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Interleaving by Parts: Join Decompositions of Interleavings and Join-Assemblage of Geodesics

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Abstract
Metrics of interest in topological data analysis (TDA) are often explicitly or implicitly in the form of an interleaving distance d_{I} between poset maps (i.e. order-preserving maps), e.g. the Gromov-Hausdorff distance between metric spaces can be reformulated in this way. We propose a representation of a poset map $\mathbf{F} : \mathcal{P} \to \mathcal{Q}$ as a join (i.e. supremum) $\bigvee_{b\in B} \mathbf{F}_b$ of simpler poset maps $\mathbf{F}_b$ (for a join dense subset $B \subset \mathcal{Q}$) which in turn yields a decomposition of d_{I} into a product metric. The decomposition of d_{I} is simple, but its ramifications are manifold: (1) We can construct a geodesic path between any poset maps $\mathbf{F}$ and $\mathbf{G}$ with $d_{\mathrm{I}}(\mathbf{F}, \mathbf{G}) < \infty$ by assembling geodesics between all $\mathbf{F}_b$s and $\mathbf{G}_b$s via the join operation. This construction generalizes at least three constructions of geodesic paths that have appeared in the literature. (2) We can extend the Gromov-Hausdorff distance to a distance between simplicial filtrations over an arbitrary poset with a flow, preserving its universality and geodesicity. (3) We can clarify equivalence between several known metrics on multiparameter hierarchical clusterings. (4) We can illuminate the relationship between the *erosion distance* by Patel and the *graded rank function* by Betthauser, Bubenik, and Edwards, which in turn takes us to an interpretation on the representation $\bigvee_b \mathbf{F}_b$ as a generalization of persistence landscapes and graded rank functions.

Keywords Persistent homology · Interleaving distance · Hierarchical clustering · Rank functions · Persistence landscapes

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1 Introduction

Persistent homology and interleaving distances.
Persistent homology plays a central role in topological data analysis (TDA) [1–3]. The most basic construction in persistent homology arises when applying the homology functor to an $\mathbb{R}$-indexed nested family of topological spaces or simplicial complexes such as the Vietoris-Rips filtration on a metric space. By utilizing homology with coefficients in a field $\mathbb{F}$, we obtain so-called *persistence modules*, which are $\mathbb{R}$-indexed functors valued in the category **vect** of vector spaces and linear maps over $\mathbb{F}$. Generalizing this notion, a poset-indexed functor valued in a certain category $\mathcal{C}$ is called a *generalized persistence module* with values in $\mathcal{C}$ [4].

One of the most prevalent metrics for quantifying the dissimilarity between two persistence modules is the *interleaving distance* d_{I}. Since d_{I} was first introduced in order to compare $\mathbb{R}$-indexed persistence modules [5], it has been generalized to various different settings [6–13]. One of the main uses of d_{I} is for comparing $\mathbb{R}^n$-indexed persistence modules, where its computation is known to be NP-hard for $n \geq 2$ [14].

While poly-time computable lower bounds for d_{I} have been studied for $n = 2$ [15–18], their extension to the case $n \geq 3$ has not received much attention. The *erosion distance* introduced by Patel [19] is an attractive alternative in this respect and will be subsequently further discussed.

Interleavings between poset maps.
Partially ordered sets are simply called *posets*. An order-preserving map $\mathcal{P} \to \mathcal{Q}$ between posets $\mathcal{P}$ and $\mathcal{Q}$ is called a *poset map*. By viewing the target poset $\mathcal{Q}$ as a category (each point $p \in \mathcal{Q}$ is an object and a unique arrow $p \to q$ exists whenever $p \leq q$ in $\mathcal{Q}$), a poset map can be viewed as a generalized persistence module. Poset maps are omnipresent in TDA, e.g. simplicial filtrations (indexed by arbitrary posets), hierarchical clusterings (indexed by arbitrary posets), and (generalized) rank functions of persistence modules. Poset maps are also utilized in discrete Morse theory, cf. [20, Theorem 11.4].

When $\mathcal{P}$ is equipped with a notion of *flow* [8, 11, 13], we can define an *interleaving distance* d_{I} between two poset maps $\mathcal{P} \to \mathcal{Q}$. Examples of such include the following.
1. Interleavings between simplicial filtrations. A distance between $\mathbb{R}$-indexed simplicial filtrations was proposed by Mémoli [21]. It turned out that this distance is a generalization of the Gromov-Hausdorff distance between finite metric spaces and thus called the Gromov-Hausdorff distance and denoted by d_{GH} [22]. This distance was proved to upper bound the bottleneck distance between persistence diagrams [21, Theorem 4.2] and even the homotopy interleaving distance by Blumberg and Lesnick [23]; see [13]. A certain variant of d_{GH} also appears in the study of metrics on *Reeb graphs* [24].
2. Interleavings between hierarchical clusterings over a poset. Hierarchical clustering methods on a metric space X encode multiscale clustering features of X [26], yielding a *dendrogram* (Fig. 1 (A)), a poset map from $[0, \infty)$ to the lattice of partitions of X. Dendrograms are widely generalized for density-sensitive hierarchical clustering [27–29], for hierarchical clustering on an asymmetric network [30], for summarizing clustering features in dynamic networks [31, 32], Fig. 1 (B)). These structures also arise in the study of phylogenetic trees [33–35] and phylogenetic networks [36–39]. We remark that these structures are finer than merge trees [40, 41] or Reeb graphs [10, 24, 42–44], as addressed in [45].
3. Interleavings between poset maps into a Grothendieck group. The *erosion distance* d_{E} was introduced for comparing *generalized persistence diagrams* of $\mathbb{R}$-indexed persistence modules valued in certain categories $\mathcal{C}$ beyond **vect** [19]. Generalized persistence diagrams

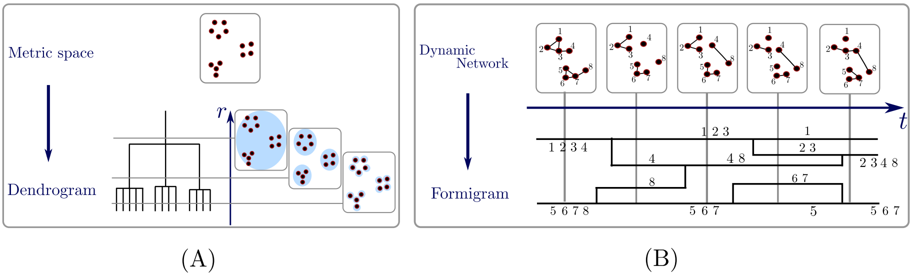

Fig. 1 (A) A dendrogram derived from a hierarchical clustering method on a metric space (Example 2.15). The dendrogram captures multiscale clustering features of the given metric space. (B) The formigram* derived from a dynamic network. The formigram tracks the evolution of connected components in the dynamic network. *The name formigram is a combination of the words formicarium (a.k.a. ant farm) and diagram. Synthetic flocking behaviors have been successfully classified via a certain lower bound for a distance between formigrams (Definition 2.22) [25]

can be encoded as certain poset maps, called *rank functions*, whose target is the *Grothendieck group* of $\mathcal{C}$ [46], equipped with a natural order. Then, d_{E} arises as an interleaving- type distance between rank functions. Even though d_{E} has been adapted to more abstract settings [47, 48], its most basic use is as a tractable lower bound for d_{I} between $\mathbb{R}^n$-indexed persistence modules *for any n* [49, Section 5]. Indeed, d_{E} has been utilized for classifying spatiotemporal persistent homology (encoded as $\mathbb{R}^3$-indexed persistence modules) of time-varying data [50].

Other related work.

The categorification of persistent homology has provided a fertile interpretation of persistence theory and the interleaving distance [4, 7, 8, 11, 13, 51]. Among others, Scoccola [13] introduced the notion of a *locally persistent category*, which is a category with a notion of approximate morphism. This enables us to define an interleaving distance between objects in the category, which encompasses many distances in TDA and facilitates a uniform treatment of those distances. For example, sufficient conditions under which an interleaving distance is geodesic have been found [13, Theorems 4.5.2 and 4.5.16].

Our contributions.

Let $(\mathcal{P}, \leq, \Omega)$ be a poset *with a flow* (Definition 2.7) and let $\mathcal{Q}$ be a poset with a *join-dense* subset $B \subset \mathcal{Q}$ (which always exists; see Section 2.1). Let $\mathbf{F}, \mathbf{G} : (\mathcal{P}, \leq, \Omega) \to (\mathcal{Q}, \leq)$ be any two poset maps and let $d_{\mathrm{I}}(\mathbf{F}, \mathbf{G})$ be their interleaving distance (Definition 2.9).

(i) We identify join representations $\mathbf{F} = \bigvee_{b \in B} \mathbf{F}_b$ and $\mathbf{G} = \bigvee_{b \in B} \mathbf{G}_b$ such that

$$d_{\mathrm{I}}(\mathbf{F}, \mathbf{G}) = \sup_{b \in B} d_{\mathrm{I}}(\mathbf{F}_b, \mathbf{G}_b),$$

where each of the $\mathbf{F}_b$s and $\mathbf{G}_b$s is structurally simple (Theorem 3.7). These join representations can be seen as a rendition of the algebraic decomposition of persistence modules (Remarks 3.8 and 3.11) as well as a generalization of *persistence landscapes* [52, 53] (Remark 4.4 (i)).

We harness item (i) in order to establish *all* of the following:

(ii) We show that d_{I} is an ℓ^∞-product[1] of multiple copies of a distance between upper sets in $\mathcal{P}$ (Theorem 3.10).

[1] Given any metric spaces (M_i, d_i), $i \in I$, the ℓ^∞-product metric is defined to be the metric $\sup_{i \in I} d_i$ on $\Pi_{i \in I} M_i$.

(iii) We show that d_{I}, as a distance between poset maps $\mathbf{F}, \mathbf{G} : \mathcal{P} \to \mathcal{Q}$, is geodesic under the assumption that $\mathcal{Q}$ is a complete lattice (Theorem 3.13). More specifically, we obtain a geodesic path between $\mathbf{F}$ and $\mathbf{G}$ by *assembling* geodesic paths between all $\mathbf{F}_b$s and $\mathbf{G}_b$s via the join operation.

