair of genus-3 fibrations on symplectic Calabi–Yau homotopy K3 surfaces were provided again in [11], and there has been no known examples for any $g \geq 4$, up to date.

In Sect. 5, we prove that there is in fact no stable range for g , where this conjecture may hold:

Theorem 1.2 *For any $g \geq 2$, there exists a genus- g fiber-sum indecomposable Lefschetz fibration without any exceptional sections.*

The most challenging part of constructing such examples is to locate all the exceptional spheres in the symplectic 4-manifold so as to know for sure that there is none that can be a section. We will produce all our genus $g \geq 4$ examples by applying the braiding lantern substitution of [11] to the explicit positive factorizations of a family of pencils with Kodaira

dimension one we obtain via unchaining (Theorem 5.4). Along with the already existing examples of $g = 2, 3$, this then completely resolves Stipsicz's conjecture (noting that by the well-known classification of genus-1 fibrations, the conjecture does hold in the remaining case).

In the course of the proof of the above theorem, we establish another result, which would be of independent interest: a complete characterization of when the total space of a Lefschetz pencil is spin, in terms of its monodromy factorization (Theorem 5.1). This extends Stipsicz's earlier work in [52] for Lefschetz fibrations (no base points), and this generalization is applicable to any symplectic 4-manifold, since a given symplectic 4-manifold may not admit a Lefschetz fibration, but it always admits a pencil (with base points) by Donaldson.

Theorem 1.3 *Let (X, f) be a genus- g Lefschetz pencil with a monodromy factorization $t_{c_1} \cdots t_{c_n} = t_{\delta_1} \cdots t_{\delta_p}$. Then X admits a spin structure if and only if there exists a quadratic form $q : H_1(\Sigma_g^p; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ with respect to the intersection pairing of $H_1(\Sigma_g^p; \mathbb{Z}_2)$ such that $q(c_i) = 1$ for any i and $q(\delta_j) = 1$ for some j .*

In the final section, Sect. 6, we will present two more applications, one regarding the topology of symplectic 4-manifolds (Theorem 6.1, and one regarding that of pencils (Theorem 6.4). All the examples we construct therein come from combining unchaining and rational blowdown surgeries that respect the fibration structures.

Constructions of small simply-connected symplectic 4-manifolds with $b_2^+ \leq 3$ via rational blowdowns has a fairly long and rich history, pioneered by the works of Fintushel–Stern and Jongil Park [23, 24, 44, 45]. Similar constructions via monodromy substitutions in positive factorizations of Lefschetz pencils was first given by Endo and Gurtas [18], who observed that lantern substitutions (see Lemma 2.1) amount to a rational blowdown of a symplectic (-4) -sphere—which since then, has been extended to many other substitutions corresponding to blowdowns of more general configurations of spheres [19, 26, 34]. The hardship of the latter approach is to have explicit positive factorizations of pencils that contain a sufficient number of lantern configurations for rational blowdowns, and was so far successfully applied to genus-2 Lefschetz pencils in [4, 18]. We demonstrate that, through unchaining, we do get such useful monodromy factorizations, from which, we can for example obtain new symplectic genus-3 Lefschetz fibrations with exotic total spaces:

Theorem 1.4 *There are genus-3 Lefschetz fibrations (X_j, f_j) , for $j = 0, 1, 2, 3$, where each X_j is a minimal symplectic 4-manifold homeomorphic but not diffeomorphic to $3\mathbb{CP}^2 \# (19 - j)\overline{\mathbb{CP}^2}$, and each f_{j+1} has a monodromy factorization obtained from that of f_j by a lantern substitution.*

Moreover, by the techniques of [11], one can explicitly describe the symplectic canonical class of each X_j as a multisection of f_j ; see Remark 6.3.

Our final application is on the diversity of Lefschetz pencils and fibrations of the same genera on a given symplectic 4-manifold (up to equivalence of pencils/fibrations through self-diffeomorphisms of the 4-manifold and the base, commuting the maps). Examples of inequivalent Lefschetz fibrations on a blow-up of $T^2 \times \Sigma_2$, whose fibers have different divisibility in homology were discussed by Ivan Smith in his thesis, and several inequivalent fibrations on homotopy elliptic surfaces, distinguished by their monodromy groups, were produced by Jongil Park and Ki-Heon Yun in [46, 47]. In [8, 9], building on Donaldson's existence result and the doubling construction for pencils, the first author established that in fact any symplectic 4-manifold, possibly after blow-ups, admits inequivalent Lefschetz pencils and fibrations of arbitrarily high genera. Also see [11, 29] for inequivalent pencils

on homotopy K3 surfaces and on ruled surfaces, respectively. Here we produce many more examples of inequivalent Lefschetz pencils and fibrations, notably with explicit positive factorizations:

Theorem 1.5 *For any $g \geq 3$ and $i = 0, 1, 2, \dots, g-1$, there are pairs of inequivalent relatively minimal genus- g Lefschetz pencils $(Y_g(i), h_g^j(i))$, $j = 1, 2$ with the same number of base points, and inequivalent Lefschetz fibrations on their blow-ups. For any $g \geq 3$, there are pairs of inequivalent relatively minimal genus- g Lefschetz pencils on once blown-up elliptic surface $E(1) \# \overline{\mathbb{CP}^2} \cong \mathbb{CP}^2 \# 10 \overline{\mathbb{CP}^2}$, with different number of reducible fibers.*

The examples of inequivalent pencils we obtain here are on a family of symplectic 4-manifolds, whose symplectic Kodaira dimensions run through $-\infty, 0$ and 1 .

2 Preliminaries

Here we review the main notions and background results used throughout the paper. Manifolds in this paper are assumed to be smooth, connected and oriented, unless otherwise stated.

2.1 Lefschetz fibrations and pencils

Let X and Σ be compact manifolds of dimension 4 and 2, respectively. A smooth map $f: X \rightarrow \Sigma$ is called a *Lefschetz fibration* if the critical locus $C = \text{Crit}(f)$ is a discrete set such that

- for any $x \in \text{Crit}(f)$ there are complex charts (U, φ) at x and (V, ψ) at $f(x)$, compatible with the orientation of X and Σ , so that

$$\psi \circ f \circ \varphi^{-1}(z, w) = zw,$$

- the restriction $f|_{f^{-1}(f(C))}$ is a surface bundle, and
- the restriction $f|_{\text{Crit}(f)}$ is injective.

For $B \subset X$ a non-empty finite set of points, a smooth map $f: X \setminus B \rightarrow \mathbb{CP}^1$ is called a *Lefschetz pencil* if $f|_{X \setminus \nu B}$ is a Lefschetz fibration, where νB is the union of balls centered at points in B , and for any $b \in B$ there exist a complex chart (U, φ) compatible with the orientation of X , and an orientation-preserving self-diffeomorphism $\Phi: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ so that:

$$\Phi \circ f \circ \varphi^{-1}(z, w) = [z : w].$$

Each point in B is called a *base point* of f .

We will denote a Lefschetz fibration or a pencil simply as the pair (X, f) . For a Lefschetz fibration or pencil (X, f) , the set of critical values $f(C) \subset \Sigma$ is discrete. So the genus of the closure of a regular fiber $\overline{f^{-1}(q)}$ does not depend on the regular value q , and is called the *genus* of (X, f) . In this paper, we will always assume that (X, f) is *relatively minimal*, that is, for any point $q \in f(X)$, the closure $\overline{f^{-1}(q)}$ does not contain any (-1) -spheres.

Lastly, an *allowable Lefschetz fibration* is a Lefschetz fibration with base $\Sigma = D^2$, where the fibers have non-empty boundaries, and for any point $q \in f(X)$, $f^{-1}(q)$ does not contain any closed surfaces.

2.2 Positive factorizations

Let Σ_g^n be a compact genus- g surface with n boundary components. We denote by Γ_g^n the mapping class group of Σ_g^n :

$$\Gamma_g^n = \pi_0 \left(\left\{ \psi \in \text{Diff}^+(\Sigma_g^n) \mid \psi|_{\partial \Sigma_g^n} = \text{id} \right\} \right).$$

Let (X, f) be a Lefschetz fibration or pencil over $\Sigma = \mathbb{CP}^1$ with l critical points and n base points (where $n = 0$ if it is a fibration). Take a regular value $p_0 \in \mathbb{CP}^1$ and simple paths $\gamma_1, \dots, \gamma_l$ from p_0 to critical values which are mutually disjoint except at p_0 . Suppose that $\gamma_1, \dots, \gamma_l$ appear in this order when we go around p_0 counterclockwise. We denote by α_i a loop with base point p_0 which first goes along γ_i , then goes around a critical value counterclockwise and goes back to p_0 along γ_i . Kas [35] showed that a monodromy of f along α_i , which can be regarded as an element of Γ_g^n under an identification $\Sigma_g^n \cong f^{-1}(p_0) \setminus \nu B$, is a *positive Dehn twist* t_{c_i} (a right-handed Dehn twist) along a simple closed curve $c_i \subset \Sigma_g^n$, called the *vanishing cycle* of f associated to c_i by γ_i .

We can deduce from the local description of f around the base points that the monodromy of f along a concatenated loop $\alpha_1 \cdots \alpha_l$ is equal to $t_{\delta_1} \cdots t_{\delta_n} \in \Gamma_g^n$, where $\delta_1, \dots, \delta_n \subset \Sigma_g^n$ are simple closed curves parallel to respective boundary components. We therefore obtain the following *positive factorization* of the boundary multi-twist $t_{\delta_1} \cdots t_{\delta_n}$ in Γ_g^n :

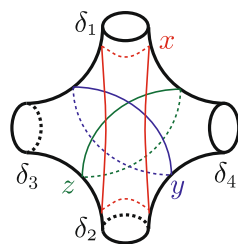
$$t_{c_l} \cdots t_{c_1} = t_{\delta_1} \cdots t_{\delta_n},$$

which is called the *monodromy factorization* of the Lefschetz pencil f . When there are no base points, i.e. when $n = 0$, this is a factorization of identity in Γ_g . Note that relative minimality of a Lefschetz fibration or pencil implies that no c_i is null-homotopic. In the case of an allowable Lefschetz fibration, where the base is D^2 instead of $\mathbb{CP}^1$, one derives a positive factorization of an element in Γ_g^n , $n > 0$, which does not need to be a boundary multi-twist, and no c_i is null-homologous.

Conversely, given such a positive factorization as above, one can build a genus- g Lefschetz pencil (X, f) with l critical points and n base points, where X is a symplectic 4-manifold [28]. Similarly, a positive factorization of any mapping class μ in Γ_g^n with $n > 0$, where no Dehn twist curve c_i is null-homologous, one can build a genus- g allowable Lefschetz fibration (X, f) , where X is a Stein domain [1, 40]. On the boundary of X , this fibration induces an open book with monodromy μ , which supports the natural contact structure induced by the Stein structure on X (either as the maximal distribution of the complex structure restricted to the boundary, or as the kernel of contact 1-form which is the primitive of the symplectic structure around the boundary). Moreover, from a given positive factorization for a Lefschetz fibration or pencil, we can obtain another one by substituting a subword of the factorization with another product of positive Dehn twists. In the following sections we will construct several Lefschetz fibrations and pencils by this procedure, sometimes called as *monodromy substitution*. In these constructions, we will repeatedly use the following well-known relations (see e.g. [22]):

Lemma 2.1 (Lantern relation) *Let $\delta_1, \delta_2, \delta_3$ and δ_4 be the four boundary curves of Σ_0^4 and let x, y and z be the interior curves as shown in Fig. 1. Then, we have the lantern relation in Γ_0^4 :*

$$t_{\delta_4} t_{\delta_3} t_{\delta_2} t_{\delta_1} = t_z t_y t_x.$$

Fig. 1 The curves x, y, z on Σ_0^4 

Lemma 2.2 (Odd chain relation) *Let $c_1, c_2, \dots, c_{2h+1}$ be simple closed curves on Σ_g^n such that c_i and c_j are disjoint if $|i - j| \geq 2$ and that c_i and c_{i+1} intersect transversely at one point. Then, a regular neighborhood of $c_1 \cup c_2 \cup \dots \cup c_{2h+1}$ is a subsurface of genus h with two boundary components, δ_1 and δ'_1 . We then have the odd chain relation in Γ_g^n :*

$$(t_{c_1} t_{c_2} \cdots t_{c_{2h+1}})^{2h+2} = t_{\delta_1} t_{\delta'_1}.$$

Lastly, we have the conjugation relation $\phi t_a \phi^{-1} = t_{\phi(a)}$, which implies the equalities $t_b t_a = t_a t_{a^{-1}(b)} = t_{b(a)} t_b$ for any t_a, t_b and $\phi \in \Gamma_g^n$. Applying these equalities to change any subfactors in a positive factorization is known as a Hurwitz move, which yields equivalent Lefschetz fibrations or pencils. Whenever we have a Hurwitz equivalence of two positive factorizations, we will use the notation $\sim$ to emphasize that the underlying fibration has not changed.

2.3 Symplectic Kodaira dimension and Calabi–Yau surfaces

A symplectic 4-manifold is said to be *minimal* if it contains no symplectic sphere with self-intersection -1 . Let $(X_{\min}, \omega_{\min})$ be a minimal symplectic 4-manifold obtained from a given closed symplectic manifold (X, ω) by blowing-down all the symplectic spheres. For $K_{\min}$ the canonical class of $(X_{\min}, \omega_{\min})$, the *symplectic Kodaira dimension* $\kappa(X)$ of (X, ω) is defined as follows:

$$\kappa(X) = \begin{cases} -\infty & \text{if } K_{\min} \cdot [\omega_{\min}] < 0 \text{ or } K_{\min}^2 < 0 \\ 0 & K_{\min} \cdot [\omega_{\min}] = K_{\min}^2 = 0 \\ 1 & K_{\min} \cdot [\omega_{\min}] > 0 \text{ and } K_{\min}^2 = 0 \\ 2 & K_{\min} \cdot [\omega_{\min}] > 0 \text{ and } K_{\min}^2 > 0. \end{cases}$$

It turns out that $\kappa(X)$ is not only independent of the associated minimal symplectic manifold $(X_{\min}, \omega_{\min})$, but also the symplectic structure ω on X —so it is a diffeomorphism invariant. Moreover, when $\kappa(X) = -\infty$, the manifold X is known to be symplectomorphic to a rational or ruled surface, so its diffeomorphism type is a blow-up of either $\mathbb{CP}^2$ or an S^2 -bundle over a Riemann surface. In particular, $b^+(X) = 1$. When $\kappa(X) = 0$, X is known to have the same rational homology as the blow-up of either a T^2 -bundle over T^2 , the Enriques surface or the K3 surface. In particular, $b^+(X) \leq 3$. (See e.g. [37, 38].)

We will make use of the following criterion for a symplectic 4-manifold to have a specific Kodaira dimension:

Theorem 2.3 (Kodaira dimension from monodromy factorizations, [10, 11, 49]) *Let X be a symplectic 4-manifold which is the total space of a genus $g \geq 3$ Lefschetz fibration with*

a monodromy factorization which lifts to Γ_g^n as a factorization of the boundary multi-twist, that is

$$t_{c_1} \cdot \dots \cdot t_{c_l} = t_{\delta_n} \cdot \dots \cdot t_{\delta_1}.$$

Then the symplectic Kodaira dimension of X is equal to

- i. $\kappa(X) = -\infty$ if $n > 2g - 2$,
- ii. $\kappa(X) = 0$ if $n = 2g - 2$ and $b^+(X) \neq 1$,
- iii. $\kappa(X) = 1$ if $n = 2g - 3$ and $b^+(X) > 3$.

Here the boundary multi-twist corresponds to n disjoint sections that are all exceptional spheres, which is a particular case of [49, Theorem 5–12] and [11, Theorem 4.1], where the positive factorizations are given for multisections (intersecting the fiber at n points) that are exceptional spheres. For part (i), observe that if the number of exceptional sections is $n > 2g - 2$, blowing them down, we get a fiber class which violates the Seiberg-Witten adjunction inequality, which is only possible in a rational or ruled surface. All parts but part (iii) also hold for $g = 2$.

