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PACMAN RENORMALIZATION AND SELF-SIMILARITY OF THE MANDELBROT SET NEAR SIEGEL PARAMETERS

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1. Introduction

Renormalization was introduced into dynamics in the mid 1970s by Feigenbaum, Coullet, and Tresser and since then has established itself as a powerful tool for penetrating into the small-scale structure of phase portraits and bifurcation loci. It turned out to be challenging to develop a rigorous mathematical theory of renormalization (for example, to prove hyperbolicity of the renormalization operator), but every time when this is achieved, plentiful deep consequences reward the effort.

The complex quadratic family provided us with several important renormalization schemes: quadratic-like, near-parabolic, and Siegel. All three are intimately related to the observed self-similarity of the Mandelbrot set M and to the celebrated MLC Conjecture on the local connectivity of M. The conjecture comes in two flavors, “primitive” and “satellite”. Development of the quadratic-like renormalization has led to substantial progress in the primitive case, while the near-parabolic renormalization has given an insight into the satellite situation.

In this paper we design a new “pacman” renormalization and prove the hyperbolicity of the corresponding renormalization operator. It implies the hyperbolicity

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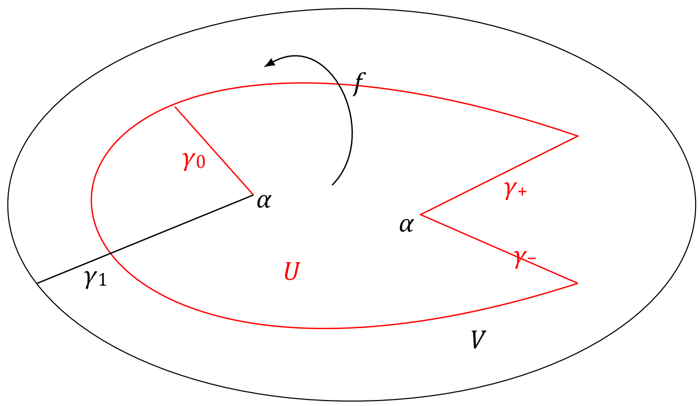

Figure 1. A (full) pacman is a 2 : 1 map $f: U \to V$ such that the critical arc γ_1 has three preimages: γ_0, γ_+, and γ_-.

of the Siegel renormalization for arbitrary periodic combinatorics (resolving a problem going back to the early 1980s) and gives an insight into the self-similarity of the Mandelbrot set near the main cardioid. In the second part of this project [DL] we will give further applications by proving local connectivity of the Mandelbrot set at some satellite parameters of bounded type (that had been previously out of reach) and showing that the corresponding Julia sets have positive area.

1.1. **Statements of the results.** Although the Mandelbrot set M is highly nonhomogeneous, it possesses some remarkable self-similarity features. Most notable is the presence of baby Mandelbrot sets inside M which are almost indistinguishable from M itself. The explanation of this phenomenon is provided by renormalization theory for quadratic-like maps, which has been a central theme in holomorphic dynamics since the mid-1980s (see [DH2,S,McM1,L1] and references therein).

By exploring the pictures, one can also observe that the Mandelbrot set has selfsimilarity features near its main cardioid. For instance, as Figure 2 indicates, near the (anti)golden mean point, the $(\mathrm{p}_n/\mathrm{p}_{n+2})$-limbs of M scale down at rate λ^{-2n}, where

$\lambda = (1+\sqrt{5})/2$ and p_n are the Fibonacci numbers. The goal of this paper is to develop a renormalization theory responsible for this phenomenon.

Our renormalization operator acts on the space of "pacmen", which are holomorphic maps $f\colon (U,\alpha) \to (V,\alpha)$ between two nested domains (see Figure 1), such that $f\colon U \setminus \gamma_0 \to V \setminus \gamma_1$ is a double branched covering, where γ_1 is an arc connecting α to ∂V. The *pacman renormalization* $\mathrm{R}f$ of f (see Figure 5) is defined by removing the sector S_1 bounded by γ_1 and its image γ_2, and taking the first return map to the remaining space; see §2 for precise definitions. Note that it acts on the rotation

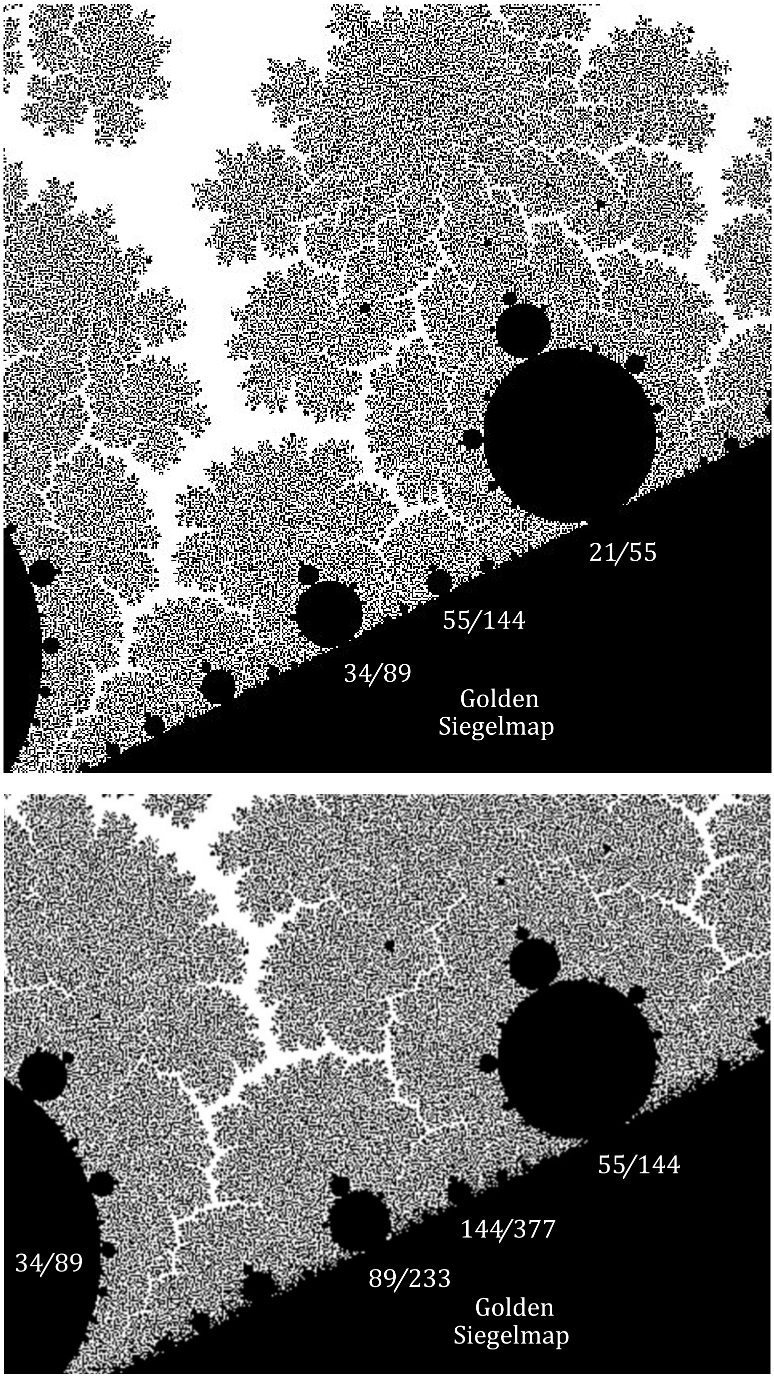

Figure 2. Limbs 8/21,21/55,55/144,144/377,... scale geometrically fast on the right-hand side of the (anti)golden Siegel parameter, while limbs 5/13,13/34,34/89,89/233,... scale geometrically fast on the left-hand side. The bottom picture is a zoom of the top picture.

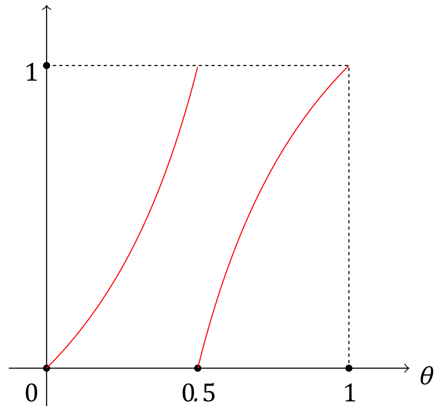

Figure 3. The map (1.1) induces a 2 : 1 map on R/Z.

numbers as

$$\theta \longrightarrow \frac{\theta}{1-\theta} \text{ if } 0 \leq \theta \leq \frac{1}{2}, \quad \theta \longrightarrow \frac{2\theta-1}{\theta} \text{ if } \frac{1}{2} \leq \theta \leq 1; \tag{1.1}$$

the graph of (1.1) is shown on Figure 3 (see Appendix A, in particular, (A.2)).

Let us denote by Θ_{per} the set of *combinatorially periodic* rotation numbers; i.e., rotation numbers periodic under (1.1). Numbers in Θ_{per} belong to the cycles of numbers with periodic continued fraction expansion.

A pacman is called *Siegel* with rotation number θ if α is a Siegel fixed point with rotation number θ whose closed Siegel disk is a quasidisk compactly contained in U (subject to extra technical assumptions; see Definition 3.1).

Theorem 1.1 (Hyperbolicity of the renormalization). *For any rotation number $\theta \in \Theta_{\text{per}}$, the pacman renormalization operator* R *has a unique periodic point f which is a Siegel pacman with rotation number θ. This periodic point is hyperbolic with one-dimensional unstable manifold. Moreover, the stable manifold of f consists of all Siegel pacmen.*

The problem of hyperbolicity goes back to the work of physicists [Wi,MN,MP]; see §1.3 for the description of the previous progress in the area.

Corollary 1.2 (Stability of Siegel maps). *Let f be a Siegel pacman with rotation number $\theta \in \Theta_{\text{per}}$. Consider the space $N_\theta(f)$ of maps sufficiently close to f whose α-fixed point has rotation number θ. Then the Siegel disk of g depends continuously on $g \in N_\theta(f)$.*

In fact, the stability of Siegel disks on the unstable manifold is one of the main steps in the proof of Theorem 1.1.

Let $c(\theta)$, $\theta \in \mathrm{R}/\mathrm{Z}$, be the parameterization of the main cardioid C by the rotation number θ. At any parabolic point $c(\mathrm{p}/\mathrm{q})$, there is a satellite hyperbolic component $\mathbf{\Delta}_{\mathrm{p}/\mathrm{q}}$

of M attached to $c(\mathrm{p/q})$. Let $a_{\mathrm{p/q}}$ be the *center* of this component, i.e., the unique superattracting parameter inside $\mathbf{\Delta}_{\mathrm{p/q}}$.

In this paper, the notation $\alpha_n \sim \beta_n$ will mean that $\alpha_n/\beta_n \to \mathrm{const} \neq 0$.

Theorem 1.3 (Scaling Theorem). *Let $\theta \in \Theta_{\mathrm{per}}$ be a rotation number, and let $\mathrm{p}_n/\mathrm{q}_n$ be its continued fraction approximands. Then*

$$|c(\theta) - a_{\mathrm{p}_n/\mathrm{q}_n}| \sim \frac{1}{\mathrm{q}_n^2}.$$

See Theorem 8.2 for a more precise version of the Scaling Theorem. In particular, Theorem 8.2 is stated for any one-parameter space witnessing the bifurcation of a Siegel map.

Self-similarity of the Mandelbrot set near the (anti)golden parameter is illustrated on Figure 2. Theorem 8.2 says that the centers of satellite hyperbolic components indeed scale as the picture predicts. In [DL] we combine methods of Theorem 1.3 with methods and ideas from transcendental dynamics to obtain a scaling law for a much larger class of parameters. The self-similarity of the whole limbs is still an open question. This problem is closely related to the problem of the realization of parameter rays for the transcendental family representing the unstable manifold.

We believe that our methods allow one to extend Theorems 1.1 and 1.3 to the case of rotation numbers of *bounded* type; the details will appear elsewhere. We conjecture that an analogous statement is true for arbitrary combinatorics, which would provide us with good geometric control of the *molecule* of the Mandelbrot set (see Appendix C).

1.2. **Outline of the proof.** We let

- $\mathbf{e}(z) = e^{2\pi i z}$;
- $p_\theta : z \mapsto \mathbf{e}(\theta) z + z^2$;
- P_θ be the set of pacmen with rotation number $\theta \in \mathbb{R}/\mathbb{Z}$;
- Θ_{bnd} be the set of *combinatorially bounded* rotation numbers (i.e., rotation numbers whose continued fraction expansions have bounded coefficients).

Let us first review Siegel renormalization theory which is the most relevant to our results; for extra historical comments on the progress in this program see §1.3.

Any holomorphic map $f : (U_f, \alpha) \to (\mathbb{C}, \alpha)$ whose fixed point α is neutral with rotation number $\theta \in \Theta_{\mathrm{per}}$ is locally linearizable near α. Its maximal completely invariant linearization domain Z_f is called the *Siegel disk* of f. If $\overline{Z}_f$ is a quasidisk compactly contained in U_f whose boundary contains exactly one critical point, then f is called a (unicritical) *Siegel map*. For any $\theta \in \Theta_{\mathrm{per}}$, the quadratic polynomial p_θ and any Siegel pacman give examples of Siegel maps (see §3, in particular Theorem 3.2).

There are two versions of the Siegel renormalization theory: *holomorphic commuting pairs renormalization* and *cylinder renormalization*. The former was developed by McMullen [McM2] (see also an earlier work by Stirnemann [St]) who proved, for any rotation number $\theta \in \Theta_{\mathrm{per}}$, the existence of a renormalization periodic point f and the exponential convergence of the renormalizations $\mathcal{R}^n_{\mathrm{cp}}(p_\theta)$ to the orbit of f. McMullen also studied the maximal domain of analyticity for f.

The cylinder renormalization $\mathrm{R}_{\mathrm{cyl}}$ was introduced by Yampolsky who showed that f can be transformed into a periodic point for $\mathrm{R}_{\mathrm{cyl}}$ with a *finite codimension* stable manifold $\mathcal{W}^s(f_\star)$ and an *at least one-dimensional* unstable manifold $\mathcal{W}^u(f_\star)$

[Ya]. Inou and Shishikura [IS] established hyperbolicity of the cylinder renormalization for high type Siegel parameters, and Gaidashev and Yampolsky [GY] proved it for the golden rotation number (see §1.3). However, the general conjecture that f is hyperbolic with dim $\mathcal{W}^u(f_\star) = 1$ remained open.

Let us now select our favorite $\theta \in \Theta_{\mathrm{per}}$; it is fixed under some iterate of (1.1). Then the corresponding iterate of the Siegel renormalization fixes f, so below we will refer to the f as "renormalization fixed points".

We start our paper (§2) by discussing an interplay between a "pacman" and a "prepacman". The latter (see Figure 6) is a piecewise holomorphic map with two branches $f_\pm : U_\pm \to S$, one of which is univalent while the other has "degree 1.5", with a single critical point. Such an object can be obtained from a pacman by cutting along the critical arc γ_1. For technical reasons, we "truncate" both pacmen and prepacmen by removing a small disk around the co-α point; see Figure 4.

Then we define, in three steps, the pacman renormalization. First we define a "pre-renormalization" (Definition 2.3) as a prepacman obtained as the first return map to an appropriate sector S. Then, by gluing the boundary arcs of S, we obtain an "abstract" pacman. Finally, we embed this pacman back into the complex plane.

There are some choices involved in this definition. We proceed to show that near any renormalizable pacman f, the choices can be made so that we obtain a holomorphic operator R in a Banach ball (Theorem 2.7).

In Section 3 we analyze the structure of Siegel pacmen f. The key result is that any Siegel map can be renormalized (in an appropriate sense) to a Siegel pacman (Corollary 3.7), where the rotation number changes as an iterate of (1.1); see Lemma 3.18.

In the case when $f = f_\star$ is the Siegel renormalization fixed point, this provides us with the pacman renormalization fixed point (§3.7). Moreover, the pacman renormalization R becomes a compact holomorphic operator in a Banach neighborhood of f, with at least one-dimensional unstable manifolds $\mathcal{W}^u(f_\star)$; see Theorem 3.16. Along these lines, we introduce and discuss associated geometric objects (§3.1): the pacman "Julia sets" K(f) and J(f), "bubble chains", and "external rays". We also use them to show, via the pullback argument, that any two combinatorially equivalent Siegel pacmen are hybrid equivalent (Theorem 3.11), i.e., there is a qc conjugacy between them which is conformal on the Siegel disk.

For a Siegel pacman f, any renormalization prepacman can be "spread around" to provide us with a dynamical tiling of a neighborhood of the Siegel disk; see §4.2 and Figures 7 and 15. Moreover, this tiling is robust under perturbations of f, even when the rotation number gets changed; see Theorem 4.6. In this case, the domain filled with the tiles can be used as the central "bubble" for the perturbed map f, replacing for many purposes the original Siegel disk $Z_\star$ of $f_\star$. In particular, it allows us to control long-term f^n-pullbacks of small disks D centered at $\partial Z_\star$ (making sure that these pullbacks are not "bitten" by the pacman's mouth). This is the crucial technical result of this paper (Key Lemma 4.8).

When f is the renormalization fixed point and the perturbed map f belongs to its unstable manifold $\mathcal{W}^u(f_\star)$, then we can apply this construction to the antirenormalizations $\mathrm{R}^{-n}f$. This allows us to show that the maximal holomorphic extension of the associated prepacman is a σ-proper map $\mathbf{F} = (\mathbf{f}_\pm : \mathbf{X}_\pm \to \mathbb{C})$, where $\mathbf{X}_\pm$ are plane domains (Theorem 5.1).

Applying this result to a parabolic map $f \in W^u(f_\star)$, we conclude that its attracting Leau-Fatou flower contains the critical point, so the critical point is non-escaping under the dynamics (Corollary 6.4).

After this preparation, we are ready to proof Theorem 1.1; see §7. Assuming for the sake of contradiction that $\dim \mathcal{W}^u(f_\star) > 1$, we can find a holomorphic curve $\Gamma_\star \subset \mathcal{W}^u(f_\star)$ through f consisting of Siegel pacmen with the same rotation number. Approximating this curve with parabolic curves $\Gamma_n \subset \mathcal{W}^u(f_\star)$, we conclude that the critical point is non-escaping for $f \in \Gamma_\star$. This allows us to apply Yampolsky's holomorphic motions argument [Ya] to show that $\dim \mathcal{W}^u(f_\star) = 1$.

Finally, using the small orbits argument of [L1], we prove that f is hyperbolic under the pacman renormalization, completing the proof.

Along these lines we prove the stability of Siegel maps (see Corollary 7.9): if a small perturbation of a Siegel map f fixes the multiplier of the α-fixed point, then the new map g is again a Siegel map. Moreover, the Siegel quasidisk $\overline{Z}_g$ is in a small neighborhood of $\overline{Z}_f$.

To derive Theorem 1.3 from Theorem 1.1, we need to show that the centers of the hyperbolic components in question are represented on the unstable manifold $\mathcal{W}^u(f_\star)$. We first show that the roots of these components are represented on $\mathcal{W}^u(f_\star)$ which requires good control of the corresponding pacmen Julia sets (see §6.5), and robustness of the renormalization with respect to a particular choice of cutting arcs; see Appendix B. Then we use quasiconformal deformation techniques to reach the desired centers from the parabolic points; see §8.

Throughout the paper we use Appendix B containing a topological preparation justifying robustness of the antirenormalizations with respect to the choice of cutting arcs.

In Appendix C we formulate the Molecule Conjecture on the existence of a pacman hyperbolic operator with the one-dimensional unstable foliation whose horseshoe is parametrized by the boundary of the main molecule of the Mandelbrot set. This conjecture would imply the MLC for all infinitely renormalizable parameters of satellite type.

1.3. **More historical comments.** Renormalization of Siegel maps appeared first in the work by physicists (see [Wi,MN,MP]) as a mechanism for self-similarity of the golden mean Siegel disk near the critical point. A few years later, Douady and Ghys discovered a surgery that reduces previously unaccessible geometric problems for Siegel disks[1] of bounded type to much better understood problems for critical circle maps. This led, in particular, to the local connectivity result for Siegel Julia sets of bounded type (Petersen [Pe]) and also became a key to the mathematical study of the Siegel renormalization. In particular, the McMullen-Yampolsky theory mentioned above (see §1.2) is based upon this machinery.

Holomorphic commuting pairs (as well as almost commuting holomorphic pairs) were studied by Stirnemannhe [St], who gave a computer-assisted proof of the existence of a renormalization fixed point with a golden-mean Siegel disk and showed that the renormalizations of the quadratic polynomial with the golden-mean Siegel disk converge to that fixed point. Recently, Gaidashev and Yampolsky gave a computer-assisted proof of the hyperbolicity of the renormalization for the golden mean rotation number [GY].

On the other hand, in the mid 2000's, Inou and Shishikura proved the existence and hyperbolicity of Siegel renormalization fixed points *of sufficiently high combinatorial type* using a completely different approach, based upon the parabolic perturbation theory [IS]. For a different viewpoint on this result see [Ya].

The proof in [IS] involves certain computer estimates. A computer-free proof of hyperbolicity in high type was presented by Cheritat [Che]. His approach also gives a proof of hyperbolicity for high type in the unicritical case $z^d + c$.

The Siegel renormalization theory achieved further prominence when it was used for constructing examples of Julia sets of positive area (see Buff-Cheritat [BC] and Avila-Lyubich [AL2]).

A different line of research emerged in the 1980s in the work of Branner and Douady who discovered a *surgery* that embeds the 1/2-limb of the Mandelbrot set into the 1/3-limb [BD]. This surgery is the prototype for the pacman renormalization that we are developing in this paper.

Note also that according to the Yoccoz inequality, the p/q-limb of the Mandelbrot set has size $O(1/\mathfrak{q})$. It is believed, though, that $1/\mathfrak{q}^2$ is the right scaling. The pacman renormalization can eventually provide an insight into this problem.

Remark 1.4. Genadi Levin has informed us about his unpublished work where it is proven, by different methods, that

$$|a_{\mathfrak{p}/\mathfrak{q}} - c(\mathfrak{p}/\mathfrak{q})| \leq \frac{C}{\mathfrak{q}^2}, C > 0 \quad (1.2)$$

where $a_{\mathfrak{p}/\mathfrak{q}}$ is the center of the p/q-satellite hyperbolic component and $c(\mathfrak{p}/\mathfrak{q})$ is its root. He has also informed us that (1.2) was independently established by Mitsuhiro Shishikura. Note that Theorem 1.3 gives a precise asymptotics for $|a_{\mathfrak{p}/\mathfrak{q}} - c(\mathfrak{p}/\mathfrak{q})|$.

1.4. **Notation.** We often write a partial map as $f: W \to W$; this means that $\mathrm{Dom} f \cup \mathrm{Im} f \subset W$.

A *simple arc* is an embedding of a closed interval. We often say that a simple arc $\ell: [0,1] \to \mathbb{C}$ *connects* $\ell(0)$ and $\ell(1)$. A *simple closed curve* or a *Jordan curve* is an

[1] The original surgery applies to Siegel polynomials only. Its extension to general Siegel maps leads to *quasicritical* circle maps; see [AL2].

[2] Unless the germ of f has finite order.

embedding of the unit circle. A *simple curve* is either a simple closed curve or a simple arc.

A *closed topological disk* is a subset of a plane homeomorphic to the closed unit disk. In particular, the boundary of a closed topological disk is a Jordan curve. A *quasidisk* is a closed topological disk qc-homeomorphic to the closed unit disk.

Given a subset U of the plane, we denote by $\mathrm{int}U$ the interior of U.

Let U be a closed topological disk. For simplicity we say that a homeomorphism $f: U \to \mathbb{C}$ is *conformal* if $f \mid \mathrm{int}U$ is conformal. Note that if U is a quasidisk, then such an f admits a qc extension through ∂U.

A *closed sector*, or *topological triangle*, S is a closed topological disk with two distinguished simple arcs γ_-, γ_+ in ∂S meeting at the *vertex v of S* satisfying $\{v\} = \gamma_- \cap \gamma_+$. Suppose further that $\gamma_-, \mathrm{int}S, \gamma_+$ have clockwise orientation at v. Then γ_- is called the *left boundary of S* while γ_+ is called the *right boundary* of S. A closed *topological rectangle* is a closed topological disk with four marked sides.

Let $f: (W,\alpha) \to (\mathbb{C},\alpha)$ be a holomorphic map with a distinguished α-fixed point. We will usually denote by λ the multiplier at the α-fixed point. If $\lambda = \mathbf{e}(\varphi)$ with $\varphi \in \mathbb{R}$, then φ is called the *rotation number of f*. If, moreover, $\varphi = \mathrm{p}/\mathrm{q} \in \mathbb{Q}$, then p/q coincides with the *combinatorial rotation number*:[2] there is a cycle of q local attracting petals at α and f maps the ith petal to i+p counting counterclockwise.

Consider a continuous map $f: U \to \mathbb{C}$ and let $S \subset \mathbb{C}$ be a connected set. An *f-lift* is a connected component of $f^{-1}(S)$. Let

$$x_0, x_1, \dots, x_n, x_{i+1} = f(x_i)$$

be an f orbit with $x_n \in S$. The connected component of $f^{-n}(S)$ containing x_0 is called the *pullback of S along the orbit $x_0, \dots, x_n$*.

To keep notation simple, we will often suppress indices. For example, we denote a pacman by $f: U_f \to V$, however a pacman indexed by i is denoted as $f_i: U_i \to V$ instead of $f_i: U_{f_i} \to V$.

Consider two partial maps $f: X \;\; X$ and $g: Y \;\; Y$. A homeomorphism $h: X \to Y$ is *equivariant* if

$$h \circ f(x) = g \circ h(x) \tag{1.3}$$

for all x with $x \in \mathrm{Dom}f$ and $h(x) \in \mathrm{Dom}g$. If (1.3) holds for all $x \in T$, then we say that h is *equivariant on T.*

We will usually denote an analytic renormalization operator as "R", i.e., Rf is a renormalization of f obtained by an analytic change of variables. A renormalization postcomposed with a straightening will be denoted by "R"; for example, $\mathsf{R}_s: \mathrm{M}_s \to \mathrm{M}$ is the Douady-Hubbard straightening map from a small copy M_s of M to the Mandelbrot set. The action of the renormalization operator on the rotation numbers will be denoted by "R".

Slightly abusing notation, we will often identify a triangulation (or a lamination) with its support.

2. Pacman renormalization operator

Definition 2.1 (Full pacman). Consider a closed topological disk $\overline{V}$ with a simple arc γ_1 connecting a boundary point of V to a point α in the interior. We will call γ_1 the *critical arc* of the pacman. A *full pacman* is a map

$$f: \overline{U} \to \overline{V}$$

such that (see Figure 1)

- $f(\alpha) = \alpha$;
- $\overline{U}$ is a closed topological disk with $U \subset \overline{V}$;
- the critical arc γ_1 has exactly three lifts $\gamma_0 \subset U$ and $\gamma_-, \gamma_+ \subset \partial U$ such that γ_0 starts at the fixed point α while γ_-, γ_+ start at the prefixed point α; we assume that γ_1 does not intersect $\gamma_0, \gamma_-, \gamma_+$ away from α;
- $f: U \to V$ is analytic and $f: U \setminus \gamma_0 \to V \setminus \gamma_1$ is a two-to-one branched covering;
- f admits a locally conformal extension through $\partial U \setminus \{\alpha'\}$.

Since $f: U \setminus \gamma_0 \to V \setminus \gamma_1$ is a two-to-one branched cover, f has a unique critical point, called $c_0(f)$, in $U \setminus \gamma_0$. We denote by $c_1(f)$ the image of c_0.

We will mostly consider truncated pacmen or simply pacmen defined as follows. Consider first a full pacman $f: U \to V$ and let O be a small closed topological disk around $\alpha \in \operatorname{int} O \not\ni c_1(f)$ and assume that γ_1 cross-intersects ∂O at a single point. Then $f^{-1}(O)$ consists of two connected components, call them $O_0 \ni \alpha$ and $O_0' \ni \alpha'$. We obtain a truncated pacman

$$f: (U \setminus O_0', O_0) \to (V, O). \tag{2.1}$$

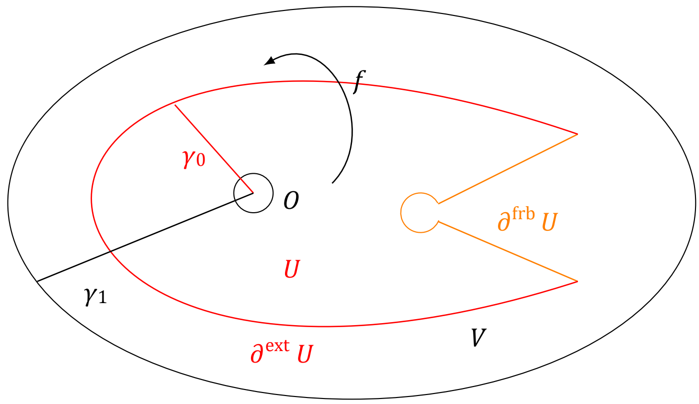

Figure 4. A pacman is a truncated version of a full pacman (see Figure 1); it is an almost 2 : 1 map $f\colon (U,O_0) \to (V,O)$ with $f(\partial U) \subset \partial V \cup \gamma_1 \cup \partial O$.

A *pacman* is an analytic map as in (2.1) admitting a locally conformal extension through ∂U such that f can be topologically extended to a full pacman; see Figure 4. In particular, every point in $V \setminus O$ has two preimages while every point in O has a single preimage.

2.1. **Dynamical objects.** Let us fix a pacman $f\colon U \to V$. Note that objects below are sensitive to small deformations of ∂U. The *non-escaping* set of a pacman is

$$\mathrm{K}_f := \bigcap_{n\geq 0} f^{-n}(\overline{U}).$$

The *escaping set* is $V \setminus \mathrm{K}_f$.

We recognize the following two subsets of the boundary of U: the *external boundary* $\partial^{\text{ext}}U := f^{-1}(\partial V)$ and *the forbidden part of the boundary* $\partial^{\text{frb}}U := \overline{\partial U \setminus \partial^{\text{ext}}U}$.

Suppose $\ell_0 : [0,1] \to \overline{V}$ is an arc connecting a point in K_f to ∂V. We define inductively images $\ell_m : [0,1] \to V$ for $m \leq M \in \{1,2,\dots,\infty\}$ as follows. Suppose

$t_m \leq 1$ is the maximal parameter such that the image of $[0,t_m]$ under $_m$ is within $\overline{U}$. If $\ell_m(t_m) \in \partial^{\text{ext}}U$, then we say ℓ_{m+1} is *defined* and we set $\ell_{m+1}(t) := f(\ell_m(t/t_m))$ for $t \leq 1$. Abusing notation, we write

$$\ell_m = f(\ell_{m-1}).$$

We define *external rays* of a pacman in the following way. Let us embed a rectangle R in $\overline{V}\setminus U$ so that the bottom horizontal side B is equal to $\partial^{\text{ext}}U$ and the top horizontal side T is a subset of ∂V. The images of the vertical lines within R

form a lamination of $\overline{V}\setminus U$. We pull back this lamination to all iterated preimages $f^{-n}(\text{R})$. Leaves of this lamination that start at ∂V are called *external ray segments* of f; infinite external ray segments are called *external rays* of f. Note that if γ is an external ray, then $f(\gamma)$, as defined in the previous paragraph, is also an external ray.

We have two maps from B to T: one is the natural identification π along the vertical lines, the other is the map $f: B \dashrightarrow T$ which is defined only on $f^{-1}(T)$. Composition thereof, $\phi = \pi^{-1} \circ f: B \dashrightarrow B$ is a partially defined two-to-one map.
We consider the set $\text{A} \subset B$ of all points for which the whole forward orbit is welldefined. Then A is completely invariant and there is a unique orientation-preserving map $\theta: \text{A} \to \text{S}^1$ which semiconjugates $\varphi: \text{A} \to \text{A}$ to the doubling map of the circle.

We say that $\theta(a)$ is the *angle* of the external ray segment passing through the point a.

An external ray segment passing through a point $a \in \text{A}$ is infinite (i.e., it is an external ray) if and only if it hits neither an iterated precritical point nor an iterated lift of $\partial^{\text{frb}}U$. The latter possibility is a major technical issue we have to deal with.

2.2. **Prime pacman renormalization.** Let us first give an example of a prime renormalization of full pacmen where we cut out the sector bounded by γ_1 and γ_2; see Figure 5. This renormalization is motivated by the surgery procedure that Branner and Douady [BD] used to construct a map between the Rabbit $L_{1/3}$ and the Basilica $L_{1/2}$ limbs of the Mandelbrot set; see Appendix C.1. Pacman renormalization will be defined in §2.3.

Recall that a sector S is a closed topological disk with two distinguished arcs in ∂S meeting at a single point, called the vertex of S. Suppose $f: U \to V$ is a full pacman and

(A) γ_0, γ_1, and $\gamma_2 := f(\gamma_1)$ are mutually disjoint except for the fixed point α.

Denote by S_1 the closed sector of V bounded by $\gamma_1 \cup \gamma_2$ and not containing γ_0. Let us further assume that

(B) S_1 does not contain the critical value; and
(C) $\gamma_- \cup \gamma_+ \subset V \setminus S_1$.

$' = f^{-1}(\gamma_2) \cap \gamma_1$ and γ_2 Let V be the Riemann surface $\overline{}$ with boundary obtained from V $\setminus \operatorname{int} S_1$ by gluing γ_1 : by means of f. This means that there is a quotient map $\overline{V} \setminus \operatorname{int} S_1 \to$

$$\psi : \quad V$$

such that ψ is conformal in $V \setminus S_1$ while $\psi(z) = \psi(f(z)) \in V$ for all $z \in \gamma_1'$. Let us select an embedding $\widehat{V} \hookrightarrow \mathbb{C}$.

The sector S_1 has two f-lifts; let S_0 be the lift of S_1 attached to α and let S_0' be the lift of S_1 attached to α. Condition (B) implies that $\gamma_- \cup \gamma_+ \subset V \setminus S_0$. Define

$$\bar{f}(z) := \begin{cases} f(z) & \text{if } z \in U \setminus (S_1 \cup S_0 \cup S_0') \\ f^2(z) & \text{if } z \in S_0 \cap f^{-1}(U). \end{cases},$$

Then the map $\bar{f}$ descends via ψ into a full pacman $\hat{f} : \widehat{U} \to \widehat{V}$ with the critical ray $\hat{\gamma}_1$.

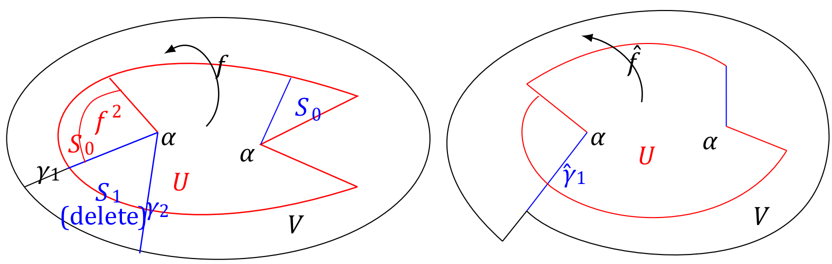

Figure 5. Prime renormalization of a pacman: delete the sector S_1, forget in U the sector S_0' attached to α, and iterate f twice on S_0. By gluing $\widehat{}$ $\widehat{\gamma_1}$ and γ_2 along $f : \gamma_1 \to \gamma_2$ we obtain a new pacman $\hat{f} : U \to V$.

2.3. **Pacman renormalization.** Let us start with defining an analogue of commuting pairs for pacmen.

A map $\psi : S \to \overline{V}$ from a closed sector (S, β_-, β_+) onto a closed topological disk

$V \subset \mathbb{C}$ is called a *gluing* if ψ is conformal in the interior of S, $\psi(\beta_-) = \psi(\beta_+)$, and ψ can be conformally extended to a neighborhood of any point in $\beta_- \cup \beta_+$ except the vertex of S.

Definition 2.2 (Prepacmen; Figure 6). Consider a sector S with boundary rays β_-, β_+ and with an interior ray β_0 that divides S into two subsectors T_-, T_+. Let $f_- : U_- \to S, f_+ : U_+ \to S$ be a pair of holomorphic maps, defined on $U_- \subset T_-, U_+ \subset T_+$. We say that $F = (S, f_-, f_+)$ is a *prepacman* if there exists a gluing ψ of S which projects (f_-, f_+) onto a pacman $f : U \to V$, where β_-, β_+ are mapped to the critical arc γ_1 and β_0 is mapped to γ_0.

The map ψ is called a *renormalization change of variables.* The definition implies that f and f_+ commute in a neighborhood of β_0. Note – that every pacman $f : U \to V$ has a prepacman obtained by cutting V along the critical arc γ_1.

Dynamical objects (such as the non-escaping set) of a prepacman F are preimages of the corresponding dynamical objects of f under ψ.

Definition 2.3 (Pacman renormalization; Figure 7). We say that a pacman $f : U \to V$ is *renormalizable* if there exists a prepacman

$$G = (g_- = f^a : U_- \to S, \qquad g_+ = f^b : U_+ \to S)$$

defined on a sector $S \subset V$ with vertex at α such that g_-, g_+ are iterates of f realizing the first return map to S and such that the f-orbits of U_-, U_+ before – they return to S cover a neighborhood of α compactly contained in U. We call G the *pre-renormalization* of f and the pacman $g : \widehat{U} \to \widehat{V}$ is the *renormalization* of f.

The numbers a,b are the *renormalization return times*.

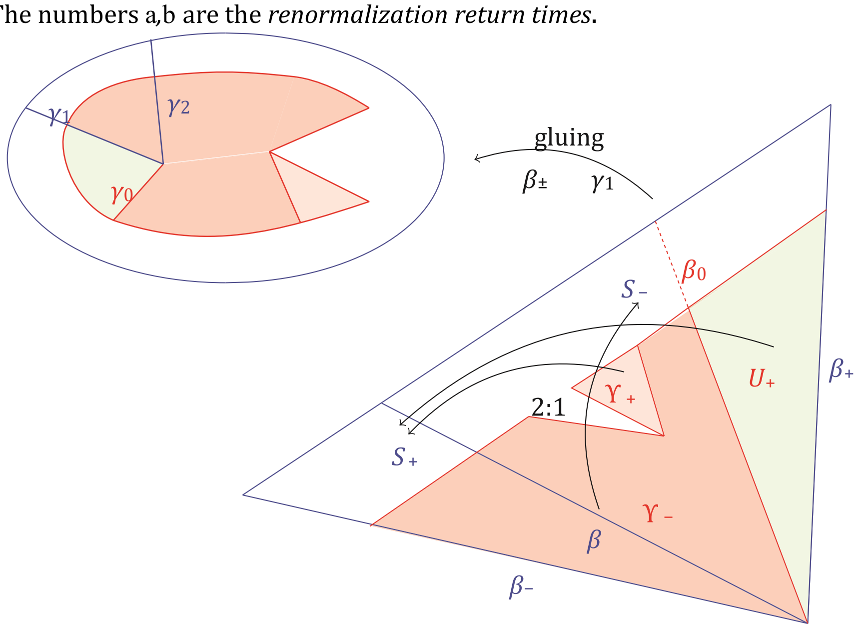

Figure 6. A (full) prepacman ($f_-: U_- \to S, f_+: U_+ \to S$). We have $U_- = \Upsilon_- \cup \Upsilon_+$ and f_- maps Υ_- two-to-one to S_- and it maps Υ_+ to S_+. The map f_+ maps U_+ univalently onto S_+. After gluing β_- and β_+ dynamically we obtain a full pacman: the arcs β_- and β_+ project to γ_1, the arc β_0 projects to γ_0, and the arc β projects to γ_2.

The renormalization of f is called *prime* if a + b = 3.

Similarly, a *pacman renormalization* is defined for any map $f : U \to V$ with a distinguished fixed point which will be called α. For example, we will show in Corollary 3.7 that any Siegel map is pacman renormalizable.

Combinatorially, a general pacman renormalization is an iteration of the prime renormalization; see details in Appendix A, in particular Lemma A.2.

We define $\Delta = \Delta_G$ to be the union of points in the f-orbits of $\overline{U}_-, \overline{U}_+$ before they return to S. Naturally, Δ is a triangulated neighborhood of α; see Figure 7. We call Δ a *renormalization triangulation* and we will often say that Δ is obtained by *spreading around* U_-, U_+.

Definition 2.4 (Conjugacy respecting prepacmen)**.** Let f and g be any two maps with distinguished α-fixed points, and let R and Q be two prepacmen in the dynamical plane of f and g defining some pacman renormalizations. Let h be a local conjugacy between f and g restricted to neighborhoods of their α-fixed points. Then *h respects R and Q* if h maps the triangulation Δ_R to Δ_Q so that the image of $(S_R, U_{R,\pm})$ is $(S_Q, U_{Q,\pm})$.

