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Injectivity of the Heisenberg X-ray transform

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ABSTRACT

We initiate the study of X-ray tomography on sub-Riemannian manifolds, for which the Heisenberg group exhibits the simplest nontrivial example. With the language of the group Fourier transform, we prove an operator-valued incarnation of the Fourier Slice Theorem, and apply this new tool to show that a sufficiently regular function on the Heisenberg group is determined by its line integrals over sub-Riemannian geodesics. We also consider the family of taming metrics g_ϵ approximating the sub-Riemannian metric, and show that the associated X-ray transform is injective for all $\epsilon > 0$. This result gives a concrete example of an injective X-ray transform in a geometry with an abundance of conjugate points.

1. Introduction

Our object of study is the geodesic X-ray transform associated to the sub-Riemannian geometry of the Heisenberg group, which is $\mathbb{H} := \mathbb{C} \times \mathbb{R}$ with the multiplication law

$$(x + iy, t)(u + iv, s) = \big(x + u + i(y + v), t + s + \tfrac{1}{2}(xv - yu)\big)\,,$$

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and a metric defined in Section 3.1. $\mathbb{H}$ is the local model for any 3-dimensional sub-Riemannian manifold of contact type, in the same sense that 3-dimensional Euclidean space is the local model for any 3-dimensional Riemannian manifold [23, Thm. 1]. This property positions $\mathbb{H}$ as the natural homogeneous starting point for studying the integral geometry of contact manifolds, just as Radon first inverted the X-ray transform in $\mathbb{R}^2$.

X-ray transforms, which integrate a function on a manifold over its geodesics, have been extensively studied on Riemannian manifolds and homogeneous spaces [10,14]. Helgason showed in [13] that the X-ray transform on symmetric spaces of noncompact type is injective. In [21] the authors prove injectivity on compact symmetric spaces excluding the n-sphere. Ilmavirts in [17] obtains injectivity on compact Lie groups excluding S^1 and S^3. (For a survey of results on Riemannian manifolds with boundary see [18].) To the author's knowledge, X-ray transforms on sub-Riemannian manifolds are virtually unexplored.

To a function $f \in L^1(\mathbb{H})$ we associate the function If, its X-ray transform, defined by

$$If(\gamma) := \int f(\gamma(s))\, ds,$$

where the geodesics γ will be cast as (projections of) integral curves of the Hamiltonian flow on $T^*\mathbb{H}$ for the degenerate fiber quadratic Hamiltonian later described (7). Related integral transforms on $\mathbb{H}$ have been studied, for example, by Rubin [33], and Strichartz [35], who consider integration over left translates of hyperplanes. We ask whether If determines f.

The sub-Riemannian setting, whose general theory is poorly understood, introduces qualitatively new features to this question. For example, fibers of the unit cotangent bundle $U^*\mathbb{H}$ (defined in Section 3.1) are now cylinders, and there is no unique Levi-Civita connection. Thus $U^*\mathbb{H}$ has noncompact fibers, and there is no canonical splitting of its tangent space into vertical and horizontal components like there is in the Riemannian case as described in [29]. See [26] for background sub-Riemannian geometry or [5] for an extensive introduction to the Heisenberg group.

A standard geometric obstacle to such inverse problems is presented by the presence of conjugate points. In [25] and [15] the authors show that conjugate points generally inhibit stable inversion of the X-ray transform on Riemannian manifolds, with unconditional loss in two dimensions. Unfortunately, the conjugate points in the Heisenberg group are ubiquitous; the cut locus to any point passes through that point—a feature generic in sub-Riemannian geometry, where the exponential map is never a local diffeomorphism at the origin [34, p. 222]. Therefore, standard tools for proving injectivity, such as Pestov energy methods, which typically require a positive-definite second fundamental form [18] do not apply without a closer look. We prove that, nonetheless, the X-ray transform on the Heisenberg group is injective.

A common recipe for inverting such integral transforms is to compute the normal operator I^*I, for I^* defined with a suitable measure in the target space, as in [1,14,32], and identify the normal operator as a function of distinguished invariant differential operators. On the Heisenberg group, I^*I is not well-behaved (or immediately well-defined on $C_c(\mathbb{H})$) due to the singular nature of the sub-Riemannian exponential map, so we focus on studying the transform I directly. We observe a convenient identification of the space of geodesics with a quotient of the Heisenberg group (12), which allows us to express I as a convolution. We then apply the group Fourier transform on the Heisenberg group and its quotient to express I essentially as a multiplication operator (Theorem 2), from which we deduce that I is injective (Theorem 1). For background material on the group Fourier transform and the harmonic analysis of the Heisenberg group, see [6], [8], or [37].

2. Main results

The Heisenberg geodesics exist for all time and are left-translates of helices and straight lines, as described in Section 3.1. Let $\mathcal{G}$ be the set of all maximal Heisenberg geodesics without orientation, and $\mathcal{G}_\lambda$ the set of all geodesics having a fixed value $\lambda \in \mathbb{R}$, for the "charge" λ, which is a constant of motion. We will parameterize $\mathcal{G}_\lambda$ using left-translates of specific model geodesics as in [17], with the caveat that Heisenberg geodesics are not one-parameter subgroups of the Heisenberg group.

Left translation by any element $(z,t) \in \mathbb{H}$ is an isometry of $\mathbb{H}$ and so $\mathbb{H}$ acts on $\mathcal{G}$. This action does not change the value of λ, and is a transitive action on each leaf $\mathcal{G}_\lambda$, $\lambda \neq 0$. (It is not transitive on $\mathcal{G}_0$ since it does not change the direction of the line in the plane which the $\lambda = 0$ geodesic projects to.) These facts are verified by inspecting the exponential map in (41). Thus we can use $\mathbb{H}$ to parameterize $\mathcal{G}_\lambda$, $\lambda \neq 0$, by fixing a particular helix $\gamma_\lambda \in \mathcal{G}_\lambda$ and left-translating it about. We take this helix to be one whose projection is a circle of radius $|R| = 1/|\lambda|$ centered at the origin and parameterized by arclength. Thus our parameterization of that part of $\mathcal{G}$ having $\lambda \neq 0$ is

$$s \mapsto (z,t)\gamma_\lambda(s), \quad \gamma_\lambda(s) = \left(Re^{i(s/R)}, \tfrac{1}{2}sR\right) \in \mathbb{H};\ R = 1/\lambda. \tag{1}$$

Using this identification, we may parameterize geodesics by (z,t,λ) as above, uniquely modulo the isotropy group $\Gamma_\lambda := \{(0, k\pi R^2) \in \mathbb{H} : k \in \mathbb{Z}\}$ stabilizing γ_λ, and write the X-ray transform concretely as

$$If(z,t,\lambda) := I_\lambda f(z,t) := \int_{\mathbb{R}} f\left((z,t)\gamma_\lambda(s)\right) ds, \quad f \in C_c(\mathbb{H}).$$

We ignore the degenerate case when $\lambda = 0$, where the geodesics are straight lines. Furthermore, since $\lambda < 0$ corresponds to a $\lambda > 0$ geodesic with opposite orientation, we will take $\lambda > 0$ unless otherwise specified. Fixing $\lambda > 0$, we prove in Proposition 11 that $I_\lambda : L^1(\mathbb{H}) \to L^1(\mathcal{G}_\lambda)$, with a natural measure on the codomain given in Section 4.1, is well-defined and bounded. Existing literature ([16] and [17] for example) profitably considers the X-ray transform as a family of operators indexed by a directional parameter in this way.

In [35, p. 392], Strichartz proves indirectly that a function on the Heisenberg group may not in general be recovered from its integrals over $\lambda = 0$ geodesics alone, but does not consider $\lambda \neq 0$ geodesics. Indeed, our main result necessarily involves geodesics with nonzero charge λ:

Theorem 1. *The Heisenberg X-ray transform $I : L^1(\mathbb{H}) \to L^1(\mathcal{G}, d\mathcal{G})$ is injective. In particular, if $f \in L^1(\mathbb{H})$, and $I_\lambda f = 0$ for all λ in a neighborhood of zero, then $f = 0$.*

The measure $d\mathcal{G}$ on the set of geodesics, $\mathcal{G}$, is defined in Section 4.1.

Thinking of the charge λ as the restricted directional parameter, Theorem 1 is an example of limited angle tomography (see [22] and [28, Ch. 6]).

We prove this result using harmonic analysis adapted to the group structure, modifying familiar results in Euclidean space. Consider, for example, the Radon and Mean Value Transforms on $\mathbb{R}^2$:

$$Rf(s,\theta) := \int_{\mathbb{R}} f\left(se^{i\theta} + ite^{i\theta}\right) dt, \qquad M^r f(z) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z + re^{i\theta}\right) d\theta \tag{2}$$

where, say, $f \in C_c(\mathbb{R}^2)$. Taking the Fourier transforms in s and z, respectively, yields

$$\mathcal{F}_{s\mapsto\sigma} Rf(\sigma,\theta) = \hat{f}(\sigma e^{i\theta}), \qquad \mathcal{F}_{z\mapsto\zeta} M^r f(\zeta) = J_0(r|\zeta|)\hat{f}(\zeta),$$

where J_0 is the zeroth-order Bessel function (43). These results are known as Fourier Slice Theorems, or Projection Slice Theorems [28]. They reveal that R, thought of as a projection onto $\{\theta\}$, becomes a restriction operator onto the "slice" $\sigma \to \sigma e^{i\theta}$ in the Fourier domain, and that M^r becomes a multiplication operator by $J_0(r|\zeta|)$ when viewed in the Fourier domain. Fourier Slice Theorems exist for more general Radon transforms as well; for example, in [11,19].

The Radon and Mean Value Transforms may be interpreted as integration over straight lines or magnetic geodesics in Euclidean space. In the case of $\mathbb{H}$—which is a "flat" sub-Riemannian geometry—we prove a corresponding Fourier Slice Theorem for Heisenberg geodesics. We use the operator-valued group Fourier transform $\mathcal{F}_{\mathbb{H}}$ associated to the Bargmann-Fock representation β_h (defined in equation (8)), which has proven a useful tool, for example, by Nachman in [27] to find the fundamental solution for the wave operator for the Heisenberg Laplacian. The theory of $\mathcal{F}_{\mathbb{H}}$ is extensively developed in [9,37]. In particular it has a Plancherel Theorem and Inversion Theorem [7,8,37].

We identify $\mathcal{G}_\lambda \cong \mathbb{H}/\Gamma_\lambda$ in Section 4.1 and so also define in equation (10) the group Fourier transform $\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}$ on the quotient. We see that in the generalized Fourier domain of $\mathcal{F}_{\mathbb{H}}$ and $\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}$, the Heisenberg X-ray transform is essentially a multiplication operator:

Theorem 2 *(Heisenberg Fourier Slice Theorem). If $f \in L^1(\mathbb{H})$, then*

$$\left(\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}\left(I_\lambda f\right)\right)(n) = (2\pi/\lambda)\,\mathcal{J}_n \circ \left(\mathcal{F}_{\mathbb{H}} f\right)(n\lambda^2), \quad \forall n \in \mathbb{Z}\setminus\{0\}, \forall \lambda > 0. \tag{3}$$

Equation (3) is an equality of operators acting on Bargmann-Fock space (originally described in [4]),

$$\mathcal{H} := \left\{ F : \mathbb{C} \to \mathbb{C},\ \text{holomorphic} : \frac{1}{\pi} \int_{\mathbb{C}} |F(\zeta)|^2 e^{-|\zeta|^2} d\zeta < \infty \right\}. \tag{4}$$

$\mathcal{J}_n : \mathcal{H} \to \mathcal{H}$ is the operator

$$\mathcal{J}_n F(\zeta) = \frac{1}{2\pi i} \left(\frac{1}{en} \right)^{n/2} \oint z^{n-1} e^{-n\zeta/z} F(\zeta + z) dz, \quad n > 0 \tag{5}$$

where the contour is a circle around the origin oriented counterclockwise (and where $\mathcal{J}_{-n} = \mathcal{J}_n$). Loosely speaking, the Heisenberg X-ray transform I is "block-diagonalized" in λ by the group Fourier transform, and each block is essentially a multiple of $\mathcal{J}_n$.