Note that the topological invariant $b^+(X)$ in the theorem can be read off from the positive factorization. Since $\pi_1(X)$ is isomorphic to the quotient of $\pi_1(\Sigma_g)$ by $N(c_1, \dots, c_l)$, the subgroup normally generated by c_i s, we can calculate $b_1(X)$ as the rank of the abelianization of $\pi_1(X)$. Then calculating $e(X) = 4 - 4g + l$ and $\sigma(X)$ algorithmically (e.g. by [20]), we can obtain $b^+(X)$ through the equality $b^+(X) = (e(X) - 2 + 2b_1 + \sigma(X))/2$.

3 Unchaining operation

In this section we will discuss the *unchaining* operation, which is a symplectic surgery that can be interpreted (at least locally) as a monodromy substitution in a Lefschetz fibration. This surgery will play a key role in all our constructions throughout the paper.

Let $c_1, \dots, c_{2g+1}, \delta_1, \delta'_1 \subset \Sigma_g^2$ be simple closed curves shown in Fig. 4. By Lemma 2.2, the following relation holds in Γ_g^2 :

$$(t_{c_1} \cdots t_{c_{2g+1}})^{2g+2} = t_{\delta_1} t_{\delta'_1}.$$

Denote the total space of the allowable Lefschetz fibration corresponding to the left and right hand sides of the relation above by V_g and V'_g , respectively. One can describe the compact manifolds V_g and V'_g by handlebody diagrams in Fig. 2.

Taking the Stein structures associated to their allowable Lefschetz fibrations above, we regard V_g and V'_g as Stein domains. (It is a straightforward exercise to turn the handle diagrams in Fig. 2 into diagrams of Stein handle decompositions following [1].) Since these fibrations induce the same open book monodromy on their boundaries, the contact structures induced on ∂V_g and $\partial V'_g$ by the underlying symplectic structures on V_g and V'_g are contactomorphic. Thus, if V_g is a Stein subdomain of a symplectic 4-manifold (X, ω_X) , we can excise V_g and glue in V'_g symplectically via any contactomorphism $\varphi : \partial V_g \rightarrow \partial V'_g$, perhaps after scaling the symplectic form on it (see e.g. [21]). We will call this particular symplectic cut-and-paste operation *unchaining*:

Definition 3.1 Let (X, ω_X) be a symplectic 4-manifold, containing V_g as a Stein submanifold. The symplectic 4-manifold (Y, ω_Y) , where $Y = (X \setminus V_g) \cup_\varphi V'_g$, with $\omega_X|_{X \setminus V_g} = \omega_Y|_{Y \setminus V'_g}$ and containing V'_g as a Stein subdomain is then said to be obtained by *unchaining* (X, ω_X)

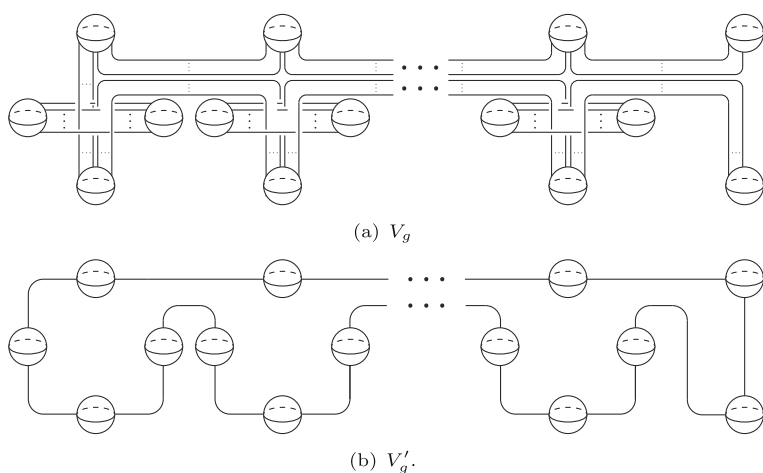


Fig. 2 Handlebody diagrams of V_g and V'_g . All the 2-handles in both figures have (-1) -framing

along $V_g \subset X$ (or by $(2g + 1)$ -*unchaining*, whenever we would like to be specific about the size).

This local surgery realizes a monodromy substitution when (X, ω_X) is the total space of a symplectic Lefschetz fibration or pencil f , whose monodromy factorization contains $(t_{c_1} \cdots t_{c_{2h+1}})^{2h+2}$ as a subword, where $c_1, \dots, c_{2h+1}$ are simple closed curves in a reference fiber of f satisfying the condition in Lemma 2.2. We can then apply a C_{2h+1} -*substitution* to the monodromy factorization of f , that is, we can substitute a subword $(t_{c_1} \cdots t_{c_{2h+1}})^{2h+2}$ in the factorization with $t_{\delta_1} t_{\delta'_1}$, where δ_1, δ'_1 are curves given in Lemma 2.2. Since the substitution does not change the right-hand side of the factorization, we obtain another symplectic Lefschetz fibration or pencil h , whose total space is a symplectic 4-manifold (Y, ω_Y) obtained by unchaining (X, ω_X) along V_g . (In this case we choose $\varphi : \partial V_g \rightarrow \partial V'_g$ to be a fiber-preserving contactomorphism between the boundary open books, which is identity along the fibers.) We will build all our examples in this paper from this perspective.

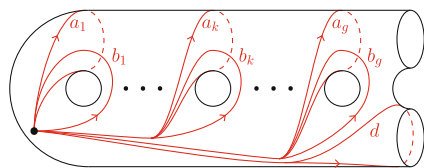
In the remainder of this section, we will explain how the Euler characteristic, the signature and the fundamental group of X and Y are related by the unchaining operation along $V_g \subset X$.

Let us start with the invariants for the subdomains V_g and V'_g . From the handle decompositions given above, it is easy to see that the Euler characteristics $e(V_g)$ and $e(V'_g)$ are respectively equal to $2(2g^2 + 2g + 1)$ and $-2g + 2$. The manifold V_g is the complement of the union U of neighborhoods of a regular fiber and two (-1) -sections in a *hyperelliptic* Lefschetz fibration over S^2 with $(2g + 1)(2g + 2)$ irreducible singular fibers and no reducible ones. Therefore, we can deduce from [20] that the signature of the hyperelliptic fibration over S^2 is $-2(g + 1)^2$. Since the signature of U is -1 , by the Novikov additivity, the signature of V_g is equal to $-2(g + 1)^2 + 1$. On the other hand, since V'_g is a disk bundle over a genus- g surface with Euler number -2 , the signature $\sigma(V'_g) = -1$. Now by the additivity of Euler characteristic and signature for gluing compact 4-manifolds along 3-manifolds, we can conclude that:

Proposition 3.2 *Let Y be obtained by unchaining X along $V_g \subset X$. Then,*

$$e(Y) = e(X) - 2g(2g + 3),$$

Fig. 3 A generating set of a fiber of the open book of ∂V_g



$$\sigma(Y) = \sigma(X) + 2g(g + 2).$$

The relation between the fundamental groups of X and Y is more involved, since in general it non-trivially depends on the fundamental group of the complement of V_g in X . The manifold ∂V_g (and thus X) contains a surface $\Sigma \cong \Sigma_g^2$ which is a fiber of the boundary open book. We take a generating set $\{a_1, b_1, \dots, a_g, b_g, d\}$ of $\pi_1(\Sigma)$ as shown in Fig. 3.

We also denote the elements in $\pi_1(X \setminus V_g)$ represented by loops in Fig. 3 by a_i, b_j , and so on. Recall that V'_g admits a handle decomposition with one 0-handle, $2g + 1$ 1-handles, and two 2-handles as in Figure 2. Now, Y is obtained from $\overline{X \setminus V_g}$ by attaching this handlebody upside down to it. Then, the cellular decomposition induced by this handle decomposition implies that Y is obtained from $\overline{X \setminus V_g}$ by attaching two 2-cells to an open book fiber Σ of $\partial(\overline{X \setminus V_g}) = \partial V'_g$ along loops freely homotopic to d and $d^{-1} \cdot \prod_{i=1}^g [a_i, b_i]$, and then attaching $2g + 1$ 3-cells and a 4-cell. By a standard consequence of Seifert Van-Kampen, the 2-cells yield further relations via their attaching maps, while the higher dimensional cells have no effect on π_1 . We therefore get:

Proposition 3.3 *The fundamental group of Y is isomorphic to*

$$\pi_1(\overline{X \setminus V_g}) / N \left(d, \prod_{i=1}^g [a_i, b_i] \right)$$

where $N(d, \prod_{i=1}^g [a_i, b_i])$ is the subgroup of $\pi_1(\overline{X \setminus V_g})$ normally generated by d , and $\prod_{i=1}^g [a_i, b_i]$. In particular, if $\overline{X \setminus V_g}$ is simply-connected, so is Y .

Remark 3.4 Propositions 3.2 and 3.3 show that in general unchaining operation is not equivalent to a sequence of rational blow-downs and blow-ups. For instance, if a simply-connected manifold Y can be obtained by applying $(2g + 1)$ -unchaining to another simply-connected manifold X (and we will see many instances in this paper), we deduce that $b^+(X) - b^+(Y)$ is equal to $g(g + 1)$. However, rational blow-downs and blow-ups do not change b^+ .

By Propositions 3.2 and 3.3, the unchaining operation decreases the second Betti number of a 4-manifold, so it is a handy operation for deriving a *smaller* symplectic 4-manifold. In addition, the unchaining might give rise to new (-1) -spheres in Y , as we will see in our examples in this paper, which we can further blow-down to obtain an even smaller manifold. The same examples will demonstrate that the symplectic Kodaira dimension is also non-increasing under the unchaining operation, i.e. $\kappa(Y) \leq \kappa(X)$, which we conjecture to be true in general:

Conjecture 3.5 *If (Y, ω_Y) is obtained from the symplectic 4-manifold (X, ω_X) by unchaining, then their symplectic Kodaira dimensions satisfy $\kappa(Y) \leq \kappa(X)$.*

Remark 3.6 (Surgery along even chains) Almost every aspect of the unchaining surgery we discuss here applies likewise to surgeries along even number of chains, which correspond to

a similar relation in the mapping class group, where one replaces a product of Dehn twists along an even number of chains with a single Dehn twist along the boundary of their tubular neighborhood [22]. Although in this article we only discuss the odd case for brevity, the even case similarly has many interesting applications, readily available in the existing literature: For instance, even chain monodromy substitution was conclusively employed in Mustafa Korkmaz and the first author's reverse engineering of small positive factorizations in [12], and the examples of genus-2 Lefschetz fibrations in [12] readily demonstrate that the analogue of the unchaining surgery on even chains, too, can decrease the Kodaira dimension. The exotic rational surfaces constructed in [12] and several symplectic Calabi–Yau homotopy K3 and homotopy Enriques surfaces in [11] can all be seen to be obtained by such a surgery from a symplectic surfaces of Kodaira dimension two. (By the inverse of the operation, one replaces a reducible fiber component with a Stein subdomain corresponding to the even chain). See also [2].

We note that one difference in the case of a surgery along an even chain is that the operation now swaps a Stein subdomain with a *symplectic filling* given by the neighborhood of a (-1) -curve, which is not a Stein filling of the contact structure on the boundary supported by the obvious open book.

4 New symplectic pencils via unchaining

Here we will produce new symplectic Lefschetz pencils from a family of pencils on complex surfaces of general type, by carefully applying the unchaining operation. As we keep track of the associated monodromies, we will then look at their lifts to detect a sufficient number of (-1) -sections to determine the Kodaira dimension of the new symplectic 4-manifolds, using Theorem 2.3.

In this and the following sections, Figs. 4–6 will be used repeatedly.

4.1 Positive factorizations for a family of holomorphic pencils

Let $c_1, \dots, c_{2g+1}, \delta_1, \delta'_1 \subset \Sigma_g^2$ be simple closed curves shown in Fig. 4. We can obtain the closed surface Σ_g by capping $\partial \Sigma_g^2$ by two disks. In this way we regard Σ_g^2 as a subsurface of Σ_g . By Lemma 2.2 we have:

$$\begin{aligned} (t_{c_1} t_{c_2} \cdots t_{c_{2g+1}})^{2g+2} &= 1 \text{ in } \Gamma_g, \text{ and} \\ (t_{c_1} t_{c_2} \cdots t_{c_{2g+1}})^{2g+2} &= t_{\delta_1} t_{\delta'_1} \text{ in } \Gamma_g^2. \end{aligned} \quad (1)$$

These positive factorizations prescribe a Lefschetz fibration (Z_g, f) and a pencil (Z'_g, f') , respectively, where (Z_g, f) is obtained from (Z'_g, f') by blowing-up the two base points. We easily calculate the Euler characteristic as $e(Z'_g) = 4g^2 + 2g + 4$, and thus

$$e(Z_g) = 4g^2 + 2g + 6.$$

Here the left-hand side of the relation (1) is obtained by lifting a braid monodromy of a non-singular projective curve of degree $2(g+1)$ in $\mathbb{CP}^2$ under the double branched covering branched at $2g+2$ points (see [42, Corollary VIII.2.3]), so f' is a holomorphic map, which is the composition of the double branched covering $p: Z'_g \rightarrow \mathbb{CP}^2$ of $\mathbb{CP}^2$ branched along a degree $2(g+1)$ non-singular curve, and the linear projection from $\mathbb{CP}^2$ to $\mathbb{CP}^1$. It is easy to see that Z'_g is simply-connected. Moreover, as shown in [31], the canonical bundle K_X of the

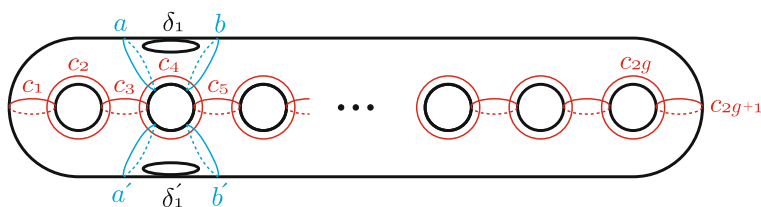


Fig. 4 The curves a, a', b, b' and c_i ($i = 1, \dots, 2g + 1$) on Σ_g^2

covering complex surface Z'_g is isomorphic to $p^*(H^{\otimes(g-2)})$ and $p^*(c_1(H)) \in H^2(Z'_g)$ is primitive, where H is the holomorphic line bundle over $\mathbb{CP}^2$ defined by a hyperplane section. In particular the signature $\sigma(Z'_g) = -2(g+1)^2 + 2$, and thus

$$\sigma(Z_g) = -2(g+1)^2,$$

and Z'_g is spin if and only if g is even. Furthermore, Z'_g (and its blow-up Z_g) is a complex surfaces of general type since H is very ample.

It is decidedly easier to identify the Stein subdomains we would like to perform unchaining surgeries along as subfactorizations in positive factorizations of Lefschetz pencils. Nonetheless, it is still often the case that the desired subfactorization (just like the desired handle decomposition) only emerges after deliberate manipulations of the original monodromy. The rest of this subsection is devoted precisely to this cause, with the sole aim of deriving a suitable positive factorization of the boundary multi-twist in Γ_g^2 , for each $g \geq 3$, which is Hurwitz equivalent the factorization (1):

Proposition 4.1 For $g \geq 3$, let $d_j = t_{c_{j-3}}^{-1} t_{c_{j-2}}^{-1} t_{c_{j-1}}^{-1}(c_j)$ and $e_j = t_{c_{j-3}} t_{c_{j-2}} t_{c_{j-1}}(c_j)$, for all $j = 4, 5, \dots, 2g + 1$, where the curves c_j, d_j, e_j are shown in Figs. 4 and 5. Also set, for a shorthand notation;

$$D_g = t_{d_4} t_{d_5} \cdots t_{d_{2g+1}} \quad \text{and} \quad E_g = t_{e_{2g+1}} \cdots t_{e_5} t_{e_4}.$$

Then, the following equations hold in Γ_g^2 :

$$t_{\delta_1} t_{\delta'_1} = (t_{c_1} t_{c_2} t_{c_3})^{4g} D_g E_g (t_{c_5} t_{c_6} \cdots t_{c_{2g+1}})^{2g-2} \quad (g : \text{odd})$$

$$t_{\delta_1} t_{\delta'_1} = (t_{c_1} t_{c_2} t_{c_3})^{4(g-1)+2} (t_{c_3} t_{c_2} t_{c_1})^2 D_g E_g (t_{c_5} t_{c_6} \cdots t_{c_{2g+1}})^{2g-2} \quad (g : \text{even})$$

which are Hurwitz equivalent to the positive factorizations in (1).