All the metrics mentioned in subsequent items can be incorporated into the framework of interleaving distances between poset maps. This enables us to prove, in a uniform way, that all metrics in the items below are geodesic.[2]

(iv) We show that computing the erosion distance between rank functions of persistence modules amounts to computing a finite number of Hausdorff distances between certain geometric signatures of *graded rank functions* [52] (Theorem 4.3). An analogous statement holds when comparing multiparameter hierarchical clusterings (Theorem 4.18).

(v) We generalize the Gromov-Hausdorff distance between metric spaces to a distance between simplicial filtrations over $\mathcal{P}$. This distance inherits the universality property of the original Gromov-Hausdorff distance (Theorem 4.11), of which the celebrated Vietoris-Rips filtration stability theorem [54, 55] becomes a consequence (Theorem 4.15 and Remark 4.16 (ii)).

(vi) We establish the equivalence between several known metrics on multiparameter hierarchical clusterings (Definition 2.22, Theorem 4.24, Remark 4.25).

(vii) We elucidate the computational complexity of the interleaving distance between formigrams (Theorem 4.30).

Organization.
Section 2 reviews the notions of lattices, subpartitions, interleaving distances, and formigrams. Section 3 addresses items (i)–(iii) above and Section 4 addresses the rest of the items. Section 5 discusses open questions.

2 Preliminaries

We review the notions of lattices (Section 2.1), subpartitions (Section 2.2), interleaving distances (Section 2.3), and formigrams (Section 2.4).

2.1 Posets, Lattices, and Poset Maps

In this section, we recall basic terminology from the theory of ordered sets and lattices [56, 57].

A **poset** $\mathcal{P} = (\mathcal{P}, \leq)$ is a nonempty set $\mathcal{P}$ equipped with a partial order, i.e. a reflexive, anti-symmetric, and transitive relation $\leq$ on $\mathcal{P}$. An element $0 \in \mathcal{P}$ is said to be a **zero element** if $0 \leq p$ for all $p \in \mathcal{P}$. If a zero element exists in $\mathcal{P}$, then it is unique. Thus we refer to 0 as *the* zero element in $\mathcal{P}$. An element $1 \in \mathcal{P}$ is said to be a **unit element** if $p \leq 1$ for all $p \in \mathcal{P}$. For $p, q \in \mathcal{P}$ with $p \leq q$, we write $[p, q]$ for the set $\{r \in \mathcal{P} : p \leq r \leq q\}$. Also, we write $p^{\uparrow}$ for the set $\{r \in \mathcal{P} : p \leq r\}$. An **upper set** in $\mathcal{P}$ is a subset $A \subset \mathcal{P}$ such that if $p \in A$ and $p \leq q$ in $\mathcal{P}$, then $q \in A$.

The **join** of a subset $A \subset \mathcal{P}$ is defined to be the least upper bound of A, i.e. an element $q \in \mathcal{P}$ is the join of A if and only if

[2] Some of those distances are already known to be geodesic, but some are not. Known results will be cited at suitable places in the paper.

(i) $p \leq q$ for all $p \in A$.
(ii) for any $s \in \mathcal{P}$, if $p \leq s$ for all $p \in A$, then $q \leq s$.

The **meet** of a subset $A \subset \mathcal{P}$ is defined to be the greatest lower bound of A. If a join and a meet of A exist, then they are unique and are written $\bigvee A$ and $\bigwedge A$, respectively. When $A = \{p_1, p_2\} \subset \mathcal{P}$, we sometimes write $p_1 \vee p_2$ and $p_1 \wedge p_2$ instead of $\bigvee\{p_1, p_2\}$ and $\bigwedge\{p_1, p_2\}$ respectively. The elements $p_1 \vee p_2$ and $p_1 \wedge p_2$ of $\mathcal{P}$, if they exist, are called the join and meet of p_1 and p_2.

The poset $\mathcal{P}$ is said to be a **join-semilattice** (resp. **meet-semilattice**) if $\mathcal{P}$ admits all finite joins (resp. meets). If $\mathcal{P}$ is both join- and meet-semilattice, then $\mathcal{P}$ is said to be a **lattice**. $\mathcal{P}$ is called a **complete lattice** if the meet and join of *any* subset (possibly infinite) $A \subset \mathcal{P}$ exist. In fact, $\mathcal{P}$ is guaranteed to be a complete lattice if either the meet of any subset exists or the join of any subset exists [57, Theorem 3.4].

If a poset $\mathcal{P}$ has a zero element 0, then any nonzero $p \in \mathcal{P}$ such that $[0, p] = \{0, p\}$ is called an **atom** of $\mathcal{P}$. A nonzero element p of a lattice $\mathcal{P}$ is **(join-)irreducible** if p is not the join of two smaller elements, that is, if $p = q \vee r$, then $p = q$ or $p = r$. Note that every atom is join-irreducible. For example, for the ordered set $\mathcal{Q} = \{p < q < r\}$, p is the zero element, q is the unique atom, and q and r are join-irreducible. A **join representation** of $p \in \mathcal{P}$ is an expression $\bigvee A$ which evaluates to p for some $A \subset \mathcal{P}$.

When $\mathcal{P}$ includes a zero element, the zero element has the join representation $\bigvee \emptyset$. A join representation $\bigvee A$ of p is **irredundant** if $\bigvee A' = \bigvee A$ implies $A' = A$ for any $A' \subset A$. We say that a subset $A \subset \mathcal{P}$ **join-refines** another subset $B \subset \mathcal{P}$ if, for each $a \in A$, there exists some element $b \in B$ such that $a \leq b$. Join-refinement defines a preorder $\preceq$ on the subsets of $\mathcal{P}$. Fix $p \in \mathcal{P}$ and let $\mathrm{ijr}(p)$ be the set of irredundant join representations of p. If $(\mathrm{ijr}(p), \preceq)$ has a unique minimum element, then the minimum element is called the **canonical join representation of** p**.**

A subset $B \subset \mathcal{P}$ is said to be **join-dense** if every element of $\mathcal{P}$ is the join of a subset of B. Trivially, $\mathcal{P}$ itself is join-dense. The poset $\mathcal{P}$ is said to be $\bigvee$**-irreducibly generated** if the set of all join-irreducible elements of $\mathcal{P}$ is join-dense. Every finite poset is $\bigvee$-irreducibly generated [58].

Remark 2.1 Any join-dense subset must contain all irreducible elements. Therefore, when $\mathcal{P}$ is finite, the set of irreducible elements in $\mathcal{P}$ is the smallest join-dense subset of $\mathcal{P}$.

Given any two posets $\mathcal{P}$ and $\mathcal{Q}$, a map $\mathbf{F} : \mathcal{P} \to \mathcal{Q}$ is called an **order-preserving map** or a **poset map** if $p \leq q$ in $\mathcal{P}$ implies $\mathbf{F}(p) \leq \mathbf{F}(q)$. The collection of all poset maps from $\mathcal{P}$ to $\mathcal{Q}$ will be denoted by $[\mathcal{P}, \mathcal{Q}]$.

Remark 2.2 We regard $[\mathcal{P}, \mathcal{Q}]$ as a poset equipped with the partial order inherited from $\mathcal{Q}$, i.e. $\mathbf{F} \leq \mathbf{G}$ in $[\mathcal{P}, \mathcal{Q}] \Leftrightarrow \mathbf{F}(p) \leq \mathbf{G}(p)$, for all $p \in \mathcal{P}$.

2.2 Lattice of Subpartitions

In this section we review the notion of subpartition [29, 30, 32] and show that the collection of all subpartitions of a set is a lattice.

Let us fix a nonempty finite set X. A **partition of** X is a collection P of nonempty disjoint $B \subset X$, called **blocks**, such that the union of all blocks B is equal to X. Every partition P of X induces the equivalence relation $\sim_P$ given by: $x \sim_P x' \Leftrightarrow x, x' \in B$ for some block $B \in P$. Reciprocally, any equivalence relation $\sim$ of X induces the partition $X/\sim$. A **subpartition** Q **of** X is a partition Q of some $X' \subset X$. The set X' is said to be the **underlying set** of Q. The equivalence relation on X' induced by Q is said to be a **subequivalence relation** on X.

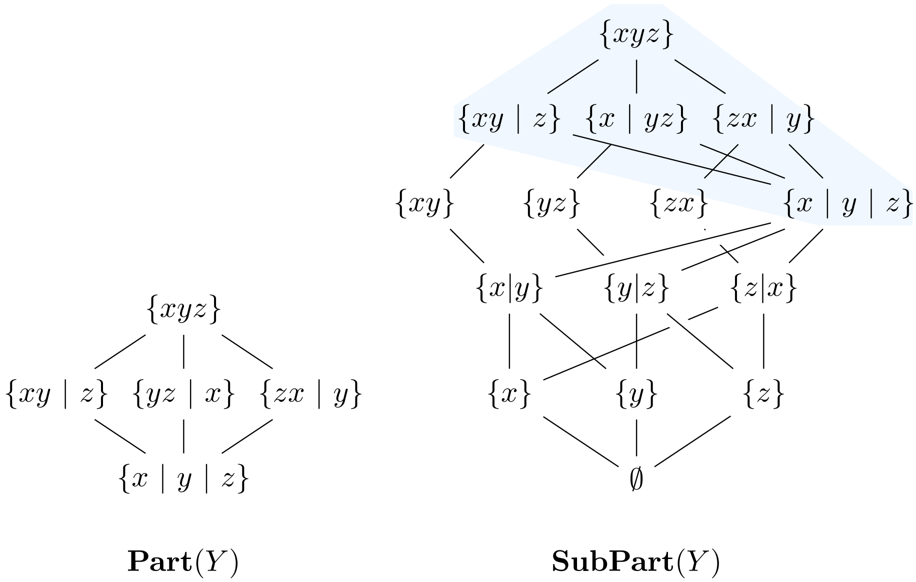

Fig. 2 The figure shows the embedding of $\mathbf{Part}(Y)$ into $\mathbf{SubPart}(Y)$ for $Y = \{x, y, z\}$ (subdiagram in the shaded region). For simplicity, distinct blocks in a partition are separated by | instead of curly brackets. In $\mathbf{Part}(Y)$, the join-irreducible elements are the atoms $\{xy \mid z\}, \{yz \mid x\}, \{zx \mid y\}$. On the other hand, in $\mathbf{SubPart}(Y)$, the join-irreducible elements are $\{x\}, \{y\}, \{z\}, \{xy\}, \{yz\}, \{zx\}$, and the first three singletons are, in particular, its atoms

Definition 2.3 (Poset of (sub)partitions) By $\mathbf{Part}(X)$ (resp. $\mathbf{SubPart}(X)$), we denote the set of all partitions (resp. subpartitions) of X. Let $P_1, P_2 \in \mathbf{Part}(X)$ (resp. let $P_1, P_2 \in \mathbf{SubPart}(X)$). P_1 is said to **refine** P_2 and write $P_1 \leq P_2$, if for any block $B_1 \in P_1$, there exists a block $B_2 \in P_2$ such that $B_1 \subset B_2$ (see Fig. 2 for an example).[3]

Note that for any $P \in \mathbf{SubPart}(X)$, we have $\emptyset \leq P \leq \{X\}$, i.e. $\emptyset$ and $\{X\}$ are the zero and unit elements of $\mathbf{SubPart}(X)$, respectively. It is well-known that $\mathbf{Part}(X)$ is a lattice [57]. We show that $\mathbf{Part}(X)$ is a sublattice of $\mathbf{SubPart}(X)$: Given any $P_1, P_2 \in \mathbf{SubPart}(X)$, let X_1 and X_2 be the underlying sets of P_1 and P_2, respectively.