2.4. **Banach neighborhoods.** Consider a pacman $f : U_f \to V$ with a non-empty truncation disk O. We assume that there is a topological disk $\widetilde{U} \Supset U_f$ with a piecewise smooth boundary such that f extends analytically to U and continuously

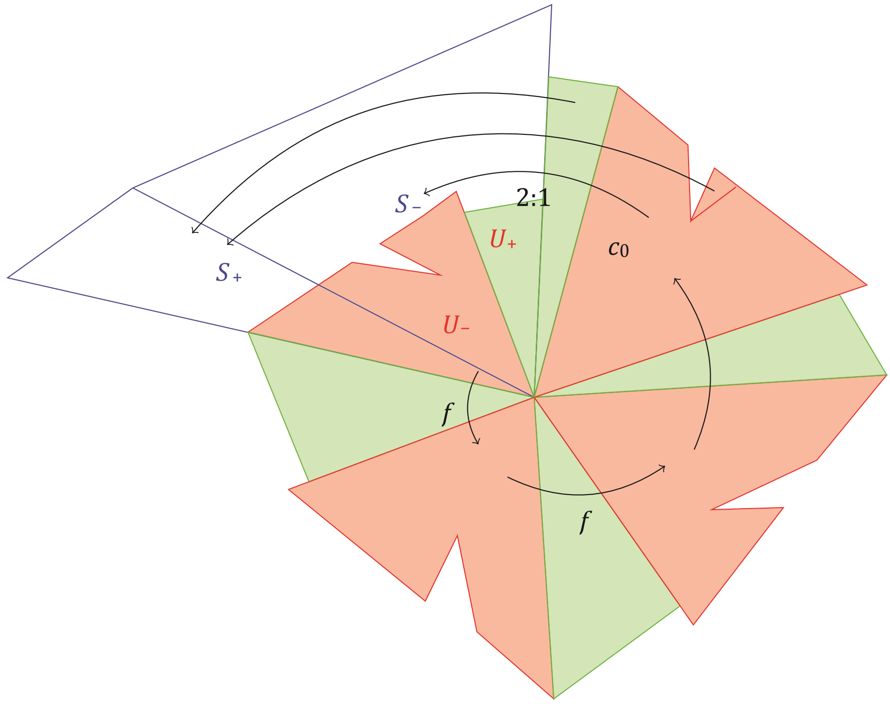

Figure 7. Pacman renormalization of f: the first return map from $U_- \cup U_+$ back to $S = S_- \cup S_+$ is a prepacman. Spreading around $U_\pm$: the orbits of U_- and U_+ before returning back to S triangulate a neighborhood Δ of α; we obtain $f : \Delta \to \Delta \cup S$, and we require that $\Delta \cup S$ is compactly contained in Dom$f$.

to its closure. Choose a small $\varepsilon > 0$ and define $N_{\widetilde{U}}(f,\varepsilon)$ to be the set of analytic maps $g : \widetilde{U} \to \mathbb{C}$ with continuous extensions to ∂U such that

$$\sup_{z\in\widetilde{U}} |f(z) - g(z)| < \varepsilon.$$

Then $N_{\widetilde{U}}(f,\varepsilon)$ is a Banach ball.

We say that a curve γ lands at α at a *well-defined angle* if there exists a tangent line to γ at α.

Lemma 2.5. *Suppose γ_0,γ_1 land at α at distinct well-defined angles. If $\varepsilon > 0$ is sufficiently small, then for every $g \in N_{\widetilde{U}}(f,\varepsilon)$ there is a domain $U_g \subset U$ such that $g : U_g \to V$ is a pacman with the same V,γ_1,O (up to translation).*

Proof. For $g \in N_{\widetilde{U}}(f,\varepsilon)$ with small ε, set

$$\begin{aligned} V(g) &:= V(f) + (\alpha(g) - \alpha(f)), \\ \gamma_1(g) &:= \gamma_1(f) + (\alpha(g) - \alpha(f)) \\ O(g) &:= O(f) + (\alpha(g) - \alpha(f)), \end{aligned}$$

and set $\gamma_0(g)$ to be the lift of $\gamma_1(g)$ landing at α. Since $\gamma_0(f), \gamma_1$ land at distinct well-defined angles, so do $\gamma_0(g), \gamma_1$ if ε is small; i.e., $\gamma_0(g), \gamma_1$ are disjoint.

Set $g_\delta = f + \delta(g - f)$ and $T_\delta(z) := z + (\alpha(g_\delta) - \alpha(f))$, where $\delta \in [0,1]$. Define $\psi_\delta(z) = g_\delta^{-1} \circ T_\delta \circ f(z)$ on ∂U_f, where the inverse branch is chosen so that $\psi_0(z) = z$ and $\psi_\delta(z)$ is continuous with respect to δ. We claim that ψ_δ is well defined and that $\psi_\delta(\partial U_f)$ is a simple closed curve for all $\delta \in [0,1]$. Indeed, let $A \Subset U$ be a closed annular neighborhood of ∂U_f that contains no critical points of f. For ε small enough, the derivative of any $g \in N_{\widetilde{U}}(f, \varepsilon)$ is uniformly bounded and non-vanishing on a slightly shrunk A; in particular g has no critical points in A.

It follows that $\psi_\delta \mid A$ has uniformly bounded derivative and (choosing yet smaller ε, if necessary) is close to the identity map, hence $\psi_\delta(\partial U_f) \subset A$ is well defined for all δ. Since f has no critical values in A, it is locally injective, which implies that $\psi_\delta(x) \neq \psi_\delta(y)$ when x is sufficiently close to y. We conclude that ψ_δ is injective on ∂U_f. Therefore $\psi_1(\partial U_f)$ is a simple closed curve; let U_g be the disk enclosed by $\psi_1(\partial U_f)$. It is straightforward to check that $g : U_g \to V$ is a pacman with critical arc γ_1 and truncation disk O.

Consider a pacman $f : U_f \to V$. Applying the λ-lemma, we can endow all $g: U_g \to V$ from a small neighborhood of f with a foliated rectangle R_g as in §2.1 such that R_g moves holomorphically and the holomorphic motion of R_g is equivariant. As a consequence, an external ray $R(g)$ with a given angle depends holomorphically on g unless $R(g)$ hits an iterated lift of $\partial^{\mathrm{frb}} U_g$ or an iterated precritical point.

Lemma 2.6 (Stability of periodic rays). *Suppose a periodic ray $R(f)$ lands at a repelling periodic point x in the dynamical plane of f. Then the ray $R(g)$ lands at x for all g in a small neighborhood of f. Moreover, the closure $\overline{R(g)}$ is contained in a small neighborhood of $\overline{R(f)}$.*

Proof. Since x is repelling periodic, it is stable by the implicit function theorem. Present $R(g)$ as a concatenation of arcs $R_1R_2R_3\cdots$ such that $R_{i+1}(g)$ is an iterated lift of $R_i(g)$. By continuity, $R_i(g)$ is stable for $i \leq n$, where n is big if g is sufficiently close to f. If n is sufficiently big and g is sufficiently close to f, then $R_n(g)$ is in a small neighborhood of $x(g)$ and, since $x(g)$ is repelling, $R_{n+1}(g)$ is in an even smaller neighborhood of $x(g)$. Proceeding by induction, we obtain that $R_{n+j}(g)$

shrinks to $x(g)$; i.e., $R(g)$ lands at $x(g)$. It also follows that $\overline{R(g)}$ is in a small

neighborhood of $\overline{R(f)}$.

2.5. **Pacman analytic operator.** Suppose that $\hat{f}: \hat{U} \to \hat{V}$ is a renormalization of $f: U_f \to V$ via a quotient map $\psi_f : S_f \to \widehat{V}$ that extends analytically through $\partial S_f \setminus \{\alpha\}$ (this actually follows from the definition of renormalization), where $S_f \subset V$ is

distinct well-defined angles. We claim that there exists an analytic renormalization

the domain of a prepacman F such that curves $\beta_0, \beta_+, \beta_-$ all land at α at pairwise

operator defined on a neighborhood of f.

We note that $\beta_\pm = f^{k_\pm}(\beta_0)$ for some integers k_+, k_-. For a map g that is sufficiently close to f, the fact that the three curves land at different angles implies that $\beta_0, g^{k_+}(\beta_0), g^{k_-}(\beta_0)$ are disjoint. Define $\tau_g : \beta_0 \cup \beta_- \cup \beta_+ \to \mathbb{C}$ by

$$\tau_g : z \mapsto z + \alpha(g) - \alpha(f) \text{ on } \beta_0 \text{ and } \tau_g = g^{k_\pm} \circ \tau_g \circ f^{-k_\pm} \text{ on } \beta_\pm.$$

Then τ_g is an equivariant holomorphic motion of $\beta_0 \cup \beta_- \cup \beta_+$ over a neighborhood of f. By the λ-lemma [BR,ST] τ_g extends to a holomorphic motion of S_f over a possibly smaller neighborhood of f. Denote by μ_g the Beltrami differential of τ_g.

Define a Beltrami differential ν_g on $\mathbb{C}$ as $\nu_g = (\psi_f)_* \mu_g \text{ on } \widehat{V}$ and $\nu_g = 0$ outside of $\hat{V}$, and let φ_g be the solution of the Beltrami equation

$$\frac{\partial \phi_g}{\partial \bar{z}} = \nu_g \frac{\partial \phi_g}{\partial z}$$

that fixes α, ∞, and the critical value. We see that $\psi_g := \varphi_g \circ \psi_f \circ \tau_g^{-1}$ is conformal on $S_g := \tau_g(S)$. It follows that ψ_g depends analytically on g (see Remark on p. 345 of [L1]).

We claim now that $\widehat{G} = (S_g, g^{k_-}, g^{k_+})$ is a prepacman. Indeed, by definition of Lemma 2.5, $\hat{g}$ restricts to a pacman with the same range as $\hat{f}$. We now have the τ_g, we have $g^{k_\pm}(\tau_g(\beta_0)) = \beta_\pm$ and ψ_g glues G to a map $\hat{g}$ which is close to $\hat{f}$. By following theorem.

Theorem 2.7 (Analytic renormalization operator). *Suppose that $\hat{f} : \widehat{U} \to \widehat{V}$ is a renormalization of $f : U_f \to V$ via a quotient map $\psi_f : S_f \to \hat{V}$. Assume that the curves $\beta_0, \beta_-, \beta_+$ (see Definition 2.2) land at α at pairwise distinct welldefined angles. Then for every sufficiently small neighborhood $N_{\widetilde{U}}(f, \varepsilon)$, there exists a compact analytic pacman renormalization operator $\mathcal{R}: g \mapsto \hat{g}$ defined on $N_{\widetilde{U}}(f, \varepsilon)$ such that $\mathrm{R}(f) = \hat{f}$. Moreover, the gluing map ψ_g, used in this renormalization, also*

depends analytically on g.

Proof. We have already shown that $\hat{g}$ depends analytically on $g \in N_{\widetilde{U}}(f, \varepsilon)$. Choose an intermediate domain $\widetilde{U}' \quad \widetilde{U}$ with $U_f \Subset \widetilde{U}' \Subset \widetilde{U}$ so that the operator R is the composition of the restriction operator $N_{\widetilde{U}}(f,\varepsilon) \to N_{\widetilde{U}'}(f,\varepsilon)$ and the pacman renormalization operator defined on $N_{\widetilde{U}'}(f,\varepsilon)$. Since the former is compact, we conclude that R is compact.

3. Siegel pacmen

We say a holomorphic map $f : U \to V$ is *Siegel* if it has a fixed point α, a Siegel quasidisk $\overline{Z}_f \ni \alpha$ compactly contained in U, and a unique critical point $c_0 \in U$ that is on the boundary of Z_f. Note that in [AL2] a Siegel map is assumed to satisfy additional technical requirements; these requirements are satisfied by restricting f to an appropriate small neighborhood of $\overline{Z}_f$.

Let us foliate a Siegel disk Z_f of f by equipotentials parametrized by their heights ranging from 0 (the height of α) to 1 (the height of ∂Z_f). Namely, if h: $Z_f \to \mathbb{D}^1$ is a linearizing map conjugating $f \mid Z_f$ to the rotation $z \mapsto \mathbf{e}(\theta)z$, then the preimage under h of the circle with radius η is the equipotential of Z_f at height η.

Definition 3.1. A pacman $f : U \to V$ is *Siegel* if

- f is a Siegel map with Siegel disk Z_f centered at α;
- the critical arc γ_1 is the concatenation of an external ray R_1 followed by an inner ray I_1 of Z_f such that the unique point in the intersection $\gamma_1 \cap \partial Z_f$ is not precritical; and
- writing $f : (U \setminus O_0', O_0) \to (V, O)$ as in (2.1), the disk O is a subset of Z_f bounded by its equipotential.

The *rotation number* of a Siegel pacman (or a Siegel map) is $\theta \in \mathbb{R}/\mathbb{Z}$ such that $\mathbf{e}(\theta)$ is the multiplier at α. It follows (see Theorem 3.2) that the rotation number of the Siegel map is in Θ_{bnd}. The level of *truncation* of f is the height of ∂O.

Since γ_1 is a concatenation of an external ray R_1 and an internal ray I_1, so is γ_0: it is a concatenation of an external ray R_0 and an internal ray I_0 with $f(R_0 \cup I_0) = R_1 \cup I_1$. Two Siegel pacmen $f : U_f \to V_f$ and g: $U_g \to V_g$ are combinatorially equivalent if they have the same rotation number and if $R_0(f_1)$ and $R_0(f_2)$ have the same external angles; see (2.1). Starting from §3.6 we will normalize γ_0 so that it passes through the critical value.

A *hybrid conjugacy* between Siegel maps $f_1 : U_1 \to V_1$ and $f_2 : U_2 \to V_2$ is a qc-conjugacy h: $U_1 \cup V_1 \to U_2 \cup V_2$ that is conformal on the Siegel disks. A hybrid conjugacy between Siegel pacmen is defined in a similar fashion. We will show in Theorem 3.11 that combinatorially equivalent pacmen are hybrid equivalent.

We will often refer to the connected component Z'_f of $f^{-1}(Z_f) \setminus Z_f$ attached to c_0 as a *co-Siegel disk*.

3.1. **Local connectivity and bubble chains.** Consider a quadratic polynomial $p_\theta : z \mapsto \mathbf{e}(\ )z + z^2 \theta$.

Theorem 3.2. *If $\theta \in \Theta_{\text{bnd}}$, then the closed Siegel disk $\overline{Z}$ of p_θ is a quasidisk containing the critical point of p_θ.*

Conversely, suppose a holomorphic map $f : U \to V$ with a single critical point has a fixed Siegel quasidisk $\overline{Z}_f \Subset U \cap V$ containing the critical point of f. Then f has a rotation number of bounded type.

In particular, p_θ is a Siegel map. The first part of Theorem 3.2 follows essentially from the Douady-Ghys surgery; see [D1]. Conversely, if $f : U \to V$ is a Siegel map, then applying the inverse Douady-Ghys surgery we obtain a quasicritical circle map is quasisymmetrically conjugate to the rigid rotation if and only if the rotation f; see [AL2, Definitions 3.1]. By [H,Sw,AL2], the restriction of f to the unit circle number of f is bounded. (Compare to [GJ].)

Let us now fix a polynomial $p = p_\theta$ with $\theta \in \Theta_{\text{bnd}}$. A *bubble* of p is either

- $Z_0 := \overline{Z}_p$, or
- $Z'_0 := \overline{Z}'_p = \overline{p^{-1}(Z_p) \setminus Z_p}$, or
- an iterated p-lift of $\overline{Z}'_p$ (see §1.4 for the definition of a lift).

The *generation* of a bubble Z_k is the smallest $n \geq 0$ such that $p^n(Z_k) \subset Z_0$. In particular, Z_0 has generation 0 and Z'_0 has generation 1. If the generation of Z_k is at least 2, then p: $Z_k \to p(Z_k)$ admits a conformal extension through ∂Z_k (because $p(Z_k) \not\ni c_1$).

We say that a bubble Z_n is *attached* to a bubble Z_{n-1} if $Z_n \cap Z_{n-1} \neq \emptyset$ and the generation of Z_n is greater than the generation of Z_{n-1}.

A *limb* of a bubble Z_k is the closure of a connected component of $K_p \setminus Z_k$ not containing the α-fixed point. A limb of $Z_0 = \overline{Z}_p$ is called *primary*.

Theorem 3.3 ([Pe]). *The filled-in Julia set K_p is locally connected. Moreover, for every $\varepsilon > 0$ there is an $n \geq 0$ such that every connected component of K_p minus all bubbles of generation at most n has diameter less than ε.*

In particular the diameter of bubbles in K_p tends to 0: for every $\varepsilon > 0$ there are at most finitely many bubbles with diameter greater than ε. Similarly, the diameter of limbs of any bubble tends to 0.

An (infinite) *bubble chain of* K_p is an infinite sequence of bubbles $B=(Z_1,Z_2,\dots)$ such that Z_1 is attached to Z_0 and Z_{n+1} is attached to Z_n; see Figure 8.

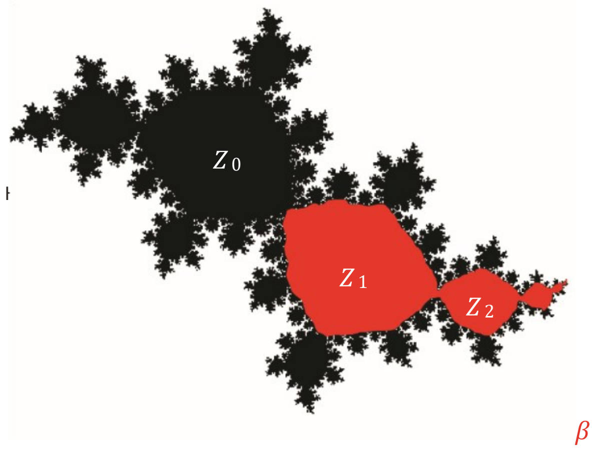

Figure 8. A bubble chain (red) landing at the β-fixed point.

As a consequence of Theorem 3.3, every bubble chain $B = (Z_1,Z_2,\dots)$ *lands*: there is a unique $x \in K_p$ such that for every neighborhood U of x there is an $m \geq 0$ such that $\bigcup_{i\geq m} Z_i$ is within U. Conversely, if $x \in K_p$ does not belong to any bubble, then there is a bubble chain $B = (Z_1,Z_2,\dots)$ landing at x. A point x is periodic if and only if B is periodic: there is an $m > 1$ and $q \geq 1$ such that p^q maps $(Z_m,Z_{m+1},\dots)$ to $(Z_1,Z_2,\dots)$.

Let $f: U \to V$ be a Siegel pacman. *Limbs, bubbles, and bubble chains for f* are defined in the same way as for quadratic polynomials with Siegel quasidisks. In particular, a bubble of f is either $\overline{Z}_f$, or $\overline{Z}'_f = \overline{f^{-1}(Z_f) \setminus Z_f}$, or an f^{n-1}-lift of $\overline{Z}'_f$, where n is the *generation* of the bubble. Since $\overline{Z}_f$ is the only bubble intersecting $\{c_1\} \cup \gamma_1$, all bubbles of positive generation are conformal lifts of $\overline{Z}'_f$. We define the *Julia set* of f as

$$\mathfrak{J}_f := \overline{\bigcup_{n\geq 0} f^{-n}(\partial Z_f)} \quad (3.1)$$

We will show in Theorem 3.12 that Theorem 3.3 holds for standard Siegel pacmen and that $\mathfrak{J}_f$ is the closure of repelling periodic points.

Limbs, bubbles, and bubble chains of a prepacman F are preimages of the corresponding dynamical objects of f.

3.2. **Siegel prepacmen.** A prepacman Q of a Siegel pacman q is also called *Siegel*; the *rotation number* and *level of truncation* of Q are those of q. Recall that Q consists of two commuting maps $q_- : U_- \to S_Q$, $q_+: U_+ \to S_Q$ such that U_- and U_+ are separated by β_0. Given a Siegel map f we say that f has a *prepacman Q around* $x \in \partial Z_f$ if q_-, q_+ are iterates of f, the vertex of S_Q is at $\alpha(f)$, and $\beta_0(Q)$ intersects ∂Z_f at x.

Lemma 3.4. *Suppose that p is a Siegel quadratic polynomial with rotation number $\theta \in \Theta_{\mathrm{bnd}}$. Consider a point $x \in \partial Z_p$ such that x is neither the critical point of p nor its iterated preimage. Then for every $r \in (0,1)$ and every $\varepsilon > 0$, the map p has a Siegel prepacman*

$$Q = (q_-: U_- \to S_Q, q_+: U_+ \to S_Q) \tag{3.2}$$

around x such that

- *the rotation number of Q is a renormalization of θ – iteration of* (A.2)*;*
- *for every $z \in U$ the orbit $z, p(z), \dots, p^k(z)$ is in the ε-neighborhood of $\overline{Z}_p$*

  $^k(z) = q_\pm(z)$*;*
  where p

  ±
- *r is the level of truncation of Q; and*
- *every external ray segment (see* §2.1*) of Q is within an external ray of p.*

Before proceeding with the proof let us define a sector renormalization of $p \mid$ $\mathbb{D}^1$.Let $h: \overline{Z}_p \to$ $\overline{Z}_p$. Consider the rotation of $\mathbb{L}_\theta: z \mapsto \mathbf{e}(z)$ of the unit disk $\mathbb{D}^1$ be the unique conformal conjugacy between $p \mid \overline{Z}_g$ and $\mathrm{L}_\theta \mid \mathbb{D}^1$ normalized such that $h(x) = 1$. A sector *pre-renormalization* of L_θ is a commuting pair $(\mathrm{L}^a \mid \mathrm{X}_-, \mathrm{L}^b \mid \mathrm{X}_+)$ realizing the first return map to $\mathrm{X}_- \cup \mathrm{X}_+$ (see Figure 27), where

$\mathrm{X}_-, \mathrm{X}_+$ are closed sectors of $\overline{\mathbb{D}^1}$ such that $\mathrm{X}_- \cap \mathrm{X}_+$ is the internal ray going towards

1; see details in→ $^{1/\delta}$ projects (§A.2. Denote by$\mathrm{L}^a \mid \mathrm{X}_-, \mathrm{L}^b \mid \mathrm{X}\delta_+$the angle of) to a new rotation.$\mathrm{X} = \mathrm{X}_- \cup \mathrm{X}_+$ at 0. The gluing map

z z

Definition 3.5. A *sector pre-renormalization of* $p \mid \overline{Z}_p$ *around* $x \in \partial Z_p$ is a commuting

pair (^{a} | X- p,^{a}L^b$_\theta$| X| X-$_+$,p) ^{b}by| Xh, where$_+$) obtained by pulling back a sector pre-

renormalization (L_θ

- $X_- := h^{-1}(\mathrm{X}_-)$, $X_+ := h^{-1}(\mathrm{X}_+)$, and $X := h^{-1}(\mathrm{X}) = X_- \cup X_+$ are closed sectors of $\overline{Z}_f$,
- the internal ray $I_0 := X_- \cap X_+$ lands at x.

The gluing map $z \mapsto z^{1/\delta}$ descents to

$$\psi_x := h^{-1} \circ [z \to z^{1/\delta}] \circ h$$

with $\psi_x(X) = \overline{Z}_{\mathcal{R}f}$.

Proof of Lemma 3.4. Consider the sector renormalization $(p^a \mid X_-, p^b \mid X_+)$ from Definition 3.5 and assume that the angle δ of X is small. We will now extend

$(p^a \mid X_-, p^b \mid X_+)$ beyond $\overline{Z}_p$ to obtain a prepacman (3.2); see Figure 9. Set

$$I_- := p^b(I_0), I_+ := p^a(I_0), I := f^{a+b}(I_0).$$

Then the sector X_- is bounded by I_-, I_0 and X_+ is bounded by I_0, I_+.

Since x is not precritical, there are unique external rays R_-, R_+, R extending I_-, I_+, I beyond $\overline{Z}_p$. Let S_Q be the closed sector bounded by $R_- \cup I_- \cup I_+ \cup R_+$ and truncated by an external equipotential E at a small height $\sigma > 0$. The curve

$R \cup I$ divides $\cup$ S into two closed sectors $\cup$ S_+ and $\cup S_-$ such that S_+ is between$_a(X) \subset RS_-$

$\cup$andI_- and$^b(XR_+) \subset I$ whileS_+. S is between R I and R_+ I_+. We note that p - - -

p

Let us next specify $U_- \supset X_-, U_+ \supset X_+$ such that

$$Q = (q_-, q_+) = (p^a \mid U_-, p^b \mid U_+) \tag{3.3}$$

is a full prepacman. Since the p-orbits of X_-, X_+ cover $\overline{Z}_p$ before they return back to X, we see that $\partial X \cap \partial Z_p$ has a unique precritical point, call it c_0', that travels through the critical point of p before it returns to X. Below we assume that

$c_0' \in X_-$; the case $c_0' \in X_+$ is analogous. Then S_+ has a conformal pullback U_+

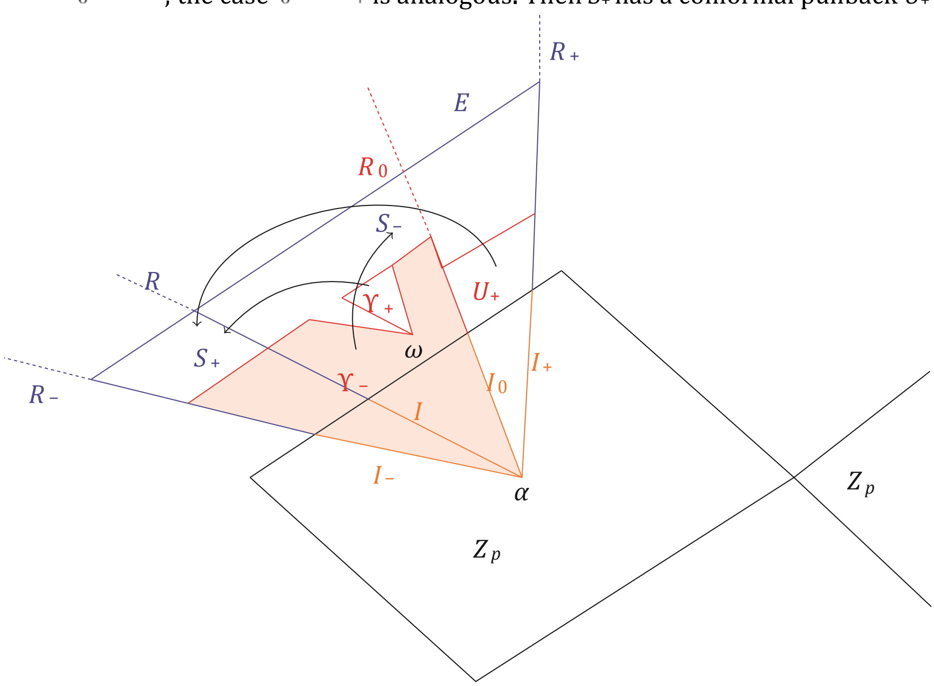

Figure 9. A full Siegel prepacman; compare with Figure 6. In the dynamical plane of a quadratic polynomial p, the sector $S_Q = S_- \cup S_+$ is bounded by $R_- \cup I_- \cup I_+ \cup R_+$ and truncated by an equipotential at small height. Pulling back S_-, S_+ along

appropriate branches of$^a \mid U, p^b \mid Up_+{}^a),p$is a full prepacman (3.3). Truncating^b we obtain $U_- = \Upsilon_- \cup \Upsilon_+$ and U_+

so that $(p^a \mid U_-, p^b \mid U_{-+})$ at ω and at the vertex where R_+ meets E (see

$(p$
Figure 11) we obtain a required prepacman (3.2).

along p^b: $X_+ \to S_+$. We have $U_+ \subset S_Q$ because rays and equipotentials bounding

S_Q enclose U_+.

The sector S has a degree two pullback Υ along p^a: $X \to S$. Under ^{a}: $\Upsilon \to S$ the fixed

point- α has two preimages, one of them is- - α, we denote-

p

– –

the other preimage by ω. Let Υ_+ be the conformal pullback of S_+ along the orbit p^a: $\{\omega\} \to \{\alpha\}$. We define $U_- := \Upsilon_- \cup \Upsilon_+ \subset S_Q$ and we observe that Q in (3.3) is a full prepacman.

By Theorem 3.3, primary limbs of K_p intersecting a small neighborhood of x have small diameters. By choosing δ and σ sufficiently small we can guarantee that $S_Q \setminus \overline{Z}_p$ is in a small neighborhood of x.

Let us now truncate Q at level r and let us show that the orbit

$$z,p(z),...,p^k(z) = q_\pm(z),k \in \{a,b\},$$

of any $z \in U$ is in a small neighborhood of $\overline{Z}_p$. The truncation of Q at level r removes points in $U = \Upsilon\ \Upsilon_+$ with p the equipotential
at height $t := r$

± – – ∪ $^\delta$. Since^a-δimages in the subdisk ofis small, we obtain thatZ$t_p$is close to 1.bounded by

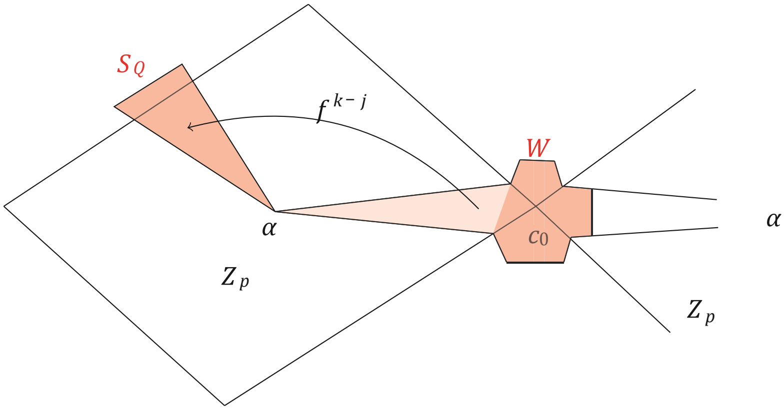

Figure 10. Since W is truncated by an equipotential of Z'_p at small height, the point $p^j(z) \in W' \setminus \overline{Z}_p$ is in a small neighborhood of c_0.

Since K_p is locally connected (Theorem 3.3), all the external rays of p land. For $z \in U_\pm \setminus K_f$, define $\rho(z) \in K_p$ to be the landing point of the external ray passing through z. Since S_Q is truncated by an equipotential at a small height, the orbit of z stays close to that of $\rho(z)$. This reduces the claim to the case $z \in K_p \cap U_\pm$.

By Theorem 3.2, there is an $\ell \geq 0$ such that all of the *big* (with diameter at least ε) primary limbs of p are attached to one of $c_0, c_{-1}, \dots, c_{-\ell}$, where c_0 is the critical point of p and c_{-i} is the unique preimage of c_0 under $p^i \mid \overline{Z}_p$. Since δ is assumed to be small, the orbit of c'_0 travels through all $c_{-\ell}, \dots, c_0$ before it returns to S_Q.

Let us denote by L the primary limb of p containing z (the case $z \in \overline{Z}_p$ is trivial). If L is not attached to c'_0, then by the above discussion all $L, p(L), \dots, p^k(L) = q$ (L) are small and the claim follows.

±

Suppose that L is attached to. c'_0 Denote by L_{-i} the connected component of $\mathfrak{K}_p \setminus \overline{Z}_p$ attached to c_{-i}. Since c'_0 travels through a critical point, we have $L = L_{-j}$ for some $j < k$.

Let W be the pullback of S_Q along

$$p^{k-j} \colon c_0 = p^j(c'_0) \to p^k(c'_0),$$

and let W' be W truncated by the equipotential of Z'_p at height t; see Figure 10. Since $t = r^\delta$ is close to 1, we obtain that $W' \cap L'_0$ is in a small neighborhood of c_0 because the angle of W at α' (the non-fixed preimage of α) is small—it is equal to δ. Therefore, $p^j(z)$ is close to c_0, and by continuity all $p^{j-1}(z), p^{j-2}(z), \dots, p^{j-\ell}(z)$ are only big limbs, we obtain that the orbitclose to $c_0, c_{-1}, \dots, c_{-\ell}$. Recall that $p_{j-z}, p_i(z()z\in), \dots, pL_{-i}$. Since$k(z)$ is in a small neighborhood $L_0, L_{-1}, \dots, L_{-\ell}$ are the of $\overline{Z}_p$.

It remains to specify external rays for Q. As shown on Figure 11 we slightly truncate S_Q at the vertex where R_+ meets the equipotential E and we slightly truncate U such that the truncations are respected dynamically and such that $\pm$

the preimage of the$^{\mathrm{ext}}U_q$, where$\partial S_Q \setminus q(:RU_{-q} \cup \to R_+ V)_q$underis the pacman of$Q$ consists

of exactly two curves that$Q$. We now can embed project to ∂

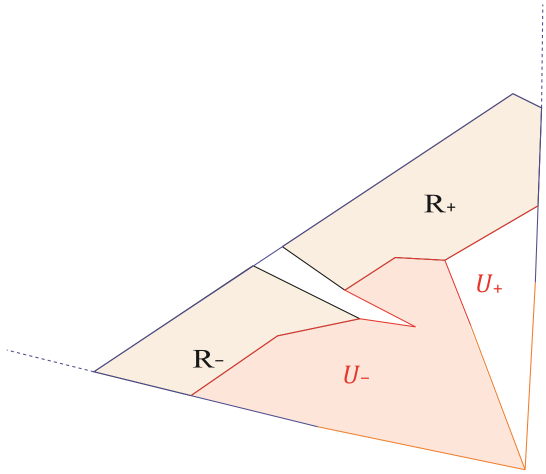

Figure 11. By truncating the prepacman from Figure 9 and embedding rectangles R±, we endow the prepacman with external rays.

in $S_Q \setminus (U_- \cup U_+)$ two rectangles R− and R+ that define external rays of Q as in §2.1.

3.3. **Pacman renormalization of Siegel maps.** An immediate consequence of [AL2, Theorem 3.19, Proposition 4.3] is the following.

Theorem 3.6. *Any two Siegel maps with the same rotation number are hybrid conjugate on neighborhoods of their closed Siegel disks.*

Proof. By [AL2, Proposition 4.3] any Siegel maps f, g can be obtained by performing the Douady-Ghys surgery on quasicritical circle maps $\tilde f, \tilde g$. By [AL2, Theorem 3.19], there is a qc map $\tilde h$ conjugating $\tilde f$ and $\tilde g$ in a small neighborhood of the unit circle. Then $\tilde h$ descends into a qc map h conjugating f and g in small neighborhoods of the boundaries of their Siegel disks. A hybrid conjugacy between f and g is obtained by setting $h \mid Z_f$ to be the canonical conformal conjugacy between $f \mid Z_f$ and $g \mid Z_g$ and running the pullback argument.

As a corollary of Theorem 3.6 and Lemma 3.4 we obtain the following.

Corollary 3.7. *Every Siegel map $f: U \to V$ is pacman renormalizable.*

Moreover the following holds. Let f be a Siegel map, and let p be the unique quadratic polynomial with the same rotation number as f. Let h be a hybrid conjugacy from a neighborhood of $\overline{Z}_f$ *to a neighborhood of* $\overline{Z}_p$*, respectively. Then there are prepacmen R and Q in the dynamical planes of f and p, respectively, such that h respects R and Q in the sense of Definition* 2.4.

Proof. Choose a small $\varepsilon > 0$ such that the ε-neighborhood of Z_f is in the domain of h. Then h pulls back a prepacman Q from Lemma 3.4 to a prepacman R in the dynamical plane of f. This shows that f is pacman renormalizable. **Lemma 3.8.** *Suppose that a Siegel pacman f is a renormalization of a quadratic polynomial. Then the non-escaping set* K_f *is locally connected.*

Moreover, for every $\varepsilon > 0$ *there is an* $n \geq 0$ *such that every connected component of* K_f *minus all the bubbles with generation at most n is less than ε. All the external rays of f land and the landing point belongs to* J_f*. Conversely, every point in* J_f *is the landing point of an external ray. The Julia set* J_f *is the closure of repelling periodic points.*

Proof. The proof follows from Theorem 3.3. Suppose that f is obtained from a of p by removing an open sector. All of the limbs of $\widetilde{Z}_\alpha$ attached to the removed quadratic polynomial p. Then every bubble Z_α of f is obtained from a bubble Z_α

sector are also removed. It follows from Theorem 3.3 that for $\varepsilon > 0$ there is an $n \geq 0$ such that every connected component of K_f minus all of the bubbles with generation at most n is less than ε. Since bubbles of f are locally connected, so is K_f. The landing property of external rays is straightforward.

3.4. **Rational rays of Siegel pacmen.** By a *rational point* we mean either a periodic or preperiodic point. Similarly, a periodic or preperiodic ray is *rational.*

Let us fix pacmen *f,p* and prepacmen *R,Q* as in Corollary 3.7. Let K_R be the non-escaping set of R. By definition, $K_R \subset K_f$; spreading around K_R we define the *local non-escaping set of f:*

$$K_f^{\mathrm{loc}} := \bigcup_{n\geq 0} f^n(\mathfrak{K}_R). \tag{3.4}$$

This is the set of points that do not escape Δ_R under $f: \Delta_R \to \Delta_R \cup S_R$; see Figure 7. Similarly we define

$$K_p^{\mathrm{loc}} := \bigcup_{n\geq 0} p^n(\mathfrak{K}_Q).$$

It is immediate that h conjugates $f \mid K^{\mathrm{loc}}_f$ and $p \mid K^{\mathrm{loc}}_p$. As a consequence, the *local Julia set*

$$J_f^{\mathrm{loc}} := \overline{\bigcup_{n\ge 0} \left(f \mid K_f^{\mathrm{loc}}\right)^{-n} (\partial Z_f)}$$

is the closure of repelling periodic points because so is J^{loc}_p. (Indeed, every $y \in J^{\mathrm{loc}}_p$ is the landing point of an external ray R_y because J_p is locally connected. Since external rays in a pacman are parametrized by angles in S^1 (see §2.1), p has a periodic external ray R_x landing at $x \in J^{\mathrm{loc}}_p$.) Moreover, for every periodic point $y \in K^{\mathrm{loc}}_f$ there is a unique periodic bubble chain B_y of K^{loc}_f landing at y.

Lemma 3.9 (External rays). *Let $y \in J^{\mathrm{loc}}_f$ be a periodic point. Then there is a periodic external ray R_y landing at y with the same period as y.*

Proof. Let $B_y = (Z_1, Z_2, \ldots)$ be the bubble chain in K^{loc}_f landing at y. Denote by x the unique point in the intersection of $\gamma_1 \cap \partial Z_0$. By Definition 3.1, the external ray R_1 lands at x. There are two iterated preimages $x_\ell, x_\rho \in \partial Z_1$ of x (by density of those) such that the rays $R_\ell,\ R_\rho$ (iterated lifts of R_1) landing at x_ℓ, x_ρ together with Z_1 separate y from $\overline{Z}_f$; see Figure 12. We denote by D the open subdisk of V bounded by $R_\ell, R_\rho,\ Z_1$ and containing y. Let D_p be the (univalent) pullback of D along $f^p: \{y\} \to \{y\}$. Then $D_p \Subset D$. By the Schwarz lemma, $f^p: D_p \to D$ expands the hyperbolic metric of D.

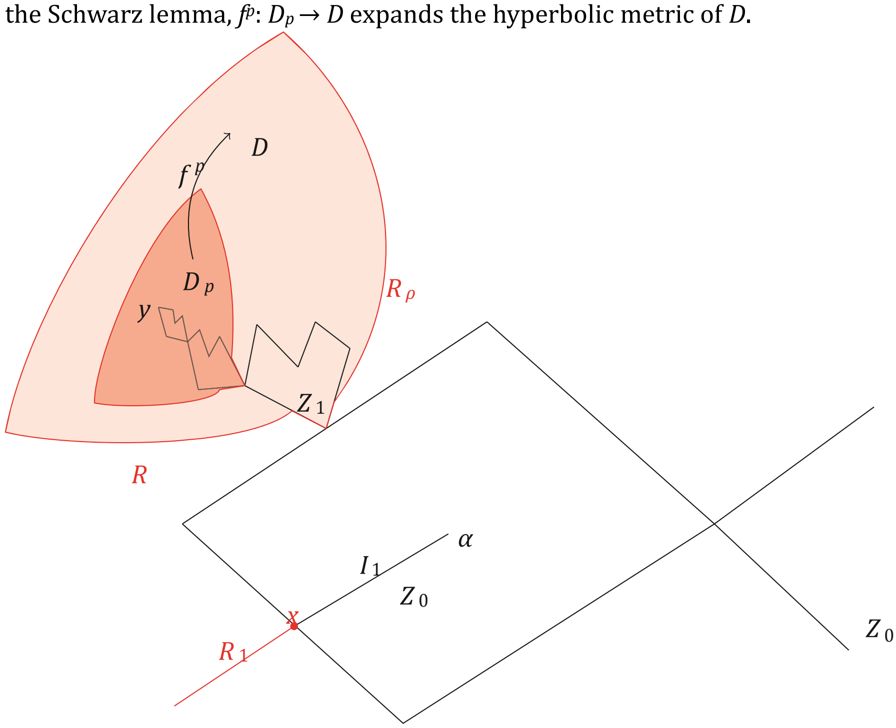

Figure 12. Illustration to the proof of Lemma 3.9. The ray R_1 has preimages R and R_ρ that land at Z_1 such that $R_\ell \cup R_\rho$ together with appropriate arcs in $\partial V \cup Z_1$ bound a disk D containing y. The disk D has a univalent lift $D_p \Subset D$. By the Schwarz lemma, $f^p : D_p \to D$ is expanding, which implies that there is an external ray landing at y.

There is a unique periodic external ray R_y in D with period p. We claim that R_y lands at y. Indeed, parametrize R as R: $\mathbb{R}_{>0} \to V$ with $f^p(R_y(t + p)) = R_y(t)$. Since all the points in D away from y escape in finite time under $\to \infty$ f^p: $D_p \to D$, the Euclidean distance between $R(t)$ and y goes to 0 as t + .

The next lemma is a preparation for a pullback argument.

Lemma 3.10 (Rational approximation of γ_1). *For every $\varepsilon > 0$, there are*

- *periodic points $x_\ell, x_\rho \in \mathfrak{J}_f^{\mathrm{loc}}$,*
- *external rays R and R_ρ landing at x_ℓ, x_ρ, respectively,*
- *periodic bubble chains B_ℓ and B_ρ in K^{loc}_f landing at x_ℓ, x_ρ, respectively, and*
- *internal rays I and I_ρ of Z_f landing at the points at which B and B_ρ are attached*

such that $R_\ell \cup B_\ell \cup I_\ell$ and $R_\rho \cup B_\rho \cup I_\rho$ are in the ε-neighborhood of γ_1 and such that $R_\ell \cup B_\ell \cup I_\ell$ is on the left of γ_1 while $R_\rho \cup B_\rho \cup I_\rho$ is on the right of γ_1.