The proof of this proposition will require some preparation, through a sequence of technical lemmas, Lemma 4.2, 4.3, and 4.4 we prove below.

For $m = 1, 2, \dots, 2g - 2$, and $l = m, m + 1, m + 2$, the braid relators amount to following Hurwitz equivalences:

$$t_{c_l} \cdot t_{c_{m+3}} t_{c_{m+2}} t_{c_{m+1}} t_{c_m} \sim t_{c_{m+3}} t_{c_{m+2}} t_{c_{m+1}} t_{c_m} \cdot t_{c_{l+1}}, \quad (2)$$

$$t_{c_m} t_{c_{m+1}} t_{c_{m+2}} t_{c_{m+3}} \cdot t_{c_l} \sim t_{c_{l+1}} \cdot t_{c_m} t_{c_{m+1}} t_{c_{m+2}} t_{c_{m+3}}. \quad (3)$$

where, and hereon, $\sim$ denotes Hurwitz equivalence of positive factorizations.

In all of the following lemmas, $g \geq 3$, and the curves c_j, d_j and e_j are the ones in Figs. 4 and 5.

Lemma 4.2 $(t_{c_1} t_{c_2} \cdots t_{c_{2g+1}})^4 \sim (t_{c_1} t_{c_2} t_{c_3})^4 (t_{c_3} t_{c_2} t_{c_1})^2 t_{d_4} t_{d_5} \cdots t_{d_{2g+1}}$ in Γ_g^2 .

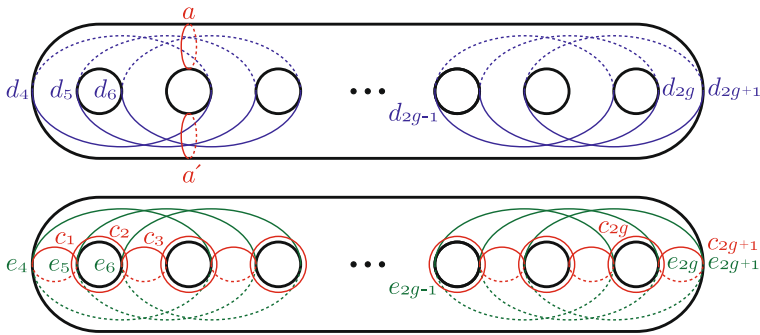


Fig. 5 The curves d_j, e_j on Σ_g ($j = 4, \dots, 2g + 1$)

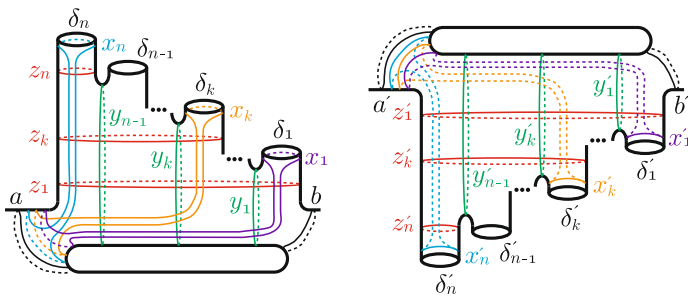


Fig. 6 The curves x_k, x'_k, z_k, z'_k and y_k, y'_k , which are contained in pairs of pants bounded by a, b, δ_1 and a', b', δ'_1 on Σ_g^2 in Fig. 4, respectively

Proof It is easy to check using the braid relators that

$$(t_{c_1} t_{c_2} \cdots t_{c_{2g+1}})^4 \sim (t_{c_1} t_{c_2} t_{c_3})^4 \cdot \prod_{i=1}^4 t_{c_{5-i}} t_{c_{6-i}} \cdots t_{c_{2g+2-i}}, \quad (4)$$

$$\prod_{i=1}^4 t_{c_{5-i}} t_{c_{6-i}} \cdots t_{c_{2g+2-i}} \sim \prod_{i=1}^{2g-2} t_{c_{i+3}} t_{c_{i+2}} t_{c_{i+1}} t_{c_i}. \quad (5)$$

Here, for $k = 1, 2, \dots, 2g - 2$, we claim the following holds in Γ_g^2 :

$$\prod_{i=k}^{2g-2} t_{c_{i+3}} t_{c_{i+2}} t_{c_{i+1}} t_{c_i} \sim (t_{c_{k+2}} t_{c_{k+1}} t_{c_k})^{2g-1-k} t_{d_{k+3}} t_{d_{k+4}} \cdots t_{d_{2g+1}}.$$

The proof of the claim will be by induction on $2g - 1 - k$. For $k = 2g - 2$, the conclusion holds, since we have the equivalence

$$t_{c_{2g+1}} \cdot t_{c_{2g}} t_{c_{2g-1}} t_{c_{2g-2}} \sim t_{c_{2g}} t_{c_{2g-1}} t_{c_{2g-2}} \cdot t_{d_{2g+1}}.$$

Assume that the relation holds for $k + 1 < 2g - 2$. By (2), we have

$$\prod_{i=k}^{2g-2} t_{c_{i+3}} t_{c_{i+2}} t_{c_{i+1}} t_{c_i}$$

$$\begin{aligned}
&= t_{c_{k+3}} t_{c_{k+2}} t_{c_{k+1}} t_{c_k} \cdot \prod_{i=k+1}^{2g-2} t_{c_{i+3}} t_{c_{i+2}} t_{c_{i+1}} t_{c_i} \\
&\sim t_{c_{k+3}} t_{c_{k+2}} t_{c_{k+1}} t_{c_k} \cdot (t_{c_{k+3}} t_{c_{k+2}} t_{c_{k+1}})^{2g-1-(k+1)} t_{d_{k+4}} t_{d_{k+5}} \cdots t_{d_{2g+1}} \\
&\sim (t_{c_{k+2}} t_{c_{k+1}} t_{c_k})^{2g-1-(k+1)} \cdot t_{c_{k+3}} t_{c_{k+2}} t_{c_{k+1}} t_{c_k} \cdot t_{d_{k+4}} t_{d_{k+5}} \cdots t_{d_{2g+1}} \\
&\sim (t_{c_{k+2}} t_{c_{k+1}} t_{c_k})^{2g-1-(k+1)} \cdot t_{c_{k+2}} t_{c_{k+1}} t_{c_k} \cdot t_{d_{k+3}} \cdot t_{d_{k+4}} t_{d_{k+5}} \cdots t_{d_{2g+1}},
\end{aligned}$$

which completes the proof of the claim.

The proof of the lemma then follows from $k = 1$ case of the claim, and the relations (4) and (5).

Lemma 4.3 *The following holds in Γ_g^2 :*

$$(t_{c_1} t_{c_2} \cdots t_{c_{2g+1}})^{2g-2} \sim t_{e_{2g+1}} \cdots t_{e_5} t_{e_4} (t_{c_1} t_{c_2} t_{c_3})^{2g-2} (t_{c_5} t_{c_6} \cdots t_{c_{2g+1}})^{2g-2}.$$

Proof Once again, it easily follows from the braid relators that

$$(t_{c_1} t_{c_2} \cdots t_{c_{2g+1}})^{2g-2} \sim \left(\prod_{i=1}^{2g-2} t_{c_{2g-1-i}} t_{c_{2g-i}} t_{c_{2g+1-i}} t_{c_{2g+2-i}} \right) (t_{c_5} t_{c_6} \cdots t_{c_{2g+1}})^{2g-2}. \quad (6)$$

Here, in Γ_g^2 , we claim that

$$\prod_{i=k}^{2g-2} t_{c_{2g-2+k-i}} t_{c_{2g-1+k-i}} t_{c_{2g+k-i}} t_{c_{2g+1+k-i}} \sim t_{e_{2g+1}} \cdots t_{e_{k+4}} t_{e_{k+3}} (t_{c_k} t_{c_{k+1}} t_{c_{k+2}})^{2g-1-k},$$

for $k = 1, 2, \dots, 2g - 2$. Once again we induct on $2g - 1 - k$. For $k = 2g - 2$ we have

$$t_{c_{2g-2}} t_{c_{2g-1}} t_{c_{2g}} \cdot t_{c_{2g+1}} \sim t_{e_{2g+1}} \cdot t_{c_{2g-2}} t_{c_{2g-1}} t_{c_{2g}}.$$

Assume that the relation holds for $k + 1 < 2g - 2$. By (3), we have

$$\begin{aligned}
&\prod_{i=k}^{2g-2} t_{c_{2g-2+k-i}} t_{c_{2g-1+k-i}} t_{c_{2g+k-i}} t_{c_{2g+1+k-i}} \\
&= \left(\prod_{i=k+1}^{2g-2} t_{c_{2g-2+k-i}} t_{c_{2g-1+k-i}} t_{c_{2g+k-i}} t_{c_{2g+1+k-i}} \right) \cdot t_{c_k} t_{c_{k+1}} t_{c_{k+2}} t_{c_{k+3}} \\
&\sim t_{e_{2g+1}} \cdots t_{e_{k+5}} t_{e_{k+4}} (t_{c_{k+1}} t_{c_{k+2}} t_{c_{k+3}})^{2g-2-k} \cdot t_{c_k} t_{c_{k+1}} t_{c_{k+2}} t_{c_{k+3}} \\
&\sim t_{e_{2g+1}} \cdots t_{e_{k+5}} t_{e_{k+4}} \cdot t_{c_k} t_{c_{k+1}} t_{c_{k+2}} t_{c_{k+3}} \cdot (t_{c_k} t_{c_{k+1}} t_{c_{k+2}})^{2g-2-k} \\
&\sim t_{e_{2g+1}} \cdots t_{e_{k+5}} t_{e_{k+4}} \cdot t_{e_{k+3}} \cdot t_{c_k} t_{c_{k+1}} t_{c_{k+2}} \cdot (t_{c_k} t_{c_{k+1}} t_{c_{k+2}})^{2g-2-k}.
\end{aligned}$$

Hence, the claim is proved.

The proof of the lemma then follows from the $k = 1$ case of the claim, and the relation (6).

Lemma 4.4 $(t_{c_3} t_{c_2} t_{c_1})^4 \sim (t_{c_1} t_{c_2} t_{c_3})^4$ in Γ_g^2 .

Proof Since $t_{c_1} t_{c_3} \sim t_{c_3} t_{c_1}$, we have

$$t_{c_3} t_{c_2} t_{c_1} t_{c_3} t_{c_2} t_{c_1} t_{c_3} t_{c_2} t_{c_1} t_{c_3} t_{c_2} t_{c_1} \sim t_{c_3} t_{c_2} t_{c_3} t_{c_1} t_{c_2} t_{c_3} t_{c_1} t_{c_2} t_{c_3} t_{c_1} t_{c_2} t_{c_1}.$$

Here, we have the following relations, again by the braid relators:

$$\begin{aligned} t_{c_1} t_{c_2} t_{c_3} \cdot t_{c_i} &\sim t_{c_{i+1}} \cdot t_{c_1} t_{c_2} t_{c_3}, \\ t_{c_i} t_{c_{i+1}} t_{c_i} &\sim t_{c_{i+1}} t_{c_i} t_{c_{i+1}} \text{ for } i = 1, 2. \end{aligned}$$

Using these relations, we have

$$\begin{aligned} (t_{c_3} t_{c_2} t_{c_3}) t_{c_1} t_{c_2} t_{c_3} t_{c_1} t_{c_2} t_{c_3} (t_{c_1} t_{c_2} t_{c_1}) &\sim t_{c_1} t_{c_2} t_{c_3} (t_{c_2} t_{c_1} t_{c_2}) (t_{c_2} t_{c_3} t_{c_2}) t_{c_1} t_{c_2} t_{c_3} \\ &\sim t_{c_1} t_{c_2} t_{c_3} (t_{c_1} t_{c_2} t_{c_1}) (t_{c_3} t_{c_2} t_{c_3}) t_{c_1} t_{c_2} t_{c_3} \\ &\sim t_{c_1} t_{c_2} t_{c_3} (t_{c_1} t_{c_2} t_{c_3} t_{c_1} t_{c_2} t_{c_3}) t_{c_1} t_{c_2} t_{c_3}. \end{aligned}$$

Therefore, we obtain $(t_{c_3} t_{c_2} t_{c_1})^4 \sim (t_{c_1} t_{c_2} t_{c_3})^4$.

We can now give our proof of the main result of this subsection:

Proof of Proposition 4.1 Applying Lemmas 4.2 and 4.3 to the odd chain relation (Lemma 2.2) we get:

$$\begin{aligned} t_{\delta_1} t_{\delta'_1} &= (t_{c_1} t_{c_2} \cdots t_{c_{2g+1}})^{2g+2} \\ &\sim (t_{c_1} t_{c_2} t_{c_3})^4 (t_{c_3} t_{c_2} t_{c_1})^{2g-2} D_g E_g (t_{c_1} t_{c_2} t_{c_3})^{2g-2} (t_{c_5} t_{c_6} \cdots t_{c_{2g+1}})^{2g-2}. \end{aligned}$$

Here, note that $2g-2 = 4k$ if $g = 2k+1$ and $2g-2 = 4k+2$ if $g = 2k+2$. Since c_1, c_2, c_3 are disjoint from $c_5, c_6, \dots, c_{2g+1}, \delta_1, \delta'_1$, by Hurwitz moves (including cyclic permutation) and Lemma 4.4, we obtain the claim.

4.2 Families of new symplectic pencils via unchaining

The new positive factorizations for Lefschetz pencils on Z'_g we obtained in Proposition 4.1 allows us to apply unchaining as a monodromy substitution, in fact, multiple times. In this way, we will derive new positive factorizations of the boundary multi-twist in Γ_g^2 , which can be further lifted to Γ_g^m for varying $m > 2$ as we move forward. The main outcomes of this subsection will be our construction of family of symplectic fibrations $(X_g(i), f_g(i))$ and pencils $(X'_g(i), f'_g(i))$ in Theorem 4.6.

Lemma 4.5 *Let a, a' and b, b' be the boundary components of regular neighborhoods of $c_1 \cup c_2 \cup c_3$ and $c_5 \cup c_6 \cup \cdots \cup c_{2g+1}$ as in Fig. 4, respectively. For $g \geq 3$, the following relations hold in Γ_g^2 .*

$$t_{\delta_1} t_{\delta'_1} = \begin{cases} t_b t_a^i t_{b'} t_{a'}^i (t_{c_1} t_{c_2} t_{c_3})^{4(g-i)} D_g E_g & (g : \text{odd}) \\ t_b t_a^i t_{b'} t_{a'}^i (t_{c_1} t_{c_2} t_{c_3})^{4(g-1-i)+2} (t_{c_3} t_{c_2} t_{c_1})^2 D_g E_g & (g : \text{even}), \end{cases} \quad (7)$$

where $0 \leq i \leq g$ if g is odd and $0 \leq i \leq g-1$ if g is even.