(i) The join $P_1 \vee P_2$ is the quotient set $(X_1 \cup X_2)/\sim$, where $\sim$ is the smallest equivalence relation on $X_1 \cup X_2$ containing $\sim_{P_1}$ and $\sim_{P_2}$. The join $P_1 \vee P_2$ is also called the **finest common coarsening** of P_1 and P_2.

(ii) The meet $P_1 \wedge P_2$ is the quotient set $(X_1 \cap X_2)/(\sim_{P_1} \cap \sim_{P_2})$. The meet $P_1 \wedge P_2$ is also called the **coarsest common refinement** of P_1 and P_2.

If $X = X_1 = X_2$, then $P_1 \vee P_2$ and $P_1 \wedge P_2$ clearly belong to $\mathbf{Part}(X)$. Hence:

Proposition 2.4 $\mathbf{SubPart}(X)$ *equipped with the refinement relation is a lattice. In particular,* $\mathbf{Part}(X)$ *is a sublattice of* $\mathbf{SubPart}(X)$.

Remark 2.5 (i) The atoms of $\mathbf{SubPart}(X)$ are $\{\{x\}\}$ for $x \in X$. In what follows, let us assume that $|X| \geq 2$. (ii) The join-irreducible elements of $\mathbf{SubPart}(X)$ consist of all the atoms and all sets of the form $\{\{x, x'\}\}$ for $x, x' \in X$ with $x \neq x'$.

[3] See [57, Section 4] for properties of $\mathbf{Part}(X)$.

Proposition 2.6 *Let X be any nonempty finite set. (i)* $\mathbf{SubPart}(X)$ *is* $\bigvee$*-irreducibly generated. (ii) Let* $P \in \mathbf{SubPart}(X)$ *be a nonzero element. If* P *includes a block* B *with* $|B| \geq 3$*, then* P *has* no *canonical join representation.*

Proof (i): Let $Q \in \mathbf{SubPart}(X)$ and let $\sim_Q$ be the subequivalence relation on X corresponding to Q. Then $Q = \bigvee A$ where $A := \left\{\{x, x'\} : x \sim_P x'\right\}$ (when $x = x'$, the set $\{x, x'\}$ is a singleton). By Remark 2.5 (ii), $Q = \bigvee A$ is a join representation by irreducible elements. (ii): Without loss of generality, assume that $\{x, y, z\} \in P$. Observe that the following three are minimal join representations of the single block partition $\{xyz\}$ and hence a minimum does not exist:

$$\bigvee\{\{xy\}, \{yz\}\}, \quad \bigvee\{\{yz\}, \{zx\}\}, \quad \bigvee\{\{zx\}, \{xy\}\}.$$

This implies that, given a minimal join representation $\bigvee A$ of P by irreducible elements, exactly one of the three subsets $\{\{xy\}, \{yz\}\}$, $\{\{yz\}, \{zx\}\}$, $\{\{zx\}, \{xy\}\}$ is a subset of A. Whatever the case is, the corresponding subset can be replaced by any of the other two, and thus there is no canonical join representation of P. □

2.3 Posets with a Flow and Interleaving Distances

We review the notion of *poset with a flow* and its associated interleaving distance [8, 11].

Flows and interleavings

For a poset $\mathcal{P}$, let $\mathbf{I}_{\mathcal{P}}$ be the identity map on $\mathcal{P}$.

Definition 2.7 A (strict) **flow** on a poset $\mathcal{P}$ is a family $\Omega := \{\Omega_\varepsilon : \mathcal{P} \to \mathcal{P}\}_{\varepsilon \in [0,\infty)}$ of poset maps on $\mathcal{P}$ such that (i) $\Omega_s \leq \Omega_t$ for all $0 \leq s \leq t$, (ii) $\Omega_t \Omega_s = \Omega_{t+s}$, for all $s, t \in [0, \infty)$, and (iii) $\mathbf{I}_{\mathcal{P}} = \Omega_0$. We call the triple $(\mathcal{P}, \leq, \Omega)$ a **poset with a flow.** [4]

It is not hard to verify that, given a poset with a flow $(\mathcal{P}, \leq, \Omega)$, $\mathcal{P}$ is uncountable unless $\Omega_\varepsilon = \mathbf{I}_{\mathcal{P}}$ for all $\varepsilon \in [0, \infty)$.

We define an extended pseudometric between points in a poset with a flow [11] as follows:

Definition 2.8 (Interleaving of poset elements) Let $(\mathcal{P}, \leq, \Omega)$ be a poset with a flow. For $\varepsilon \in [0, \infty)$, $p, q \in \mathcal{P}$ are said to be ε**-interleaved** if $p \leq \Omega_\varepsilon(q)$ and $q \leq \Omega_\varepsilon(p)$. The **interleaving distance between** p **and** q is defined as:

$$d_\Omega(p, q) := \inf\{\varepsilon \in [0, \infty) : p, q \text{ are } \varepsilon\text{-interleaved}\}.$$

If p, q are not ε-interleaved for any $\varepsilon \in [0, \infty)$, then we declare that $d_\Omega(p, q) = \infty$.

By [11, Lem. 3.7] and [8, Theorem 3.21], we know that d_Ω is an extended pseudometric on $\mathcal{P}$. For example, let $\mathbb{R}^n$ be equipped with the product order given as $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \Leftrightarrow x_i \leq y_i$ for each $i = 1, \ldots, n$. Then, the supremum norm distance $||\cdot - \cdot||_\infty$ in $\mathbb{R}^n$ coincides with the interleaving distance with the flow Ω [11] given as

$$\Omega := \left(\Omega_\varepsilon : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n) + \varepsilon(1, \ldots, 1)\right)_{\varepsilon \in [0,\infty)}. \tag{1}$$

Next, we introduce two special examples of Definition 2.8.

Interleaving distance between poset maps

Let $(\mathcal{P}, \leq, \Omega)$ be a poset with a flow, and let $(\mathcal{Q}, \leq)$ be another poset. Recall that $[\mathcal{P}, \mathcal{Q}]$ is a poset (Remark 2.2). The flow Ω on $(\mathcal{P}, \leq)$ yields the flow $- \cdot \Omega$, given by pre-composition with Ω, on $[\mathcal{P}, \mathcal{Q}]$. Thus, Definition 2.8 is specialized to:

[4] By weakening conditions (ii) and (iii), we obtain the notion of *coherent flow* [11] (or *superlinear family of translations* [8]). This level of generality is not required for the purpose of this paper.

Definition 2.9 The **interleaving distance between poset maps F**, **G** : $(\mathcal{P}, \leq, \Omega) \to (\mathcal{Q}, \leq)$ is defined as:

$$d_{\mathrm{I}}(\mathbf{F}, \mathbf{G}) := \inf \{\varepsilon \in [0, \infty) : \mathbf{F}, \mathbf{G} \text{ are } \varepsilon\text{-interleaved w.r.t. the flow } - \cdot \Omega\}.$$

Interleaving distance between upper sets

Let $(\mathcal{P}, \leq, \Omega)$ be a poset with a flow. Let $(U(\mathcal{P}), \subset)$ be the poset of upper sets of $\mathcal{P}$ with the inclusion relation. Then, the flow Ω on $\mathcal{P}$ gives rise to a family $\widehat{\Omega} = (\widehat{\Omega}_\varepsilon)_{\varepsilon \in [0,\infty)}$ of poset maps $U(\mathcal{P}) \to U(\mathcal{P})$ given, for each $A \in U(\mathcal{P})$, as

$$\widehat{\Omega}_\varepsilon(A) := \{p \in \mathcal{P} : \Omega_\varepsilon(p) \in A\}.$$

Indeed $\widehat{\Omega}_\varepsilon(A)$ is an upper set. To see this let $x \in \widehat{\Omega}_\varepsilon(A)$ and let $x \leq y$ in $\mathcal{P}$. Then $\Omega_\varepsilon(x) \in A$ and $\Omega_\varepsilon(x) \leq \Omega_\varepsilon(y)$. Since A is an upper set, $\Omega_\varepsilon(y) \in A$, implying that $y \in \widehat{\Omega}_\varepsilon(A)$.

Proposition 2.10 *$\widehat{\Omega}$ is a flow on $U(\mathcal{P})$.*

Proof Let $A \in U(\mathcal{P})$. The equality $A = \widehat{\Omega}_0(A)$ is clear. Let $t \leq s$ in $[0, \infty)$. Then,

$$\begin{aligned}\widehat{\Omega}_t(A) &= \{p \in \mathcal{P} : \Omega_t(p) \in A\} \\ &\subset \{p \in \mathcal{P} : \Omega_s(p) \in A\} \qquad \text{since } \Omega_t(p) \leq \Omega_s(p) \text{ and } A \text{ is an upper set.} \\ &= \widehat{\Omega}_s(A).\end{aligned}$$

Also, for any $s, t \in [0, \infty)$,

$$\begin{aligned}\widehat{\Omega}_t(\widehat{\Omega}_s(A)) &= \left\{p \in \mathcal{P} : \Omega_t(p) \in \widehat{\Omega}_s(A)\right\} \\ &= \{p \in \mathcal{P} : \Omega_s(\Omega_t(p)) \in A\} = \{p \in \mathcal{P} : \Omega_{t+s}(p) \in A\} = \widehat{\Omega}_{t+s}(A).\end{aligned}$$

□

Definition 2.11 (Interleaving of upper sets) Let $(\mathcal{P}, \leq, \Omega)$ be a poset together with a flow. Then, $d_{\widehat{\Omega}}$ will denote the induced interleaving distance on the poset $(U(\mathcal{P}), \subset, \widehat{\Omega})$ of upper sets of $\mathcal{P}$.

