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Dynamics of a stochastic prey–predator system with prey refuge, predation fear and its carry-over effects

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ABSTRACT

This paper proposes and studies the dynamics of a Holling-type II predator–prey interaction system that incorporates the following three components: (1) a prey refuge; (2) predation fear and its carry-over effects; and (3) environmental noise in both prey and predator populations. The impacts of those three components are studied through both rigorous analysis and numerical simulations. Analytical results show that the introduction of prey refuge, predation fear, and its carry-over effects can generate Hopf bifurcation. It is found that increasing prey refuge and predation fear effect in a reasonable region can stabilize the system, while excessive refuge strength would lead to the extinction of predators. The theoretical results of the corresponding system with environmental noise include (1) sufficient conditions for the existence of a unique ergodic stationary distribution of the SDE system by constructing appropriate stochastic Lyapunov functions; (2) the explicit probability density function of the distribution by solving the Fokker–Planck equation; and (3) the extinction conditions of prey and/or predator species at an exponential rate in the long run. Our work shows that the proposed model, incorporating prey refuge, predation fear, carry-over effect, and environmental noise, exhibits rich and complex dynamic behaviors. Moreover, our results indicate that small environmental noise can save the prey and predator from extinction, while large environmental noise can drive the species to extinction. These interesting findings provide more perspectives on the protection and control of species in complex communities.

1. Introduction

The dynamic interplay between prey and predator has been a major subject of interest in ecology for a long time and will continue to be so, given their ubiquitous presence and relevance in the field [1]. Prey animals use various anti-predator tactics to avoid being eaten, such as seeking safe shelter [2], changing their habitat, reducing foraging frequency [3], and altering their metabolism [4]. The non-lethal effects of predation, caused by the predator's influence on prey behavior and characteristics, have the potential to impact the entire prey population over time [5]. For instance, prey animals that are fearful of being eaten may forage less and undergo physiological changes that affect their reproductive success (see [6–16]). Research by Zanette et al. [6] has shown that the fear of predators can reduce offspring generation in song sparrows, as anti-predator behavior lowers both birth rates and survival rates. In an experiment with Drosophila melanogaster and mantid predators, Elliott et al. [8] found that fear disrupted population dynamics and increased the likelihood of extinction by sevenfold, highlighting the impact of fear on fitness in species with limited sociality.

Many researchers constructed mathematical models to investigate prey and predator interaction, how the fear of predators and group defense [13,17], or prey refuge [14,18,19], or additional food [15] are interrelated and affect dynamical behaviors of the populations. The mathematical analyses of [7] revealed that large levels of fear can maintain the predator–prey system by eliminating the presence of periodic solutions, but comparatively low levels of fear can create numerous limit cycles via subcritical Hopf bifurcations, resulting in bi-stability phenomena. Halder et al. [11] studied the effects of indirect prey–predator contact on the prey population by analyzing the influence of predation fear combined with the Allee effect caused by predation. Cong et al. [16] developed a three-species food chain system that incorporates the cost and utility of anti-predator behaviors. They showed that the predator's fear impact can cause the system to transition from chaotic to stable dynamics. To assess the impact of fear on prey demography, Zhang et al. [18] developed a Holling-type-II predation model including a prey refuge. They revealed that the model contains a Hopf bifurcation and a limit cycle and that the fear effect has

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no influence on prey population density. On the other side, the predation fear's carry-over effects [20] have lately been used in ecological and evolutionary aspects, and they may be applied to a wide range of scenarios, e.g. [5,21,22]. Predation fear and its carry-over effects in prey–predator interaction with Holling-type-I functional response were investigated by Sasmal and Takeuchi [5] using continuous time population models.

Moreover, environmental noise is an inevitable factor that affects species models in the natural world [23–25]. The noises can be classified as additive or multiplicative depending on the source. The former is not governed by the system and may be brought directly into it, whereas the latter is connected to system parameters and variables. One advantage of using multiplicative noises is that they can ensure the nonnegativity of the solution. Additionally, the presence of multiplicative noise can lead to two main peculiarities: an absorbing barrier at zero population density and the occurrence of anomalous fluctuations [26,27]. Fiasconaro et al. [23] investigated noise-induced pattern formation in a population dynamical model of three interacting species in coexistence and found nonmonotonic patterns influenced by multiplicative noise intensity and evolution time. Chichigina et al. [24] studied dynamical system stability in multiplicative one-sided pulse noise with hidden periodicity, finding high periodicity stable and low periodicity unstable. By adding a source of multiplicative noise to the diffusion equation for the picophytoplankton biomass concentration, Valenti et al. [25] can take into account random fluctuations of environmental variables.

Stochastic noises are being employed more and more in the transient dynamics of multidisciplinary physics models [28–35]. Noise not only leads to the disorder of the dynamics but also can have a positive impact, such as noise induced resonances and noise enhanced stability (NES) [36–39]. For instance, Mikhaylov et al. [33] investigated the stochastic resonance phenomenon in a metal-oxide memristive device and analyzed the effect of white Gaussian noise on the sub-threshold sinusoidal driving signal. Guarcello et al. [34] proposed a threshold detector for noise-distributed fluctuations based on a Josephson junction. In [35], the authors studied non-Gaussian noise sources' impact on transient Josephson junction dynamics, finding nonmonotonic escape time behavior influenced by intensity and frequency. Furthermore, the NES phenomenon is that the presence of steady-state and multiplicative noise sources causes the non-monotonic behavior of the mean first passage time (MFPT) as a function of the noise intensity. Bonanno et al. [36] analyzed NES effect on escape time statistical properties, examining the correlation between noise sources and stochastic volatility in a model with stochastic volatility. Valenti et al. [37] investigated ohmic dissipation and monochromatic driving's impact on quantum system stability, and studied the quantum noise-enhanced stability phenomenon. In [38], the study examined volatility, risk, and instability in financial markets, using mean first hitting time as a price stability indicator, showing nonmonotonic behavior.

It is obvious that the essential structure and factors of species growth, such as resources and vital rates—birth, death, and immigration, vary non-deterministically as a result of environmental variations (e.g., seasonal impacts, temperature, social cycles, etc.) [40]. It is a crucial aspect to consider when constructing mathematical models since it adds an additional degree of realism compared to its deterministic version (see [41–47] and references therein). Cai and Mao [44] explored the dynamics of a two-dimensional foraging arena model in the face of environmental disruption. They investigated the existence and uniqueness of a positive solution of the model, as well as the long-term dynamical behaviors. Majumder et al. [46] built a stochastic predator–prey-parasite model by randomly perturbing three biologically significant parameters, and investigated the routes of population extinction in the model. In [47], the authors presented a stochastic prey–predator model with a fear function based on predators' induced dread, in which the predators compete intra-specifically. They obtained sufficient conditions for both species' extinction, non-persistence, and weak persistence. All of these findings show that environmental noise has a significant impact on ecosystems. Therefore, it is an interesting problem to try to qualitatively describe the combined impacts of predation fear, its carry-over effects, prey refuge, and environmental noise on population survival.

Motivated by the explanation above and thanks to the insightful works, we constructed a stochastic predation system with prey refuge subject to predation fear and its carry-over effects, and explore, in a stochastic environment, how the interplay among prey refuge, predation fear, and its carry-over effects can affect the dynamic pattern of the system, especially for the impact on the permanence of populations. Here, we assume that stochastic perturbations are of multiplicative white noise type that is directly proportional to the prey and predator species, then the stochastic system can be described as the following form:

$$\begin{cases} \mathrm{d}N(t) = \left(\underbrace{\gamma N}_{\text{birth}} \underbrace{\dfrac{1+cN}{1+cN+fP}}_{\text{fear and carry-over effect}} - \underbrace{d_1}_{\text{natural death}} N \right. \\ \qquad \left. - \underbrace{d_2}_{\text{density-dependent death}} N^2 - \dfrac{\beta(1-m)NP}{1+a(1-m)N} \right) \mathrm{d}t + \sigma_1 N \mathrm{d}B(t), \\ \mathrm{d}P(t) = \left(\alpha P \underbrace{\dfrac{\beta(1-m)N}{1+a(1-m)N}}_{\text{type-II response incorporating a prey refuge}} - \underbrace{d_3}_{\text{natural death}} P \right) \mathrm{d}t + \sigma_2 P \mathrm{d}B(t), \end{cases} \tag{1}$$

its state space is the first quadrant

$$\mathbb{R}^2_+ = \{(N(t), P(t)) : N(t) > 0,\ P(t) > 0\}, \tag{2}$$

and $N(t)$, $P(t)$ are the densities of prey and predator populations at time t, respectively. The parameters γ, c, f, d_1, d_2, d_3, α and β are all positive constants, γ the maximum birth rate of prey without predators, c the fear-related carry-over effect parameter, quantified by the parameter f, d_2 the density-dependent death rate of prey, d_1 and d_3 the natural death rates of prey and predator, respectively, α the conversion factor that indicates the amount of newly born predators for each captured prey, $\frac{\beta}{a}$ the maximum quantity of prey that can be devoured by a predator per unit time, $\frac{1}{a}$ the density of prey required to reach one half that rate.

The term $\frac{\beta N}{1+aN}$ is the Holling-type II functional response. $(1-m)N$ denotes the number of prey available to predators, where m is in the range $[0,1)$ and it is the prey refuge protection rate. The standard one-dimensional independent Brownian motion described over the complete probability space $\left(\mathbb{R}^2_+, \mathbf{B}(\mathbb{R}^2_+), \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P}\right)$ is denoted by $B(t)$ with $B(0) = 0$, where $\mathbf{B}(\mathbb{R}^2_+)$ denotes the Borel σ-algebra on $\mathbb{R}^2_+$, and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions (i.e., it is rightly continuous and increasing while $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets). $\sigma_i^2 > 0\,(i = 1,2)$ denote the intensities of the white noise $B(t)$.

The remaining sections of our paper are organized as follows. In Section 2, we investigate the fundamental dynamical results of the deterministic system of (1), such as the boundedness and asymptotic stability of equilibria, as well as the existence of Hopf bifurcation. Section 3 proves that the stochastic system (1) has a unique ergodic stationary distribution; we obtain the exact expression of the probability density function of the system (1) by solving the corresponding Fokker–Planck equation; and we provide the conditions that lead to the extinction of both prey and predator species. In Section 4, we present numerical simulation findings that show some of the potential roles that predation fear, its carry-over effects, prey refuge, and environmental noises may play in predator–prey interactions. Section 5 ends with a summary of the findings. The deterministic system's results are detailed in Appendix.

2. Dynamics of the deterministic version of system (1)

Without the environmental noises, i.e., $\sigma_1 = \sigma_2 = 0$, two-species prey–predator system becomes,

$$\begin{cases} \dfrac{\mathrm{d}N(t)}{\mathrm{d}t} = \dfrac{\gamma N(1+cN)}{1+cN+fP} - d_1 N - d_2 N^2 - \dfrac{\beta(1-m)NP}{1+a(1-m)N}, \\ \dfrac{\mathrm{d}P(t)}{\mathrm{d}t} = \dfrac{\alpha\beta(1-m)NP}{1+a(1-m)N} - d_3 P, \end{cases} \tag{3}$$

and the fundamental dynamical properties of (3) are summarized as follows.

Theorem 2.1. *For the system (3), $\mathbb{R}^2_+$ is positively invariant, and every solution of the system is ultimately bounded in $\mathbb{R}^2_+$ with the properties as follows:*

$$\lim_{t\to\infty}\sup N(t) \le \frac{\gamma - d_1}{d_2},$$

$$\lim_{t\to\infty}\sup N(t) + \frac{1}{\alpha}P(t) \le \begin{cases} \dfrac{\gamma - d_1}{d_2} & \text{if } d_3 > \gamma - d_1, \\ \dfrac{(\gamma - d_1 + d_3)^2}{4 d_2 d_3} & \text{if } d_3 \le \gamma - d_1. \end{cases}$$

In addition, we have the following:

(i) If $\frac{\gamma - d_1}{d_2} < \frac{d_3}{\alpha\beta(1-m)}$, then $\lim\sup_{t\to\infty} P(t) = 0$.

(ii) If $\gamma < \min\{d_1, d_3\}$, then $\lim_{t\to\infty} N(t) + \frac{1}{\alpha}P(t) = 0$.

(iii) If there exists a positive number $w < \dfrac{\frac{\gamma - \max\{d_1, d_3\}}{d_2} - \frac{1}{c}}{\max\{\frac{f}{c}, \frac{1}{\alpha}\}}$ such that

$$\lim_{t\to\infty}\inf N(t) + \frac{1}{\alpha}P(t) \ge w \quad \textit{for any} \quad N(0) + \frac{1}{\alpha}P(0) > w.$$

The detailed proof of Theorem 2.1 can be found in Appendix.

Based on the existence of c, f, m and d_1, we discuss the dynamics of the system (3) in the following three cases:

- **Case (a)**: $c = f = m = d_1 = 0$. In the absence of predation fear, its carry-over effects, prey refuge, and natural death rate, the predation system (3) is the classical Holling type II predator–prey model with a logistic growth in prey population. If $\frac{\gamma}{d_2} < \frac{d_3}{\alpha\beta - a d_3}$, the boundary equilibrium E_1 is globally asymptotically stable; if $\frac{d_3}{\alpha\beta - a d_3} < \frac{\gamma}{d_2} < \frac{\alpha\beta + a d_3}{a(\alpha\beta - a d_3)}$, the system has a unique positive equilibrium E^* which is globally asymptotically stable.
- **Case (b)**: $c = f = m = 0$, $d_1 \neq 0$. In the absence of predation fear, its carry-over effects, and prey refuge, the predator–prey relationship in the predation system (3) follows the Holling type II model. If $0 < \frac{\gamma - d_1}{d_2} < \frac{d_3}{\alpha\beta - a d_3}$, the boundary equilibrium E_1 is globally asymptotically stable; if $\frac{d_3}{\alpha\beta - a d_3} < \frac{\gamma - d_1}{d_2} < \frac{\alpha\beta + a d_3}{a(\alpha\beta - a d_3)}$, the system has a unique positive equilibrium E^* which is globally asymptotically stable.
- **Case (c)**: $c = f = 0$, $m \neq 0$, $d_1 \neq 0$. In this case, the predation system (3) has no predation fear and its carry-over effects in prey. The boundary equilibrium E_1 is globally asymptotically stable if $0 < \frac{\gamma - d_1}{d_2} < \frac{d_3}{(\alpha\beta - a d_3)(1-m)}$; if $\frac{d_3}{(\alpha\beta - a d_3)(1-m)} < \frac{\gamma - d_1}{d_2} < \frac{\alpha\beta + a d_3}{a(\alpha\beta - a d_3)(1-m)}$, the unique positive equilibrium E^* is globally asymptotically stable.

The dynamics of Cases (a)–(c) can be summarized by the following theorem:

Theorem 2.2. *The system (3) always has a trivial equilibrium $E_0 = (0,0)$, and a boundary $E_1 = (N^*, 0)$ and the unique positive equilibrium $E^* = (N^*, P^*)$ in Cases (a)–(c), respectively. Sufficient conditions for the existence and stability of these equilibria are summarized in Table 1.*

For proof of this theorem, see Appendix.

Since the system (3) can be reduced to the system given by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N(\gamma - d_1 - d_2 N), \quad P = 0,$$

and

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -d_3 P, \quad N = 0.$$

Therefore, the system (3) always has a trivial extinction equilibrium $E_0 = (0,0)$ and a boundary equilibrium $E_1 = (\frac{\gamma - d_1}{d_2}, 0)$ if $\frac{\gamma}{d_1} > 1$.

Theorem 2.3. *For the system (3), a sufficient condition for the persistence of predator is $\frac{\gamma - d_1}{d_2} > \frac{d_3}{(\alpha\beta - a d_3)(1-m)} > 0$.*

For proof of this theorem, see Appendix.

Notice that the equilibria of the system (3) satisfy the following equations:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N\left(\frac{\gamma(1+cN)}{1+cN+fP} - d_1 - d_2 N - \frac{\beta(1-m)P}{1+a(1-m)N}\right) = 0 \Rightarrow N = 0$$

or

$$l_1 P^2 + l_2 P + l_3 = 0, \tag{4}$$

where

$$\begin{aligned} l_1 &= \beta f(1-m) > 0, \\ l_2 &= \beta(1-m)(1+cN) + f(d_1 + d_2 N)(1 + a(1-m)N), \\ l_3 &= (d_1 + d_2 N - \gamma)(1+cN)(1+a(1-m)N), \end{aligned} \tag{5}$$

and

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P\left(\frac{\alpha\beta(1-m)N}{1+a(1-m)N} - d_3\right) = 0 \Rightarrow N = \frac{d_3}{(\alpha\beta - a d_3)(1-m)} \text{ or } P = 0.$$

As $l_1 > 0$, Eq. (4) has exactly one positive root if $l_3 < 0$. A necessary condition for N^* to be positive is $d_3 < \frac{\alpha\beta}{a}$. Then the coexistence equilibrium $E^* = (N^*, P^*)$ is feasible provided,

$$\frac{\gamma - d_1}{d_2} > \frac{d_3}{(\alpha\beta - a d_3)(1-m)} > 0. \tag{6}$$

In case, the former inequality is not satisfied, then Eq. (4) has either two or no positive roots. In fact, $l_2 > 0$ always holds as $N^* > 0$. Thus, the system (3) has a unique interior equilibrium $E^* = (N^*, P^*)$, where

$$N^* = \frac{d_3}{(\alpha\beta - a d_3)(1-m)}, \quad P^* = \frac{-l_2 + \sqrt{l_2^2 - 4 l_1 l_3}}{2 l_1}. \tag{7}$$

Remark 1. The density of the prey population does not rely on the cost of predation fear f and its carry-over effect c when the prey and predator populations coexist at $E^* = (N^*, P^*)$, but the density of the predator population is influenced by both the variables (see Fig. 4(a)). Moreover, the strength of prey refuges m influences the density of two populations (see Fig. 1).