Proof. Consider a finite set of periodic points $y_1, y_2, \dots, y_p \in J^{\mathrm{loc}}_f$. By Lemma 3.9 each y_i is the landing point of an external periodic ray, call it $R_{y,i}$, and the landing point of a periodic bubble chain, call it $B_{y,i}$. Let $\{W_1, W_2, \dots, W_p\}$ be the set of connected components of

$$U \setminus \overline{Z_f \bigcup_i (B_{y,i} \cup R_{y,i}});$$

we assume that ∂W_p contains $\partial^{\mathrm{frb}} U_f$. By adding more periodic points we can also assume that $c_0 \notin \partial W_p$. Set

$$W := W_1 \cup W_2 \cup \cdots \cup W_{p-1}.$$

By the Schwarz lemma, $f \mid W$ is expanding with respect to the hyperbolic metric of W. Since $c_0 \notin \partial W_p$, there is a sequence of periodic points $x_{\ell,j} \in \mathfrak{J}_f^{\mathrm{loc}}$ such that the orbit of x_ℓ is in $\overline{W}$ and such that $x_{,j}$ converges from the left to the unique point x_1 in $\gamma_1 \cap \partial Z_f$.

We claim that the external rays $R_{\ell,j}$ landing at $x_{,j}$ converge to the external ray landing at x_1. Indeed, since $x_{\ell,j} \to x_1$, the external angle of $R_{\ell,j}$ (see § 2.1) converges to the external angle of R_1. By continuity, $R_{\ell,j}([0,T])$ converges to $R_1([0,T])$ for any $T \in$

$\mathbb{R}_{>0}$. Since $f \mid W$ is expanding, $R_{\ell,j}([T,+\infty))$ is in a small neighborhood of x_j which converges to x_1.

The bubble chains $B_{\ell,j}$ of $\mathfrak{K}_f^{\mathrm{loc}}$ landing at x_j shrink because there are no big limbs in a neighborhood of x_1. Define I_j to be the internal ray of Z_f landing at the point where B_j is attached. Then $R_{\ell,j} \cup B_{\ell,j} \cup I_{\ell,j}$ is a required approximation for sufficiently big j. Similarly, $R_\rho \cup B_\rho \cup I_\rho$ is constructed.

3.5. **Hybrid equivalence.** Recall from §2 that a pacman $f: U_f \to V_f$ is required to have a locally analytic extension through ∂U_f. By means of the pullback argument, we will now show the following.

Theorem 3.11. *Let $f: U_f \to V_f$ and $g: U_g \to V_g$ be two combinatorially equivalent Siegel pacmen and suppose that f and g have the same truncation level. Then f and g are hybrid equivalent.*

Proof. Let p be the unique quadratic polynomial with the same rotation number as f and g. Let h_f and h_g be hybrid conjugacies from neighborhoods of $\overline{Z}_f$ and $\overline{Z}_g$ to a neighborhood of Z_p, respectively. As in Corollary 3.7, there are prepacmen Q, R, and S in the dynamical planes of p, f, and g, respectively, such that h_f and h_g are conjugacies respecting prepacmen R,Q and S,Q, respectively; see Definition 2.4. The composition $h := h_g^{-1} \circ h_f$ is a conjugacy respecting R,S.

We define $\mathfrak{K}^{\mathrm{loc}}_f$ as in (3.4); $\mathfrak{K}^{\mathrm{loc}}_g$ and $\mathfrak{K}^{\mathrm{loc}}_p$ are similarly defined. Then h conjugates $f \mid \mathfrak{K}^{\mathrm{loc}}_f$ and $g \mid \mathfrak{K}^{\mathrm{loc}}_g$.

As in Lemma 3.10 let $R_\ell(f) \cup B_\ell(f) \cup I_\ell(f)$ and $R_\rho(f) \cup B_\rho(f) \cup I_\rho(f)$ be approximations of $\gamma_1(f)$ from the left and from the right, respectively. Similarly, let $R_\ell(g) \cup B_\ell(g) \cup I_\ell(g)$ and $R_\rho(g) \cup B_\rho(g) \cup I_\rho(g)$ be approximations of $\gamma_1(g)$. We choose the approximations in compatible ways:

- $B_\ell(g), I_\ell(g), B_\rho(g), I_\rho(g)$ are the images of $B_\ell(f), I_\ell(f), B_\rho(f), I_\rho(f)$ under h;
- $R_\ell(g), R_\rho(g)$ have the same external angles as $R_\ell(f), R_\rho(f)$.

Write

$$T_f := \mathfrak{K}_f^{\mathrm{loc}} \bigcup_{n\ge 0} f^n(R_\rho \cup R_\ell) \quad \text{and} \quad T_g := \mathfrak{K}_g^{\mathrm{loc}} \bigcup_{n\ge 0} g^n(R_\rho \cup R_\ell).$$

Then T_f and T_g are forward invariant sets such that $V_f \setminus T_f$ and $V_g \setminus T_g$ consist of finitely many connected components. Since $R_\ell(g), R_\rho(g)$ have the same external angles, we can extend h to a qc map $h: V_f \to V_g$ such that h is equivariant on $T_f \cup \partial^{\mathrm{ext}} U_f$.

We now slightly increase U_f by moving $\partial^{\mathrm{frb}} U_f$ so that the new disk $\mathfrak{U}_f$ satisfies

$$f(\partial^{\mathrm{frb}}\mathfrak{U}_f) \subset Z_f \cup \overline{B_\ell \cup R_\ell \cup B_\rho \cup R_\rho}.$$

and we can slightly move(Indeed, we can slightly move $\gamma_+ \subset \gamma_- \subset^{\mathrm{frb}} \partial U^{\mathrm{frb}}_f U$ so that its image is within$_f$ so that its image is within $R_\ell \cup B_\ell \cup Z_f$ $R_\ell \cup B_\ell \cup Z_f$.)

∂

Similarly, we slightly increase U_g by moving $\partial^{\mathrm{frb}} U_g$ so that the new disk $\mathfrak{U}_g$ satisfies

$$g(\partial^{\mathrm{frb}}\mathfrak{U}_g) \subset Z_g \cup \overline{B_\ell \cup R_\ell \cup B_\rho \cup R_\rho}$$

and such that $h \mid T_f$ lifts to a conjugacy between $f \mid \partial\mathfrak{U}_f$ and $g \mid \partial\mathfrak{U}_g$. This allows us to apply the *pullback argument*: we set $h_0 := h$ and we construct qc maps

$$h_n : V_f \to V_g \text{ by } h_n(x) := \begin{cases} g^{-1} \circ h_{n-1} \circ f(x) & \text{if } x \in \mathfrak{U}_f \\ h_{n-1}(x) & \text{if } x \notin \mathfrak{U}_f. \end{cases},$$

We can choose h such that h_0 and h_1 are connected by an isotopy $\tilde{h}$: $[0,1]\times V_f \to V_g$ that is constant on T_f and uniformly continuous in the hyperbolic metrics of $V_f \setminus T_f$, $V_g \setminus T_g$. This implies that the Euclidean distance between h_n and h_{n+1} tends to 0. Since the space of qc maps with uniformly bounded dilatation is compact, we may pass to the limit and construct a hybrid conjugacy between f and g.

3.6. **Standard Siegel pacmen.** We say a Siegel pacman is *standard* if γ_0 passes through the critical value.

A *standard prepacman* R in the dynamical plane of a Siegel map g is a prepacman around the critical value of g (see §3.2). Then the pacman r obtained from R is standard and the renormalization change of variables ψ_R respects the internal ray landing at the critical value:

$$\psi_R(I_1(g)) = I_1(r). \tag{3.5}$$

The pacman renormalization associated with R is called a *standard pacman renormalization of* g.

By Theorem 3.11, two standard Siegel pacmen are hybrid equivalent if and only if they have the same rotation number.

Theorem 3.12. *Let f be a standard Siegel pacman. Then K_f is locally connected. Moreover, for every $\varepsilon > 0$ there is an $n \geq 0$ such that every connected component of K_f minus all of the bubbles with generation at most n is less than ε.*

As a consequence, every periodic point of J_f is the landing point of a bubble chain.

Proof. For every $\theta \in \Theta_{\mathrm{bnd}}$, there is a standard pacman g with rotation number θ such that g is a renormalization of a quadratic polynomial. The statement now follows from Theorem 3.11 combined with Lemma 3.8.

3.7. **A fixed point under renormalization.** Consider a Siegel map f with rotation number $\theta \in \Theta_{\mathrm{per}}$ and consider $x \in \partial Z_f$ such that x is neither the critical point nor its iterated preimage. Let $(f^{\mathfrak a} \mid X_{-,x}, f^{\mathfrak b} \mid X_{+,x})$ be the sector prerenormalization of $f \mid \overline{Z}_f$ as in Definition 3.5. Since $\theta \in \Theta_{\mathrm{per}}$, we can assume

(see §A.4) that the renormalization fixes $f \mid \overline{Z}_f$: the gluing map $\psi_x : X_x \to \overline{Z}_f$ projects $(f^{\mathfrak a} \mid X_{-,x}, f^{\mathfrak b} \mid X_{+,x})$ back to $f \mid Z_f$. For $x \in \{c_0, c_1\}$ we write

$$\psi_0 = \psi_{c_0}, X_0 = X_{c_0}, X_{\pm,0} = X_{\pm,c_0}$$

and $\psi_1 = \psi_{c_1}, X_1 = X_{c_1}, X_{\pm,1} = X_{\pm,c_1}$.

Theorem 3.13 ([McM2]). *For every $\theta \in \Theta_{\mathrm{per}}$, there is a Siegel map $g_\star : U_\star \to V_\star$ with rotation number θ such that for a certain sector pre-renormalization of $g_\star \mid \overline{Z}_{g_\star}$ as above the gluing map ψ_0 extends analytically through $\partial Z_\star \cap \partial X_0$ to a gluing map ψ_0 projecting $(g_\star^{\mathfrak a} \mid S_{-,0}, g_\star^{\mathfrak b} \mid S_{+,0})$ back to $g_\star : U_\star \to V_\star$, where $S_{\pm,0} \subset U_\star$.*
Moreover, there is an improvement of the domain*: the forward orbits*

$$\bigcup_{i \in \{0,1,\dots,\mathfrak a\}} g_\star^i(S_{-,0}) \cup \bigcup_{j \in \{0,1,\dots,\mathfrak b\}} g_\star^j(S_{+,0})$$

are compactly contained in $U_\star \cap V_\star$.

Up to conformal conjugacy, g is unique in a neighborhood of $\overline{Z}_{g_\star}$. We note that the improvement of the domain follows from complex a priori bounds for quasicritical circle maps [AL2, §3.3] after applying the inverse Douady-Ghys surgery; see also [Ya, Proposition 3.2]. It will allow us in Theorem 3.16 to construct a pacman analytic self-operator $\mathcal{R} : \mathcal{B} \dashrightarrow \mathcal{B}$.

Corollary 3.14. *The gluing map ψ_1 extends analytically through $\partial Z_{g_\star} \cap \partial X_1$ and, up to replacing ψ_1 with its iterate, satisfies the same properties as ψ_0 in Theorem* 3.13*; in particular, the improvement of the domain holds for ψ_1.*

Proof. We need to check that $\psi_1 := g_\star \circ \psi_0 \circ g_\star^{-1}$ is well defined. Since ψ_0 projects $(g_\star^{\mathfrak a}, g_\star^{\mathfrak b})$ to $g_\star$ and since the maps $g_\star^{\mathfrak a}, g^{\mathfrak b}$ are two-to-one in a neighborhood of c_0, we obtain that for z close to c_1 the gluing map ψ_0 maps $g_\star^{-1}(z)$ to a pair of points that have the same g-image. This shows that ψ_1 is well defined. Up to replacing ψ_1 with its iterate we can guarantee that the improvement of the domain holds for ψ_1.

Note that ψ_1 is expanding on $\overline{Z}_{g_\star} \cap \partial X_1$ because $\psi_1 \mid \overline{Z}_{g_\star}$ is conjugate to

$$\overline{\mathbb{D}^1} \to \overline{\mathbb{D}^1}, z \to z^{1/t}, t > 1.$$

Lemma 3.15 (Fixed Siegel pacman). *For any $\theta \in \Theta_{\mathrm{per}}$ there is a standard Siegel pacman $f_\star : U_\star \to V_\star$ that has a standard Siegel prepacman*

$$F = (f^{\mathfrak a} \mid U_- \to S, f^{\mathfrak b} \mid U_+ \to S)$$

together with a gluing map $\psi : S \to \overline{V}$ projecting F back to f. Moreover, the improvement of the domain holds for the renormalization:

$$\Delta_{F_\star} \Subset f_\star^{-1}(U_\star). \tag{3.6}$$

(See §2.3 for the definition of $\Delta_{F_\star}$.*)*

The pacman f is conformally conjugate to g in a neighborhood of $Z_\star := Z_{f_\star}$.

Proof. Consider a Siegel map g from Theorem 3.13 and ψ_1 from Corollary 3.14.

By Corollary 3.7, g has a standard prepacman Q: $U_{Q\pm} \to S_Q$ such that $S_Q \setminus Z_{g_\star}$ is in a small neighborhood of c_1. Since θ is of periodic type, we can prescribe Q to have rotation number θ. Since ψ_1 is expanding on $\partial \overline{Z}_{g_\star}$, for a sufficiently big integer $t \geq 1$ the prepacman

$$(\psi_1^t)^*(Q) := (\psi_1^{-t} \circ q_\pm \circ \psi_1^t \colon \psi_1^{-t}(U_{Q,\pm}) \to \psi_1^{-t}(S_Q))$$

has the property that $\psi_1^{-t}(S_Q) \setminus Z_{g_\star}$ is in a much smaller neighborhood of c_1.

Let $f_\star \colon U_\star \to V_\star$ be a pacman obtained from Q. The prepacman $(\psi_1^t)_*(Q)$ projects to the standard prepacman, call it

$$F_\star \colon (f_{\star,\pm} \colon U_{\star,\pm} \to S_\star)$$

such that $S_\star \setminus Z_{f_\star}$ is in a small neighborhood of c_1. The map ψ_1^t descends to a gluing map, call it ψ, projecting F back to f.

If t is sufficiently big, then $\mathbb{\Delta}_{F_\star}$ is compactly contained in $f_\star^{-1}(U_\star)$.

3.8. **Analytic renormalization self-operator.** Applying Theorem 2.7 to f from Lemma 3.15 we obtain the following.

Theorem 3.16 (Analytic operator $\mathcal{R} \colon \mathcal{B} \dashrightarrow \mathcal{B}$). *Let $f_\star \colon U_\star \to V_\star$ be a pacman, and let F be a prepacman from Lemma* 3.15*. Then there are small neighborhoods $N_{\widetilde{U}}(f_\star, \varepsilon), N_{\widetilde{U}}(f_\star, \delta)$ of $f_\star$ with $\varepsilon < \delta$ and there is an analytic pacman renormalization operator $\mathcal{R} \colon N_{\widetilde{U}}(f_\star, \varepsilon) \to N_{\widetilde{U}}(f_\star, \delta)$ such that $\mathcal{R} f_\star = f_\star$. Moreover, the operator* R *is compact, so its spectrum is a sequence converging to* 0*. The prerenormalization of $\mathcal{R} f_\star$ is $F_\star$.*

Proof. Let $f_\star \colon U' \to V'$ be a pacman obtained form $f_\star \colon U_\star \to V_\star$ by slightly decreasing U so that $U' \Subset U_\star$ and $\mathbb{\Delta}_{F_\star} \Subset f_\star^{-1}(U')$. Since the renormalization is defined on $\mathbb{\Delta}_{F_\star}$, by Theorem 2.7 there is a compact analytic pacman renormalization operator $N_{\widetilde{U}'}(f_\star, \varepsilon) \to \widetilde{N}_{\widetilde{U}}(f_\star, \widetilde{\delta})$, where U' and U are small neighborhoods of the closures of U' and $U_\star$. Precomposing with the restriction operator $N_{\widetilde{U}}(f_\star, \varepsilon) \to N_{\widetilde{U}'}(f_\star, \varepsilon)$, we obtain the required operator R.

To simplify notation, we will often write an operator in Theorem 3.16 as R: B B with $\mathcal{B} = N_{\widetilde{U}}(f_\star, \delta)$. We can assume (by Lemma 3.4) that f has any given truncation level between 0 and 1.

Corollary 3.17. *In a small neighborhood of f, the operator* R: B B *has an analytic finite-dimensional unstable submanifold* W^u *tangent to the unstable direction of* R*.*

We will show in Theorem 7.7 that W^u has dimension 1.

Proof. Since R is compact, it has a finite-dimensional unstable direction.

[HPS, Corollary (5.4)] asserts that W^u exists as a C^∞-smooth submanifold. The corollary is proven by showing that the graph transform on the space of submanifolds in a sufficiently small cone-neighborhood of the unstable direction of R (i.e., "candidates" to W^u) has the unique fixed point W^u. In our analytic setup, the graph transform iteratively applied to an analytic submanifold gives a sequence of analytic submanifolds converging exponentially fast and uniformly to W^u (see the estimate on p. 55 of [HPS]). Therefore, W^u is an analytic submanifold. An *indifferent pacman* is a pacman with indifferent α-fixed point. The *rotation number* of an indifferent pacman f is $\theta \in \mathbb{R}/\mathbb{Z}$ so that $\mathbf{e}(\theta)$ is the multiplier at $\alpha(f)$.

If, moreover, $\theta \in \mathbb{Q}$, then f is *parabolic*.

We denote by θ the multiplier of f.

Lemma 3.18. *Let* $R_{\mathrm{prm}} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ *be the map defined by*

$$R_{\mathrm{prm}}(\theta) = \begin{cases} \frac{\theta}{1-\theta} & \text{if } 0 \le \theta \le \frac{1}{2}, \\ \frac{2\theta-1}{\theta} & \text{if } \frac{1}{2} \le \theta \le 1; \end{cases} \tag{3.7}$$

see (A.2). *Then there is a* $k \ge 1$ *such that the following holds. Let* $f \in \mathcal{B}$ *be an indifferent pacman with rotation number* θ. *Then* $\mathcal{R}f$ *is again an indifferent pacman with rotation number* $R_{\mathrm{prm}}^k(\theta)$.

In particular, $R^{\mathfrak{k}}_{\mathrm{prm}}(\theta_\star) = \theta_\star$.

Proof. Recall that the renormalization $\mathcal{R}$ of $f_\star$ is an extension of a sector renormalization of $f \mid \overline{Z}_\star$; see Definition 3.5 and Appendix A. By Lemma A.2, a sector renormalization is an iteration of the prime renormalization. Therefore, $\mathcal{R}$ is an iteration of the prime pacman renormalization $\mathcal{R}_{\mathrm{prm}}$; see Definition 2.3. We need to check that if f is an indifferent pacman with rotation number θ, then $\mathcal{R}_{\mathrm{prm}}f$ is again an indifferent pacman with rotation number $R_{\mathrm{prm}}(\theta)$. By continuity, it is sufficient to assume that f is a parabolic pacman with rotation number p/q. The statement is clear if the germ of f has finite order. So assume that f has a local attracting flower at α with q petals. If $p \le q/2$, then $\mathcal{R}_{\mathrm{prm}}$ deletes p local attracting petals; otherwise $\mathcal{R}_{\mathrm{prm}}$ deletes $q-p$ local attracting petals. We see that $\mathcal{R}_{\mathrm{prm}}f$ has rotation number $R_{\mathrm{prm}}(p/q)$.

Remark 3.19. We will show in [DL] that $\mathcal{R} : \mathcal{B} \dashrightarrow \mathcal{B}$ can be constructed so that $\mathfrak{k}$ is the minimal period of θ under R_{prm}.

4. Control of pullbacks

Let us fix the renormalization operator

$$\mathcal{R}\colon \mathcal{B} \dashrightarrow \mathcal{B}, \mathcal{R}f_\star = f_\star$$

from Theorem 3.16 around a fixed Siegel pacman f. By Corollary 3.17 $\mathcal{R}$ has an unstable manifold $\mathcal{W}^u$ at $f_\star$.

4.1. **Renormalization triangulations.** Suppose that $f_0 \in \mathcal{B}$ is renormalizable $n \geq 0$ times (this is always the case if f_0 is sufficiently close to f) and antirenormalizable $-m \geq 0$ times. We write $[f_k : U_k \to V] := \mathcal{R}^k f_0$ for the kth (anti)renormalization of f_0, where $m \leq k \leq n$. We denote by $\psi_k : S_k \to V$ the renormalization change of variables realizing the renormalization of f_{k-1} (compare with the left side of Figure 21). We write

$$\varphi_k := \psi_{k-1}.$$

Let us cut the dynamical plane of $f_k : U_k \to V$, with $k \in \{m,\dots,n\}$, along γ_1; we denote the resulting prepacman by

$$F_k = (f_{k,\pm} : U_{k,\pm} \to V \setminus \gamma_1). \tag{4.1}$$

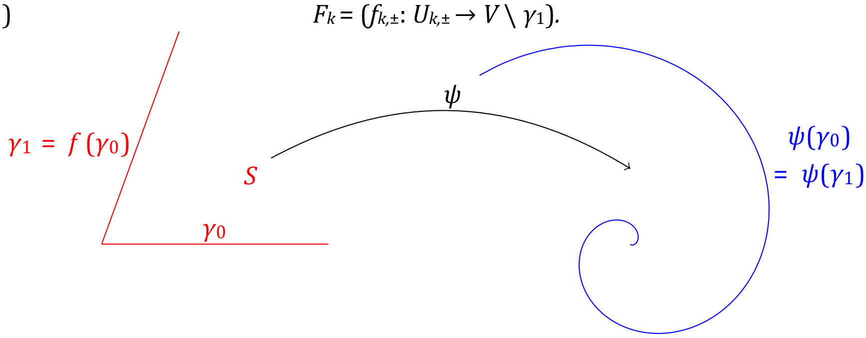

Figure 13. Suppose $f(z) = \lambda z$ with $\lambda \notin \mathbb{R}_+$, and let S be the sector between γ_0 and $\gamma_1 = f(\gamma_0)$. Let $\psi : S \to \mathbb{C}$ be the gluing map identifying γ_0 and γ_1 dynamically. If $|\lambda| \neq 1$, then $\psi(\gamma_0)$ does not land at 0 at a well-defined angle.

Lemma 4.1. *By restricting $\mathcal{R}$ to a smaller neighborhood of f, the following is true. Suppose f_0 is renormalizable $n \geq 1$ times. Then the map*

$$\Phi_n := \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n$$

admits a conformal extension from a neighborhood of $c_1(f_n)$ (where Φ_n is defined canonically) to $V \setminus \gamma_1$. The map $\Phi_n : V \setminus \gamma_1 \to V$ embeds the prepacman F_n (4.1) to the dynamical plane of f_0; we denote the embedding by

$$\begin{aligned} F_n^{(0)} &= \left(f_{n,\pm}^{(0)} : U_{n,\pm}^{(0)} \to S_n^{(0)}\right) \\ &= \left(f_0^{\mathfrak{a}_n} : U_{n,-}^{(0)} \to S_n^{(0)},\ f_0^{\mathfrak{b}_n} : U_{n,+}^{(0)} \to S_n^{(0)}\right), \end{aligned}$$

where the numbers $\mathfrak{a}_n, \mathfrak{b}_n$ are the renormalization return times satisfying (A.4).

Let Δ_n be the triangulation obtained by spreading around $U_{n,-}^{(0)}$ and $U_{n,+}^{(0)}$; see §2.3 *and* Figure 7. *In the dynamical plane of f_0 we have*

$$\Delta_0 := \overline{U}_0 \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_n,$$

$\Delta_1(f_0)$ is close in the Hausdorff topology to $\Delta_1(f_\star)$, and moreover $f_0(\Delta_n) \Subset \Delta_{n-1}$.

We call Δ_n the *nth renormalization triangulation.* Examples of $\Delta_0,\Delta_1,\Delta_2$ are shown in Figures 14 and 15. We say that $\Delta_n(f_0)$ is the *full lift* of $\Delta_{n-1}(f_1)$. Similarly (i.e., by lifting and then spreading around), a *full lift* will be defined for other objects.

In the proof of Lemma 4.1 we need to deal with the fact that $\psi_1(\gamma_1)$ can spiral around α; see Figure 13 for illustration. We will first show in Lemma 4.2 that Lemma 4.1 holds in a neighborhood of $\partial Z_\star$. By the topological robustness of antirenormalization (Theorem B.8), Lemma 4.1 holds also inside Z.

4.1.1. *Combinatorics of triangles.* Before giving the proof of Lemma 4.1, let us introduce additional notation. For consistency, we set $\Phi_0 := \mathrm{id}$; then $\Delta_0 = \overline{U_0}$ is a triangulation consisting of two closed triangles—the closures of the connected components of $U_0 \setminus (\gamma_0 \cup \gamma_1)$. We denote these triangles by $\Delta_0(0)$ and $\Delta_0(1)$ so that $\mathrm{int}(\Delta_0(0)),\gamma_0,\mathrm{int}(\Delta_0(1))$ have counterclockwise orientation around α; see Figure 14. The triangulation $\Delta_0(f_n)$ is defined similarly.

Let $\Delta_n(0,f_0),\Delta_n(1,f_0)$ be the images of $\Delta_0(0,f_n),\Delta_0(1,f_n)$ via the map Φ_n from Lemma 4.1. By definition, Δ_n is a triangulated neighborhood of α obtained by spreading around $\Delta_n(0,f_0),\Delta_n(1,f_0)$. We enumerate in counterclockwise order these triangles as $\Delta_n(i)$ with $i \in \{0,1,\dots,\mathfrak{q}_n-1\}$. By construction, $\Delta_n(0)\cup\Delta_n(1) \ni c_1(f_n)$.

We remark that $f_0 \mid \Delta_n$ is an antirenormalization of $f_n\colon U_n \to V$ in the sense of Appendix A. Moreover, there is a $\mathfrak{p}_n$ such that

$$f_0 : \Delta_n(i) \to \Delta_n(i + \mathfrak{p}_n) \tag{4.2}$$

is conformal for $i \notin \{-\mathfrak{p}_n, -\mathfrak{p}_n + 1\}$ allwith the index taken modulo $\mathfrak{q}_n$. For the exceptional triangles, we have an almost two-to-one map

$$f_0 \colon \Delta_n(-\mathfrak{p}_n) \cup \Delta_n(-\mathfrak{p}_n + 1) \to S_0^{(n)} \supset \Delta_n(0) \cup \Delta_n(1). \tag{4.3}$$

We will show in Theorem 4.6 that if f_0 is close to f, then $\mathbb{\Delta}_n = \bigcup_i \Delta_n(i)$ approximates $\overline{Z}_\star$ dynamically and geometrically.

By construction, for every triangle $\Delta_n(i,f_0)$ there exist $t \geq 0$ and $j \in \{0,1\}$ such that a certain branch of f_0^{-t} maps conformally $\Delta_n(i,f_0)$ to $\Delta_n(j,f_0)$. We define $\Psi_{n,i}$ on $\Delta_n(i,f_0)$ by

$$\Psi_{n,i} := \Phi_n^{-1} \circ f_0^{-t}\colon \Delta_n(i,f_0) \to \Delta_0(j,f_n). \tag{4.4}$$

4.1.2. *Walls.* Let A be a closed annulus, and let O be the bounded component of $\mathbb{C}\setminus A$. We say that A is a *univalent N-wall* if the restriction $f_0 \mid O\cup A$ is univalent and for all $z \in O$ and all j with $|j| \leq N$ we have

$$(f_0 \mid O \cup A)^j(z) \subset O \cup A.$$

More generally, we say that A is an *N-wall* if A contains a univalent N-wall A such that O is in the bounded component of $\mathbb{C}\setminus A'$.

Fix a small $r > 0$ and denote by $Z_\star^r$ the open subdisk of Z bounded by the equipotential at height r. Set $\Pi_0 := \overline{U}_0 \setminus Z_\star^r$. It is a closed annulus enclosing α. We decompose Π_0 into two closed rectangles $\Pi_0(0) = \Pi_0 \cap \Delta_0(0)$ and $\Pi_0(1) = \Pi_0 \cap \Delta_0(1)$; they are the closures of the connected components of $\Pi_0 \setminus (\gamma_0 \cap \gamma_1)$. The following lemma proves that the *wall of* Δ_n exists.

Lemma 4.2 (The wall of Δ_n). *Suppose all $f_0, f_1, \dots, f_n$ are sufficiently close to f. Then there exists a wall $\Pi_n(f_0)$ with the following properties:*

(1) *The map Φ_n extends from a neighborhood of $c_1(f_n)$ to $\Pi_0 \setminus \gamma_1$.*

(2) *Let $\Pi_n(0,f_0)$ and $\Pi_n(1,f_0)$ be the images of $\Pi_0(0,f_n)$ and $\Pi_0(1,f_n)$ under Φ_n. Then, by spreading around $\Pi_n(0,f_0)$ and $\Pi_n(1,f_1)$, we obtain an annulus Π_n enclosing α. We enumerate counterclockwise rectangles in Π_n as $\Pi_n(i)$ with $i \in \{0,1,\dots,q_n - 1\}$.*

(3) *We have $\Pi_0 \Supset \Pi_1 \Supset \cdots \Supset \Pi_n$ with $\Pi_0(f_0)$ close to $\Pi_0(f_\star)$.*

(4) *For every $\Pi_n(i)$, there is a $t \geq 0$ such that a certain branch of f_n^{-t} maps $\Pi_n(i)$ onto $\Pi_n(j)$ with $j \in \{0,1\}$. Then*

$$\Psi_{n,i} := \Phi_{n-1} \circ f_0^{-t} \colon \Pi_n(i,f_0) \to \Pi_0(j,f_n) \tag{4.5}$$

is conformal. If n is sufficiently big, then all the $\Psi_{n,i}$ expand the Euclidean metric and the expanding constant is at least η^n for a fixed $\eta > 1$. In particular, the diameters of the rectangles in Π_n tend to 0.

(5) *The wall $\Pi_n(f_0)$ approximates $\partial Z_\star$ in the following sense: $\partial Z_\star$ is a concatenation of arcs $J_0 J_1 \cdots J_{q_n - 1}$ such that $\Pi_n(i)$ and J_i are close in the Hausdorff topology.*

As in the case of renormalization triangulation, we say that $\Pi_n(f_0)$ is the *full lift* of $\Pi_{n-1}(f_1)$.

Proof. The proof follows from the robustness of the renormalization change of variables in a neighborhood of $\partial Z_\star$. Such change of variables is eventually expanding.

Consider first the case $f_0 = f_1 = \cdots = f_n = f_\star$. It follows from the improvement of the domain that the wall $\Pi_n(f_\star)$ is well defined and, moreover, the diameters of the rectangles in $\Pi_n(f_\star)$ tend to 0 as n increases. Choosing a sufficiently big k and applying the Schwarz lemma (after a slight enlargement of the rectangles), we obtain that all the $\Psi_{k,i} \colon \Pi_k(i, f_\star) \to \Pi_0(j, f_\star)$ expand the Euclidean metric.

By continuity and the assumption that $f_0, f_1, \dots, f_n$ are sufficiently close to f, the maps $\Psi_{k,i} \colon \Pi_k(i,f_s) \to \Pi_0(j,f_{s+k})$ also expand the Euclidean metric. Decomposing $\Psi_{n,i} \colon \Pi_n(i,f_0) \to \Pi_0(j,f_n)$ into a composition of $\lfloor \frac{n}{k} \rfloor$ maps of the form $\Psi_{k,t}$ and one remaining map, we see that $\Psi_{n,i} \colon \Pi_n(i,f_0) \to \Pi_0(j,f_n)$ is a required expanding map. This implies claim (4); other claims are consequences of claim (4).

4.1.3. *Proof of Lemma* 4.1. We will now apply Theorem B.8 to show that the full lift $\Delta_n(f_0)$ of $\Delta_0(f_n)$ exists.

Let $Q_0 \subset Z_\star$ be the closed annulus bounded by the equipotentials at heights r and $2r$. Then $Q_0 \subset \Pi_0$ and we decompose Q_0 into two rectangles $Q_0(0) = \Pi_0^{(0)} \cap Q_0$ and $Q_0(1) = \Pi_0(1) \cap Q_0$. Let $Q_n(0,f_0)$ and $Q_n(1,f_0)$ be the images of $Q_0(0,f_n)$ and $Q_0(1,f_n)$ under Φ_n. By spreading around $Q_n(0,f_0)$ and $Q_n(1,f_0)$, we obtain (by Lemma 4.2) an annulus Q_n enclosing α. We enumerate counterclockwise rectangles in Q_n as $Q_n(i)$ with $i \in \{0,1,\dots,\mathrm{q}_n - 1\}$. We have $Q_n(i) \subset \Pi_n(i)$.

Denote by Ω_n the open topological disk enclosed by Q_n. Then $f_0 \mid \Omega_n \cup Q_n$ is an antirenormalization of $f_1 \mid \Omega_{n-1} \cup Q_{n-1}$ (in the sense of Appendix B) with respect to the dividing pair of curves γ_0,γ_1.

By induction, we will now extend the wall Π_n to the triangulation Δ_n. Suppose the statement is verified for $n-1$. In the dynamical plane of f_1, we denote by $\gamma_0^{(n-1)}$ the lift of $\gamma_0(f_n)$ under the $(n-1)$-antirenormalization specified so that $\gamma_0^{(n-1)}$ crosses Q_{n-1} at $Q_{n-1}(0) \cap Q_{n-1}(1)$. Note that $Q_n(0) \cup Q_n(1)$ is in a small neighborhood of c_1 because Φ_n is contracting. Therefore, $\gamma_0^{(n-1)} \cap Q_{n-1}$ is uniformly close to $\gamma_0 \cap Q_{n-1}$. We can slightly adjust γ_0 in a neighborhood of Q_{n-1}, such that the new γ_0^{new} crosses Q_{n-1} at $Q_{n-1}(0) \cap Q_{n-1}(1)$. Let $\gamma_1^{(n-1)}$ and γ_1^{new} be the images of $\gamma_0^{(n-1)}$ and γ_0^{new}, respectively. Since a wall contains a fence (see Remark B.11), by Theorem B.8 the antirenormalization of $f_1 \mid \Omega_{n-1} \cup Q_{n-1}$ with respect to $\gamma_0^{(n-1)}, \gamma_1^{(n-1)}$ is naturally conjugate to the corresponding antirenormalization of $f_1 \mid \Omega_{n-1} \cup Q_{n-1}$ with respect to $\gamma_0^{\mathrm{new}}, \gamma_1^{\mathrm{new}}$. Therefore, the full lift $\Delta_n(f_0)$ of $\Delta_{n-1}(f_1)$ exists; $\Delta_n(f_0)$ is a required triangulation of $\Pi_n \cup \Omega_n$.

By (3.6) combined with continuity, we have $f_0(\Delta_1) \subset \Delta_0$. Applying induction on n, we obtain $f_0(\mathbb{A}_{n+1}) \Subset \mathbb{A}_n$.

We can now define $F_n^{(0)} = \left(f_{n,\pm}^{(0)} : U_{n,\pm}^{(0)} \to S_n^{(0)}\right)$ as the lift of F_n (see (4.1)) to the dynamical plane of f_0, where

$$S_n^{(0)} := f_0\left(\Delta_n(-\mathrm{p}_n) \cup \Delta_n(-\mathrm{p}_n + 1)\right)$$

(compare with (4.3)).

4.1.4. *Changing* γ_1. In fact, the exact behavior of γ_1 in a small neighborhood of α is irrelevant in the proof of Lemma 4.1. We have

Lemma 4.3. *Let* $\gamma_0^{\mathrm{new}}, \gamma_1^{\mathrm{new}} = f_n(\gamma_0^{\mathrm{new}})$ *be a new pair of curves in the dynamical plane of* f_n *such that*

- $\gamma_0 \setminus Z_\star^r = \gamma_0^{\mathrm{new}} \setminus Z_\star^r$ *and* $\gamma_1 \setminus Z_\star^r = \gamma_1^{\mathrm{new}} \setminus Z_\star^r$*; and*
- γ_0^{new} *and* γ_1^{new} *are disjoint away from* α.

Then Lemma 4.1 *still holds after replacing* γ_0,γ_1 *with* $\gamma_0^{\mathrm{new}}, \gamma_1^{\mathrm{new}}$. *More precisely, let* $\Delta_0^{\mathrm{new}}(0,f_n), \Delta_0^{\mathrm{new}}(1,f_n)$ *be the closures of the connected components of* $U_0 \setminus (\gamma_0^{\mathrm{new}} \cup \gamma_1^{\mathrm{new}})$ *in the dynamical plane of* f_n. *As in Lemma* 4.1 *the map* Φ_n

extends from a neighborhood of $c_1(f_n)$ *to* $V \setminus \gamma_1^{\mathrm{new}}$*; let* $\Delta^{\mathrm{new}}_n(0,f_0), \Delta^{\mathrm{new}}_n(1,f_0)$ *be the images of* $\Delta_0^{\mathrm{new}}(0,f_n), \Delta_0^{\mathrm{new}}(1,f_n)$ *under the new* Φ_n*. By spreading around* $\Delta^{\mathrm{new}}_n(0,f_0), \Delta^{\mathrm{new}}_n(1,f_0)$ *we obtain a new triangulated neighborhood* Δ^{new}_n *of* α*.*

Note that Δ^{new}_n and Δ_n triangulate the same neighborhood of α.

Proof. Since $\gamma_1^{\mathrm{new}}, \gamma_0^{\mathrm{new}}$ coincide with γ_1, γ_0 outside Z^r, the wall Π_n is unaffected; thus we can repeat the proof of Lemma 4.1 for γ_1^{new}.

4.1.5. *Siegel triangulations.* We will also consider triangulations that are perturbations of Δ_n. Let us introduce appropriate notation. Consider a pacman $f \in \mathrm{B}$. A *Siegel triangulation* Δ is a triangulated neighborhood of α consisting of closed triangles, each with a vertex at α, such that

- triangles of Δ are $\{\Delta(i)\}_{i\in\{0,\dots,\mathfrak{q}\}}$ enumerated counterclockwise around α so that $\Delta(i)$ is attached to $\Delta(i-1)$ (on the right) and to $\Delta(i+1)$ (on the left); all other pairs of triangles are disjoint away from α;
- there is a $\mathfrak{p} > 0$ such that f maps $\Delta(i)$ to $\Delta(i+\mathfrak{p})$ for all $i \notin \{-\mathfrak{p}, -\mathfrak{p}+1\}$, while $f(\Delta(-\mathfrak{p},-\mathfrak{p}+1)) \cap \Delta = \Delta(0,1)$;
- Δ has a distinguished 2-wall Π enclosing α and containing $\partial\Delta$ such that each $\Pi(i) := \Pi \cap \Delta(i)$ is connected and f maps $\Pi(i)$ to $\Pi(i+\mathfrak{p})$ for all $i \notin \{-\mathfrak{p}, -\mathfrak{p}+1\}$; and
- Π contains a univalent 2-wall Q such that each $Q(i) := Q \cap \Pi(i)$ is connected and f maps $Q(i)$ to $Q(i+\mathfrak{p})$ for all $i \notin \{-\mathfrak{p}, -\mathfrak{p}+1\}$.

The nth renormalization triangulation is an example of a Siegel triangulation.

Similar to Lemma 4.2(5), we say that Π *approximates* $\partial Z_\star$ if $\partial Z_\star$ is a concatenation of arcs $J_0 J_1 \cdots J_{\mathfrak{q}-1}$ such that $\Pi(i)$ and J_i are close in the Hausdorff topology.

Lemma 4.4. *Let* $f \in \mathrm{B}$ *be a pacman such that all* $f, \mathcal{R}f, \dots, \mathcal{R}^n f$ *are in a small neighborhood of* f*. Let* $\Delta(\mathcal{R}^n f)$ *be a Siegel triangulation in the dynamical plane of* $\mathcal{R}^n f$ *such that* $\Pi(\mathcal{R}^n f)$ *approximates* $\partial Z_\star$*. Then* $\Delta(\mathcal{R}^n f)$ *has a full lift* $\Delta(f)$ *which is again a Siegel triangulation. Moreover,* $\Pi(f)$ *also approximates* $\partial Z_\star$*.*

Proof. The proof is similar to the proof of Lemma 4.1. Suppose first $n = 1$. Since all $\Pi(i, \mathcal{R}f)$ are small, the arc γ_0 can be slightly adjusted[3] in a neighborhood of Π so that γ_0 crosses Π along $\Pi(i,\mathcal{R}f) \cap \Pi(i+1,\mathcal{R}f)$ with $i \notin \{-\mathfrak{p}, -\mathfrak{p}+1\}$. This allows us to construct a full lift $\Pi(f)$ of $\Pi(\mathcal{R}f)$. By Corollary B.14, the annuli $\Pi(f)$ and $Q(f)$ are again 2-walls. Applying Theorem B.8 from Appendix B we construct a full lift $\Delta(f)$ of

[3] The lift of the triangulation will depend on this adjustment.

$\Delta(Rf)$. Lemma 4.2(4) allows us to apply induction on n: for big n, the wall $\Pi(f)$ approximates $\partial Z_\star$ better than $\Pi(R^n f)$ approximates $\partial Z_\star$.

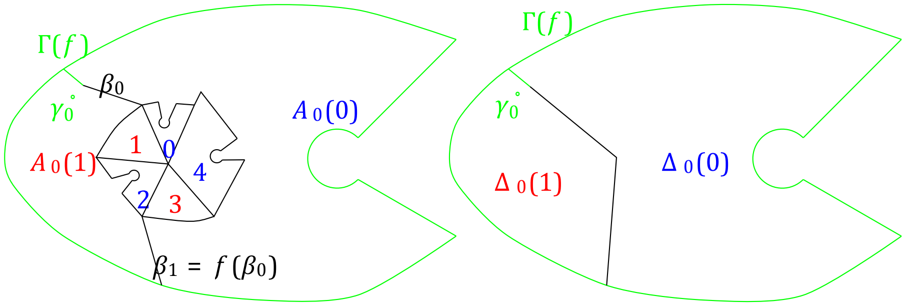

Figure 14. Renormalization tiling. Right: triangles $\Delta_0(0), \Delta_0(1)$ are the closures of the connected components of $U_f \setminus (\gamma_0 \cup \gamma_1)$. They form a renormalization tiling of level 0. Left: the triangles labeled by 0 and 1, i.e., $\Delta_1(0,f)$ and $\Delta_1(1,f)$, respectively, are antirenormalization embeddings of $\Delta_0(0,f_1), \Delta_0(1,f_1)$; the forward orbit of $\Delta_1(0,f), \Delta_1(1,f)$ triangulates a neighborhood of α. Together with $A_0(0) \cup A_0(1)$, this gives a tiling of U_f of level 1.