The following remark and example will be useful in later sections.

Remark 2.12 (i) Arbitrary unions and intersections of upper sets in $\mathcal{P}$ yield upper sets, implying that $U(\mathcal{P})$ is a complete lattice.

(ii) For any family $(A_i)_i$ of upper sets in $\mathcal{P}$, we have

$$\widehat{\Omega}_\varepsilon\left(\bigcap_i A_i\right) = \bigcap_i \widehat{\Omega}_\varepsilon(A_i) \text{ and } \widehat{\Omega}_\varepsilon\left(\bigcup_i A_i\right) = \bigcup_i \widehat{\Omega}_\varepsilon(A_i).$$

In other words, $\widehat{\Omega}_\varepsilon$ preserves arbitrary meets and joins.

Example 2.13 Consider the poset **Int** $:= \{(a, b) \in \mathbb{R}^2 : a \leq b\}$, the upper-half plane in $\mathbb{R}^2$ above the diagonal line $y = x$, equipped with the order $(a, b) \leq (a', b') \Leftrightarrow a' \leq a < b \leq b'$. Let us define the flow Ω on **Int** by

$$\Omega := \left(\Omega_\varepsilon : (a, b) \mapsto (a - \varepsilon, b + \varepsilon)\right)_{\varepsilon \in [0,\infty)}. \tag{2}$$

For the poset $(U(\mathbf{Int}), \subset, \widehat{\Omega})$ of upper sets in **Int**, the distance $d_{\widehat{\Omega}}$ coincides with the Hausdorff distance d_{H} in $(\mathbf{Int}, ||-||_\infty)$ (Definition A4). This follows from the observation that for any $A \in U(\mathbf{Int})$, the ε-thickened set

$$A^\varepsilon := \left\{(a, b) \in \mathbf{Int} : \exists (a', b') \in A, \text{ such that } ||(a, b) - (a', b')||_\infty \leq \varepsilon\right\}, \tag{3}$$

coincides with $\widehat{\Omega}_\varepsilon(A)$.

2.4 Formigrams and their Interleavings

In this section we review the notion of formigrams and their specialized interleaving and Gromov-Hausdorff distances [32, 45] (note: Sections 3 and 4.1 can be read without reading this section).

Formigrams

We begin by reviewing the definition of dendrogram. Let us fix a nonempty finite set X.

Definition 2.14 ([26]) A **dendrogram** over a finite set X is any function $\theta : \mathbb{R}_{\geq 0} \to \mathbf{Part}(X)$ such that the following properties hold: (1) $\theta(0) = \{\{x\} : x \in X\}$, (2) if $t_1 \leq t_2$, then $\theta(t_1) \leq \theta(t_2)$, (3) there exists $T > 0$ such that $\theta(t) = \{X\}$ for $t \geq T$, (4) for all t there exists $\varepsilon > 0$ s.t. $\theta(s) = \theta(t)$ for $s \in [t, t+\varepsilon]$ (right-continuity) (see Fig. 1 (A)). The **ultrametric induced by** θ is the distance function $u_\theta : X \times X \to \mathbb{R}_{\geq 0}$ defined as

$$u_\theta(x, x') := \min \left\{ t \in \mathbb{R} : x \sim_{\theta(t)} x' \right\}.$$

Note that we have the strong-triangle inequality: $u_\theta(x, x'') \leq \max \left\{ u_\theta(x, x'), u_\theta(x', x'') \right\}$ for every $x, x', x'' \in X$.

Example 2.15 The **single linkage hierarchical clustering** method on a finite metric space (X, d_X) induces the dendrogram $\theta : \mathbb{R}_{\geq 0} \to \mathbf{Part}(X)$ given as $t \mapsto X/\sim_t$, where $\sim_t$ is the transitive closure of the relation $R_t := \{(x, x') \in X \times X : d_X(x, x') \leq t\}$ (Fig. 1 (A)). The mapping $(X, d_X) \mapsto (X, u_\theta)$ is known to be 1-Lipschitz with respect to the Gromov-Hausdorff distance (Definition A5) [26, Proposition 2], i.e.

$$d_{\mathrm{GH}}((X, u_{\theta_X}), (Y, u_{\theta_Y})) \leq d_{\mathrm{GH}}((X, d_X), (Y, d_Y)). \tag{4}$$

Formigrams, a mathematical model for time-varying clusters in dynamic networks [45], are a generalization of dendrograms. Formigrams are defined as *constructible cosheaves over* $\mathbb{R}$ [10, 59] with values in $\mathbf{SubPart}(X)$, which amounts to a (*costalk*-)function $\mathbb{R} \to \mathbf{SubPart}(X)$ described as follows.

Definition 2.16 A **formigram** over X is a *function*[5] $\theta : \mathbb{R} \to \mathbf{SubPart}(X)$ satisfying the following: there exists a finite set $\mathbf{crit}(\theta) = \{t_1, \ldots, t_n\} \subset \mathbb{R}$ of **critical points** s.t. (i) θ is constant on (t_i, t_{i+1}) for each $i = 0, \ldots, n$, where $t_0 = -\infty$ and $t_{n+1} = \infty$. (ii) At each critical point, θ is locally maximal, i.e.

$$\text{for } i \in \{1, \ldots, n\} \text{ and for } \varepsilon \in \left[0, \min_{j \in \{i, i+1\}} (t_j - t_{j-1})\right), \;\; \theta(t_i - \varepsilon) \leq \theta(t_i) \geq \theta(t_i + \varepsilon). \tag{5}$$

See Figs. 1 (B), 3, and 5 (A) for illustrative examples. Note that $\mathbf{crit}(\theta)$ is not necessarily unique nor nonempty. We also remark that, given a dendrogram $\theta : \mathbb{R}_{\geq 0} \to \mathbf{Part}(X)$, θ can be seen as a formigram by trivially extending its domain to $\mathbb{R}$, i.e. $\theta(t) := \emptyset$ for $t \in (-\infty, 0)$.

Since $\mathbf{SubPart}(X)$ is a poset, the collection of all formigrams on X can be regarded as a poset in its own right when endowed with the partial order defined as $\theta \leq \theta' \Leftrightarrow \theta(t) \leq \theta'(t)$, for all $t \in \mathbb{R}$.

Definition 2.17 By $\mathbf{Formi}(X)$, we denote the poset of all formigrams over X.

Interleaving distance on $\mathbf{Formi}(X)$

For $t \in \mathbb{R}$ and $\varepsilon \in [0, \infty)$ we denote the closed interval $[t - \varepsilon, t + \varepsilon]$ of $\mathbb{R}$ by $[t]^\varepsilon$.

[5] But not necessarily a poset map.

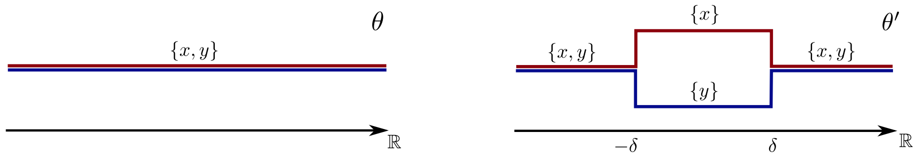

Fig. 3 The two formigrams in Example 2.21

Definition 2.18 ([45]) Let $\theta : \mathbb{R} \to \mathbf{SubPart}(X)$ be a formigram and let $\varepsilon \in [0, \infty)$. We define the *$\varepsilon$-smoothing* $\mathbf{S}_\varepsilon(\theta) : \mathbb{R} \to \mathbf{SubPart}(X)$ *of* θ as

$$\mathbf{S}_\varepsilon(\theta)(t) := \bigvee \left\{\theta(s) : s \in [t]^\varepsilon\right\}$$

, for $t \in \mathbb{R}$.

The family $\mathbf{S} = (\mathbf{S}_\varepsilon)_{\varepsilon \in [0,\infty)}$ constitutes a flow on the poset $\mathbf{Formi}(X)$ (Definition 2.7):

Proposition 2.19 ([45]) *Let $\varepsilon \in [0, \infty)$, and let θ be a formigram over X. Then, (i) the ε-smoothing $\mathbf{S}_\varepsilon(\theta)$ of θ is also a formigram over X and $\mathbf{S}_0(\theta) = \theta$. (ii) Also, for any $a, b \in [0, \infty)$, we have: $\mathbf{S}_a(\mathbf{S}_b(\theta)) = \mathbf{S}_{a+b}(\theta)$. (iii) Let θ' be another formigram over X. For any $\varepsilon \in [0, \infty)$, we have: $\theta \leq \theta' \Rightarrow \mathbf{S}_\varepsilon(\theta) \leq \mathbf{S}_\varepsilon(\theta')$.*

By Definition 2.8 and Proposition 2.19, we have:

Definition 2.20 The **interleaving distance between** $\theta, \theta' \in \mathbf{Formi}(X)$ is

$$d_\mathrm{F}(\theta, \theta') := \inf\left\{\varepsilon \in [0, \infty) : \theta, \theta' \text{ are } \varepsilon\text{-interleaved w.r.t. the flow } \mathbf{S}\right\}.$$

Since Definition 2.20 is a special case of Definition 2.8, we readily know that d_F is an extended pseudometric. In fact, d_F is an extended *metric*, not just a pseudometric. This can be proved by exploiting the fact that a formigram has finitely many critical points; similar ideas can be found in [45, Theorem 4.5]. We remark that when restricted to *treegrams* [30] (cf. Remark 4.23 (ii)) d_F agrees with what's been called the *ℓ^∞-cophenetic metric on phylogenetic trees* [60, 61].