In order to illustrate the impact of prey refuge m on the change of coexistence equilibrium $E^* = (N^*, P^*)$ of the system (3), we also present the diagram with varying m as shown in Fig. 1. This suggests that: (a) Large values of m can increase the size of the prey population; (b) Decreasing the values of m can increase the size of the predator population.

The classification for the existence and stability of these equilibria of the system (3) is summarized in the following theorem.

Theorem 2.4. *For the system (3), it always has a trivial extinction equilibrium $E_0 = (0,0)$ and the boundary equilibrium $E_1 = (\frac{\gamma - d_1}{d_2}, 0)$ if $\frac{\gamma}{d_1} > 1$, and can have up to a unique positive equilibrium $E^* = (N^*, P^*)$. The necessary and sufficient conditions for their existence and stability are listed in Table 2. The global dynamics of (3) can be classified into the following scenarios:*

(i) If $\frac{\gamma}{d_1} < 1$, then the system (3) has only the extinction equilibrium E_0 which is globally stable. Under this condition, the prey is unable to survive and the predator dies out.

Table 1
Existence and stability of equilibria where GAS is an abbreviation for globally asymptotically stable.

Cases	Equilibria	Existence condition	Stability condition
(a)	$E_0 = (0,0)$	Always	Unstable
	$E_1 = \left(\frac{\gamma}{d_2}, 0\right)$	Always	GAS if $\frac{\gamma}{d_2} < \frac{d_3}{\alpha\beta - ad_3}$
	$E^* = \left(\frac{d_3}{\alpha\beta - ad_3}, \frac{\alpha(\gamma(\alpha\beta - ad_3) - d_2 d_3)}{(\alpha\beta - ad_3)^2}\right)$	$\frac{\gamma}{d_2} > \frac{d_3}{\alpha\beta - ad_3}$ and $d_3 < \frac{\alpha\beta}{a}$	GAS if $\frac{d_3}{\alpha\beta - ad_3} < \frac{\gamma}{d_2} < \frac{\alpha\beta + ad_3}{a(\alpha\beta - ad_3)}$
(b)	$E_0 = (0,0)$	Always	Unstable
	$E_1 = \left(\frac{\gamma - d_1}{d_2}, 0\right)$	$\frac{\gamma}{d_1} > 1$	GAS if $\frac{\gamma - d_1}{d_2} < \frac{d_3}{\alpha\beta - ad_3}$
	$E^* = \left(\frac{d_3}{\alpha\beta - ad_3}, \frac{\alpha((\alpha\beta - ad_3)(\gamma - d_1) - d_2 d_3)}{(\alpha\beta - ad_3)^2}\right)$	$\frac{\gamma - d_1}{d_2} > \frac{d_3}{\alpha\beta - ad_3}$ and $d_3 < \frac{\alpha\beta}{a}$	GAS if $\frac{d_3}{\alpha\beta - ad_3} < \frac{\gamma - d_1}{d_2} < \frac{\alpha\beta + ad_3}{a(\alpha\beta - ad_3)}$
(c)	$E_0 = (0,0)$	Always	Unstable
	$E_1 = \left(\frac{\gamma - d_1}{d_2}, 0\right)$	$\frac{\gamma}{d_1} > 1$	GAS if $\frac{\gamma - d_1}{d_2} < \frac{d_3}{(\alpha\beta - ad_3)(1-m)}$
	$E^* = \left(\frac{d_3}{(\alpha\beta - ad_3)(1-m)}, \frac{\alpha((\alpha\beta - ad_3)(\gamma - d_1)(1-m) - d_2 d_3)}{(\alpha\beta - ad_3)^2(1-m)^2}\right)$	$\frac{\gamma - d_1}{d_2} > \frac{d_3}{(\alpha\beta - ad_3)(1-m)}$ and $d_3 < \frac{\alpha\beta}{a}$	GAS if $\frac{d_3}{(\alpha\beta - ad_3)(1-m)} < \frac{\gamma - d_1}{d_2} < \frac{\alpha\beta + ad_3}{a(\alpha\beta - ad_3)(1-m)}$

Table 2
Equilibria and their stability of the system (3), where $N^* = \frac{d_3}{(\alpha\beta - ad_3)(1-m)}$, $P^* = \frac{-l_2 + \sqrt{l_2^2 - 4l_1 l_3}}{2l_1}$, where l_1, l_2 and l_3 are given in (5).

Equilibria	Existence condition	Stability condition
$E_0 = (0,0)$	Always	Globally asymptotically stable if $\frac{\gamma}{d_1} < 1$
$E_1 = \left(\frac{\gamma - d_1}{d_2}, 0\right)$	$\frac{\gamma}{d_1} > 1$	Globally asymptotically stable if $\frac{\gamma - d_1}{d_2} < \frac{d_3}{(\alpha\beta - ad_3)(1-m)}$
$E^* = (N^*, P^*)$	$\frac{\gamma - d_1}{d_2} > \frac{d_3}{(\alpha\beta - ad_3)(1-m)} > 0$	Globally asymptotically stable if $d_2 > \frac{\gamma c f P^*}{(1 + cN^* + fP^*)^2} + \frac{a\beta(1-m)^2 P^*}{(1 + a(1-m)N^*)^2}$.

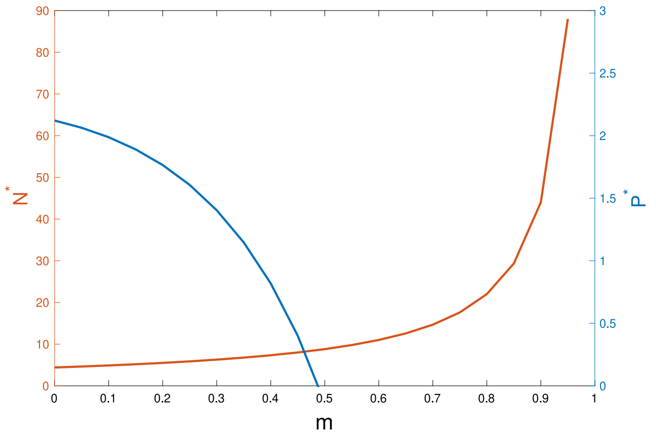


Fig. 1. When $\gamma = 2.25$, $c = 0.8$, $f = 1$, $\alpha = 0.8$, $\beta = 0.5$, $a = 0.5$, $d_1 = 0.1$, $d_2 = 0.25$, $d_3 = 0.55$, N^* increases with an increase in the strength of prey refuge m while P^* gradually approach zero as m goes to large.

(ii) *If* $\frac{\gamma}{d_1} > 1$ *and* $\frac{\gamma - d_1}{d_2} < \frac{d_3}{(\alpha\beta - ad_3)(1-m)}$, *then the system* (3) *has the extinction equilibrium* E_0 *and the boundary equilibrium* E_1, *where* E_0 *is unstable while* E_1 *is globally asymptotically stable.*

(iii) *If* $\frac{\gamma}{d_1} > 1$ *and* $\frac{\gamma - d_1}{d_2} > \frac{d_3}{(\alpha\beta - ad_3)(1-m)} > 0$, *then the system* (3) *has the extinction equilibrium* E_0 *and the boundary equilibrium* E_1 *and one interior equilibrium* E^*, *where both* E_0 *and* E_1 *are unstable while* E^* *is globally asymptotically stable if* $d_2 > \frac{\gamma c f P^*}{(1 + cN^* + fP^*)^2} + \frac{a\beta(1-m)^2 P^*}{(1 + a(1-m)N^*)^2}$.

Remark 2. The detailed proof of Theorem 2.4 can be found in Appendix. The number of equilibria and the dynamical behavior of the system (3) are determined by the values of its parameters γ, d_1, d_2 and the relationship of $\frac{\gamma - d_1}{d_2}$ and $\frac{d_3}{(\alpha\beta - ad_3)(1-m)}$. Moreover, the local stability of the equilibria of (3) is determined by the eigenvalues $\lambda_i(E^*)$ $(i = 1, 2)$ of its associated Jacobian matrix (54). Theorem 2.4 implies that: (1) the predator dies out if the maximum birth rate of prey is smaller than its natural death rate $\gamma < d_1$; (2) the predator dies out while the prey still exists if $0 < \frac{\gamma - d_1}{d_2} < \frac{d_3}{(\alpha\beta - ad_3)(1-m)}$; (3) the system (3) has one stable interior equilibrium which occurs when $\frac{\gamma - d_1}{d_2} > \frac{d_3}{(\alpha\beta - ad_3)(1-m)} > 0$ and $d_2 > \frac{\gamma c f P^*}{(1 + cN^* + fP^*)^2} + \frac{a\beta(1-m)^2 P^*}{(1 + a(1-m)N^*)^2}$.

Bifurcations explain how the system dynamics can change fundamentally dramatically when a parameter changes. A Hopf bifurcation is a crucial location where the stability of a system changes and a periodic solution appears. We consider here some parameters: the fear-related carry-over effect (c), the maximum birth rate of prey (γ), a prey refuge factor (m), and the fear factor (f) as bifurcation parameters. Analytically we take the fear-related carry-over effect c of the prey as a bifurcation parameter to investigate the instability of the positive equilibrium $E^* = (N^*, P^*)$ and keep other parameters fixed.

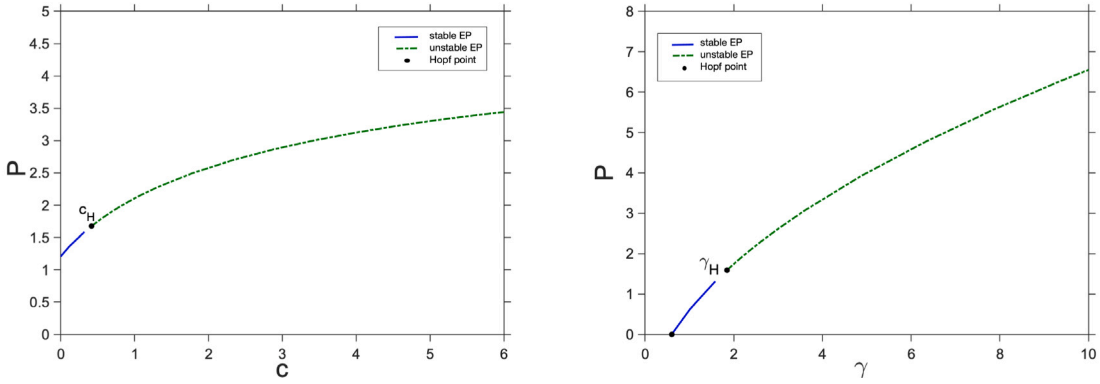
Theorem 2.5. *Assume that* $\frac{\mathrm{d}(\lambda(c))}{\mathrm{d}c}\big|_{c=c_H} \neq 0$ *holds, the positive equilibrium* $E^* = (N^*, P^*)$ *of the system* (3) *is locally asymptotically stable when* $c < c_H$, *and undergoes Hopf bifurcation at* E^* *when* $c = c_H$, *where*

$$\begin{aligned} c_H = {} & \frac{(\alpha\beta - ad_3)(1-m)}{2d_3^2(a(\alpha\beta - ad_3)^2(1-m)^2 P^* - \alpha^2\beta d_2)} \\ & \times \Big((\alpha^2\beta\gamma f(\alpha\beta - ad_3)(1-m)P^*(4ad_3 f(\alpha\beta - ad_3)^2(1-m)^2 P^{*2} \\ & + (4ad_3(\alpha\beta - ad_3)^2(1-m)^2 + \alpha^2\beta\gamma f(\alpha\beta - ad_3)(1-m) \\ & - 4d_2 d_3\alpha^2\beta f)P^* - 4d_2 d_3\alpha^2\beta))^{\frac{1}{2}} \\ & - 2ad_3 f(\alpha\beta - ad_3)^2(1-m)^2 P^{*2} - (2ad_3(\alpha\beta - ad_3)^2(1-m)^2 \\ & + \alpha^2\beta\gamma f(\alpha\beta - ad_3)(1-m) - 2d_2 d_3\alpha^2\beta f)P^* + 2d_2 d_3\alpha^2\beta \Big). \end{aligned} \tag{8}$$

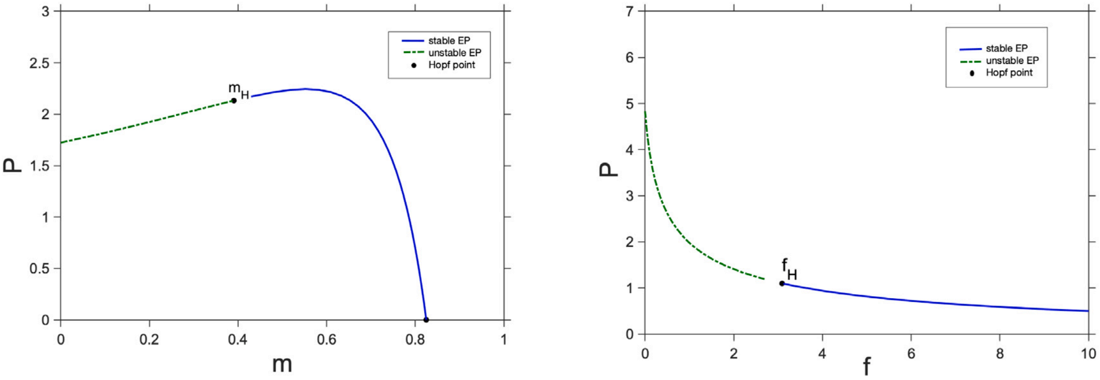
For proof of this theorem, see Appendix.

We present the bifurcation diagrams with varying c, γ, m, and f in Fig. 2 to explore these parameters' impact on the predator's dynamics. The bifurcation diagram in Fig. 2(a) corresponds to Theorem 2.5, which provides the additional information that: (a) increasing the values of c can increase the size of the predator population; (b) too large value of c destabilizes the system and leads to a Hopf bifurcation. And Fig. 2(b) displays similar results for the intrinsic growth rate γ on predators. Fig. 2(c) shows that increasing the values of m can increase first and then reduce the size of the predator population to zero, and Fig. 2(d) shows the size of the predator population reduces with increasing the values of f. However, the parameters m and f can lead to a Hopf bifurcation to stabilize the system.

For the above analytical results, we also offer some numerical simulation in Section 4.1, see Fig. 5 that depicts the dynamics of the convergence to equilibria; Figs. 6, 7 and 8 that respectively display the effects of parameters γ, α, β, c, f, d_1, d_2, d_3 and a in the form of bifurcation graph.

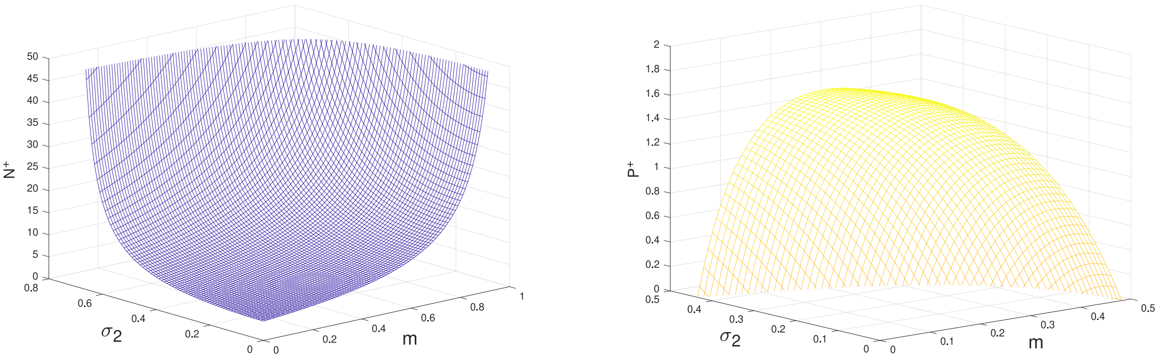


(a) $\gamma = 2.25,\ m = 0.25,\ f = 1$ (b) $c = 0.8,\ m = 0.25,\ f = 1$



(c) $\gamma = 2.25,\ c = 0.8,\ f = 1$ (d) $\gamma = 2.25,\ c = 0.8,\ m = 0.25$

Fig. 2. One parameter bifurcation diagrams with varying γ, c, m and f, respectively. Other parameters are $\alpha = 0.8$, $\beta = 0.8$, $a = 0.5$, $d_1 = 0.1$, $d_2 = 0.25$, $d_3 = 0.55$. Fig. 2(b), 2(a), 2(c) and 2(d) describe positive equilibrium's changes in stability as increasing γ, c, m and f, respectively, where the blue solid line represents the stable equilibrium point, the green dotted line represents the unstable equilibrium point and the black dot denotes Hopf bifurcation.