The classical Fourier Slice Theorem for R in (2) states that knowledge of Rf for a fixed θ_0 determines the Fourier transform $\hat{f}(\zeta)$ for all $\zeta \parallel \theta_0$. Similarly, the Heisenberg Fourier Slice Theorem says that knowledge of $I_\lambda f$ for fixed λ determines the group Fourier transform $\mathcal{F}_{\mathbb{H}} f(h)$, up to multiplication by the operator $\mathcal{J}_n$, for all $h \in \lambda^2 \mathbb{Z}^*$. Therefore, injectivity of I follows once we show that $\mathcal{J}_n$ is an injective operator at least whenever n is an odd integer (Proposition 22).

Finally, in Section 5, we consider the ray transform I^ϵ (defined in (27)) associated to a special family of left-invariant taming metrics g_ϵ parameterized by $\epsilon > 0$:

$$g_\epsilon := dx^2 + dy^2 + (1/\epsilon)^2 \Theta^2; \quad \Theta := dt - \tfrac{1}{2}(xdy - ydx).$$

First, we prove a Heisenberg Fourier Slice Theorem for g_ϵ geodesics:

Theorem 3 *(g_ϵ Heisenberg Fourier Slice Theorem). If $f \in L^1(\mathbb{H})$, and $\epsilon > 0$ then*

$$\mathcal{F}_{\mathbb{H}/\Gamma_\lambda^\epsilon} \left(I_\lambda^\epsilon f \right)(n) = (2\pi/\lambda) \mathcal{J}_n \left(\frac{1}{\sqrt{1 + 2\epsilon^2 \lambda^2}} \right) \circ (\mathcal{F}_{\mathbb{H}} f) \left(\frac{n\lambda^2}{1 + 2\epsilon^2 \lambda^2} \right), \quad \forall n \in \mathbb{Z}^*,\ \forall \lambda > 0.$$

Here $\mathcal{J}_n(r)$, $r > 0$, is defined in (21).

We then use Theorem 3 in the same way with Proposition 33 to show that I^ϵ is injective:

Theorem 4. *For all $\epsilon > 0$, the Heisenberg taming X-ray transform $I^\epsilon : L^1(\mathbb{H}) \to L^1(\mathcal{G}^\epsilon, d\mathcal{G}^\epsilon)$ is injective. In particular, if $f \in L^1(\mathbb{H})$ and $I_\lambda^\epsilon f = 0$ for all λ in a neighborhood of zero, then $f = 0$.*

The measure $d\mathcal{G}^\epsilon$ on the set of g_ϵ-geodesics, $\mathcal{G}^\epsilon$, is defined in (30).

The first part of Theorem 4 is not new. In [30] the authors prove a support theorem for geodesics of left-invariant metrics on the Heisenberg group, which implies injectivity of the associated X-ray transform. However, to the author's knowledge, the second part of Theorem 4 is new.

3. Preliminaries

3.1. Heisenberg geometry

We define the sub-Riemannian metric on $\mathbb{H}$ by declaring the left-invariant vector fields

$$X = \partial_x - \frac{1}{2} y \partial_t, \qquad Y = \partial_y + \frac{1}{2} x \partial_t, \tag{6}$$

to be orthonormal, and the length of $T = \partial_t$ to be infinite. Then any finite length smooth path in $\mathbb{H}$ must be tangent to the nonintegrable distribution $\mathcal{D}_q := \mathrm{Span}\{X_q, Y_q\}$, $q \in \mathbb{H}$. We call such a path *horizontal.* The length of a horizontal path equals the length of its projection to the plane by the map

$$\pi(x, y, t) = (x, y).$$

A minimizing Heisenberg geodesic is a shortest horizontal path joining two points of $\mathbb{H}$. That any two points in $\mathbb{H}$ are connected by a horizontal path is guaranteed by Chow's Theorem and the fact that $\mathcal{D}$ satisfies the Hörmander condition (i.e. $\mathcal{D}$ is bracket-generating).

The fiber quadratic Hamiltonian $H : T^*\mathbb{H} \to \mathbb{R}$ given in canonical coordinates by

$$H(x, y, t, p_x, p_y, p_t) = \tfrac{1}{2}\left((p_x - \tfrac{1}{2} y p_t)^2 + (p_y + \tfrac{1}{2} x p_t)^2\right) \tag{7}$$

generates the Heisenberg geodesics. By 'generate' we mean that any solution to Hamilton's equations for H projects, via the canonical projection $T^*\mathbb{H} \to \mathbb{H}$, to a sub-Riemannian geodesic, and conversely, all Heisenberg geodesics arise this way [26, Sec 1.5]. If we want geodesics parameterized by arclength we only take solutions for which $H = 1/2$. (Thus, we define the unit cotangent bundle $U^*\mathbb{H}$ as the set of all $(q, p) \in T^*\mathbb{H}$ for which $H(q, p) = 1/2$.) These geodesics can be best understood by their projection under π to the plane: they are circles or lines. Indeed

$$\dot{p}_t = -\frac{\partial H}{\partial t} = 0,$$

so that $\lambda := p_t$ is a constant of motion. If we interpret λ as the charge of a particle, then H, viewed as a Hamiltonian on $T^*\mathbb{R}^2$, is the Hamiltonian for a particle of charge λ travelling in the plane under the influence of a constant unit strength magnetic field. These solutions are well-known and easy to derive [26, p. 12]. When $H = 1/2$ they are circles of radius $R = 1/|\lambda|$ for $\lambda \neq 0$, and lines when $\lambda = 0$. See eq. (1) for a concrete representation of all geodesics with $\lambda \neq 0$.

3.2. The group Fourier transforms

We start by giving a brief description of the representation theory of the Heisenberg group. A more detailed discussion can be found in [6]. Denote by $\mathcal{U}(\mathcal{H})$ the set of unitary operators on Bargmann-Fock space, defined in (4). For each $h \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, the map (motivated in Section 6.4)

$$\beta_h : \mathbb{H} \to \mathcal{U}(\mathcal{H})$$

given by

$$\beta_h(z,t)F(\zeta) := e^{2iht - \sqrt{h}\zeta\overline{z} - \frac{h}{2}|z|^2} F(\zeta + \sqrt{h}z), \quad F \in \mathcal{H},\ h > 0, \tag{8}$$

and $\beta_h(z,t) = \beta_{|h|}(\overline{z}, -t)$ for $h < 0$, is a strongly continuous unitary representation of the Heisenberg group on $\mathcal{H}$. Moreover, it is known that these representations are irreducible, and by the Stone-von Neumann Theorem, up to unitary equivalence, these are all of the irreducible unitary representations on $\mathbb{H}$ that are nontrivial on the center of $\mathbb{H}$ [6].

We define the group Fourier transform of an integrable function on $\mathbb{H}$. Denote by $\mathcal{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$. The Heisenberg Fourier transform of $f \in L^1(\mathbb{H})$ is the operator-valued function

$$\mathcal{F}_{\mathbb{H}} f : \mathbb{R}^* \to \mathcal{B}(\mathcal{H})$$
$$\mathcal{F}_{\mathbb{H}} f(h) := \int_{\mathbb{H}} f(q)\beta_h(q)^* dq$$

where the integral is taken in the Bochner sense [37, p. 11]. Think of h as a semi-classical parameter.

Remark 5. Many authors define $\mathcal{F}_{\mathbb{H}}$ alternatively with the Schrödinger representations. Our definition seems more natural for studying the X-ray transform due to the simplicity of (5), and is equivalent by conjugation with a unitary intertwining map; the choice is largely a personal preference. We also normalize the representations β_h in such a way that they all act on the same space $\mathcal{H}$, rather than a family of spaces parameterized by $h \in \mathbb{R}^*$, as in [6].

If $f \in L^1(\mathbb{H}) \cap L^2(\mathbb{H})$, then $\mathcal{F}_{\mathbb{H}}(f)(h)$ is a Hilbert-Schmidt operator on $\mathcal{H}$ [8,9]. Let S_2 denote the space of Hilbert-Schmidt operators on $\mathcal{H}$, and define the Hilbert Space $L^2(\mathbb{R}^*, S_2; d\mu) = L^2(S_2)$ via the inner product

$$\langle A, B\rangle_{L^2(S_2)} := \int_{\mathbb{R}^*} \operatorname{tr}\left(A(h)B(h)^*\right) d\mu(h), \quad d\mu = \pi^{-2}|h| dh.$$

We will need the following theorems from Geller, normalized to account for the slightly different group law for $\mathbb{H}$ used here and in [8].

Theorem 6 *([8, Plancherel Theorem]). If $f \in L^1(\mathbb{H}) \cap L^2(\mathbb{H})$, then $||f||_{L^2(\mathbb{H})} = ||\mathcal{F}_{\mathbb{H}} f||_{L^2(S_2)}$.*

Theorem 7 *([8, Fourier Inversion Theorem]). If $f \in S(\mathbb{H})$, Schwartz space on $\mathbb{R}^3$, then*

$$f(q) = \int_{\mathbb{R}^*} \operatorname{tr}\left(\beta_h(q)\mathcal{F}_{\mathbb{H}} f(h)\right) d\mu(h), \quad q \in \mathbb{H}. \tag{9}$$

Thus $\mathcal{F}_{\mathbb{H}}$ extends to an isometry from $L^2(\mathbb{H})$ into $L^2(S_2)$. In fact, it is onto as well. Furthermore if $f \in L^1(\mathbb{H})$, then by convolving f with an approximation of identity, we may use (9) to prove $\mathcal{F}_{\mathbb{H}}$ is injective on $L^1(\mathbb{H})$.

While the definition above is sufficient for our purposes, we remark that $\mathcal{F}_{\mathbb{H}}$ has been extended to much more general classes of function such as tempered distributions [2]. In [3,36], and much more generally in [20] the authors use the group Fourier transform to develop theory of pseudo-differential operators.

Finally, $\Gamma_\lambda := \{(0, k\pi R^2) \in \mathbb{H} : k \in \mathbb{Z}\}$, where $R = 1/\lambda$, is a discrete subgroup of the center of $\mathbb{H}$. Since $\beta_h(z,t) = e^{2iht}\beta_h(z,0)$, the representation β_h descends to the so-called reduced Heisenberg group $\mathbb{H}/\Gamma_\lambda$ if and only if $h \in \lambda^2\mathbb{Z}^*$. To a function $g \in L^1(\mathbb{H}/\Gamma_\lambda)$, we associate the *reduced Fourier transform*, defined as

$$\begin{aligned} &\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}(g) : \mathbb{Z}^* \to \mathcal{B}(\mathcal{H}) \\ &\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}(g)(n) := \int_{\mathbb{H}/\Gamma_\lambda} g(q)\beta_{n\lambda^2}(q)^* dq, \end{aligned} \tag{10}$$

where $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$.