4.2. **Renormalization tilings.** In this subsection we will show the robustness of renormalization triangulations. Along these lines we will also extend $\Delta_n(f)$ to a tiling of U_f.

Let γ_0° be the preimage of $\overline{\gamma_1 \setminus U_f}$ under $f\colon \bar\gamma_0 \to \bar\gamma_1$; see Figure 14. In other words, γ_0° is the subcurve of $\bar\gamma_0$ consisting of points that escape U_f after one iteration. Set $\Gamma(f) := \partial U_f \cup \gamma_0^\circ(f)$.

Lemma 4.5. *For every i we have*

$$\Psi_{1,i}(\partial\Delta_1(f_0) \cap \partial\Delta_1(i,f_0)) \subset \Gamma(f).$$

Moreover, there is an i such that $\gamma_0^\circ \subset \Psi_{1,i}(\partial\Delta_1(f_0) \cap \partial\Delta_1(i,f_0))$. The set $\Gamma(f)$ is disjoint from $\Delta_0(f)$.

There are disjoint arcs β_0 and $\beta_1 = f(\beta_0)$ such that

- *the concatenation of γ_0° and β_0 connects $\partial\Delta_0$ to $\partial\Delta_1$; and*
- *β_1 connects $\partial\Delta_0$ to $\partial\Delta_1$.*

In a small neighborhood of f the curves β_0, β_1 can be chosen so that there is a holomorphic motion of

$$[\Delta_1 \cup \partial\Delta_0 \cup \gamma_0^\circ \cup \beta_0 \cup \beta_1](f_0) \tag{4.6}$$

that is equivariant *with the following maps:*

(1) $f_0\colon \beta_0(f_0) \to \beta_1(f_0)$;
(2) $f_0\colon \Delta_1(i,f_0) \to \Delta_1(i+\mathfrak{p}_1, f_0)$ *for* $i \notin \{-\mathfrak{p}_1, -\mathfrak{p}_1+1\}$
(3) $\Psi_{1,i}\colon \partial\Delta_1(f_0) \cap \Delta_1(i,f_0) \to \Gamma(f_1)$. ;

Proof. Each triangle $\Delta_1(i)$ has three distinguished closed sides; we denote them by $\lambda(i)$, $\rho(i)$, and $\ell(i)$ such that $\lambda(i)$ and $\rho(i)$ are the left and right sides meeting at the α-fixed point while $\ell(i)$ is the side opposite to α. We have:

$$\lambda(i)\triangle\rho(i+1) \text{ is the symmetric difference between}$$
$$\partial\Delta_1(f_0)\cap\partial\Delta_1(i,f_0)) = \ell_i \cup \left(\overline{\lambda(i)\triangle\rho(i+1)}\right) \cup \left(\overline{\rho(i)\triangle\lambda(i-1)}\right),$$

where $\lambda(i)$ and $\rho(i+1)$. Note that $\Psi_{1,i}(\ell(i)) \subset \partial\Delta_0$ and, moreover, $\bigcup_i \Psi_{1,i}(\ell(i)) = \partial\Delta_0$.

Let us analyze $\overline{\lambda(i)\triangle\rho(i+1)}$. We assume that $\lambda(i) \neq \rho(i+1)$. Then one of the curves in $\{\lambda(i),\rho(i+1)\}$ is a Ψ-preimage of $\gamma_0(f_1)$ while the other is a preimage of $\gamma_1(f_1)$. We have:

$$\Psi_{1,i}\left(\overline{\lambda(i)\triangle\rho(i+1)}\right) = \gamma_0^\circ.$$

It is clear (see Appendix A) that $\lambda(i) \neq \rho(i+1)$ for at least one i.

The property $\Gamma(f) \cap \partial\Delta_1(f) = \emptyset$ follows from $\partial\Delta_0 \cap f(\Delta_1) = \emptyset$; see Lemma 4.1. Since $\Gamma(f)\cap\partial\Delta_1(f) = \emptyset$, we can find β_0 such that $\gamma_0^\circ \cup \beta_0$ is in a small neighborhood of γ_0 and $\gamma_0^\circ \cup \beta_0$ connects $\partial\Delta_0$ to $\partial(\Delta_1 \setminus (\Delta_1(-\mathfrak{p}_1)\cup\Delta_1(-\mathfrak{p}_1-1)))$. Then $\beta_1 = f(\beta_0)$ is disjoint from $\gamma_0^\circ \cup \beta_0$ and β_1 connects $\partial\Delta_0$ to $\partial\Delta_1$.

In a small neighborhood of f we have a holomorphic motion of $\partial\Delta_0(f_0)$. Applying the λ-lemma, we obtain a holomorphic motion of the triangulation Δ_0 that is equivariant with $f_0 \mid \gamma_0$. Lifting this motion via $\Psi_{1,i}$, we obtain a holomorphic motion of $\Delta_1 \cup \Gamma$ equivariant with (2) and (3). Applying again the λ-lemma, we extend the latter motion to the motion of (4.6) that is also equivariant with (1).

Let A_0 be the closed annulus between $\partial\Delta_0$ and $\partial\Delta_1$. The arcs $\gamma_0^\circ \cup \beta_0, \beta_1$ split A_0 into two closed *rectangles* $A_0(0), A_0(1)$ (see Figure 14) enumerated such that $\operatorname{int}(A_0(0)), \gamma_0^\circ \cup \beta_0, \operatorname{int}(A_0(1)), \beta_1$ have counterclockwise orientation.

Let A_n be the closed annulus between $\partial\Delta_n$ and $\partial\Delta_{n+1}$. Define

$$A_n(0,f_0) := \Phi_n(A_0(0,f_n)) \text{ and } A_n(1,f_0) := \Phi_n(A_0(1,f_n))$$

and spread $A_n(0,f_0), A_n(1,f_0)$ dynamically (compare with the definition of $\Delta_n(i)$ in §4.1); we obtain the partition of $A_n(f_0)$ by rectangles $\{A_n(i,f_0)\}_{0\le i<\mathfrak{q}_n}$ enumerated counterclockwise. Similar to (4.5) we define the map $\Psi_{n,i}: A_n(i,f_0) \to A_0(j,f_n)$ with $j \in \{0,1\}$.

The nth renormalization tiling is the union of all the triangles of Δ_n and the union of all the rectangles of all A_m for all $m < n$. The nth renormalization tiling is defined as long as $f_0,\dots,f_n$ are in a small neighborhood of f.

A *qc combinatorial pseudoconjugacy of level n between f_0 and f* is a qc map $h\colon \overline{U}_0 \to \overline{U}_\star$ that is compatible with the nth renormalization tilings as follows:

- h maps $\Delta_n(i,f_0)$ to $\Delta_n(i,f_\star)$ for all i;
- h maps $A_m(i,f_0)$ to $A_m(i,f_\star)$ for all i and $m < n$;
- h is equivariant on $\Delta_n(i,f_0)$ for all $i \in \{-\mathfrak{p}_n, -\mathfrak{p}_n+1\}$; and h is equivariant on $A_m(i,f_0)$ for all $i \in \{-\mathfrak{p}_m, -\mathfrak{p}_m+1\}$ and $m<n$ $f \mid \overline{Z}$

The following theorem says that $f \mid \Delta_n(f)$ *approximates* both dynamically and geometrically.

Theorem 4.6 (Combinatorial pseudoconjugacy)**.** *Consider an nth renormalizable pacman f and set*

$$d := \max_{i\in\{0,1,\dots,n\}} \operatorname{dist}(\mathcal{R}^i f, f_\star).$$

If d is sufficiently small, then there is a qc combinatorial pseudoconjugacy h of level n between f and f and, moreover, the following properties hold. The qc dilatation and the distance between $h \mid \Delta_n(f)$ and the identity on $\Delta_n(f)$ are bounded by constants $K(d), M(d)$, respectively, with $K(d) \to 1$ and $M(d) \to 0$ as $d \to 0$.

Proof. By Lemma 4.5, the set (4.6) moves holomorphically with f in a small neighborhood of f. Applying the λ-lemma, we obtain a holomorphic motion τ of the first renormalization tiling with f in a small neighborhood $\mathcal{U}$ of $f_\star$.

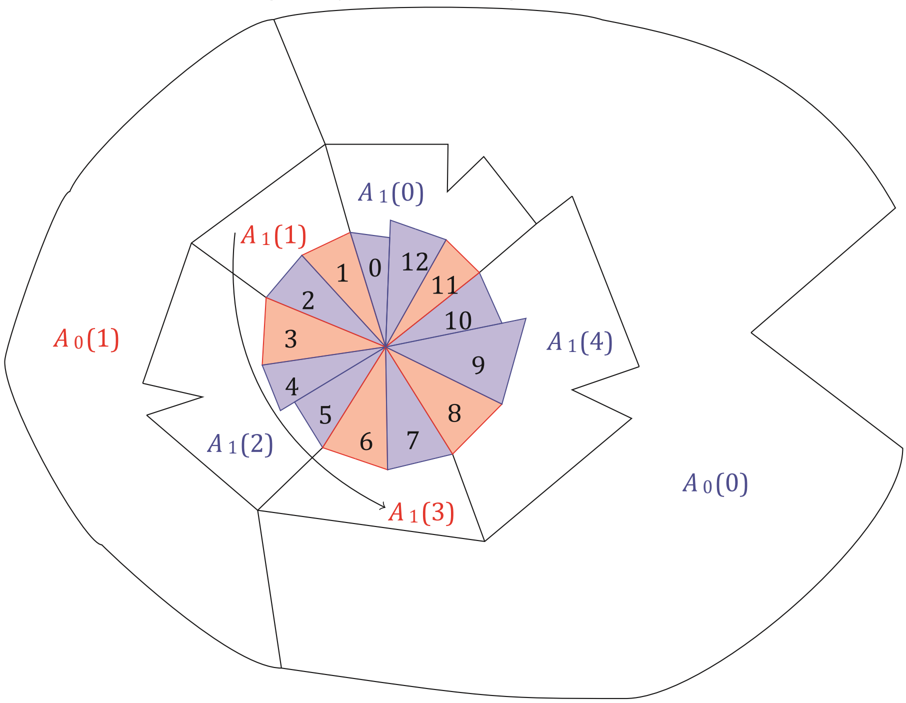

Figure 15. Renormalization tiling of level 2; tilings of smaller levels are depicted on Figure 14. There are $q_2 = 12$ triangles in Δ_2 with rotation number $p_2/q_2 = 5/12$. Geometry of triangles in Δ_2 is simplified. The image of $\Delta_2(8) \cup \Delta_2(9)$ is a sector slightly bigger than $\Delta_2(0) \cup \Delta_2(1)$; compare with Figure 7.

Suppose now that d is so small that all $f_i := \mathrm{R}^i f$ are in U. For every $\Delta_n(i)$ of f_0 or of f consider the map $\Psi_{n,i}\colon \Delta_n(i) \to \Delta_0(j)$, where $\Delta_0(j)$ is the corresponding triangle of f_n or of f. Then h on $\Delta_n(i,f_0)$ is defined by applying first

$\Psi_{n,i}\colon \Delta_n(i,f_0) \to \Delta_0(j,f_n)$ (see (4.4)), then applying the motion τ from $\Delta_0(j,f_n)$ to $\Delta_0(j, f_\star)$, and then applying $\Psi_{n,i}^{-1}\colon \Delta_0(j, f_\star) \to \Delta_n(i, f_\star)$.

Similarly, for every $A_m(i)$ of f_0 or of $f_\star$ consider the map $\Psi_{m,i}\colon A_m(i) \to A_0(j)$, where $A_0(j)$ is the corresponding rectangle of f_m or of f. Then h on $A_m(i,f_0)$ is defined by applying first $\Psi_{m,i}\colon A_m(i,f_0) \to A_0(j,f_m)$, then applying the motion τ from $A_m(j, f_0)$ to $A_m(j, f_\star)$, and then applying $\Psi_{m,i}^{-1}\colon A_0(j, f_\star) \to A_m(i, f_\star)$.

Observe now that h is well defined for all the points on the boundaries of all the rectangles and all the triangles because τ is equivariant with (1), (2), (3) of Lemma 4.5. Therefore, all points have well-defined images under h.

The qc dilatation of h is bounded by the qc dilatation of τ at f_i with $i \in \{0,1,\dots,n\}$. This bounds the qc dilatation of h by $K(d)$ as above with $K(d) \to 1$ as $d \to 0$.

If $n = 1$, then since τ is continuous, the distance between $h \mid \Delta_1(f_0)$ and the identity on $\Delta_1(f_0)$ is bounded by $M(d)$ as required. If $n > 1$, then $\Delta_n(f_0) \Subset U_0$ and the claim follows from the compactness of qc maps with bounded dilatation.

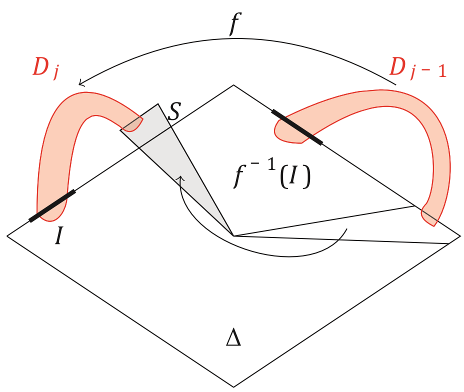

Figure 16. If D_j intersects $S = \Delta(0) \cup \Delta(1)$ and S is disjoint from $\Delta(I)$, then D_{j-1} may intersect $\Delta \backslash \Delta(f^{-1}(I))$ because $f(\Delta) = \Delta \cup S$.

Corollary 4.7. *There is an $\varepsilon > 0$ with the following property. Suppose that $f \in$ B is infinitely renormalizable and that all $\mathrm{R}^n f$ for $n \geq 0$ are in the ε-neighborhood of f. Then there is a qc map $h\colon U_f \to U_\star$ such that h^{-1} is a conjugacy on $\overline{Z}_\star$.*

Therefore, a certain restriction of f is a Siegel map and $f, f_\star$ are hybrid conjugate on neighborhoods of their Siegel disks.

Proof. If ε is sufficiently small, then by Theorem 4.6, for every $n \geq 0$ there exists qc combinatorial pseudoconjugacy h_n of level n between f and f such that the dilation of h_n is uniformly bounded for all n. By compactness of qc maps, we may pass to the limit and construct a qc map $h\colon U_f \to U_\star$ such that h^{-1} is a conjugacy on $\overline{Z}_\star$ and $c_0(f) \in h^{-1}(\overline{Z}_\star)$. It follows, in particular, that f is a Siegel map. By Theorem 3.6, the maps $f, f_\star$ are hybrid conjugate on neighborhoods of their Siegel disks.

4.3. **Control of pullbacks.** Recall from Lemma 4.1 that $\mathfrak{a}_n, \mathfrak{b}_n$ denote the closest renormalization return times computed by (A.4). By definition, $\mathfrak{a}_n + \mathfrak{b}_n = \mathfrak{q}_n$. We now restrict our attention to $f \in \mathcal{W}^u$.

Key Lemma 4.8. *There is a small open topological disk D around $c_1(f_\star)$ and there is a small neighborhood $\mathcal{U} \subset \mathcal{W}^u$ of $f_\star$ such that the following property holds. For every sufficiently big $n \geq 1$, for each $\mathfrak{t} \in \{\mathfrak{a}_n, \mathfrak{b}_n\}$, and for all $f \in \mathcal{R}^{-n}(\mathcal{U})$, we have $c_{1+\mathfrak{t}}(f) := f^{\mathfrak{t}}(c_1) \in D$ and D can be pulled back along the orbit $c_1(f), c_2(f), \dots, c_{1+\mathfrak{t}}(f) \in D$ to a disk D_0 such that $f^{\mathfrak{t}}\colon D_0 \to D$ is a branched covering; moreover, $D_0 \subset U_f \setminus \gamma_1$.*

Proof. The main idea of the proof is to block the forbidden part of the boundary $\partial^{\text{frb}} U_f$ from the backward orbit of D. The proof is split into short subsections. We start the proof by introducing conventions and additional terminology. The central argument will be presented in Claim 10(4).

4.3.1. *The triangulated disk Δ approximates $\overline{Z}_\star$.* Throughout the proof we will often say that a certain object is *small* if it has small size independently of n. Choose a big $s \gg 0$ and choose a small neighborhood $\mathcal{U}$ of $f_\star$ such that every $f \in \mathcal{R}^{-n}(\mathcal{U})$ is at least $m := n + s$ renormalizable and each $f_i := \mathcal{R}^i f$ with $i \in \{0, 1, \dots, m\}$ is close to f.

Consider the mth renormalization triangulation $\Delta_m(i)$ of f. Let h be a qc combinatorial pseudoconjugacy of level m as in Theorem 4.6. To keep notation simple, we sometimes drop the subindex m and write $\Delta(i), \Delta, \mathfrak{q}, \mathfrak{p}$ for $\Delta_m(i), \Delta_m, \mathfrak{q}_m, \mathfrak{p}_m$.

Since f_i with $i \in \{0, 1, \dots, m\}$ are close to f, the map $h \mid \Delta$ is close to the identity (by Theorem 4.6). In particular, $\mathbb{\Delta}(f) = h^{-1}(\mathbb{\Delta}(f_\star))$ approximates $\overline{Z}_\star$. Since s is big and since $\mathfrak{a}_i, \mathfrak{b}_i$ have exponential growth with the same exponent (A.4), we have (4.7) $\mathfrak{t}/\mathfrak{q}_m \in \{\mathfrak{a}_n/\mathfrak{q}_{n+s}, \mathfrak{b}_n/\mathfrak{q}_{n+s}\}$ is arbitrary small.

4.3.2. *Disks $D_k \ni f^k(c_1)$.* For convenience, we will write $c_0^\star = c_0(f_\star)$ and $c_1^\star = c_1(f_\star)$. Let us show that $D \ni f^{\mathfrak{t}}(c_1)$. Consider first the dynamical plane of f. Since n is big, we see that $f_\star^{\mathfrak{a}_n}(c_1^\star), f_\star^{\mathfrak{b}_n}(c_1^\star)$ are arbitrarily close to $c_1^\star$; i.e., $D \ni f_\star^{\mathfrak{t}}(c_1)$. It follows from (4.7) that

$$\min\{\mathfrak{a}_m, \mathfrak{b}_m\} - 1 > \max\{\mathfrak{a}_n, \mathfrak{b}_n\} \geq \mathfrak{t}. \tag{4.8}$$

This shows that $c_1^\star, \dots, f_\star^{\mathfrak{t}}(c_1^\star)$ do not visit triangles $\Delta(-\mathfrak{p}_m, f_\star) \cup \Delta(-\mathfrak{p}_m + 1, f_\star)$ as it takes either $\mathfrak{a}_m - 1$ or $\mathfrak{b}_m - 1$ iterations for a point in $\Delta(0, f_\star) \cup \Delta(1, f_\star)$ to visit them. Since h is a conjugacy away from $\Delta(-\mathfrak{p}) \cup \Delta(-\mathfrak{p} + 1)$, we obtain that h^{-1} maps $c_1^\star, \dots, f_\star^{\mathfrak{t}}(c_1^\star)$ to $c_1, \dots, f^{\mathfrak{t}}(c_1)$. Since h is close to the identity, $f^{\mathfrak{t}}(c_1)$ is close to $f_\star^{\mathfrak{t}}(c_1^\star)$; thus $f^{\mathfrak{t}}(c_1) \in D$.

Let $D_0, D_1, \dots, D_t = D$ be the pullbacks of D along the orbit $c_1, \dots, f^t(c_1) \in D$; i.e., $D_t := D \ni f^t(c_1)$ and D_i is the connected component of $f^{-1}(D_{i+1})$ containing $f^i(c_1)$. Our main objective is to show that the D_i do not intersect $\partial^{\mathrm{frb}} U_f$; this will imply that the maps $f\colon D_i \to D_{i+1}$ are branched coverings for all $i \in \{0, \dots, t-1\}$.

4.3.3. *Sectors* $\Delta(I)$ *and* $\Lambda(I)$. An *interval* I of $\mathbb{Z}/\mathfrak{q}\mathbb{Z}$ is a set of consecutive numbers $i, i+1, \dots, i+j$ taken modulo $\mathfrak{q}$. We define the *sector parametrized by* I as $\Delta(I) := \bigcup_{i \in I} \Delta(i)$. Furthermore, we let

$$(4.9)\qquad f^{-1}(I) := \begin{cases} I - \mathfrak{p} & \text{if } I \cap \{0,1,\mathfrak{p},\mathfrak{p}+1\} = \emptyset, \\ (I - \mathfrak{p}) \cup \{-\mathfrak{p}, -\mathfrak{p}+1\} & \text{if } I \cap \{0,1\} \neq \emptyset, \\ (I - \mathfrak{p}) \cup \{0,1\} & \text{if } I \cap \{\mathfrak{p}, \mathfrak{p}+1\} \neq \emptyset. \end{cases}$$

In other words, we require that if $I - \mathfrak{p}$ contains one of $\mathfrak{p}, \mathfrak{p}+1$, then it also contains the other number; and similarly with the pair $0,1$. By (4.2) and (4.3) the following holds.

Claim 1. *The preimage of* $\Delta(I)$ *under* $f \mid \Delta$ *is within* $\Delta(f^{-1}(I))$.

Unfortunately, we do not have the property that

$$\text{if } D_j \cap \Delta \subset \Delta(I), \text{ then } D_{j-1} \cap \Delta \subset \Delta(f^{-1}(I))$$

because the image of $\Delta(-\mathfrak{p}) \cup \Delta(-\mathfrak{p}+1)$ is slightly bigger than $\Delta(0) \cup \Delta(1)$; see (4.3). To handle this issue, we will adjust Δ to a slightly smaller triangulated neighborhood Λ such that

$$(4.10)\qquad \Lambda \subseteq f^i(\Lambda) \subseteq \Delta \text{ for all } i \in \{0, 1, \dots, \min\{\mathfrak{a}_m, \mathfrak{b}_m\}\}.$$

Consider the dynamical plane of $f_m = \mathcal{R}^m f$ and let $\Lambda_0(0, f_m)$ and $\Lambda_0(1, f_m)$ be the closures of the connected components of $f_m^{-1}(U_m) \setminus (\gamma_1 \cup \gamma_0)$ attached to α such that $\Lambda_0(0, f_m) \subset \Delta_0(0, f_m)$ and $\Lambda_0(1, f_m) \subset \Delta_0(1, f_m)$; see Figure 17. Writing $\Lambda_0(f_m) = \Lambda_0(0, f_m) \cup \Lambda_1(1, f_m)$ we obtain a shrunken version of $\Delta_0(f_m)$. The map Φ_m embeds $\Lambda_0(0, f_m)$ and $\Lambda_1(1, f_m)$ to the dynamical plane of f_0; spreading

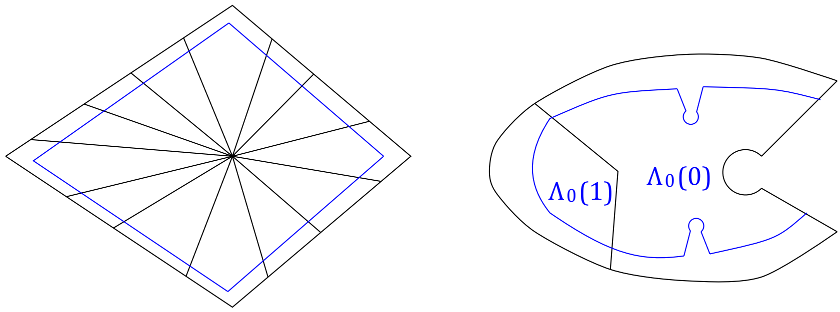

Figure 17. Right: $\Lambda_0(0,f_m)$ and $\Lambda_0(1,f_m)$ are shrunken versions of $\Delta_0(0,f_m)$ and $\Delta_0(1,f_m)$. Left: by transferring $\Lambda_0(0,f_m)$ and $\Lambda_0(1,f_m)$ to $\Lambda_m(0,f_0)$ and $\Lambda_m(1,f_0)$ by $\Phi_{m,0}$, and spreading these triangles dynamically, we obtain the triangulated neighborhood Λ_m of α such that Λ_m is a slightly shrunken version of Δ_m; compare with Figures 14 and 15.

around the embedded triangles, we obtain a triangulated neighborhood Λ of α such that (4.10) holds.

Let us also give a slightly different description of Λ. Recall (4.4) that $\Psi_{m,i}$ maps each $\Delta_m(i,f_0)$ conformally to some $\Delta_0(j,f_m)$. Then $\Lambda(i) = \Lambda_m(i,f_0) \subset \Delta_m(i,f_0)$ is the preimage of $\Lambda_0(j,f_m)$ under that map. We define

$$\Lambda := \bigcup_{0\le i<q} \Lambda(i) \text{ and } \Lambda(I) = \bigcup_{i\in I} \Lambda(i).$$

For the same reason as for $\Delta(f_\star)$, the triangulation $\Lambda(f_\star)$ approximates $\overline{Z}_\star$. And since $h \mid \Delta$ is close to the identity, $\Lambda(f)$ also approximates $\overline{Z}_\star$ in the sense of Theorem 4.6. For the same reason as for Claim 1, we have the following.

Claim 2. *We have* $\Lambda(i) = \Lambda \cap \Delta(i)$ *for every i. The preimage of* $\Lambda(I)$ *under* $f \mid \Lambda$ *is within* $\Lambda(f^{-1}(I))$.

The following claim is a refinement of (4.10). This will help us to control the intersections of D_k with Λ.

Claim 3. *Let I be an interval. Consider* $z \in \Lambda$. *If* $f^i(z) \in \Delta(I)$ *for* $i < \min\{a,b\}$, *then* $z \in \Lambda(f^{-i}(I))$.

As a consequence, if $T \cap \Delta \subset \Delta(I)$ *for an interval I and a set* $T \subset V$, *then*

$$f^{-i}(T) \cap \Lambda \subset \Lambda(f^{-i}(I))$$

for all $i < \min\{a,b\}$.

Proof. Since $f^i(z) \in \Delta(I)$, every preimage of $f^i(z)$ under the ith iterate of $f \mid \Delta$ is within $\Delta(f^{-1}(I))$ by Claim 1. By Claim 2, $z \in \Delta(f^{-i}(I)) \cap \Lambda \subset \Lambda(f^{-i}(I))$.

The second statement follows from the first because points in Λ do not escape Δ under f^i for all $i < \min\{a,b\}$; see (4.10).

4.3.4. *Truncated sectors S_k and disks* $\mathfrak{D}_k \supset \mathfrak{D}'_k \supset D_k$. Let I_t be the smallest interval containing $\{0,1\}$ such that $\Delta(I_t, f) \supset D_t \cap \Delta(f)$ for all f subject to the condition of the Key Lemma. Set $I_{t-j} := f^{-j}(I_t)$. By Claim 3 we have $D_k \cap \Lambda \subset$

$\Lambda(I_k)$.

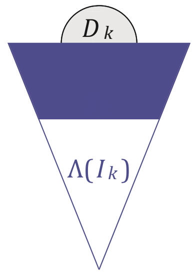

Figure 18. The rectangle S_k is an appropriate truncation of $\Lambda(I_k)$ such that $S_k \supset D_k \cap \Lambda$ and $S_k \supset (f \mid \Lambda)^{-1}(S_{k+1})$.

Recall that the intersection of each $\Delta(i, f_\star)$ with $\overline{Z}_\star$ is a closed sector of $\overline{Z}_\star$ bounded by two closed internal rays of $\overline{Z}_\star$. Let us fix $p > 1$ which will be specified in §4.3.5 as a certain period. Since D_t is small, we obtain the following.

Claim 4. (1) *All $|I_k|/q$ are small. All $\Delta(I_k, f_\star)$ have a small angle at the α-fixed point.*

(2) *For every $j \le t - 3 - p$, the intervals $I_j, I_{j+1}, \dots, I_{j+p+3}$ are pairwise disjoint.*

(3) *Moreover, the intervals $I_0, I_1, \dots, I_{p+1}$ are disjoint from $\{-p, -p+1\}$.*

Proof. It is sufficient to prove the statement for f; the map h transfers the result to the dynamical plane of f.

All $\Delta(i, f_\star)$ have comparable angles (see Lemma A.3): there are $x < y$ independent of n such that the angle of $\Delta(i)$ at α is between x/q and y/q.

Let χ be the angle of $\Delta(I_t)$ at α. The angle χ is small because $D = D_t$ is small. By definition of $I_k = f^{-1}(I_{k+1})$ (see (4.9)) the angle of $\Delta(I_{k+1})$ at α is bounded by the angle of $\Delta(I_k)$ at α plus y/q. Therefore, the angle at α of every $\Delta(I_k)$ is bounded by $\chi + (2+t)y/q$, where the number $(2+t)y/q$ is still small by (4.7). We obtain that all $\Delta(I_k)$ have small angles.

Since $f_\star \mid \overline{Z}_\star$ is an irrational rotation and $|I_k|/q$ are small, we see that $I_j, I_{j+1}, \dots, I_{j+p+3}$ are disjoint. Since I_0 contains $\{0,1\}$ we see that $I_0, I_1, \dots, I_{p+1}$ do not intersect $\{-p, -p+1\}$ $\subset f^{-1}(I_0)$.

Recall from §3 that the Siegel disk $Z_\star$ of $f_\star$ is foliated by equipotentials parametrized by their heights ranging from 0 (the height of α) to 1 (the height of $\partial Z_\star$). We denote by $Z^r_\star$ the open subdisk of Z bounded by the equipotential at height r.

Next we will construct a rectangle S_k by truncating $\Lambda(I_k)$ by a curve in

$h^{-1}(Z^r_\star \setminus Z^{r-\varepsilon}_\star)$ such that the family S_k is backward invariant in the following sense: $S_k \supset (f \mid \Lambda)^{-1}(S_{k+1})$ for all $k \in \{0,1,\dots,\mathfrak{t}-1\}$; see Figure 18. Assume that $r < 1$ is close to 1 and choose $\varepsilon > 0$ such that $1-r$ is much bigger than ε. Consider an interval I_k for $k \le \mathfrak{t}$ and consider $i \in I_k$.

- If for all $\ell \in \{k, k+1, \dots, \mathfrak{t}\}$ we have $i + \mathfrak{p}(\ell - k) \notin \{-\mathfrak{p}, -\mathfrak{p}+1\}$, then define
$$S_k(i) := \Lambda(i) \setminus h^{-1}(Z^r_\star);$$
- otherwise define
$$S_k(i) := \Lambda(i) \setminus h^{-1}(Z^{r-\varepsilon}_\star).$$

Set $S_k := \bigcup_{i \in I_k} S_k(i)$. Since $\Lambda(I_k, f_\star)$ has a small angle at α (see Claim 4) and the truncation level r is close to 1, we have the following.

Claim 5. *All S_k are small.*

Claim 6. *For every $k \le \mathfrak{t}$, the preimage of S_{k+1} under $f \mid \Lambda$ is within S_k.*

Proof. By Claim 2 we only need to check that the truncation is respected by backward dynamics. The proof is based on the fact that points in S_k pass at most once through the critical sector $\Lambda(-\mathfrak{p}) \cup \Lambda(-\mathfrak{p}+1)$ under the iteration of $f, f^2, \dots, f^{\mathfrak{t}-k}$.

The sector S_k consists of subsectors $S_k(i)$, where $i \in I_k$. If $i \notin \{-\mathfrak{p}, -\mathfrak{p}+1\}$, then
$$f : S_k(i) \to S_{k+1}(i+\mathfrak{p})$$
is a homeomorphism. Suppose $i \in \{-\mathfrak{p}, -\mathfrak{p}+1\}$. Then $S_{k+1}(i+\mathfrak{p}) = \Lambda(i) \setminus h^{-1}(Z^r_\star)$ because $i+\mathfrak{p}, \dots, i+\mathfrak{p}(\mathfrak{t}-k)$ are disjoint from $\{-\mathfrak{p}, -\mathfrak{p}+1\}$ by (4.8). On the other hand, by definition of S_k,
$$S_k \supset h^{-1}\left(\left(\Lambda(-\mathfrak{p}, f_\star) \cup \Lambda(-\mathfrak{p}+1, f_\star)\right) \setminus Z^{r-\varepsilon}_\star\right).$$
Since h is close to identity (see §4.3.1), the preimage of $S_{k+1}(i+\mathfrak{p})$ under $f \mid \Lambda$ is within $S_k(i) \subset S_k$.

We can assume that $D_\mathfrak{t}$ is so small that it does not intersect $h^{-1}(Z^r_\star)$. Then $D_\mathfrak{t} \cap \Lambda \subset S_\mathfrak{t}$; using Claims 3 and 6 we obtain $D_k \cap \Lambda \subset S_k$.

Next let us inductively enlarge D_k as $\mathfrak{D}_k \supset \mathfrak{D}'_k \supset D_k$. Set
$$\mathfrak{D}_\mathfrak{t} = \mathfrak{D}' := D_\mathfrak{t}$$
and define $\mathfrak{D}'_k$ to be the connected component of $f^{-1}(\mathfrak{D}_{k+1})$ containing D_k. We define $\mathfrak{D}_k$ to be the filled-in $\mathfrak{D}'_k \cup \operatorname{int} S_k$; i.e., $\mathfrak{D}_k$ is $\mathfrak{D}'_k \cup \operatorname{int} S_k$ plus all of the bounded components of $\mathbb{C} \setminus (\mathfrak{D}'_k \cup \operatorname{int} S_k)$.

Claim 7. *For all $k \le \mathfrak{t}$ the intersection $\mathfrak{D}_k \cap \Lambda$ is connected and we have $S_k = \overline{\mathfrak{D}_k} \cap \Lambda$.*

Proof. The claim follows from $D_k \cap \Lambda \subset S_k$, Claim 6, and the definition of $\mathfrak{D}_k$.

4.3.5. *Bubble chains.* Below we will separate the forbidden part of the boundary $\partial^{\mathrm{frb}} U_f$ from all D_j by external rays and bubble chains (see Figure 19). Recall from §3.1 that for f a bubble chain is a sequence of iterated lifts of $\overline{Z}_\star$; for f the role of $\overline{Z}_\star$ will be played by Λ.

Consider first the dynamical plane of f. Recall from Theorem 3.12 that the non-escaping set $K_\star$ of $f_\star$ is locally connected and that K has at most finitely many limbs (and bubbles) with diameter at least ε for every $\varepsilon > 0$. Let Z_a and Z_b be two bubbles attached to $\overline{Z}'_\star$ such that Z_a and Z_b are close to γ_- and γ_+, respectively. Let $f_\star^{p_a}$ be the iterate of f such that $f^{p_a}(Z_a) = \overline{Z}$. Recall from Definition 3.1 that γ_1, as well as γ_0, is a concatenation of an external ray and an internal ray. We also recall from §3.6 that we normalized γ_0 to pass through the critical value. Therefore, the critical point $c_0^\star$ is the landing point of two external rays R_-, R_+; we denote by W the open wake of $\overline{Z}'_\star$: the connected component of $V \setminus (R_- \cup R_+)$ containing $Z'_\star$. Let W_a be the univalent pullback of W under $f_\star^{p_a} : \operatorname{int}(Z_a) \to Z'_\star$. Then W_a is the wake of Z_a and we have $W_a \Subset W'$. By the Schwarz lemma, the map $f^{p_a} : W_a \to W'$ has a unique fixed point; we call it x.

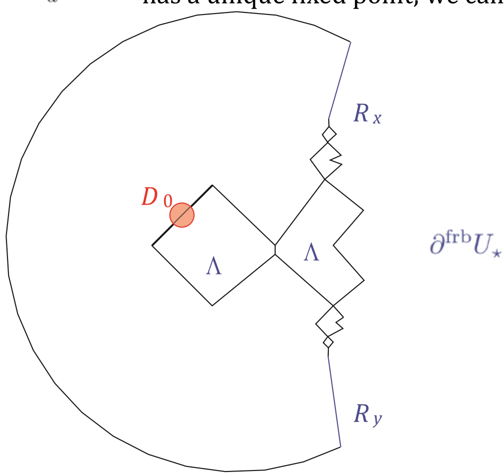

Figure 19. Separation of $\partial^{\mathrm{frb}} U_f$ from α. Disks Λ and Λ approximate $\overline{Z}_\star$ and $\overline{Z}'_*$. Iterated lifts of Λ form periodic bubble chains B_x and B_y landing at periodic points x and y. Together with external rays R_x, R_y the bubble chains B_x, B_y separate $\partial^{\mathrm{frb}} U_f$ from the critical value. The configuration is stable because of the stability of local dynamics at x and y. Disks D_k may intersect Λ but, by Claim 10, they do not intersect $B_x \cup B_y \setminus \Lambda'$.

Set $Z_2 := Z_a$ and for $i \geq 2$ define Z_{i+1} to be the unique preimage of Z_i under $f^{p_a} : W_a \to W'$. By the expansion of $f_\star^{p_a} : W_a \to W'$, the Z_i shrink to x. We have constructed the bubble chain B_x

$$Z_1 = \overline{Z}'_*, Z_2, Z_3, \dots \tag{4.11}$$

landing at x. Write $p_x := p_a$ and denote by $B'_x \subset B_x$ the subchain obtained by removing $\overline{Z}'$ in B_x. Then p_x is the minimal period of B_x because $f^{p_x}(B'_x) = B_x$.

Similarly, we define y to be the unique fixed point of $f_\star^{p_b} : W_b \to W' \Supset W_b$ and B_y to be the bubble chain landing at y. The minimal period of B_y is p_y. We have

(4.12) $$B_x = f_*^{p_x}(B'_x) \quad \text{and } B_y = f_*^{p_y}(B'_y), p_x,\ p_y \geq 2.$$

We denote by p the least common period of B_x and B_y. Let R_x and R_y be the external rays landing at x, y.

Since f is close to f, by Lemma 2.6, periodic rays R_x,R_y exist in the dynamical plane of f and are close to the corresponding rays in the dynamical plane of f.

Set Λ to be the closure of the connected component of $f^{-1}(\Lambda) \setminus \Lambda$ that has a non-empty intersection with Λ. Then Λ is connected and

(4.13) $$\mathbb{\Lambda} \cap \mathbb{\Lambda}' \subset \Lambda(-\mathfrak{p}) \cup \Lambda(-\mathfrak{p}+1) \text{ and } f(\mathbb{\Lambda}') \subset \mathbb{\Lambda}.$$

We say that Λ is *attached* to Λ, or more specifically that Λ is *attached* to $\Lambda(-p) \cup \Lambda(-p+1)$.

Observe also that Λ approximates $\overline{Z}'_*$ because Λ is close to $\overline{Z}_*$ and f is close to f.

A *bubble of generation* $e + 1 \geq 1$ for f is an f^e-lift of Λ. Fix a big $M \gg 1$.

We assume that the neighborhood $U \subset W^u$ in the statement of Key Lemma 4.8 is selected sufficiently small, depending in particular on M. Since Λ is close to $\overline{Z}'_*$, the map f is close to f, and $\partial\Lambda \cap (\Lambda(0) \cup \Lambda(1))$ is small, we obtain the following.

Claim 8. *Every bubble Z_δ of $f_\star$ of generation up to M is approximated by a bubble Λ_δ of f such that*

(1) *Λ_δ is close to Z_δ and $f \mid \Lambda_\delta$ is close to $f_\star \mid Z_\delta$;*
(2) *if Z_δ is attached to Z_Y, then Λ_δ is attached to Λ_Y; and*
(3) *if Z_δ is attached to Z, then Λ_δ is attached to $\Lambda \setminus (\Lambda(0) \cup \Lambda(1))$..*

Using Claim 8, we approximate the bubbles Z_k in B_x (see (4.11)) with $k \leq M$ by the corresponding bubbles Λ_k. We can assume that the remaining Z_{M+j} are within the linearization domain of x. Taking pullbacks within the linearization domain of x, we construct the *bubble chain* $B_x(f)$ landing at x as a sequence $\mathbb{\Lambda}' = \mathbb{\Lambda}_1, \mathbb{\Lambda}_2, \dots$. Similarly, $B_y(f)$ is constructed. The chains $B_x \cup B_y(f)$ are close to $B_x \cup B_y(f_\star)$ in the following sense: there are continuous maps

$$g_1 : B_x \cup B_y(f) \to B_x \cup B_y(f_\star) \text{ and } g_2 : B_x \cup B_y(f_\star) \to B_x \cup B_y(f)$$

close to the identities. Equation (4.12) holds in the dynamical planes of f. Thus we have constructed $(R_x \cup B_x \cup B_y \cup R_y)(f)$ that is close to $(R_x \cup B_x \cup B_y \cup R_y)(f_\star)$.

Assume that D is so small that it is disjoint from the forward orbit of $R_x \cup R_y$. As a consequence, we obtain the following.

Claim 9. *All D_k are disjoint from $R_x \cup R_y$.*

Proof. We proceed by induction on $k \in \{t, t-1, \dots, 0\}$. Since D_{k+1} is disjoint from the forward orbit of $R_x \cup R_y$, so is $\mathrm{D}'_k \cup \operatorname{int}(S_k)$; the latter surrounds D_k.

4.3.6. *Control of* D_k.

Claim 10. *For all* $k \in \{0,1,\dots,t\}$ *the following hold:*

(1) D_k *intersects* Λ *if and only if* $I_k \supset \{-p,-p+1\}$.

(2) *If* D_k *intersects* Λ, *then*

$$\mathrm{D}_k \cap \mathbb{A}' \subset f^{-1}(S_{k+1})$$

is contained in a small neighborhood of c_0.