(a) (b)

Fig. 3. When $\gamma = 2.25$, $c = 0.8$, $f = 1$, $\alpha = 0.8$, $\beta = 0.5$, $a = 0.5$, $d_1 = 0.1$, $d_2 = 0.25$, $d_3 = 0.55$, $\sigma_1 = 0.01$, the effect of changing the values of prey refuge m and noise intensity σ_2 in the equations of N^+ and P^+, respectively.

3. Dynamics of the stochastic system

In this section, we focus on the dynamics of the stochastic system (1). Consider a Markov process $X(t) = (N(t), P(t)) \triangleq (X_1(t), X_2(t))$ in the state space $(\mathbb{R}_+^2, \mathbf{B}(\mathbb{R}_+^2), \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ that satisfies the Itô SDE as follows:

$$\mathrm{d}X(t) = \boldsymbol{f}(X(t))\mathrm{d}t + \boldsymbol{h}(X(t))\mathrm{d}B(t), \qquad X(0) = X_0, \tag{9}$$

where $\boldsymbol{f}(\cdot) : \mathbb{R}_+^2 \longmapsto \mathbb{R}_+^2$ and $\boldsymbol{h}(\cdot) : \mathbb{R}_+^2 \longmapsto \bar{\mathbb{R}}_+^{2\times 2}$ are locally Lipschitz functions, and the diffusion matrix of $X(t)$ is defined as $\boldsymbol{A}(X) := \boldsymbol{h}(X)\boldsymbol{h}^T(X) = (a_{ij}(X))$. For system (1), the diffusion matrix is $\boldsymbol{A}(X) = \mathrm{diag}(\sigma_1^2 N^2, \sigma_2^2 P^2)$.

The generator $\mathcal{L}$ which is uniformly elliptical in $\mathbb{R}_+^2$ related with system (1) is introduced in the following. Define an operator $\mathcal{L}V$ for each twice continuously differentiable function $V(X(t))$ as

$$\mathcal{L}V(X(t)) = \sum_{i=1}^{2} \boldsymbol{f}_i(X)\frac{\partial V(X)}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}(X)\frac{\partial^2 V(X)}{\partial X_i \partial X_j}.$$

3.1. Stationary distribution

For further study of the stochastic system (1)'s dynamics, we assume that Ξ is a bounded set:

$$\Xi = \left\{ (N,P) \in \mathbb{R}_+^2 : 0 < N(t) + \tfrac{1}{\alpha} P(t) \right. \\ \left. \le \begin{cases} \frac{\gamma - d_1}{d_2} & \text{if } d_3 > \gamma - d_1 \\ \frac{(\gamma - d_1 + d_3)^2}{4 d_2 d_3} & \text{if } d_3 \le \gamma - d_1 \end{cases} \right\} \subset \mathbb{R}_+^2,$$

and the following lemma proves the existence of a positive solution for system (1).

Lemma 3.1. *For any given initial value* $(N(0), P(0)) \in \Xi$*, there is a unique positive solution* $(N(t), P(t))$ *of the system* (1) *for all* $t \ge 0$ *and will remain in* Ξ *with probability one.*

Since the detailed proof is virtually identical to Lemma 2.4 in [45] and Theorem 2.6 in [43], we omit it.

Remark 3. From Lemma 3.1, it can be obtained that Ξ is almost surely positively invariant of the stochastic system (1), that is, if $(N(0), P(0)) \in \Xi$, then $\mathbf{P}\{N(t), P(t) \in \Xi\} = 1$ for all $t \ge 0$.

One major concern in biology is the long-term persistence of the species. In this section, we will analyze sufficient conditions for the existence of a unique ergodic stationary distribution via Theorem 4.1 and Assumption (B) in [48].

Theorem 3.2. *Assume that* $\gamma - d_1 > \frac{\sigma_1^2}{2}$, $d_3 > \max\{\frac{\sigma_1^2}{2}, \frac{\sigma_2^2}{2}\}$ *and*

$$\mathcal{R}_s := \frac{\alpha\beta(1-m)(\gamma - d_1 - \frac{\sigma_1^2}{2})^2}{(\gamma - d_1)(d_2 + a(1-m)(\gamma - d_1))(d_3 + \frac{\sigma_2^2}{2})} > 1$$

hold, then the system (1) *has a unique stationary distribution* $\boldsymbol{\pi}(\cdot)$ *with initial value* $(N(0), P(0)) \in \mathbb{R}_+^2$ *and it has ergodic property.*

Proof. For the stochastic system (1), the diffusion matrix is

$$G = \begin{pmatrix} \sigma_1^2 N^2 & 0 \\ 0 & \sigma_2^2 P^2 \end{pmatrix}, \tag{10}$$

which is positive definite for any compact subset of $\mathbb{R}_+^2$, then there is a positive number $g_0 = \min_{(N,P)\in\Pi} \left\{\sigma_1^2 N^2, \sigma_2^2 P^2\right\}$ such that

$$\sum_{i,j=1}^{2} g_{ij}\eta_i\eta_j = \sigma_1^2 N^2 \eta_1^2 + \sigma_2^2 P^2 \eta_2^2 \ge g_0 |\eta|^2,\ (N,P) \in \Pi \subset \mathbb{R}_+^2,\ \eta = (\eta_1, \eta_2) \in \mathbb{R}_+^2. \tag{11}$$

This means that the condition (B.1) of Assumption (B) in [48] holds.

Define a C^2-functions $\mathcal{V} : \mathbb{R}_+^2 \to \mathbb{R}_+$

$$\mathcal{V}(N,P) = C_0\left(V_1(N,P) + 2\sqrt{\tfrac{k_1\alpha\beta(\gamma - d_1)^2}{a d_2}} V_2(N) + V_3(P)\right) + V_4(N,P), \tag{12}$$

where

$$\begin{aligned} V_1(N,P) &= k_1 N - \ln P, \\ V_2(N) &= \tfrac{d_2}{2(\gamma - d_1)^2} N + \tfrac{1}{\gamma - d_1}(-\ln N), \\ V_3(P) &= \tfrac{2(\gamma f + \beta(1-m))}{d_3} \sqrt{\tfrac{k_1\alpha\beta}{a d_2}} P, \\ V_4(N,P) &= \tfrac{1}{\theta+2}\left(N + \tfrac{P}{\alpha}\right)^{\theta+2}, \quad 0 < \theta < 1, \end{aligned} \tag{13}$$

and $C_0 > 0$ is a sufficiently large number satisfying

$$-C_0\mathcal{R}_s + F_1^\mu + F_2^\mu \le -2,$$

where $F_1^\mu = \sup_{N>0} F_1(N)$, $F_2^\mu = \sup_{P>0} F_2(P)$, and $F_1(N)$, $F_2(P)$ are defined in (24). It can be seen that $\mathcal{V}(N,P)$ is continuous and it tends to ∞ as (N,P) approaches the boundary of $\mathbb{R}_+^2$. Then, there is a unique minimum point $(\underline{N}, \underline{P})$ of $\mathcal{V}(N,P)$ in the interior of $\mathbb{R}_+^2$. Hence, we construct a non-negative C^2-function $V : \mathbb{R}_+^2 \to \mathbb{R}_+ \cup \{0\}$ as follows:

$$V(N,P) = \mathcal{V}(N,P) - \mathcal{V}(\underline{N}, \underline{P}). \tag{14}$$

Applying the Itô̂'s formula, we derive

$$\begin{aligned} &\mathcal{L}V_1(N,P) = k_1\mathcal{L}(N) + \mathcal{L}(-\ln P) \\ &= k_1\left(\frac{\gamma N(1+cN)}{1+cN+fP} - d_1 N - d_2 N^2 - \frac{\beta(1-m)NP}{1+a(1-m)N}\right) \\ &\quad - \frac{\alpha\beta(1-m)N}{1+a(1-m)N} + d_3 + \tfrac{\sigma_2^2}{2} \\ &\le k_1\left((\gamma - d_1)N - d_2 N^2\right) - \frac{\alpha\beta(1-m)N}{1+a(1-m)N} + d_3 + \frac{\sigma_2^2}{2} \\ &\le -\left(\frac{k_1(\gamma - d_1)}{a(1-m)}(1 + a(1-m)N) + \frac{\alpha\beta(1-m)N}{1+a(1-m)N}\right) \\ &\quad + \frac{k_1(\gamma - d_1)(d_2 + a(1-m)(\gamma - d_1))}{a(1-m)d_2} + d_3 + \frac{\sigma_2^2}{2} \\ &\le -2\sqrt{\frac{k_1\alpha\beta(\gamma - d_1)}{a}}\sqrt{N} + \frac{k_1(\gamma - d_1)(d_2 + a(1-m)(\gamma - d_1))}{a(1-m)d_2} + d_3 + \frac{\sigma_2^2}{2}, \end{aligned} \tag{15}$$

$$\begin{aligned} &\mathcal{L}V_2(N) = \frac{d_2}{2(\gamma - d_1)^2}\mathcal{L}(N) + \frac{1}{\gamma - d_1}\mathcal{L}(-\ln N) \\ &\le \frac{d_2}{2(\gamma - d_1)^2}\left((\gamma - d_1)N - d_2 N^2\right) \\ &+ \frac{1}{\gamma - d_1}\left(-(\gamma - d_1) + d_2 N + (\gamma f + \beta(1-m))P + \frac{\sigma_1^2}{2}\right) \\ &\le \frac{d_2 N}{2(\gamma - d_1)}\left(1 - \frac{d_2 N}{\gamma - d_1}\right) + \frac{d_2 N}{\gamma - d_1} - \frac{1}{\gamma - d_1}\left(\gamma - d_1 - \frac{\sigma_1^2}{2}\right) \\ &+ \frac{\gamma f + \beta(1-m)}{\gamma - d_1}P \\ &\le \sqrt{\frac{d_2}{\gamma - d_1}}\sqrt{N} - \frac{1}{\gamma - d_1}\left(\gamma - d_1 - \frac{\sigma_1^2}{2}\right) + \frac{\gamma f + \beta(1-m)}{\gamma - d_1}P, \end{aligned} \tag{16}$$

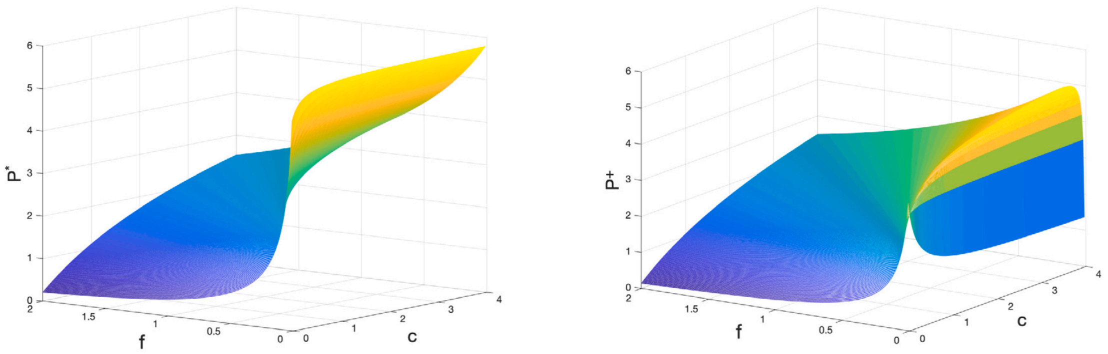
$$\begin{aligned} &\mathcal{L}\left(V_1(N,P) + 2\sqrt{\frac{k_1\alpha\beta(\gamma - d_1)^2}{a d_2}}V_2(N) + V_3(P)\right) \\ &\le -2\sqrt{\frac{k_1\alpha\beta(\gamma - d_1)}{a}}\sqrt{N} + \frac{k_1(\gamma - d_1)(d_2 + a(1-m)(\gamma - d_1))}{a(1-m)d_2} \\ &+ \frac{2}{d_3}\sqrt{\frac{k_1\alpha\beta}{a d_2}}(\gamma f + \beta(1-m))\left(\frac{\alpha\beta(1-m)NP}{1+a(1-m)N} - d_3 P\right) + d_3 + \frac{\sigma_2^2}{2} \\ &+ 2\sqrt{\frac{k_1\alpha\beta(\gamma - d_1)^2}{a d_2}}\left(\sqrt{\frac{d_2}{\gamma - d_1}}\sqrt{N} - \frac{1}{\gamma - d_1}\left(\gamma - d_1 - \frac{\sigma_1^2}{2}\right)\right. \\ &\left. + \frac{\gamma f + \beta(1-m)}{\gamma - d_1}P\right) \\ &\le -2\sqrt{\frac{k_1\alpha\beta}{a d_2}}\left(\gamma - d_1 - \frac{\sigma_1^2}{2}\right) + \frac{k_1(\gamma - d_1)(d_2 + a(1-m)(\gamma - d_1))}{a(1-m)d_2} \\ &+ \frac{2}{d_3}\sqrt{\frac{k_1\alpha\beta}{a d_2}}\,\frac{\alpha\beta(1-m)(\gamma f + \beta(1-m))}{1+a(1-m)N}NP + d_3 + \frac{\sigma_2^2}{2}. \end{aligned} \tag{17}$$

Taking

$$k_1 = \frac{\alpha\beta a d_2 (1-m)^2(\gamma - d_1 - \frac{\sigma_1^2}{2})^2}{(\gamma - d_1)^2(d_2 + a(1-m)(\gamma - d_1))^2}, \tag{18}$$

then we get

$$\begin{aligned} &\mathcal{L}\left(V_1(N,P) + 2\sqrt{\frac{k_1\alpha\beta(\gamma - d_1)^2}{a d_2}}V_2(N) + V_3(P)\right) \\ &\le -\frac{\alpha\beta(1-m)(\gamma - d_1 - \frac{\sigma_1^2}{2})^2}{(\gamma - d_1)(d_2 + a(1-m)(\gamma - d_1))} + d_3 + \frac{\sigma_2^2}{2} \\ &+ \frac{2\alpha^2\beta^2(1-m)^2(\gamma f + \beta(1-m))(\gamma - d_1 - \frac{\sigma_1^2}{2})}{d_3(\gamma - d_1)(d_2 + a(1-m)(\gamma - d_1))}NP \\ &= -\left(d_3 + \tfrac{\sigma_2^2}{2}\right)(\mathcal{R}_s - 1) + \frac{2\alpha^2\beta^2(1-m)^2(\gamma f + \beta(1-m))(\gamma - d_1 - \frac{\sigma_1^2}{2})}{d_3(\gamma - d_1)(d_2 + a(1-m)(\gamma - d_1))}NP, \end{aligned} \tag{19}$$



(a) $\sigma_1 = \sigma_2 = 0$ (b) $\sigma_1 = \sigma_2 = 0.01$

Fig. 4. When $\gamma = 2.25$, $\alpha = 0.8$, $\beta = 0.5$, $a = 0.5$, $d_1 = 0.1$, $d_2 = 0.25$, $d_3 = 0.55$, the effects of changing the values of c and f in the equation of P^+ with or without noise.

where

$$\mathcal{R}_s := \frac{\alpha\beta(1-m)(\gamma - d_1 - \frac{\sigma_1^2}{2})^2}{(\gamma - d_1)(d_2 + a(1-m)(\gamma - d_1))\left(d_3 + \frac{\sigma_2^2}{2}\right)}. \tag{20}$$

And

$$\begin{aligned}
\mathcal{L}V_4(N,P) &= \left(N + \frac{P}{\alpha}\right)^{\theta+1}\left(\frac{\gamma N(1+cN)}{1+cN+fP} - d_1 N - d_2 N^2 - \frac{\beta(1-m)P}{1+a(1-m)N}\right.\\
&\left.+\frac{\beta(1-m)N}{1+a(1-m)N} - \frac{d_3}{\alpha}P\right) + \frac{\theta+1}{2}\left(N + \frac{P}{\alpha}\right)^{\theta}\left(\sigma_1^2 N^2 + \sigma_2^2\left(\frac{P}{\alpha}\right)^2\right)\\
&\le \left(N + \frac{P}{\alpha}\right)^{\theta+1}\left((\gamma + d_3 - d_1)N - d_2 N^2 - d_3\left(N + \frac{P}{\alpha}\right)\right)\\
&+\frac{\theta+1}{2}\left(N + \frac{P}{\alpha}\right)^{\theta}\sigma_{\max}^2\left(N + \frac{P}{\alpha}\right)^2\\
&\le \left(N + \frac{P}{\alpha}\right)^{\theta+1}\left(\frac{(\gamma + d_3 - d_1)^2}{4d_2} - d_3\left(N + \frac{P}{\alpha}\right)\right) + \frac{\sigma_{\max}^2(\theta+1)}{2}\left(N + \frac{P}{\alpha}\right)^{\theta+2}\\
&= \frac{(\gamma + d_3 - d_1)^2}{4d_2}\left(N + \frac{P}{\alpha}\right)^{\theta+1} - \frac{1}{2}\left(d_3 - \frac{\sigma_{\max}^2(\theta+1)}{2}\right)\left(N + \frac{P}{\alpha}\right)^{\theta+2}\\
&-\frac{1}{2}\left(d_3 - \frac{\sigma_{\max}^2(\theta+1)}{2}\right)\left(N + \frac{P}{\alpha}\right)^{\theta+2}\\
&\le C_1 - \frac{1}{2}\left(d_3 - \frac{\sigma_{\max}^2(\theta+1)}{2}\right)\left(N + \frac{P}{\alpha}\right)^{\theta+2}\\
&\le C_1 - \frac{1}{2}\left(d_3 - \frac{\sigma_{\max}^2(\theta+1)}{2}\right)\left(N^{\theta+2} + \left(\frac{P}{\alpha}\right)^{\theta+2}\right),
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
C_1 = \sup_{(N,P)\in\mathbb{R}_+^2} &\left\{ \frac{(\gamma+d_3-d_1)^2}{4d_2}\left(N + \frac{P}{\alpha}\right)^{\theta+1}\right.\\
&\left.-\frac{1}{2}\left(d_3 - \frac{\sigma_{\max}^2(\theta+1)}{2}\right)\left(N + \frac{P}{\alpha}\right)^{\theta+2}\right\} < \infty.
\end{aligned} \tag{22}$$