Proof Take the following chain relators:

$$\begin{aligned} C_3 &:= (t_{c_1} t_{c_2} t_{c_3})^4 t_a^{-1} t_{a'}^{-1}, \\ C_{2g-3} &:= (t_{c_5} t_{c_6} \cdots t_{c_{2g+1}})^{2g-2} t_{b'}^{-1} t_b^{-1}. \end{aligned}$$

Suppose g is even. By applying C_3^{-1} substitutions i times and C_{2g-3}^{-1} substitution once to $(t_{c_1} t_{c_2} t_{c_3})^{4(g-1)}$ and $(t_{c_5} t_{c_6} \cdots t_{c_{2g+1}})^{2g-2}$ on the right-hand side of our relation in Proposition 4.1, respectively, we obtain

$$t_{\delta_1} t_{\delta'_1} = t_a^i t_{a'}^i (t_{c_1} t_{c_2} t_{c_3})^{4(g-1-i)+2} (t_{c_3} t_{c_2} t_{c_1})^2 D_g E_g t_b t_{b'}.$$

Since a, a', b, b' are disjoint from each other, the elements $t_a, t_{a'}, t_b$ and $t_{b'}$ all commute. Moreover, $t_{\delta_1} t_{\delta'_1}$ is central in Γ_g^2 , so we can get the claimed relation after conjugating both sides with the inverse of $t_b t_{b'}$.

The proof for the case of odd g is very similar.

We will now show that there is a lift of the relations in Lemma 4.5 from Γ_g^2 to $\Gamma_g^{2(i+1)}$ as a factorization of the boundary multi-twist. These lifts play a crucial role in our calculation of the symplectic Kodaira dimension of the underlying manifolds.

Theorem 4.6 *There are symplectic genus- g Lefschetz pencils $(X'_g(i), f'_g(i))$, for each $0 \leq i \leq g$ if g is odd, and for $0 \leq i \leq g-1$ if g is even, with monodromy factorizations in $\Gamma_g^{2(i+1)}$ as follows:*

$$\begin{aligned} & t_{\delta_{i+1}} \cdots t_{\delta_2} t_{\delta_1} t_{\delta'_{i+1}} \cdots t_{\delta'_2} t_{\delta'_1} \\ &= \begin{cases} t_{x_{i+1}} \cdots t_{x_2} t_{x_1} t_{x'_{i+1}} \cdots t_{x'_2} t_{x'_1} (t_{c_1} t_{c_2} t_{c_3})^{4(g-i)} D_g E_g & (g : \text{odd}) \\ t_{x_{i+1}} \cdots t_{x_2} t_{x_1} t_{x'_{i+1}} \cdots t_{x'_2} t_{x'_1} (t_{c_1} t_{c_2} t_{c_3})^{4(g-1-i)+2} (t_{c_3} t_{c_2} t_{c_1})^2 D_g E_g & (g : \text{even}) \end{cases} \end{aligned}$$

where the curves x_k, x'_k (for $k = 1, 2, \dots, n$) on the surface Σ_g^{2n} , $n = i+1$, are as in Fig. 6. Blowing up all the base points of each $(X'_g(i), f'_g(i))$ yields symplectic Lefschetz fibrations $(X_g(i), f_g(i))$, whose monodromy factorizations in Γ_g are obtained from the above after capping off all the boundary components.

Proof Let y_k, y'_k (for $k = 1, 2, \dots, n$) and z_k, z'_k (for $k = 1, 2, \dots, n-1$) be the auxiliary curves in Fig. 6, where $n = i+1$. Note that $y_{n-1} = x_n, y'_{n-1} = x'_n$ and $z_n = \delta_n, z'_n = \delta'_n$. Set $t_{y_0} = t_b, t_{y'_0} = t_{b'}$ and $t_{x_0} = t_{x'_0} = t_{\delta_0} = t_{\delta'_0} = \text{id}$ in Γ_g^{2n} .

We claim that whenever we have a relation in Γ_g^{2n} of the form

$$t_{z_1} t_{z'_1} = t_a^{n-1} t_b t_{a'}^{n-1} t_{b'} \cdot T,$$

where T is an element in Γ_g^{2n} , then the following relation holds in Γ_g^{2n} as well for $k = 1, 2, \dots, n$:

$$\begin{aligned} & t_{z_k} t_{\delta_{k-1}} \cdots t_{\delta_2} t_{\delta_1} \cdot t_{z'_k} t_{\delta'_{k-1}} \cdots t_{\delta'_2} t_{\delta'_1} \\ &= t_a^{n-k} t_{y_{k-1}} t_{x_{k-1}} \cdots t_{x_2} t_{x_1} \cdot t_{a'}^{n-k} t_{y'_{k-1}} t_{x'_{k-1}} \cdots t_{x'_2} t_{x'_1} \cdot T. \end{aligned}$$

In particular, we obtain the following relation

$$t_{\delta_n} \cdots t_{\delta_1} \cdot t_{\delta'_n} \cdots t_{\delta'_1} = t_{x_n} \cdots t_{x_1} \cdot t_{x'_n} \cdots t_{x'_1} \cdot T.$$

in Γ_g^{2n} from $y_{n-1} = x_n, y'_{n-1} = x'_n$ and $z_n = \delta_n, z'_n = \delta'_n$.

We will prove the claim by induction on k . This relation holds for $k = 1$ from $t_{y_0} = t_b, t_{y'_0} = t_{b'}, t_{x_0} = t_{x'_0} = t_{\delta_0} = t_{\delta'_0} = \text{id}$ and the assumption $t_{z_1} t_{z'_1} = t_a^{n-1} t_b t_{a'}^{n-1} t_{b'} \cdot T$ in Γ_g^{2n} . Assume that the relation holds for each $1 < k$. Then, we multiply both sides of the above relation by $t_{z_{k+1}} t_{\delta_k} t_{z'_{k+1}} t_{\delta'_k}$. Since the curves $z_{k+1}, \delta_k, z'_{k+1}, \delta'_k$ are disjoint from each other and any curves appeared in the above relation, we obtain:

$$\begin{aligned} & t_{z_k} \cdot t_{z_{k+1}} t_{\delta_k} t_{\delta_{k-1}} \cdots t_{\delta_2} t_{\delta_1} \cdot t_{z'_k} \cdot t_{z'_{k+1}} t_{\delta'_k} t_{\delta'_{k-1}} \cdots t_{\delta'_2} t_{\delta'_1} \\ &= t_a^{n-k-1} t_{z_{k+1}} t_{a'} t_{y_{k-1}} t_{\delta_k} t_{x_{k-1}} \cdots t_{x_2} t_{x_1} \cdot t_{a'}^{n-k-1} t_{z'_{k+1}} t_{a'} t_{y'_{k-1}} t_{\delta'_k} t_{x'_{k-1}} \cdots t_{x'_2} t_{x'_1} \cdot T. \end{aligned}$$

By the lantern relations $t_{z_{k+1}} t_a t_{y_{k-1}} t_{\delta_k} = t_{z_k} t_{y_k} t_{x_k}$ and $t_{z'_{k+1}} t_{a'} t_{y'_{k-1}} t_{\delta'_k} = t_{z'_k} t_{y'_k} t_{x'_k}$, we have

$$\begin{aligned} & t_{z_k} \cdot t_{z_{k+1}} t_{\delta_k} t_{\delta_{k-1}} \cdots t_{\delta_2} t_{\delta_1} \cdot t_{z'_k} \cdot t_{z'_{k+1}} t_{\delta'_k} t_{\delta'_{k-1}} \cdots t_{\delta'_2} t_{\delta'_1} \\ &= t_a^{n-(k+1)} t_{z_k} t_{y_k} t_{x_k} t_{x_{k-1}} \cdots t_{x_2} t_{x_1} \cdot t_{a'}^{n-(k+1)} t_{z'_k} t_{y'_k} t_{x'_k} t_{x'_{k-1}} \cdots t_{x'_2} t_{x'_1} \cdot T. \end{aligned}$$

Because z_k is disjoint from a and z'_k is disjoint from $x_i, y_i, a, a', z_{k+1}, \delta_i$, we can remove t_{z_k} and $t_{z'_k}$ from both sides of this relation. This concludes the proof of the claim.

By considering a subsurface of Σ_g^{2n} of genus g with two boundary components z_1 and z'_1 , we get the positive factorization of $t_{z_1} t'_{z_1}$ given in Lemma 4.5. The proof of the theorem now follows from applying the above argument to this positive factorization. Corresponding to it is the promised family of symplectic Lefschetz pencils $(X'_g(i), f'_g(i))$. Capping all the boundary components induces a well-known homomorphism $\Gamma_g^{2(i+1)} \rightarrow \Gamma_g$, under which the monodromy factorization of this pencil maps to a monodromy factorization of a symplectic Lefschetz fibration $(X_g(i), f_g(i))$, which clearly realizes the blow-up of the pencil at all its base points.

4.3 Topology of the symplectic 4-manifolds $X_g(i)$ and $X'_g(i)$

Here we will investigate the topology of the symplectic manifolds $X'_g(i)$ and therefore that of $X_g(i) \cong X'_g(i) \# 2(i+1)\overline{\mathbb{CP}}^2$, which are the total spaces of the new pencils and fibrations we produced above. We will record all the results for $X'_g(i)$, which then translate to those of $X_g(i)$ through blow-ups.

Lemma 4.7 *For $0 \leq i \leq g$ when g is odd, and $0 \leq i \leq g-1$ when g is even,*

$$e(X'_g(i)) = 12(g-i) \text{ and } \sigma(X'_g(i)) = -8(g-i).$$

When $i \leq g-1$, $X'_g(i)$ is simply-connected, whereas for $i = g$ (when g is odd), we have $\pi_1(X'_g(g)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof The positive factorization $t_{\delta_1} t_{\delta'_1}$ in Theorem 4.6, which gives rise to $X_g(i)$ and $X'_g(i)$, is obtained by applying C_3 -substitutions i times and C_{2g-3} -substitution once to the factorization of $t_{\delta_1} t_{\delta'_1}$ in Proposition 4.1. (Note that $X_g(0) \neq Z_g$ since we still have the latter substitution in effect.) It means that for $g \geq 3$, $X_g(i)$ is obtained by applying 3-unchaining i times and $(2g-3)$ -unchaining once to Z_g , and then $X'_g(i)$ is obtained by blowing-down $X_g(i)$ $2(i+1)$ -times. The Euler characteristics and signature calculations then follow directly from those of Z_g (Subsection 4.1), and Proposition 3.2.

To calculate $\pi_1(X_g(i))$ we can invoke Proposition 3.3, or rely on the handle decomposition induced by the global Lefschetz fibration to carry out a more direct calculation as follows: Let $a_1, b_1, \dots, a_g, b_g$ be generators of $\pi_1(\Sigma_g)$ as shown in Fig. 3 (here we regard Σ_g^2 as a subsurface of Σ_g). Set $\gamma_k = [a_1, b_1] \cdots [a_k, b_k]$, which is represented by a separating curve, and $a_0 = b_0 = a_{g+1} = \gamma_{g+1} = \gamma_0 = \gamma_{-1} = 1$ in $\pi_1(\Sigma_g)$. Here, by $[x, y]$ we mean the commutator $xyx^{-1}y^{-1}$. Note that the curves x_i and x'_i are isotopic to a and a' after filling all boundary components of $\Sigma_g^{2(i+1)}$ (see Fig. 6) and that the curves as in Figs. 4 and 5 are used repeatedly. Up to conjugation in $\pi_1(\Sigma_g)$, we have:

$$\begin{aligned} c_1 &= a_1, \quad c_2 = b_1, \quad c_3 = b_1 a_1^{-1} b_1^{-1} a_2, \quad a = a_2, \quad a' = \gamma_1 a_2, \\ d_{2k} &= b_k a_k^{-1} b_{k-1} b_{k-2} a_{k-2} b_{k-2}^{-1}, \quad 2 \leq k \leq g, \end{aligned}$$

$$\begin{aligned}d_{2k+1} &= a_{k+1}^{-1} b_k b_{k-1} a_{k-1}, \quad 2 \leq k \leq g, \\e_{2k} &= \gamma_{k-1} a_k b_k b_{k-1} a_{k-2}^{-1} \gamma_{k-3}^{-1}, \quad 2 \leq k \leq g, \\e_{2k+1} &= \gamma_k a_{k+1} b_k b_{k-1} a_{k-1}^{-1} \gamma_{k-2}^{-1}, \quad 2 \leq k \leq g\end{aligned}$$

Recall that $\pi_1(X) \cong \pi_1(\Sigma_g) / N$, where N is the subgroup of $\pi_1(\Sigma_g)$ normally generated by the vanishing cycles of the Lefschetz fibration $(X_g(i), f_g(i))$, which are the Dehn twist curves in its monodromy factorization. We then see that the collection of Dehn twist curves in the factorization of kill all the generators through the above identities as follows. The relations $a_2 = d_4 = 1$ give $b_2 b_1 = 1$, especially $b_1 b_2 = b_2 b_1 = 1$. By $d_{2k+2} = d_{2k+1} = 1$ for $2 \leq k \leq g-1$, we obtain $b_{k+1} b_k^{-1} = 1$. Combining two relations, we have $b_k b_{k-1} = 1$ for $2 \leq k \leq g$, and therefore $a_{k+1}^{-1} a_{k-1} = 1$ from $d_{2k+1} = 1$. Moreover, we see that $a_{2j} = 1$ for any j by $a_2 = 1$. The above argument means that $\pi_1(X)$ is generated by a_1 and b_1 . In particular, since the relations $a_2 = \gamma_1 a_2 = 1$ gives $\gamma_1 = 1$, we see that a_1 commutes with b_1 and that the relations $e_{2k} = e_{2k+1} = 1$ are not needed. Finally, by the relations $a_1 = b_1 = 1$, we obtain $\pi_1(X_g(i)) = 1$ for $i = 0, 1, 2, \dots, g-1$.

When g is odd, we no longer have the Dehn twist curves c_1, c_2, c_3 , and without the implied relations $c_1 = c_2 = c_3 = 1$ (yielding $a_1 = b_1 = a_2 = 1$, and in turn $b_2 = 1$), the remaining relations give $\pi_1(X_g(g)) \cong \langle a_1, b_1 \mid \gamma_1 = 1 \rangle = \mathbb{Z} \oplus \mathbb{Z}$.

Finally, armed with our deeper knowledge of the presence of —sufficiently many— exceptional sections in each Lefschetz fibration $(X_g(i), f_g(i))$, we can utilize Theorem 2.3 to determine the symplectic Kodaira dimension of majority of these manifolds:

Theorem 4.8 (Symplectic Kodaira dimension of $X'_g(i)$) *Let $g \geq 3$, $0 \leq i \leq g$ when g is odd, and $0 \leq i \leq g-1$ when g is even. In all of the following cases, $X'_g(i)$ is a symplectic 4-manifold with Kodaira dimension*

$$\kappa(X'_g(i)) = \begin{cases} -\infty & \text{if } i > g-2 \\ 0 & i = g-2 \\ 1 & i = g-3 \text{ or } i = g-2n, \end{cases}$$

where n is any integer satisfying $2 \leq n \leq \lfloor \frac{g}{2} \rfloor$. In all of the cases above with $i \leq g-2$, $X'_g(i)$ is spin, and thus minimal. Moreover, $X'_g(g-1) \cong E(1)$ and $X'_g(g) \cong S^2 \times T^2$ when g is odd.

Proof First of all, let us observe that, since the Euler characteristic and signature satisfy $e = 2 - 2b_1 + b_2^+ + b_2^-$ and $\sigma = b_2^+ - b_2^-$, our π_1 calculations in Lemma 4.7 imply that for $g \geq 3$, and $i = 0, 1, 2, \dots, g-2$, $b_1(X'_g(i)) = 0$, $b_2^+(X'_g(i)) = 2(g-i) - 1$ and $b_2^-(X'_g(i)) = 10(g-i) - 1$. So $X'_g(i)$ (and $X_g(i)$) is neither a rational nor a ruled surface when $i = 0, 1, 2, \dots, g-2$.

With this observation, the cases of $\kappa(X'_g(i)) = -\infty$ and 0 are now immediate from Theorem 2.3. It remains to prove that when $i = g-2n$ or $i = g-3$, $\kappa(X'_g(i)) = 1$.