Remark 8. The reduced Fourier transform defined above is not invertible unless we also consider the representations $(z,t) \mapsto e^{iz\cdot\xi}$; $\xi \in \mathbb{C}$, which are trivial on the center, in the definition. (Indeed, if $\partial_t g(z,t) = 0$, then $\mathcal{F}_{\mathbb{H}/\Gamma_\lambda} g = 0$.) This extension is not necessary for our purposes.

4. Proof of Theorems 1 and 2

4.1. The space of geodesics

Recall that $\mathbb{H}$ acts transitively on $\mathcal{G}_\lambda$ on the left. Since

$$(0, \pi R^2)\gamma_\lambda(s) = \left(Re^{is/R}, \tfrac{R}{2}(s + 2\pi R)\right) = \gamma_\lambda\left(s + 2\pi R\right);\ R = 1/\lambda, \tag{11}$$

the subgroup $\Gamma_\lambda := \{(0, k\pi R^2) \in \mathbb{H} : k \in \mathbb{Z}\}$ stabilizes $\mathcal{G}_\lambda$. Upon fixing γ_λ, we have the identification

$$\begin{aligned} \mathcal{G}_\lambda &\cong \mathbb{H}/\Gamma_\lambda \\ (z,t)\gamma_\lambda &\mapsto (z,t)\Gamma_\lambda. \end{aligned} \tag{12}$$

When $\lambda = 1$, we omit subscripts and write $\Gamma = \Gamma_1$.

Let $\{d\mu_\lambda(z,t) \cong dx \wedge dy \wedge dt$ be the Haar measure on $\mathbb{H}/\Gamma_\lambda$, and let $\mathcal{G}_\lambda$ inherit a multiple of the Haar measure, $d\mathcal{G}_\lambda := \lambda dx \wedge dy \wedge dt$, normalized to satisfy (16). Furthermore, let $d\mathcal{G} := \lambda e^{-\lambda} dx \wedge dy \wedge dt \wedge d\lambda$, with a weight chosen to ensure boundedness in Proposition 11.

4.2. Simplification to the reduced X-ray transform

The dilation map, $\delta_\lambda(z,t) := (\lambda z, \lambda^2 t)$, is an automorphism of the Heisenberg group for $\lambda \neq 0$. Furthermore,

$$\delta_\lambda : \Gamma_\lambda \ni (0, k\pi\lambda^{-2}) \mapsto (0, k\pi) \in \Gamma,$$

so $\delta_\lambda : \mathbb{H}/\Gamma_\lambda \to \mathbb{H}/\Gamma$ is well-defined. Denote by δ_λ^* the pullback operator (sometimes called the pullback relation) defined on functions:

$$\begin{aligned} \delta_\lambda^* &: L^1(\mathbb{H}) \to L^1(\mathbb{H}) & \delta_\lambda^* f(z,t) &= f(\lambda z, \lambda^2 t) \\ \delta_\lambda^* &: L^1(\mathbb{H}/\Gamma) \to L^1(\mathbb{H}/\Gamma_\lambda) & \delta_\lambda^* g((z,t)\Gamma_\lambda) &= g\left((\lambda z, \lambda^2 t)\Gamma\right). \end{aligned}$$

Remark 9. In the sequel, we write any function $g : \mathbb{H}/\Gamma_\lambda \to \mathbb{C}$ as $g(z,t)$, in place of $g((z,t)\Gamma_\lambda)$, understanding that the t variable is taken mod $\pi\lambda^{-2}$.

The dilation map δ_λ is relevant because it is a conformal map for the sub-Riemannian metric (with constant conformal factor λ). Consequently, we have the following homogeneity of the ray transform:

Proposition 10 *(Homogeneity of I). For $f \in C_c(\mathbb{H})$,*

$$I_\lambda f(z,t) = (1/\lambda)\delta_\lambda^* \left(I_1(\delta_{1/\lambda}^* f) \right)(z,t). \tag{13}$$

Proof. Note that dilation preserves geodesics but rescales their speed:

$$\delta_{1/\lambda}\gamma_1(s) = \gamma_\lambda(s/\lambda). \tag{14}$$

Then

$$\begin{aligned}
\delta_\lambda^*\left(I_1(\delta_{1/\lambda}^* f)\right)(z,t) &= I_1\left(\delta_{1/\lambda}^* f\right)(\lambda z, \lambda^2 t) \\
&= \int_{\mathbb{R}} \delta_{1/\lambda}^* f\left((\lambda z, \lambda^2 t)\gamma_1(s)\right) ds \\
&= \int_{\mathbb{R}} f\left(\delta_{1/\lambda}(\lambda z, \lambda^2 t)\delta_{1/\lambda}\left(\gamma_1(s)\right)\right) ds, \qquad \text{because } \delta_\lambda \in \mathrm{Aut}(\mathbb{H}), \\
&= \int_{\mathbb{R}} f\left((z,t)\gamma_\lambda(s/\lambda)\right) ds, \qquad \text{by (14)}, \\
&= \lambda \int_{\mathbb{R}} f\left((z,t)\gamma_\lambda(s)\right) ds = \lambda I_\lambda f(z,t). \quad \square
\end{aligned}$$

Next, we exploit the periodic symmetry of Heisenberg geodesics to reduce the X-ray transform to one period.

Proposition 11. *For any $\lambda > 0$, $I_\lambda : L^1(\mathbb{H}) \to L^1(\mathcal{G}_\lambda)$ is well-defined, bounded, and factors in the following way:*

$$\begin{array}{ccc}
L^1(\mathbb{H}) & \xrightarrow{\;I_\lambda\;} & L^1(\mathcal{G}_\lambda \cong \mathbb{H}/\Gamma_\lambda) \\
\big\downarrow{\scriptstyle P_\lambda} & \nearrow{\scriptstyle I_\lambda^{\mathrm{red}}} & \\
L^1(\mathbb{H}/\Gamma_\lambda) & &
\end{array}$$

where the maps which we call Central Periodization and the reduced X-ray transform are given by

$$P_\lambda f\,(z,t) = \sum_{k\in\mathbb{Z}} f\left(z, t + k\pi R^2\right), \qquad I_\lambda^{\mathrm{red}} g(z,t) = \int_0^{2\pi R} g\left((z,t)\gamma_\lambda(s)\right) ds;\ R = 1/\lambda.$$

Furthermore, $I : L^1(\mathbb{H}) \to L^1(\mathcal{G}, d\mathcal{G})$ is well-defined and bounded.

Proof. By homogeneity (13), and since pullback by δ_λ is bounded in the above L^1 spaces for $\lambda \neq 0$, it suffices to prove the proposition for $\lambda = 1$. For this case, we omit subscripts and write P and I^{red}. The map

$$C_c(\mathbb{H}) \ni f \mapsto \int_{\mathbb{H}/\Gamma} Pf\,(z,t)\, d\mu_1(z,t)$$

is a left-invariant positive linear functional on $C_c(\mathbb{H})$. By uniqueness of the Haar measure on $\mathbb{H}$ (which is just the Lebesgue measure), and the Riesz-Representation theorem, $\exists c > 0$ such that

$$\int_{\mathbb{H}/\Gamma} Pf\,(z,t)\,d\mu_1(z,t) = c\int_{\mathbb{H}} f(z,t)d(z,t), \tag{15}$$

and one may check that $c = 1$ (see [7, Thm. 2.49] for the general statement). So in particular, $||Pf||_{L^1(\mathbb{H}/\Gamma)} \leq ||f||_{L^1(\mathbb{H})}$.

For $g \in C_c(\mathbb{H}/\Gamma)$,

$$\begin{aligned}
&||I^{\text{red}}g||_{L^1(\mathcal{G}_1)} \\
&\quad= \int_{\mathcal{G}_1} |I^{\text{red}}g\,(z,t)\,|d\mathcal{G}_1 \\
&\quad= \int_{\mathbb{H}/\Gamma} \left| \int_0^{2\pi} g\left((z,t)\gamma_1(s)\right) ds \right| d\mu_1(z,t) \\
&\quad\leq \int_0^{2\pi}\int_{\mathbb{H}/\Gamma} |g\left((z,t)\gamma_1(s)\right)|d\mu_1(z,t)ds \\
&\quad= \int_0^{2\pi}\int_{\mathbb{H}/\Gamma} |g\left((z,t)\right)|d\mu_1\left((z,t)\gamma_1(s)^{-1}\right)ds \\
&\quad= \int_0^{2\pi}\int_{\mathbb{H}/\Gamma} |g\left((z,t)\right)|d\mu_1(z,t)ds, \qquad \text{since } \mathbb{H}/\Gamma \text{ is unimodular (i.e. } \mu_1 \text{ is bi-invariant)} \\
&\quad= 2\pi||g||_{L^1(\mathbb{H}/\Gamma)}.
\end{aligned}$$

Thus P and I^{red} extend to L^1 bounded maps. Given $f \in C_c(\mathbb{H})$, since $Pf \in C_c(\mathbb{H}/\Gamma)$ and

$$\begin{aligned}
I^{\text{red}}Pf(z,t) &= \int_0^{2\pi}\sum_{k\in\mathbb{Z}} f\left((z,t+k\pi)\gamma_1(s)\right)ds = \int_0^{2\pi}\sum_{k\in\mathbb{Z}} f\left((z,t)\gamma_1(s+2\pi k)\right)ds, \quad \text{by (11)}, \\
&= \sum_{k\in\mathbb{Z}}\int_0^{2\pi} f\left((z,t)\gamma_1(s+2\pi k)\right)ds = \sum_{k\in\mathbb{Z}}\int_{2\pi k}^{2\pi(k+1)} f\left((z,t)\gamma_1(s)\right)ds = I_1f(z,t),
\end{aligned}$$

we have $||I_1f||_{L^1(\mathcal{G}_1)} \leq 2\pi||f||_{L^1(\mathbb{H})}$. The third equality follows from uniform convergence of the integrand on the interval $[0,2\pi] \ni s$. Therefore I_1 extends to a bounded map from $L^1(\mathbb{H})$ to $L^1(\mathcal{G}_1)$. In particular one may check, using (13), that $||I_\lambda f||_{L^1(\mathcal{G}_\lambda)} = ||I_1 f||_{L^1(\mathcal{G}_1)} \leq 2\pi ||f||_{L^1(\mathbb{H})}$.

Finally, for $f \in L^1(\mathbb{H})$, we have

$$\begin{aligned} ||If||_{L^1(\mathcal{G})} :=& \int_{\mathcal{G}} |If(z,t,\lambda)| d\mathcal{G} \\ =& \int_0^\infty \int_{\mathcal{G}_\lambda} |I_\lambda f(z,t)| d\mathcal{G}_\lambda e^{-\lambda} d\lambda \\ \leq& 2\pi ||f||_{L^1(\mathbb{H})} \int_0^\infty e^{-\lambda} d\lambda = 2\pi ||f||_{L^1(\mathbb{H})} \end{aligned}$$

as desired. □

Remark 12. The reduced X-ray transform $I^{\mathrm{red}} : L^1(\mathbb{H}/\Gamma) \to L^1(\mathcal{G}_1)$ is not injective. In fact, if

$$g(z,t) = z^2 e^{-|z|^2} e^{4it},$$

then $I^{\mathrm{red}} g = 0$. In Appendix 6.1 we give essentially a Singular Value Decomposition of I^{red} and characterize its kernel on $L^2(\mathbb{H}/\Gamma)$.