Therefore, according to (19) and (21), we deduce

$$\begin{aligned}
\mathcal{L}V(N,P) &= \mathcal{L}(\mathcal{V}(N,P) - \mathcal{V}(\underline{N},\underline{P}))\\
&= \mathcal{L}\left(C_0\left(V_1(N,P) + 2\sqrt{\frac{k_1\alpha\beta(\gamma - d_1)^2}{ad_2}}V_2(N) + V_3(P)\right) + V_4(N,P) - \mathcal{V}(\underline{N},\underline{P})\right)\\
&\le -C_0\mathcal{R}_s + \frac{2C_0\alpha^2\beta^2(1-m)^2(\gamma f + \beta(1-m))(\gamma - d_1 - \frac{\sigma_1^2}{2})}{d_3(\gamma - d_1)(d_2 + a(1-m)(\gamma - d_1))}NP + C_1\\
&-\frac{1}{2}\left(d_3 - \frac{\sigma_{\max}^2(\theta+1)}{2}\right)\left(N^{\theta+2} + \left(\frac{P}{\alpha}\right)^{\theta+2}\right)\\
&= -C_0\mathcal{R}_s + \frac{2C_0\alpha^2\beta^2(1-m)^2(\gamma f + \beta(1-m))(\gamma - d_1 - \frac{\sigma_1^2}{2})}{d_3(\gamma - d_1)(d_2 + a(1-m)(\gamma - d_1))}NP\\
&+C_1 + F_1(N) + F_2(P),
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
F_1(N) &:= -\frac{1}{2}\left(d_3 - \frac{\sigma_{\max}^2(\theta+1)}{2}\right)N^{\theta+2},\\
F_2(P) &:= -\frac{1}{2}\left(d_3 - \frac{\sigma_{\max}^2(\theta+1)}{2}\right)\left(\frac{P}{\alpha}\right)^{\theta+2}.
\end{aligned} \tag{24}$$

Set

$$\begin{aligned}
F(N,P) &= -C_0\mathcal{R}_s + \frac{2C_0\alpha^2\beta^2(1-m)^2(\gamma f+\beta(1-m))(\gamma-d_1-\frac{\sigma_1^2}{2})}{d_3(\gamma-d_1)(d_2+a(1-m)(\gamma-d_1))}NP\\
&+C_1 + F_1(N) + F_2(P)
\end{aligned} \tag{25}$$

and assume that $d_3 > \frac{\sigma_{\max}^2(\theta+1)}{2}$, $\sigma_{\max} = \max\{\sigma_1, \sigma_2\}$, thus we obtain

$$F(N,P) \le \begin{cases} F(\infty, P) \to -\infty, & \text{if } N \to \infty,\\ F(N, \infty) \to -\infty, & \text{if } P \to \infty,\\ -C_0\mathcal{R}_s + F_1^{\mu} + F_2^{\mu} \le -2, & \text{if } N \to 0^+ \text{ or } P \to 0^+. \end{cases}$$

Therefore, for a sufficiently small $0 < \epsilon < 1$, we can get

$$\mathcal{L}V(N,P) \le -1 \ \text{ for any } (N,P) \in \mathbb{R}_+^2 \setminus \Pi_\epsilon\ ,$$

where $\Pi_\epsilon = \left[\epsilon, \frac{1}{\epsilon}\right] \times \left[\epsilon, \frac{1}{\epsilon}\right]$. This implies that the condition (B.2) of Assumption (B) in [48] also holds. Thereupon, the stochastic system (1) has a unique stationary distribution that is ergodic.

Remark 4. Theorem 3.2 indicates that the system (1)'s solution can have an asymptotically stationary distribution, suggesting stochastic stability, instead of exploding to infinity. As a result of Theorem 3.2, the system (1) oscillates around the positive equilibrium $E^* = (N^*, P^*)$ of the corresponding deterministic system, and it has the ergodic property where the positive solution converges to a unique stationary distribution if the conditions $\gamma - d_1 > \frac{\sigma_1^2}{2}$, $d_3 > \max\{\frac{\sigma_1^2}{2}, \frac{\sigma_2^2}{2}\}$ and $\mathcal{R}_s := \frac{\alpha\beta(1-m)(\gamma-d_1-\frac{\sigma_1^2}{2})^2}{(\gamma-d_1)(d_2+a(1-m)(\gamma-d_1))(d_3+\frac{\sigma_2^2}{2})} > 1$ are met. In this way, the prey and predator species have shown their capacity to sustain themselves under specific conditions.

3.2. Probability density function

As a result of the foregoing study, it can be concluded that the global solution (N,P) of the stochastic system (1) follows a unique stationary distribution. Next, we will derive the distribution's explicit probability density function. It is necessary to show the system's two fundamental transformations to accomplish this goal effectively.

Let $(x_1, x_2)^T = (\ln N, \ln P)^T$ and using Itô's formula, the system (1) depicts that

$$\begin{cases} \mathrm{d}x_1 = \left(\dfrac{\gamma(1+c\mathrm{e}^{x_1})}{1+c\mathrm{e}^{x_1}+f\mathrm{e}^{x_2}} - d_1 - d_2\mathrm{e}^{x_1} - \dfrac{\beta(1-m)\mathrm{e}^{x_2}}{1+a(1-m)\mathrm{e}^{x_1}} \right. \\ \left. -\dfrac{\sigma_1^2}{2} \right) \mathrm{d}t + \sigma_1 \mathrm{d}B(t), \\ \mathrm{d}x_2 = \left(\dfrac{\alpha\beta(1-m)\mathrm{e}^{x_1}}{1+a(1-m)\mathrm{e}^{x_1}} - d_3 - \dfrac{\sigma_2^2}{2} \right) \mathrm{d}t + \sigma_2 \mathrm{d}B(t). \end{cases} \tag{26}$$

A crucial value is obtained by taking into consideration the random effect, that is,

$$\widetilde{\mathcal{R}}_s = \frac{\alpha\beta}{a\left(d_3 + \frac{\sigma_2^2}{2}\right)}, \tag{27}$$

which is related to $\mathcal{R}_0 = \frac{\alpha\beta}{ad_3}$. From the expressions of $\widetilde{\mathcal{R}}_s$ and $\mathcal{R}_0$, we get $\widetilde{\mathcal{R}}_s \le \mathcal{R}_0$, and the sign holds if and only if $\sigma_2 = 0$. If $\widetilde{\mathcal{R}}_s > 1$, the following equation

$$\begin{cases} \dfrac{\gamma(1+c\mathrm{e}^{x_1})}{1+c\mathrm{e}^{x_1}+f\mathrm{e}^{x_2}} - d_1 - d_2\mathrm{e}^{x_1} - \dfrac{\beta(1-m)\mathrm{e}^{x_2}}{1+a(1-m)\mathrm{e}^{x_1}} - \dfrac{\sigma_1^2}{2} = 0, \\ \dfrac{\alpha\beta(1-m)\mathrm{e}^{x_1}}{1+a(1-m)\mathrm{e}^{x_1}} - d_3 - \dfrac{\sigma_2^2}{2} = 0, \end{cases} \tag{28}$$

determines the quasi-equilibrium $E^+ = (N^+, P^+) := (\mathrm{e}^{x_1^+}, \mathrm{e}^{x_2^+}) \in \mathbb{R}_+^2$. From (28), it can be calculated that

$$N^+ = \frac{1}{a(1-m)(\widetilde{\mathcal{R}}_s - 1)}, \tag{29}$$

and the equation

$$\begin{aligned} \mathcal{H}(P) = &\left(\beta(1-m)(1+cN^+) + f(1+a(1-m)N^+)\left(d_2N^+ + d_1 + \frac{\sigma_1^2}{2} \right) \right) P \\ &+ f\beta(1-m)P^2 + (1+cN^+)(1+a(1-m)N^+) \\ &\times \left(d_1 + \frac{\sigma_1^2}{2} + d_2N^+ - \gamma \right) = 0 \end{aligned}$$

has a unique positive solution

$$\begin{aligned} P^+ = &\frac{1}{2f(2d_3+\sigma_2^2)} \Big[\Big(\left(2\alpha f d_2 N^{+2}\right)^2 + 4\alpha f d_2 \left(2\alpha f d_1 + \alpha f \sigma_1^2 - c\sigma_2^2 - 2cd_3\right) N^{+3} \\ &+ \left(c^2 \left(2d_3+\sigma_2^2\right)^2 + 2\alpha f \left(4c\gamma - 2cd_1 - 2d_2 - cd_1^2\right)\left(2d_3+\sigma_2^2\right) \right. \\ &\left. + \alpha^2 f^2 \left(2d_1+\sigma_1^2\right)^2 \right) N^{+2} \\ &+ 2\left(2d_3+\sigma_2^2\right)\left(2cd_3 + 4\gamma\alpha f - 2\alpha f d_1 - \alpha f\sigma_1^2 + c\sigma_2^2\right) N^+ + \left(2d_3+\sigma_2^2\right)^2 \Big)^{\frac{1}{2}} \\ &- \left(2\alpha f d_2 N^{+2} + \left(2\alpha f d_1 + 2cd_3 + c\sigma_2^2 + \alpha f \sigma_1^2\right) N^+ + 2d_3 + \sigma_2^2 \right) \Big] \end{aligned} \tag{30}$$

if $\frac{\gamma - d_1}{d_2} > N^+ + \frac{\sigma_1^2}{2d_2}$.

Eqs. (29) and (30) determine the relationship between $E^+ = (N^+, P^+)$ and the strength of prey refuge m and noise intensity in the predator species σ_2. The expressions show how the values of N^+ and P^+ are influenced by the parameter m and noise intensity σ_2, respectively. Fig. 3 illustrates this relationship. The quantity of prey N^+ may grow as the parameters m and σ_2 increase, but the number of predators P^+ may decrease. If the system (1) does not contain stochastic noise (i.e., $\sigma_1 = \sigma_2 = 0$), then $E^+ = E^* = (N^*, P^*)$. Figs. 4(a) and 4(b) display the connections between these points P^*, P^+ and the carry-over effect parameter c, as well as the fear level induced by predator f with or without noise intensity. The number of predators P^+ impacted by both parameters c and f varies more complicated due to environmental noise.

In light of the foregoing, let $(y_1, y_2)^T = (x_1 - x_1^+, x_2 - x_2^+)$, where $x_1^+ = \ln N^+$, $x_2^+ = \ln P^+$, then the corresponding linearized system of (26) is

$$\begin{cases} \mathrm{d}y_1 = (-c_{11}y_1 - c_{12}y_2)\mathrm{d}t + \sigma_1 \mathrm{d}B(t), \\ \mathrm{d}y_2 = c_{21}y_1 \mathrm{d}t + \sigma_2 \mathrm{d}B(t), \end{cases} \tag{31}$$

where

$$\begin{aligned} c_{11} &= d_2N^+ - \frac{a\beta(1-m)^2N^+P^+}{(1+a(1-m)N^+)^2} - \frac{\gamma c f N^+P^+}{(1+cN^+ + fP^+)^2}, \\ c_{12} &= \frac{\beta(1-m)P^+}{1+a(1-m)N^+} + \frac{\gamma f(1+cN^+)P^+}{(1+cN^++fP^+)^2} > 0, \\ c_{21} &= \frac{\alpha\beta(1-m)N^+}{(1+a(1-m)N^+)^2} > 0. \end{aligned} \tag{32}$$

The probability density function of the system (1) may be found in the following theorem.

Theorem 3.3. *Assume that $\widetilde{\mathcal{R}}_s = \frac{\alpha\beta}{a(d_3+\frac{\sigma_2^2}{2})} > 1$ and $c_{11} > 0$, for any initial value $(N(0), P(0)) \in \mathbb{R}_+^2$, the system (1)'s stationary distribution around $E^+ = (N^+, P^+)$ follows a unique log-normal density function $\mathcal{P}(N, P)$, that is,*

$$\mathcal{P}(N,P) = (2\pi)^{-1}|\Sigma|^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}\left(\ln\frac{N}{N^+}, \ln\frac{P}{P^+}\right)\Sigma^{-1}\left(\ln\frac{N}{N^+}, \ln\frac{P}{P^+}\right)^T}, \tag{33}$$

where the positive definite matrix $\Sigma = \varrho_1^2\mathcal{Z}_1^{-1}\Sigma_0(\mathcal{Z}_1^{-1})^T + \varrho_2^2\mathcal{Z}_2^{-1}\Sigma_0(\mathcal{Z}_2^{-1})^T$ with $\varrho_1 = c_{21}\sigma_1$, $\varrho_2 = c_{12}\sigma_2$,

$$\Sigma_0 = \begin{pmatrix} \frac{1}{2c_{11}} & 0 \\ 0 & \frac{1}{2c_{11}c_{12}c_{21}} \end{pmatrix}, \quad \mathcal{Z}_1 = \begin{pmatrix} c_{21} & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{Z}_2 = \begin{pmatrix} -c_{11} & -c_{12} \\ 1 & 0 \end{pmatrix},$$

and c_{11}, c_{12}, c_{21} are described in Eq. (32).

Proof. As a matter of simplicity, let $\mathcal{Y} = (y_1, y_2)^T$, $\hat{G} = \mathrm{diag}(\sigma_1, \sigma_2)$ and

$$A = \begin{pmatrix} -c_{11} & -c_{12} \\ c_{21} & 0 \end{pmatrix}. \tag{34}$$

The system (31) then can be rewritten as $\mathrm{d}\mathcal{Y} = A\mathcal{Y}\mathrm{d}t + \hat{G}\mathrm{d}B(t)$. The unique density function $\mathcal{P}(\mathcal{Y})$ around the equilibrium $E^+ = (N^+, P^+)$, according to [49], fulfills the Fokker–Planck equation as follows

$$-\sum_{i=1}^{2} \frac{\sigma_i^2}{2}\frac{\partial^2}{\partial y_i^2}\mathcal{P} + \frac{\partial}{\partial y_1}\left[(-c_{11}y_1 - c_{12}y_2)\mathcal{P}\right] + \frac{\partial}{\partial y_2}\left[c_{21}y_1\mathcal{P}\right] = 0. \tag{35}$$

$\mathcal{P}(\mathcal{Y})$ can be characterized by a quasi-Gaussian distribution [50], i.e., $\mathcal{P}(\mathcal{Y}) = c\mathrm{e}^{-\frac{1}{2}\mathcal{Y}W\mathcal{Y}^T}$, where $c > 0$ is given by the normalized condition $\int_{\mathbb{R}_+^2}\mathcal{P}(\mathcal{Y})\mathrm{d}\mathcal{Y} = 1$ and W is a real symmetric matrix, since the diffusion matrix $\hat{G}$ is a constant matrix. Putting these results into Eq. (35) yields the fact that W obeys $W\hat{G}^2W + A^TW + WA = 0$ which is an algebraic equation. If W is an inverse matrix, an equivalent equation is given by setting $\Sigma = W^{-1}$,

$$\hat{G}^2 + A\Sigma + \Sigma A^T = 0. \tag{36}$$

According to the principle of finite independent superposition [51], Eq. (36) is equal to combining two equations, as below,

$$\hat{G}_i^2 + A\Sigma_i + \Sigma_i A^T = 0, \quad i = 1, 2,$$

where $\hat{G}_1 = \mathrm{diag}(\sigma_1, 0)$, $\hat{G}_2 = \mathrm{diag}(0, \sigma_2)$, $\hat{G}^2 = \hat{G}_1^2 + \hat{G}_2^2$ and $\Sigma = \Sigma_1 + \Sigma_2$.

The matrix A's character polynomials are

$$J_A(\lambda) = \lambda^2 + c_{11}\lambda + c_{12}c_{21}, \tag{37}$$

then if $c_{11} > 0$, A has all negative real-part eigenvalues. Σ of Eq. (36) is positive definite and can be deduced from Lemma 2.6 of [52]. To obtain the special expression for Σ, two processes are necessary.