Example 2.21 Let $\delta > 0$. Consider the formigrams $\theta, \theta' : \mathbb{R} \to \mathbf{SubPart}(\{x, y\})$ defined as:

$$\theta(t) = \{\{x, y\}\}, \text{ for all } t \in \mathbb{R}, \qquad \theta'(t) = \begin{cases} \{\{x, y\}\}, & \text{if } t \leq -\delta \\ \{\{x\}, \{y\}\}, & \text{if } -\delta < t < \delta \\ \{\{x, y\}\}, & \text{if } t \geq \delta \end{cases}$$

(see Fig. 3). We prove that $d_\mathrm{F}(\theta, \theta') = \delta$. Let $\varepsilon \in [0, \infty)$. Because the partition $\{\{x\}, \{y\}\}$ refines the partition $\{\{x, y\}\}$, we have $\theta' \leq \mathbf{S}_\varepsilon(\theta)$. Let us assume $\varepsilon \in [0, \delta)$: Then, we have $\mathbf{S}_\varepsilon(\theta')(0) = \bigvee_{s \in [0]^\varepsilon} \theta'(s) = \{\{x\}, \{y\}\}$ and $\theta(0) = \{\{x, y\}\}$. Thus, $\theta' \not\leq \mathbf{S}_\varepsilon(\theta)$, and θ, θ' are not ε-interleaved. On the other hand, $\mathbf{S}_\varepsilon(\theta') = \theta$ for $\varepsilon \in [\delta, \infty)$. and thus $d_\mathrm{F}(\theta, \theta') = \delta$.

Gromov-Hausdorff distance between formigrams

We may wish to compare formigrams over different sets. To this end, we revisit the Gromov-Hausdorff distance d_GH between formigrams [32, 45]; the naming of the distance is based on the fact that it generalizes the Gromov-Hausdorff distance between finite ultrametric spaces [32] (see Remark 4.25 (ii)). See Definition A5 for the original definition of the Gromov-Hausdorff distance between compact metric spaces. (*Note*: All the Gromov-Hausdorff distances in this paper will turn out to be special instances of the generalized Gromov-Hausdorff distance in Definition 4.9, as proved in Proposition A13 in the appendix.)

Let X, Z be two nonempty sets, let $P \in \mathbf{SubPart}(X)$, and let $\varphi : Z \twoheadrightarrow X$ be a surjective map. The **pullback** of P via φ is the subpartition of Z defined as $\varphi^* P := \{\varphi^{-1}(B) \subset Z : B \in P\}$, implying:

$$\begin{aligned} &z, z' \in Z \text{ belong to the same block of } \varphi^* P \\ &\qquad \Leftrightarrow \varphi(z), \varphi(z') \in X \text{ belong to the same block of } P. \end{aligned} \tag{6}$$

Let θ_X be a formigram over X. The pullback of θ_X via φ is the formigram

$$\varphi^*\theta_X : \mathbb{R} \to \mathbf{SubPart}(Z) \text{ such that } \left(\varphi^*\theta_X\right)(t) := \varphi^*\left(\theta_X(t)\right).$$

Let X and Y be any two nonempty sets. A *tripod R between X and Y* is a pair of surjections $R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ from any set Z onto X and Y [21].

Definition 2.22 Let θ_X,θ_Y be any two formigrams over X and Y, respectively. The **Gromov-Hausdorff distance between θ_X and θ_Y** is defined as:

$$d_{\mathrm{GH}}(\theta_X, \theta_Y) := \frac{1}{2} \min_R d_{\mathrm{F}}(\varphi_X^*\theta_X, \varphi_Y^*\theta_Y),$$

where the minimum is taken over all tripods between X and Y.[6] [7]

It directly follows from Definition 2.20 and 2.22 that, given any two formigrams θ_X and θ'_X over the *same* underlying set X, we have $2\, d_{\mathrm{GH}}(\theta_X, \theta'_X) \le d_{\mathrm{F}}(\theta_X, \theta'_X)$.

3 Interleaving by Parts and Geodesicity

We *decompose* the interleaving distance d_{I} between poset maps (Section 3.1) and harness this decomposition to show the geodesicity of d_{I} in certain settings (Section 3.2).

3.1 Join Representations of Interleavings between Poset Maps

The goal of this section is to establish Theorems 3.7 and 3.10.
Join representations for poset maps
Recall that any poset $\mathcal{P}$ contains at least one trivial join-dense subset ($\mathcal{P}$ itself). In this section, we adhere to:

Convention 3.1 $(\mathcal{P}, \le)$ denotes a poset and $(\mathcal{Q}, \le)$ denotes a poset with a distinguished join-dense subset $B \subset \mathcal{Q}$ and containing a zero element[8].

Proposition 3.2 *Let $q \in \mathcal{Q}$. Then, $q = \bigvee (B \cap [0, q])$.*

[6] Because X and Y are finite, the minimum is always achieved by at least one tripod R. Also, it suffices to consider subsets $Z \subset X \times Y$ which project *onto* X and Y via the canonical projections with φ_X, φ_Y being the canonical projections.

[7] In this paper the Gromov-Hausdorff distance between formigrams is actually called the *formigram interleaving distance* in [32], [45, Definition 4.11]. In this paper, we reserve the name *interleaving distance* for Definition 2.20 because d_{GH} is not the interleaving distance in the sense of Definition 2.8. A close relationship between the Gromov-Hausdorff distance and the interleaving distance is highlighted in [13, 29, 51].

[8] If, a priori, $\mathcal{Q}$ does not contain a zero element, one can simply add one to $\mathcal{Q}$. Namely, replace $\mathcal{Q}$ by $\mathcal{Q} \cup \{0\}$ where 0 is forced to be the smallest element in $\mathcal{Q} \cup \{0\}$, by definition.

Proof Clearly, we have $\bigvee[0, q] = q$. As B is join-dense, there exists $B' \subset B$ s.t. $\bigvee B' = q$. Since $B' \subset B \cap [0, q] \subset [0, q]$, we have $\bigvee (B \cap [0, q]) = q$, as desired. □

The following proposition is straightforward.

Proposition 3.3 *Given any subfamily $\{\mathbf{F}_a : a \in A\}$ of $[\mathcal{P}, \mathcal{Q}]$, assume that for all $p \in \mathcal{P}$, the join $\bigvee_{a\in A} \mathbf{F}_a(p)$ exists in $\mathcal{Q}$. Then the join $\bigvee\{\mathbf{F}_a : a \in A\}$ exists in the poset $[\mathcal{P}, \mathcal{Q}]$, which is given by $p \mapsto \bigvee_{a\in A} \mathbf{F}_a(p)$.*

Let $q \in \mathcal{Q}$ and let $U \subset \mathcal{P}$ be an upper set of $\mathcal{P}$. We define the upper set indicator map $\mathbf{I}_q^U : (\mathcal{P}, \leq) \to (\mathcal{Q}, \leq)$ as

$$(\mathbf{I}_q^U)(p) := \begin{cases} q, & p \in U \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Definition 3.4 Let $\mathbf{F} \in [\mathcal{P}, \mathcal{Q}]$ and let $b \in B$. For the upper set $\mathbf{F}^{-1}(b^{\uparrow}) := \{p \in \mathcal{P} : b \leq \mathbf{F}(p)\}$, we define the b-**part** of $\mathbf{F}$ as $\mathbf{F}_b := \mathbf{I}_b^{\mathbf{F}^{-1}(b^{\uparrow})}$.

We introduce a certain type of join representation of a poset map which is the source of many results in this paper.

Proposition 3.5 (Join representation via indicator maps) *For any poset map $\mathbf{F} : \mathcal{P} \to \mathcal{Q}$,*

$$\mathbf{F} = \bigvee \{\mathbf{F}_b : b \in B\}. \tag{8}$$

Proof Fix $p \in \mathcal{P}$. Let us define $A_p, B_p \subset \mathcal{Q}$ as $A_p := B \cap [0, \mathbf{F}(p)]$ and $B_p := \{\mathbf{F}_b(p) : b \in B\}$. Then,

$$\begin{aligned}
\mathbf{F}(p) &= \bigvee A_p && \text{by Proposition 3.2} \\
&= \bigvee \left(A_p \cup \{0\}\right) \\
&= \bigvee \left(B_p \cup \{0\}\right) && \text{since } A_p \cup \{0\} = B_p \cup \{0\} \\
&= \bigvee B_p \\
&= \left(\bigvee \{\mathbf{F}_b : b \in B\}\right)(p) && \text{by Proposition 3.3,}
\end{aligned}$$

which proves the equality in Eq. 8. □

Remark 3.6 (i) The join representation in Eq. 8 is functorial in the sense that $\mathbf{F} \leq \mathbf{G}$ in $[\mathcal{P}, \mathcal{Q}] \Leftrightarrow$ for all $b \in B$, $\mathbf{F}_b \leq \mathbf{G}_b$.

(ii) One can establish "duals" of Convention 3.1, Propositions 3.2 and 3.3 and Definition 3.4 which permit representing $\mathbf{F}$ as a *meet* of $\mathbf{F}$'s "dual" b-parts, instead of the join representation given in Eq. 8. We have not found any significant use of this dual statement for the purpose of this paper.

(iii) The $\mathbf{F}_b$s in Eq. 8 are *not* necessarily join-irreducible in $[\mathcal{P}, \mathcal{Q}]$. However, there is a special case: Let $\mathcal{P}$ be a totally ordered set and assume that B consists solely of join-irreducible elements of $\mathcal{Q}$ (cf. Remark 2.1). Then, each $\mathbf{F}_b$ is join-irreducible and thus, in that case, equality Eq. 8 could be regarded as a join *decomposition* of $\mathbf{F}$.

Interleaving by parts

We show that the interleaving distance between poset maps admits a join representation which is compatible with the join representation in Eq. 8.

Theorem 3.7 (Interleaving by parts) *Let $(\mathcal{P}, \leq, \Omega)$ be a poset with a flow, and let $\mathcal{Q}$ be a poset with zero and a join-dense subset B. For any $\mathbf{F}, \mathbf{G} : (\mathcal{P}, \leq) \to (\mathcal{Q}, \leq)$,*

$$d_{\mathrm{I}}(\mathbf{F}, \mathbf{G}) = \sup_{b \in B} d_{\mathrm{I}}(\mathbf{F}_b, \mathbf{G}_b). \tag{9}$$

Note that, in the standard ordered set $(\mathbb{R}, \leq)$, we have $\bigvee A = \sup A$ for any $A \subset \mathbb{R}$. Therefore, invoking Eq. 8, we can rewrite Eq. 9 as:

$$d_{\mathrm{I}}\left(\bigvee_{b \in B} \mathbf{F}_b, \bigvee_{b \in B} \mathbf{G}_b\right) = \bigvee_{b \in B} d_{\mathrm{I}}(\mathbf{F}_b, \mathbf{G}_b),$$

which shows that the join operation of Eq. 8 commutes with d_{I}.