Remark 13. From these computations, we may also deduce a sub-Riemannian Santaló formula:

$$\int_{\mathcal{G}_\lambda} I_\lambda f(z,t) d\mathcal{G}_\lambda = 2\pi \int_{\mathbb{H}} f(z,t) d(z,t), \quad f \in L^1(\mathbb{H}). \tag{16}$$

This is an example of a Santaló formula like those proven in [31], but without the latter's restriction to the "reduced unit cotangent bundle."

4.3. Lemmas on the group Fourier transform

We now prove a few general properties of the group Fourier transform. The first is a Poisson Summation Formula for $\mathbb{H} \to \mathbb{H}/\Gamma$ - a quick consequence of the classical version. The author has not found a reference for this version, but does not believe it is new.

Lemma 14 *(Poisson Summation Formula). If $f \in L^1(\mathbb{H})$, then*

$$\mathcal{F}_{\mathbb{H}/\Gamma}(Pf)(n) = \mathcal{F}_{\mathbb{H}} f(n), \quad \forall n \in \mathbb{Z}^*.$$

Proof. For $F, G \in \mathcal{H}$,

$$\begin{aligned}
&\langle \mathcal{F}_{\mathbb{H}/\Gamma}\left(Pf\right)(n)F, G\rangle_{\mathcal{H}} \\
&\quad := \int\limits_{\mathbb{H}/\Gamma} \sum_{k\in\mathbb{Z}} f(z, t+k\pi)\langle \beta_n(z,t)^* F, G\rangle_{\mathcal{H}} d\mu_1(z,t) \\
&\quad = \int\limits_{\mathbb{H}/\Gamma} \sum_{k\in\mathbb{Z}} f(z, t+k\pi)\langle \beta_n(z,t+k\pi)^* F, G\rangle_{\mathcal{H}} d\mu_1(z,t), \quad \text{since } \beta_n(z,t) = e^{2int}\beta_n(z,0), \\
&\quad = \int\limits_{\mathbb{H}} f(z,t)\langle \beta_n(z,t)^* F, G\rangle_{\mathcal{H}} d(z,t),
\end{aligned}$$

where the third equality follows from (15), and the fact that $f(z,t)\langle \beta_n(z,t)^* F, G\rangle_{\mathcal{H}} \in L^1(\mathbb{H})$ by the Cauchy-Schwartz inequality. Since F and G were arbitrary, the identity follows from the definition of the Bochner integral. □

Next, we observe how the Fourier transforms behave with respect to dilations.

Lemma 15 *(Dilation property). Fix $\lambda > 0$.*
If $f \in L^1(\mathbb{H})$, then

$$\mathcal{F}_{\mathbb{H}}\left(\delta_\lambda^* f\right)(h) = \lambda^{-4}\mathcal{F}_{\mathbb{H}} f(h/\lambda^2), \quad \forall h \in \mathbb{R}^*.$$

And if $g \in L^1(\mathbb{H}/\Gamma)$, then

$$\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}\left(\delta_\lambda^* g\right)(n) = \lambda^{-4}\mathcal{F}_{\mathbb{H}/\Gamma}(g)(n), \quad \forall n \in \mathbb{Z}^*.$$

We expect the above exponent of λ because the homogeneous dimension of the Heisenberg group is 4.

Proof.
$$\begin{aligned}
\mathcal{F}_{\mathbb{H}}\left(\delta_\lambda^* f\right)(h) &= \int\limits_{\mathbb{H}} f(\lambda z, \lambda^2 t)\beta_h(z,t)^* d(z,t) \\
&= \lambda^{-4}\int\limits_{\mathbb{H}} f(z,t)\beta_h\left(\lambda^{-1}z, \lambda^{-2}t\right)^* d(z,t) \\
&= \lambda^{-4}\int\limits_{\mathbb{H}} f(z,t)\beta_{h/\lambda^2}(z,t)^* d(z,t) = \lambda^{-4}\mathcal{F}_{\mathbb{H}} f(h/\lambda^2),
\end{aligned}$$

and the proof for $\mathcal{F}_{\mathbb{H}/\Gamma}$ is nearly identical. □

4.4. Proof of Theorem 2

The reduced X-ray transform I^{red} is equivariant with respect to left translation by $\mathbb{H}$ in the sense that

$$I^{\mathrm{red}}\left(L^*_{(w,s)}g\right)(z,t) = \int_0^{2\pi} L^*_{(w,s)}g\left((z,t)\gamma_1(\theta)\right)d\theta = \int_0^{2\pi} g\left((w,s)(z,t)\gamma_1(\theta)\right)d\theta$$
$$= I^{\mathrm{red}}g\left((w,s)(z,t)\right) = \left(L^*_{(w,s)}I^{\mathrm{red}}g\right)(z,t).$$

Thus, I^{red} is a convolution operator. In fact, if we define the compactly supported distribution $\kappa \in \mathcal{E}'(\mathbb{H}/\Gamma)$ by $\kappa(g) := \int_0^{2\pi} g\left(\gamma_1(\theta)^{-1}\right)d\theta$ then $I^{\mathrm{red}}g = \kappa * g$, where $f * g\,(z,t) := \int_{\mathbb{H}/\Gamma} f\left((z,t)(w,s)^{-1}\right)g(w,s)d(w,s)\Gamma$. Therefore, by an analogous Paley-Wiener theory [37, Ch. 1], we expect $\mathcal{F}_{\mathbb{H}/\Gamma}(\kappa)(n) \in \mathcal{B}(\mathcal{H})$, and $\mathcal{F}_{\mathbb{H}/\Gamma}\left(I^{\mathrm{red}}g\right)(n) = \mathcal{F}_{\mathbb{H}/\Gamma}(\kappa)(n) \circ \mathcal{F}_{\mathbb{H}/\Gamma}(g)(n)$. The next proposition makes this heuristic explicit.

Proposition 16. *If $g \in L^1(\mathbb{H}/\Gamma)$, then for all $n \in \mathbb{Z}^*$,*

$$\mathcal{F}_{\mathbb{H}/\Gamma}\left(I^{\mathrm{red}}g\right)(n) = (2\pi)\mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}/\Gamma}(g)(n)$$

with $\mathcal{J}_n$ defined in (5).

Proof.

$$\begin{aligned}
&\mathcal{F}_{\mathbb{H}/\Gamma}\left(I^{\mathrm{red}}g\right)(n)\\
&\quad := \int_{\mathbb{H}/\Gamma}\int_0^{2\pi} g\left((z,t)\gamma_1(s)\right)\beta_n(z,t)^* ds\, d\mu_1(z,t)\\
&\quad = \int_0^{2\pi}\int_{\mathbb{H}/\Gamma} g\left(z,t\right)\beta_n\left((z,t)\gamma_1(s)^{-1}\right)^* d\mu_1(z,t)ds, && \text{since } \mathbb{H}/\Gamma \text{ is unimodular,}\\
&\quad = \int_0^{2\pi}\int_{\mathbb{H}/\Gamma} g\left(z,t\right)\beta_n\left(\gamma_1(s)\right)\circ\beta_n(z,t)^* d\mu_1(z,t)ds, && \text{since } \beta_n(z,t) \text{ is a unitary rep,}\\
&\quad = \int_0^{2\pi}\beta_n\left(\gamma_1(s)\right)ds \circ \int_{\mathbb{H}/\Gamma} g\left(z,t\right)\beta_n(z,t)^* d\mu_1(z,t)\\
&\quad = (2\pi)\mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}/\Gamma}\left(g\right)(n)
\end{aligned}$$

where the “multiplier”

$$\mathcal{J}_n := \frac{1}{2\pi}\int_0^{2\pi} \beta_n\left(\gamma_1(s)\right) ds \tag{17}$$

is given explicitly on $F \in \mathcal{H}$ by

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \beta_n\left(\gamma_1(s)\right) F(\zeta) ds &= \frac{1}{2\pi}\int_0^{2\pi} \beta_n\left(e^{is}, s/2\right) F\left(\zeta\right) ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{ins-\sqrt{n}\zeta e^{-is}-\frac{n}{2}} F\left(\zeta+\sqrt{n}e^{is}\right) ds, \qquad z=\sqrt{n}e^{is}, \\
&= \frac{1}{2\pi i}\left(\frac{1}{en}\right)^{n/2} \oint z^{n-1} e^{-n\zeta/z} F\left(\zeta+z\right) dz
\end{aligned}$$

which is the same as (5). □

Remark 17. $\mathcal{J}_n$ is similar to the "representation integral" considered in [17], though $s \mapsto \beta_n\left(\gamma_1(s)\right)$ is not a homomorphism. Such integration of representations over geodesics also appears in [12], where the authors used the Principal Series representations of $SL(2,\mathbb{R})$ to show that the normal operator I^*I associated to the X-ray transform on constant negative curvature surfaces is a nontrivial function of the Laplace-Beltrami operator.

Together with Proposition 11, these imply the Heisenberg Fourier Slice Theorem:

Proof of Theorem 2. Let $f \in L^1(\mathbb{H})$, $\lambda > 0$ and $n \in \mathbb{Z}^*$. By Proposition 11 and 16, we have

$$\mathcal{F}_{\mathbb{H}/\Gamma}\left(I_1 f\right)(n) = \mathcal{F}_{\mathbb{H}/\Gamma}\left(I^{\mathrm{red}} P f\right)(n) = (2\pi)\mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}/\Gamma}\left(P f\right)(n) = (2\pi)\mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}} f(n). \tag{18}$$

Exploiting homogeneity of I,

$$\begin{aligned}
\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}\left(I_\lambda f\right)(n) &= \lambda^{-1}\mathcal{F}_{\mathbb{H}/\Gamma_\lambda}\left(\delta_\lambda^* I_1\left(\delta_{1/\lambda}^* f\right)\right)(n), && \text{Proposition 10} \\
&= \lambda^{-5}\mathcal{F}_{\mathbb{H}/\Gamma}\left(I_1\left(\delta_{1/\lambda}^* f\right)\right)(n), && \text{Lemma 15} \\
&= 2\pi\lambda^{-5}\mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}}\left(\delta_{1/\lambda}^* f\right)(n), && \text{by (18)}, \\
&= 2\pi\lambda^{-1}\mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}} f(n\lambda^2), && \text{Lemma 15}
\end{aligned}$$

as desired. □

Remark 18. In the special case when $n = 0$ or $h = 0$, the group Fourier transforms are qualitatively different; they are the Euclidean Fourier transform in the z variable (the precise sense in which this limiting behavior occurs is formalized by Geller in [8]). In this case, the Fourier Slice theorem takes the form

$$\widetilde{(I_\lambda f)}(\lambda\zeta, 0) = (2\pi/\lambda) J_0(|\zeta|) \widehat{f}(\lambda\zeta, 0); \quad \forall \lambda > 0,\ f \in L^1(\mathbb{H}),$$

where J_0 is the classical Bessel function of order zero, and

$$\widehat{f}(\zeta, 0) = \int_{\mathbb{C}} \int_{\mathbb{R}} f(z,t) e^{-i\zeta\cdot z} dt dz,\ f \in L^1(\mathbb{H}),$$

$$\widetilde{g}(\zeta, 0) = \int_{\mathbb{C}} \int_0^{\pi\lambda^{-2}} g(z,t) e^{-i\zeta\cdot z} dt dz;\ g \in L^1(\mathbb{H}/\Gamma_\lambda).$$

4.5. Proof of Theorem 1

We now make use of the Heisenberg Fourier Slice theorem to prove injectivity of I. First, we describe an important class of functions which are the cylindrical harmonics of the Heisenberg group.