Firstly, for the algebraic equation

$$\hat{G}_1^2 + A\Sigma_1 + \Sigma_1 A^T = 0, \tag{38}$$

where $\hat{G}_1 = \mathrm{diag}(\sigma_1, 0)$, we can calculate a matrix as

$$\mathcal{B}_1 = \begin{pmatrix} -c_{11} & -c_{12}c_{21} \\ 1 & 0 \end{pmatrix} = \mathcal{Z}_1 A \mathcal{Z}_1^{-1}, \tag{39}$$

where $\mathcal{Z}_1 = \begin{pmatrix} c_{21} & 0 \\ 0 & 1 \end{pmatrix}$. Then, Eq. (38) can be transformed into the following equation

$$\mathcal{Z}_1\hat{G}_1^2\mathcal{Z}_1^T + \mathcal{B}_1\mathcal{Z}_1\Sigma_1\mathcal{Z}_1^T + \mathcal{Z}_1\Sigma_1\mathcal{Z}_1^T\mathcal{B}_1^T = 0,$$

that equals

$$G_0^2 + \mathcal{B}_1 \Sigma_0 + \Sigma_0 \mathcal{B}_1^T = 0,$$

where $G_0 = \mathrm{diag}(1,0)$, $\Sigma_0 = \varrho_1^{-2} \mathcal{Z}_1 \Sigma_1 \mathcal{Z}_1^T$, $\varrho_1 = c_{21}\sigma_1$, and we can derive the positive definite matrix

$$\Sigma_0 = \begin{pmatrix} \frac{1}{2c_{11}} & 0 \\ 0 & \frac{1}{2c_{11}c_{12}c_{21}} \end{pmatrix}. \tag{40}$$

Thus, $\Sigma_1 = \varrho_1^2 \mathcal{Z}_1^{-1} \Sigma_0 (\mathcal{Z}_1^{-1})^T$ can be calculated and it is positive definite.

Secondly, consider the algebraic equation

$$\hat{G}_2^2 + A\Sigma_2 + \Sigma_2 A^T = 0, \tag{41}$$

where $\hat{G}_2 = \mathrm{diag}(0, \sigma_2)$. Similarly, we can get the following matrix

$$\mathcal{B}_2 = \mathcal{Z}_2 A \mathcal{Z}_2^{-1} = \mathcal{B}_1, \tag{42}$$

where $\mathcal{Z}_2 = \begin{pmatrix} -c_{11} & -c_{12} \\ 1 & 0 \end{pmatrix}$. We also transform Eq. (41) into the equation as follows

$$\mathcal{Z}_2 \hat{G}_2^2 \mathcal{Z}_2^T + \mathcal{B}_2 \mathcal{Z}_2 \Sigma_2 \mathcal{Z}_2^T + \mathcal{Z}_2 \Sigma_2 \mathcal{Z}_2^T \mathcal{B}_2^T = 0,$$

which means $G_0^2 + \mathcal{B}_2 \Sigma_0 + \Sigma_0 \mathcal{B}_2^T = 0$, where $\Sigma_0 = \varrho_2^{-2} \mathcal{Z}_2 \Sigma_2 \mathcal{Z}_2^T$, $\varrho_2 = c_{12}\sigma_2$. As indicated in the above step, Σ_0 is positive definite. Thus, we get the positive definite matrix $\Sigma_2 = \varrho_2^2 \mathcal{Z}_2^{-1} \Sigma_0 (\mathcal{Z}_2^{-1})^T$.

As a result, the real symmetric matrix $\Sigma = \Sigma_1 + \Sigma_2$ is positive definite, and the probability density function $\mathcal{P}(N, P)$ is approximately normal around the equilibrium point E^+. The processing is done.

Remark 5. According to Theorem 3.3, a unique log-normal density function, $\mathcal{P}(N, P)$, exists in the presence of $\widetilde{\mathcal{R}}_s > 1$ and $c_{11} > 0$, which also illustrates the system (1)'s stochastic permanence. That is to say, in the sense of probability, the quasi-equilibrium point $E^+ = (N^+, P^+)$ is stable with a higher probability. That means that the prey and predator trajectories will spend more time in the vicinity of E^+.

3.3. Extinction

In this subsection, we investigate the conditions that lead to the extinction of the prey and predator populations. For convenience, we define a constant

$$\widehat{\mathcal{R}}_s := \frac{2\gamma}{\min\left\{\left(d_1 + \frac{\sigma_1^2}{2}\right), \left(d_3 + \frac{\alpha^2\sigma_2^2}{2}\right)\right\}},$$

and then have the following extinction result.

Theorem 3.4. *Assume that $\widehat{\mathcal{R}}_s < 1$, then for any initial value $(N(0), P(0)) \in \mathbb{R}_+^2$, the solution $(N(t), P(t))$ of system (1) has the following property*

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left(N(t) + \frac{1}{\alpha}P(t)\right) \le \frac{\min\left\{\left(d_1 + \frac{\sigma_1^2}{2}\right), \left(d_3 + \frac{\alpha^2\sigma_2^2}{2}\right)\right\}}{2}(\widehat{\mathcal{R}}_s - 1) \quad \textit{a.s.}$$

which means that the prey and predator populations will go to extinction almost surely, respectively.

Proof. From the stochastic system (1), let $U(t) = N(t) + \frac{1}{\alpha}P(t)$, and using Itô's formula to $\ln U(t)$, we obtain

$$\begin{aligned} \mathrm{d}\ln U(t) = {} & \Bigg[\frac{1}{N+\frac{1}{\alpha}P}\left(\frac{\gamma N(1+cN)}{1+cN+fP} - d_1 N - d_2 N^2 - \frac{\beta(1-m)NP}{1+a(1-m)N} + \frac{1}{\alpha}\left(\frac{\alpha\beta(1-m)NP}{1+a(1-m)N} - d_3 P\right)\right) \\ & - \frac{\sigma_1^2 N^2 + \sigma_2^2 P2}{2(N+\frac{1}{\alpha}P)^2}\Bigg]\mathrm{d}t \\ & + \frac{\sigma_1 N + \sigma_2 P}{N+\frac{1}{\alpha}P}\mathrm{d}B(t) \\ \le {} & \left[\gamma - \left(\frac{d_1 N + \frac{d_3}{\alpha}P}{N+\frac{1}{\alpha}P} + \frac{\sigma_1^2 N^2 + \sigma_2^2 P^2}{2(N+\frac{1}{\alpha}P)^2}\right)\right]\mathrm{d}t + \frac{\sigma_1 N + \sigma_2 P}{N+\frac{1}{\alpha}P}\mathrm{d}B(t) \\ \le {} & \gamma \mathrm{d}t - \frac{1}{(N+\frac{1}{\alpha}P)^2}\left((d_1 + \frac{\sigma_1^2}{2})N^2 + (\frac{d_3}{\alpha^2} + \frac{\sigma_2^2}{2})P^2\right)\mathrm{d}t + \frac{\sigma_1 N + \sigma_2 P}{N+\frac{1}{\alpha}P}\mathrm{d}B(t) \\ \le {} & r\mathrm{d}t - \frac{1}{(N+\frac{1}{\alpha}P)^2}\min\{(d_1 + \frac{\sigma_1^2}{2}), (d_3 + \frac{\alpha^2\sigma_2^2}{2})\}(N^2 + \frac{1}{\alpha^2}P^2)\mathrm{d}t \\ & + \frac{\sigma_1 N + \sigma_2 P}{N+\frac{1}{\alpha}P}\mathrm{d}B(t) \\ \le {} & \gamma \mathrm{d}t - \frac{\min\{(d_1 + \frac{\sigma_1^2}{2}), (d_3 + \frac{\alpha^2\sigma_2^2}{2})\}}{2}\mathrm{d}t + \frac{\sigma_1 N + \sigma_2 P}{N+\frac{1}{\alpha}P}\mathrm{d}B(t). \end{aligned} \tag{43}$$

Table 3
Definitions of variables and parameters in the system (1).

Parameters	Description	Value	Resource
γ	Intrinsic growth rate	2.25	[5]
c	Carry-over effect parameter	0.8	[5]
f	Fear level induced by predator	1	[5]
α	Energy conversion rate	0.8	[18]
β	Search rate	0.8	[18]
a	The half-saturation constant	0.5	[18]
m	Strength of prey refuge	0.25	[18]
d_1	Natural death rate of prey	0.1	[5]
d_2	The density-dependent death rate of prey.	0.25	[5]
d_3	Natural death rate of predator	0.55	[18]
σ_1, σ_2	Intensities of the noise	Variables	Estimate

Integrating the above inequality from 0 to t and then dividing by t into both sides, we derive

$$\begin{aligned} \frac{\ln U(t) - \ln U(0)}{t} \le {} & \gamma - \frac{\min\left\{\left(d_1 + \frac{\sigma_1^2}{2}\right), \left(d_3 + \frac{\alpha^2\sigma_2^2}{2}\right)\right\}}{2} \\ & + \frac{1}{t}\int_0^t \frac{\sigma_1 N(s) + \sigma_2 P(s)}{N(s) + \frac{1}{\alpha}P(s)}\mathrm{d}B(s), \end{aligned} \tag{44}$$

then

$$\begin{aligned} \limsup_{t\to\infty} \frac{\ln U(t)}{t} & \le \gamma - \frac{\min\left\{\left(d_1 + \frac{\sigma_1^2}{2}\right), \left(d_3 + \frac{\alpha^2\sigma_2^2}{2}\right)\right\}}{2} \\ & = \frac{\min\left\{\left(d_1 + \frac{\sigma_1^2}{2}\right), \left(d_3 + \frac{\alpha^2\sigma_2^2}{2}\right)\right\}}{2}(\widehat{\mathcal{R}}_s - 1) \quad \text{a.s.} \end{aligned} \tag{45}$$

When $\widehat{\mathcal{R}}_s < 1$, it can be obtained that $\lim_{t\to\infty} U(t) = \lim_{t\to\infty}(N(t) + \frac{1}{\alpha}P(t)) = 0$ a.s., which is shown that

$$\lim_{t\to\infty} N(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} P(t) = 0 \quad \text{a.s.} \tag{46}$$

This completes the proof.

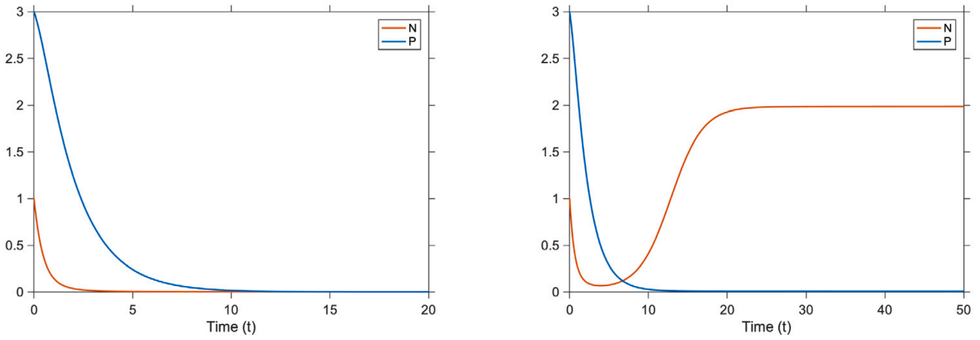
Remark 6. Theorem 3.4 discusses the dynamics of a no-species situation for the stochastic system (1), that is, $\widehat{\mathcal{R}}_s = \frac{2\gamma}{\min\left\{(d_1 + \frac{\sigma_1^2}{2}), (d_3 + \frac{\alpha^2\sigma_2^2}{2})\right\}} < 1$ is satisfied, $N(t)$ and $P(t)$ in system (1) go to extinction with probability one, respectively. From the expression of $\widehat{\mathcal{R}}_s$, we can conclude that the maximum birth rate of prey γ, the natural death rate of prey and predator populations (d_1, d_3), and environmental noise (σ_1, σ_2) have a critical influence on species extinction in system (1). In addition, we obtain that large values of the natural death rates and environmental disturbances and small values of γ will lead to the species extinction of system (1).

4. Numerical simulations

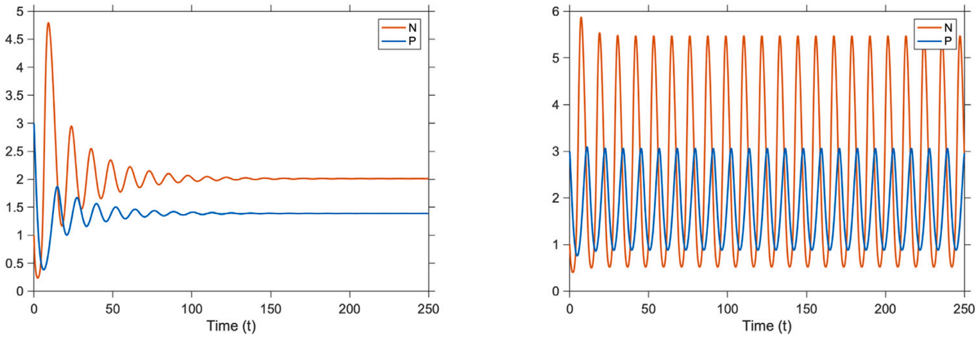
In this section, we make some numerical simulations to verify the dynamic results of the system (1) with or without noises. The corresponding biological parameters of the system (1) are shown in Table 3.

4.1. Deterministic version of the system (1)

Fig. 5 illustrates the stability of three equilibria ($E_0 = (0,0), E_1 = (\frac{\gamma - d_1}{d_2}, 0)$, $E^* = (N^*, P^*)$) of the system (1) with $\sigma_1 = \sigma_2 = 0$ (i.e., the system (3)), and the dynamics of evolution to various equilibria are depicted. By setting the parameters' values (see Table 3), in Fig. 5(a), both prey and predator species become extinct when the intrinsic growth rate γ of the prey N is smaller and satisfies $\gamma < d_1$. In Figs. 5(b), 5(c) and 5(d), let the parameters satisfy the conditions of Theorems 2.4 and 2.5, one can see that the development of equilibria E_0, E_1 and E^* is illustrated in its dynamics, respectively.

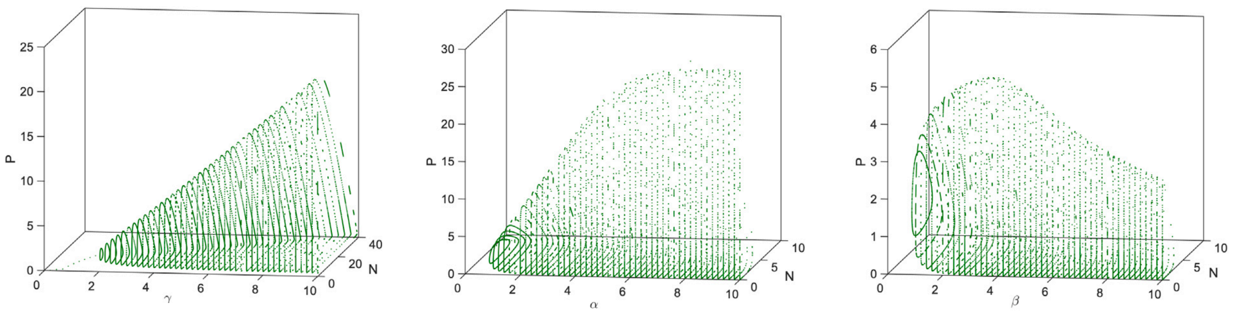


(a) $\gamma = 0.6$, $d_1 = 0.7$ (b) $\gamma = 0.6$



(c) $\gamma = 1.65$ (d) $\gamma = 2.25$

Fig. 5. Dynamics of the convergence to equilibrium with initial value $(N(0), P(0)) = (1, 3)$ and different values of γ, d_1. Table 3 lists other parameter values for (a), (b), (c), and (d), respectively. (a) Both N and P cannot survive; (b) N can survive but P cannot; (c) N and P can survive; (d) N and P coexist cyclically.

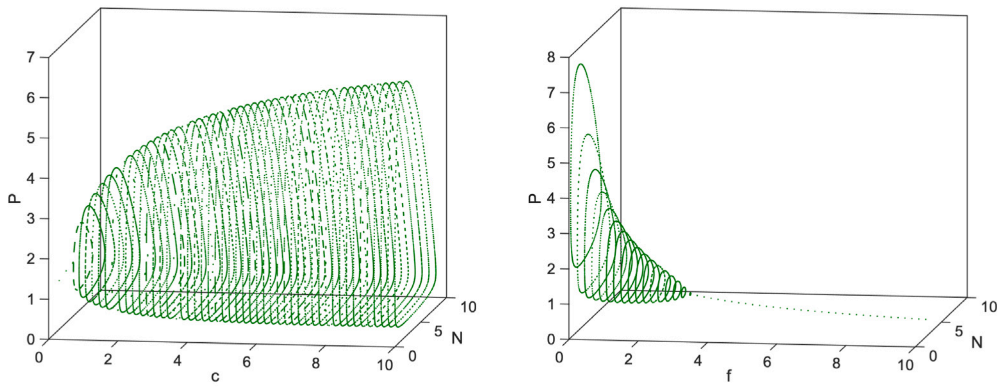


(a) (b) (c)

Fig. 6. Bifurcation diagram of the system (1) without noises, with γ, α and β as bifurcation parameters, respectively. At (a) $\gamma = 1.848$, (b) $\alpha = 0.6876$ and (c) $\beta = 0.696$, respectively, Hopf bifurcation occurs where the steady state becomes unstable and the system begins to exhibit limit cycles.