By Lemma 5.6 in the next section, $X'_g(g-2n)$ is spin, and in particular, minimal. We can further deduce from the Euler characteristic and signature calculations in Lemma 4.7 that $K_{X'_g(g-2n)}^2 = 2e(X'_g(g-2n)) + 3\sigma(X'_g(g-2n)) = 0$. Moreover, $b_2^+(X'_g(g-2n)) > 3$ by the above calculation, so it cannot have Kodaira dimension zero by [38] (see also [6]). It follows that $\kappa(X'_g(g-2n)) = 1$.

By the above calculations again, $X'_g(g-3)$ is neither rational nor ruled, and since $b_2^+(X'_g(g-3)) > 3$, it cannot have Kodaira dimension zero either. Next, consider the corresponding Lefschetz fibration $(X_g(g-3), f_g(g-3))$ obtained by blowing up all the base points

of the pencil in hand, so it has $2g - 4$ exceptional sections. If there is any other exceptional sphere in $(X_g(g-3), f_g(g-3))$, disjoint from these $2g - 4$ exceptional spheres, it would necessarily intersect the fiber positively, at least once. So a maximal collection of disjoint exceptional spheres would be intersecting the fiber at least $2g - 3$ times. By [49][Theorem 5–12], this would imply that $\kappa(X_g(g-3)) < 2$, so the Kodaira dimension here would be 1. On the other hand, if there is no such exceptional sphere, that is, if $X'_g(g-3)$ is minimal, by Lemma 4.7, we have $K_{X'_g(g-3)}^2 = 2e(X'_g(g-3)) + 3\sigma(X'_g(g-3)) = 0$, which also implies $\kappa(X_g(g-3)) = 1$.

For the diffeomorphism statements, recall that when $\kappa = -\infty$ we either have a rational or ruled surface in hand. So from our calculation of the topological invariants in Lemma 4.7, we deduce that $X'_g(g-1) \cong E(1)$, and $X'_g(g)$ is diffeomorphic to either $S^2 \times T^2$ or $S^2 \tilde{\times} T^2$. As we will see that $X'_g(g)$ is spin in Lemma 5.6, we conclude that $X'_g(g) \cong S^2 \times T^2$.

Remark 4.9 For the values of i that are not covered in the theorem, we expect that $\kappa(X'_g(i)) = 1$, which would be immediate once these manifolds are seen to be minimal.

Among our families, the pencils $(K_g, k_g) := (X'_g(g-2), f'_g(g-2))$ are perhaps the most interesting. It follows from our calculations of the topological invariants and by Freedman's theorem that each K_g is a symplectic Calabi–Yau surface homeomorphic to the K3 surface. Moreover, any symplectic homotopy K3 surface with trivial canonical class is known to have only one Seiberg–Witten basic class [43], and so does K_g in particular.

Corollary 4.10 (Pencils on symplectic Calabi–Yau homotopy K3 surfaces) *For each $g \geq 3$, there is a symplectic genus- g Lefschetz pencil (K_g, k_g) , where K_g is a symplectic Calabi–Yau homotopy K3 surface, and k_g has the monodromy factorization:*

$$\begin{aligned} & t_{\delta_{g-1}} \cdots t_{\delta_2} t_{\delta_1} t_{\delta'_{g-1}} \cdots t_{\delta'_2} t_{\delta'_1} \\ &= \begin{cases} t_{x_{g-1}} \cdots t_{x_1} t_{x'_g} \cdots t_{x'_1} (t_{c_1} t_{c_2} t_{c_3})^8 t_{d_4} \cdots t_{d_{2g+1}} t_{e_{2g+1}} \cdots t_{e_4} & (g : \text{odd}) \\ t_{x_{g-1}} \cdots t_{x_1} t_{x'_g} \cdots t_{x'_1} (t_{c_1} t_{c_2} t_{c_3})^6 (t_{c_3} t_{c_2} t_{c_1})^2 t_{d_4} \cdots t_{d_{2g+1}} t_{e_{2g+1}} \cdots t_{e_4} & (g : \text{even}) \end{cases} \end{aligned}$$

in Γ^{2g-2} , where the curves $\delta_j, \delta'_j, c_j, d_j, e_j, x_j, x'_j$ are as in Figs. 4, 5 and 6.

A natural question, which might be tractable through the explicit handle diagrams prescribed by the factorizations above is:

Question 4.11 *Are the symplectic Calabi–Yau homotopy K3 surfaces $K_g, g \geq 3$, all diffeomorphic to the standard K3 surface?*

Remark 4.12 Observe that, we can view K_g to be obtained from the rational surface $E(1)$ by the natural inverse of the unchaining surgery. We speculate that when this inverse operation can be performed in the complex category, it would correspond to contracting a (-2) -curve (of genus $(g-2)$ for these examples) and then smoothing the singularity. This leads to another interesting question: what are the symplectic Calabi–Yau surfaces one can get by contracting a symplectic (-2) -curve in a rational surface?

5 Pencils on spin 4-manifolds and Stipsicz's conjecture

In this section, we will first extend a result of Stipsicz in [52], which gave a characterization of a Lefschetz fibration to be a spin 4-manifold, to that of Lefschetz pencils (Theorem 5.1).

These are given in terms of the $\mathbb{Z}_2$ -homology classes of the vanishing cycles and exceptional sections of the associated positive factorization. We will then address Stipsicz's conjecture on the existence of (-1) -sections in fiber sum indecomposable Lefschetz fibrations [51] by providing counter-examples of any genus $g \geq 2$ (Theorem 5.4).

5.1 Spin structures on Lefschetz pencils

A $\mathbb{Z}_2$ -valued function $q : H_1(\Sigma_g^p; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ is said to be a *quadratic form with respect to the intersection pairing* if $q(a+b)$ is equal to $q(a) + q(b) + \langle a, b \rangle_2$ for any $a, b \in H_1(\Sigma_g^p; \mathbb{Z}_2)$, where $\langle a, b \rangle_2 \in \mathbb{Z}_2$ is the mod 2 intersection pairing. Here is our characterization of whether the total space of a Lefschetz pencil is spin, in terms of the associated positive factorization:

Theorem 5.1 (Spin characterization from pencil monodromies) *Let (X, f) be a genus- g Lefschetz pencil with a monodromy factorization $t_{c_1} \cdots t_{c_n} = t_{\delta_1} \cdots t_{\delta_p}$. Then X admits a spin structure if and only if there exists a quadratic form $q : H_1(\Sigma_g^p; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ with respect to the intersection pairing such that $q(c_i) = 1$ for any i and $q(\delta_j) = 1$ for some j .*

Since we can obtain the total space of a Lefschetz pencil by attaching 4-handles to the complement of (-1) -sections of a Lefschetz fibration, Theorem 5.1 will follow from the following lemma:

Lemma 5.2 *Let (X, f) be a Lefschetz fibration, $p \geq 1$ be an integer, $S_1, \dots, S_p \subset X$ be disjoint sections of f and $t_{c_1} \cdots t_{c_n} = t_{\delta_1}^{a_1} \cdots t_{\delta_p}^{a_p}$ be the corresponding monodromy factorization (note that $a_i \in \mathbb{Z}$ is the self-intersection of S_i). For $S = \sqcup_i S_i$, the disjoint union of S_i 's, the complement $X \setminus S$ admits a spin structure if and only if there exists a quadratic form $q : H_1(\Sigma_g^p; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ with respect to the intersection pairing, which satisfies:*

- (A) $q(c_i) = 1$ for any i ,
- (B) $q(\delta_j) \equiv a_j$ for some j .

The lemma above still holds for $p = 0$ provided that there exists a “homological dual” of a regular fiber with even self-intersection. See [52] for details.

Proof Define the sets S_1 and S_2 as follows:

- S_1 is the set of isomorphism classes of spin structures on Σ_g^p .
- S_2 is the set of quadratic forms on $H_1(\Sigma_g^p; \mathbb{Z}_2)$ with respect to the intersection pairing.

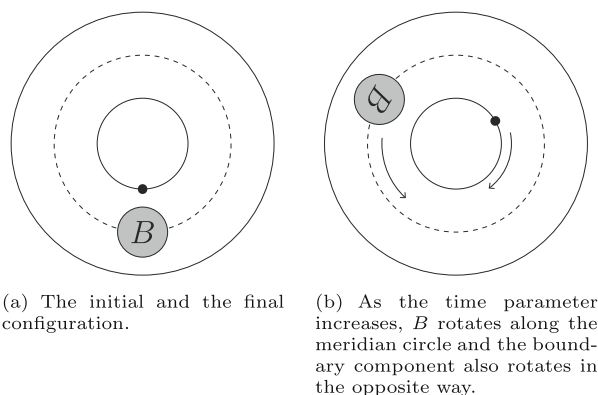
Note that both of the sets S_1 and S_2 admit free and effective actions of $H^1(\Sigma_g^p; \mathbb{Z}_2)$. In the same way as that in [33] (in which the author dealt with the spin structures on Σ_g), one can define a map $Q : S_1 \rightarrow S_2$ as follows: for a spin structure $\mathfrak{s}$ of Σ_g^p , define a function $Q(\mathfrak{s})$ so that it takes the value 0 on $c \in H_1(\Sigma_g^p; \mathbb{Z}_2)$ if the restriction of $\mathfrak{s}$ on a circle representing c can be extended to a spin structure over a 2-manifold and takes the value 1 otherwise. One can further show that the map Q is equivariant with respect to the action of $H^1(\Sigma_g^p; \mathbb{Z}_2)$, in particular Q is bijective. (See [33] for details.)

Following the definition of spin structures given in [52], one can easily deduce that there exists a one-to-one correspondence between S_1 and the set of isomorphism classes of spin structures on $D^2 \times \Sigma_g^p$. In what follows, we will identify these two sets by this correspondence.

The complement $X \setminus S$ can be decomposed as follows:

$$X \setminus S = D^2 \times \Sigma_g^p \cup (h_1 \sqcup \cdots \sqcup h_n) \cup D^2 \times \Sigma_g^p, \quad (8)$$

Fig. 7 The inner circles are the boundary components near δ_j , while the outer circles are parallel to ∂A



where h_i is a 2-handle attached along the vanishing cycle c_i . We denote the subset $D^2 \times \Sigma_g^p \cup (h_1 \sqcup \dots \sqcup h_n) \subset X \setminus S$ by X' . As is shown in the proof of [52, Theorem 1.1], the condition (A) in Lemma 5.2 holds for a quadratic form if and only if the associated spin structure on $D^2 \times \Sigma_g^p$ can be extended to that on X' . The latter $D^2 \times \Sigma_g^p$ in the decomposition (8) can be regarded as the union of a 2-handle $D^2 \times B$, where B is a small ball close to the boundary component near δ_j , and $2g + p - 1$ 3-handles. Thus, $X \setminus S$ admits a spin structure if and only if there exists a spin structure on X' which can be extended to that on $X' \cup (D^2 \times B)$.

In what follows, we identify a surface Σ_g^p with a fiber in $\partial X'$. Let A be an annulus neighborhood of the boundary component near δ_j which contains B and is away from any of the vanishing cycles $c_1, \dots, c_n$. We take a parallel transport self-diffeomorphism φ of Σ_g^p along the boundary of $f(X')$ so that it preserves $\partial \Sigma_g^p$ and A point-wise. We can then obtain the following diffeomorphism:

$$\partial X' \cong ([0, 1] \times \Sigma_g^p)/(1, x) \sim (0, \varphi(x)).$$

By the assumption φ is isotopic (relative to A) to the a_j -th power of the Dehn twist diffeomorphism along a simple closed curve near ∂A . The latter diffeomorphism is further isotopic to the identity via the isotopy (supported on a neighborhood of A) described in Fig. 7.

As the figure shows, the isotopy makes B rotate around the boundary component a_j times keeping the inward tangent vector at the center of B inward, and also makes the boundary component $-a_j$ times. We denote the concatenation of the two isotopies by φ_t ($t \in [0, 1]$). Identifying $\partial D^2 \times \Sigma_g^p$ with $([0, 1] \times \Sigma_g^p)/(1, x) \sim (0, x)$, we can explicitly give an attaching map of $D^2 \times \Sigma_g^p$ to X' as follows:

$$([0, 1] \times \Sigma_g^p)/(1, x) \sim (0, x) \ni (t, x) \mapsto (t, \varphi_t(x)) \in ([0, 1] \times \Sigma_g^p)/(1, x) \sim (0, \varphi(x)).$$

We denote this attaching map by Φ . The attaching map of the 2-handle $D^2 \times B$ is the restriction $\Phi|_{\partial D^2 \times B}$. For a given $\mathbb{Z}_2$ -valued quadratic function q on $H_1(\Sigma_g^p; \mathbb{Z}_2)$, the restriction of the associated spin structure on the image $\Phi(\partial D^2 \times B)$ bounds a spin 2-manifold if $q(a_j \delta_j) = 0$ and does not bound otherwise. Moreover, it is easy to deduce from the definition of Φ that the pull-back of a spin structure $\mathfrak{s}$ on $\Phi(\partial D^2 \times B)$ by $\Phi|_{\partial D^2 \times B}$ bounds a spin 2-manifold if and only if either a_j is odd and $\mathfrak{s}$ does not bound a spin 2-manifold, or a_j is even and $\mathfrak{s}$ bounds a spin 2-manifold. The latter condition is equivalent to the condition (B) in Lemma 5.2.

Remark 5.3 The proof of Lemma 5.2 also shows that for a quadratic form q with the condition (A), the condition (B) holds if and only if $q(\delta_j) \equiv a_j$ for any j . Furthermore, if we can find

a quadratic form with the condition (A) and one of the section $S_1, \dots, S_p$ has even self-intersection, the condition (B) is automatically satisfied. Indeed, q induces a quadratic form on $H_1(\Sigma_g; \mathbb{Z}_2)$ and it can be extended to that on $H_1(\Sigma_g^p; \mathbb{Z}_2)$ so that the values of δ_i 's are all zero. (An alternative way to deduce this observation is to find a spin structure on X by applying Stipsicz's result to the induced quadratic form on $H_1(\Sigma_g; \mathbb{Z}_2)$.)

5.2 Stipsicz's conjecture on exceptional sections

We will now prove that:

Theorem 5.4 (Counter-examples to Stipsicz's conjecture) *For any $g \geq 3$, there exists a genus- g fiber-sum indecomposable Lefschetz fibration without any exceptional sections.*

Unlike the handful of earlier counter-examples with $g = 2, 3$ [3, 11, 48], these examples will have Kodaira dimension 1 when $g \geq 4$.

We will describe these counter-examples with explicit monodromy factorizations. The spin characterization in Theorem 5.1 will play a vital role here to pin down the exact number of exceptional spheres in the total spaces.

Let $c_1, \dots, c_{2g+1}, a, b, a', b' \subset \Sigma_g^2$ be the simple closed curves shown in Fig. 4. As we did in the proof of Theorem 4.6, for any $n \geq 1$ we regard Σ_g^2 as a subsurface of Σ_g^{2n} with two boundary components z_1, z'_1 so that the curves a, b, a', b' are embedded in Σ_g^{2n} as shown in Fig. 6. In what follows, we use symbols representing simple closed curves in Σ_g^{2n} (such as c_i, a, b) to represent homology classes in $H_1(\Sigma_g^{2n}; \mathbb{Z}_2)$ represented by the corresponding curves.

Lemma 5.5 *For any $g \geq 3$ and $n \geq 1$ such that $g + n$ is odd, there exists a quadratic form $q : H_1(\Sigma_g^{2n}; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ with respect to the intersection pairing which satisfies*

1. $q(c_1) = \dots = q(c_{2g+1}) = 1$,
2. $q(d_4) = \dots = q(d_{2g+1}) = q(e_4) = \dots = q(e_{2g+1}) = 1$,
3. $q(x_1) = \dots = q(x_n) = q(x'_1) = \dots = q(x'_n) = 1$, and
4. $q(\delta_1) = \dots = q(\delta_n) = q(\delta'_1) = \dots = q(\delta'_n) = 1$.