(3) *If* D_k *intersects* $\mathbb{A}'$ *for* $k < \mathfrak{t} - 1$, *then* $k < t - 1 - p$ *and* $\mathrm{D}_{k+1}, \mathrm{D}_{k+2},\dots, \mathrm{D}_{k+p+1}$ *are disjoint from* Λ.

(4) *If* D_k *intersects* $B_x \cup B_y$, *then the intersection is within* Λ *and, in particular,* $I_k \supset \{-p,-p+1\}$. $f : \mathfrak{D}'_k \to \mathfrak{D}_{k+1}$

(5) D_k *is an open disk disjoint from* $\partial^{\mathrm{frb}} U_f$; *in particular,is a branched covering (of degree one or two) for* $k < t$.

Proof. We proceed by induction. Suppose that all of the statements are proven for moments $\{t,\dots,k+2,k+1\}$; let us prove them for k.

If $I_k \supset \{-p,$ $-\mathfrak{p}+1\}$, then $D_{k+1} \supset S_{k+1}(0) \cup S_{k+1}(1) \ni c_1$ (see §4.3.4). Since D_{k+1} $\mathbb{A}'$. contains either $S_{k+1}(-1)$ or S (2), we see that $\mathfrak{D}' = f^{-1}(\mathfrak{D}_{k+1}$ $_{k+1}$ $_k$) intersects

Suppose $I_k \cap \{-p,-p+1\} = \emptyset$. Then D_{k+1} does not contain c_1. Hence every point in D_{k+1} has at most one preimage under $f \mid \mathfrak{D}'_k$. Since $\mathrm{D}_{k+1} \cap \Lambda$ is connected, every preimage of $\mathrm{D}_{k+1} \cap \Lambda$ under $f \mid \mathfrak{D}'_k$ is in $\mathbb{A}$. By (4.13), D_k has empty intersection with Λ. Since $\mathfrak{D}'_k \cup S_k$ does not surround $\mathbb{A}'$, we also obtain $\mathrm{D}_k \cap \mathbb{A}' = \emptyset$. This proves part (1).

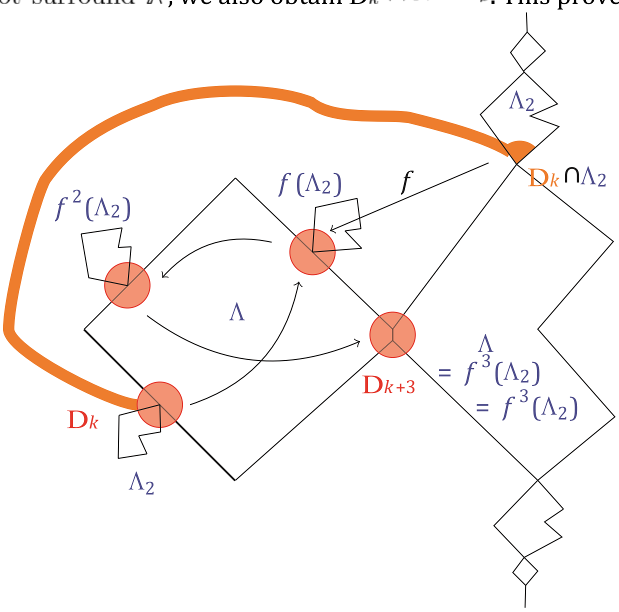

Figure 20. Illustration to the proof of Claim 10, part (4), in case $p_x = 3$. Suppose that f^3 maps the bubble $\mathbb{A}_2$ (in B_x) to $\mathbb{A}'$ and suppose that $D_k \cap \mathbb{A}_2 \neq \emptyset$. Let $\mathbb{A}'_2$ be the lift of $f(\Lambda_2)$ attached to Λ. Since $f(\Lambda_2)$ is attached to $S_{k+1} = \overline{\mathfrak{D}}_{k+1} \cap \mathbb{A}$ and since $D_{k+1} \cup S_{k+1}$ does not surround the critical value, we obtain that the pullback of $f(\Lambda_2)$ along $f\colon \mathfrak{D}'_k \to \mathfrak{D}_{k+1}$ is attached to S_k, contradicting $D_k \cap \mathbb{A}_2 \neq \emptyset$.

Part (2) follows from $D_{k+1} \cap \Lambda \subset S_{k+1}$ (see Claim 7), the definition of D_k, and the fact that S_{k+1} is a small neighborhood of c_1; see Claim 5.

Part (3) follows from part (1) combined with Claim 4 (part (2)).

Let us now prove part (4); see Figure 20 for illustration. By continuity, part (4) holds for $k \in \{t,\dots,t-p\}$. Below we assume that $k < t-p$. Assume that part (4) does not hold; let $D_k \cap (B_x \setminus \Lambda') \neq \emptyset$; the case $\mathfrak{D}_k \cap (B_y \setminus \Lambda') \neq \emptyset$ is similar. Write

$$B_x = (\mathbb{A}' = \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \dots),$$

where Λ_i is a bubble attached to Λ_{i-1}. Then there is a Λ_i with $i \geq 2$ such that

$$D_k \cap \mathbb{A}_i \neq \emptyset.$$

We assume that $i \geq 2$ is minimal and we claim that $i = 2$. Recall from §4.3.5 that f^p maps Λ_{j+1} to Λ_j, where $p_x \leq p$ is the minimal period of x. Suppose $i > 2$ and consider $B_x^{(i)} = \bigcup_{j\geq i} \mathbb{A}_j$. Since $D_k \cap B_x^{(i)} \neq \emptyset$ and $(D'_k \cup S_k) \cap R_x = \emptyset$ (the latter follows from Claim 9), we obtain that $D'_k \cap B_x^{(i)} \neq \emptyset$; hence $\mathfrak{D}_{k+1} \cap f(B_x^{(i)}) \neq \emptyset$. Applying induction we obtain $D_{k+p_x} \cap B_x^{(i-1)} = \mathfrak{D}_{k+p_x} \cap f^{p_x}(B_x^{(i)}) \neq \emptyset$, contradicting the induction assumption that part (4) holds for $k+p_x$.

Consider the bubbles

$$f(\mathbb{A}_2), f^2(\mathbb{A}_2), \dots, f^{p_x}(\mathbb{A}_2) = \mathbb{A}'.$$

By Claim 8, part (3), they are attached to $\Lambda \setminus (\Lambda(0) \cup \Lambda(1))$. Observe that D_{k+p_x} intersects Λ. Indeed, since $\mathfrak{D}'_k \cup S_k$ is disjoint from R_x, the disk D_k intersects $B'_x = \bigcup_{j\geq 2} \mathbb{A}_j$; hence $D_{k+1} \cap f(B'_x) \neq \emptyset$. Applying induction we obtain $D_{k+p_x} \cap B_x = \mathfrak{D}_{k+p_x} \cap f^{p_x}(B'_x) \neq \emptyset$. Therefore, $D_{k+p_x} \cap \mathbb{A}' \neq \emptyset$ because D_{k+p_x} is disjoint from B'_x by the induction assumption that part (4) holds for $k + p_x$.

Since D_{k+p_x} intersects $\mathbb{A}'$ we have $I_{k+p_x} \supset \{-\mathfrak{p}, -\mathfrak{p}+1\}$. Therefore, each $f^j(\Lambda_2)$ with $j \in \{1,\dots,p_x\}$ is attached to $S_{k+j} \subset D_{k+j}$. Moreover, every point in D_{k+j} has at most one preimage under $f \mid \mathfrak{D}'_{k+j-1}$ for $j \in \{1,\dots,p_x\}$ because $D_{k+j} \cap \Lambda \subset S_{k+j}$ does not contain c_1.

Let $\mathbb{A}'_2$ be the lift of $f(\Lambda_2)$ attached to S_k. We note that $\mathbb{A}'_2 \neq \mathbb{A}_2$ and $f(\mathbb{A}_2) \neq \mathbb{A}'$ (by (4.12)). Recall that every point in D_{k+1} has at most one preimage under $f \mid \mathfrak{D}'_k$. We claim that the lift of $f(D_k \cap \Lambda_2)$ under $f \mid \mathfrak{D}'_k$ is in $\mathbb{A}'_2$ and not in Λ_2.

Indeed, since $D_{k+1} \cup f(\Lambda_2)$ neither contains nor surrounds the critical value, the lift of $f(D_k \cap \Lambda_2)$ under $f \mid \mathfrak{D}'_k$ agrees with the lift of $f(D_k \cap \Lambda_2)$ under $f \mid \Lambda'_2$. This proves part (4).

By part (2) D_{k+1} is disjoint from $\partial^{\mathrm{frb}} U_f$ because D_{k+1} can intersect $B_x \cup B_y$ only in a small neighborhood of c_0. Therefore, $f\colon \mathfrak{D}'_k \to \mathfrak{D}_{k+1}$ is a branched covering.

This shows f: $D_0 \to D_t$ is a branched covering. Observe next that $D_0 \cap \Lambda \subset S_0$ is a small neighborhood of c_1 that is disjoint from γ_1. We can easily separate $D_0 \setminus \Lambda$ from $\gamma_1 \setminus \Lambda$ using Λ and finitely many backward iterated lifts of $B_x \cup B_y \cup R_x \cup R_y$.

This finishes the proof of the Key Lemma.

5. Maximal prepacmen

Let $g : X \to Y$ be a holomorphic map between Riemann surfaces. Recall that g is

- proper, if $g^{-1}(K)$ is compact for each compact $K \subset Y$;
- σ-proper (see [McM2, §8]) if each component of $g^{-1}(K)$ is compact for each compact $K \subset Y$; or equivalently if X and Y can be expressed as increasing unions of subsurfaces X_i, Y_i such that $g : X_i \to Y_i$ is proper.

A proper map is clearly σ-proper.

A prepacman $\mathbf{F} = (\mathbf{f}_-, \mathbf{f}_+)$ of a pacman f is called *maximal* if both $\mathbf{f}_-$ and $\mathbf{f}_+$ extend to σ-proper maps $\mathbf{f}_- : \mathbf{X}_- \to \mathbb{C}$ and $\mathbf{f}_+ : \mathbf{X}_+ \to \mathbb{C}$. We will usually normalize $\mathbf{F}$ such that $0 = \psi_{\mathbf{F}}^{-1}$(critical value), where $\psi_{\mathbf{F}}$ is a quotient map from $\mathbf{F}$ to $\mathbf{f}$; see §2.3. Under this assumption $\mathbf{F}$ is defined uniquely up to rescaling.

Theorem 5.1 (Existence of maximal prepacmen). *Every $f \in W^u$ sufficiently close to f has a maximal prepacman* $\mathbf{F}$ *that depends analytically on f.*

A refined statement will be proven as Theorem 5.5. The analytic dependence means that the restriction of a map to a disk compactly contained in the domain depends analytically on f in the associated Banach space. Note that analytic dependence is sufficient to check for one-parameter families. In the proof, we will show that $\mathbf{F}$ is obtained from f by an analytic change of variables.

5.1. **Linearization of ψ-coordinates.** Consider again $[f_0 : U_0 \to V] \in W^u$ close to f. By definition of W^u, the map f_0 can be antirenormalized infinitely many times. We define the *tower of antirenormalizations* as

$$\mathrm{t}(f_0) = (F_k)_{k \le 0}.$$

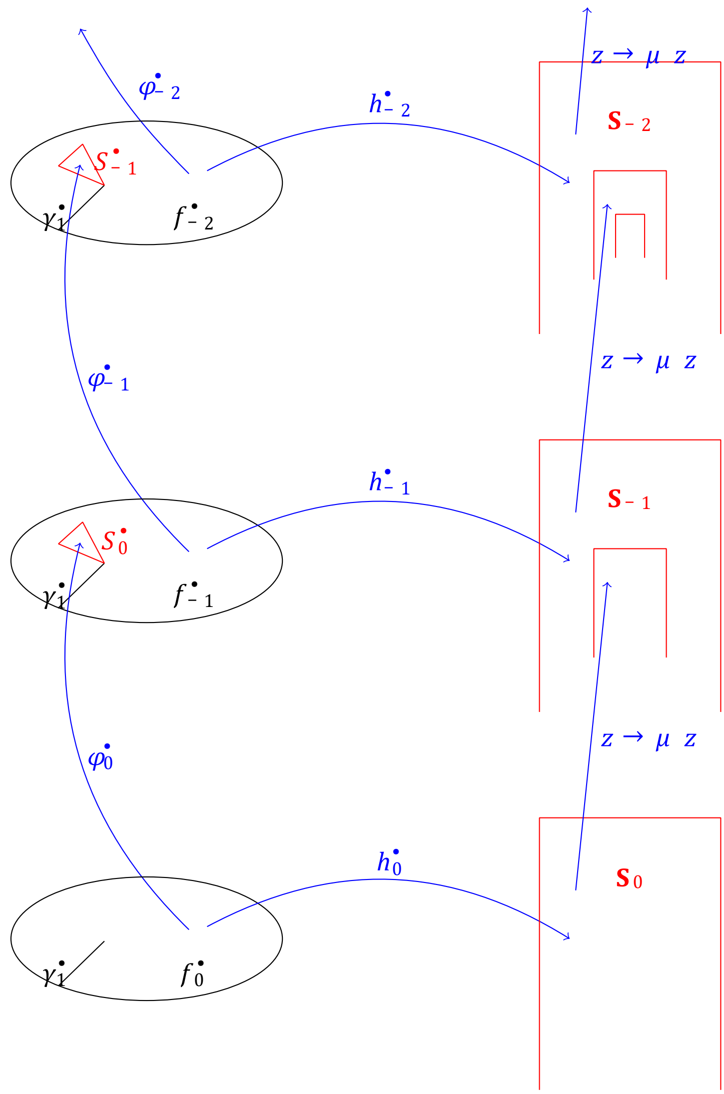

Figure 21. Left: each pacman $f_i^\bullet$ embeds as a prepacman to the dynamical plane of $f^\bullet_{i-1}$ via $\varphi^\bullet{}_i$. Right: sectors $S_i^\bullet$ after linearization of ψ-coordinates. Note that $S_i^\bullet$ can intersect $\gamma_1^\bullet$ in a small neighborhood of $\alpha^\bullet = T_i(\alpha)$.

Each f_k embeds to the dynamical plane of f_{k-1} as a prepacman $F_k^{(k-1)}$ such that $f_{k,\pm}^{(k-1)}$ are iterates of f_{k-1}.

Let us specify the following translation:

$$T_k : z \to z - c_1(f_k).$$

Let us now translate each f_k so that $c_1(f_k) = 0$. *We mark the translated objects with "•".* For $k \leq 0$, set

$$\varphi_k^\bullet(z) := T_{k-1} \circ \varphi_k \circ T_{k-1}$$

so that $\phi_k^\bullet(0) = 0$. Similarly, define $U_k^\bullet := T_k(U_k)$ and $V_k^\bullet := T_k(V)$, and conjugate all f_k and all $F_k = F_k^{(k)}$ by T_k; the resulting maps are denoted by $f_k^\bullet : U_k^\bullet \to V_k^\bullet$ and by

$$F_k^\bullet = (f_{k,\bullet\pm} : U_{k,\bullet\pm} \to S_k^\bullet).$$

We also write $\gamma_1^\bullet(f_k) := T_k(\gamma_1)$. The tower $(F_k^\bullet)_{k\le 0}$ is illustrated on Figure 21. Denote by

$$\mu_* := \phi_*'(c_1(f_\star)) = (\phi_*^\bullet)'(0), \ |\mu_\star| < 1,$$

the self-similarity coefficient of $\overline{Z}_\star$.

Lemma 5.2 (Linearization). *For every $f_0 \in W^u$ sufficiently close to f, the limit*

$$h_0^\bullet(z) = h_{f_0}^\bullet := \lim_{i\to-\infty} \frac{\phi_{i+1}^\bullet \circ \cdots \circ \phi_0^\bullet(z)}{\mu_*^{-i}} \tag{5.1}$$

is a univalent map on a certain neighborhood of 0 *(independent on f_0).*

We remark that the linearization is normalized in such a way that $(h_0^\bullet)'(0) = 1$ if $f_0 = f_\star$.

Proof. The proof follows from a standard linearization argument. Write $\varphi_i^\bullet(z) = \mu_i z + O(z^2)$; since $\varphi_i^\bullet$ tends exponentially fast to $\varphi_*^\bullet$ we see that μ_i tends exponentially fast to μ_* and that the constant in the error term does not depend on i. For z in a small neighborhood of 0, we have

$$\left|\phi_{i+1}^\bullet \circ \cdots \circ \phi_0^\bullet(z)\right| \le C(|\mu_*| + \varepsilon)^i |z|$$

for some constants C and $\varepsilon > 0$ such that $|\mu_*| + 2\varepsilon < 1$. Write

$$h^{(i)}(z) := \frac{\phi_{i+1}^\bullet \circ \cdots \circ \phi_0^\bullet(z)}{\mu_*^{-i}}.$$

Then

$$\frac{h^{(i-1)}(z)}{h^{(i)}(z)} = \frac{\phi_i^\bullet\left(\phi_{i+1}^\bullet \circ \cdots \circ \phi_0^\bullet(z)\right)}{\mu_*\phi_{i+1}^\bullet \circ \cdots \circ \phi_0^\bullet(z)} = \frac{\mu_{-i+1}}{\mu_*} + O\left(\phi_{i+1}^\bullet \circ \cdots \circ \phi_0^\bullet(z)\right)$$

tends exponentially fast to 1 in some neighborhood of 0. This implies that $h^{(i)}(z)$ converges to a univalent map in some neighborhood of 0.

Let us write $h_i^\bullet = h_{f_i}$. *We will use bold symbols for objects in the linearizing coordinates.* By construction (5.1), the maps $h_i^\bullet$ satisfy the linearization equation (see Figure 21)

$$h_{i-1}^\bullet \circ \phi_i^\bullet = [z \to \mu_\star z] \circ h_i^\bullet. \tag{5.2}$$

For $i \le 0$, set

$$h_i^\#(z) := \mu_*^{-i} h_0^\bullet(z). \tag{5.3}$$

It follows from (5.2) that

$$h_0^\bullet(z) = h_{-1}^{\#}\left(\phi_0^\bullet(z)\right) = \cdots = h_i^{\#}\left(\phi_{i+1}^\bullet \circ \cdots \circ \phi_0^\bullet(z)\right). \tag{5.4}$$

We will usually use "#" to mark linearized objects rescaled by $\mu_\star^{-i}$.

Lemma 5.3 (Extension of $h_0^\bullet$). *Under the above assumptions* $h_0^\bullet$ *extends to a univalent map* $h_0^\bullet \colon \operatorname{int}(V_0^\bullet \setminus \gamma_1^\bullet) \to \mathbb{C}$.

Proof. By Lemma 4.1 the map $\phi_{i-1}^\bullet \circ \cdots \circ \phi_0^\bullet$ extends to a conformal map defined on $\operatorname{int}(V_0^\bullet \setminus \gamma_1^\bullet)$. Since $\phi_{i-1}^\bullet \circ \cdots \circ \phi_0^\bullet$ is contracting, for every $z \in \operatorname{int}(V_0^\bullet \setminus \gamma_1^\bullet)$ there is an $i < 0$ such that $\phi_{i-1}^\bullet \circ \cdots \circ \phi_0^\bullet(z)$ is within a neighborhood of 0 where $h_i^\bullet$ is defined (this is easily true if $f_0^\bullet = f_\star^\bullet$; applying Theorem 4.6 we obtain this property for all $f_0^\bullet$). Therefore, (5.4) extends $h_0^\bullet$ dynamically to $\operatorname{int}(V_0^\bullet \setminus \gamma_1^\bullet)$.

We set $\mathbf{S}_i := \overline{h_i^\bullet(V_i^\bullet \setminus \gamma_1^\bullet)}$. Let us now conjugate every map $F_k^\bullet$ by $h_k^{\#}$; we define $\mathbf{F}_k^{\#} := h_k^{\#} \circ F_k^\bullet \circ \left(h_k^{\#}\right)^{-1}$. We construct the *tower in the linearizing coordinates*

$$\mathfrak{t}^{\#}(\mathbf{F}_0) = \left(\mathbf{F}_k^{\#}\right)_{k\le 0} = \left(\mathbf{f}_{k,\pm}^{\#} \colon \mathbf{U}_{k,\pm}^{\#} \to \mathbf{S}_k^{\#}\right)_{k \le 0}, \tag{5.5}$$

where

$$\operatorname{int}\left(\mathbf{S}_k^{\#}\right) = h_k^{\#}\left(V_k^\bullet \setminus \gamma_1^\bullet\right) = h_k^{\#} \circ T_k\left(V_k \setminus \gamma_1\right), \tag{5.6}$$

and other objects marked by "#" are similarly defined.

The next lemma follows from (A.4).

Lemma 5.4. *There are numbers* $m_{1,1}, m_{1,2}, m_{2,1}, m_{2,2}$ *such that for* $k < 0$ *we have*

$$\mathbf{f}_{k+1,-}^{\#} = \left(\mathbf{f}_{k,-}^{\#}\right)^{m_{1,1}} \circ \left(\mathbf{f}_{k,+}^{\#}\right)^{m_{1,2}}, \quad \mathbf{f}_{k+1,+}^{\#} = \left(\mathbf{f}_{k,-}^{\#}\right)^{m_{2,1}} \circ \left(\mathbf{f}_{k,+}^{\#}\right)^{m_{2,2}}.$$

Note also that

$$\mathbf{f}_{k,\pm}^{\#} = \frac{1}{\mu_*^k}\mathbf{f}_{k,\pm}\left(\mu_*^k z\right). \tag{5.7}$$

5.2. **Global extension of prepacmen in** $\mathcal{W}^u$. Using Key Lemma 4.8 we deduce the following.

Theorem 5.5 (Existence of a maximal prepacman). *If* $f_0 \in \mathcal{W}^u$ *is sufficiently close to* $f_\star$, *then every pair* $\mathbf{F}_i^{\#} = \left(\mathbf{f}_{k,\pm}^{\#}\right)$ *in the tower* $\mathfrak{t}^{\#}(\mathbf{F}_0)$ *(see (5.5)) extends to σ-proper branched coverings*

$$\mathbf{f}_{k,\pm}^{\#} \colon \mathbf{X}_{k,\pm}^{\#} \to \mathbb{C},$$

where $\mathbf{X}_{k,\pm}^{\#}$ *are open connected subsets of* $\mathbb{C}$.

Note that the case $f_0 = f_\star$ follows from [McM2, Theorem 8.1].

Proof. Let

$$F_0 = \left(f_{0,\pm} \colon U_{0,\pm} \to S := V \setminus (\gamma_1 \cup O)\right)$$

be a commuting pair obtained from\ $F_0 = (f_{0,\pm} : U_{0,\pm} \to V^{-}{}_{,\pm 1} \setminus (\gamma O_1))$ fromby removing

a$U_{0,\pm}$. By small neighborhood O of α from $V\gamma_1$ and by removing f_0

Lemma 4.1 the map $\varphi_{k-1} \circ \cdots \circ \varphi_0$ embeds F_0 to the dynamical plane of f_k as a commuting pair denoted by

(5.8) $$F_0^{(k)} = \left(f_k^{\mathfrak{a}-k}, f_k^{\mathfrak{b}-k}\right) : \mathfrak{U}_{0,-}^{(k)} \cup \mathfrak{U}_{0,+}^{(k)} \to \mathfrak{S}_0^{(k)}.$$

Since φ_k is contracting at the critical value, the diameter of $U_{0,-}^{(k)} \cup U_{0,+}^{(k)} \cup \mathfrak{S}_0^{(k)} \ni c_1(f_n)$ tends to 0. By Key Lemma 4.8, for a sufficiently big $k < 0$ there is a small open topological disk D around the critical value of f_k such that the pair (5.8) extends into a pair of commuting branched coverings

$$F_0^{(k)} = \left(f_k^{\mathfrak{a}-k}, f_k^{\mathfrak{b}-k}\right) : W_-^{(k)} \cup W_+^{(k)} \to D,$$

(5.9) $$W_-^{(k)} \cup W_+^{(k)} \cup D \subset V \setminus \gamma_1.$$

with

Conjugating (5.9) by $h_k^{\#} \circ T_k$ we obtain the commuting pair

$$(\mathbf{f}_{0,-}, \mathbf{f}_{0,+}) : \mathbf{W}_-^{(k)} \cup \mathbf{W}_+^{(k)} \to \mathbf{D}^{(k)}.$$

Since for a sufficiently big t and all $m \le 0$ the modulus of the annulus $\mathbf{D}^{(tm-t)} \setminus \mathbf{D}^{(tm)}$ is uniformly bounded from 0 we obtain $\bigcup_{k \ll 0} \mathbf{D}^{(k)} = \mathbb{C}$. Setting

(5.10) $$\mathbf{X}_{0,-} := \bigcup_{k \ll 0} \mathbf{W}_-^{(k)}, \quad \mathbf{X}_{0,+} := \bigcup_{n \ll 0} \mathbf{W}_+^{(k)}$$

we obtain# σ-proper maps $\mathbf{f}_{0,\pm} : \mathbf{X}_{0,\pm} \to \mathbb{C}$, where $\mathbf{X}_{0,\pm}$ are connected. Similarly,

$(\mathbf{f}_{k,\pm})$ extends to a pair of σ-proper maps.

6. Maximal parabolic prepacmen

Since the multiplier of the α-fixed point is expanded under $\mathcal{R}$ at $f_\star$ (by Lemma 3.18), we can consider a parabolic pacman $f_0 \in W^u$ close to f such that Theorem 5.5

applies for $R^n f_0 f$ with$_0$. As in$^n \le$ §05and bywe denote by $\mathbf{F}^{\#}{}_n$ the rescaled version of $\mathbf{F}_n =$

$(\mathbf{f}_{n,\pm})$ the maximal prepacmen of $\mathbf{F}_n$ so that $\mathbf{F}_0$ is an f_n = iteration of $\mathbf{F}^{\#}{}_n$; see Lemma 5.4.

6.1. **The post-critical set of a maximal prepacman.** The *forward orbit of* $z \in \mathbb{C}$ under $\mathbf{F}_n$ is

$$\mathrm{orb}_z(\mathbf{F}_n) := \{\mathbf{f}_{n,-}^{s} \circ \mathbf{f}_{n,+}^{r}(z) \mid s,r \ge 0\};$$

we do not require that $\mathbf{f}_{n,-}^{s} \circ \mathbf{f}_{n,+}^{r}(z)$ is defined for all pairs s,r. A *finite orbit of* z is $\mathrm{orb}_z^{\le q}(\mathbf{F}_n) := \left\{\mathbf{f}_{n,-}^{s} \circ \mathbf{f}_{n,+}^{r}(z) \mid s,r \in \{0,1,\dots,q\}\right\}.$

Similarly, $\mathrm{orb}_z(\mathbf{F}^\#_n)$ and $\mathrm{orb}^{\le q}_z(\mathbf{F}^\#_n)$ are defined. Since $\mathbf{F}_0$ is an iteration of $\mathbf{F}_n^\#$, there is a $k > 1$ such that

$$\mathrm{orb}_z^{\le q}(\mathbf{F}_0) \subseteq \mathrm{orb}_z^{\le k^{-n} q}\left(\mathbf{F}_n^\#\right)$$

for all $n \le 0$ and $z \in \mathbb{C}$.

An *orbit path* of $\mathbf{F}_m$ is a sequence $x_0, x_1, \dots, x_n$ such that either$^\#$, an orbit path of $x_{i+1} = \mathbf{f}\mathbf{F}_{m,0-}$(is a x_i)

or $x_{i+1} = \mathbf{f}_{m,+}(x_i)$. Since $\mathbf{F}_0$ is an iteration of $\mathbf{F}_n$ "suborbit" path of $\mathbf{F}^\#_n$.

Let us denote by

$$C(\mathbf{F}_k) := \{z \in \mathbb{C} \mid \mathbf{f}'_{k,-}(z) = 0 \text{ or } \mathbf{f}'_{k,+}(z) = 0\},$$

the set of critical points of $\mathbf{F}_k$; its *post-critical set* is

$$P(\mathbf{F}_k) = \bigcup_{\substack{n+m>0\\ n,m\ge 0}} \mathbf{f}^n_{k,-} \circ \mathbf{f}^m_{k,+}(C(\mathbf{F}_k)).$$

Similarly $P(\mathbf{F}^\#_n)$ is defined. Clearly,

$$P(\mathbf{F}_0) \subset P\left(\mathbf{F}_n^\#\right) = \mu_*^n P(\mathbf{F}_n).$$

Recall that 0 is a critical value of $\mathbf{F}^\#_n$ for all $n \le 0$; we denote by $o^\#_n$ the critical point of $\mathbf{F}^\#_n$ such that $o^\#_n$ is identified with the critical point $c_0(f_n)$ under the homeomorphism $\mathrm{int}\mathbf{S}_n^\# \simeq V \setminus \gamma_1$; see (5.6).

Lemma 6.1 (Every critical orbit "passes" through 0)**.** *For any critical point x_0 of $\mathbf{f}_{0,\iota}$ with $\iota \in \{-,+\}$ the following holds. For all sufficiently big $n < 0$ there is an orbit path of $\mathbf{F}^\#_n$*

$$x_0, x_1, x_2, \dots, x_k;\quad x_i = \mathbf{f}_{n,j^\#(i)}(x_{i-1}) \tag{6.1}$$

such that

- $\mathbf{f}_{0,\iota} = \mathbf{f}_{n,j^\#(k)} \circ \mathbf{f}_{n,j^\#(k-1)} \circ \cdots \circ \mathbf{f}_{n,j^\#(1)}$*, in particular* $x_k = \mathbf{f}_{0,\iota}(x_0)$*;*
- $x_i = o^\#_n$ *and* $x_{i+1} = 0$ *for some i.*

Therefore,

$$P(\mathbf{F}_0) \subset \bigcup_{n\le 0} \mathrm{orb}_0\left(\mathbf{F}_n^\#\right).$$

Proof. Clearly, the second statement follows from the first. We will use notation from the proof of Theorem 5.5. Suppose for definiteness ι = "$-$". Recall (5.10) that $\mathrm{Dom}\,\mathbf{f}_{0,-} = \bigcup_{n\ll 0} \mathbf{W}_-^{(n)}$; thus $x_0 \in \mathbf{W}_-^{(n)}$ for some $n < 0$. The map $\mathbf{f}_{0,-} \mid \mathbf{W}_-^{(n)}$ is conformally conjugate to $f_n^{\mathfrak{a}_{-n}} \mid W_-^{(n)} \to D$ (see (5.9)) after identifying $\mathbf{W}_-^{(n)}$ with $W_-^{(n)}$. This shows that $x_0, \mathbf{f}_{0,-}(x_0)$ is within an actual orbit $x_0, x_1, \dots, x_k$ of

$$\left(\mathbf{f}^\#_{n,\pm} \colon \mathbf{U}^\#_{n,\pm} \to \mathbf{S}^\#_n\right),$$

which is a prepacman of f_n. We deduce that one of x_i is $o^\#_n$ and $x_{i+1} = 0$.

6.2. **Global attracting basin of a parabolic pacman.** Since $\mathbf{f}_{0,\pm} : \mathrm{Dom}\mathbf{f}_{0,\pm} \to \mathbb{C}$ are σ-proper commuting maps with maximal domain we have

(6.2) $$\mathrm{Dom}(\mathbf{f}_{0,-} \circ \mathbf{f}_{0,+}) = \mathrm{Dom}(\mathbf{f}_{0,+} \circ \mathbf{f}_{0,-}) \subset \mathrm{Dom}\mathbf{F}_0 := \mathrm{Dom}\mathbf{f}_{0,-} \cap \mathrm{Dom}\mathbf{f}_{0,+}.$$

Note that for every $q \geq 1$ we have $f_0^q \neq \mathrm{id}$ in any small neighborhood of $\alpha(f_0)$ because, otherwise, considering a lift $\mathbf{f}_{0,-}^{\mathfrak{r}} \circ \mathbf{f}_{0,+}^{\mathfrak{s}}$ of f_0^q we would obtain $\mathbf{f}_{0,-}^{\mathfrak{r}} \circ \mathbf{f}_{0,+}^{\mathfrak{s}} = \mathrm{id}$ in $\mathbb{C}$ which is impossible. Therefore, there is a small open attracting parabolic flower H_0 around the α-fixed point of f_0. Each petal of H_0 lands at α at a well-defined angle. Assume H_0 is small enough so that $H_0 \subset V \setminus \gamma_1$, possibly up to a slight rotation of γ_1. By Lemma 4.3 the flower H_0 lifts to the dynamical plane of $\mathbf{F}_0$ via the identification $V \setminus \gamma_1 \simeq \mathrm{int}\mathbf{S}_0$; we denote by $\mathbf{H}_0$ the lift.

Let $\mathbf{e}(\mathfrak{p}_0/\mathfrak{q}_0) \neq 1$ be the multiplier of the α-fixed point of f_0. Since f_0 is close to f, we have $\mathfrak{q}_0 > 1$. By replacing H_0 with its subflower we can assume that there are exactly $\mathfrak{q}_0$ connected components of H_0 with combinatorial rotation number $\mathfrak{p}_0/\mathfrak{q}_0$. We enumerate them counterclockwise as $H_0^0, H_0^1, \dots, H_0^{\mathfrak{q}_0-1}$. Then f_0 maps H_0^i to $H_0^{i+\mathfrak{p}_0}$. We will show in Corollary 6.4 that H_0 is in fact unique; i.e., f_0 has exactly $\mathfrak{q}_0$ attracting directions at α. Denote by $\mathbf{H}^i_0$ the lift of H_0^i to the dynamical plane of $\mathbf{F}_0$.

Lemma 6.2. *There are* $\mathfrak{r},\mathfrak{s} \geq 1$ *with* $\mathfrak{r} + \mathfrak{s} = \mathfrak{q}_0$ *such that*

$$\mathbf{f}_{0,-}^{\mathfrak{r}} \circ \mathbf{f}_{0,+}^{\mathfrak{s}}(\mathbf{H}_0^i) \subset \mathbf{H}_0^i.$$

The set $\mathbf{H}_0$ *is in* $\mathrm{Dom}(\mathbf{f}_{0,-}^a \circ \mathbf{f}_{0,+}^b)$ *for all $a,b \geq 0$.*

It will follow from Proposition 6.5 that $\mathbf{f}_{0,-}^{\mathfrak{r}} \circ \mathbf{f}_{0,+}^{\mathfrak{s}} : \mathbf{H}_0^i \to \mathbf{H}_0^i$ is the first return map.

Proof. We have $f_0^{\mathfrak{q}_0}(H_0^i) \subset H_0^i$. Cutting the prepacman f_0 along γ_1 we see that there are $\mathfrak{r},\mathfrak{s} \geq 1$ with $\mathfrak{r} + \mathfrak{s} = \mathfrak{q}_0$ such that $f_{0,-}^{\mathfrak{r}} \circ f_{0,+}^{\mathfrak{s}}(H_0^i) \subset H_0^i$. This implies the first claim. As a consequence, $\mathbf{H}_0$ is in $\mathrm{Dom}(\mathbf{f}_{0,-}^{\mathfrak{r}j} \circ \mathbf{f}_{0,+}^{\mathfrak{s}j})$ for all $j \geq 0$. Combined with (6.2), we obtain the second claim.

As a consequence, all of the branches of $\mathbf{f}_{0,-}^a \circ \mathbf{f}_{0,+}^b$ with $a,b \in \mathbb{Z}$ are well defined for points in $\mathbf{H}_0$. Set

$$\mathbf{H} := \bigcup_{a,b\in\mathbb{Z}} (\mathbf{f}_{0,-})^a \circ (\mathbf{f}_{0,+})^b\,(\mathbf{H}_0)$$

to be the full orbit of $\mathbf{H}_0$. Since $\mathbf{f}_{0,-}, \mathbf{f}_{0,+}$ commute and $\mathbf{H}_0$ is forward invariant under $\mathbf{f}_{0,-}^{\mathfrak{r}} \circ \mathbf{f}_{0,+}^{\mathfrak{s}}$, the set $\mathbf{H}$ is an open fully invariant subset of $\mathbb{C}$ within $\mathrm{Dom}\mathbf{f}_{0,-} \cap \mathrm{Dom}\mathbf{f}_{0,+}$. We call $\mathbf{H}$ the *global attracting basin* of the α-fixed point.

A connected component $\mathbf{H}'$ of $\mathbf{H}$ is *periodic* if there are $s,r \in \mathbb{N}_{>0}$ such that $\mathbf{f}_{0,-}^s \circ \mathbf{f}_{0,+}^r(\mathbf{H}') = \mathbf{H}'$. A pair (s,r) is called a period of $\mathbf{H}$. We will show in Corollary 6.6 that there is no component $\mathbf{H}'$ of $\mathbf{H}$ such that $\mathbf{f}_{0,-}^r(\mathbf{H}') = \mathbf{H}'$ or $\mathbf{f}_{0,+}^r(\mathbf{H}') = \mathbf{H}'$ for some $r > 0$.

By Lemma 6.2, the components of **H** intersecting $\mathbf{H}_0$ are (r,s)-periodic. Observe next that for any periodic component **H** and any component **H** of **H** there are $a,b \geq 1$ with $\mathbf{f}^{a}_{0,-} \circ \mathbf{f}^{b}_{0,+}(\mathbf{H}'') = \mathbf{H}'$; i.e., **H** and **H** are dynamically related. Indeed, by definition there are $a', b' \in \mathbb{Z}$ such that a certain branch of $\mathbf{f}^{a'}_{0,-} \circ \mathbf{f}^{b'}_{0,+}$ maps **H** to **H**. Applying $\mathbf{f}^{st}_{0,-} \circ \mathbf{f}^{rt}_{0,+}$ with $t \gg 1$, we obtain the required $a,b \geq 1$. As a consequence, all the periodic components have the same periods; in particular they are (r,s)-periodic.

6.3. **Attracting Fatou coordinates.** It is classical that $f_0^{q_0} : H_0^0 \to H_0^0$ admits *attracting Fatou coordinates:* a univalent map $h : H_0^0 \to \mathbb{C}$ such that

- $h \circ f_0^{q_0}(z) = h(z) + 1$; and
- there is an $L > 1$ such that

$$h(H_0^0) \supset \{z \in \mathbb{C} \mid \mathrm{Re}(z) > L\}. \tag{6.3}$$

There is a unique dynamical extension $h: H_0 \to \mathbb{C}$ such that

$$h \circ f_0(z) = h(z) + 1/q_0. \tag{6.4}$$

Lifting h to the dynamical plane of $\mathbf{F}_0$ we obtain $\mathbf{h}: \mathbf{H}_0 \to \mathbb{C}$.

Lemma 6.3 (Fatou coordinates of **H**). *The map* $\mathbf{h}: \mathbf{H}_0 \to \mathbb{C}$ *extends uniquely to a map* $\mathbf{h}: \mathbf{H} \to \mathbb{C}$ *satisfying*

$$\mathbf{h} \circ \mathbf{f}_{0,\pm}(z) = \mathbf{h}(z) + 1/q_0 \tag{6.5}$$

for any choice of "±". For every component **H** *of* **H**, *the map* $\mathbf{h}|\mathbf{H}'$ *is* σ*-proper. The singular values of* **h** *are exactly the* **h***-images of the critical points of* $\mathbf{F}_0$ *and their iterated preimages.*

Moreover, components of $\mathbf{H}_0$ *are in different components of* **H**. *The set* **H** *is a proper subset of* $\mathbb{C}$. *By postcomposing* **h** *with a translation we can assume that*

$$\mathbf{h}(0) = 0. \tag{6.6}$$

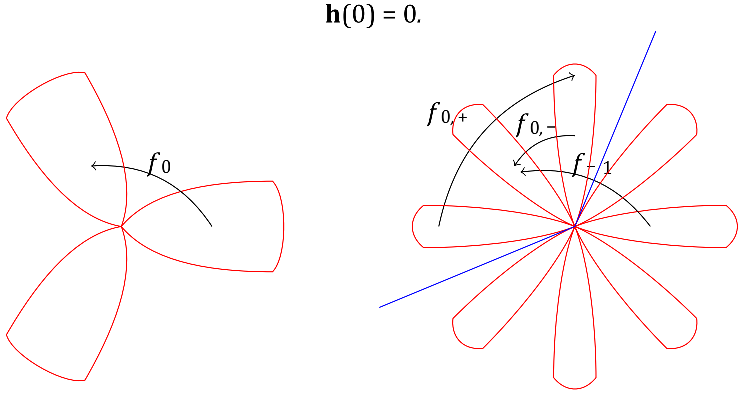

Figure 22. A parabolic pacman f_0 with rotation number 1/3 embeds as a prepacman into the dynamical plane of a parabolic pacfman=f-f_{12} with rotation number 3. /8. We have $f_{0,-} = f$-$^3{}_1$ and

0,+ −1

Proof. On $\mathbf{H}_0$ equation (6.5) is just a lift of(6.4). Applying $\mathbf{f}_{0^-,\pm}^{1}$ and using commutativity of $\mathbf{f}_{0,-}, \mathbf{f}_{0,+}$, we obtain a unique extension of $\mathbf{h}$ to $\mathbf{H}$ such that (6.5) holds.