Remark 3.8 Eq. 9 is strongly analogous to the well-known decomposability of the interleaving distance between persistence modules $\mathbb{R} \to \mathbf{vect}$ (Definition A3 in the appendix).

Assume for simplicity that B is finite. Then, one can check that the RHS of Eq. 9 is equal to

$$\min_{\tau : B \to B} \max_{b \in B} d_{\mathrm{I}}(\mathbf{F}_b, \mathbf{G}_{\tau(b)}) \tag{10}$$

where the minimum is taken over all bijections $\tau : B \to B$. Now consider any two persistence modules $M, N : \mathbb{R} \to \mathbf{vect}$ with indecomposable decompositions $M \cong \bigoplus_{i \in I} M_i$ and $N \cong \bigoplus_{j \in J} N_j$ where $|I|, |J| < \infty$. Let $K := I \sqcup J$. Then $d_{\mathrm{I}}(M, N)$ is equal to the following (which is in the same form as Eq. 10):

$$\min_{\tau : K \to K} \max_{k \in K} d_{\mathrm{I}}(M_k, N_{\tau(k)}),$$

where we define. $M_j := 0$ for $j \in J$ and $N_i := 0$ for $i \in I$.

Proof of Theorem 3.7 Let $\varepsilon \in [0, \infty)$.

$$\begin{aligned}
& \mathbf{F}, \mathbf{G} \text{ are } \varepsilon\text{-interleaved} \\
\Leftrightarrow\ & \mathbf{F} \leq \mathbf{G}\Omega_\varepsilon \text{ and } \mathbf{G} \leq \mathbf{F}\Omega_\varepsilon. \\
\Leftrightarrow\ & \text{For any } b \in B,\ \mathbf{F}_b \leq (\mathbf{G}\Omega_\varepsilon)_b \text{ and } \mathbf{G}_b \leq (\mathbf{F}\Omega_\varepsilon)_b && \text{by Remark 3.6 (i)} \\
\overset{(*)}{\Leftrightarrow}\ & \text{For any } b \in B,\ \mathbf{F}_b \leq \mathbf{G}_b\Omega_\varepsilon \text{ and } \mathbf{G}_b \leq \mathbf{F}_b\Omega_\varepsilon && \text{see below} \\
\Leftrightarrow\ & \text{For any } b \in B,\ \mathbf{F}_b, \mathbf{G}_b \text{ are } \varepsilon\text{-interleaved} && \text{by Definition 2.9.}
\end{aligned}$$

For $(*)$, it suffices to show that $(\mathbf{F}\Omega_\varepsilon)_b = \mathbf{F}_b\Omega_\varepsilon$. To this end, we prove $p \in (\mathbf{F}\Omega_\varepsilon)^{-1}(b^\uparrow) \Leftrightarrow \Omega_\varepsilon(p) \in \mathbf{F}^{-1}(b^\uparrow)$. Indeed, for $p \in \mathcal{P}$,

$$p \in (\mathbf{F}\Omega_\varepsilon)^{-1}(b^\uparrow) \;\Leftrightarrow\; b \leq (\mathbf{F}\Omega_\varepsilon)(p) \;\Leftrightarrow\; b \leq \mathbf{F}(\Omega_\varepsilon(p)) \;\Leftrightarrow\; \Omega_\varepsilon(p) \in \mathbf{F}^{-1}(b^\uparrow).$$

Interleaving distance between upper set indicator maps

We represent the RHS of Eq. 9 as the interleaving distance between upper sets of $\mathcal{P}$ (Definition 2.11). Recall Definition 3.4.

Proposition 3.9 *Let $(\mathcal{P}, \leq, \Omega)$ be a poset with flow. Let $U, V \in U(\mathcal{P})$ and let $x \in \mathcal{Q}$. Then, we have:*

$$d_{\mathrm{I}}(\mathbf{I}_x^U, \mathbf{I}_x^V) = d_{\widehat{\Omega}}(U, V).$$

Proof

$$
\begin{aligned}
& \mathbf{I}_x^U, \mathbf{I}_x^V \text{ are } \varepsilon\text{-interleaved.} \\
\Leftrightarrow\ & \mathbf{I}_x^U \leq \mathbf{I}_x^V \Omega_\varepsilon \text{ and } \mathbf{I}_x^V \leq \mathbf{I}_x^U \Omega_\varepsilon. \\
\Leftrightarrow\ & \text{For any } p \in \mathcal{P}, \mathbf{I}_x^U(p) \leq \mathbf{I}_x^V(\Omega_\varepsilon(p)) \text{ and } \mathbf{I}_x^V(p) \leq \mathbf{I}_x^U(\Omega_\varepsilon(p)). \\
\Leftrightarrow\ & \text{For any } p \in \mathcal{P}, (p \in U \Rightarrow \Omega_\varepsilon(p) \in V) \text{ and } (p \in V \Rightarrow \Omega_\varepsilon(p) \in U). \\
\Leftrightarrow\ & (\text{For any } p \in \mathcal{P} : p \in U \Rightarrow \Omega_\varepsilon(p) \in V) \text{ and } (\text{For any } p \in \mathcal{P} : p \in V \Rightarrow \Omega_\varepsilon(p) \in U). \\
\Leftrightarrow\ & U \subset \widehat{\Omega}_\varepsilon(V) \text{ and } V \subset \widehat{\Omega}_\varepsilon(U) \\
\Leftrightarrow\ & U, V \text{ are } \varepsilon\text{-interleaved.}
\end{aligned}
$$

□

By Theorem 3.7 and Proposition 3.9, d_{I} coincides with the ℓ^∞-product metric of $|B|$ copies of $d_{\widehat{\Omega}}$ on $U(\mathcal{P})$:

Theorem 3.10 *(Decomposition of d_{I}) Let $(\mathcal{P}, \leq, \Omega)$ be a poset with a flow, and let $\mathcal{Q}$ be a poset with zero and a join-dense subset B. For any $\mathbf{F}, \mathbf{G} : (\mathcal{P}, \leq) \to (\mathcal{Q}, \leq)$ we have*

$$d_{\mathrm{I}}(\mathbf{F}, \mathbf{G}) = \sup_{b \in B} d_{\widehat{\Omega}}\big(\mathbf{F}^{-1}(b^{\uparrow}), \mathbf{G}^{-1}(b^{\uparrow})\big).$$

This theorem implies that being able to find a “good" join dense subset $B \in \mathcal{Q}$ can allow us to efficiently compute $d_{\mathrm{I}}(\mathbf{F}, \mathbf{G})$. The join-dense subset B would be “good" from a computational viewpoint if B includes a small number of elements and, for each $b \in B$, $d_{\widehat{\Omega}}\big(\mathbf{F}^{-1}(b^{\uparrow}), \mathbf{G}^{-1}(b^{\uparrow})\big)$ is easy to compute.

Remark 3.11 Theorem 3.10 is analogous to the celebrated *isometry theorem* in TDA [5, 12, 62] in the following sense: By Theorem 3.10, we can make use of the collections

$$B(\mathbf{F}) := \left\{\mathbf{F}^{-1}(b^{\uparrow})\right\}_{b \in B} \text{ and } B(\mathbf{G}) := \left\{\mathbf{G}^{-1}(b^{\uparrow})\right\}_{b \in B},$$

for computing $d_{\mathrm{I}}(\mathbf{F}, \mathbf{G})$. Analogously, for any persistence modules $M, N : \mathbb{R} \to \mathbf{vect}$ (Definition A1), their *barcodes* (or equivalently *persistence diagrams*) are utilized for computing the interleaving distance between M and N via the bottleneck distance [5, 62].

3.2 Geodesicity of Interleavings between Poset Maps

The goal of this section is to prove that the interleaving distance between poset maps $\mathcal{P} \to \mathcal{Q}$ is geodesic *whenever $\mathcal{Q}$ is a complete lattice* (Theorem 3.13). This proves, in a uniform way, that many of the well known metrics that will be recalled in later sections are geodesic.

In the rest of the paper, any extended pseudometric space (M, d) is simply referred to as a metric space. Any $x \in M$ will be identified with the class $[x] := \{y : d(x, y) = 0\}$. Let $x, y \in M$ with $d(x, y) < \infty$. A continuous map $g : [0, 1] \to M$ is called a **path** from x to y if $g(0) = x$ and $g(1) = y$. The path g is called **geodesic** if

$$d(g(s), g(t)) = |s - t| \cdot d(x, y)$$

for all $s, t \in [0, 1]$. If there exists a geodesic path between every pair of points in M within a finite distance, M is called **geodesic**.