Proof Since the elements $c_1, \dots, c_{2g}, \delta_1, \dots, \delta_n, \delta'_1, \dots, \delta'_n$ form a basis of $H_1(\Sigma_g^{2n}; \mathbb{Z}_2)$, there exists a quadratic form $q : H_1(\Sigma_g^{2n}; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ which assigns the value 1 to all the elements in the basis. The following equalities (in $H_1(\Sigma_g^{2n}; \mathbb{Z}_2)$) can be verified easily:

$$\begin{aligned} c_{2g+1} &= c_1 + c_3 + \dots + c_{2g-1} + \delta_1 + \delta_2 + \dots + \delta_n, \\ d_j &= e_j = c_{j-3} + c_{j-2} + c_{j-1} + c_j, \\ x_i &= c_1 + c_3 + \delta_i, \quad x'_i = c_1 + c_3 + \delta'_i, \\ \delta'_1 &= \delta_1 + \dots + \delta_n + \delta'_2 + \dots + \delta'_n. \end{aligned}$$

Using these equalities, we can calculate values of q as follows:

$$\begin{aligned} q(c_{2g+1}) &= \sum_{i=1}^g q(c_{2i-1}) + \sum_{j=1}^n q(\delta_j) = g + n \equiv 1, \\ q(d_j) &= q(e_j) = \sum_{i=0}^3 q(c_j - i) + \sum_{0 \leq i < k \leq 3} \langle c_{j-i}, c_{j-k} \rangle_2 = 4 + 3 \equiv 1, \end{aligned}$$

$$q(x_i) = q(c_1) + q(c_3) + q(\delta_i) = 3 \equiv 1, \quad q(x'_i) = q(c_1) + q(c_3) + q(\delta'_i) = 3 \equiv 1,$$

$$q(\delta'_1) = \sum_{i=1}^n q(\delta_i) + \sum_{i=2}^n q(\delta'_i) = 2n - 1 \equiv 1.$$

We can thus conclude that the form q satisfies the desired conditions.

Let $X'_g(i)$ be the symplectic genus- g Lefschetz pencil with $2(i+1)$ base points we constructed in Sect. 4, which has the following monodromy factorization:

$$t_{\delta_{i+1}} \cdots t_{\delta_1} t_{\delta'_{i+1}} \cdots t_{\delta'_1} \\ = \begin{cases} t_{x_{i+1}} \cdots t_{x_1} t_{x'_{i+1}} \cdots t_{x'_1} (t_{c_1} t_{c_2} t_{c_3})^{4(g-i)} D_g E_g & (g : \text{odd}), \\ t_{x_{i+1}} \cdots t_{x_1} t_{x'_{i+1}} \cdots t_{x'_1} (t_{c_1} t_{c_2} t_{c_3})^{4(g-i)-2} (t_{c_3} t_{c_2} t_{c_1})^2 D_g E_g & (g : \text{even}). \end{cases}$$

Lemma 5.6 *The manifold $X'_g(i)$ is spin if and only if $g+i$ is even.*

Proof Since the signature of $X'_g(i)$ is equal to $-8(g-i)$ (see Lemma 4.7), $X'_g(i)$ is not spin if $g+i$ is odd. On the other hand, when $g+i$ is even, the quadratic form obtained in Lemma 5.5 above satisfies the necessary conditions in Theorem 5.1, so $X'_g(i)$ is spin in this case.

We are now ready to prove the main result of this subsection:

Proof of Theorem 5.4 Suppose that g is even. The manifold $X'_g(0)$ is spin by Lemma 5.6, and in particular minimal. Furthermore, we can deduce from Theorem 4.8 that the Kodaira dimension of $X_g(0) = X'_g(0) \# 2\overline{\mathbb{CP}}^2$ is equal to 1. The monodromy factorization of the Lefschetz fibration on $X_g(0)$ can be changed by Hurwitz moves as follows:

$$t_{x_1} t_{x'_1} (t_{c_1} t_{c_2} t_{c_3})^{4g-2} (t_{c_3} t_{c_2} t_{c_1})^2 D_g E_g \\ \sim \underline{t_{x_1} t_{x'_1} t_{c_1}^2 t_{c_1}^{-1} t_{c_2}} t_{c_3} t_{c_2} t_{c_3} (t_{c_1} t_{c_2} t_{c_3})^{4g-4} (t_{c_3} t_{c_2} t_{c_1})^2 D_g E_g.$$

The curves $x_1, x'_1, \delta_1, \delta'_1$ and two disjoint curves parallel to c_1 bound a sphere with six boundary components. Thus, we can apply the *braiding lantern substitution* ([11, Lemma 5.1]) to the underlined part above. This substitution replaces two disjoint exceptional sections with an exceptional bisection. Furthermore, by [18, Theorem 3.1]), this amounts to a rational blowdown of a symplectic (-4) -sphere which can be viewed as the union of the four holed sphere on the fiber and the four disjoint vanishing cycles in the lantern configuration, whereas by [27, Lemma 5.1] (also see [11, Proposition 6.1]), such a rational blowdown has the same effect as regular blowdown whenever this (-4) -sphere intersects an exceptional sphere once, which is the case here. (Either one of the exceptional spheres corresponding to the two boundary twists in the braiding lantern configuration hits the (-4) -sphere once.) So the resulting symplectic 4-manifold is $X'_g(0) \# \overline{\mathbb{CP}}^2$. Since $X'_g(0)$ is minimal and has Kodaira dimension 1 (especially not rational nor ruled), the homology class represented by the bisection produced by the braiding lantern substitution above is the only class in $H_2(X'_g(0) \# \overline{\mathbb{CP}}^2; \mathbb{Z})$ (up to sign) represented by a (-1) -sphere (see [36, Corollary 3]), in particular there is no (-1) -spheres intersecting a fiber of the Lefschetz fibration on $X'_g(0) \# \overline{\mathbb{CP}}^2$ at one point transversely. Thus, the Lefschetz fibration we obtained after the braiding lantern substitution cannot admit any exceptional sections.

Next, assume g is odd. Once again, the manifold $X'_g(1)$ is spin by Lemma 5.6, and therefore minimal. Furthermore, it follows from Theorem 4.8 that $\kappa(X_g(1)) = 1$ if $g \geq 5$,

and $\kappa(X_3(1)) = 0$. The manifold $X_g(1)$ admits a Lefschetz fibration with the following monodromy factorization:

$$\begin{aligned} t_{\delta_2} t_{\delta_1} t_{\delta'_2} t_{\delta'_1} &= t_{x_2} t_{x_1} t_{x'_2} t_{x'_1} (t_{c_1} t_{c_2} t_{c_3})^{4g-4} D_g E_g \\ &\sim \overline{t_{x_2} t_{x'_2} t_{c_1}^2} \cdot \overline{t_{x_1} t_{x'_1} t_{c_3}^2 t_{c_1}^{-1} t_{c_3}^{-1} (c_2) t_{c_3}^{-1} (c_2)} (t_{c_1} t_{c_2} t_{c_3})^{4g-6} D_g E_g. \end{aligned}$$

As we did above, we can apply the braiding lantern substitution at the underlined parts. The resulting Lefschetz fibration has two exceptional bisections, and its total space is $X'_g(1) \# 2\mathbb{CP}^2$, by the same arguments as above. Since $X'_g(1)$ is minimal and has non-negative Kodaira dimension (especially not rational nor ruled), one can again deduce from [36, Corollary 3] that the homology classes represented by the two bisections (produced by the braiding lantern substitutions) are the only classes (up to sign) represented by exceptional spheres. Thus, the Lefschetz fibration on $X'_g(1) \# 2\mathbb{CP}^2$ cannot admit any exceptional sections.

Finally, by Usher's theorem on minimality of symplectic fiber sums [54, Corollary 1.2] (also see [7, Theorem 1]) the presence of exceptional spheres in total spaces of all the Lefschetz fibrations above imply that none can be a fiber sum of nontrivial Lefschetz fibrations.

Remark 5.7 There are many more counter-examples one can produce using similar arguments and ingredients. Note that the product $t_{x_{i+1}} \cdots t_{x_1} t_{x'_{i+1}} \cdots t_{x'_1}$ is Hurwitz equivalent to $t_{x_{i+1}} t_{x'_{i+1}} \cdots t_{x_1} t_{x'_1}$. We can change the product

$$(t_{c_1} t_{c_2} t_{c_3})^{4(g-i)-l} (t_{c_3} t_{c_2} t_{c_1})^l \quad (\text{where } l = 0 \text{ or } 2)$$

to $t_{c_1}^{4(g-i)} t_{c_3}^{4(g-i)} W$ via Hurwitz moves, where W is some product of Dehn twists. Thus, as we did in the proof of Theorem 5.4, we can apply braiding lantern substitution to the Lefschetz fibration on $X_g(i)$ obtained in Sect. 4 so that the resulting Lefschetz fibration has $i + 1$ exceptional bisections, when $i + 1$ is less than or equal to $4(g - i)$. If $g + i$ is even and $i \leq g - 2$, then we can further prove that the resulting fibration cannot admit any exceptional sections. Hence, we record that at least for any $g \leq 9$, there exists a genus- g Lefschetz fibration (X, f) , which is a counter-example to the Stipsicz conjecture, where $\kappa(X) = 0$.

6 Further applications

In this last section, we will show that by combining unchaining surgery with rational blow-downs, we can produce further interesting examples. The two applications we present here, one regarding the topology of symplectic 4-manifolds and one regarding that of pencils, will utilize the Lefschetz pencils $X'_g(i)$ we constructed using unchaining.

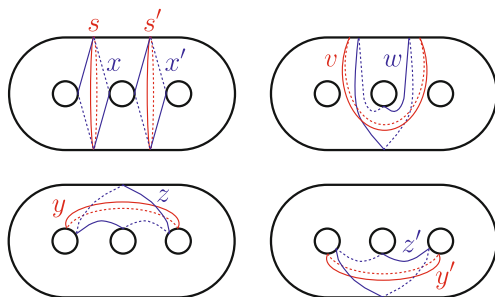
6.1 Exotic 4-manifolds with $b^+ = 3$ via genus-3 fibrations

In this subsection we will prove the following theorem:

Theorem 6.1 *There are genus-3 Lefschetz fibrations (X_j, f_j) , for $j = 0, 1, 2, 3$, where each X_j is a minimal symplectic 4-manifold homeomorphic but not diffeomorphic to $3\mathbb{CP}^2 \# (19 - j)\mathbb{CP}^2$, and each f_{j+1} has a monodromy factorization obtained from that of f_j by a lantern substitution.*

We will need the following lemma in our proof:

Fig. 8 The curves in a genus-3 surface



Lemma 6.2 *Let the curves c_j, a, a' be as in Figs. 4 and 5. One can perform 7 consecutive lantern substitutions within the product $t_a^2 t_{a'}^2 t_{c_1}^6 t_{c_3}^3 t_{c_5}^6 t_{c_7}^6$ in Γ_3 , up to Hurwitz moves.*

Proof Let $x, x', y, y', z, z', w, s, s', v$ be the curves given in Σ_3 as in Fig. 8. Note that x, x', y, y', z, z', w are non-separating and s, s', v are separating. We have the following lantern relators L_1, L'_1, L_2, L'_2 and L_3 :

$$\begin{aligned} L_1 &= t_{c_3} t_s t_x t_a^{-1} t_{c_1}^{-1} t_{c_1}^{-1} t_{a'}^{-1}, & L'_1 &= t_{c_5} t_{s'} t_{x'} t_{c_7}^{-1} t_a^{-1} t_{a'}^{-1} t_{c_7}^{-1}, \\ L_2 &= t_a t_y t_z t_{c_1}^{-1} t_{c_5}^{-1} t_{c_3}^{-1} t_{c_7}^{-1}, & L'_2 &= t_{a'} t_{y'} t_{z'} t_{c_1}^{-1} t_{c_5}^{-1} t_{c_3}^{-1} t_{c_7}^{-1}, \\ L_3 &= t_{a'} t_v t_w t_a^{-1} t_s^{-1} t_{s'}^{-1} t_a^{-1}. \end{aligned}$$

For W_1 and W_2 any two products of Dehn twists in Γ_g , we will write $W_1 \xrightarrow{L} W_2$ when W_2 is obtained by applying a lantern substitution to W_1 using the lantern relator L . Since

$$t_a^2 t_{a'}^2 t_{c_1}^6 t_{c_3}^3 t_{c_5}^6 t_{c_7}^6 \sim t_{c_1}^4 t_{c_3}^3 t_{c_5}^4 t_{c_7}^4 \cdot t_{c_7} t_{a'} t_a t_{c_7} \cdot t_{a'} t_{c_1} t_{c_1} t_a,$$

applying L_1 -substitutions gives

$$t_{c_1}^4 t_{c_3}^3 t_{c_5}^4 t_{c_7}^4 \cdot t_{c_7} t_{a'} t_a t_{c_7} \cdot t_{a'} t_{c_1} t_{c_1} t_a \xrightarrow{L_1} t_{c_1}^4 t_{c_3}^3 t_{c_5}^4 t_{c_7}^4 \cdot t_{c_7} t_{a'} t_a t_{c_7} \cdot t_{c_3} t_s t_x.$$

Moreover, because

$$t_{c_1}^4 t_{c_3}^3 t_{c_5}^4 t_{c_7}^4 \cdot t_{c_7} t_{a'} t_a t_{c_7} \cdot t_{c_3} t_s t_x \sim (t_{c_7} t_{c_5} t_{c_3} t_{c_1})^3 \cdot t_{c_1} t_{c_3} t_{c_7} \cdot t_{c_7} t_{a'} t_a t_{c_7} \cdot t_s t_x,$$

by applying L_2 -substitutions twice and L'_2 -substitution once we obtain

$$\begin{aligned} & (t_{c_7} t_{c_5} t_{c_3} t_{c_1})^3 \cdot t_{c_1} t_{c_3} t_{c_7} \cdot t_{c_7} t_{a'} t_a t_{c_7} \cdot t_s t_x \\ & \xrightarrow{L_2, L_2, L'_2} (t_a t_y t_z)^2 (t_{a'} t_{y'} t_{z'}) \cdot t_{c_1} t_{c_3} t_{c_7} \cdot t_{c_7} t_{a'} t_a t_{c_7} \cdot t_s t_x. \end{aligned}$$

We can further apply L'_1 and L'_2 -substitutions as follows:

$$\begin{aligned} & (t_a t_y t_z)^2 (t_{a'} t_{y'} t_{z'}) \cdot t_{c_1} t_{c_3} t_{c_7} \cdot t_{c_7} t_{a'} t_a t_{c_7} \cdot t_s t_x \\ & \xrightarrow{L'_1} (t_a t_y t_z)^2 (t_{a'} t_{y'} t_{z'}) \cdot t_{c_1} t_{c_3} t_{c_7} \cdot t_{c_5} t_{s'} t_{x'} \cdot t_s t_x \\ & \xrightarrow{L'_2} (t_a t_y t_z)^2 (t_{a'} t_{y'} t_{z'})^2 \cdot t_{s'} t_{x'} \cdot t_s t_x. \end{aligned}$$

Here, note that $t_a t_y t_z \sim t_y t_z t_a$, which follows from the lantern relation $t_a t_y t_z = t_{c_1} t_{c_3} t_{c_5} t_{c_7}$, as a is disjoint from c_1, c_3, c_5, c_7 . Therefore, we may further rewrite the product above as

$$(t_a t_y t_z)^2 (t_{a'} t_{y'} t_{z'})^2 \cdot t_{s'} t_{x'} \cdot t_s t_x \\ \sim (t_y t_z)^2 (t_{a'} t_{y'} t_{z'})^2 \cdot t_a^2 t_s t_{s'} \cdot t_x t_{x'}.$$

Finally, by applying L_3 we obtain

$$(t_y t_z)^2 (t_{a'} t_{y'} t_{z'})^2 \cdot t_a^2 t_s t_{s'} \cdot t_x t_{x'} \xrightarrow{L_3} (t_y t_z)^2 (t_{a'} t_{y'} t_{z'})^2 \cdot t_a t_v t_w \cdot t_x t_{x'}.$$

This finishes the proof.