Since $\mathbf{f}_{0,-}, \mathbf{f}_{0,+}$ are σ-proper maps, so is $\mathbf{h} \mid \mathbf{H}$. Indeed, suppose that $\mathbf{H} \subset \mathbf{H}$ is a periodic component intersecting $\mathbf{H}_0$; the other cases follow by applying a certain branch of $\mathbf{f}^{a}_{0,-} \circ \mathbf{f}^{b}_{0,+}$, where $a,b \in \mathbb{Z}$. Recall from Lemma 6.2 that $\mathbf{H}$ is (r,s)periodic. Consider a compact set $K \subset \mathbb{C}$. We denote by $\mathbf{K}$ a connected component of the preimage of K under $\mathbf{h}| \mathbf{H}'$. Then for a sufficiently big $i \gg 1$, we have $\mathrm{Re}(K + i) > L$ and $\mathbf{K}^2 := \mathbf{f}^{\mathfrak{r}i}_{0,-} \circ \mathbf{f}^{\mathfrak{s}i}_{0,+}(\mathbf{K})$ intersects $\mathbf{H}_0$, where L is a constant from (6.3). Then $\mathbf{K}_2 \subset \mathbf{H}_0$ and $\mathbf{K}_2$ is compact as a connected component of the preimage of $K + i$ under $\mathbf{h} \mid \mathbf{H}' \cap \mathbf{H}_0$. We obtain that $\mathbf{K} \subset \mathbf{f}^{-\mathfrak{r}i}_{0,-} \circ \mathbf{f}^{-\mathfrak{s}i}_{0,+}(\mathbf{K}_2)$ is compact because $\mathbf{f}^{\mathfrak{r}i}_{0,-} \circ \mathbf{f}^{\mathfrak{s}i}_{0,+}$ is σ-proper. This also shows that singular values of $\mathbf{h}$ are the $\mathbf{h}$-images of either critical points of $\mathbf{F}_0$ or their iterated preimages. (We recall a σ-proper map has no asymptotic values.)

Let $\mathbf{H}^x_0$ and $\mathbf{H}^y_0$ be two different components of $\mathbf{H}_0$, and let $\mathbf{H}^x$ and $\mathbf{H}^y$ be the periodic components of $\mathbf{H}$ containing $\mathbf{H}^x_0$ and $\mathbf{H}^y_0$. Since all points in $\mathbf{H}^x$ and $\mathbf{H}^y$ escape eventually to $\mathbf{H}^x_0$ and $\mathbf{H}^y_0$ under the iteration of $\mathbf{f}^{\mathfrak{r}}_{0,-} \circ \mathbf{f}^{\mathfrak{s}}_{0,+}$ we have $\mathbf{H}^x \neq \mathbf{H}^y$.

As a consequence $\mathbf{H} =$ C. The claim concerning (6.6) is immediate.

From now on we assume that (6.6) holds. Denote by $\mathbf{H}^{\mathrm{per}} \subset \mathbf{H}$ the union of periodic components of $\mathbf{H}$.

Corollary 6.4 (Critical point). *The set* $\mathbf{H}^{\mathrm{per}}$ *contains* $P(\mathbf{F}_0)$ *and at least one critical point. In particular,* $0 \in \mathbf{H}^{\mathrm{per}}$*. All the critical points of* $\mathbf{F}_0$ *are within* $\mathbf{H}$*. In the dynamical plane of* f_0 *the flower* H_0 *is unique:* f_0 *has exactly* $\mathfrak{q}_0$ *attracting directions at* α *cyclically permuted by* f_0*.*

Proof. Since $\mathbf{h}\colon \mathbf{H}^{\mathrm{per}} \to \mathbb{C}$ is not a covering map, $\mathbf{H}^{\mathrm{per}}$ contains at least one critical point of $\mathbf{F}_0$. Since $\mathbf{H}^{\mathrm{per}}$ is forward invariant, $\mathbf{H}^{\mathrm{per}}$ contains $\mathrm{o}^{\#}_{n}$ for all sufficiently big $n < 0$; see Lemma 6.1. Therefore, $\mathbf{H}^{\mathrm{per}}$ contains all of the critical values of $\mathbf{F}_0$. Since $\mathbf{H}$ is fully invariant, it contains all of the critical points. As a consequence, H_0

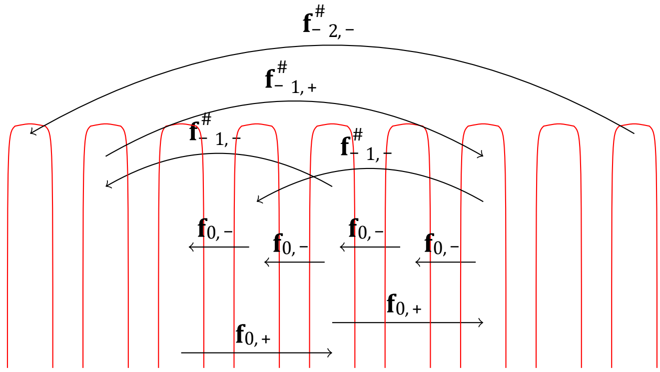

Figure 23. The maximal prepacman $\mathbf{F}_0 = (\mathbf{f}_{0,\pm})$ of a parabolic pacman f_0 with rotation number 1/3; see Figure 22. The map $\mathbf{f}_{0,-}$ shifts periodic components of $\mathbf{H}$ to the left while $\mathbf{f}_{0,+}$ shifts the periodic components of $\mathbf{H}$ to the right. We have $\mathbf{f}_{n,-} = \mathbf{f}_{n}{}^{2}{}_{-1,-} \circ$

and$\mathbf{f}_{n-1,}f_{+0,}$and$_{+}\mathbf{f}_{n,2+}$in Figure 22).$= \mathbf{f}_{n-1,-} \circ \mathbf{f}_{n-1,+}$ for all n (obtained from $f_{0,-} = f_1^3$

$= f_1$

is unique because the global attracting basin of another flower would also contain 0.

6.4. **Dynamics of periodic components.** It follows from Lemma 6.2 that

$$\mathbf{H} = \bigcup_{a,b\in \mathbb{Z}} (\mathbf{f}^{\#}_{n,-})^a \circ (\mathbf{f}^{\#}_{n,+})^b (\mathbf{H}_0)$$

for all $n \le 0$. It is also clear that $\mathbf{H}^{\mathrm{per}}$ is the union of $\mathbf{F}^{\#}_n$-periodic components.

Let H_n be a small parabolic attracting flower of f_n admitting a lift to the dynamical plane of $\mathbf{F}^{\#}_n$; we denote this lift by $\mathbf{H}^{\#}_n \to H_n$. We denote by $\mathrm{p}_n/\mathrm{q}_n$ the combinatorial rotation number of f_n.

Let I_n be an index set enumerating clockwise the connected components of H_n starting with the component closest to γ_1. Since H_n embeds naturally to the dynamical plane of f_{n-1} (see Figure 22), we have a natural embedding of I_n to

I_{n-1}.

Let us write

$$I_0 = \{-a_0, -a_0 + 1, \dots, b_0 - 1, b_0\}$$

with $a_0, b_0 > 0$ and $a_0 + b_0 + 1 = \mathrm{q}_0$. The component of H_0 indexed by $i+1$ follows in clockwise order the component of H_0 indexed by i. Then f_0 maps the component of H_0 indexed by i to the component of H_0 indexed by either $i - \mathrm{p}_0$ or $i + \mathrm{q}_0 - \mathrm{p}_0$ depending on whether $i - \mathrm{p}_0 \ge -a_0$.

For every $n < 0$, choose a parameterization $I_n = \{-a_n, -a_n + 1, \dots, b_n - 1, b_n\}$ so that the natural embedding of I_n to I_{n-1} is viewed as $I_n \subset I_{n-1}$. Set $I_{-\infty} := \bigcup_{n\le 0} I_n = \mathbb{Z}$.

Recall (see §6.2) that a connected component $\mathbf{H}'$ of $\mathbf{H}$ is periodic if $\mathbf{f}^s_{0,-} \circ \mathbf{f}^r_{0,+}(\mathbf{H}') = \mathbf{H}'$ for some $s,r \in \mathbb{N}_{>0}$.

Proposition 6.5 (Parameterization of $\mathbf{H}^{\text{per}}$). *The connected components of* $\mathbf{H}^{\text{per}}$ *are uniquely enumerated as* $(\mathbf{H}^i)_{i\in\mathbb{Z}}$ *so that for every sufficiently big* $n \ll 0$ *the component* $\mathbf{H}^i$ *contains the image of the component of* H_n *indexed by i under* $H_n \simeq \mathbf{H}^{\#}_n \subset \mathbf{H}^{\text{per}}$.

The actions of $\mathbf{f}^{\#}_{n,\pm}$ *on* $(\mathbf{H}^i)_{i\in\mathbb{Z}}$ *are given (up to interchanging* $\mathbf{f}^{\#}_{n,-}$ *and* $\mathbf{f}^{\#}_{n,+}$*) by*

$$\mathbf{f}^{\#}_{n,-}(\mathbf{H}^i) = (\mathbf{H}^{i-\mathfrak{p}_n}) \text{ and } \mathbf{f}^{\#}_{n,+}(\mathbf{H}^i) = \mathbf{H}^{i+\mathfrak{q}_n-\mathfrak{p}_n}. \tag{6.7}$$

Moreover, by re-enumerating components of $\mathbf{H}_0$ *we can assume that* $\mathbf{H}^0$ *contains* 0.

Proof. By construction, $I_{-\infty} \simeq \mathbb{Z}$ enumerates all of the periodic components of $\mathbf{H}$ intersecting $\bigcup_{n\le 0} \mathbf{H}^{\#}_n$ with actions given by (6.7). Since $\bigcup_{i\in\mathbb{Z}}\mathbf{H}^i$ is forward invariant and since every periodic component is in the forward orbit of $\mathbf{H}^0$ (see §6.2), we obtain $\bigcup_{i\in\mathbb{Z}}\mathbf{H}^i = \mathbf{H}^{\text{per}}$. We can re-enumerate $(\mathbf{H}^i)_{i\in\mathbb{Z}}$ in a unique way so that $\mathbf{H}^0 \ni 0$.

Corollary 6.6. *There is no component* $\mathbf{H}'$ *of* $\mathbf{H}$ *such that* $\mathbf{f}^r_{0,-}(\mathbf{H}') = \mathbf{H}'$ *or* $\mathbf{f}^r_{0,+}(\mathbf{H}') = \mathbf{H}'$ *for some* $r > 0$.

Proof. Suppose the converse and consider such $\mathbf{H}'$, say $\mathbf{f}^r_{0,-}(\mathbf{H}') = \mathbf{H}'$. Choose $a,b \in \mathbb{Z}$ such that a certain branch of $\mathbf{f}^a_{0,-} \circ \mathbf{f}^b_{0,+}$ maps $\mathbf{H}'$ to $\mathbf{H}^0$. Recall that $(\mathfrak{r},\mathfrak{s})$ is a period of $\mathbf{H}^0$. By postcomposing $\mathbf{f}^a_{0,-} \circ \mathbf{f}^b_{0,+}$ with an iterate of $\mathbf{f}^{\mathfrak{r}}_{0,-} \circ \mathbf{f}^{\mathfrak{s}}_{0,+}$ we can assume that $a,b \ge 0$. It now follows from Proposition 6.5 that applying first $\mathbf{f}^r_{0,-} \mid \mathbf{H}'$ and then $\mathbf{f}^a_{0,-} \circ \mathbf{f}^b_{0,+}$ is different from applying $\mathbf{f}^a_{0,-} \circ \mathbf{f}^b_{0,+} \mid \mathbf{H}'$ and then $\mathbf{f}^r_{0,-}$. This is a contradiction.

Corollary 6.7. *For* $a,b,c,d \ge 0$ *and* $n \le 0$,

$$\left(\mathbf{f}^{\#}_{n,-}\right)^a \circ \left(\mathbf{f}^{\#}_{n,+}\right)^b (0) = \left(\mathbf{f}^{\#}_{n,-}\right)^c \circ \left(\mathbf{f}^{\#}_{n,+}\right)^d (0)$$

if and only if $a = c$ *and* $b = d$.

Proof. It is sufficient to prove it for $n = 0$. Suppose $\mathbf{f}^a_{0,-} \circ \mathbf{f}^b_{0,+}(0) = \mathbf{f}^c_{0,-} \circ \mathbf{f}^d_{0,+}(0)$. It follows from (6.5) that $a+b = c+d$. If (a,b) is not equal to (c,d), then $\mathbf{f}^a_{0,-} \circ \mathbf{f}^b_{0,+}(0)$, $\mathbf{f}^c_{0,-} \circ \mathbf{f}^d_{0,+}(0)$ are in different connected components of $\mathbf{H}^{\text{per}}$; see (6.7). Therefore, $a = c$ and $b = d$.

6.5. **Valuable flowers of parabolic pacmen.** This subsection is a preparation for proving the Scaling Theorem (§8); it will not be used in proving the Hyperbolicity Theorem (§7).

Definition 6.8 (Valuable flowers)**.** Let f be a parabolic pacman with rotation number p/q. A *valuable flower* (see Figure 26) is an open forward invariant set H such that

(A) $H \cup \{\alpha(f)\}$ is connected;

(B) H has q connected components $H^0, H^1, \ldots, H^{q-1}$, called *petals*, enumerated counterclockwise at α; every H^i is an open topological disk with a single access to α;

(C) $f(H^i) \subset H^{i+p}$;

(D) all of the points in H are attracted by α; (E) H^{-p} contains the critical point of f.

We remark that a local flower (see §6.2) satisfies (A)–(D).

We say a Siegel triangulation (see §4.1.5) Δ *respects* a flower H if every petal of H is within a triangle of Δ.

Theorem 6.9 (Parabolic valuable flowers)**.** *Let* $f_0 \in W^u$ *be a parabolic pacman. Then for all sufficiently big* $n \ll 0$ *the pacman* $f_n = R^n f_0$ *has a valuable flower* H_n *and a Siegel triangulation* $\Delta(f_n)$ *respecting* H_n *such that:*

- $\Delta(f_n)$ *has a wall* $\Pi(f_n)$ *approximating* $\partial Z_\star$*;* • $\Delta(f_{n-1})$

and H_{n-1} *are full lifts of* $\Delta(f_n)$ *and* H_n*.*

Moreover, for a given closed disk $\mathbf{D} \subset \mathbf{H}^0$ *the flower* H_n *with* $n \ll 0$ *can be constructed in such a way that* $\mathbf{D}$ *projects via* $\mathrm{int}(\mathbf{S}_n^{\#}) \simeq V \setminus \gamma_1$ *(see (5.6)) to a subset of* H^0_n*.*

Proof. Let us recall (see §6.2) that a local flower H_0 was chosen sufficiently small such that $H_0 \subset V \setminus \gamma_1$, possibly up to a slight rotation of γ_1 in a small neighborhood of α. We denote by $\Delta^{new}{}_0$ the triangulation obtained from Δ_0 by this slight adjustment of γ_1. By Lemma 4.3, the triangulation≤ $\Delta^{new}{}_0$ admits a full lift^{new}, the flower$\Delta^{new}{}_{-n}$ to the dynamical plane of f_n for all n 0. Since H_0 is respected by Δ_0

H_0 also admits a full lift H_n to the dynamical plane of f_n such that H_n is respected by Δ_{new-n}.

6.5.1. *Valuable petals.* Recall that p_n/q_n denotes the rotation number of f_n. A *valuable petal* H^j_n of f_n is an open connected set attached to α such that

- $f_n^{q_n}$ extends analytically from a neighborhood of α to $f_n^{q_n}: H^j_n \to H^j_n$ (in particular, H^j_n is $f_n^{q_n}$-invariant);
- $f_n^{q_n}: H^j_n \to H^j_n$ has a critical point; and all points in H^j are attracted to α under f^q.
- n nn

Claim 1. *For $n \ll 0$ the map f_n has a valuable petal H^0_n containing the critical value 0 such that $\mathrm{H}^0_n = H_n^0 \cup D$, where H_n^0 is a petal of H_n and D is a small neighborhood of c_1 containing the projection of* $\mathbf{D}$ *via* (5.6). *Moreover, there is an $M > 0$ such that $f_n^{\mathfrak{q}_n M}(\mathrm{H}^0_n) \subset H_n$.*

Proof. In the dynamical plane of $\mathbf{F}_0$ consider the petal $\mathbf{H}^0 \ni 0$. Recall from §6.4 that $\mathbf{H}^\#_n$ denotes the lift of H_n to the dynamical plane of $\mathbf{F}^\#_n$. If $n \ll 0$ is sufficiently big, then $\mathbf{H}^0$ contains a unique connected component of $\mathbf{H}^\#_n$, call it $(\mathbf{H}^\#_n)^0$. Note also that $(\mathbf{H}^\#_n)^0 = (\mathbf{H}^\#_m)^0$ for all sufficiently big $n, m \ll 0$; see Proposition 6.5.

Enlarge $\mathbf{D}$ to a bigger closed disk $\mathbf{D} \subset \mathbf{H}^0$ such that

- $(\mathbf{H}^\#_n)^0 \cup \mathbf{D}$ is forward invariant under the first return map $\mathbf{f}^{\mathfrak{r}}_{0,-} \circ \mathbf{f}^{\mathfrak{s}}_{0,+}$ (see Lemma 6.2); and
- $\mathbf{f}^{\mathfrak{r}}_{0,-} \circ \mathbf{f}^{\mathfrak{s}}_{0,+}\left((\mathbf{H}^\#_n)^0 \cup \mathbf{D}\right) \ni 0.$

Since $\mathbf{D}$ is compact, we have $\mathbf{D} \subset \mathbf{S}^\#_n$ for all sufficiently big $n \ll 0$. For such n we can project $\mathbf{D}$ to the dynamical plane of f_n; we denote this projection by $D \ni c_1$. By construction, $D \cup H_n^0$ is $f_n^{\mathfrak{q}_n}$-invariant: $f_n^{\mathfrak{q}_n} : H_n^0 \to H_n^0$ has an analytic extension to $f_n^{\mathfrak{q}_n} : D \cup H_n^0 \to D \cup H_n^0$. For $n \ll 0$, the disk D is a small neighborhood of c_1.

For $n \ll 0$, we enumerate petals of H_n counterclockwise so that $H_n^0 \subset \mathrm{H}^0_n$. Choose a big K (we will specify K in §6.5.3). For $k \in \{0,1,\dots,K\}$ we define D_k to be the image of $D_0 = D$ under f_n^k, and for $k \in \{-K,-K+1,\dots,-1\}$ we define D_k to be the lift of D_0 along the orbit of $f_n^{-k} : H_n^{\mathfrak{p}_n k} \to H_n^0$. Then

$$\mathrm{H}^{\mathfrak{p}_n k}_n := H_n^{\mathfrak{p}_n k} \cup D_k \tag{6.8}$$

is a valuable petal extending $H_n^{\mathfrak{p}_n k}$ for all $k \in \{-K,\dots,K\}$. For $n \ll 0$, all $\mathrm{H}^{\mathfrak{p}_n k}_n$ are in a small neighborhood of $\overline{Z}_\star$.

6.5.2. *Walls respecting H_n.* Set $N := M + 3$, where M is defined in Claim 1. Let us consider the dynamical plane of f_0. In a small neighborhood of α we can choose a univalent $(N+1)\mathfrak{q}_0$-wall A_0 *respecting* H_0 in the following way:

(a) α is in the bounded component O_0 of $\mathbb{C} \setminus A_0$ while the critical point and the critical value of f_0 are in the unbounded component of $\mathbb{C} \setminus A_0$;

(b) each petal H_0^i intersects A_0 at a connected set; and by enlarging H_0, we can also guarantee

(c) H_0 contains all $z \in A_0 \cup O_0$ with forward orbits in $A_0 \cup O_0$.

We can also assume that the intersection of A_0 with each triangle of Δ^{new}_0 is a closed topological rectangle. Lifting these rectangles to the dynamical plane of f_n and spreading around them, we obtain a *full lift* A_n of A_0. Then A_n is a univalent $N\mathfrak{q}_n$-wall (see Lemma B.13) enclosing an open topological disk $O_n \ni \alpha$ such that A_n respects H_n as above (see (a)–(c)). Naturally, A_n consists of closed topological rectangles: each rectangle is in a certain triangle of Δ^{new}_n.

Claim 2. *For $n \ll 0$, the wall A_n approximates $\partial Z_\star$ (compare to Lemma 4.2, part (5)): $\partial Z_\star$ is a concatenation of arcs $J_0 J_1 \cdots J_{m-1}$ such that J_i is close to the ith rectangle of A_n counting counterclockwise.*

Proof. By Theorem 4.6, it is sufficient to prove such a statement in the dynamical plane of $f_\star$: if A_0 is an annulus bounded by two equipotentials of Z, then a full lift A_n approximates $\partial Z_\star$ for a big n. Since the antirenormalization change of variables for f is conjugate to $z \to z^t$ with $t < 1$, the claim follows.

Consider the dynamical plane of f. Recall that $f_\star \mid \overline{Z}_\star$ is a homeomorphism. For $k \in \mathbb{Z}$, we define

$$c_k := \left(f_\star \mid \overline{Z}_\star\right)^k (c_0).$$

Consider now the dynamical plane of f_n. For $k \in \{-K,-K+1,\dots,K\}$, we define $c_k(f_n) \in f_n^k\{c_0\}$ to be the closest point to $c_k(f_\star)$. The point $c_k(f_n)$ is well defined as long as f_n is in a small neighborhood of f.

Claim 3. *For $k \in \{-K,-K+1,\dots,K\}$, we have*

- $c_{k+1}(f_n) \in H_{\mathfrak{p}_n n k}$*; and*
- $H^{\mathfrak{p}_n k}_n \setminus O_n$ *is in a small neighborhood of* c_{k+1}

Proof. The first statement follows from $c_{k-1}(f_n) \in D_k \subset H^{\mathfrak{q}_n k}_n$; see (6.8). The second statement follows from the improvement of the domain.

Claim 4. *Let P be a connected component of $O_n \setminus H_n$. Then $f_n^{\mathfrak{q}_n i} \mid P$ is univalent for all $i \in \{1,\dots,N\}$. Moreover,*

$$f_n^{\mathfrak{q}_n i}(P) \subset f_n^{\mathfrak{q}_n j}(P) \text{ for all } i < j \text{ in } \{0,1,\dots,N\}.$$

Proof. The first claim follows from the assertion that A_n is an $N\mathfrak{q}_n$-wall. The second claim follows from (c).

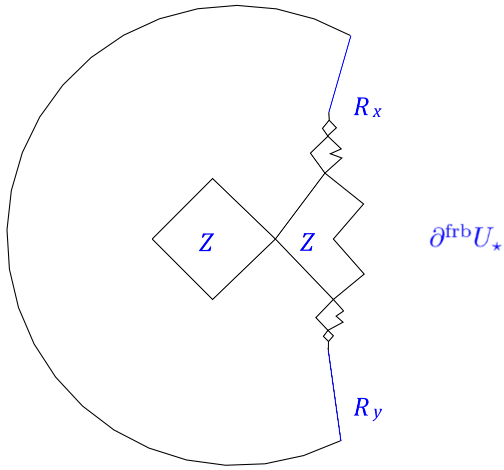

Figure 24. Separation of $\partial^{\mathrm{frb}} U_\star$. Co-Siegel disk $Z'_\star$ together with its iterated lifts form two periodic bubble chains landing at periodic points x and y. The bubble chains together with external rays R_x and R_y separate $\partial^{\mathrm{frb}} U_\star$ from α.

6.5.3. *Julia rays in* $\partial\mathfrak{J}_\star$. Consider the dynamical plane of $f_\star : U_\star \to V$. By Theorem 3.12, we can choose (see Figure 24) two periodic points $x, y \in \mathfrak{J}_\star$ together with two periodic external rays R_x, R_y landing at x, y and two periodic bubble chains B_x, B_y landing at x, y so that x and y are close to $\partial^{\mathrm{frb}} U_\star$ and $R_x \cup B_x \cup B_y \cup R_y$ separates $\partial^{\mathrm{frb}} U_\star$ from c_1 as well as from all the remaining points in the forward orbits of x, y. Let p be a common period of x, y. Set $K := 4p$.

A *Julia ray* J of $\mathfrak{J}_\star$ is a simple arc in J starting at a point in $\partial Z_\star$.

Claim 5. *There are Julia rays $J_x \subset B_x$ and $J_y \subset B_y$ such that J_x and J_y start at the critical point c_0 and land at x and y, respectively. Moreover, J_x and J_y are periodic with period p: the rays J_x and J_y decompose as concatenations $J_x^1 J_x^2 J_x^3 \cdots$ and $J_y^1 J_y^2 J_y^3 \cdots$ such that $f_\star^p$ maps J_x^k and J_y^k to J_x^{k-1} and J_y^{k-1}, respectively.*

Proof. Write $B_x = (Z_1, Z_2, \ldots)$; since x is close to $\partial^{\mathrm{frb}} U_\star$ we see that $Z_1 = \overline{Z}'_\star$. Since x is periodic with period p, there is an $a > 0$ such that f^p maps Z_{a+i} to Z_i for all i.

Let $J_x^1 \subset \mathfrak{J}_\star$ be a simple arc in $\partial Z_1 \cup \partial Z_2 \cup \cdots \cup \partial Z_a$ connecting the critical point c_0 to the point where ∂Z_{a+1} is attached to ∂Z_a. We inductively define J_x^j to be the iterated lift of J_x^{j-1} such that J_x^j starts where J_x^{j-1} terminates. This constructs $J_x = J_x^1 J_x^2 J_x^3 \cdots$; $J_y = J_y^1 J_y^2 J_y^3 \cdots$ is similarly constructed.

6.5.4. *Julia rays for f_n.* Recall that in Claim 5 we specified Julia rays $J_x(f_\star)$ and $J_y(f_\star)$. Since f_0 is sufficiently close to f, the periodic points x, y exist in the dynamical plane of f_0 and are close to those of f. For $n \ll 0$ let us now construct *Julia rays* $J_x(f_n) = J_x^1 J_x^2 J_x^3 \cdots$ and $J_y(f_n) = J_y^1 J_y^2 J_y^3 \cdots$ such that

(1) f_n^p maps J_x^k to J_x^{k-1} and J_y^k to J_y^{k-1} (compare with Claim 5);

(2) $J_x^k(f_n)$ and $J_y^k(f_n)$ are in small neighborhoods of $J_x^k(f_*)$ and $J_y^k(f_*)$, respectively;

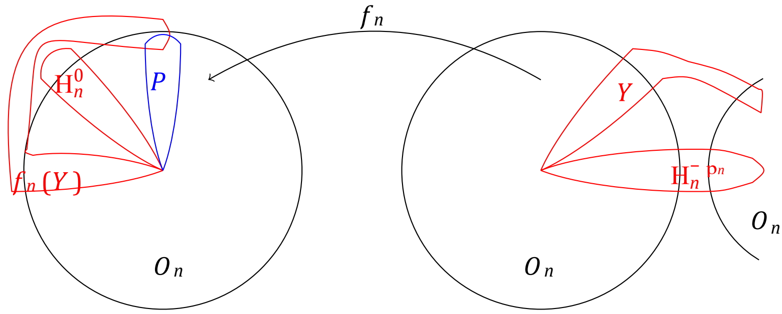

Figure 25. Illustration to the proof of Claim 6. If Y intersects O'_n, then applying f_n we obtain that $P \cup f_n(Y_n)$ encloses H^0_n. Since P is surrounded by the wall A_n, the set $f_n^{q_n M}(P \cup f_n(Y))$ also encloses H^0_n. Then $f_n^{q_n} \mid f_n^{q_n M}(P \cup f_n(Y))$ has degree one while $f_n^{q_n} \mid \mathrm{H}^0_n$ has degree 2; this is a contradiction.

(3) for $z \in J_x^1 \cup J_x^2 \cup J_y^1 \cup J_y^2$ there is a $q \le 2p$ such that either $f_n^q(z) \in O_n$ or $f_n^q(z) \in \bigcup_{|k|\le 2p} \mathbb{H}_n^{k\mathrm{p}_n}$. In the former case we can assume that $f_n^\ell(z) \notin A_n \cup O_n$ for $\ell \in \{0,1,\dots,q-1\}$.

Construction of J_x and J_y. We will use notation from the proof of Claim 5. By stability of periodic points, x,y exist for f_n and are close to $x(f_\star), y(f_\star)$. The curve J_x^1 is a simple arc in $\partial Z_1 \cup \partial Z_2 \cup \dots \cup \partial Z_a$. We split J_1 as the concatenation $\ell_1 \cup \ell_2 \cup \dots \cup \ell_a$ with $\ell_j = J_x^1 \cap \partial Z_j$. Let $f_\star^{d(j)}$ be the smallest iterate mapping Z_j to $\overline{Z}_\star$. Since $J_x \subset \mathfrak{J}_\star$, the curve

$$\tilde{\ell}_j := f_\star^{d(j)}(\ell_j)$$

is a simple arc in $\partial Z_\star$ connecting c_1 and a certain $c_{t(j)}$. Using Claims 2 and 3, we approximate each $\tilde{\ell}_j(f_\star)$ by a curve $\tilde{\ell}_j(f_n)$ within $O_n \cup \mathrm{H}_n^{\mathrm{p}_n(t(j)-1)} \cup \mathrm{H}^0_n$. Lifting $\tilde{\ell}_j(f_n)$ along the branch of $f_n^{d(j)}$ that is close to $f_\star^{d(j)} \mid \ell_j(f_\star)$, we construct $\ell_j(f_n)$ that is close to $\ell_j(f_\star)$. Assembling all ℓ_j, we construct $J_x^1(f_n)$. By continuity, pulling back $J_x^1(f_n)$ we construct finitely many $J_x^k(f_n)$ approximating $J_x^k(f_\star)$ such that the remaining curves $J_x^k(f_\star)$ are within the linearization domain of x. Taking pullbacks within the linearization domain of x, we construct a ray $J_x(f_n)$ landing at x. Similarly, J_y is constructed. Property (3) follows from $|t(j)| \le p$.

6.5.5. *Blocking $\partial^{\mathrm{frb}} U_n$.* Recall from Claim (1) that $f_n^{q_n M}(\mathrm{H}^0_n) \subset H_n^0$. For $t \in \{M, M-1, M-2, \dots, 0\}$ we set $H_n^{(t)}$ to be the forward f_n-orbit of $f_n^{q_n t}(\mathrm{H}_n^0)$.

Claim 6. *The flower $H_n^{(t)}$ does not intersect $\partial^{\mathrm{frb}} U_n$ for all $t \in \{M,\dots,0\}$.*

As a consequence, H_n extends to a required $\mathbb{H}_n := H_n^{(0)}$ for $n \ll 0$.

Proof. Recall that valuable petals $\mathbb{H}_n^{k\mathfrak{p}_n} \subset U_n$ with $|k| \le K$ are already constructed.

Set

$$H_n'^{(t)} := H_n^{(t)} \setminus \bigcup_{|k|\le K} \mathbb{H}_n^{k\mathfrak{p}_n}. \tag{6.9}$$

Let us show that $H_n'^{(t)}$ does not hit $R_x \cup J_x \cup J_y \cup R_y$; this would imply that $H_n^{(t)}$ does not intersect $\partial^{\mathrm{frb}} U_n$. Suppose the converse; since $H_n'^{(t)}$ does not intersect $R_x \cup R_y$, we can consider the first moment t (i.e., t is the closest to M) when $H_n'^{(t)}$ hits $J_x \cup J_y$. Denote by X a petal of $H_n'^{(t)}$ intersecting $J_x \cup J_y$. Choose $z \in X \cap (J_x \cup J_y)$; we can assume that $z \in J_x^1 \cup J_x^2 \cup J_y^1 \cup J_y^2$, otherwise t is not the first moment when $H_n'^{(t)}$ hits $J_x \cup J_y$. By property (3) from §6.5.4, there is a $q \le 2p$ such that either $f_n^q(z) \in O_n$ or $f_n^q(z) \in \bigcup_{|k|\le 2p} \mathbb{H}_n^{k\mathfrak{p}_n}$. The latter would imply that X is a petal in $\bigcup_{|k|\le 4p} \mathbb{H}_n^{k\mathfrak{p}_n}$; this contradicts (6.9). Therefore, $f_n^q(z) \in O_n$.

Write

$$O_n' := f_n^{-1}(O_n) \setminus (A_n \cup O_n) \ni f_n^{q-1}(z)$$

and set $Y := f^{q-1}(X)$. We have $O_n' \cap Y \ni f_n^{q-1}(z)$; see Figure 25. Since $\mathbb{H}_n^{-\mathfrak{p}_n}$ contains a critical point, we see that $f_n^q(z)$ is within a connected component P of $O_n \setminus (H_n \cup \{\alpha\})$ and, moreover, $P \cup f_n(Y)$ surrounds $\mathbb{H}_n^0$.

Let us apply $f_n^{\mathfrak{q}\,M}$ to $f_n(Y) \cup P$. By Claim 4 (recall that $N > M + 1$; see §6.5.2), we have $f_n^{\mathfrak{q}_n M}(P) \subset (A_n \cup O_n) \setminus H_n$ and $f_n^{\mathfrak{q}\,M}(P)$ does not contain a critical point of $f_n^{\mathfrak{q}_n}$. On the other hand, $f_n^{\mathfrak{q}\,M+1}(Y)$ does not contain a critical point of $f_n^{\mathfrak{q}_n}$ as a subset of H_n. Note that $f_n^{\mathfrak{q}_n M}(P \cup f_n(Y))$ still surrounds $\mathbb{H}_n^0$. This is a contradiction: $f_n^{\mathfrak{q}} \mid f_n^{\mathfrak{q}\,M}(P \cup f_n(Y))$ has degree one while $f_n^{\mathfrak{q}} \mid \mathbb{H}_n^0$ has degree 2.

6.5.6. *Siegel triangulation.* It remains to construct a Siegel triangulation $\Delta(f_n)$ respecting $\mathbb{H}_n$ for $n \ll 0$. In the dynamical plane of f_n, let us choose a curve $\ell_1 \subset V$ connecting ∂V to α such that ℓ_1 enters U_n in O_n', then reaches $\partial \mathbb{H}_n^{-\mathfrak{p}_n}$, then travels to α within $\partial \mathbb{H}_n^{-\mathfrak{p}_n}$. We can assume that $\ell_1 \setminus O_n$ is disjoint from $\gamma_1 \setminus O_n$. Observe that ℓ_1 is liftable to the dynamical planes f_m for all $m \le n$. Indeed, $\ell_1 \cap \partial \mathbb{H}_n^{-\mathfrak{p}_n}$ is liftable because so is $\partial \mathbb{H}_n^{-\mathfrak{p}_n}$, while $\ell_1 \setminus \partial \mathbb{H}_n^{-\mathfrak{p}_n}$ is liftable because it is disjoint from γ_1.

Let us slightly perturb ℓ_1 so that the new ℓ_1 is disjoint from $\mathbb{H}_n$. Define ℓ_0 to be the preimage of ℓ_1 connecting ∂U_n to α. Then $\ell_1 \cup \ell_0$ splits U_n into two closed sectors; they form the triangulation denoted by $\Delta(f_n)$. We can assume that ℓ_1 was chosen so that $\ell_1 \setminus O_n$ and $\ell_0 \setminus O_n$ are connected. We define the wall $\Pi(f_n)$ to be the closures of two connected components of $U_n \setminus (O_n \cup \ell_0 \cup \ell_1)$.

For $m \le n$ we define $\Delta(f_m)$ and $\Pi(f_m)$ to be the full lifts of $\Delta(f_n)$ and $\Pi(f_n)$. Then $\Delta(f_m)$ is a required triangulation for $m \ll n$.

7. Hyperbolicity Theorem

Recall that by λ we denote the multiplier of the α-fixed point of f. For λ close to λ set

$$F(\lambda) := \{f \in W^u \mid \text{the multiplier of } \alpha \text{ is } \lambda\}$$

to be the analytic submanifold of W^u obtained by fixing the multiplier at α. Then $F(\lambda)$ forms a foliation of a neighborhood of f.

7.1. **Holomorphic motion of $P(\mathbf{F}_0)$.** Let $U \subset W^u$ be a small neighborhood of f such that every $f \in U$ has a maximal prepacmen; see Theorem 5.5.

Lemma 7.1 (Holomorphic motion of the critical orbits)**.** *For every* p/q, *the set*

$$\bigcup_{n \le 0} \operatorname{orb}_0(\mathbf{F}_n^{\#})$$

moves holomorphically with $f_0 \in F(\mathbf{e}(p/q)) \cap U$.

Recall from Lemma 6.1 that $P(\mathbf{F}_0) \subset \bigcup_{n \le 0} \operatorname{orb}_0(\mathbf{F}^{\#}_n)$; thus $P(\mathbf{F}_0)$ also moves holomorphically with $f_0 \in F(\mathbf{e}(p/q)) \cap U$.

Proof. By Corollary 6.7, points in $\operatorname{orb}_0(\mathbf{F}^{\#}_n)$ do not collide with each other when $f_0 \in F(\mathbf{e}(p/q)) \cap U$ is deformed. This gives a holomorphic motion of $\operatorname{orb}_0(\mathbf{F}_0) \subset \operatorname{orb}_0(\mathbf{F}^{\#}_1) \subset \operatorname{orb}_0(\mathbf{F}^{\#}_2) \subset \cdots$ and we can take the union.

Let $\mathcal{U}' \subset \mathcal{U}$ be a neighborhood of f such that every non-empty $\mathcal{F}(\lambda) \cap \mathcal{U}'$ has radius at least three times less than those of $F(\lambda) \cap U$.

Corollary 7.2 (Extended holomorphic motions)**.** *For* $f_0 \in \mathcal{F}(\mathbf{e}(\mathfrak{p}/\mathfrak{q})) \cap \mathcal{U}'$ *there is*

dynamics of $(\mathbf{F}^{\#}_n)_n$) *on*

a holomorphic motion $\tau(f_0)$ *of* C *such that* $\tau(f_0)$ *is equivariant (with respect to the*

$$\bigcup_{n \le 0} \operatorname{orb}_0(\mathbf{F}_n^{\#}).$$

Proof. The proof follows by applying the λ-lemma to the holomorphic motion from Lemma 7.1.

Corollary 7.3 (Passing to the limit of holomorphic motions). *For* $f_0 \in \mathcal{F}(\lambda_\star) \cap \mathcal{U}'$ *there is a holomorphic motion* $\tau(f_0)$ *of* $\mathbb{C}$ *such that* $\tau(f_0)$ *is equivariant on*

$$\bigcup_{n\le 0} \mathrm{orb}_0(\mathbf{F}_n^{\#}).$$

Proof. Choose a sequence $\mathfrak{p}_n/\mathfrak{q}_n$ such that $\mathbf{e}(\mathfrak{p}_n/\mathfrak{q}_n) \to \mathbf{e}(\theta_\star)$. By passing to the limit in Corollary 7.2 we obtain the desired property.

Corollary 7.4. *The dimension of* $\mathcal{F}(\lambda_\star)$ *is* 0.

Proof. Suppose the dimension of $\mathcal{F}(\lambda_\star)$ is greater than 0. Consider the space $\mathcal{F}(\lambda_\star) \cap \mathcal{U}'$. By Corollary 7.3 the set $\overline{P(\mathbf{F}_0)} \subset \overline{\bigcup_{n\le 0} \mathrm{orb}_0(\mathbf{F}_n^{\#})}$ moves holomorphically with $f_0 \in \mathcal{F}(\lambda_\star) \cap \mathcal{U}'$. Projecting this holomorphic motion to the dynamical plane of f_0, we obtain a holomorphic motion of the post-critical set of $f_0 \in \mathcal{F}(\lambda_\star) \cap \mathcal{U}'$. Therefore, there is a small neighborhood of $f_\star$ in $\mathcal{F}(\lambda_\star) \cap \mathcal{U}'$ consisting of Siegel maps. But all such maps must be in the stable manifold of f by Theorem 7.5.

7.2. **The exponential convergence.** The following theorem follows from [McM2, Theorem 8.1].

Theorem 7.5. *Let* $f \in \mathcal{B}$ *be a Siegal pacman with the same rotation number as* f *which is sufficiently close to* f*. Then* $\mathcal{R}^n f$ *converges exponentially fast to* f*.*

Remark 7.6. The proof of [McM2, Theorem 8.1] is based on a "deep point argument". Alternatively, the exponential convergence follows from a variation of the Schwarz lemma following the lines of [L1,AL1].

7.3. **The hyperbolicity theorem.**

Theorem 7.7 (Hyperbolicity of $\mathcal{R}$). *The renormalization operator* $\mathcal{R}\colon \mathcal{B} \to \mathcal{B}$ *is hyperbolic at* f *with one-dimensional unstable manifold* $\mathcal{W}^u$ *and codimension-one stable manifold* $\mathcal{W}^s$*.*

In a small neighborhood of f *the stable manifold* $\mathcal{W}^s$ *coincides with the set of pacmen in* $\mathcal{B}$ *that have the same multiplier at the* α*-fixed point as* f*. Every pacman in* $\mathcal{W}^s$ *is Siegel.*

In a small neighborhood of f *the unstable manifold* $\mathcal{W}^u$ *is parametrized by the multipliers of the* α*-fixed points of* $f \in \mathcal{W}^u$*.*

Proof. It was already shown in Corollary 7.4 that the dimension of $\mathcal{W}^u$ is one. Let us show that $\mathcal{W}^s$ has codimension one. Denote by $\mathcal{B}^*$ the submanifold of $\mathcal{B}$ consisting of all the pacmen with the same multiplier at the α-fixed point as f. Then $\mathcal{R}$ naturally restricts to R: $B^* \to B^*$. Consider the derivative Diff(R | B^*); by Corollary 7.4 the spectrum of Diff(R | B^*) is within the closed unit disk. Suppose that the spectrum of Diff(B^*) intersects the unit circle. By [L1, Small orbits theorem] R | B^* has a small slow orbit: there is an $f \in B^*$ such that f is infinitely many times renormalizable but

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\mathcal{R}^n f\| = 0 .$$

Moreover, it can be assumed thatof f. By Corollary 4.7, f is a Siegel pacman and by Theorem 7.5,$\{R^n f\}_{n\geq 0}$ is in a sufficiently small neighborhood$R_n f$ converges

exponentially fast to f. This is a contradiction. Therefore, the spectrum of R is compactly contained in the unit disk, and all of the pacmen in B^* are infinitely renormalizable and thus are Siegel (Corollary 4.7). The submanifold B^* coincides with W^s in a small neighborhood of f.

7.4. **Control of Siegel disks.** The following lemma follows from [McM2, Theorem 8.1] combined with Theorem 3.6 and Lemma 3.4.