Proposition 3.12 *For any poset $(\mathcal{P}, \leq, \Omega)$ with a flow, the distance $d_{\widehat{\Omega}}$ on $U(\mathcal{P})$ (Definition 2.11) is geodesic.*

Proof Let $A, B \in U(\mathcal{P})$ with $d_{\widehat{\Omega}}(A, B) = \rho \in (0, \infty)$. For $t \in [0, 1]$, let $A_t := \widehat{\Omega}_{\rho t}(A) \cap \widehat{\Omega}_{\rho(1-t)}(B)$. We claim that $t \mapsto A_t$ for $t \in [0, 1]$ is a geodesic path from A to B.[9] First, we claim $d_{\widehat{\Omega}}(A, A_0) = 0$ (the equality $d_{\widehat{\Omega}}(B, B_0) = 0$ is proved similarly). To this end, we show that, for *any* $\delta > 0$, $A_0 \subset \widehat{\Omega}_\delta(A)$ and $A \subset \widehat{\Omega}_\delta(A_0)$. Let $\delta > 0$. By construction we have $A_0 \subset A \subset \widehat{\Omega}_\delta(A)$. Next,

$$\begin{aligned}
A &\subset \widehat{\Omega}_\delta(A) \cap \widehat{\Omega}_{\delta+\rho}(B) && \text{since } A \subset \widehat{\Omega}_{\rho+\delta}(B) \text{ from } d_{\widehat{\Omega}}(A, B) = \rho \\
&\subset \widehat{\Omega}_\delta(A \cap \widehat{\Omega}_\rho(B)) && \text{by Remark 2.12 (ii)} \\
&= \widehat{\Omega}_\delta(A_0) && \text{by definition.}
\end{aligned}$$

Now fix any $s < t$ in $[0, 1]$ and we show that $d_{\widehat{\Omega}}(A_s, A_t) = (t - s)\rho$. We claim that $d_{\widehat{\Omega}}(A_s, A_t) \leq (t - s)\rho$. By Remark 2.12 (ii),

$$\widehat{\Omega}_{\rho(t-s)}(A_s) = \widehat{\Omega}_{\rho t}(A) \cap \widehat{\Omega}_{\rho(1+t-2s)}(B) \supset \widehat{\Omega}_{\rho t}(A) \cap \widehat{\Omega}_{\rho(1-t)}(B) = A_t.$$

Similarly, one has $\widehat{\Omega}_{\rho(t-s)}(A_t) \supset A_s$. This shows that $d_{\widehat{\Omega}}(A_s, A_t) \leq (t - s)\rho$. □

Theorem 3.13 *Let $(\mathcal{P}, \leq, \Omega)$ be a poset with a flow and $(\mathcal{Q}, \leq)$ be a poset such that any of its subsets admits a join (and $\mathcal{Q}$ is thus a complete lattice). The interleaving distance d_{I} on $[\mathcal{P}, \mathcal{Q}]$ is geodesic.*

Proof Let B be any join-dense subset of $\mathcal{Q}$. By Proposition 3.12, for each $b \in B$, there exists a geodesic path $g_b : [0, 1] \to U(\mathcal{P})$ from $\mathbf{F}^{-1}(b^\uparrow)$ to $\mathbf{G}^{-1}(b^\uparrow)$. We *embed* these paths into $[\mathcal{P}, \mathcal{Q}]$ and *assemble* the embedded paths via $\bigvee$: For each $t \in [0, 1]$, define $\mathbf{H}_t : (\mathcal{P}, \leq) \to (\mathcal{Q}, \leq)$ by

$$p \mapsto \bigvee_{b \in B} \mathbf{I}_b^{g_b(t)}(p),$$

which is well-defined since $\mathcal{Q}$ is a complete lattice.

Now we claim that $t \mapsto \mathbf{H}_t$ is a geodesic path from $\mathbf{F}$ to $\mathbf{G}$. It is clear that $\mathbf{H}_0 = \mathbf{F}$ and $\mathbf{H}_1 = \mathbf{G}$. For every $s, t \in [0, 1]$:

$$\begin{aligned}
d_{\mathrm{I}}(\mathbf{H}_s, \mathbf{H}_t) &= d_{\mathrm{I}}\left(\bigvee_{b \in B} \mathbf{I}_b^{g_b(s)}, \bigvee_{b \in B} \mathbf{I}_b^{g_b(t)}\right) && \text{by definition} \\
&= \sup_{b \in B} d_{\widehat{\Omega}}(g_b(s), g_b(t)) && \text{by Theorem 3.10} \\
&= \sup_{b \in B} |s - t| \cdot d_{\widehat{\Omega}}\left(\mathbf{F}^{-1}(b^\uparrow), \mathbf{G}^{-1}(b^\uparrow)\right) && \text{since } g_b \text{ is a geodesic path} \\
&= |s - t| \cdot \sup_{b \in B} d_{\widehat{\Omega}}\left(\mathbf{F}^{-1}(b^\uparrow), \mathbf{G}^{-1}(b^\uparrow)\right) \\
&= |s - t| \cdot d_{\mathrm{I}}(\mathbf{F}, \mathbf{G}) && \text{Theorem 3.10.}
\end{aligned}$$

□

We remark that when the target poset $\mathcal{Q}$ is not a complete lattice, one may replace $\mathcal{Q}$ by its *Dedekind-MacNeille completion* $\bar{\mathcal{Q}}$ *(i.e. the smallest complete lattice containing* $\mathcal{Q}$*) and then consider geodesic paths in* $[\mathcal{P}, \bar{\mathcal{Q}}]$.

[9] This construction is similar to the construction of Hausdorff geodesic paths given in [63].

Remark 3.14 The construction of the geodesic path above is analogous to the construction of a geodesic path between persistence diagrams of persistence modules $\mathbb{R} \to \mathbf{vect}$ via a linear interpolation guided by an optimal matching [64]. In the proof above, the collection $\{\mathbf{F}^{-1}(b^{\uparrow}) : b \in B\}$ can be viewed as a proxy for the "persistence diagram" of $\mathbf{F}$.

Example 3.15 (Realization of geodesics for the erosion distance.) From Theorem 3.13, we know that any two poset maps $\mathbf{F}, \mathbf{G} : \mathbf{Int} \to \mathbb{Z}_{\geq 0}^{\mathrm{op}}$ at finite erosion distance can be joined by a geodesic path in $[\mathbf{Int}, \mathbb{Z}_{\geq 0}^{\mathrm{op}}]$. Assume that $\mathbf{F}$ and $\mathbf{G}$ are the rank functions of two persistence modules M and N, respectively. The geodesic path $g : [0, 1] \to [\mathbf{Int}, \mathbb{Z}_{\geq 0}^{\mathrm{op}}]$ between $\mathbf{F}$ and $\mathbf{G}$ constructed in the proof of Theorem 3.13 is actually *not* always *realizable* by a path in $\mathbf{vect}^{\mathbb{R}}$, i.e. there is sometimes no continuous map $h : [0, 1] \to (\mathbf{vect}^{\mathbb{R}}, d_{\mathrm{E}} \circ \mathrm{rk})$ such that $\mathrm{rk} \circ h = g$.[10] This implies that, we do not know, at this point, whether or not $d_{\mathrm{E}}(\mathrm{rk}(-), \mathrm{rk}(-))$ is a geodesic distance on the space of $\mathbb{R}$-indexed persistence modules. Studying this can potentially be useful for clarifying the relationship between the bottleneck distance and $d_{\mathrm{E}}(\mathrm{rk}(-), \mathrm{rk}(-))$.

4 Consequences

This section describes a number of consequences of Theorems 3.7 and 3.10.

In Section 4.1 we establish a connection between the erosion distance [19] and graded rank functions [52]. In Section 4.2 we show that the Gromov-Hausdorff distance can be recast within the framework of interleaving distances and thereby obtain a far reaching generalization of this distance. Therein, basic properties of this distance are established including its universality and geodesicity. In Sections 4.3 and 4.4 we prove that computing the interleaving distance between multiparameter hierarchical clusterings reduces to computing Hausdorff distances in Euclidean spaces. This result is useful to establish the equivalence between several known metrics for comparing multiparameter hierarchical clusterings. In Section 4.4 we determine the computational complexity of the interleaving distance between formigrams from Definition 2.20.

Convention 4.1 In the rest of the paper, the posets $\mathbb{R}_{\geq 0}$ and $\mathbb{R}^n$ (for any $n \in \mathbb{N}$) are equipped with the flow described in Eq. 1. Also, the poset $\mathbf{Int}$ is equipped with the flow from Example 2.13.

4.1 Erosion Distance and Graded Rank Functions

The goal of this section is to establish a connection between the erosion distance and *graded rank functions* [52] and to provide an interpretation of the join representation given in Eq. 8.

Although the erosion distance has a fairly general form [19, 47, 48], its most basic use is for comparing two monotonically decreasing integer-valued maps as in [49, Section 5]. We restrict ourselves to this basic setting. Let $\mathbb{Z}_{\geq 0}^{\mathrm{op}}$ be the opposite poset of the nonnegative integers, i.e. $a \leq b$ in $\mathbb{Z}_{\geq 0}^{\mathrm{op}} \Leftrightarrow b \leq a$ in $\mathbb{Z}_{\geq 0}$.

[10] *Example:* For $a, b \in \mathbb{R}$ with $a < b$, let $\mathbb{I}[a, b) : \mathbb{R} \to \mathbf{vect}$ the *interval module* with support $[a, b)$ [65]. Let $M, N : \mathbb{R} \to \mathbf{vect}$ be defined as $M = \mathbb{I}[0, 10) \oplus \mathbb{I}[10, 16)$ and $N = \mathbb{I}[4, 14)$. Now define $\mathbf{F}, \mathbf{G}$ to be $\mathrm{rk}(M)$ and $\mathrm{rk}(N)$ respectively.

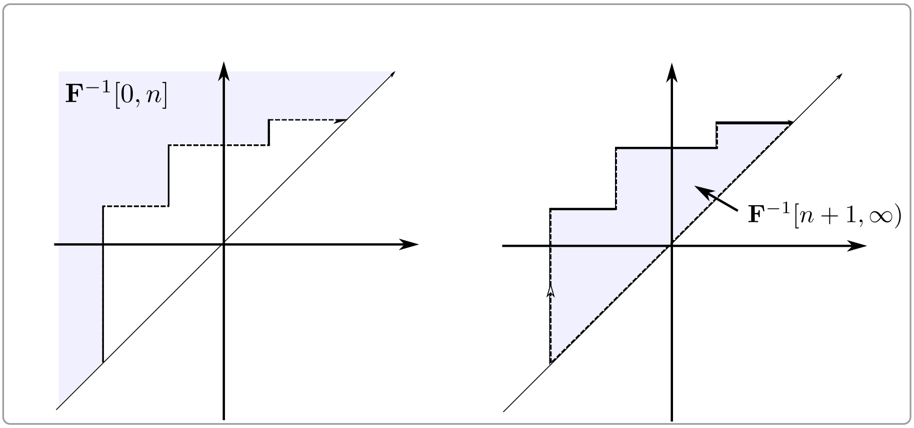

Fig. 4 Illustration of $\mathbf{F}^{-1}[0,n]$ and $\mathbf{F}^{-1}[n+1,\infty)$ for the case when $\mathbf{F}$ is the rank function of a generic constructible persistence module M (Definition A1 and A2). The boundary line between $\mathbf{F}^{-1}[0,n]$ and $\mathbf{F}^{-1}[n+1,\infty)$ is a visual representation of the $(n+1)$-th persistence landscape of M

Definition 4.2 Let $(\mathcal{P},\leq,\Omega)$ be a poset with a flow. Given $\mathbf{F},\mathbf{G}:(\mathcal{P},\leq,\Omega)\to\mathbb{Z}_{\geq 0}^{\mathrm{op}}$, the interleaving distance $d_{\mathrm{I}}(\mathbf{F},\mathbf{G})$ is called the ($\mathcal{P}$-)**erosion distance**.