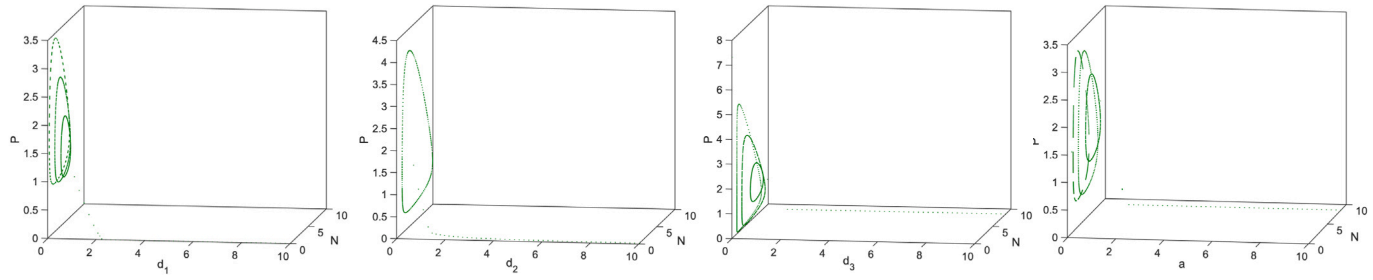
Bifurcation analysis with γ, α, β (see Fig. 6), c, f (see Fig. 7), d_1, d_2, d_3, a (see Fig. 8) as the bifurcation parameters respectively present the stability of the system (3)'s states. Figs. 6(a), 6(b), 6(c) and Fig. 7(a) display that for smaller γ, α, β and c, the prey-only equilibrium $E_1 = (\frac{\gamma - d_1}{d_2}, 0)$ is stable which means the predator population vanishes. And for bigger γ, α, β and c, Hopf bifurcation occurs that implies the prey and predator population coexist and show periodic behavior. It is indicated that these parameters have a destabilizing influence on the system (3). Furthermore, the amplitude of the oscillations grows as γ, α and c increase respectively (see Figs. 6(a), 6(b) and Fig. 7(a)), while the rise in amplitude does not grow monotonically as β increases (see Fig. 6(c)). Fig. 7(b) and Fig. 8 show that the periodic oscillations gradually disappear as the parameters f, d_1, d_2, d_3 and a increase. For larger values of f (Fig. 7(b)), d_3 (Fig. 8(c)) and a (Fig. 8(d)), the predator population cannot persist and goes extinct, and for greater values of d_1 (Fig. 8(a)) and d_2 (Fig. 8(b)) both the prey and predator populations cannot survive.

Fig. 9 depicts the impact of the strength of prey refuge via the parameter m, and other parameters are taken as in Table 3. With an increase in m, the system (3) respectively exhibits periodic oscillations (see Figs. 9(a), $m = 0.15$), a limit cycle around the interior equilibrium point $E^* = (N^*, P^*)$ (Figs. 9(b), $m = 0.3955$), and a stable positive equilibrium E^* (Figs. 9(c), $m = 0.6$). The maximum Lyapunov exponent graphs in Fig. 10 with respect to the parameter m illustrate the system (3)'s transitions between the three forms of dynamics mentioned in Fig. 9. The negative Lyapunov exponent demonstrates that the system corresponds to a stable state (see Fig. 10(c)).



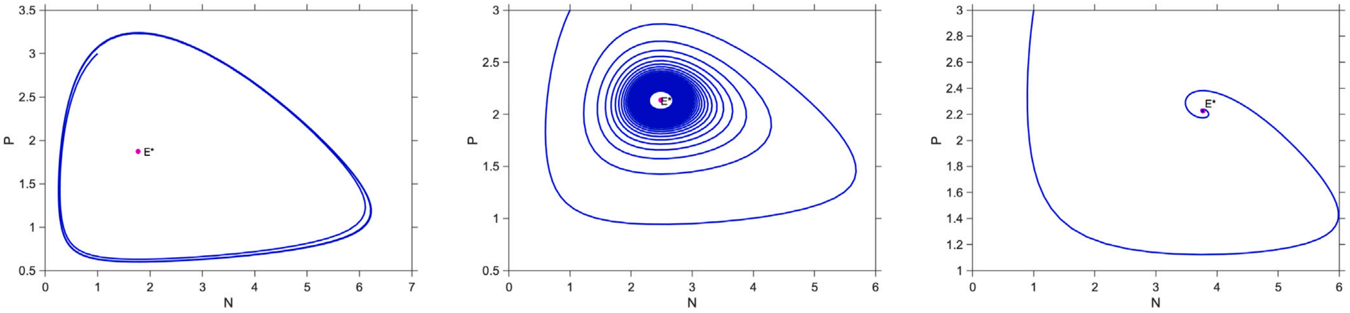
(a) (b)

Fig. 7. Bifurcation diagram of the system (1) without noises, respectively, with c and f as bifurcation parameters. (a) At $c = 0.4164$, there is a Hopf bifurcation; (b) at $f = 3.088$, a bifurcation disappears and the predator population P dies out with an increase in f.



(a) (b) (c) (d)

Fig. 8. Bifurcation diagram of the system (1) without noises, respectively, with d_1, d_2, d_3 and a as bifurcation parameters. At (a) $d_1 = 0.544$, (b) $d_2 = 0.3002$, (c) $d_3 = 0.6399$ and (d) $a = 0.6828$, the bifurcation and periodic oscillation are no longer present.



(a) $m = 0.15$ (b) $m = 0.3955$ (c) $m = 0.6$

Fig. 9. Solutions of the system (3) with different values for m. Other parameters are fixed in Table 3. (a) Periodic oscillations when $m = 0.15$; (b) limit cycle when $m = 0.3955$; (c) stable equilibrium when $m = 0.6$, respectively.

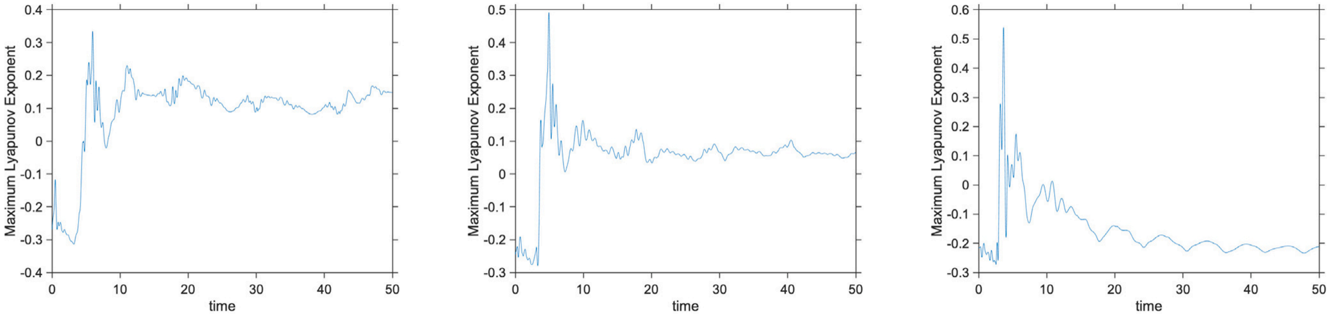
4.2. The stochastic system (1)

Numerical simulations are performed in this section to validate the theoretical conclusions. We use the Milstein higher-order approach introduced in [53] for the stochastic system (1), and the discretization form of the stochastic system is provided by:

$$\begin{cases} N^{k+1} = N^k + \left(\dfrac{\gamma N^k (1 + cN^k)}{1 + cN^k + fP^k} - d_1 N^k - d_2 {N^k}^2 - \dfrac{\beta(1-m)N^k P^k}{1 + a(1-m)N^k} \right) \Delta t \\ +N^k \left(\sigma_1 \sqrt{\Delta t}\omega_k + \dfrac{\sigma_1^2}{2}(\omega_k^2 - 1)\Delta t \right), \\ P^{k+1} = P^k + \left(\dfrac{\alpha\beta(1-m)N^k P^k}{1 + a(1-m)N^k} - d_3 P^k \right) \Delta t + P^k \left(\sigma_2 \sqrt{\Delta t}\omega_k + \dfrac{\sigma_2^2}{2}(\omega_k^2 - 1)\Delta t \right), \end{cases} \tag{47}$$

where $\Delta t > 0$ is the time step, σ_i^2 $(i = 1, 2)$ denote the intensity of white noises, ω_k $(k = 1, 2, \ldots, n)$ are mutually independent Gaussian random variables following the distribution $N(0, 1)$.

Fig. 11 depicts the time series and stationary distributions of prey $N(t)$ and predator $P(t)$ for the stochastic system (1) at $t = 1000$ from 1000000 simulations with different values of σ_i $(i = 1, 2)$ and keeping all parameters fixed as in Table 3, with the probability density functions of $N(t)$ and $P(t)$ represented by red smoothed curves. Figs. 11(a), 11(b) and 11(c) show a visible difference in the stationary distributions as the magnitude of σ_i $(i = 1, 2)$ increases. It can be seen that the average values and skewness of the distribution for $N(t)$ and $P(t)$ change when the intensity of noise σ_i $(i = 1, 2)$ increases. When $\sigma_i = 0.01$ $(i = 1, 2)$, in Fig. 11(a), the distribution is similar to a normal distribution. With



(a) $m = 0.15$, $\lambda_{max} = 0.148$ (b) $m = 0.3955$, $\lambda_{max} = 0.0653$ (c) $m = 0.6$, $\lambda_{max} = -0.211$

Fig. 10. The maximum Lyapunov exponent for m with different values and other parameters are the same as those in Fig. 9. (a) $m = 0.15$, $\lambda_{max} = 0.148$; (b) $m = 0.3955$, $\lambda_{max} = 0.0653$; (c) $m = 0.6$, $\lambda_{max} = -0.211$.

increasing σ_i $(i = 1, 2)$ to 0.2, the distribution is positively skewed, see Fig. 11(c).

Fixing the noise intensity $\sigma_1 = 0.01$ and other parameters are taken as in Table 3, Fig. 12 illustrates the influence of noise intensity σ_2 which respect to predator population to the stochastic system (1). We can observe that as the amount of σ_2 increases, the prey population N survives and the density of N increases, while the predator population P becomes extinct (see Figs. 12(a), 12(b) and 12(c)).

Studying the transition time from an initial state to extinction is crucial for both predator and prey species. As an example, we conduct 10,000 simulations using the stochastic system (1) with equal values for $\sigma := \sigma_1 = \sigma_2$. We vary σ within the range of 0.1 to 1 and calculate the extinction time for $N(t)$ and $P(t)$ as a function of σ. This analysis allows us to observe how changes in σ affect the duration until $N(t)$, $P(t)$ reach extinction. Fig. 13 depicts the relationship between the mean first passage time (MFPT) and the intensity of multiplicative noise σ. Initially, the mean first passage time increases, but then it starts to decrease as the noise intensity increases further. At a certain point, a maximum value is reached, creating a resonance peak in the curve. This resonance phenomenon implies that the system (1)'s stability is enhanced, making it less susceptible to phase transitions.

Figs. 13(a) and 13(b) show the relationship of MFPT with the carry-over effect parameter c or fear level induced by predator f and the noise intensity σ in the system (1), respectively. The peak height in Fig. 13(a) decreases as the carry-over effect parameter c increases, whereas in Fig. 13(b), the peak height increases with a higher predator-induced fear level f. That is, when the noise intensity value is kept constant, increasing the carry-over effect parameter accelerates the mean first passage time (see Fig. 13(a)). In this case, the acceleration of time implies that the likelihood of extinction increases. Higher levels of predator-induced fear resulted in a slower mean first passage time, indicating a decreased probability of extinction for the population (see Fig. 13(b)).

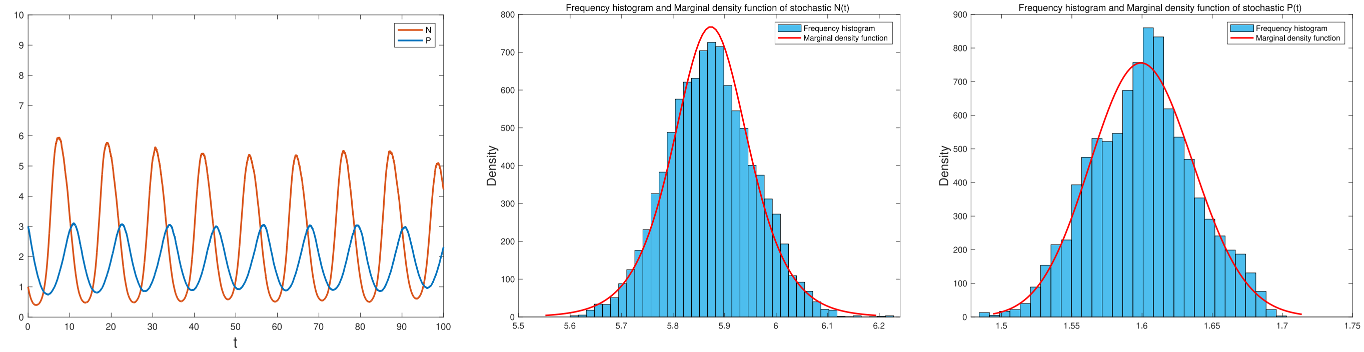
According to Theorem 3.3, the system (1) has a unique log-normal probability density function $\mathcal{P}(N, P)$ around the quasi-endemic equilibrium $E^+ = (N^+, P^+)$. Then, we illustrate how the probability density function graph changes when the environmental noise factors vary. Fig. 14 shows graphs of $\mathcal{P}(N, P)$ derived using parameters $\sigma_1 = 0.1$, $\sigma_2 = 0.6$ (see Fig. 14(a)), $\sigma_1 = \sigma_2 = 0.6$ (Fig. 14(b)), $\sigma_1 = 0.6$, $\sigma_2 = 0.72$ (Fig. 14(c)), and $\sigma_1 = 1.25$, $\sigma_2 = 0.8$ (Fig. 14(d)), respectively, other parameters are kept unchanged as given in Table 3. The two bigger environmental noises work to the shape parameter $\mathcal{P}(N, P)$.

Furthermore, the relationship between the parameters d_3, σ_2 and the critical values $\mathcal{R}_s$ and $\widetilde{\mathcal{R}}_s$ of the stochastic system (1) is shown in Fig. 15, and the remaining parameter values are the same in Table 3. Figs. 15(a) and 15(b) delineate a full look at the gradual changes of $\mathcal{R}_s$ and $\widetilde{\mathcal{R}}_s$ by both the natural death rate of predator d_3 and the noise intensity σ_2.

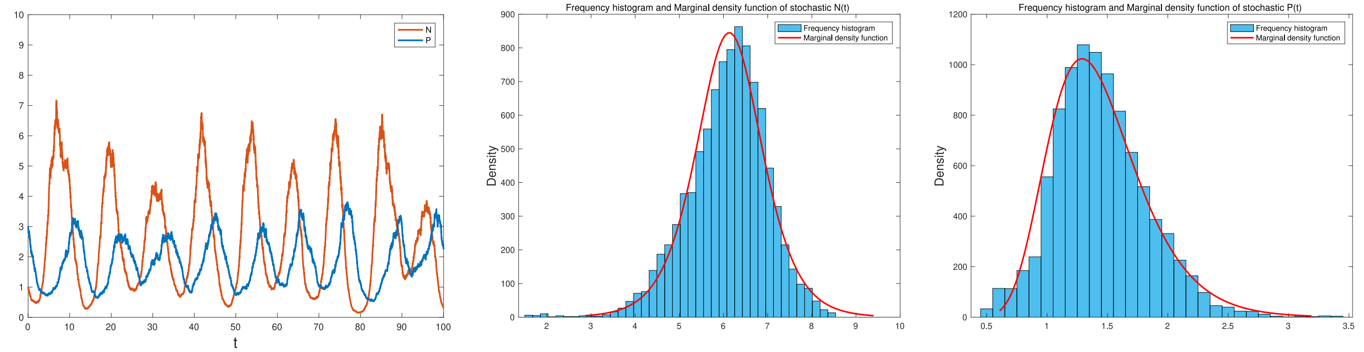
5. Conclusions

The prey–predator system with Holling-type II functional response is investigated in this paper in the presence of environmental noise, a prey refuge, predation fear, and its carry-over effects. The deterministic version of system (1) (i.e., the system (3)) demonstrates complicated dynamics in terms of both fear and its carry-over effects, as well as refuge characteristics, as demonstrated by theoretical results and numerical simulations. By altering the parameter values of γ, α, β, c, f, d_1, d_2, d_3, a or m, the deterministic system (i.e., $\sigma_1 = \sigma_2 = 0$) can undergo supercritical or subcritical bifurcation behavior (see Figs. 6, 7, 8 and 9). As a result, we see that the prey refuge m, predation fear f, and its carry-over effect c play a critical part in the system's stability. We also discover that prey refuge or predation fear can cause shifts between the species' survival and extinction states.