Lemma 7.8. *Every Siegel map f has a pacman renormalization* $R_2 f$ *such that*

$R_2 f$ *is in* B *and is sufficiently close to f.*

We say that a holomorphic map $f : U \to V$ is *locally Siegel* if it has a distinguished Siegel fixed point. The following corollary follows from Theorem 7.7 combined with Lemma 7.8.

Corollary 7.9. *Let* $f : U \to W$ *be a Siegel map with rotation number* $\theta \in \Theta_{\text{per}}$, *and let* $N(f)$ *be a small Banach neighborhood of f. Then every locally Siegel map*

$g \in N(f)$ *with rotation number* θ *is a Siegel map. Moreover, the Siegel disk* $\overline{Z}_g$

depends continuously on g.

8. Scaling Theorem

In this section we prove a refined version of Theorem 1.3. Consider$\theta_\star \in \Theta_{\text{per}}$ and let f be a Siegel map with rotation number θ. Let $\mathcal{U} \ni f$ be a small Banach neighborhood of f, and let $W \subset U$ be a one-dimensional slice containing f such that W is transverse to the hybrid class of f; i.e., in a small neighborhood of $f \in W$ all maps have different multipliers at their α-fixed points.

We say a map $g \in U$ is *satellite* if it has a satellite valuable flower.

Definition 8.1 (Satellite valuable flowers)**.** A *satellite valuable flower* of g is an open forward invariant set H such that (see Figure 26)

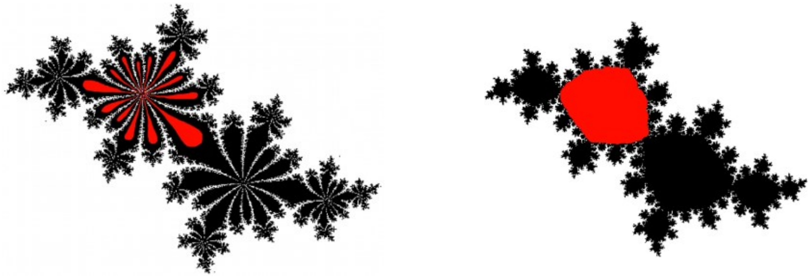

Figure 26. A satellite valuable flower (red) of the 5/13 Rabbit approximates the golden Siegel disk (also red).

(A) $H \cup \{\alpha(g)\}$ is connected;

(B) H has q connected components $H^0, H^1, \dots, H^{q-1}$, called *petals*, enumerated counterclockwise at α; every H^i is an open topological disk with a single access to α;

(C) $g(H^i) \subset H^{i+p}$, where p is coprime to q;

(D) there is an attracting periodic cycle $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{q-1})$ with $\gamma_i \in H^i$ attracting all points in H;

(E) H^{-p} contains the critical point of g.

The number p/q is called the *combinatorial rotation number* of H. The *multiplier* of H is the multiplier of γ.

For convenience, let us say that a parabolic valuable flower (see Definition 6.8) with rotation number p/q is a satellite valuable flower with rotation number p/q and multiplier 1.

By Lemma 3.18, R acts on the rotation numbers of indifferent pacmen as R_{prm}^{k} for a certain $k \geq 1$; see also Remark 3.19.

Theorem 8.2. *Suppose a sequence* $(\mathfrak{p}_n/\mathfrak{q}_n)_{n=0}^{-\infty}$ *converges to* θ *so that* $R_{\mathrm{prm}}^{\mathfrak{k}}(\mathfrak{p}_n/\mathfrak{q}_n) = \mathfrak{p}_{n+1}/\mathfrak{q}_{n+1}$. *Fix* $\lambda_1 \in \mathbb{D}^1$ *and a small neighborhood of* $\overline{Z}_f$. *Then there is a continuous path* $\lambda_t \in \mathbb{D}^1 \cup \{1\}$ *with* $t \in [0,1]$ *emerging from* $1 = \lambda_0$ *such that for every sufficiently big* $n \ll 0$ *there is a unique path* $g_{n,t} \in \mathcal{W}$, *where* $t \in [0,1]$, *with the following properties:*

- $g_{n,t}$ *has a valuable flower* $H_{n,t}$ *with rotation number* $\mathfrak{p}_n/\mathfrak{q}_n$; *moreover for* $t \neq 0$ *the multiplier of the corresponding attracting cycle* $\gamma_{n,t}$ *is equal to* λ_t;
- *all* $H_{n,t}$ *are contained in the given small neighborhood of* $\overline{Z}_f$ *and depend continuously on t; and*

- dist$(f, g_{n,t}) \sim \left((R^{\mathfrak{k}}_{\mathrm{prm}})'(\theta_\star)\right)^n$ *for every t.*

Note that the path $g_{n,t}$ starts at a unique parabolic map in W with rotation number p_n/q_n.

8.1. **Proof of Theorem 8.2.** The proof is split into short subsections. Consider a pacman hyperbolic renormalization operator $\mathcal{R}\colon \mathcal{B} \dashrightarrow \mathcal{B}$ around a fixed point $f_\star = \mathcal{R}(f_\star$ of $\mathcal{R}$ at $f_\star$.) with rotation number θ. As before, W^u denotes the unstable manifold

8.1.1. *Perturbation of parabolic pacmen.* By shifting the sequence $(p_n/q_n)_n$ we can assume that p_0/q_0 is close to θ. Then there is a unique parabolic pacman $f_0 \in W^u$ with rotation number p_0/q_0. Then $f_n := R^n f_0, n \le 0$, has rotation number p_n/q_n.

By Theorem 6.9 and possibly by further shifting $(p_n/q_n)_n$, we can assume that

- each f_n has a valuable flower $H(f_n)$ at the α-fixed point;
- each f_n has a triangulation $\Delta(f_n)$ respecting $H(f_n)$: every petal of $H(f_n)$ is within a triangle of $\Delta(f_n)$;
- $\Delta(f_n)$ has a wall $\Pi(f_n)$ approximating $\partial Z_\star$;
- $\Delta(f_n)$ and $H(f_n)$ are the full lifts of $\Delta(f_{n+1})$ and $H(f_{n+1})$.

Let $g_0 \in W^u$ be a slight perturbation of f_0 that splits α into a repelling fixed point α and an attracting cycle $\gamma(g_0)$ such that α is on the boundary of the immediate attracting basin of $\gamma(g_0)$. Then $\Delta(f_0)$, $\Pi(g_0)$, $H(f_0)$ are perturbed to $\Delta(g_0)$, $\Pi(g_0)$, $H(g_0)$ such that all points in $H(g_0)$ are attracted by $\gamma(g_0)$. We can assume that the perturbation is sufficiently small such that $\Pi(g_0)$ still approximates $\partial Z_\star$. By Lemma 4.4, there are full lifts $\Delta(g_n)$, $H(g_n)$ of $\Delta(g_0)$, $H(g_0)$.

As before, we denote by $\mathbf{F}_n$ and $\mathbf{G}_n$ the maximal prepacmen of f_n and g_n and we denote by $\mathbf{G}^\#_n$ the rescaled version of $\mathbf{G}_n$ such that $\mathbf{G}_0 = \mathbf{G}^\#_0$ is an iteration of $\mathbf{G}^\#_n$. Recall from §6.2 that $H(f_0)$ admits a global extension $\mathbf{H}(\mathbf{F}_0)$ in the dynamical plane of $\mathbf{F}_0$. Similarly, we now define the maximal extension $\mathbf{H}(\mathbf{G}_n)$ of $H(g_n)$.

Each $H(g_n)$ lifts to the dynamical plane of $\mathbf{G}^\#_n$; denote by $\mathbf{H}(g_0)$ the lift of $H(g_0)$. Similar to (6.2), we set

$$\mathbf{H}(\mathbf{G}_0) := \bigcup_{a,b\in\mathbb{Z}} (\mathbf{g}_{0,-})^a \circ (\mathbf{g}_{0,+})^b \left(\mathbf{H}(g_0)\right)$$

to be the full orbit of $\mathbf{H}(g_0)$. The same argument as in the proof of Lemma 6.2 shows that $\mathbf{H}(\mathbf{G}_0)$ is fully invariant and is within $\mathrm{Dom}\,\mathbf{G}_{0,-} \cap \mathrm{Dom}\,\mathbf{G}_{0,+}$.

Denote by $\mathbf{H}^{\mathrm{per}}(\mathbf{G}_0)$ the union of periodic components of $\mathbf{H}(\mathbf{G}_0)$. The same argument as in the proof of Proposition 6.5 shows the following.

Proposition 8.3 (Parameterization of $\mathbf{H}^{\mathrm{per}}(\mathbf{G}_0)$). *The connected components of* $\mathbf{H}^{\mathrm{per}}(\mathbf{G}_0)$ *are uniquely enumerated as* $(\mathbf{H}^i)_{i\in\mathbb{Z}}$ *such that* $\mathbf{H}^0 \ni 0$ *and such that the* *actions of* $(\mathbf{g}_{n,\#0\pm}\ i)_{i\in\mathbb{Z}}$ *are given (up to interchanging* $\mathbf{g}_{n,-}$ *and* $\mathbf{g}_{n,+}$*) by on* $(\mathbf{H}$

(8.1) $$\mathbf{g}_{n,\#-}(\mathbf{H}_i) = (\mathbf{H}_{i-p_n}) \textit{ and } \mathbf{g}_{n,\#+}(\mathbf{H}_i) = \mathbf{H}_{i+q_n-p_n}.$$

8.1.2. *QC-deformation of* g_n. Suppose first that $\lambda_1 \neq 0$. Denote by λ_0 the multiplier of $\gamma(g_0)$. Let $\mathbf{g}^{\mathfrak{r}}_{0,-}\circ \mathbf{g}^{\mathfrak{s}}_{0,+}: \mathbf{H}^0(\mathbf{G}_0) \to \mathbf{H}^0(\mathbf{G}_0)$ be the first return map (compare with Lemma 6.2). There is a semiconjugacy $\mathbf{h}$: $\mathbf{H}^0(\mathbf{G}_0) \to \mathbb{C}$ from $\mathbf{g}^{\mathfrak{r}}_{0,-} \circ \mathbf{g}^{\mathfrak{s}}_{0,+}$ to the linear map $z \to \lambda_0 z$. Choose a continuous path of qc maps τ_t: $\mathbb{C} \to \mathbb{C}$ with $t \in [0,1]$ such that $\tau_0 = \mathrm{id}$ and τ_t conjugates $z \to \lambda_0 z$ to $z \to \lambda_t z$.

Lifting τ_t under $\mathbf{h}$ and spreading the associated Beltrami form dynamically, we obtain a qc map τ_t: $\mathbb{C} \to \mathbb{C}$ conjugating $\mathbf{G}_0$ to a maximal prepacman $\mathbf{G}_{0,t}$; similarly τ_t conjugates $\mathbf{G}^\#_{n,t}$ to a maximal prepacman $\mathbf{G}^\#_{n,t}$ for $n \le 0$.

Define now $\tau_{n,t}$ to be the projection of τ_t to the dynamical plane of g_n via $\operatorname{int}\mathbf{S}^\#_k \simeq V \setminus \gamma_1$ (see (5.6)); we normalize $\tau_{n,t}$ to preserve $\alpha(g_n)$ and $c_1(g_n)$. By compactness of qc-maps, there is a small $T > 0$ such that all $g_{n,t}$ are in $\mathcal{B}$ for $t \le T$. For $m \le 0$ consider the sequence $\mathcal{R}^{-n+m}(g_{n,t})$. All pacmen in this sequence are qcconjugate with uniform dilatation. Moreover, the conjugacies preserve the critical value and the α-fixed point because of the normalization for renormalization change of variables; see §2.5. By compactness of qc-maps, $\mathcal{R}^{-n+m}(g_{n,t})$ has an accumulated point $q_{m,t} \in \mathcal{B}$, and, moreover, we can assume that $\mathcal{R}q_{m,t} = q_{m+1,t}$; i.e., $q_{m,t} \in \mathcal{W}^u$ and $q_{m,t}$ tends to $f_\star$ as m tends to $-\infty$. We define $\Delta(q_{n,t})$,$\Pi(q_{n,t})$,$H(q_{n,t})$ to be the images of $\Delta(g_{n,t})$,$\Pi(g_{n,t})$,$H(g_{n,t})$ via the qc-conjugacy from $g_{n,t}$ to $q_{n,t}$. By improvement of the domain, $\Delta(q_{n,t})$ is in a small neighborhood of Z and $\Pi(q_{n,t})$ approximates $\partial Z_\star$ for $n \ll 0$. By shifting the sequence $(p_n/q_n)_n$ we can assume that this already occurs for $n = 0$. We can now repeat the above argument and construct $q_{n,t}$ for $t \in [T,2T]$. After finitely many repetitions, we construct $q_{n,t}$ for all t in $[0,1]$.

8.1.3. *QC-surgery towards the center.* Suppose now $\lambda_1 = 0$. In this case we apply a qc-surgery. As in §8.1.2 we denote by λ_0 the multiplier of $\gamma(g_0)$.

Consider the first return map

$$\mathbf{w}^0 := \mathbf{g}^{\mathfrak{r}}_{0,-} \circ \mathbf{g}^{\mathfrak{s}}_{0,+}: \mathbf{H}^0(\mathbf{G}_0) \to \mathbf{H}^0(\mathbf{G}_0).$$

It has a unique attracting fixed point γ^0 and a unique critical value at 0. Thus $\mathbf{w}_0$ also has a unique critical point. We can choose a small disk $\mathbf{D}$ around γ^0 such that

- $0 \in \mathbf{w}_0(\mathbf{D})\ \mathbf{D}$;
- $\mathbf{w}_0 : \mathbf{H}^0(\mathbf{G}_0) \setminus \mathbf{D} \to \mathbf{H}^0(\mathbf{G}_0) \setminus \mathbf{w}_0(\mathbf{D})$ is a 2-to-1 covering map.

By Theorem 6.9, we can project $\mathbf{D}$ to a disk within $H(g_0)$. We claim that there is a continuous path of qc maps $\tau'_t: \mathbf{H}^0(\mathbf{G}_0) \to \mathbf{H}^0(\mathbf{G}_0)$ and a continuous path $\mathbf{w}_t$: $\mathbf{H}^0(\mathbf{G}_0) \to \mathbf{H}^0(\mathbf{G}_0)$ such that • τ'_t is equivariant (with respect to the actions of $\mathbf{w}_0$ and $\mathbf{w}_t$) on $\mathbf{H}^0(\mathbf{G}_0)\setminus\mathbf{D}$;

- $\mathbf{w}_t$ has a unique critical value at 0 and a unique attracting fixed point at $\gamma_{0,t}$;

- $\gamma_{0,1} = 0$; i.e., 0 is a supperattracting fixed point of $\mathbf{w}_1$.

Indeed, it is sufficient to construct $\mathbf{w}_t \mid \mathbf{D}$ and $\tau'_t \mid \mathbf{D}$ equivariant on $\partial\mathbf{D}$; pulling back the Beltrami differential of $\tau'_t \mid \mathbf{D}$ via the covering map $\mathbf{w}_0 \mid \mathbf{H}^0(\mathbf{G}_0)\setminus\mathbf{D}$ gives the Beltrami differential for $\tau'_t \mid \mathbf{H}^0(\mathbf{G}_0)$.

Applying $\mathbf{G}_0$, we spread the Beltrami form of τ'_t dynamically to obtain a global qc map $\tau_t\colon \mathbb{C} \to \mathbb{C}$ which is unique up to affine rescaling. Spreading the surgery dynamically, we obtain a continuous path of maximal prepacmen $\mathbf{G}^\#_{n,t}$. Define now $\tau_{n,t}$ to be the projection of τ_t to the dynamical plane of g_n via $\operatorname{int}\mathbf{S}^\#_n \simeq V \setminus \gamma_1$; similarly, $g_{n,t}$ is the projection of $\mathbf{G}^\#_{n,t}$. The argument now continues in the same way as in §8.1.2.

8.1.4. *Lamination around $f_\star$.* In §§8.1.1, 8.1.2, and 8.1.3 we constructed continuous paths $q_{n,t} \in \mathcal{W}^u, n \ll 0$, with $\mathcal{R}(q_{n,t}) = q_{n+1,t}$ so that each $q_{n,t}$ has a valuable flower $\mathrm{H}(q_{n,t})$ with multiplier λ_t, where $\lambda_0 = 1$. Moreover, $\mathrm{H}(q_{n,t})$ is within a triangulation $\Delta(q_{n,t})$ respecting $\mathrm{H}(q_{n,t})$ such that the wall $\Pi(q_{n,t})$ approximates $\partial Z_\star$.

For a big $m \ll 0$, we define $\mathcal{F}_{m,t}$ to be the set of all pacmen close to $q_{m,t}$ such that the multiplier of $\gamma(q_{m,t})$ is λ_t. Locally $(\mathcal{F}_{m,t})_t$ is a codimension-one lamination of $\mathcal{B}$. Since $\mathcal{F}_{m,t}$ is in a small neighborhood of $q_{m,t}$, every pacman $g \in \mathcal{F}_{m,t}$ has a valuable flower $\mathrm{H}(g)$ and a triangulation $\Delta(g)$ respecting $\mathrm{H}(g)$ such that $\Delta(g)$ and $\mathrm{H}(g)$ depend continuously on g. The wall $\Pi(g)$ approximates $\partial Z_\star$. For $n \le m$, we define

$$\mathcal{F}_{n,t} := \{g \in \mathcal{B} \mid \mathcal{R}^{m-n}(g) \in \mathcal{F}_{m,t}\}.$$

Since $\mathcal{R}$ is hyperbolic,

$$\mathfrak{F} := \{\mathcal{F}_{n,t}\}_{n,t} \cup \{\mathcal{W}^s\} \tag{8.2}$$

forms a codimension-one lamination in a neighborhood of $f_\star$. A pacman $g \in \mathcal{F}_{n,t}$ has $\mathrm{H}(g)$ and $\Delta(g)$ having the same properties as above. In particular, all the pacmen in $\mathcal{F}_{n,t}$ are hybrid conjugate in neighborhoods of their valuable flowers.

8.1.5. *Scaling.* By Corollary 3.7, the Siegel map f can be renormalized to a pacman. By Lemma 7.8 we can assume that the renormalization of f is within a small neighborhood of $f_\star$. This allows us to define an analytic renormalization operator $\mathcal{R}_2 : \mathcal{U} \dashrightarrow \mathcal{B}$ from a small neighborhood of f to a small neighborhood of $f_\star$. Since maps in $\mathcal{W}$ have different multipliers, the image of $\mathcal{W}$ under $\mathcal{R}_2$ is transverse to the lamination $\mathfrak{F}$; see (8.2).

We define $f_{n,t}$ to be the unique intersection of $\mathcal{F}_{n,t}$ with the image of $\mathcal{W}$ under $\mathcal{R}_2$, and we define $g_{n,t} \in \mathcal{W}$ to be the preimage of $f_{n,t}$ via $\mathcal{R}_2$. Since $\Pi(f_{n,t})$ approximates $\partial Z_\star$, the triangulation $\Delta(f_{n,t})$ and the valuable flower $\mathrm{H}(f_{n,t})$ have full lifts $\Delta(g_{n,t})$ and $\mathrm{H}(g_{n,t})$; see

Lemma 4.4. Since the holonomy along F is asymptotically conformal [L1, Lemma 7.3], we obtain the scaling result for $g_{n,t}$.

8.1.6. *Uniqueness of* $g_{n,t}$. Recall (Theorem 7.7) that W^u is parametrized by the multipliers of the α-fixed points. Therefore, parabolic pacmen with rotation numbers $p_n/q_n, n \ll 0$, are unique. As a consequence the paths of satellite pacmen emerging from these parabolic pacmen are unique. Similarly, parabolic maps $g_{n,0} \in W$ with rotation numbers p_n/q_n are unique; thus the paths $g_{n,t}$ are unique.

Appendix A. Sector renormalizations of a rotation

Consider $\theta \in \mathbb{R}/\mathbb{Z}$ and let

$$L_\theta : \overline{\mathbb{D}^1} \to \overline{\mathbb{D}^1}, z \to \mathbf{e}(\theta)z$$

be the corresponding rotation of the closed unit disk by angle θ.

A.1. **Prime renormalization of a rotation.** Assume that $\theta \neq 0$ and consider a closed internal ray I of $\overline{\mathbb{D}^1}$. A *fundamental sector* $Y \subset \overline{\mathbb{D}^1}$ of L_θ is the smallest closed sector bounded by I and $L_\theta(I)$. If $\theta = 1/2$, then $I \cup L_\theta(I)$ is a diameter and both sectors of $\overline{\mathbb{D}^1}$ bounded by $I \cup L_\theta(I)$ are fundamental. The angle ω at the vertex of Y is θ if $\theta \in [0,1/2]$ or $1-\theta$ if $1-\theta \in [0,1/2]$.

A fundamental sector is defined uniquely up to rotation; let us first rotate it such that $1 \in \overline{\mathbb{D}^1} \setminus Y$. Set $Y_- := L_\theta^{-1}(Y)$ and set Y_+ to be the closure of $\mathbb{D}^1 \setminus (Y \cup Y_-)$; see Figure 27. Then

$$(L_\theta \mid Y_+, L^2_\theta \mid Y_-) \tag{A.1}$$

is the first return of points in $Y_- \cup Y_+$ back to $Y_- \cup Y_+$. The *prime renormalization of* L_θ is the rotation $L_{R_{\mathrm{prm}}(\theta)} : \overline{\mathbb{D}^1} \to \mathbb{D}^1$ obtained from (A.1) by applying the gluing map $\psi_{\mathrm{prm}} : Y_- \cup Y_+ \to \mathbb{D}^1, z \to z^{1/(1-\omega)}$.

Lemma A.1. *We have*

$$R_{\mathrm{prm}}(\theta) = \begin{cases} \frac{\theta}{1-\theta} & \text{if } 0 \le \theta \le \frac{1}{2}, \\ \frac{2\theta-1}{\theta} & \text{if } \frac{1}{2} \le \theta \le 1 \end{cases} \tag{A.2}$$

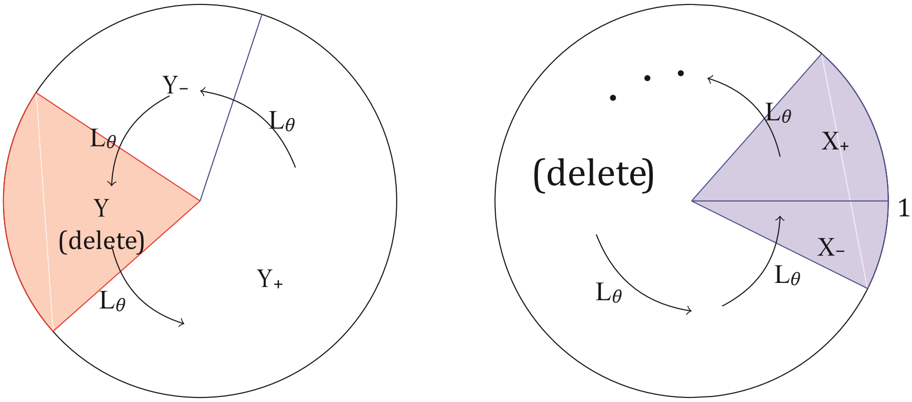

Figure 27. Left: the prime renormalization deletes a fundamental sector Y and projects $(L^2_\theta \mid Y_-, L_\theta \mid Y_+)$ to a new rotation.
Right: $\left(\mathbb{L}_\theta^{q+1} \mid \mathbb{X}_-, \mathbb{L}_\theta^{q} \mid \mathbb{X}_+\right)$ is the first return map to a fundamental sector $Y = X_- \cup X_+$.

Present θ using continued fractions in the following ways:

$$\theta = [0;a_1,a_2,\dots] = 1 - [0;b_1,b_2,\dots]$$

with $a_i,b_i \in \mathbb{N}_{>0}$. Then

$$R_{\mathrm{prm}}([0;a_1,a_2,\dots]) = \begin{cases} [0;a_1-1,a_2,\dots] & \text{if } a_1>1,\\ 1-[0;a_2,a_3,\dots] & \text{if } a_1=1,\end{cases}$$

and

$$R_{\mathrm{prm}}(1-[0;b_1,b_2,\dots]) = \begin{cases} 1-[0;b_1-1,b_2,\dots] & \text{if } b_1>1,\\ [0;b_2,b_3,\dots] & \text{if } b_1=1.\end{cases}$$

As a consequence, θ is periodic under R_{prm} if and only if there is a θ with periodic continued fraction expansion such that $\theta = R^n_{\mathrm{prm}}(\theta')$ for some $n \ge 0$.

Proof. The proof follows by routine calculations. If $\theta \in [0,1/2]$, then projecting $z \to \mathbf{e}(\theta)z$ by ψ_{prm} we obtain

$$z \to \left(\mathbf{e}(\theta)z^{1-\theta}\right)^{1/1-\theta} = \mathbf{e}\left(\frac{\theta}{1-\theta}\right)z.$$

If $\theta \in [1/2,1]$, then projecting $z \to \mathbf{e}(\theta-1)z$ by ψ_{prm} we obtain

$$z \to \left(\mathbf{e}(\theta-1)z^{\theta}\right)^{1/\theta} = \mathbf{e}\left(\frac{\theta-1}{\theta}\right)z.$$

Observe that $\frac{\theta-1}{\theta} \in [-1,0]$; adding $+1$ we obtain $\frac{2\theta-1}{\theta}$.

Write

$$\theta = \frac{1}{a_1 + [0; a_2, a_3, \dots]}$$

and observe that $\theta \in [0,1/2]$ if and only if $a_1 > 1$ (with the exception $\theta = [0;1,1]$). If $a_1 > 1$, then

$$\frac{\theta}{1-\theta} = \frac{1}{a_1 + [0; a_2, a_3, \dots] - 1} = R_{\mathrm{prm}}(\theta).$$

If $a_1 = 1$, then

$$\frac{2\theta - 1}{\theta} = 2 - a_1 - [0; a_2, a_3, \dots] = R_{\mathrm{prm}}(\theta).$$

Similarly $R_{\mathrm{prm}}(1 - [0; b_1, b_2, \dots])$ is verified.

A.2. **Sector renormalization.** A *sector renormalization* $\mathcal{R}$ of $\mathbb{L}_\theta$ is

- a *renormalization* sector $\mathbb{X}$ presented as a union of two subsectors $\mathbb{X}_- \cup \mathbb{X}_+$ normalized so that $1 \in \mathbb{X}_- \cap \mathbb{X}_+$;
- a pair of iterates, called a sector *pre-renormalization*,

(A.3) $$\left(\mathbb{L}_\theta^{\mathfrak{a}} \mid \mathbb{X}_-, \mathbb{L}_\theta^{\mathfrak{b}} \mid \mathbb{X}_+\right)$$

  realizing the first return of points in $\mathbb{X}_- \cup \mathbb{X}_+$ back to $\mathbb{X}$; and
- the gluing map

$$\psi\colon \mathbb{X}_- \cup \mathbb{X}_+ \to \overline{\mathbb{D}^1}, z \to z^{1/\omega},$$

  projecting (A.3) to a new rotation $\mathbb{L}_\mu$, where ω is the angle of $\mathbb{X}$ at 0.

We write $\mathcal{R}\mathbb{L}_\theta = \mathbb{L}_\mu$, and we call $\mathfrak{a}$ and $\mathfrak{b}$ the *renormalization return times*. We allow one of the sectors $\mathbb{X}_\pm$ to degenerate, but not both. Note that the assumption $1 \in \mathbb{X}_- \cap \mathbb{X}_+$ can always be achieved using rotation.

The prime renormalization is an example of a sector renormalization.

Suppose two sector renormalizations $\mathcal{R}_1(\mathbb{L}_\theta) = \mathbb{L}_\mu$ and $\mathcal{R}_2(\mathbb{L}_\mu) = \mathbb{L}_\nu$ are given. The *composition* $\mathcal{R}_2 \circ \mathcal{R}_1(\mathbb{L}_\theta) = \mathbb{L}_\nu$ is obtained by lifting the pre-renormalization of $\mathcal{R}_2$ to the dynamical plane of $\mathbb{L}_\theta$.

Lemma A.2. *A sector renormalization is an iteration of the prime renormalization.*

Proof. Suppose $\mathcal{R}$ is a sector renormalization with renormalization return times $\mathfrak{a}$ and $\mathfrak{b}$ as above. Proceed by induction on $\mathfrak{a} + \mathfrak{b}$. If $\mathfrak{a} + \mathfrak{b} = 3$, then $\mathcal{R}$ is the prime renormalization. Otherwise, we project the pre-renormalization of $\mathcal{R}$ to the dynamical plane of $\mathcal{R}_{\mathrm{prm}}(\mathbb{L}_\theta)$ and obtain the new sector renormalization $\mathcal{R}$ of $\mathcal{R}_{\mathrm{prm}}(\mathbb{L}_\theta)$ so that

$$\mathcal{R}' \circ \mathcal{R}_{\mathrm{prm}}(\mathbb{L}_\theta) = \mathcal{R}(\mathbb{L}_\theta).$$

The renormalization return times a', b' of $\mathcal{R}'$ satisfy $a' + b' < a + b$.

Consider again the fundamental sector Y bounded by I and $L_\theta(I)$. There is a minimal a > 0 such that $L^{-a}(I) \subset Y$. Up to rotation, we can assume that $L^{-a}(I)$ lands at 1. We define X_+ to be the subsector of Y bounded by I and $L^{-a}(I)$ and we define X_- to be the subsector of Y bounded by L(I) and $L^{-a}(I)$. Then

$$\left(\mathbb{L}_\theta^{\mathfrak{a}} \mid \mathbb{X}_-, \mathbb{L}_\theta^{\mathfrak{a}+1} \mid \mathbb{X}_+\right)$$

is a sector pre-renormalization, called the *first return to the fundamental sector*; see Figure 27. We denote by R_{fast} the associated sector renormalization and we write $\mu = R_{\mathrm{fast}}(\theta)$ if $R_{\mathrm{fast}}(L_\theta) = L_\mu$.

By Lemma A.2, for every $\theta \neq 0$ there is a unique $n(\theta)$ such that $R_{\mathrm{fast}}(\theta) = R_{\mathrm{prm}}^{n(\theta)}(\theta)$. We note that if $\theta \in \{1/m, 1-1/m\}$ with $m > 1$, then $n(\theta) = m - 1$. (In this case the sector X_- is degenerate.)

A.3. **Renormalization triangulation.** Given a sector pre-renormalization (A.3), the set of sectors

$$\bigcup_{i=0}^{\mathfrak{a}-1} \mathbb{L}_\theta(\mathbb{X}_-) \bigcup_{i=0}^{\mathfrak{b}-1} \mathbb{L}_\theta(\mathbb{X}_+)$$

is called a *renormalization triangulation* of D^1. Alternatively, consider the associated renormalization $L_\mu = R(L_\theta)$. The internal rays towards 1 and $L_\mu(1)$ split D^1 into two closed sectors T_0 and T_1. We call $\{T_-, T_+\}$ the *basic triangulation of* L_μ. Lifting the sectors T_-, T_+ via the gluing map, and spreading them dynamically we obtain the renormalization triangulation. We also say that the renormalization triangulation is the *full lift* of the basic triangulation.

Let Θ_N be the set of angles θ such that $\theta = [0; a_1, a_2, \ldots]$ with $|a_i| \le N$ or $\theta = 1 - [0; a_1, a_2, \ldots]$ with $|a_i| \le N$. By Lemmas A.1 and A.2, the set Θ_N is invariant under any sector renormalization.

Lemma A.3. *For every N there is a* $t > 1$ *with the following property. Consider the renormalization triangulation associated with some sector renormalization of* L_θ, *where* $\theta \in \Theta_N$. *Then any two triangles have comparable angles at* 0*: the ratio of the angles is between* $1/t$ *and t.*

Proof. There is a neighborhood U of 1 such that for all $\theta \in \Theta_N$ we have $L_\theta(1) \notin U$. Therefore, both sectors in the basic triangulation have comparable angles at 0 uniformly on $\theta \in \Theta_N$. Since a renormalization triangulation is the full lift of a basic triangulation, the lemma is proven.

A.4. **Periodic case.** It follows from Lemmas A.1 and A.2 that L_θ is a fixed point of some sector renormalization if and only if $\theta \in \Theta_{per}$. Suppose $\theta \in \Theta_{per}$ and choose a sector renormalization R_1 such that $R_1(L_\theta) = L_\theta$. Write $\mathcal{R}_n := \mathcal{R}_1^n$ and denote by $\mathfrak{a}_n, \mathfrak{b}_n$, and ψ_n the renormalization return times and the gluing map of R_n. Then $\psi_n = \psi_1^n$ and there is a matrix M with positive entries such that

$$\begin{pmatrix} \mathfrak{a}_n \\ \mathfrak{b}_n \end{pmatrix} = \mathbb{M}^{n-1} \begin{pmatrix} \mathfrak{a}_1 \\ \mathfrak{b}_1 \end{pmatrix}. \tag{A.4}$$

As a consequence, $\mathfrak{a}_n, \mathfrak{b}_n$ have exponential growth with the same exponent. We also note that

$$\mathfrak{a}_1, \mathfrak{b}_1 \geq 2 \tag{A.5}$$

because $R_1 = R^t_{prm}$ with $t > 1$.

Appendix B. Lifting of curves under antirenormalization

In this appendix we give a sufficient condition for liftability of arcs under a sector antirenormalization. This implies that the sector antirenormalization is robust with respect to a particular choice of cutting arcs; see Theorem B.8.

B.1. **Robustness of antirenormalization.** Consider a closed pointed topological disk $(W, 0)$ and let U, V be two closed topological subdisks of W such that $0 \in \mathrm{int}(U \cap V)$. A homeomorphism $f: U \to V$ fixing 0 is called a *partial homeomorphism* of $(W, 0)$ and is denoted by $f: W \dashrightarrow W$ or $f: (W, 0)$ $(W, 0)$. If $U = V = W$, then f is an actual self-homeomorphism of $(W, 0)$.

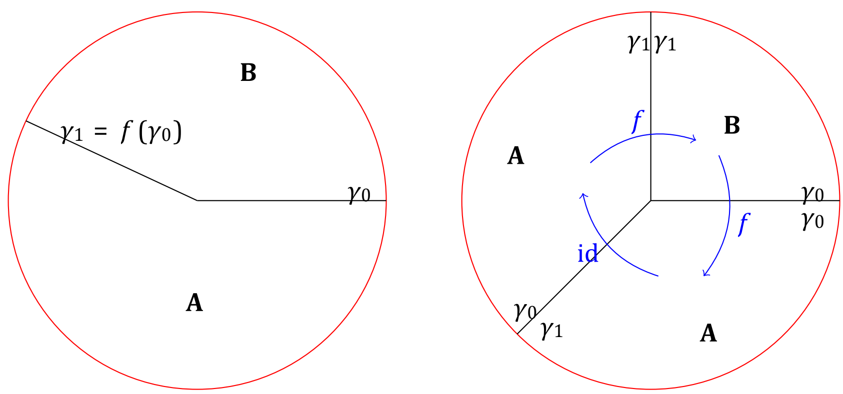

Figure 28. Left: a homeomorphism $f: W \to W$ and a dividing pair γ_0, γ_1. Right: the 1/3 antirenormalization of f (with respect to the clockwise orientation).

B.1.1. *Leaves over* $f: (W, 0) \dashrightarrow (W, 0)$. Let γ_0, γ_1 be two simple arcs connecting 0 to points in ∂W such that γ_0 and γ_1 are disjoint except for 0 and such that γ_1 is the image of γ_0 in the following sense: $\gamma'_0 := \gamma_0 \cap U$ and $\gamma'_1 := \gamma_1 \cap V$ are simple closed curves such that f maps γ'_0 to γ'_1. Such a pair γ_0, γ_1 is called *dividing*. Then $\gamma_0 \cup \gamma_1$ splits W into two closed sectors **A** and **B** denoted so that int**A**,γ_1,int**B**,γ_0 are clockwise oriented around 0; see the left side of Figure 28.

We say that $\gamma_0 = \ell(\mathbf{A}) = \rho(\mathbf{B})$ is the *left boundary of* **A** and the *right boundary of* **B** and we say that $\gamma_1 = \rho(\mathbf{A}) = \ell(\mathbf{B})$ is the *right boundary of* **A** and *the left boundary of* **B**.

Let X,Y be topological spaces, and let $g\colon X \dashrightarrow Y$ be a partially defined continuous map. We define

$$X \sqcup_g Y := X \sqcup Y/(\operatorname{Dom} g \ni x \sim g(x) \in \operatorname{Im} g).$$

Consider a (finite or infinite) sequence $(S_k)_k$, where each S_k is a copy of either **A** or **B**. Define the partial map $g_k\colon \rho(S_k) \dashrightarrow \ell(S_{k+1})$ by

$$g_k := \begin{cases} \mathrm{id}\colon \gamma'_1 \to \gamma'_1 & \text{if } (S_k,S_{k+1}) =\sim (\mathbf{A},\mathbf{B}), \\ \mathrm{id}\colon \gamma'_0 \to \gamma'_0 & \text{if } (S_k,S_{k+1}) =\sim (\mathbf{B},\mathbf{A}), \\ f^{-1}\colon \gamma'_1 \to \gamma'_0 & \text{if } (S_k,S_{k+1}) =\sim (\mathbf{A},\mathbf{A}), \\ f\colon \gamma'_0 \to \gamma'_1 & \text{if } (S_k,S_{k+1}) =\sim (\mathbf{B},\mathbf{B}). \end{cases} \tag{B.1}$$

The *dynamical gluing of* $(S_k)_k$ is

$$\cdots S_{k-1} \sqcup_{g_{k-1}} S_k \sqcup_{g_k} S_{k+1} \sqcup_{g_{k+1}} \cdots .$$

The *jump* $\iota(k)$ *from* S_k *to* S_{k+1} is set to be $0, 1, 1, -1$ if (S_k, S_{k+1}) is a copy of (**A**,**B**), (**B**,**A**), (**A**,**A**), (**B**,**B**), respectively.

For a sequence $\mathbf{s} = (a_i)_{i\in I}$ we denote by $\mathbf{s}[i]$ the ith element in $\mathbf{s}$; i.e., $\mathbf{s}[i] = a_i$.

Definition B.1 (Leaves of $f\colon W \dashrightarrow W$)**.** Suppose $\mathbf{s} \in \{\mathbf{A},\mathbf{B}\}^{\mathbb{Z}}$. Set $W_\mathbf{s}[i]$ to be a copy of the closed sector $\mathbf{s}[i]$. The *leaf* $W_\mathbf{s}$ is the surface obtained by the dynamical gluing of $(W_\mathbf{s}[i])_{i\in\mathbb{Z}}$.

The *projection* $\pi\colon W_\mathbf{s} \to W$ maps each $W_\mathbf{s}[i] \setminus \rho(W_\mathbf{s}[i-1])$ to $\mathbf{s}[i]$. By $\pi^{-1}_{\mathbf{s},i}\colon \mathbf{s}[i] \to W_\mathbf{s}[i]$ we denote the corresponding inverse branch.

Note that if $\mathbf{s}[i] = \mathbf{s}[i+1]$, then π is discontinuous at $W_\mathbf{s}[i] \cap W_\mathbf{s}[i+1]$. As z approaches $W_\mathbf{s}[i] \cap W_\mathbf{s}[i+1]$ from int$W_\mathbf{s}[i]$, respectively, int$W_\mathbf{s}[i+1]$, its image $\pi(z)$ approaches $\rho(\mathbf{s}[i])$, respectively, $\ell(\mathbf{s}[i+1]) \neq \rho(\mathbf{s}[i])$.

For every $\mathbf{s}$, there is a unique point $\tilde{0} \in W_\mathbf{s}$ such that $\pi(\tilde{0}) = 0$. By construction, $W_\mathbf{s} \setminus \{\tilde{0}\}$ is topologically a closed half-plane.

For $J \subset \mathbb{Z}$ we write $W_\mathbf{s}[J] = \bigcup_{j\in J} W_\mathbf{s}[j]$. To keep notation simple, we write $W_\mathbf{s}[\geq i] = W_\mathbf{s}[\{k \mid k \geq i\}]$ and similarly for ">", "≤", "<".

B.1.2. *Lifts of curves.* Let α: $[0,1] \to W \setminus \{0\}$ be a curve in W. A *lift of α to $W_{\mathbf{s}}$* is a curve $\widetilde{\alpha}\colon [0,1] \to W_{\mathbf{s}}$ such that

- n
- for every $t \in [0,1]$, there is an $n(t) \in \mathbb{Z}$ such that $\pi(\alpha(t)) = f^{n(t)}(\alpha(t))$; $\sim$

  $W_{\mathbf{s}}[i] \setminus \rho(W_{\mathbf{s}}[i -$

  (0) = 0; $n(t)$ is constant for all t for which $\widetilde{\alpha}(t)$ is within some 1]); and
- if $\widetilde{\alpha}(t') \in \operatorname{int} W_{\mathbf{s}}[i]$ while $\widetilde{\alpha}(t) \in \operatorname{int} W_{\mathbf{s}}[i+1]$, then $n(t) - n(t')$ is equal to the jump from $W_{\mathbf{s}}[i]$ to $W_{\mathbf{s}}[i+1]$.

In other words, whenever α crosses the boundary of $\mathbf{s}[i]$, the lift of α is adjusted to respect the dynamical gluing. A lift of a curve parametrized by $[0,1)$ is similarly defined. Note that $\pi(\alpha)$ is, in general, discontinuous.

For every curve α as above, there is at most one lift of $\to$ α starting at a given

preimage of $\alpha(0)$ under π: $W_{\mathbf{s}}$ W. It is easy to see that there is an $\varepsilon > 0$ such that all lifts (specified by the starting points) of α: $[0,\varepsilon] \to W$ exist, and are thus unique. The main question we address is the existence of the global lifts.

If α: $[0,1) \to W$ $\sim$ $\sim$ $\setminus \{0\}$ is such that $\alpha(1) = \lim_{t\to 1} \alpha(t) = 0$, then we say that a lift α of α *lands at* 0 if $\pi(\alpha(t)) \to 0$ as $t \to 1$.

B.1.3. *Antirenormalizations.* We will now show that for every p/q there is a unique antirenormalization with rotation number p/q.