With respect to the standard orthonormal basis $\{\omega_k(\zeta) = \zeta^k/\sqrt{k!} \in \mathbb{C} : k = 0, 1, ...\}$ of $\mathcal{H}$ the matrix coefficients of the Bargmann-Fock representation, (8), $M^h_{jk}(z,t) := \langle \beta_h(z,t)\omega_j, \omega_k \rangle_{\mathcal{H}}$ are given for $h > 0$ via a brute force computation by

$$M^h_{jk}(z,t) = \begin{cases} \sqrt{\frac{k!}{j!}} \left(+\sqrt{h} z\right)^{j-k} L^{(j-k)}_k \left(h|z|^2\right) e^{-h|z|^2/2} e^{2iht} & j \geq k \\ \sqrt{\frac{j!}{k!}} \left(-\sqrt{h}\overline{z}\right)^{k-j} L^{(k-j)}_j \left(h|z|^2\right) e^{-h|z|^2/2} e^{2iht} & j \leq k \end{cases}, \tag{19}$$

and $M^h_{jk}(z,t) = M^{|h|}_{jk}(\overline{z}, -t)$ for $h < 0$ (see Appendix 6.5 for conversion between Folland's [6, p. 64] and our conventions).

Here $L^{(\alpha)}_j(x)$ is the generalized Laguerre polynomial, defined recursively by

$$\begin{aligned} L^{(\alpha)}_0(x) &= 1 \\ L^{(\alpha)}_1(x) &= 1 + \alpha - x \\ (j+1) L^{(\alpha)}_{j+1}(x) &= (2j + 1 + \alpha - x) L^{(\alpha)}_j(x) - (j + \alpha) L^{(\alpha)}_{j-1}(x). \end{aligned} \tag{20}$$

The following mild generalization of (17) will be useful for subsequent computations.

Definition 19. For $n \in \mathbb{Z}^*$, let

$$\mathcal{J}_n(r) := \frac{1}{2\pi} \int_0^{2\pi} \beta_n \left(re^{i\theta}, \theta/2\right) d\theta, \quad r > 0. \tag{21}$$

In particular, $\mathcal{J}_n(1) = \mathcal{J}_n$, defined in (17).

Proposition 20 *(SVD of $\mathcal{J}_n(r)$). For every $n \in \mathbb{Z}^*$ and $r > 0$, the operator $\mathcal{J}_n(r) : \mathcal{H} \to \mathcal{H}$ is bounded in the operator-norm topology. Furthermore, $\mathcal{J}_{-n}(r) = \mathcal{J}_n(r)$, and, with respect to the orthonormal basis $\{\omega_j = \zeta^j/\sqrt{j!} : j = 0, 1, 2, ...\}$ of $\mathcal{H}$, we have*

$$\mathcal{J}_n(r)\omega_j = \sqrt{\frac{j!}{(j+n)!}} \left(nr^2\right)^{n/2} e^{-nr^2/2} L_j^{(n)}\left(nr^2\right)\omega_{j+n}, \quad \forall j \in \mathbb{N},\ n > 0. \tag{22}$$

Proof. $\mathcal{J}_n(r) : \mathcal{H} \to \mathcal{H}$ is bounded in the operator-norm topology for any $n \in \mathbb{Z}^*$ since

$$||\mathcal{J}_n(r)||_{\text{op}} \leq \frac{1}{2\pi} \int_0^{2\pi} ||\beta_n\left(re^{i\theta}, \theta/2\right)||_{\text{op}} d\theta = 1. \tag{23}$$

Note that, for $n \in \mathbb{Z}^*$,

$$\mathcal{J}_{-n} = \frac{1}{2\pi} \int_0^{2\pi} \beta_n(e^{-i\theta}, -\theta/2) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \beta_n\left(e^{i\theta}, \theta/2\right) d\theta = \mathcal{J}_n.$$

For $n > 0$,

$$\begin{aligned}
\langle \mathcal{J}_n(r)\omega_j, \omega_k\rangle_{\mathcal{H}} &= \frac{1}{2\pi} \int_0^{2\pi} \langle \beta_n\left(re^{i\theta}, \theta/2\right)\omega_j, \omega_k\rangle_{\mathcal{H}} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} M_{jk}^n\left(re^{i\theta}, \theta/2\right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{i(j-k+n)\theta} d\theta M_{jk}^n(r, 0) \qquad \text{observing symmetry in (19)} \\
&= \delta(j - k + n) M_{jk}^n(r, 0) \qquad (24) \\
&= M_{j,j+n}^n(r, 0),
\end{aligned}$$

in which case,

$$\mathcal{J}_n(r)\omega_j = M_{j,j+n}^n(r, 0)\omega_{j+n},$$

and, by (19),

$$M_{j,j+n}^n(r, 0) = \sqrt{\frac{j!}{(j+n)!}} \left(nr^2\right)^{n/2} e^{-nr^2/2} L_j^{(n)}\left(nr^2\right), \quad n > 0. \quad \square$$

Corollary 21. *Let $r > 0$ and $n \in \mathbb{Z}^*$ be fixed. The operator $\mathcal{J}_n(r)$ is injective if and only if $L_j^{(n)}\left(nr^2\right)$ is nonzero for all $j \in \mathbb{N}$.*

Proof. Since $\mathcal{J}_n(r)$ is bounded (by (23)), it is injective if and only if $L_j^{(n)}\left(nr^2\right)$ is nonzero for all $j \in \mathbb{N}$. $\square$

Proposition 22. *The operator $\mathcal{J}_n : \mathcal{H} \to \mathcal{H}$ is injective whenever n is an odd integer.*

Proof. Given $n \in 2\mathbb{Z}+1$, by Corollary 21, the operator $\mathcal{J}_n$ is injective if and only if the sequence $\{L_j^{(n)}(n)\}_{j=0}^{\infty}$ is nonvanishing.

Set $a_j^{(n)} = j!L_j^{(n)}(n) \in \mathbb{Z}$. Then $a_0^{(n)} = a_1^{(n)} = 1$, and by (20),

$$\begin{aligned} a_{j+1}^{(n)} =& (2j+1)a_j^{(n)} - j(j+n)a_{j-1}^{(n)} \\ =& a_j^{(n)} \pmod 2 \end{aligned}$$

since n is odd. Therefore $a_j^{(n)} = a_0^{(n)} = 1 \pmod 2$ for all $j = 0, 1, 2, \ldots$. In particular, $L_j^{(n)}(n) = a_j^{(n)}/j! \neq 0$ for $j \in \mathbb{N}$. Therefore $\mathcal{J}_n$ is injective whenever n is an odd integer. $\square$

Remark 23. We know that $\mathcal{J}_2$ is not injective since $L_2^{(2)}(2) = 0$. However, the author is not currently aware of a general statement characterizing all $(j, n) \in \mathbb{N} \times \mathbb{N}^*$ for which $L_j^{(n)}(n) = 0$. While knowing this is not essential for proving injectivity of I, it would provide more ways to invert I. This is because the space of geodesics is four dimensional, and so we only need a subset of the overdetermined data to reconstruct f from If.

The proof of Theorem 1, injectivity of the X-ray transform, is now almost immediate.

Proof of Theorem 1. Suppose $I_\lambda f = 0$ for all $\lambda \in (0, \eta)$, where $\eta > 0$. By the Heisenberg Fourier Slice Theorem (Theorem 2),

$$0 = \mathcal{J}_n \circ \mathcal{F}_{\mathbb{H}} f(n\lambda^2), \quad \forall n \in \mathbb{Z}^*, \ \forall \lambda \in (0, \eta).$$

By Proposition 22

$$0 = \mathcal{F}_{\mathbb{H}} f(n\lambda^2), \quad \forall n \in 2\mathbb{Z}+1, \ \forall \lambda \in (0, \eta). \tag{25}$$

In which case

$$0 = \mathcal{F}_{\mathbb{H}} f(h), \quad \forall h \in \bigcup_{n \in 2\mathbb{Z}+1} n\left(0, \eta^2\right) = \mathbb{R}^*.$$

Therefore $f = 0$ by the Fourier Inversion theorem for $\mathcal{F}_{\mathbb{H}}$. $\square$

5. X-ray transform for the taming metric g_ϵ

We use the same machinery to prove injectivity of the X-ray transform associated to the family of left-invariant *taming metrics* on $\mathbb{H}$. A taming metric on a sub-Riemannian manifold is a Riemannian metric whose restriction to the horizontal distribution equals the sub-Riemannian metric. See [26, Sec. 1.9].

Consider the family of left-invariant Riemannian metrics for $\epsilon > 0$:

$$g_\epsilon := dx^2 + dy^2 + (1/\epsilon)^2\,\Theta^2,$$

where $\Theta := dt - \frac{1}{2}(xdy - ydx)$ is a contact form for the Heisenberg distribution $\mathcal{D}$, defined in Section 3.1. Then g_ϵ is a taming metric for the sub-Riemannian metric $g = dx^2 + dy^2|_{\mathcal{D}}$. Indeed, since $\mathcal{D}_q = \ker\,\Theta_q, q \in \mathbb{H}$, we have $g_\epsilon|_{\mathcal{D}} = g$.

Geodesics of $(\mathbb{H}, g_\epsilon)$ converge uniformly to the sub-Riemannian geodesics as $\epsilon \to 0$, [5, p. 33]. The explicit expression for g_ϵ geodesics is derived in [5, Sec. 2.4.4]. We record the exponential map for g_ϵ in (42).

Remark 24. To avoid quantifying ϵ in every proposition of this section, with the exception of Theorems 3 and 4, we will assume that we have chosen a fixed $\epsilon > 0$.

Let $\mathcal{G}^\epsilon$ be the set of geodesics for g_ϵ without orientation and $\mathcal{G}^\epsilon_\lambda$ the subset of geodesics having charge λ (which is still a constant of motion). Geodesics with $\lambda \neq 0$ still project to circles in the plane, and those with $\lambda = 0$ project to lines; g_ϵ-geodesics differ from sub-Riemannian geodesics only by an ϵ-dependent velocity in the $T = \partial_t$ direction. Left translation by any element $(z,t) \in \mathbb{H}$ is a g_ϵ-isometry, and so $\mathbb{H}$ acts on $\mathcal{G}^\epsilon$ by pointwise left multiplication. This action does not change the value of λ and is a transitive action on each leaf $\mathcal{G}^\epsilon_\lambda$ when $\lambda \neq 0$.