The proof of the theorem will now follow from applying the above lemma to the positive factorization for the Lefschetz fibration $X_3(1)$ we obtained by unchaining.

Proof of Theorem 6.1 Let (X, f) be the genus-3 Lefschetz fibration $(X_3(1), f_3(1))$ on 4-fold blow-up of a symplectic Calabi–Yau homotopy K3-surface, whose monodromy is:

$$t_{\delta_2} t_{\delta_1} t_{\delta'_2} t_{\delta'_1} = t_{x_2} t_{x_1} t_{x'_2} t_{x'_1} (t_{c_1} t_{c_2} t_{c_3})^8 t_{d_4} t_{d_5} t_{d_6} t_{d_7} t_{e_7} t_{e_6} t_{e_5} t_{e_4}.$$

We will show that this monodromy factorization is Hurwitz equivalent to a positive factorization which contains the product in Lemma 6.2.

We can check at once that $t_{d_4} t_{d_5} t_{d_6} t_{d_7} (c_{i+4}) = c_i$ for $i = 1, 2, 3$. This gives

$$t_{c_1} t_{c_2} t_{c_3} \cdot t_{d_4} t_{d_5} t_{d_6} t_{d_7} \sim t_{d_4} t_{d_5} t_{d_6} t_{d_7} \cdot t_{c_5} t_{c_6} t_{c_7}.$$

Therefore, in Γ_3 , the following relation holds

$$1 = t_a^2 t_{a'}^2 (t_{c_1} t_{c_2} t_{c_3})^8 t_{d_4} t_{d_5} t_{d_6} t_{d_7} t_{e_7} t_{e_6} t_{e_5} t_{e_4} \\ \sim t_a^2 t_{a'}^2 (t_{c_1} t_{c_2} t_{c_3})^4 t_{d_4} t_{d_5} t_{d_6} t_{d_7} (t_{c_5} t_{c_6} t_{c_7})^4 t_{e_7} t_{e_6} t_{e_5} t_{e_4}. \quad (9)$$

Here, using the braid relators and the disjointness of c_1 and c_3 we obtain

$$t_{c_1} t_{c_2} t_{c_3} \cdot t_{c_1} t_{c_2} t_{c_3} \cdot t_{c_1} t_{c_2} t_{c_3} \cdot t_{c_1} t_{c_2} t_{c_3} \sim t_{c_1} \cdot t_{c_1} t_{c_2} t_{c_3} \cdot t_{c_1} t_{c_2} \cdot t_{c_1} t_{c_2} t_{c_3} \cdot t_{c_1} t_{c_2} t_{c_3} \\ \sim t_{c_1} \cdot t_{c_1} t_{c_2} t_{c_3} \cdot t_{c_1} \cdot t_{c_1} t_{c_2} t_{c_3} \cdot t_{c_1} \cdot t_{c_1} t_{c_2} t_{c_3}.$$

That is, we have

$$(t_{c_1} t_{c_2} t_{c_3})^4 \sim (t_{c_1}^2 t_{c_2} t_{c_3})^3 \sim t_{c_1}^6 t_{c_3}^3 t_{c_1}^{-4} t_{c_3}^{-3} (c_2) t_{c_1}^{-2} t_{c_3}^{-2} (c_2) t_{c_3}^{-1} (c_2).$$

Similarly, we get

$$(t_{c_5} t_{c_6} t_{c_7})^4 \sim (t_{c_5} t_{c_6} t_{c_7}^2)^3 \sim t_{c_5} (c_6) t_{c_5}^2 t_{c_7}^2 (c_6) t_{c_5}^3 t_{c_7}^4 (c_6) t_{c_5}^3 t_{c_7}^6.$$

Since c_5 and c_7 are disjoint from a, a', c_1, c_2, c_3 , the above arguments, the relation (9) and a cyclic permutation then give

$$1 = t_a^2 t_{a'}^2 t_{c_1}^6 t_{c_3}^3 t_{c_5}^3 t_{c_7}^6 \cdot t_{c_1}^{-4} t_{c_3}^{-3} (c_2) t_{c_1}^{-2} t_{c_3}^{-2} (c_2) t_{c_3}^{-1} (c_2) \cdot t_{d_4} t_{d_5} t_{d_6} t_{d_7} \cdot T_2,$$

where $T_2 = t_{c_5} (c_6) t_{c_5}^2 t_{c_7}^2 (c_6) t_{c_5}^3 t_{c_7}^4 (c_6) \cdot t_{c_5}^3 t_{c_7}^6 (c_7) t_{c_5}^3 t_{c_7}^6 (e_6) t_{c_5}^3 t_{c_7}^6 (e_5) t_{c_5}^3 t_{c_7}^6 (e_4)$. As we applied in the proof of Lemma 6.2, we can apply 7 consecutive Lantern substitutions to the monodromy of (X, f) , which yield new symplectic genus-3 Lefschetz fibrations (X_k, f_k) , for $k = 1, \dots, 7$, where X_k is obtained from X by rationally blowing-down k symplectic (-4) -spheres. (Note that we apply the 7 Lantern substitutions in the same order as we did in the proof of Lemma 6.2.)

Let us first determine the homeomorphism type of each X_k . From the algebraic topological invariants of $(X, f) = (X_3(1), f_3(1))$ we calculated earlier, we deduce that $e(X_k) = 28 - k$ and $\sigma(X_k) = -20 + k$. Moreover, we claim that $\pi_1(X_k) = 1$ for each $k = 1, 2, \dots, 7$. When

$1 \leq k \leq 5$, we see that the monodromy of f_k contains Dehn twists $t_{d_4}, t_{d_5}, t_{d_6}, t_{d_7}, t_{c_1}, t_{c_2}, t_a$. These Dehn twist curves alone give enough relations to kill the fundamental group as in the proof of Theorem 4.7. If $k = 6, 7$, the monodromy of f_k contains the Dehn twists $t_{a'}, t_x, t_{t_3^{-1}(c_2)}, t_{d_4}, t_{d_5}, t_{d_6}, t_{d_7}$. Since $x = a_1 a_2 = 1$, $t_{c_3}^{-1}(c_2) = b_1^{-1} a_2^{-1} b_1 a_1 b_1^{-1} = 1$ and $a' = [a_1, b_1] a_2$, we again have $\pi_1(X_j) = 1$. None of these manifolds have even intersection forms, because they all contain reducible fibers, which always yield surfaces of odd self-intersection. (And none other than X_4 has signature divisible by 16 anyway.) Hence, by Freedman's theorem, we see that each X_k is homeomorphic to $3\mathbb{CP}^2 \# (23 - k)\overline{\mathbb{CP}}^2$.

On the other hand, the standard 4-manifolds $3\mathbb{CP}^2 \# (23 - k)\overline{\mathbb{CP}}^2$ do not admit symplectic structures: any 4-manifold that is a smooth connected sum of two 4-manifolds with $b_2^+ > 0$ has vanishing Seiberg–Witten invariants [56], whereas by Taubes, any symplectic 4-manifold with $b^+ > 0$ has non-trivial Seiberg–Witten invariants [53]. So, none of the X_k is diffeomorphic to $3\mathbb{CP}^2 \# (23 - k)\overline{\mathbb{CP}}^2$.

The more interesting cases will be for $k = 4, 5, 6, 7$. (When $k \leq 3$, $c_1^2(X_k) = -4 + k < 0$, whereas by Taubes' seminal work, a minimal symplectic 4-manifold with $b_2^+ > 0$ always has $c_1^2 \geq 0$.) The rest of our proof is devoted to showing that each X_k , for $k = 4, 5, 6, 7$, is indeed minimal. To do so, we will need to go over our construction one more time, this time paying attention to how exceptional spheres intersect the spheres cobounded by the 4 lantern curves in each lantern substitution. (After a small perturbation, each one of these spheres can be contained in a singular fiber with multiple nodes, and it is this symplectic (-4) -sphere one rationally blowdowns in the course of the lantern substitution.) Below we will simply refer to these as *lantern spheres*, and denote them using the corresponding lantern substitutions (while pointing out any potential ambiguities).

The initial genus-3 Lefschetz fibration (X, f) we started with, where X is the 4-fold blow-up of a symplectic Calabi–Yau homotopy K3 surface we will simply denote by X' , had a total of 4 exceptional sections S_1, S_2, S'_1, S'_2 , which hit the fibers at the marked points obtained by collapsing the boundary components $\delta_1, \delta_2, \delta'_1, \delta'_2$ in Fig. 6. The reader may want to refer to this figure and Fig. 4 for the rest of our discussion, where the curves x_1, x_2 and x'_1, x'_2 are the lifts of a and a' in our lantern substitutions.

The first lantern sphere L_1 we rationally blowdown can be seen to intersect once with S_2, S'_2 . So by [27, Lemma 5.1], X_1 is still diffeomorphic to 3-fold blow-up of X' . Moreover, by [11, Lemma 6.1], the substitution, when the marked points are taken into account, correspond to a braiding lantern substitution, which turns the pair of exceptional sections the lantern sphere intersects to an exceptional bisection of the resulting fibration. Denote this exceptional bisection by $S_{22'}$. Let us denote the next three lantern spheres, two of which correspond to the lantern substitution L_2 while one of which corresponds to L'_2 , as $L_2(1), L'_2(2)$ and $L_2(3)$ in the order they will be blown down.

The next lantern sphere $L_2(1)$ intersects the exceptional spheres S_1 and $S_{22'}$ only once, and therefore we once again conclude that X_2 is diffeomorphic to a twice blow-up of X' , whereas the spheres S_1 and $S_{22'}$ now descend to an exceptional triple-section $S_{122'}$. Similarly, we can verify that X_3 is diffeomorphic to a single blow-up of X' and the exceptional sphere, denoted by $S_{11'22'}$, becomes a quadruple-section. Now, the lantern sphere $L_2(3)$ intersects $S_{11'22'}$ algebraically and geometrically twice. So by [13, Lemma 1.1], the symplectic 4-manifold X_4 obtained by rationally blowing down $L_2(3)$ in X_3 is minimal, and moreover, the symplectic 4-manifolds X_5, X_6, X_7 we obtain by further rational blowdowns along the (-4) -spheres remain to be minimal. Lastly, for the statement of the theorem, we only record the minimal cases (when $k = 4, 5, 6, 7$) by resetting the index of X_j as $j = k - 4$.

Remark 6.3 With all the additional information we have on the exceptional (multi)sections of the Lefschetz fibrations (X_k, f_k) , we can easily calculate the Seiberg-Witten invariants of X_k , and moreover, represent their basic classes as multisections. The symplectic Calabi–Yau surface X' has trivial canonical class, which is its only Seiberg-Witten basic class [43]. Therefore the Seiberg-Witten invariants of each $X_k \cong X' \# (4-k)\overline{\mathbb{CP}}^2$, for $k = 0, 1, 2, 3$, are determined by the blow-up formula [56], and the 2^{4-k} basic classes of $X_0 := X, X_1, X_2$ and X_3 are all represented by the collection of exceptional (multi)sections $\{S_1, S'_1, S_2, S'_2\}$, $\{S_1, S_{1'2'2'}\}$, $\{S_1, S_{122'}\}$, and $\{S_{1122'}\}$, respectively, where each sphere is taken with either orientation. In particular, $X_3 \cong X' \# \overline{\mathbb{CP}}^2$ has only two basic classes: $\pm E$, where E is represented by $S_{11'2'2'}$. In turn, by the Seiberg-Witten formula for rational blowdown [24], the only basic class that descends from X_3 to X_4 is E , since it intersects the (-4) -sphere of the rational blowdown exactly 4 times, and the class it descends to, up to sign, is the only basic class of X_4 . Using the machinery of [11], we can conclude that this new class is represented by a quadruple-section (represented by the same 4 marked points on the fiber), which is now a torus R_0 of self-intersection 0. Iterating the same arguments, we see that each X_{4+j} , for $j = 0, 1, 2, 3$, has only one basic class up to sign, which is represented by a quadruple-section R_j of genus j , with $R_j^2 = j$. By Taubes, the multisection R_j represents the symplectic canonical class (and its Seiberg-Witten value is $+1$).

6.2 Inequivalent pencils with explicit monodromies

Here we will construct inequivalent relatively minimal genus- g Lefschetz fibrations for each $g \geq 3$, on a family of symplectic 4-manifolds of Kodaira dimension $-\infty, 0$ and 1 , and especially on rational surfaces.

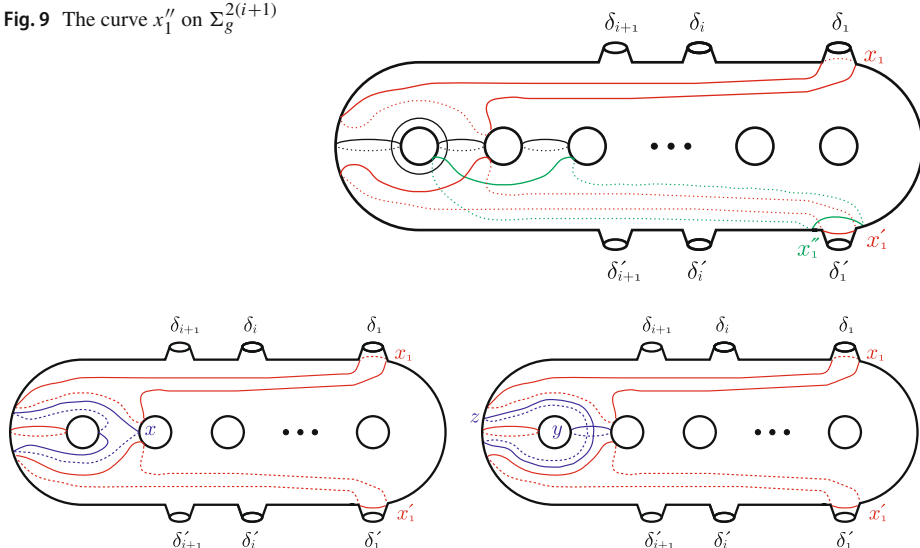
Theorem 6.4 *For any $g \geq 3$ and $i = 0, 1, 2, \dots, g-1$, there are pairs of inequivalent relatively minimal genus- g Lefschetz pencils $(Y_g(i), h_g^j(i))$, $j = 1, 2$, and inequivalent Lefschetz fibrations on their blow-ups. For any $g \geq 3$, there are pairs of inequivalent relatively minimal genus- g Lefschetz pencils on once blown-up elliptic surface $E(1) \# \overline{\mathbb{CP}}^2 \cong \mathbb{CP}^2 \# 10\overline{\mathbb{CP}}^2$, with different number of reducible fibers.*

These inequivalent pencils and fibrations will be produced following the recipe given in [11], and will be distinguished by the number of their reducible fibers. Therefore they are even inequivalent up to *fibered Luttinger surgeries* (see [5, 8, 9]), which never change the topological type of fibers.

We will need the following lemma:

Lemma 6.5 *Let x_1'' be the simple closed curve on $\Sigma_g^{2(i+1)}$ as in Fig. 9, and let $D'_g := t_{d_6} t_{d_7} \cdots t_{d_{2g+1}}$ and $E'_g := t_{e_{2g+1}} \cdots t_{e_7} t_{e_6}$. For each $i = 0, 1, 2, \dots, g-1$, the following positive factorization is Hurwitz equivalent to the monodromy factorization of the pencil $(X'_g(i), f'_g(i))$ given in Theorem 4.6: when g is odd,*

$$\begin{aligned} & t_{\delta_{i+1}} \cdots t_{\delta_2} t_{\delta_1} t_{\delta'_{i+1}} \cdots t_{\delta'_2} t_{\delta'_1} \\ &= \prod_{k=i+2}^2 t_{x_k} t'_{x'_k} \cdot t_{c_2} t_{c_3} (t_{c_1} t_{c_2} t_{c_3})^{4(g-1-i)+1} (t_{c_1} t_{c_2} t_{c_3})^2 \cdot t_{x_1} t_{d_4} t_{d_5} t'_{x_1''} \cdot D'_g E'_g \cdot t_{c_5} \cdot t_{e_5} t_{e_4} \end{aligned}$$

Fig. 9 The curve x_1'' on $\Sigma_g^{2(i+1)}$ **Fig. 10** The curves x, y, z on $\Sigma_g^{2(i+1)}$

when g is even,

$$\begin{aligned}
 & t_{\delta_{i+1}} \cdots t_{\delta_2} t_{\delta_1} t_{\delta'_{i+1}} \cdots t_{\delta'_2} t_{\delta'_1} \\
 &= \prod_{k=i+2}^2 t_{x_k} t_{x'_k} \cdot t_{c_2} t_{c_3} (t_{c_1} t_{c_2} t_{c_3})^{4(g-1-i)+1} (t_{c_3} t_{c_2} t_{c_1})^2 \cdot t_{x_1} t_{d_4} t_{d_5} t_{x_1''} \cdot D'_g E'_g \cdot t_{c_5} \cdot t_{e_5} t_{e_4}.
 \end{aligned}$$

Proof Let us prove the case of even g , and leave the odd case to the reader, the proof of which is very similar.