Lemma B.2. *For every* $\mathfrak{q} \in \mathbb{N}_{>2}$ *and every* $\mathfrak{p} \in \{1,2,\dots,\mathfrak{q}-1\}$ *coprime with* $\mathfrak{q}$ *there exists a unique* $\mathfrak{q}$*-periodic sequence* $\mathbf{s} \in \{\mathbf{A},\mathbf{B}\}^{\mathbb{Z}}$ *such that*

- $\mathbf{s}[0] = \mathbf{A}$ *and* $\mathbf{s}[-1] = \mathbf{B}$*;*
- *for every* $j \in \mathbb{Z}$ *with* $(j \mod \mathfrak{q}) \notin \{-\mathfrak{p}, -\mathfrak{p}-1\}$*, we have* $\mathbf{s}[j+\mathfrak{p}] = \mathbf{s}[j]$*;*
- $\mathbf{s}[-\mathfrak{p}-1] = \mathbf{A}$ *and* $\mathbf{s}[-\mathfrak{p}] = \mathbf{B}$*.*

Proof. Since p and q are coprime, there are unique $\mathfrak{a},\mathfrak{b} \in \{1,2,\dots,\mathfrak{q}-1\}$ such that

$$\mathfrak{p}\mathfrak{a} = -1 \mod \mathfrak{q}, \quad \mathfrak{p}\mathfrak{b} = 1 \mod \mathfrak{q}.$$

Note that $\mathfrak{a} + \mathfrak{b} = \mathfrak{q}$. We have

- $\mathbf{s}[i\mathfrak{p} + j\mathfrak{q}] = \mathbf{A}$ for all $i \in \{0,1,\dots,\mathfrak{a}-1\}$ and all $j \in \mathbb{Z}$; and
- $\mathbf{s}[-1 + i\mathfrak{p} + j\mathfrak{q}] = \mathbf{B}$ for all $i \in \{0,\dots,\mathfrak{b}-1\}$ and all $j \in \mathbb{Z}$.

The numbers $\mathfrak{a},\mathfrak{b}$ appearing in the proof of Lemma B.2 are called the *renormalization return times*.

For a sequence $\mathbf{s}$ as in Lemma B.2, let

$$\mathbf{s}/\mathfrak{q} \in \{\mathbf{A},\mathbf{B}\}^{\mathbb{Z}/\mathfrak{q}\mathbb{Z}}, (\mathbf{s}/\mathfrak{q})[i] := \mathbf{s}[i + \mathbb{Z}\mathfrak{q}]$$

be the quotient sequence, and let $W_{\mathbf{s}/\mathfrak{q}}$ be the quotient of the leaf $W_{\mathbf{s}}$ by identifying each $W_{\mathbf{s}}[k]$ with $W_{\mathbf{s}}[k + \mathfrak{q}]$. We denote by $\pi\colon W_{\mathbf{s}/\mathfrak{q}} \to W$ the natural projection.
Then the $\mathfrak{p}/\mathfrak{q}$-*antirenormalization* $f_{-1}\colon W_{\mathbf{s}/\mathfrak{q}} \dashrightarrow W_{\mathbf{s}/\mathfrak{q}}$ is defined as follows (see Figure 28):

- for every $j \notin \{-\mathfrak{p} - 1, -\mathfrak{p}\}$, the map $f_{-1}\colon W_{\mathbf{s}/\mathfrak{q}}[j] \to W_{\mathbf{s}/\mathfrak{q}}[j + \mathfrak{p}]$ is the natural isomorphism;
- the map $f_{-1}\colon W_{\mathbf{s}/\mathfrak{q}}[-\mathfrak{p} - 1, -\mathfrak{p}] \dashrightarrow W_{\mathbf{s}/\mathfrak{q}}[-1, 0]$ is $f\colon W \setminus \gamma_0 \dashrightarrow W \setminus \gamma_1$.

Note that $\mathfrak{p}/\mathfrak{q}$ is the clockwise rotation number.

By construction, $(f_{-1}^{\mathfrak{a}} \mid W_{\mathbf{s}/\mathfrak{q}}[0], f_{-1}^{\mathfrak{b}} \mid W_{\mathbf{s}/\mathfrak{q}}[-1])$ is the first return of f_{-1} back to $W_{\mathbf{s}/\mathfrak{q}}[-1, 0]$. After appropriate gluing of arcs in $\partial\left(W_{\mathbf{s}/\mathfrak{q}}[-1, 0]\right)$, the map $(f_{-1}^{\mathfrak{a}} \mid W_{\mathbf{s}/\mathfrak{q}}[0], f_{-1}^{\mathfrak{b}} \mid W_{\mathbf{s}/\mathfrak{q}}[-1])$ is $f\colon W \dashrightarrow W$.

Denote by

$$\gamma_0^{\mathbf{s}/\mathfrak{q}}, \gamma_p^{\mathbf{s}/\mathfrak{q}} \tag{B.2}$$

the left boundaries of $W_{\mathbf{s}/\mathfrak{q}}[0]$ and $W_{\mathbf{s}/\mathfrak{q}}[p]$, respectively. Then $\gamma_0^{\mathbf{s}/\mathfrak{q}}, \gamma_p^{\mathbf{s}/\mathfrak{q}}$ is a dividing pair for $f_{-1}\colon W_{\mathbf{s}/\mathfrak{q}} \dashrightarrow W_{\mathbf{s}/\mathfrak{q}}$ and the antirenormalization procedure can be iterated.

Let β be a curve in W, and let $\tilde\beta$ be a lift of β to $W_{\mathbf{s}}$. The image of $\tilde\beta$ in $W_{\mathbf{s}/\mathfrak{q}} \simeq W_{\mathbf{s}}/\sim$ is called a *lift of β to $W_{\mathbf{s}/\mathfrak{q}}$*. For example, $\gamma_0^{\mathbf{s}/\mathfrak{q}}$ is a lift of γ_0.

B.1.4. *Prime antirenormalization.* The $1/3$ and $2/3$-antirenormalizations are called *prime*. It is easy to check that

- if $\mathfrak{p}/\mathfrak{q} = 1/3$, then $\mathbf{s}/\mathfrak{q} = (\mathbf{A},\mathbf{A},\mathbf{B})$;
- if $\mathfrak{p}/\mathfrak{q} = 2/3$, then $\mathbf{s}/\mathfrak{q} = (\mathbf{A},\mathbf{B},\mathbf{B})$.

Lemma B.3 (Compare with Lemma A.2). *Any antirenormalization is an iteration of prime antirenormalizations.*

Proof. We proceed by induction on $\mathfrak{q}$. Assume $\mathfrak{q} > 3$, define $\mathfrak{p}'/\mathfrak{q}' := R_{\mathrm{prm}}(\mathfrak{p}/\mathfrak{q})$ (see (A.2)), and observe that $\mathfrak{q}' < \mathfrak{q}$.

- If $0 < \mathfrak{p} < \mathfrak{q}/2$, then the $\mathfrak{p}/\mathfrak{q}$-antirenormalization is the $1/3$-antirenormalization of the $\mathfrak{p}'/\mathfrak{q}'$-antirenormalization.
- If $\mathfrak{q}/2 < \mathfrak{p} < \mathfrak{q}$, then the $\mathfrak{p}/\mathfrak{q}$-antirenormalization is the $2/3$-antirenormalization of the $\mathfrak{p}'/\mathfrak{q}'$-antirenormalization.

Denote by $\mathbf{s}_\bullet := (\mathbf{A},\mathbf{B})^{\mathbb{Z}}$ the sequence in $\{\mathbf{A},\mathbf{B}\}^{\mathbb{Z}}$ with even entries equal to $\mathbf{A}$ and odd entries equal to $\mathbf{B}$. Simplifying notation, we write $W_{\mathbf{s}_\bullet} = W_\bullet$.

Suppose that $f : W \to W$ is a homeomorphism. In this case antirenormalizations of f can be defined canonically (i.e., independent of the choice of γ_0, γ_1) as follows. Observe first that

(B.3) $$\pi : W_\bullet \setminus \{\widetilde{0}\} \to W \setminus \{0\}$$

is a universal cover. Let $(\widetilde{\gamma}_i^\bullet \subset W_\bullet)_{i \in \mathbb{Z}}$ be all the lifts of γ_0 and γ_1 enumerated from left to right such $\widetilde{\gamma}_i^\bullet$ $\widetilde{\gamma}_{i+1}^\bullet$ that $W[i]$ is $\widetilde{\gamma}_i^\bullet$ betweenand; in particular,is a lift of

$\gamma_{i \bmod 2}$. Let

$$f_-, f_+ : W_\bullet \to W_\bullet$$

be the lifts of $f : W \to W$ specified so that

$-(\widetilde{\gamma}_{2i}^\bullet) = \widetilde{\gamma}_{2i-1}^\bullet$ ~ ~ f and

$f_+(\gamma_{2i}^\bullet) = \gamma_{2i+1}^\bullet$. $_+$ commute. Write

Observe

that f and f

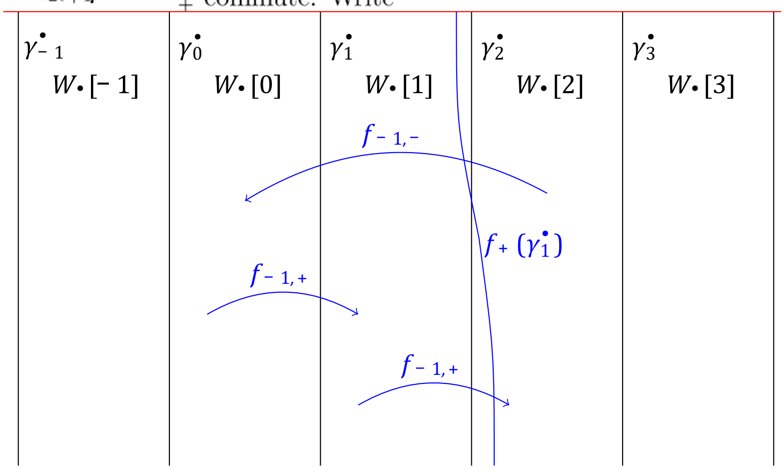

Figure 29. IllustrationtoLemmaB.3: $f_{-1,-} : W_\bullet[2] \to W_\bullet[0]$

$-\tau := f_-^{-1} \circ f_+ : W_\bullet \to W_\bullet$;

and f after gluing $\gamma^\bullet$ and $\widetilde{\gamma}_3^\bullet$ $f_{-1,+} : W_\bullet[0,1] \to W_\bullet[1,2]$ become the 1/3-antirenormalization of $_0$; see also Figure 28.

then $\tau \mid (W_\bullet \setminus \{\widetilde{0}\})$ is a deck transformation of W and we can rewrite (B.3) as

$$W \setminus \{0\} \simeq \left(W_\bullet \setminus \{\widetilde{0}\}\right) / \langle \tau \rangle$$

$$W \simeq W_\bullet / \langle \tau \rangle.$$

We also write

Lemma B.4 (The 1/3-antirenormalization)**.** *Suppose $f : W \to W$ is a selfhomeomorphism. Set*

$$f_{-1,+} := f_+,\ f_{-1,-} := \tau^{-1} = f_- \circ f_+^{-1},\ \tau_{-1} := f_{-1,-}^{-1} \circ f_{-1,+} = f_-^{-1} \circ f_+^2.$$

Then τ_{-1} acts properly discontinuously on $W_\bullet \setminus \{\widetilde{0}\}$. We view

$$\left(W_\bullet \setminus \{\widetilde{0}\}\right) / \langle \tau_{-1} \rangle$$

as a punctured closed topological disk and we view $W_\bullet / \langle \tau_{-1} \rangle$ as a closed topological disk.

Let $f_{-1} : W_{-1} \to W_{-1}$ be the 1/3*-antirenormalization of f. Then f_{-1} is conjugate to*

$$f_{-1,-}/\langle \tau_{-1} \rangle = f_{-1,+}/\langle \tau_{-1} \rangle \colon W_\bullet/\langle \tau_{-1} \rangle \to W_\bullet/\langle \tau_{-1} \rangle$$

by the conjugacy

$$h\colon W_{-1} \to W_\bullet[0,1,2]/\langle \tau_{-1} \rangle$$

mapping

$$W_{-1}[0], W_{-1}[1], W_{-1}[2]$$

respectively to

$$W_\bullet[2]/\langle \tau_{-1} \rangle, W_\bullet[0]/\langle \tau_{-1} \rangle, W_\bullet[1]/\langle \tau_{-1} \rangle$$

which are copies of **A**,**A**,**B**.

Proof. Clearly, $W_\bullet[0,1,2]$ is a fundamental domain for τ_{-1}. It is easy to see (see Figure 29) that h identifies

- $f_{-1} : W_{-1}[0] \to W_{-1}[1]$ (which is id: **A** → **A**) with

$$f_{-1,-} = \tau^{-1}\colon W_\bullet[2] \to W_\bullet[0];$$

- $f_{-1} : W_{-1}[1,2] \to W_{-1}[2,0]$ (which is $f : W \setminus \gamma_0 \to W \setminus \gamma_1$) with

$$f_{-1,+} : W_\bullet[0,1] \to W_\bullet[1,2].$$

Remark B.5. The proof of Lemma B.4 shows also that h is uniquely characterized by the following properties:

- h maps $\gamma_0^{\mathbf{s}/3}$ to $\widetilde{\gamma}_2^\bullet/\langle \tau \rangle$ (see (B.2));
- if $\ell \subset W \setminus \{0\}$ is a curve starting at γ_0 and $\widetilde{\ell} \subset W_{-1}$ is the unique lift of ℓ starting at some point of $\gamma_0^{\mathbf{s}/3}$, then $h(\widetilde{\ell}/\langle \tau_{-1} \rangle) \subset W_\bullet$ is the unique lift of ℓ starting at $\widetilde{\gamma}_2^\bullet$.

Similar to Lemma B.4 we have

Lemma B.6 (The 2/3-antirenormalization)**.** *Suppose $f : W \to W$ is a self-homeomorphism. Set*

$$f_{-1,+} := \tau = f_+ \circ f_-^{-1}, \quad f_{-1,-} := f_-, \quad \tau_{-1} := f_{-1,-}^{-1} \circ f_{-1,+} = f_-^{-2} \circ f_+.$$

Then τ_{-1} acts properly discontinuously on $W_\bullet \setminus \{\widetilde{0}\}$. We view $\left(W_\bullet \setminus \{0\}\right)/\langle \tau_{-1}\rangle$ as a punctured closed topological disk and we view $W_\bullet/\langle\tau_{-1}\rangle$ as a closed topological disk.

Let $f_{-1} : W_{-1} \to W_{-1}$ be the 2/3*-antirenormalization of f. Then f_{-1} is conjugate to*

$$f_{-1,-}/\langle\tau_{-1}\rangle = f_{-1,+}/\langle\tau_{-1}\rangle \colon W_\bullet/\langle\tau_{-1}\rangle \to W_\bullet/\langle\tau_{-1}\rangle$$

by the conjugacy

$$h \colon W_{-1} \to W_\bullet[-1,0,1]/\langle\tau_{-1}\rangle$$

mapping

$$W_{-1}[0], W_{-1}[1], W_{-1}[2]$$

respectively to

$$W_\bullet[0]/\langle\tau_{-1}\rangle, W_\bullet[1]/\langle\tau_{-1}\rangle, W_\bullet[-1]/\langle\tau_{-1}\rangle$$

which are the copies of **A**,**B**,**B***.*

B.1.5. *Fences.* Consider again a partial homeomorphism $f \colon W \dashrightarrow W$ and let **s** be an antirenormalization sequence from Lemma B.2. We view W as a subset of $\mathbb{C}$.

A *fence* is a simple closed curve $Q \subset \operatorname{Dom} f \cap \operatorname{Im} f$ such that

- 0 is in the bounded component Ω of $\mathbb{C} \setminus Q$; and
- Q intersects γ_0 at a single point x and Q intersects γ_1 at $f(x)$.

Let $f_{-1} \colon W_{\mathbf{s}/\mathfrak{q}} \dashrightarrow W_{\mathbf{s}/\mathfrak{q}}$ be an antirenormalization of f_{-1} as in §B.1.3. We denote by $Q_{\mathbf{s}}$ the lift of Q to $W_{\mathbf{s}}$ and we denote by $Q_{\mathbf{s}/\mathfrak{q}}$ the projection of $Q_{\mathbf{s}}$ to $W_{\mathbf{s}/\mathfrak{q}}$.

Lemma B.7. *The curve $Q_{\mathbf{s}/\mathfrak{q}}$ is again a fence respecting $\gamma_0^{\mathbf{s}/\mathfrak{q}}, \gamma_p^{\mathbf{s}/\mathfrak{q}}$; see* (B.2)*.*

Proof. Every $Q_{\mathbf{s}/\mathfrak{q}} \cap W_{\mathbf{s}/\mathfrak{q}}[i]$ is an arc connecting a point on the left boundary of $W_{\mathbf{s}/\mathfrak{q}}[i]$ to a point on the right boundary of $W_{\mathbf{s}/\mathfrak{q}}[i]$. Moreover, $Q_{\mathbf{s}/\mathfrak{q}} \cap W_{\mathbf{s}/\mathfrak{q}}[i]$ meets $Q_{\mathbf{s}/\mathfrak{q}} \cap W_{\mathbf{s}/\mathfrak{q}}[i+1]$ because $g_k \colon \rho(S_i) \dashrightarrow \ell(S_{i+1})$ (see (B.1)) respects the intersection of Q with γ_0, γ_1.

B.1.6. *Robustness of antirenormalization.*

Theorem B.8. *Let $f\colon W \dashrightarrow W$ be a partial homeomorphism, let $\gamma_0,\gamma_1 \subset W$ be a dividing pair of arcs, let $Q \subset \operatorname{Dom} f$ be a fence respecting γ_0,γ_1 and enclosing $\Omega \ni 0$, and let*

$$f_{-1} : W_{-1} \quad W_{-1}$$

be the p/q*-antirenormalization of f; see* §B.1.3*.*

Assume that $\gamma_0^{\mathrm{new}},\gamma_1^{\mathrm{new}}$ is another pair of dividing arcs such that $\gamma_0^{\mathrm{new}} \setminus \Omega$, $\gamma_1^{\mathrm{new}} \setminus \Omega$ coincides with $\gamma_0 \setminus \Omega,\gamma_1 \setminus \Omega$. Denote by

$$f_{-1,\mathrm{new}} : W_{-1,\mathrm{new}} \quad W_{-1,\mathrm{new}}$$

the p/q*-antirenormalization of f relative to the pair $\gamma_0^{\mathrm{new}},\gamma_1^{\mathrm{new}}$. Then f_{-1} and $f_{-1,\mathrm{new}}$ are naturally conjugate by $h\colon W_{-1} \to W_{-1,\mathrm{new}}$ uniquely specified by the following properties:*

(1) $\pi \circ h(z) = \pi(z)$ *for every $z \in W_{-1} \setminus \Omega_{-1}$, where Ω_{-1} is the topological disk enclosed by Q_{-1} (see Lemma* B.7*); and* (2) *if $\widetilde{\beta} \subset W_{-1}$ is a lift of a curve $\beta \subset W$, then $h(\widetilde{\beta})$ is a lift of β to $W_{-1,\mathrm{new}}$.*

Proof. Since the pair $\gamma_0^{\mathrm{new}} \setminus \Omega, \gamma_1^{\mathrm{new}} \setminus \Omega$ coincides with $\gamma_0 \setminus \Omega,\gamma_1 \setminus \Omega$, condition (1) uniquely specifies $h \mid W_{-1} \setminus \Omega_{-1}$.

Let us now extend $f\colon W \dashrightarrow W$ to a homeomorphism $f\colon W \to W$ mapping γ_0 to γ_1. The extension changes $f_{-1} \mid W_{-1} \setminus \Omega_{-1}$ and $f_{-1,\mathrm{new}} \mid W_{-1,\mathrm{new}} \setminus \Omega_{-1,\mathrm{new}}$ but does not affect $f_{-1} \mid \Omega_{-1}$ and $f_{-1,\mathrm{new}} \mid \Omega_{-1,\mathrm{new}}$. Therefore, it is sufficient to prove the theorem under the assumption that $f\colon W \to W$ is a homeomorphism.

Since every antirenormalization is an iteration of prime antirenormalizations (see Lemma B.3), we can further assume that f_{-1} and $f_{-1,\mathrm{new}}$ are prime antirenormalizations. By Lemmas B.4 and B.6 both f_{-1} and $f_{-1,\mathrm{new}}$ are naturally conjugate to

$$f_{-1,\bullet}/\langle\tau_{-1}\rangle = f_{-1,+}/\langle\tau_{-1}\rangle : W_\bullet/\langle\tau_{-1}\rangle \to W_\bullet/\langle\tau_{-1}\rangle,$$

which is independent of the choice of γ_0,γ_1. It remains to observe that the conjugacy between f_{-1} and $f_{-1,\mathrm{new}}$ satisfies condition (2); see Remark B.5.

Corollary B.9 (Lifting condition)**.** *The curves $\gamma_{0,\mathrm{new}}$ and $\gamma_{1,\mathrm{new}}$ have unique lifts*

$$h^{-1}\left(\gamma_0^{\mathbf{s}/\mathfrak{q},\mathrm{new}}\right), h^{-1}\left(\gamma_{\mathfrak{p}}^{\mathbf{s}/\mathfrak{q},\mathrm{new}}\right) \subset W_{-1} \tag{B.4}$$

(see (B.2)*) such that the pair* (B.4) *coincides with $\gamma_0^{\mathbf{s}/\mathfrak{q}}, \gamma_{\mathfrak{p}}^{\mathbf{s}/\mathfrak{q}}$ in $W_{-1} \setminus \Omega$. Moreover,* (B.4) *is a dividing pair.*

Remark B.10 (General lifting condition)*.* Suppose that **s** is a sequence such that (**A**,**B**) or (**B**,**A**) appears infinitely many times in both **s**[≥ 0] and **s**[≤ 0]. In the arXiv version of the paper we proved that if $\beta_0,\beta_1 = f(\beta_0)$ is a pair of curves respecting (Q,γ_0,γ_1), then all lifts of β_0,β_1 in $W_{\mathbf{s}}$ exist, are pairwise disjoint, and land at 0. In particular, this implies Corollary B.9 and Theorem B.8.

B.1.7. *Walls.* Let us view W as a subset of C. A *wall around* 0 *respecting* γ_0,γ_1 is either a closed annulus or a simple closed curve $Q \subset U \cap V$ such that

(1) $\mathbb{C} \setminus Q$ has two connected components, and, moreover, denoting by Ω the bounded component of $\mathbb{C} \setminus Q$, we have $0 \in \Omega$;

(2) $\gamma_0 \cap Q$ and $\gamma_1 \cap Q$ are connected;

(3) if $x \in \Omega$, then $f^{\pm 1}(x) \in Q \cup \Omega$.

In other words, points in W do not jump over Q under the iteration of f. If Q is a simple closed curve, then f restricts to an actual homeomorphism $f : \Omega \to \Omega$.

Remark B.11. Note that a wall contains a fence; see §B.1.5. Therefore, in the statement of Theorem B.8 we can replace a fence with a wall.

For a sequence $\mathbf{s} \in \{\mathbf{A},\mathbf{B}\}^{\mathbb{Z}}$ we denote by $Q_{\mathbf{s}}[i]$ the closure of the preimage of Q under $\pi\colon W_{\mathbf{s}}[i] \to W$; we set $Q_{\mathbf{s}} := \bigcup_{i\in\mathbb{Z}} Q_{\mathbf{s}}[i]$.

Lemma B.12. *The set $Q_{\mathbf{s}}$ is connected. The closure of the connected component of $W_{\mathbf{s}} \setminus Q_{\mathbf{s}}$ containing* 0 *is $\Omega_{\mathbf{s}}$.*

Proof. The proof follows from the definition: since points in Ω do not jump over Q every $Q_{\mathbf{s}}[i]$ intersects $Q_{\mathbf{s}}[i+1]$, therefore $Q_{\mathbf{s}}$ is connected and the claim follows.

Suppose $f_{-1} : W_{\mathbf{s}/q} \to W_{\mathbf{s}/q}$ is an antirenormalization of f and suppose W has a wall Q (respected by γ_0,γ_1,f) enclosing Ω. The image of $Q_{\mathbf{s}}$ in $W_{\mathbf{s}/q}$ is called the *full lift $Q_{\mathbf{s}/q}$ of Q*. Similarly, we denote by $\Omega_{\mathbf{s}/q}$ the image of $\Omega_{\mathbf{s}}$ in $W_{\mathbf{s}/q}$. We say that Q is an N-*wall* if it takes at least N iterates of $f^{\pm 1}$ for points in Ω to cross Q. The next lemma follows by definition.

Lemma B.13. *If Q is an N-*wall*, then $Q_{\mathbf{s}/q}$ is an* $(N-1)\min\{a,b\}$-*wall.*

Since for a periodic combinatorics $\min\{a,b\} \ge 2$ (see (A.5)), we have the following.

Corollary B.14. *Suppose $f_{-1} : W_{\mathbf{s}/q} \to W_{\mathbf{s}/q}$ is an antirenormalization of f associated with a periodic combinatorics; see* §A.4. *Then a lift of a* 2-*wall (respected by γ_0,γ_1,f) is again a* 2-*wall.*

Remark B.15. Antirenormalization can easily be defined for a partial branched covering $f_0 : (W,0)\ \ (W,0)$ of any degree. In this case it is natural to assume that γ_0 does not contain a critical point of f. To apply Theorem B.8, it is sufficient to assume that there is a *univalent* fence Q (respected by γ_0,γ_1,f) enclosing Ω such that $f \mid Q \cup \Omega$ has degree one. The antirenormalization is robust with respect to replacing γ_0,γ_1 with a new pair $\gamma_0^{\mathrm{new}}, \gamma_1^{\mathrm{new}}$ as above.

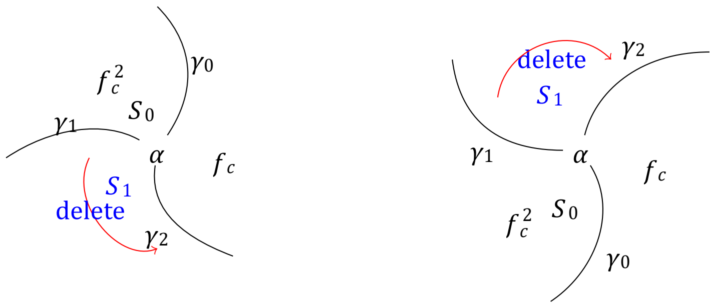

Figure 30. Possible local dynamics at the α-fixed point.

Appendix C. The Molecule Conjecture

Let us denote by Mol the main molecule of the Mandelbrot set; i.e., the smallest closed subset of M containing the main hyperbolic component as well as all hyperbolic components obtained from the main component via parabolic bifurcations; see [DH1,L2] for the background on the Mandelbrot set. In this appendix we write $f_c(z) = z^2 + c$.

C.1. **Branner-Douady maps.** Let us denote by $L_{p/q}$ the primary p/q-limb of the Mandelbrot set, and let us denote by $M_{p/q} \subset L_{p/q}$ the p/q-satellite small copy of M. We also write $L_{0/1} = M_{0/1} = M$.

In [BD] Branner and Douady constructed a partial surjective continuous map $R_{\mathrm{prm}} : L_{1/3}\ \ L_{1/2}$ such that its inverse $R_{\mathrm{prm}}^{-1} : \mathcal{L}_{1/2} \to L_{1/3}$ is an embedding.

This construction could be easily generalized to a continuous map $R_{\mathrm{prm}} : L_{p/q}$ $L_{R_{\mathrm{prm}}(p/q)}$, where (compare to (A.2))

$$R_{\mathrm{prm}}(p/q) = \begin{cases} \frac{p}{q-p} & \text{if } 0 < \frac{p}{q} \leq \frac{1}{2}, \\ \frac{2p-q}{p} & \text{if } \frac{1}{2} \leq \frac{p}{q} < 1, \end{cases}$$

as follows. Recall that $c \in L_{p/q}$ if and only if in the dynamical plane of f_c there are exactly q external rays landing at the α-fixed point and the rotation number of these rays is p/q; i.e., if γ is a ray landing at α, then there are $p-1$ rays landing at α between γ and $f_c(\gamma)$ counting counterclockwise.

Choose an external ray γ_0 landing at α in the dynamical plane of f_c with $c \in L_{p/q}$. Define $\gamma_1 = f_c(\gamma_0)$ and $\gamma_2 = f_c(\gamma_1)$. Denote by S_0 the open sector between γ_0 and γ_1 not containing γ_2; see Figure 30. Similarly, let S_1 be the open sector between γ_1 and γ_2 not containing γ_0. We assume that γ_0 is chosen such that S_1 does not contain the critical value; thus S_1 has two conformal lifts, one of them is S_0 and we denote by S_0' the other. If $S_1 \supset S_0'$, then replace S_0' by its unique lift in $\mathbb{C} \setminus S_1$.

Let us delete S_1, glue γ_1 and γ_2 dynamically so $\gamma_1 \ni x \sim f(x) \in \gamma_2$, and iterate f_c twice on S_0. We obtain a new map denoted by $\bar f_c : \mathbb{C} \setminus S_0' \to \mathbb{C}$. The *filled-in Julia set* $\overline{K}_c$ of $\bar f_c$ is the set of points with bounded orbits that do not escape to S_0'. The set $\overline{K}_c$ is connected if and only if 0 does not escape to S_0'; in this case the new local dynamics of $\bar f_c$ at α has rotation number $\mathcal{R}_{\text{prm}}(p/q)$ and, moreover, $\bar f_c$ is hybrid equivalent to a quadratic polynomial $f_{\mathcal{R}_{\text{prm}}(c)}$ with $c \in L_{\mathcal{R}_{\text{prm}}(p/q)}$. This defines the map $\mathcal{R}_{\text{prm}} : L_{p/q} \to L_{\mathcal{R}(p/q)}$. Note that $\mathcal{R}_{\text{prm}} : M_{1/2} \to M_{0/1} = M$ becomes the Douady-Hubbard straightening map $M_{1/2} \to M$ of the basilica satellite copy of the Mandelbrot set.

In general, $\mathcal{R}_{\text{prm}} : L_{p/q} \to L_{\mathcal{R}_{\text{prm}}(p/q)}$ depends on the choice of γ_0. However, if $c \in M_{p/q}$, then $\mathcal{R}_{\text{prm}}(c) \in M_{\mathcal{R}_{\text{prm}}(p/q)}$ and $\mathcal{R}_{\text{prm}} : M_{p/q} \to M_{\mathcal{R}_{\text{prm}}(p/q)}$ coincides with the canonical homeomorphism between small copies of the Mandelbrot set.

Remark C.1. The Branner–Douady surgery has also been studied by Riedl [R]; he showed, in particular, that every dyadic Misiurewicz parameter is connected through a simple arc (vein) in the Mandelbrot set to the origin.

C.2. **The molecule and the fast molecule maps.** Denote by $\mathbf{\Delta}$ the main hyperbolic component of M. Recall that a parameter $c \in \partial\mathbf{\Delta}$ is parametrized by the multiplier $\mathbf{e}(\theta(c))$ of its non-repelling fixed point. We define *the molecule map* $\mathcal{R}_{\text{prm}} : \mathcal{M} \dashrightarrow \mathcal{M}$ such that

- $\mathcal{R}_{\text{prm}} : L_{p/q} \to L_{\mathcal{R}_{\text{prm}}(p/q)}$ is the Branner–Douady renormalization map for $p/q \neq 0/1$ and for some choice of γ_0; and
- if $c \in \partial\mathbf{\Delta}$, then $\mathcal{R}_{\text{prm}}(c)$ is such that

$$\theta(\mathcal{R}_{\text{prm}}(c)) = \begin{cases} \frac{\theta(c)}{1-\theta(c)} & \text{if } 0 \leq \theta(c) \leq \frac{1}{2}, \\ \frac{2\theta(c)-1}{\theta(c)} & \text{if } \frac{1}{2} \leq \theta(c) \leq 1. \end{cases}$$

Siegel parameters of periodic type are exactly periodic points of $\mathsf{R}_{\mathrm{prm}} \mid \partial\mathbf{\Delta}$ (Lemma A.2). Furthermore, for a satellite copy of the Mandelbrot set M_s, there is an $n \geq 1$ such that $\mathsf{R}^n_{\mathrm{prm}} : \mathrm{M}_s \to \mathrm{M}$ is the Douady-Hubbard straightening map.

The map $\mathsf{R}_{\mathrm{prm}} : \mathcal{M} \dashrightarrow \mathcal{M}$ is combinatorially modeled by $Q(z) := z(z+1)^2$; see Figure 31. The latter map has a unique parabolic fixed point at 0. The attracting basin of 0 contains exactly one critical point of Q. The second critical point is a preimage of 0. Denote by F the invariant Fatou component of Q. We can extend $\mathsf{R}_{\mathrm{prm}}$ to $\mathbf{\Delta}$ so that $\mathsf{R}_{\mathrm{prm}} \mid \overline{\mathbf{\Delta}}$ is conjugate, say by π, to $Q \mid F$. Then π extends uniquely to a monotone continuous map $\pi : \mathsf{Mol} \to K_Q$ semiconjugating $\mathsf{R}_{\mathrm{prm}} \mid \mathsf{Mol}$ and $Q \mid K_Q$, where K_Q is the filled-in Julia set of Q:

$$\begin{array}{ccc} \mathsf{Mol} & \xrightarrow{\mathsf{R}_{\mathrm{prm}}} & \mathsf{Mol} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ K_Q & \xrightarrow{Q} & K_Q \end{array}$$

If the MLC-conjecture holds, then π is a homeomorphism.

For every $c \in \partial\Delta \setminus \{\text{cusp}\}$ define $\mathrm{n}(c) := \mathrm{n}(\theta_c)$, where θ_c is the rotation number of f_c and $\mathrm{n}(\theta)$ is specified by $R_{\mathrm{fast}}(\theta) = R_{\mathrm{prm}}^{\mathrm{n}(\theta)}(\theta)$; see §A.2. For every $c \in \mathrm{L}_{p/q}$ define $\mathrm{n}(c) := \mathrm{n}(c_{p/q})$, where $c_{p/q}$ is the root of $\mathrm{L}_{p/q}$. The *fast Molecule map* is a partial map on M defined by

$$\mathsf{R}_{\mathrm{fast}}(c) = \mathsf{R}^{\mathrm{n}(c)}_{\mathrm{prm}}(c).$$

The restriction $\mathsf{R}_{\mathrm{fast}} \mid \partial\mathsf{Mol} \setminus \{\text{cusp}\}$ is continuous but it does not extend continuously to the cusp: $\mathsf{R}_{\mathrm{fast}}(\partial\mathrm{M}_{1/n}) = \partial\mathrm{M}$.

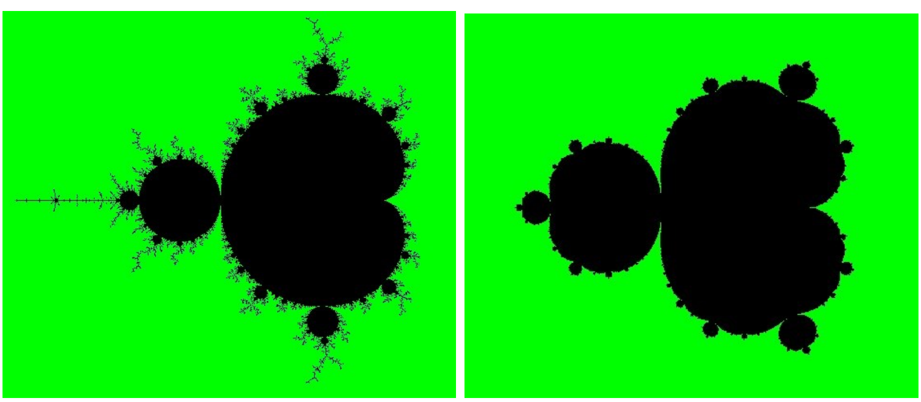

Figure 31. Left: the Mandelbrot set. Right: the filled Julia set of $Q(z) = z(z+1)^2$.

C.3. **The molecule conjecture.** Given a renormalization operator $\mathcal{R}\colon \mathcal{B} \quad \mathcal{B}$, its *renormalization horseshoe* is the set of points in $\mathcal{B}$ with bi-infinite precompact orbits. We conjecture that there is a pacman renormalization operator $\mathcal{R}_{\text{fast}}\colon \mathcal{B}_{\mathcal{M}ol} \to \mathcal{B}_{\mathcal{M}ol}$ with the following properties. The operator $\mathcal{R}_{\text{fast}}$ is hyperbolic and piecewise analytic with one-dimensional unstable direction such that its renormalization horseshoe $\mathcal{R}_{\text{fast}}\colon \mathcal{H}_{\mathcal{M}ol} \to \mathcal{H}_{\mathcal{M}ol}$ is compact and combinatorially associated with $\mathcal{R}_{\text{fast}} \mid \mathcal{M}ol \setminus \{\text{cusp}\}$ as follows.

There is a continuous surjective map $\rho\colon \mathcal{H}_{\mathcal{M}ol} \to \mathcal{M}ol$ that is a semiconjugacy away from the cusp:

$$\begin{array}{ccc}
\mathcal{H}_{\mathcal{M}ol} \setminus \rho^{-1}(\text{cusp}) & \xrightarrow{\ \mathcal{R}_{\text{fast}}\ } & \mathcal{H}_{\mathcal{M}ol} \setminus \rho^{-1}(\text{cusp}) \\
\downarrow & \rho\rho & \downarrow \\
\partial\mathcal{M}ol \setminus \{\text{cusp}\} & \xrightarrow{\ \mathcal{R}_{\text{fast}}\ } & \partial\mathcal{M}ol \setminus \{\text{cusp}\}
\end{array}$$

Denote by $\partial^{\text{irr}}\mathcal{M}ol$ the set of non-parabolic parameters in $\partial\mathcal{M}ol$. Conjecturally, $\mathcal{R}_{\text{fast}} \mid \mathcal{H}_{\mathcal{M}ol}$ is the natural extension of $\mathcal{R}_{\text{fast}} \mid \partial\mathcal{M}ol \setminus \{\text{cusp}\}$ compactified by adding limits to parabolic parameters at all possible directions. Such a construction is known as a parabolic enrichment; see [La,D2].

all pacmen inThe space $\mathcal{B}\mathcal{F}_{\mathcal{M}sol}$are hybrid conjugate tohas a codimension-one stable lamination (f_c in neighborhoods of their “mother$\mathcal{F}_{cs})_{c\in\mathcal{M}ol}$ such that

c

hedgehogs”; see §C.4. For every $f \in \mathcal{H}_{\mathcal{M}ol}$, the leaf $\mathcal{F}^s_{\rho(f)}$ is a stable manifold of $\mathcal{R}_{\text{fast}}$ at f. The unstable manifold of $\mathcal{R}_{\text{fast}}$ at f is parametrized by a parabolic enrichment of a neighborhood of $\rho(f)$. Locally, $\mathcal{R}_{\text{fast}}$ can be factorized as an iterate

of $\mathcal{R}_{\text{prm}}\colon \mathcal{B}_{\mathcal{M}ol} \to \mathcal{B}$
$\rho^{-1}(\text{cusp})$. $_{\mathcal{M}ol}$; however, the latter operator has parabolic behavior at

The Molecule Conjecture contains both Theorem 7.7 (for periodic type parameters from $\partial\boldsymbol{\Delta}$) and the Inou-Shishikura theory [IS] (for high type parameters from $\partial\boldsymbol{\Delta}$). It also implies the local connectivity of the Mandelbrot set for all parameters on the main (and thus any) molecule.

C.4. **Conjecture on the upper semicontinuity of the mother hedgehog.** A closely related conjecture is the upper semicontinuity of the mother hedgehog. For a Siegel parameter $c \in \partial\boldsymbol{\Delta}$, consider the closed Siegel disk $\overline{Z}_c$ of f_c; if f_c has a Cremer point, then let $\overline{Z}_c := \{\alpha\}$. If $\overline{Z}_c$ contains a critical point, then we set $H_c := \overline{Z}_c$. Otherwise, f_c has a *hedgehog* (see [PM]): a compact closed connected filled-in forward invariant set $H' \supsetneqq \overline{Z}_c$ such that $f_c\colon H' \to H'$ is a homeomorphism.

We define H_c to be the *mother hedgehog* (see [Chi]): the closure of the union of all of the hedgehogs of f_c.

Recall that the filled-in Julia set K_g of a polynomial depends upper semicontinuously on g. Viewing H_c as an indifferent-dynamical analogue of K_g, we conjecture the following.

Conjecture C.2. *The mother hedgehog H_c depends upper semicontinuously on c.*

For bounded type parameters (i.e., when H_c is a Siegel quasidisk) Conjecture C.2 follows from the continuity of the Douady-Ghys surgery.

Conjecture C.2 can be adjusted for parabolic parameters $c \in \mathbf{\Delta}$ as follows. Let A_c be the immediate attracting basin of the parabolic fixed point α. Then there is a choice of a valuable flower H_c with $\overline{H}_c \subset A_c \cup \{\alpha\}$ such that H_c depends upper semicontinuously on $c \in \partial\mathbf{\Delta}$. In particular, H_c contains the union of all limiting mother hedgehogs for perturbations of f_c.

Similarly, Conjecture C.2 can be adjusted for all parameters in ∂Mol. Our result on the control of the valuable flower (see Theorem 8.2) can be thought of as a step towards this general conjecture.

Conjecture C.2 and its generalizations describe in a convenient way how an attracting fixed point bifurcates into repelling. An important consequence is control of the post-critical set: if a perturbation of f_c is within Mol, then the new postcritical set is within a small neighborhood of H_c. A statement of this sort (for parabolic parameters approximating a Siegel polynomial) was proven by Buff and Ch´eritat; see [BC, Corollary 4]. This was a key ingredient in constructing a Julia set with positive measure.

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Figures 2, 8, 26, 31 are made with W. Jung's program *Mandel*.

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