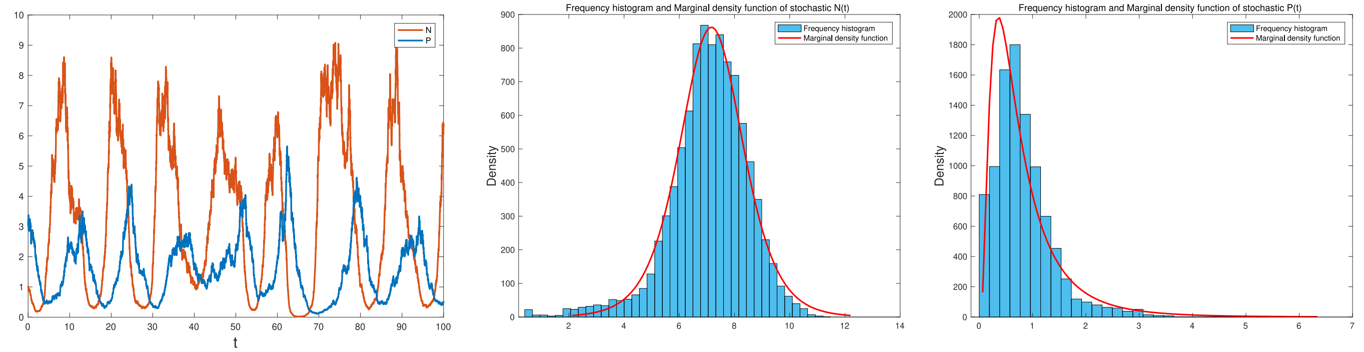
On the other side, our assumption is that the environmental disturbances are modeled as white noise and that they directly influence prey and predator populations. Based on this, we investigated if the stochastic system (1) is stable and how environmental fluctuations affect it to understand how the species might survive in the long run. By Theorem 3.2, under $\mathcal{R}_s = \frac{\alpha\beta(1-m)(\gamma-d_1-\frac{\sigma_1^2}{2})^2}{(\gamma-d_1)(d_2+a(1-m)(\gamma-d_1))(d_3+\frac{\sigma_2^2}{2})} > 1$ and extra conditions, the system (1) has a unique ergodic stationary distribution $\boldsymbol{\pi}(\cdot)$, which contributes to the two species' stochastic persistence. Moreover, by considering the two dimensional Fokker–Planck equation, Theorem 3.3 demonstrates that under the condition $\widetilde{\mathcal{R}}_s = \frac{\alpha\beta}{a(d_3+\frac{\sigma_2^2}{2})} > 1$, the stationary distribution $\boldsymbol{\pi}(\cdot)$ around $E^+ = (N^+, P^+)$ has a log-normal density function $\mathcal{P}(N, P)$. We are concerned with persistence, which not only indicates that the species is not on the verge of extinction but also indicates that it is settling into a form of random equilibrium. Theorem 3.4 further demonstrates the extinction of both prey and predator species for a long time when $\widehat{\mathcal{R}}_s = \frac{2\gamma}{\min\left\{(d_1+\frac{\sigma_1^2}{2}), (d_3+\frac{\alpha^2\sigma_2^2}{2})\right\}} < 1$ holds. Environmental noises may alter the dynamic behavior of the species, and if the disturbance is severe enough, it is very destructive to the persistence of the population and can result in the extinction of prey and predator populations. In other words, multiplicative noise leads to the presence of absorbing barriers at $N = 0$ and $P = 0$, causing ecological systems to converge towards these states as noise intensities increase. Anomalous fluctuations occur during the decay process. We also analyze that the mean first passage time is influenced by both the intensity of multiplicative noise and the carry-over effect parameter c, or the fear level induced by the predator f. Interestingly, these factors exhibit a similar effect on MFPT. Specifically, the curve of MFPT demonstrates a maximum value, indicating the presence of a resonance phenomenon. The study's findings indicated that the survival of the population is adversely affected by high carry-over effect parameters, while high levels of fear induced by predators promote population



(a) $\sigma_1 = \sigma_2 = 0.01$, $\mathscr{R}_s = 1.776 > 1$, $\widetilde{\mathscr{R}}_s = 2.327 > 1$



(b) $\sigma_1 = \sigma_2 = 0.1$, $\mathscr{R}_s = 1.752 > 1$, $\widetilde{\mathscr{R}}_s = 2.306 > 1$



(c) $\sigma_1 = \sigma_2 = 0.2$, $\mathscr{R}_s = 1.682 > 1$, $\widetilde{\mathscr{R}}_s = 2.246 > 1$

Fig. 11. Time series and histogram of the probability density function for $N(t)$ and $P(t)$ at $t = 1000$ for the stochastic system (1) with three different values of σ_i $(i = 1, 2)$: (a) $\sigma_1 = \sigma_2 = 0.01$; (b) $\sigma_1 = \sigma_2 = 0.1$; (c) $\sigma_1 = \sigma_2 = 0.2$, the probability density functions of $N(t)$ and $P(t)$ are represented by the red smoothed curves. Keeping all parameters fixed as in Table 3. Number of simulations for the frequency histogram fitting density curves of N and P of system (1) with 1000000 iteration points, and the run time of our code is about 6.9547 s.

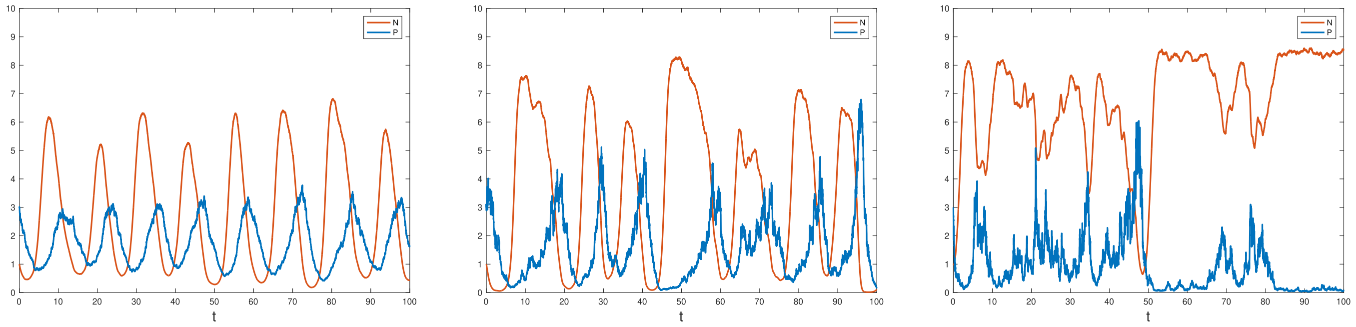
survival. All of these results might influence population control and conservation biology.

Finally, numerical simulations confirm our mathematical findings that include: (1) the existence of predation fear and prey refuge can end the oscillations and bring the deterministic system back to a stable state, even to a boundary state; (2) the system undergoes a Hopf bifurcation as the level of predation fear's carry-over effects gradually increases; and (3) the environmental noise can affect the growth of predator and prey populations, and when it is big enough, the random disturbance will alter the dynamics of the stochastic system.

There are some interesting topics that could be explored using the proposed system. For example, the system could be extended to investigate how different types of disturbances, such as multiplicative noise or human disturbance, impact population dynamics. Additionally, the proposed model could be applied to inform conservation and management strategies by predicting the effects of various management actions, such as habitat restoration or invasive species removal, on population dynamics. These topics are expected to be addressed in future research.

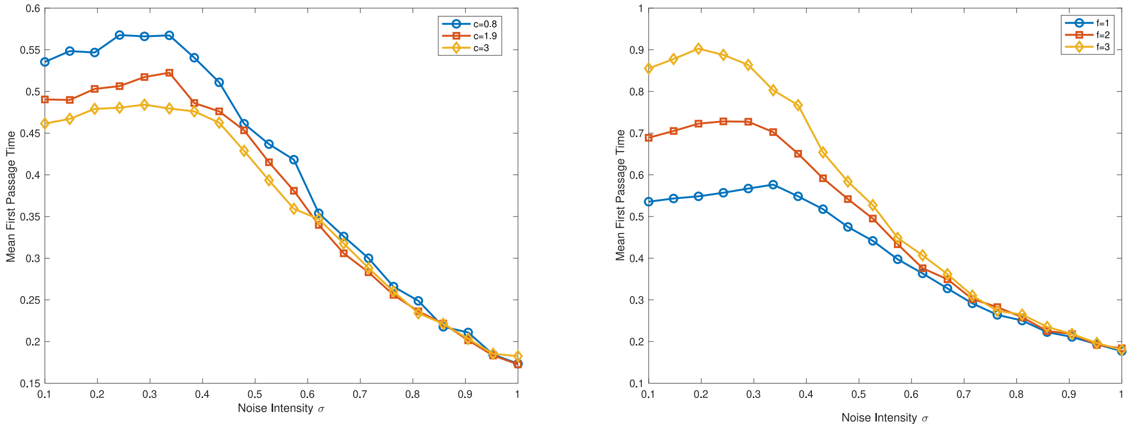
CRediT authorship contribution statement

Feng Rao: Conceptualization, Methodology, Mathematical analysis, Software, Writing – original draft & editing. **Yun Kang:** Supervision, Methodology, Writing – review & editing.



(a) $\sigma_2 = 0.1$ (b) $\sigma_2 = 0.3$ (c) $\sigma_2 = 0.6$

Fig. 12. Time series of the stochastic system (1) with different noise intensity σ_2. Other parameters are taken as in Table 3 and $\sigma_1 = 0.01$.



(a) (b)

Fig. 13. Mean first passage time as a function of multiplicative noise intensity σ for transferring from the initial value state $(N(0), P(0))$ to the state of the extinction is plotted respectively for different values of the carry-over effect parameter c and predator-induced fear level f. (a) $c = 0.8, 1.9, 3$ and (b) $f = 1, 2, 3$. The other fixed parameter values are the same as in Table 3.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

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Appendix

Proof of Theorem 2.2

Proof. The proof of the existence of equilibria is straightforward, so it is omitted here. To prove the stability of equilibria, let E^* be the corresponding equilibrium of the above three cases, $J|_{(E^*)}$ the corresponding Jacobian matrix evaluated at E^* and $\lambda_i(E^*)\,(i = 1, 2)$ the eigenvalues of $J|_{(E^*)}$.

Case (a) The system (48) always has an equilibrium $E^* = (\frac{d_3}{\alpha\beta - ad_3}, \frac{\alpha(\gamma(\alpha\beta - ad_3) - d_2 d_3)}{(\alpha\beta - ad_3)^2})$. The Jacobian matrix of (48)

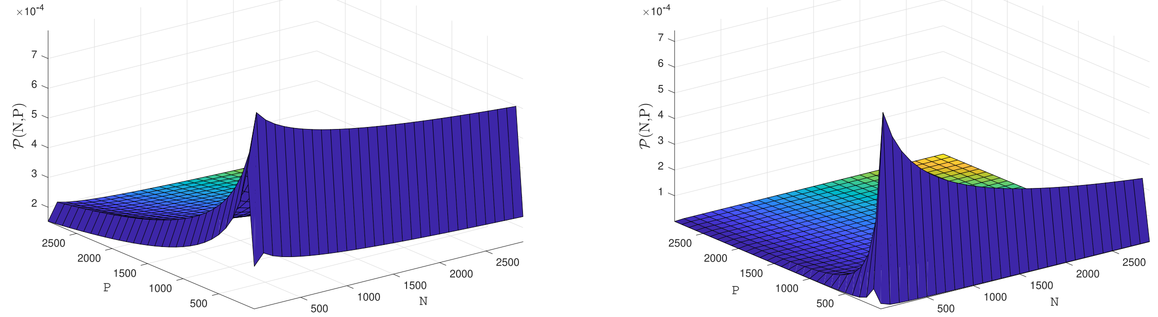
$$\begin{cases} \dfrac{\mathrm{d}N(t)}{\mathrm{d}t} = \gamma N - d_2 N^2 - \dfrac{\beta N P}{1 + aN}, \\ \dfrac{\mathrm{d}P(t)}{\mathrm{d}t} = \dfrac{\alpha\beta N P}{1 + aN} - d_3 P, \end{cases} \tag{48}$$

evaluated at E^* is

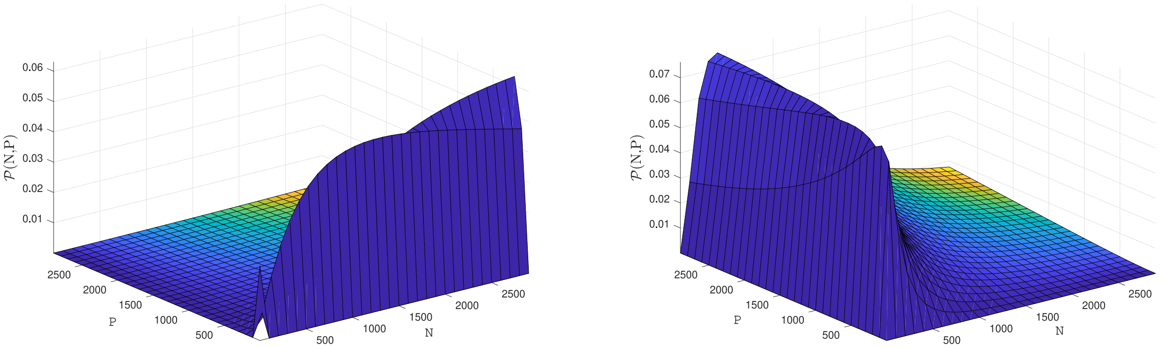
$$J|_{(E^*)} = \begin{pmatrix} \frac{-d_3(a\gamma(\alpha\beta - ad_3) - d_2(\alpha\beta + ad_3))}{-\alpha\beta(\alpha\beta - ad_3)} & -\frac{d_3}{\alpha} \\ \frac{(\alpha\beta - ad_3)\gamma - d_2 d_3}{\beta} & 0 \end{pmatrix}. \tag{49}$$

Direct computation yields

$$\lambda_1(E^*) + \lambda_2(E^*) = \frac{d_3(a\gamma(\alpha\beta - ad_3) - d_2(\alpha\beta + ad_3))}{\alpha\beta(\alpha\beta - ad_3)},$$

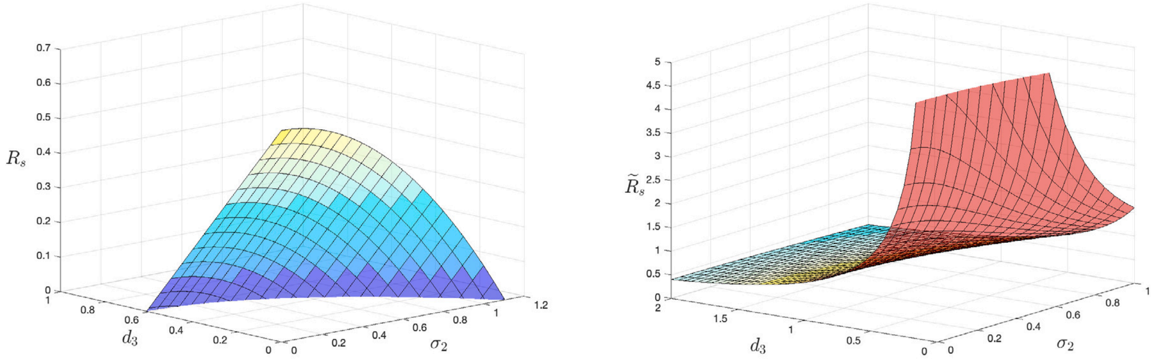


(a) $\sigma_1 = 0.1$, $\sigma_2 = 0.6$ (b) $\sigma_1 = \sigma_2 = 0.6$



(c) $\sigma_1 = 0.6$, $\sigma_2 = 0.72$ (d) $\sigma_1 = 1.25$, $\sigma_2 = 0.8$

Fig. 14. Graphs of the probability density function of (N, P) showing the effect of noises in the system (1). Other parameters are taken as in Table 3.



(a) (b)

Fig. 15. The combined influence of parameters d_3 and σ_2 on $\mathcal{R}_s$ and $\widetilde{\mathcal{R}}_s$ of the system (1), respectively. Other parameters are taken as in Table 3.

and

$$\lambda_1(E^*)\lambda_2(E^*) = \frac{d_3((\alpha\beta - ad_3)\gamma - d_2 d_3)}{\alpha\beta}.$$

Then the unique equilibrium E^* is locally asymptotically stable if $\frac{d_3}{\alpha\beta-ad_3} < \frac{\gamma}{d_2} < \frac{\alpha\beta+ad_3}{a(\alpha\beta-ad_3)}$. It follows from the Poincare–Bendixson theorem [54] that all solutions of (48) converge to E^*, i.e., E^* is globally asymptotically stable.

Case (b) The system (50) always has an equilibrium $E^* = (\frac{d_3}{\alpha\beta-ad_3}, \frac{\alpha((\alpha\beta-ad_3)(\gamma-d_1)-d_2d_3)}{(\alpha\beta-ad_3)^2})$. The Jacobian matrix of (50)

$$\begin{cases} \dfrac{\mathrm{d}N(t)}{\mathrm{d}t} = \gamma N - d_1 N - d_2 N^2 - \dfrac{\beta N P}{1+aN}, \\ \dfrac{\mathrm{d}P(t)}{\mathrm{d}t} = \dfrac{\alpha\beta N P}{1+aN} - d_3 P, \end{cases} \tag{50}$$

evaluated at E^* is

$$J|_{(E^*)} = \begin{pmatrix} \frac{d_3(a(\alpha\beta-ad_3)(\gamma-d_1)-d_2(\alpha\beta+ad_3))}{\alpha\beta(\alpha\beta-ad_3)} & -\frac{d_3}{\alpha} \\ \frac{(\alpha\beta-ad_3)(\gamma-d_1)-d_2d_3}{\beta} & 0 \end{pmatrix}. \tag{51}$$

By calculating the eigenvalues, we have

$$\lambda_1(E^*)+\lambda_2(E^*)=\frac{d_3(a(\alpha\beta-ad_3)(\gamma-d_1)-d_2(\alpha\beta+ad_3))}{\alpha\beta(\alpha\beta-ad_3)},$$

and

$$\lambda_1(E^*)\lambda_2(E^*)=\frac{d_3((\alpha\beta-ad_3)(\gamma-d_1)-d_2d_3)}{\alpha\beta}.$$

It follows that the unique equilibrium E^* of system (50) is globally asymptotically stable if $\frac{d_3}{\alpha\beta-ad_3}<\frac{\gamma-d_1}{d_2}<\frac{\alpha\beta+ad_3}{a(\alpha\beta-ad_3)}$.