We choose a particular geodesic γ^ϵ_λ to be the one whose projection to the plane is a unit-speed circular path of radius $R = 1/|\lambda|$ centered at the origin, and parameterize the set of g_ϵ geodesics having charge λ by

$$s \to (z,t)\gamma_\lambda(s), \quad \gamma^\epsilon_\lambda(s) = \left(Re^{is/R}, s\frac{(R^2 + 2\epsilon^2)}{2R}\right) \in \mathbb{H}; \quad R = 1/\lambda. \tag{26}$$

Remark 25. The geodesics described by (26) are not arclength parameterized; indeed, $g_\epsilon(\dot\gamma^\epsilon_\lambda(s), \dot\gamma^\epsilon_\lambda(s)) = 1 + \epsilon^2\lambda^2$. Instead, we insist that their projections to the plane are unit-speed.

We define the X-ray transform associated to the taming metric g_ϵ by

$$I^\epsilon f(z,t,\lambda) := I^\epsilon_\lambda f(z,t) := \int_{\mathbb{R}} f\left((z,t)\gamma^\epsilon_\lambda(s)\right) ds, \quad f \in C_c(\mathbb{H}). \tag{27}$$

Note that

$$\gamma_\lambda^\epsilon(s+2\pi R) = \gamma_\lambda^\epsilon(s)(0, \pi R^2 + 2\pi\epsilon^2). \tag{28}$$

Therefore the isotropy group of γ_λ^ϵ for the action of $\mathbb{H}$ by left translation on $\mathcal{G}_\lambda^\epsilon$ is

$$\Gamma_\lambda^\epsilon := \{(0, k\pi(R^2 + 2\epsilon^2)) \in \mathbb{H} : k \in \mathbb{Z}\}.$$

We have the identification

$$\begin{gathered}\mathcal{G}_\lambda^\epsilon \cong \mathbb{H}/\Gamma_\lambda^\epsilon \\ (z,t)\gamma_\lambda^\epsilon \mapsto (z,t)\Gamma_\lambda^\epsilon.\end{gathered}$$

Remark 26. Again, when $\lambda = 1$, we omit subscripts and write $\Gamma^\epsilon = \Gamma_1^\epsilon$. We will also write $g(z,t)$, for any function $g : \mathbb{H}/\Gamma_\lambda^\epsilon \to \mathbb{C}$, in place of $g\left((z,t)\Gamma_\lambda^\epsilon\right)$.

Let $\{d\mu_\lambda^\epsilon(z,t) \cong dx \wedge dy \wedge dt$ be the Haar measure on $\mathbb{H}/\Gamma_\lambda^\epsilon$, and let $\mathcal{G}_\lambda^\epsilon$ inherit a multiple of the Haar measure

$$d\mathcal{G}_\lambda^\epsilon := \lambda dx \wedge dy \wedge dt. \tag{29}$$

Furthermore, let

$$d\mathcal{G}^\epsilon := \lambda e^{-\lambda} dx \wedge dy \wedge dt \wedge d\lambda. \tag{30}$$

Note the homogeneity of geodesics with respect to dilation:

$$\delta_{1/\lambda}\gamma_1^{\epsilon\lambda}(s) = \gamma_\lambda^\epsilon(s/\lambda); \quad R = 1/\lambda. \tag{31}$$

Proposition 27 *(Homogeneity of I^ϵ). For $f \in C_c(\mathbb{H})$, we have*

$$I_\lambda^\epsilon(f)(z,t) = \lambda^{-1}\delta_\lambda^* I_1^{\epsilon\lambda}\left(\delta_{1/\lambda}^* f\right)(z,t). \tag{32}$$

Proof. This is essentially the same proof as (13):

$$\begin{aligned}\delta_\lambda^* I_1^{\epsilon\lambda}\left(\delta_{1/\lambda}^* f\right)(z,t) &= I_1^{\epsilon\lambda}\left(\delta_{1/\lambda}^* f\right)(\lambda z, \lambda^2 t) \\ &= \int\limits_{\mathbb{R}} \delta_{1/\lambda}^* f\left((\lambda z, \lambda^2 t)\gamma_1^{\epsilon\lambda}(s)\right) ds \\ &= \int\limits_{\mathbb{R}} f\left((z,t)\delta_{1/\lambda}\gamma_1^{\epsilon\lambda}(s)\right) ds\end{aligned}$$

$$= \int_{\mathbb{R}} f\left((z,t)\gamma^\epsilon_\lambda(s/\lambda)\right) ds, \qquad \text{by (31)},$$

$$= \lambda \int_{\mathbb{R}} \delta^*_{1/\lambda} f\left((\lambda z, \lambda^2 t)\gamma_1^{\epsilon\lambda}(\lambda s)\right) ds$$

$$= \lambda I^\epsilon_\lambda f(z,t). \qquad \square$$

Furthermore, in virtually the same way as Proposition 11, we reduce the X-ray transform I^ϵ to one period:

Proposition 28. *For any $\lambda > 0$, $I^\epsilon_\lambda : L^1(\mathbb{H}) \to L^1(\mathcal{G}^\epsilon_\lambda)$ is well-defined, bounded, and factors in the following way:*

$$\begin{array}{ccc} L^1(\mathbb{H}) & \xrightarrow{\;I^\epsilon_\lambda\;} & L^1(\mathcal{G}^\epsilon_\lambda \cong \mathbb{H}/\Gamma^\epsilon_\lambda) \\ {\scriptstyle P^\epsilon_\lambda}\big\downarrow & \nearrow {\scriptstyle I^{\epsilon,\mathrm{red}}_\lambda} & \\ L^1(\mathbb{H}/\Gamma^\epsilon_\lambda) & & \end{array}$$

where

$$P^\epsilon_\lambda f\,(z,t) = \sum_{k\in\mathbb{Z}} f\left(z, t + k\pi(R^2 + 2\epsilon^2)\right),$$
$$I^{\epsilon,\mathrm{red}}_\lambda g(z,t) := \int_0^{2\pi R} g\left((z,t)\gamma^\epsilon_\lambda(s)\right) ds;\; R = 1/\lambda. \tag{33}$$

Furthermore, $I^\epsilon : L^1(\mathbb{H}) \to L^1(\mathcal{G}^\epsilon, d\mathcal{G}^\epsilon)$ is well-defined and bounded.

Proof. By homogeneity (32), and since pullback by δ_λ is bounded in the above L^1 spaces for $\lambda \neq 0$, it suffices to prove the proposition for $\lambda = 1$. For this case, we omit subscripts and write P^ϵ and $I^{\epsilon,\mathrm{red}}$.

For exactly the same reason as (15), P^ϵ maps $C_c(\mathbb{H})$ to $C_c(\mathbb{H}/\Gamma^\epsilon)$, and

$$\int_{\mathbb{H}/\Gamma^\epsilon} P^\epsilon f\,(z,t)\, d\mu^\epsilon_1(z,t) = \int_{\mathbb{H}} f(z,t) d(z,t). \tag{34}$$

So in particular, $||P^\epsilon f||_{L^1(\mathbb{H}/\Gamma^\epsilon)} \leq ||f||_{L^1(\mathbb{H})}$.

For $g \in C_c(\mathbb{H}/\Gamma^\epsilon)$,

$$||I^{\epsilon,\mathrm{red}} g||_{L^1(\mathcal{G}^\epsilon_1)} = \int_{\mathcal{G}^\epsilon_1} |I^{\epsilon,\mathrm{red}} g\,(z,t)\,| d\mathcal{G}^\epsilon_1$$

$$= \int_{\mathbb{H}/\Gamma^\epsilon} \left| \int_0^{2\pi} g\left((z,t)\gamma_1^\epsilon(s)\right) ds \right| d\mu_1^\epsilon(z,t)$$

$$\leq \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^\epsilon} \left|g\left((z,t)\gamma_1^\epsilon(s)\right)\right| d\mu_1^\epsilon(z,t) ds$$

$$= \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^\epsilon} \left|g\left((z,t)\right)\right| d\mu_1^\epsilon \left((z,t)\gamma_1^\epsilon(s)^{-1}\right) ds$$

$$= \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^\epsilon} \left|g\left(z,t\right)\right| d\mu_1^\epsilon(z,t) ds, \qquad \text{since } \mathbb{H}/\Gamma^\epsilon \text{ is unimodular,}$$

$$= 2\pi ||g||_{L^1(\mathbb{H}/\Gamma^\epsilon)}.$$

Thus P^ϵ and $I^{\epsilon,\mathrm{red}}$ extend to L^1 bounded maps. Given $f \in C_c(\mathbb{H})$, since $Pf \in C_c(\mathbb{H}/\Gamma^\epsilon)$ and

$$I^{\epsilon,\mathrm{red}} P^\epsilon f(z,t)$$

$$= \int_0^{2\pi} \sum_{k\in\mathbb{Z}} f\left((z, t + k\pi(1+2\epsilon^2))\gamma_1^\epsilon(s)\right) ds = \int_0^{2\pi} \sum_{k\in\mathbb{Z}} f\left((z,t)\gamma_1^\epsilon(s + 2\pi k)\right) ds, \quad \text{by (28)},$$

$$= \sum_{k\in\mathbb{Z}} \int_0^{2\pi} f\left((z,t)\gamma_1^\epsilon(s+2\pi k)\right) ds = \sum_{k\in\mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} f\left((z,t)\gamma_1^\epsilon(s)\right) ds = I_1^\epsilon f(z,t),$$

we have $||I_1^\epsilon f||_{L^1(\mathcal{G}_1^\epsilon)} \leq 2\pi ||f||_{L^1(\mathbb{H})}$. The third equality follows from uniform convergence of the integrand on the interval $[0, 2\pi] \ni s$. Therefore I_1^ϵ extends to a bounded map from $L^1(\mathbb{H})$ to $L^1(\mathcal{G}_1^\epsilon)$. In particular, one may check, using (32), that $||I_\lambda^\epsilon f||_{L^1(\mathcal{G}_\lambda^\epsilon)} = ||I_1^\epsilon f||_{L^1(\mathcal{G}_1^\epsilon)} \leq 2\pi ||f||_{L^1(\mathbb{H})}$.

Finally we have, for $f \in L^1(\mathbb{H})$,

$$||I^\epsilon f||_{L^1(\mathcal{G}^\epsilon)} := \int_{\mathcal{G}^\epsilon} |I^\epsilon f(z,t,\lambda)| d\mathcal{G}^\epsilon$$

$$= \int_0^\infty \int_{\mathcal{G}_\lambda^\epsilon} |I_\lambda^\epsilon f(z,t)| d\mathcal{G}_\lambda^\epsilon e^{-\lambda} d\lambda$$

$$\leq 2\pi ||f||_{L^1(\mathbb{H})} \int_0^\infty e^{-\lambda} d\lambda = 2\pi ||f||_{L^1(\mathbb{H})}$$

as desired. □

Remark 29. From these computations, we may also deduce a Santaló formula for g_ϵ:

$$\int_{\mathcal{G}_\lambda^\epsilon} I_\lambda^\epsilon f(z,t) d\mathcal{G}_\lambda^\epsilon = 2\pi \int_{\mathbb{H}} f(z,t) d(z,t), \quad f \in L^1(\mathbb{H})$$

which refines the usual Santaló formula.