Note that since c_1, c_2, c_3 are disjoint from $x_1, \dots, x_i$ and $x'_1, \dots, x'_i$, and x_ℓ is disjoint from x'_j for any ℓ and j , we have the following relation:

$$\begin{aligned}
 & t_{x_{i+1}} \cdots t_{x_2} t_{x_1} t_{x'_{i+1}} \cdots t_{x'_2} t_{x'_1} (t_{c_1} t_{c_2} t_{c_3})^{4(g-1-i)+2} (t_{c_3} t_{c_2} t_{c_1})^2 D_g E_g \\
 & \sim t_{c_1} \cdot \prod_{k=i+1}^2 t_{x_k} t_{x'_k} \cdot t_{c_2} t_{c_3} (t_{c_1} t_{c_2} t_{c_3})^{4(g-1-i)+1} (t_{c_3} t_{c_2} t_{c_1})^2 \cdot t_{x_1} t_{x'_1} \cdot D_g E_g \\
 & \sim \prod_{k=i+1}^2 t_{x_k} t_{x'_k} \cdot t_{c_2} t_{c_3} (t_{c_1} t_{c_2} t_{c_3})^{4(g-1-i)+1} (t_{c_3} t_{c_2} t_{c_1})^2 \cdot t_{x_1} t_{x'_1} \cdot D_g E_g \cdot t_{c_1}
 \end{aligned}$$

where $D_g = t_{d_4} t_{d_5} \cdots t_{d_{2g+1}}$ and $E_g = t_{e_{2g+1}} \cdots t_{e_5} t_{e_4}$. Here, it is easy to check that $x_1'' = (t_{d_4} t_{d_5})^{-1} (x'_1)$ and $c_5 = (t_{e_5} t_{e_4}) (c_1)$. This gives

$$\begin{aligned}
 t_{x'_1} \cdot t_{d_4} t_{d_5} & \sim t_{d_4} t_{d_5} \cdot t_{x_1''} \\
 t_{e_5} t_{e_4} \cdot t_{c_1} & \sim t_{c_5} \cdot t_{e_5} t_{e_4}.
 \end{aligned}$$

This completes the proof.

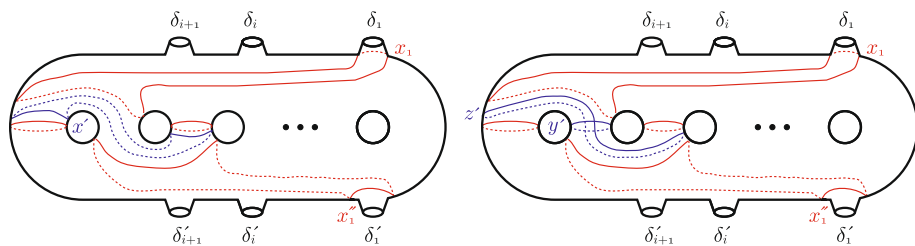


Fig. 11 The curves x' , y' , z' on $\Sigma_g^{2(i+1)}$

Proof of Theorem 6.4 We will run our constructions for any $i = 0, 1, 2, \dots, g - 1$.

It is easy to see that, after Hurwitz moves, the monodromy factorization of $(X'_g(i), f_g(i))$ yields a positive factorization with the subword $t_{c_1}^2 t_{x_1} t'_{x_1}$. Note that $x_1, x'_1, \delta_1, \delta'_1$ and two disjoint curves parallel to c_1 bound a sphere with six boundary components. Applying the lantern substitution to this subword $t_{c_1}^2 t_{x_1} t'_{x_1}$ replaces it with the new subword $t_x t_y t_z$ in the factorization, where x, y, z are the curves as in Fig. 10. Akin to the proof of Theorem 5.4, this then gives a genus- g Lefschetz pencil $(Y_g^1(i), h_g^1(i))$. This pencil $h_g^1(i)$ has one reducible fiber corresponding to the Dehn twist along the separating curve y .

On the other hand, if we take the Hurwitz equivalent monodromy factorization of $(X'_g(i), f_g(i))$ given in Lemma 6.5, it is again easy to see that, after further Hurwitz moves, we get a positive factorization with the subword $t_{c_1} t_{x_1} t_{x'_1} t_{c_5}$. Note that $c_1, x_1, x'_1, c_5, \delta_1, \delta'_1$ bound a sphere with six boundary components. When we apply a braiding lantern substitution along this subword $t_{c_1} t_{x_1} t_{x'_1} t_{c_5}$, we replace it with the new subword $t_{x'} t_{y'} t_{z'}$ in the factorization, where x', y', z' are the nonseparating curves in Fig. 11. This gives a genus- g Lefschetz pencil $(Y_g^2(i), h_g^2(i))$ where as well, where all singular fibers of $h_g^2(i)$ are irreducible.

By the arguments we employed in earlier proofs, since the lantern spheres of for these substitutions intersect an exceptional section once, we have $Y_g^j(i) \cong X'_g(i) \# \overline{\mathbb{CP}^2}$, for each $j = 1, 2$. Therefore, the two pencils $h_g^j(i)$ can be viewed on the same symplectic 4-manifold, but have different number of reducible fibers (one versus none), which means they are not equivalent.

The examples in the last part of the theorem are derived in the particular case when $i = g - 1$, where $Y_g(g - 1) \cong X'_g(g - 1) \# \overline{\mathbb{CP}^2}$, and $X'_g(g - 1) \cong E(1)$ by Theorem 4.7.

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References

1. Akbulut, S., Ozbagci, B.: Lefschetz fibrations on compact Stein surfaces. *Geom. Topol.* **5**, 319–334 (2001)
2. Akhmedov, A., Katzarkov, L.: Symplectic surgeries along certain singularities and new Lefschetz fibrations. *Adv. Math.* **360**, 106920 (2020)
3. Akhmedov, A., Monden, N.: The existence of an indecomposable minimal genus two Lefschetz fibration. *Osaka J. Math.* **58**(1), 29–36 (2021)
4. Akhmedov, A., Park, J.-Y.: Lantern substitution and new symplectic 4-manifolds with $b_2^+ = 3$. *Math. Res. Lett.* **21**(1), 1–17 (2014)

5. Auroux, D.: Mapping class group factorizations and symplectic 4-manifolds: some open problems, Problems on mapping class groups and related topics, 123–132, Proc. Sympos. Pure Math. **74**, Amer. Math. Soc., Providence, RI, 2006
6. Bauer, S.: Almost complex 4-manifolds with vanishing first Chern class. J. Differ. Geom. **79**(1), 25–32 (2008)
7. Baykur, R.I.: Minimality and fiber sum decompositions of Lefschetz fibrations. Proc. Am. Math. Soc. **144**(5), 2275–2284 (2016)
8. Baykur, R.I.: Inequivalent Lefschetz fibrations and surgery equivalence of symplectic 4-manifolds. J. Symp. Geom. **14**(3), 671–686 (2016)
9. Baykur, R.I.: Inequivalent Lefschetz fibrations on rational and ruled surfaces. Proc. Sympos. Pure Math. **102**, 21–28 (2019)
10. Baykur, R. I.: Small symplectic Calabi-Yau surfaces and exotic 4-manifolds via genus-3 pencils, to appear in the Proceedings of the 2020 BIRS Workshop on Interactions of gauge theory and contact and symplectic topology (Open Book Series)
11. Baykur, R.I., Hayano, K.: Multisections of Lefschetz fibrations and topology of symplectic 4-manifolds. Geom. Topol. **20**(4), 2335–2395 (2016)
12. Baykur, R.I., Korkmaz, M.: Small Lefschetz fibrations and exotic 4-manifolds. Math. Annalen **367**(3–4), 1333–1361 (2017)
13. Dorfmeister, J.G.: Minimality of symplectic fiber sums along spheres. Asian J. Math. **17**(3), 423–442 (2013)
14. Dorfmeister, J.G.: Kodaira dimension of symplectic fiber sums along spheres. Geom. Dedicata. **177**, 1–25 (2015)
15. Donaldson, S.K.: Lefschetz pencils on symplectic manifolds. J. Differ. Geom. **53**(2), 205–236 (1999)
16. Donaldson, S. K.: Lefschetz pencils and mapping class groups, Problems on mapping class groups and related topics, 151–163, Proc. Sympos. Pure Math., **74**, Amer. Math. Soc., Providence, RI, 2006
17. Donaldson, S.K.: Some problems in differential geometry and topology. Nonlinearity **21**(9), 157–164 (2008)
18. Endo, H., Gurtas, Y.Z.: Lantern relations and rational blowdowns. Proc. Am. Math. Soc. **138**(3), 1131–1142 (2010)
19. Endo, H., Mark, T., Van Horn-Morris, J.: Monodromy substitutions and rational blowdowns. J. Topol. **4**(1), 227–253 (2011)
20. Endo, H., Nagami, S.: Signature of relations in mapping class groups and non-holomorphic Lefschetz fibrations. Trans. Am. Math. Soc. **357**(8), 3179–3199 (2005)
21. Etnyre, J.: Lectures on open book decompositions and contact structures, Floer homology, gauge theory, and low-dimensional topology, 103–141, Clay Math. Proc., **5**, Amer. Math. Soc., Providence, RI, 2006
22. Farb, B., Margalit, D.: A primer on mapping class groups, Princeton Mathematical Series, **49**, Princeton University Press, Princeton, NJ, 2012, xiv+472
23. Fintushel, R., Park, J., Stern, R.: Rational surfaces and symplectic 4-manifolds with one basic class. Algebr. Geom. Topol. **2**, 391–402 (2002)
24. Fintushel, R., Stern, R.: Rational blowdowns of smooth 4-manifolds. J. Differ. Geom. **46**(2), 181–235 (1997)
25. Friedl, S., Vidussi, S.: On the topology of symplectic Calabi-Yau 4-manifolds. J. Topol. **6**(4), 945–954 (2013)
26. Gay, D., Mark, T.E.: Convex plumbings and Lefschetz fibrations. J. Symplectic Geom. **11**(3), 363–375 (2013)
27. Gompf, R.E.: A new construction of symplectic manifolds. Ann. Math. (2) **142**(3), 527–595 (1995)
28. Gompf, R. E., Stipsicz, A.I.: 4-Manifolds and Kirby Calculus, Graduate Studies in Mathematics **20**, American Mathematical Society, 1999
29. Hamada, N.: Sections of the Matsumoto-Cadavid-Korkmaz Lefschetz fibration, preprint; [arXiv:1610.08458](https://arxiv.org/abs/1610.08458)
30. Hamada, N., Hayano, K.: Topology of holomorphic Lefschetz pencils on the four-torus. Algebr. Geom. Topol. **18**(3), 1515–1572 (2018)
31. Hitchin, N.: Harmonic spinors. Adv. Math. **14**, 1–55 (1974)
32. Ho, C.I., Li, T.J.: Luttinger surgery and Kodaira dimension. Asian J. Math. **16**(2), 299–318 (2012)
33. Johnson, D.: Spin structures and quadratic forms on surfaces. J. Lond. Math. Soc. **22**(2), 365–373 (1980)
34. Karakurt, C., Starkston, L.: Surgery along star-shaped plumbings and exotic smooth structures on 4-manifolds. Algebr. Geom. Topol. **16**(3), 1585–1635 (2016)
35. Kas, A.: On the handlebody decomposition associated to a Lefschetz fibration. Pac. J. Math. **89**, 89–104 (1980)

36. Li, T.J.: Smoothly embedded spheres in symplectic 4-manifolds. *Proc. Am. Math. Soc.* **127**(2), 609–613 (1999)
37. Li, T.J.: Symplectic 4-manifolds with Kodaira dimension zero. *J. Differ. Geom.* **74**(2), 321–352 (2006)
38. Li, T.J.: Quaternionic bundles and Betti numbers of symplectic 4-manifolds with Kodaira dimension zero, *Int. Math. Res. Not.*, 2006,
39. Li, T.J.: Symplectic Calabi-Yau surfaces, *Handbook of geometric analysis*, No. 3, 231–356, *Adv. Lect. Math. (ALM)*, **14**, Int. Press, Somerville, MA, 2010
40. Loi, A., Piergallini, R.: Compact Stein surfaces with boundary as branched covers of B^4 . *Invent. Math.* **143**(2), 325–348 (2001)
41. McDuff, D., Salamon, D.: A survey of symplectic 4-manifolds with $b^+ = 1$. *Turk. J. Math.* **20**, 47–60 (1996)
42. Moishezon, B., Teicher, M.: Braid group technique in complex geometry. I. Line arrangements in $\mathbb{CP}^2$, *Braids* (Santa Cruz, CA, 1986), *Amer. Math. Soc., Providence, RI, Contemp. Math.*, **78**(1988), 425–555
43. Morgan, J., Szabó, Z.: Homotopy $\mathbb{K}$ surfaces and mod 2 Seiberg-Witten invariants. *Math. Res. Lett.* **4**(1), 17–21 (1997)
44. Park, J.: Seiberg-Witten invariants of generalised rational blow-downs. *Bull. Austral. Math. Soc.* **56**(3), 363–384 (1997)
45. Park, J.: Simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $c_1^2 = 2$. *Invent. Math.* **159**(3), 657–667 (2005)
46. Park, J., Yun, K.-H.: Nonisomorphic Lefschetz fibrations on knot surgery 4-manifolds. *Math. Ann.* **345**(3), 581–597 (2009)
47. Park, J., Yun, K.-H.: Lefschetz fibrations on knot surgery 4-manifolds via Stallings twist. *Michigan Math. J.* **66**(3), 481–498 (2017)
48. Sato, Y.: The necessary condition on the fiber-sum decomposability of genus-2 Lefschetz fibrations. *Osaka J. Math.* **47**(4), 949–963 (2010)
49. Sato, Y.: Canonical classes and the geography of nonminimal Lefschetz fibrations over S^2 . *Pac. J. Math.* **262**(1), 191–226 (2013)
50. Smith, I.: Torus fibrations on symplectic four-manifolds. *Turk. J. Math.* **25**, 69–95 (2001)
51. Stipsicz, A.I.: Indecomposability of certain Lefschetz fibrations. *Proc. Am. Math. Soc.* **129**(5), 1499–1502 (2001)
52. Stipsicz, A.I.: Spin structures on Lefschetz fibrations. *Bull. Lond. Math. Soc.* **33**, 466–472 (2001)
53. Taubes, C.H.: The Seiberg–Witten invariants and symplectic forms. *Math. Res. Lett.* **1**, 809–822 (1994)
54. Usher, M.: Minimality and symplectic sums. *Int. Math. Res. Not.* **17**, 49857 (2006)
55. Usher, M.: Kodaira dimension and symplectic sums. *Comment. Math. Helv.* **84**(1), 57–85 (2009)
56. Witten, E.: Monopoles and four-manifolds. *Math. Res. Lett.* **1**, 769–796 (1994)
57. Xiao, G.: Surfaces fibrées en courbes de genre deux. *Lecture Notes in Math*, vol. 1137. Springer-Verlag, Berlin (1985)

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