Case (c) The system (52) always has an equilibrium $E^* = (\frac{d_3}{(\alpha\beta-ad_3)(1-m)}, \frac{\alpha((\alpha\beta-ad_3)(\gamma-d_1)(1-m)-d_2d_3)}{(\alpha\beta-ad_3)^2(1-m)^2})$. The Jacobian matrix of (52)

$$\begin{cases}\dfrac{\mathrm{d}N(t)}{\mathrm{d}t}=\gamma N-d_1N-d_2N^2-\dfrac{\beta(1-m)NP}{1+a(1-m)N},\\[2mm] \dfrac{\mathrm{d}P(t)}{\mathrm{d}t}=\dfrac{\alpha\beta(1-m)NP}{1+a(1-m)N}-d_3P,\end{cases} \tag{52}$$

evaluated at E^* is

$$J|_{(E^*)}=\begin{pmatrix}\frac{d_3(a(\alpha\beta-ad_3)(\gamma-d_1)(1-m)-d_2(\alpha\beta+ad_3))}{\alpha\beta(\alpha\beta-ad_3)(1-m)} & -\frac{d_3}{\alpha}\\ \frac{(\alpha\beta-ad_3)(\gamma-d_1)(1-m)-d_2d_3}{\beta(1-m)} & 0\end{pmatrix}. \tag{53}$$

By computing the eigenvalues, we obtain that

$$\lambda_1(E^*)+\lambda_2(E^*)=\frac{d_3(a(\alpha\beta-ad_3)(\gamma-d_1)(1-m)-d_2(\alpha\beta+ad_3))}{\alpha\beta(\alpha\beta-ad_3)(1-m)},$$

and

$$\lambda_1(E^*)\lambda_2(E^*)=\frac{d_3((\alpha\beta-ad_3)(\gamma-d_1)(1-m)-d_2d_3)}{\alpha\beta(1-m)}.$$

Then the unique equilibrium E^* of system (52) is globally asymptotically stable if $\frac{d_3}{(\alpha\beta-ad_3)(1-m)}<\frac{\gamma-d_1}{d_2}<\frac{\alpha\beta+ad_3}{a(\alpha\beta-ad_3)(1-m)}$.

Proof of Theorem 2.1

Proof. We skip the analytical detail of the forward part of Theorem 2.1 since it is similar to the proof of Theorem 2.1 in [13]. We only show the items (i), (ii), and (iii).

If $\frac{\gamma-d_1}{d_2}<\frac{d_3}{\alpha\beta(1-m)}$, then from the system (3) and $\lim_{t\to\infty}\sup N(t)\le\frac{\gamma-d_1}{d_2}$, we have

$$\begin{aligned}\frac{\mathrm{d}P(t)}{\mathrm{d}t}&=\frac{\alpha\beta(1-m)NP}{1+a(1-m)N}-d_3P\\&=\left(\frac{\alpha\beta(1-m)N}{1+a(1-m)N}-d_3\right)P\\&\le\left(\frac{\alpha\beta(1-m)\frac{\gamma-d_1}{d_2}}{1+a(1-m)N}-d_3\right)P\\&\le\left(\frac{\alpha\beta(1-m)(\gamma-d_1)}{d_2}-d_3\right)P<0\end{aligned}$$

which indicates that $\lim_{t\to\infty}\sup P(t)=0$.

The system (3) gives the following equation:

$$\frac{\mathrm{d}(N+\frac{1}{\alpha}P)}{\mathrm{d}t}=\frac{\gamma N(1+cN)}{1+cN+fP}-d_1N-d_2N^2-\frac{d_3}{\alpha}P.$$

If $\gamma<\min\{d_1,d_3\}$, then

$$\frac{\mathrm{d}(N+\frac{1}{\alpha}P)}{\mathrm{d}t}\le\gamma N-d_1N-\frac{d_3}{\alpha}P\le\gamma(N+\frac{1}{\alpha}P)-\min\{d_1,d_3\}(N+\frac{1}{\alpha}P).$$

which indicates that $\lim_{t\to\infty}N+\frac{1}{\alpha}P=0$. Moreover, we have the following

$$\begin{aligned}\frac{\mathrm{d}(N+\frac{1}{\alpha}P)}{\mathrm{d}t}&=\frac{\gamma N(1+cN)}{1+cN+fP}-d_1N-d_2N^2-\frac{d_3}{\alpha}P\\&\ge(\frac{\gamma c}{1+cN+fP}-d_2)N^2-(d_1N+\frac{d_3}{\alpha}P)\\&\ge\frac{1}{c}(\gamma c-d_2-d_2(cN+fP))N-(d_1N+\frac{d_3}{\alpha}P)\\&\ge\frac{1}{c}(\gamma c-d_2-d_2c(N+\frac{f}{c}P))N-(d_1N+\frac{d_3}{\alpha}P)\\&\ge\frac{1}{c}(\gamma c-d_2-d_2c\max\{\frac{f}{c},\frac{1}{\alpha}\}(N+\frac{1}{\alpha}P))N-\max\{d_1,d_3\}(N+\frac{1}{\alpha}P)\\&\ge(\frac{1}{c}(\gamma c-d_2-d_2c\max\{\frac{f}{c},\frac{1}{\alpha}\}(N+\frac{1}{\alpha}P))-\max\{d_1,d_3\})(N+\frac{1}{\alpha}P).\end{aligned}$$

If there exists a positive number $w<\frac{\gamma c-d_2-d\max\{d_1,d_3\}}{d_2c\max\{\frac{f}{c},\frac{1}{\alpha}\}}$ such that $\frac{\mathrm{d}(N+\frac{1}{\alpha}P)}{(N+\frac{1}{\alpha}P)\mathrm{d}t}\Big|_{N+\frac{1}{\alpha}P=w}>0$. Then, $\lim\inf_{t\to\infty}N+\frac{1}{\alpha}P\ge w$ for any $N(0)+\frac{1}{\alpha}P(0)>w$.

Proof of Theorem 2.3

Proof. Let $\Omega_N=\{(N,P)\in\mathbb{R}_+^2 : w\le N+\frac{1}{\alpha}P\le\max\{\frac{\gamma-d_1}{d_2},\frac{(\gamma-d_1+d_3)^2}{4d_2d_3}\}\}\cap\{P=0\}$. Referring to the results in [55], the persistence of predator is determined by the sign of $\frac{\mathrm{d}P}{P\mathrm{d}t}|_{\Omega_N}=\frac{\mathrm{d}P}{P\mathrm{d}t}|_{E_1=(\frac{\gamma-d_1}{d_2},0)}$. The dynamics of P-class are governed by

$$\frac{\mathrm{d}P}{\mathrm{d}t}=P\left(\frac{\alpha\beta(1-m)N}{1+a(1-m)N}-d_3\right),$$

which gives

$$\begin{aligned}\frac{\mathrm{d}P}{P\mathrm{d}t}\Big|_{\Omega_N}&=\left(\frac{\alpha\beta(1-m)N}{1+a(1-m)N}-d_3\right)\Big|_{E_1=(\frac{\gamma-d_1}{d_2},0)}\\&=\frac{\alpha\beta(\gamma-d_1)(1-m)}{d_2+a(\gamma-d_1)(1-m)}-d_3>0\end{aligned}$$

if $\frac{\gamma-d_1}{d_2}>\frac{d_3}{(\alpha\beta-ad_3)(1-m)}$, which means that the persistence of predator for the system (3).

Proof of Theorem 2.4

Proof. Straightforward computation yields that the system (3) always has an extinction equilibrium $E_0=(0,0)$ and the prey-only equilibrium $E_1=(\frac{\gamma-d_1}{d_2},0)$ if $\frac{\gamma}{d_1}>1$. The stability of the boundary equilibria is obtained from the signs of eigenvalues of the corresponding Jacobian matrices. Thus, E_1 is locally asymptotically stable if $0<\frac{\gamma-d_1}{d_2}<\frac{d_3}{(\alpha\beta-ad_3)(1-m)}$, and it is unstable if $\frac{\gamma-d_1}{d_2}>\frac{d_3}{(\alpha\beta-ad_3)(1-m)}>0$.

The Jacobian matrix of the system (3) at $E^*=(N^*,P^*)$ is

$$J(E^*)=\begin{pmatrix}\frac{\gamma cfN^*P^*}{(1+cN^*+fP^*)^2}-d_2N^*+\frac{a\beta(1-m)^2N^*P^*}{(1+a(1-m)N^*)^2} & -\frac{\gamma f(1+cN^*)N^*}{(1+cN^*+fP^*)^2}-\frac{\beta(1-m)N^*}{1+a(1-m)N^*}\\ \frac{\alpha\beta(1-m)P^*}{(1+a(1-m)N^*)^2} & 0\end{pmatrix}, \tag{54}$$

where

$$\begin{aligned}J_{11}&=\frac{\gamma cfN^*P^*}{(1+cN^*+fP^*)^2}-d_2N^*+\frac{a\beta(1-m)^2N^*P^*}{(1+a(1-m)N^*)^2},\\J_{12}&=-\frac{\gamma f(1+cN^*)N^*}{(1+cN^*+fP^*)^2}-\frac{\beta(1-m)N^*}{1+a(1-m)N^*}<0,\\J_{21}&=\frac{\alpha\beta(1-m)P^*}{(1+a(1-m)N^*)^2}>0,\end{aligned}$$

and the characteristic equation at E^* is

$$\mathcal{F}(\lambda)=\lambda^2-J_{11}\lambda-J_{12}J_{21}=0. \tag{55}$$

Let $\lambda_1(E^*)$, $\lambda_2(E^*)$ be the roots of (55) and assume that $\Re\lambda_1(E^*) \leq \Re\lambda_2(E^*)$, we have

$$\lambda_1(E^*) + \lambda_2(E^*) = J_{11},$$
$$\lambda_1(E^*)\lambda_2(E^*) = -J_{12}J_{21} > 0.$$

Both roots of Eq. (55) are either negative or with negative real parts since if $J_{11} < 0$, i.e., $d_2 > \frac{\gamma c f P^*}{(1+cN^*+fP^*)^2} + \frac{a d_3^2 P^*}{\alpha^2 \beta N^{*2}}$, then we have $\Re\lambda_1(E^*) \leq \Re\lambda_2(E^*) < 0$, and the positive equilibrium $E^* = (N^*, P^*)$ is locally asymptotically stable; if $J_{11} > 0$, i.e., $d_2 < \frac{\gamma c f P^*}{(1+cN^*+fP^*)^2} + \frac{a d_3^2 P^*}{\alpha^2 \beta N^{*2}}$, then we have $0 < \Re\lambda_1(E^*) \leq \Re\lambda_2(E^*)$, and the positive equilibrium E^* is unstable.

From the first equation of system (3), we have

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= \frac{\gamma N(1+cN)}{1+cN+fP} - d_1 N - d_2 N^2 - \frac{\beta(1-m)NP}{1+a(1-m)N}\\ &\leq (\gamma - d_1)N - d_2 N^2 - \frac{\beta(1-m)NP}{1+a(1-m)N}.\end{aligned}$$

When $\frac{\gamma}{d_1} < 1$, then $\frac{\mathrm{d}N}{\mathrm{d}t} < 0$ which shows that $N(t)$ goes to zero as $t \to \infty$. It also causes $P(t)$ to go to zero. Thus, E_0 is globally asymptotically stable.

From the second equation of system (3), we get

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\alpha\beta(1-m)NP}{1+a(1-m)N} - d_3 P < \left(\frac{\alpha\beta(1-m)(\gamma-d_1)}{d_2 + a(1-m)(\gamma-d_1)} - d_3\right)P.$$

If $\frac{\gamma-d_1}{d_2} < \frac{d_3}{(\alpha\beta - a d_3)(1-m)}$, then $P(t) \to 0$ as $t \to \infty$. Hence, the system (3) reduces to a limiting system [56] as follows

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \gamma N - d_1 N - d_2 N^2,$$

which shows that $N(t) \to \frac{\gamma-d_1}{d_2}$. Thus $E_1 = (\frac{\gamma-d_1}{d_2}, 0)$ is globally attractive.

Let $\mathcal{F}_1(N,P) = \frac{\gamma N(1+cN)}{1+cN+fP} - d_1 N - d_2 N^2 - \frac{\beta(1-m)NP}{1+a(1-m)N}$, $\mathcal{F}_2(N,P) = \frac{\alpha\beta(1-m)NP}{1+a(1-m)N} - d_3 P$ and choose Dulac function $\mathcal{D}(N,P) = \frac{1}{NP}$, then we have

$$\begin{aligned}\frac{\partial \mathcal{F}_1(N,P)\mathcal{D}(N,P)}{\partial N} &= \frac{\partial\left(\frac{\gamma(1+cN)}{P(1+cN+fP)} - \frac{d_2 N}{P} - \frac{\beta(1-m)}{1+a(1-m)N}\right)}{\partial N}\\ &= \frac{\gamma c f}{(1+cN+fP)^2} + \frac{a\beta(1-m)^2}{(1+a(1-m)N)^2} - \frac{d_2}{P},\\ \frac{\partial \mathcal{F}_2(N,P)\mathcal{D}(N,P)}{\partial P} &= \frac{\partial\left(\frac{\alpha\beta(1-m)}{1+a(1-m)N} - \frac{d_3}{N}\right)}{\partial P} = 0.\end{aligned}$$

Therefore, if $d_2 > \frac{\gamma c f P}{(1+cN+fP)^2} + \frac{a\beta(1-m)^2 P}{(1+a(1-m)N)^2}$, the system (3) has no limit cycle according to Dulac–Bendixson theorem [57], which implies that this system has only equilibrium dynamics. That is, $E^* = (N^*, P^*)$ is globally asymptotically stable.

Proof of Theorem 2.5

Proof. Set $\lambda(c) = \xi(c) + \mathrm{i}\omega(c)$ be an eigenvalue of Eq. (55), then we obtain

$$\begin{aligned}&\xi^2 - \omega^2 - J_{11}\xi - J_{12}J_{21} = 0,\\ &2\xi\omega - J_{11}\omega = 0.\end{aligned} \tag{56}$$

There is $\lambda(c) = 0$ at the Hopf bifurcation point, then we let $c = c_H$,

$$\begin{aligned}&\omega^2(c_H) + J_{12}J_{21} = 0,\\ &J_{11}\omega(c_H) = 0,\end{aligned} \tag{57}$$

where $\omega(c_H) \neq 0$. Thus, $J_{11}(c_H) = 0$ and $\omega(c_H) = \sqrt{-J_{12}(c_H)J_{21}(c_H)} > 0$, where

$$\begin{aligned}c_H =\ & \frac{(\alpha\beta - a d_3)(1-m)}{2d_3^2(a(\alpha\beta - a d_3)^2(1-m)^2 P^* - \alpha^2\beta d_2)}\\ &\times\Big(\ (\alpha^2\beta\gamma f(\alpha\beta - a d_3)(1-m)P^*(4a d_3 f(\alpha\beta - a d_3)^2(1-m)^2 P^{*2}\\ &+(4a d_3(\alpha\beta - a d_3)^2(1-m)^2 + \alpha^2\beta\gamma f(\alpha\beta - a d_3)(1-m) - 4d_2 d_3\alpha^2\beta f)P^* - 4d_2 d_3\alpha^2\beta))^{\frac{1}{2}}\\ &-2a d_3 f(\alpha\beta - a d_3)^2(1-m)^2 P^{*2} - (2a d_3(\alpha\beta - a d_3)^2(1-m)^2 + \alpha^2\beta\gamma f(\alpha\beta - a d_3)(1-m)\\ &-2d_2 d_3\alpha^2\beta f)P^* + 2d_2 d_3\alpha^2\beta\ \Big).\end{aligned} \tag{58}$$

Differentiating Eq. (55) and $\xi(c_H) = 0$, we derive

$$\begin{aligned}&-2\omega\frac{\mathrm{d}\omega}{\mathrm{d}c} - J_{11}\frac{\mathrm{d}\xi}{\mathrm{d}c} = \frac{\mathrm{d}J_{12}J_{21}}{\mathrm{d}c},\\ &2\omega\frac{\mathrm{d}\xi}{\mathrm{d}c} - J_{11}\frac{\mathrm{d}\omega}{\mathrm{d}c} = \omega\frac{J_{11}}{\mathrm{d}c},\end{aligned}$$

thus, if $\left(2\omega^2\frac{\mathrm{d}J_{11}}{\mathrm{d}c} - J_{11}\frac{\mathrm{d}J_{12}J_{21}}{\mathrm{d}c}\right)\big|_{c=c_H} \neq 0$, we get

$$\left.\frac{\mathrm{d}\xi}{\mathrm{d}c}\right|_{c=c_H} = \left.\frac{2\omega^2\frac{\mathrm{d}J_{11}}{\mathrm{d}c} - J_{11}\frac{\mathrm{d}J_{12}J_{21}}{\mathrm{d}c}}{J_{11}^2 + 4\omega^2}\right|_{c=c_H} \neq 0.$$

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