We note a Poisson Summation Formula for P^ϵ:

Lemma 30. *For $f \in L^1(\mathbb{H})$,*

$$\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}\left(P^\epsilon f\right)(n) = \mathcal{F}_{\mathbb{H}} f\left(\frac{n}{1+2\epsilon^2}\right), \quad \forall n \in \mathbb{Z}^*. \tag{35}$$

Proof. This is just a rescaling of Lemma 14. Observe that $\Gamma^\epsilon = (1+2\epsilon^2)\Gamma$. Using Lemma 15 with $\lambda = 1/\sqrt{1+2\epsilon^2}$, and noting that $\delta^*_{\sqrt{1+2\epsilon^2}} P^\epsilon f = P^1 \delta^*_{\sqrt{1+2\epsilon^2}} f$, we are done. □

Observe how the Fourier transform respects dilations:

Lemma 31. *For $g \in L^1(\mathbb{H}/\Gamma^\epsilon)$, $\lambda > 0$,*

$$\mathcal{F}_{\mathbb{H}/\Gamma_\lambda^\epsilon}\left(\delta_\lambda^* g\right) = \lambda^{-4} \mathcal{F}_{\mathbb{H}/\Gamma^{\epsilon\lambda}}(g)(n), \quad \forall n \in \mathbb{Z}^*. \tag{36}$$

Proof. Observe that $\Gamma_\lambda^\epsilon = \lambda^{-2}(1+2\epsilon^2)\Gamma$, and $\Gamma^{\epsilon\lambda} = (1+2\epsilon^2\lambda^2)\Gamma$. Then apply Lemma 15. □

As before, $I^{\epsilon,\mathrm{red}}$ is a convolution operator by a compactly supported distribution. We compute its generalized Fourier multiplier:

Proposition 32. *For $g \in L^1(\mathbb{H}/\Gamma^\epsilon)$,*

$$\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}\left(I^{\epsilon,\mathrm{red}} g\right)(n) = 2\pi \mathcal{J}_n\left(\frac{1}{\sqrt{1+2\epsilon^2}}\right) \circ \mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(g)(n), \quad \forall n \in \mathbb{Z}^*.$$

Proof.

$$\begin{aligned}
\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}\left(I^{\epsilon,\mathrm{red}} g\right)(n) &= \int_{\mathbb{H}/\Gamma^\epsilon} \int_0^{2\pi} g\left((z,t)\gamma_1^\epsilon(s)\right) \beta_{n/(1+2\epsilon^2)}(z,t)^* ds d\mu_1^\epsilon(z,t) \\
&= \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^\epsilon} g(z,t)\, \beta_{n/(1+2\epsilon^2)}\left((z,t)\gamma_1^\epsilon(s)^{-1}\right)^* d\mu_1^\epsilon(z,t) ds \\
&= \int_0^{2\pi} \int_{\mathbb{H}/\Gamma^\epsilon} g(z,t)\, \beta_{n/(1+2\epsilon^2)}\left(\gamma_1(s)\right) \circ \beta_{n/(1+2\epsilon^2)}(z,t)^* d\mu_1^\epsilon(z,t) ds
\end{aligned}$$

$$= \int_0^{2\pi} \beta_{n/(1+2\epsilon^2)}\left(\gamma_1^\epsilon(s)\right) ds \circ \mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(g)(n).$$

$$=: 2\pi \mathcal{J}_n\left(\frac{1}{\sqrt{1+2\epsilon^2}}\right) \circ \mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}(g)(n). \quad \square$$

We may now prove the Heisenberg Fourier Slice Theorem for g_ϵ:

Proof of Theorem 3. Combining Proposition 28 and 32,

$$\mathcal{F}_{\mathbb{H}/\Gamma^\epsilon}\left(I^\epsilon f\right) = 2\pi \mathcal{J}_n\left(\frac{1}{\sqrt{1+2\epsilon^2}}\right) \circ \mathcal{F}_{\mathbb{H}} f\left(\frac{n}{1+2\epsilon^2}\right). \tag{37}$$

Now, exploiting homogeneity of I^ϵ,

$$\begin{aligned}
&\mathcal{F}_{\mathbb{H}/\Gamma_\lambda^\epsilon}\left(I_\lambda^\epsilon f\right)(n) \\
&\quad = \lambda^{-1}\mathcal{F}_{\mathbb{H}/\Gamma_\lambda^\epsilon}\left(\delta_\lambda^* I_1^{\epsilon\lambda}\left(\delta_{1/\lambda}^* f\right)\right)(n), && \text{by Proposition 27,} \\
&\quad = \lambda^{-5}\mathcal{F}_{\mathbb{H}/\Gamma^{\epsilon\lambda}}\left(I_1^{\epsilon\lambda}\left(\delta_{1/\lambda}^* f\right)\right)(n), && \text{by Lemma 31,} \\
&\quad = 2\pi\lambda^{-5}\mathcal{J}_n\left(\frac{1}{\sqrt{1+2\epsilon^2\lambda^2}}\right)\circ \mathcal{F}_{\mathbb{H}}(\delta_{1/\lambda}^* f)\left(\frac{n}{1+2\epsilon^2\lambda^2}\right), && \text{by (37),} \\
&\quad = (2\pi/\lambda)\mathcal{J}_n\left(\frac{1}{\sqrt{1+2\epsilon^2\lambda^2}}\right)\circ \mathcal{F}_{\mathbb{H}} f\left(\frac{n\lambda^2}{1+2\epsilon^2\lambda^2}\right), && \text{by Lemma 31.} \quad \square
\end{aligned}$$

Proposition 33. *Let $\epsilon > 0$ and $n \in \mathbb{Z}^*$ be fixed. Then $\mathcal{J}_n\left(\frac{1}{\sqrt{1+2\epsilon^2\lambda^2}}\right) : \mathcal{H} \to \mathcal{H}$ is injective for almost all $\lambda > 0$.*

Proof. Set $r = \frac{1}{\sqrt{1+2\epsilon^2\lambda^2}}$. By Corollary 21, the operator $\mathcal{J}_n(r)$ is injective if and only if nr^2 is not a zero of $L_j^{(n)}$ for any $j \in \mathbb{N}$. Since there are only countably many such zeros, the proposition follows. $\square$

We now have the tools to prove injectivity of the taming X-ray transform I^ϵ:

Proof of Theorem 4. Suppose, $I_\lambda^\epsilon f = 0$ for all $\lambda \in (0, \eta)$, where $\eta > 0$. Then by Theorem 3 and Proposition 33,

$$0 = \mathcal{F}_{\mathbb{H}} f\left(\frac{n\lambda^2}{1+2\epsilon^2\lambda^2}\right)$$

for almost all $\lambda \in (0, \eta)$, and all $n \in \mathbb{Z}^*$. Let A be the set of all such $\lambda \in (0, \eta)$, and $B = \{\lambda^2/(1+2\epsilon^2\lambda^2) : \lambda \in A\}$. Then in other words

$$0 = \mathcal{F}_{\mathbb{H}} f(h) \qquad \forall h \in \bigcup_{n\in\mathbb{Z}} nB.$$

Since B has full measure on the interval $\left(0, \frac{\eta^2}{1+2\epsilon^2\eta^2}\right)$, we know $\mathcal{F}_{\mathbb{H}} f = 0$ almost everywhere. Therefore $f = 0$ by the Fourier Inversion Theorem. □

6. Appendix

6.1. SVD of $I^{\mathrm{red}}|_{{}^0L^2(\mathbb{H}/\Gamma)}$

While not strictly necessary for our main result, the computation in Proposition 20 also gives us the SVD of I^{red} when restricted to a specific subspace. Here, similarly with [24], we implicitly exploit the fact that I^{red} intertwines the Heisenberg Laplacian on $\mathbb{H}$ with another differential operator on $\mathbb{H}/\Gamma$ for which the functions M^h_{jk}, $h \in \mathbb{R}^*$, and M^n_{jk}, $n \in \mathbb{Z}^*$, are eigenfunctions, respectively.

Consider the subspaces of $L^2(\mathbb{H}/\Gamma)$

$$L^2(\mathbb{C}) \cong \{f \in L^2(\mathbb{H}/\Gamma) : f(z,t) = f(z,0),\ \forall (z,t) \in \mathbb{H}/\Gamma\}$$

$${}^0L^2(\mathbb{H}/\Gamma) := \{f \in L^2(\mathbb{H}/\Gamma) : \int_0^\pi f(z,t)dt = 0,\ \forall z \in \mathbb{C}\}.$$

Lemma 34. *We have the orthogonal decomposition*

$$L^2(\mathbb{H}/\Gamma) \cong L^2(\mathbb{C}) \oplus {}^0L^2(\mathbb{H}/\Gamma). \tag{38}$$

Proof. Given $f \in L^2(\mathbb{H}/\Gamma)$, let

$$f_0(z,t) := \frac{1}{\pi}\int_0^\pi f(z,t)dt \qquad \text{and} \qquad g = f - f_0.$$

Then $f_0 \in L^2(\mathbb{C})$ and $g \in {}^0L^2(\mathbb{H}/\Gamma)$.

Furthermore, for arbitrary $f_0 \in L^2(\mathbb{C})$, and $g \in {}^0L^2(\mathbb{H}/\Gamma)$,

$$\int_{\mathbb{H}/\Gamma} f_0(z,t)\overline{g(z,t)}d\mu_1(z,t) = \int_{\mathbb{C}} f_0(z)\int_0^\pi \overline{g(z,t)}dtdz = 0.$$

The orthogonal decomposition (38) follows. □

In what follows, set

$$\psi^n_{jk} := \frac{\sqrt{|n|}}{\pi}M^n_{jk}; \quad j,k \in \mathbb{N},\ n \in \mathbb{Z}^* \tag{39}$$

for M^n_{jk} defined in (19). The functions ψ^n_{jk} for, $n \in \mathbb{Z}^*$ and $j, k \in \mathbb{N}$, form an orthonormal basis for ${}^0L^2(\mathbb{H}/\Gamma)$. (See [37, Ch. 4], where the author uses slightly different notation.)

Proposition 35.

$$I^{\mathrm{red}} : L^2(\mathbb{H}/\Gamma) \to L^2(\mathbb{H}/\Gamma)$$

is well-defined and bounded.

Proof. For $g \in C_c(\mathbb{H}/\Gamma)$, the Cauchy-Schwartz Inequality yields

$$|I^{\mathrm{red}} g(z,t)|^2 = \left(\int_0^{2\pi} \left| g\left((z,t)(e^{i\theta}, \theta/2)\right) \right| d\theta \right)^2 \leq 2\pi \int_0^{2\pi} \left| g\left((z,t)(e^{i\theta}, \theta/2)\right) \right|^2 d\theta. \tag{40}$$

Then

$$\begin{aligned}
&||I^{\mathrm{red}} g||^2_{L^2(\mathbb{H}/\Gamma)} \\
&= \int_{\mathbb{H}/\Gamma} |I^{\mathrm{red}} g(z,t)|^2 \{d\mu_1(z,t) \\
&\leq (2\pi) \int_0^{2\pi} \int_{\mathbb{H}/\Gamma} \left| g\left((z,t)(e^{i\theta}, \theta/2)\right) \right|^2 d\mu_1(z,t) d\theta, && \text{by (40)}, \\
&= (2\pi)^2 \int_{\mathbb{H}/\Gamma} \left| g\left((z,t)\right) \right|^2 d\mu_1(z,t), && \text{by left-invariance of } \mu_1, \\
&= (2\pi)^2 ||g||^2_{L^2(\mathbb{H}/\Gamma)},
\end{aligned}$$

so I^{red} extends to a bounded function from $L^2(\mathbb{H}/\Gamma)$ to itself. □

Proposition 36. I^{red} *preserves the orthogonal decomposition in Lemma 34. i.e.,*

$$\begin{aligned}
I^{\mathrm{red}}|_{L^2(\mathbb{C})} &: L^2(\mathbb{C}) \to L^2(\mathbb{C}) \\
I^{\mathrm{red}}|_{{}^0L^2(\mathbb{H}/\Gamma)} &: {}^0L^2(\mathbb{H}/\Gamma) \to {}^0L^2(\mathbb{H}/\Gamma).
\end{aligned}$$

Furthermore, the restriction $I^{\mathrm{red}}|_{L^2(\mathbb{C})}$ *is essentially* 2π *times the Mean Value Transform* M^1.

Proof. For $f \in L^2(\mathbb{C})$,

$$I^{\mathrm{red}}|_{L^2(\mathbb{C})} f(z,t) = \int_0^{2\pi} f\left((z,t)(e^{i\theta}, \theta/2)\right) d\theta = \int_0^{2\pi} f\left(z + e^{i\theta}, t + \theta/2 + \tfrac{1}{2}\mathrm{Im}\left(\overline{z} e^{i\theta}\right)\right) d\theta$$

$$= \int_0^{2\pi} f(z+e^{i\theta})d\theta = 2\pi M^1 f(z),$$

and so $I^{\text{red}} f \in L^2(\mathbb{C})$.

For $g \in {}^0L^2(\mathbb{H}/\Gamma)$,

$$\begin{aligned}\int_0^{\pi} I^{\text{red}} g(z,t)dt &= \int_0^{\pi}\int_0^{2\pi} g\left(z+e^{i\theta}, t+\theta/2+\tfrac{1}{2}\text{Im}\left(\overline{z}e^{i\theta}\right)\right) d\theta dt \\ &= \int_0^{2\pi}\int_0^{\pi} g(z+e^{i\theta},t)dtd\theta = 0,\end{aligned}$$

so that $I^{\text{red}} g \in {}^0L^2(\mathbb{H}/\Gamma)$. □

We know that $I^{\text{red}}|_{L^2(\mathbb{C})} = 2\pi M^1$ has a continuous spectrum (see (2), or Remark 18), so we restrict the reduced X-ray transform to ${}^0L^2(\mathbb{H}/\Gamma)$, where it has a discrete spectrum, and compute the Singular Value Decomposition there.

Theorem 37 *(SVD of $I^{red}|_{{}^0L^2(\mathbb{H}/\Gamma)}$). For all $n \in \mathbb{Z}^*$ and $j, k \in \mathbb{N}$,*

$$I^{\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma)} \psi^n_{jk} = 2\pi\sqrt{\frac{j!}{(j+|n|)!}} (|n|/e)^{|n|/2} L_j^{(|n|)}(|n|)\psi^n_{j+|n|,k}.$$

Proof. Note that, for $(w,s), (z,t) \in \mathbb{H}$

$$\begin{aligned}M^n_{jk}\left((w,s)(z,t)\right) &= \langle \beta_n\left((w,s)(z,t)\right)\omega_j, \omega_k\rangle_{\mathcal{H}} = \langle \beta_n((w,s)) \circ \beta_n((z,t))\omega_j, \omega_k\rangle_{\mathcal{H}} \\ &= \sum_{l=0}^{\infty} \langle \beta_n((w,s))\omega_l, \omega_k\rangle_{\mathcal{H}} \langle \beta_n((z,t))\omega_j, \omega_l\rangle_{\mathcal{H}} \\ &= \sum_{l=0}^{\infty} M^n_{jl}((z,t))M^n_{lk}((w,s)).\end{aligned}$$

Then

$$\begin{aligned}&I^{\text{red}}|_{{}^0L^2(\mathbb{H}/\Gamma)} \psi^n_{jk}(z,t) \\ &= \frac{\sqrt{|n|}}{\pi} \int_0^{2\pi} M^n_{jk}\left((z,t)(e^{i\theta},\theta/2)\right) d\theta \\ &= \frac{\sqrt{|n|}}{\pi} \sum_{l=0}^{\infty} \int_0^{2\pi} M^n_{jl}(e^{i\theta},\theta/2) M^n_{lk}(z,t) d\theta\end{aligned}$$

$$= \frac{\sqrt{|n|}}{\pi} \sum_{l=0}^{\infty} \delta(j - l + |n|) M^n_{jl}(1,0) M^n_{lk}(z,t), \qquad \text{by (24) in Proposition 22,}$$

$$= M^n_{j,j+n}(1,0)\psi^n_{j+|n|,k}(z,t)$$

$$= 2\pi \sqrt{\frac{j!}{(j+|n|)!}} (|n|/e)^{|n|/2} L_j^{(|n|)}(|n|)\psi^n_{j+|n|,k}(z,t). \qquad \square$$

In view of Proposition 36 and Theorem 37, the kernel of I^{red} on $L^2(\mathbb{H}/\Gamma)$ is given by the L^2-closure of

$$\text{Span}\{\psi^n_{jk} : j,k \in \mathbb{N}, n \in \mathbb{Z}^*, L_j^{(|n|)}(|n|) = 0\}$$

We know this kernel contains at least the closure of $\{\psi^2_{2,k} : k = 0,1,2...\}$ since $L_2^{(2)}(2) = 0$. Determining the entire kernel will require a number-theoretic argument (see Remark 23).

6.2. Exponential map for Heisenberg geodesics

The sub-Riemannian flow maps from the unit cotangent bundle $U^*\mathbb{H} := H^{-1}(\frac{1}{2})$ to itself. We work in the left-trivialization of the unit cotangent bundle: $U^*\mathbb{H} \cong \mathbb{H} \times U(1) \times \mathbb{R} \ni (z,t,e^{i\phi},\lambda)$. The exponential map $\exp : \mathbb{R} \times U^*\mathbb{H} \to \mathbb{H}$ is given in these coordinates by

$$\exp_{(z,t)}\left(s(e^{i\phi},\lambda)\right) = (z,t) \begin{cases} \left(e^{i\phi}\frac{(e^{i\lambda s}-1)}{i\lambda}, \frac{\lambda s - \sin(\lambda s)}{2\lambda^2}\right) & \lambda \neq 0 \\ \left(se^{i\phi}, 0\right) & \lambda = 0 \end{cases} \tag{41}$$

(see [26, Ch. 1]). As a function of s, this describes the unit-speed geodesic with initial point (z,t) whose projection to the plane is a counterclockwise-parameterized circle of radius $R = 1/|\lambda|$ with initial velocity in the direction of ϕ if $\lambda > 0$, and $\phi + \pi$ if $\lambda < 0$. If $\lambda = 0$ the projection is a straight line in the direction ϕ. The geodesics in (1) are obtained by rotations and left translation of (41).

The Riemannian exponential map $\exp^\epsilon$ for g_ϵ is given in the same coordinates by

$$\exp^\epsilon_{(z,t)}\left(s(e^{i\phi},\lambda)\right) = \exp_{(z,t)}\left(s(e^{i\phi},\lambda)\right)(0, \epsilon^2\lambda s) \tag{42}$$

(see [26, Thm. 11.8] for an explanation). Because we are using cylindrical coordinates in the fibers, neither of these exponential maps describe geodesics with initial condition strictly in the λ direction. In the case of g, these geodesics are fixed points in $\mathbb{H}$, and in the case of g_ϵ these geodesics are integral curves of the Reeb vector field $\epsilon^2\lambda T$. In both cases, the X-ray transforms are inverted without considering these geodesics.

6.3. Bessel functions

The classical Bessel function of order n is defined by

$$J_n(r) := \frac{1}{2\pi i^n}\int_0^{2\pi} e^{ir\cos\theta}e^{-in\theta}d\theta. \tag{43}$$

6.4. Infinitesimal representation

Define the complex vector fields on $\mathbb{H}$:

$$Z := \frac{1}{2}(X - iY), \qquad \overline{Z} := \frac{1}{2}(X + iY)$$

where X and Y are given in (6). Then $\beta_h : \mathbb{H} \to \mathcal{U}(\mathcal{H})$ as defined in (8) is the unique strongly continuous unitary group homomorphism for which, on the level of Lie algebras,

$$(\beta_h)_* Z = \sqrt{h}\partial_\zeta, \qquad (\beta_h)_* \overline{Z} = -\sqrt{h}\zeta, \qquad (\beta_h)_* T = 2h.$$

Fix $F \in \mathcal{H}$ and $(z, t) \in \mathbb{H}$. To obtain (8), let $G_h(\tau, \zeta)$ be unique solution to the differential equation

$$\frac{d}{d\tau}G_h(\tau,\zeta) = (\beta_h)_*\left(tT + zZ + \overline{z}\overline{Z}\right)G_h(\tau,\zeta) = \left(2iht + \sqrt{h}(z\partial_\zeta - \overline{z}\zeta)\right)G_h(\tau,\zeta)$$

subject to the condition $G_h(0,\zeta) = F(\zeta)$. Then $\beta_h(z,t)F(\zeta) := G_h(1,\zeta)$. See [6, Ch. 1 Sec 3] to see this worked out for the Schrödinger representation.

6.5. Alternate conventions

Folland [6] defines the Bargmann-Fock representation on the 1-parameter family of Hilbert spaces

$$\mathcal{H}^h := \left\{F : \mathbb{C} \to \mathbb{C},\ \text{holomorphic } : h\int_{\mathbb{C}} |F(\zeta)|^2 e^{-\pi h|\zeta|^2}d\zeta < \infty\right\}, \quad h > 0,$$

and $\mathcal{H}^h := \left\{F : \overline{F} \in \mathcal{H}^{|h|}\right\}$ for $h < 0$.

For $h \in \mathbb{R}^*$ and $\lambda > 0$, the maps

$$\begin{aligned} S_\lambda : \mathcal{H}^h \to \mathcal{H}^{\lambda h}; \qquad & S(F)(\zeta) := F(\sqrt{\lambda}\zeta) \\ c : \mathcal{H}^h \to \mathcal{H}^{-h}; \qquad & c(F) := \overline{F} \end{aligned}$$

are all isometries.

Folland defines the Fock representation, for $h > 0$, as

$$\beta_h^{\mathrm{Fol}}(z,t)F(\zeta) := e^{2\pi hit - \pi h\zeta\overline{z} - \pi h|z|^2/2}F(\zeta + z), \quad F \in \mathcal{H}^h$$

and $\beta_h^{\mathrm{Fol}}(z,t) = c \circ \beta_{|h|}^{\mathrm{Fol}}(\overline{z}, -t) \circ c$ for $h < 0$.

Our definition is rescaled so that every β_h acts on the same space $\mathcal{H} = \mathcal{H}^{1/\pi}$. Folland's definition, β_h^{Fol}, is related to ours via

$$\beta_h^{\mathrm{Fol}}(z,t) = S_{\pi h} \circ \beta_{\pi h}(z,t) \circ S_{\pi h}^{-1}, \quad h > 0.$$

An advantage of this convention is that as h varies, β_h varies by precomposition with automorphisms of $\mathbb{H}$:

$$\beta_h(z,t) = \beta_1(\sqrt{h}z, ht), \quad \text{for } h > 0$$
$$\beta_h(z,t) = \beta_{|h|}(\overline{z}, -t), \quad \text{for } h < 0.$$

Granted, an advantage of Folland's definition is that the Fourier transform defined with β_h^{Fol} does "converge" to the Euclidean Fourier transform as $h \to 0$.

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