In $\mathbb{Z}_{\geq 0}^{\mathrm{op}}$, we have $\vee\{m,n\}=\min\{m,n\}$ and each element of $\mathbb{Z}_{\geq 0}^{\mathrm{op}}$ is join-irreducible. This implies that $\mathbb{Z}_{\geq 0}^{\mathrm{op}}$ itself is the unique join-dense subset of $\mathbb{Z}_{\geq 0}^{\mathrm{op}}$. By virtue of Theorem 3.10, we have[11]

$$d_{\mathrm{I}}(\mathbf{F},\mathbf{G})=\sup_{n\in\mathbb{Z}_{\geq 0}} d_{\widehat{\Omega}}\left(\mathbf{F}^{-1}[0,n],\mathbf{G}^{-1}[0,n]\right).$$

Recall the Hausdorff distance in $(\mathbf{Int},||-||_\infty)$ (Example 2.13 and Definition A4). We establish the following relationship between the erosion distance and *graded rank functions* [52], which directly follows from the above equality and Example 2.13.

Theorem 4.3 *The erosion distance between* $\mathbf{F},\mathbf{G}:(\mathbf{Int},\leq)\to\mathbb{Z}_{\geq 0}^{\mathrm{op}}$ *is equal to*

$$d_{\mathrm{I}}(\mathbf{F},\mathbf{G})=\sup_{n\in\mathbb{Z}_{\geq 0}} d_{\mathrm{H}}\left(\mathbf{F}^{-1}[0,n],\mathbf{G}^{-1}[0,n]\right). \tag{11}$$

Remark 4.4 (i) If $\mathbf{F}$ is the *rank function* of a persistence module $M:\mathbb{R}\to\mathbf{vect}$ (Definitions A1 and A2), then $\mathbf{F}^{-1}[n+1,\infty)=\mathbf{Int}\setminus\mathbf{F}^{-1}[0,n]$ is the support of the $(n+1)$-th *graded rank function* of M [52, Definition 4.2] (Fig. 4). In light of this, graded rank functions (or *persistence landscapes* [53]) are special instances of the join-representation (of rank functions) described in Proposition 3.5. Stability properties of graded rank functions have been discussed in [52].

(ii) The interleaving distance between persistence modules $M,N:\mathbb{R}^d\to\mathbf{vect}$ (Definition A3) is known to be bounded from below by the erosion distance between the rank functions of M and N; see [53, Theorem 17] [19, Theorem 8.2] for $d=1$ and [49, Theorem 6.2] for arbitrary d.

(iii) An optimal algorithm for computing $d_{\mathrm{I}}(\mathbf{F},\mathbf{G})$ in Eq. 11 and more general results are given in [49, Section 5].

[11] Although $\mathbb{Z}_{\geq 0}^{\mathrm{op}}$ does not contain a zero element, as noted in Convention 3.1, we can apply Theorem 3.10 in this case by viewing the codomain of a poset map $\mathcal{P}\to\mathbb{Z}_{\geq 0}^{\mathrm{op}}$ as $(\mathbb{Z}_{\geq 0}\cup\{\infty\})^{\mathrm{op}}$.

4.2 Generalization of the Gromov-Hausdorff Distance

In this section we show the following:

(i) The Gromov-Hausdorff distance d_{GH} (between metric spaces [66] or between $\mathbb{R}$-indexed simplicial filtrations [21, 22]) can be incorporated into the framework of interleaving distances of Section 2.3; cf. Theorem 4.8. This leads to the next item.
(ii) We obtain a far reaching generalization of d_{GH}, which will be also denoted by d_{GH} (Definition 4.9); although the instance of d_{GH} described in Definition 4.9 is a distance between simplicial filtrations over a poset with a flow, there is a precise sense in which it generalizes all the other Gromov-Hausdorff distances mentioned in this paper (Remark 4.10).
(iii) This generalized d_{GH} inherits the universality property satisfied by the original Gromov-Hausdorff distance (Theorem 4.11), of which the celebrated Vietoris-Rips filtration stability theorem [54, 55] becomes a consequence (Theorem 4.15 and Remark 4.16 (ii)).
(iv) Using Theorem 3.13 we show that the generalized d_{GH} is also geodesic. Interestingly, our construction of geodesic paths generalizes a known one between compact metric spaces [67] in a precise sense; see Remark 4.13.

Some basic properties of the generalized d_{GH} are deferred to the appendix.

Throughout this section, X, Y, and Z will denote nonempty finite sets. By $\mathbf{Simp}(X)$, we denote the collection of *abstract simplicial complexes* [68] over a vertex set $A \subset X$ ordered by inclusion. $\mathbf{Simp}(X)$ is a complete lattice whose joins are unions and meets are intersections. By $\mathbf{pow}_{\geq 1}(X)$, we denote the the collection of nonempty subsets of X ordered by inclusion.

Definition 4.5 (Simplexization) Let us define the poset map $s : \mathbf{pow}_{\geq 1}(X) \to \mathbf{Simp}(X)$ as $\sigma \mapsto \mathbf{pow}_{\geq 1}(\sigma)$. In words, nonempty $\sigma \subset X$ is sent to the simplicial complex consisting solely of the $(|\sigma| - 1)$-simplex σ and all of its nonempty subsets.

Remark 4.6 Let $\Delta(X)$ be the image of the map s. The collection of all join-irreducible elements of $\mathbf{Simp}(X)$ equals $\Delta(X)$. Hence, $\Delta(X)$ is join-dense in $\mathbf{Simp}(X)$ (Remark 2.1).

A poset map $\mathbf{F}_X : (\mathcal{P}, \leq) \to \mathbf{Simp}(X)$ is said to be an ($\mathcal{P}$-indexed simplicial) **filtration** (over X). The filtration is called **full** if there exists $p \in \mathcal{P}$ such that $X \in \mathbf{F}_X(p)$.

Assume that $(\mathcal{P}, \leq) = (\mathbb{R}, \leq)$. For any $\sigma \in \mathbf{pow}_{\geq 1}(X)$, the **birth time of** σ is defined as

$$b_{\mathbf{F}_X}(\sigma) := \inf \{t \in \mathbb{R} : \sigma \in \mathbf{F}_X(t)\}.$$

If σ does not belong to $\mathbf{F}_X(t)$ for any $t \in \mathbb{R}$, then $b_{\mathbf{F}_X}(\sigma)$ is defined to be ∞.

We review the Gromov-Hausdorff distance between $\mathbb{R}$-indexed filtrations (introduced by Mémoli [21] and further studied by Mémoli and Okutan [22]):

Definition 4.7 ([21, p.4]) Given any $\mathbf{F}_X : (\mathbb{R}, \leq) \to \mathbf{Simp}(X)$ and $\mathbf{G}_Y : (\mathbb{R}, \leq) \to \mathbf{Simp}(Y)$, the **Gromov-Hausdorff distance** between $\mathbf{F}_X$ and $\mathbf{G}_Y$ is defined by

$$d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{G}_Y) := \frac{1}{2} \min_{R} \max_{\sigma \in \mathbf{pow}_{\geq 1}(Z)} \left| b_{\mathbf{F}_X}(\varphi_X(\sigma)) - b_{\mathbf{G}_Y}(\varphi_Y(\sigma)) \right|,$$

where the minimum is taken over all tripods $R : X \xtwoheadleftarrow{\varphi_X} Z \xtwoheadrightarrow{\varphi_Y} Y$.

At first glance, this distance may not appear to be related to an interleaving type distance between poset maps. However, we will see that such a relation exists.

Given a surjective map $\varphi_X : Z \twoheadrightarrow X$ and a simplicial filtration $\mathbf{F}_X : (\mathcal{P}, \leq) \to \mathbf{Simp}(X)$, we define the *pullback* $\varphi_X^*\mathbf{F}_X$ of $\mathbf{F}_X$ via φ_X as the filtration $(\mathcal{P}, \leq) \to \mathbf{Simp}(Z)$ as follows. The filtration $\varphi_X^*\mathbf{F}_X$ sends each $p \in \mathcal{P}$ to the smallest simplicial complex $K \in \mathbf{Simp}(Z)$ that contains *all* the preimages $\varphi_X^{-1}(\tau)$ for $\tau \in \mathbf{F}_X(p)$. Hence, we have the commutative diagram:

$$\begin{tikzcd}
 & \mathbf{Simp}(Z) \\
\mathcal{P} \arrow[r, "\mathbf{F}_X"'] \arrow[ur, dotted, "\varphi_X^*\mathbf{F}_X"] & \mathbf{Simp}(X) \arrow[u, "\varphi_X^*"']
\end{tikzcd}$$

where, for all $K \in \mathbf{Simp}(X)$, $\varphi_X^*(K)$ is the smallest subcomplex of the full complex $\mathbf{pow}_{\geq 1}(Z)$ on Z that contains $\varphi_X^{-1}(\sigma) \subset Z$ for all $\sigma \in K$.

Let $d_{\mathrm{I}}^{\mathbf{Simp}(Z)}$ be the interleaving distance on $[\mathbb{R}, \mathbf{Simp}(Z)]$. By Theorem 3.10 we can reformulate Definition 4.7 as follows.

Theorem 4.8 *Given any filtrations* $\mathbf{F}_X : (\mathbb{R}, \leq) \to \mathbf{Simp}(X)$ *and* $\mathbf{G}_Y : (\mathbb{R}, \leq) \to \mathbf{Simp}(Y)$*,*

$$d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{G}_Y) = \frac{1}{2} \min_R d_{\mathrm{I}}^{\mathbf{Simp}(Z)}(\varphi_X^*\mathbf{F}_X, \varphi_Y^*\mathbf{G}_Y),$$

where the minimum is taken over all tripods $R : X \xleftarrow{\varphi_X}\!\!\!\!\leftarrow Z \xrightarrow{\varphi_Y}\!\!\!\!\rightarrow Y$.

Proof Let d_{H} be the Hausdorff distance in $\mathbb{R}$ (Definition A4). By Theorem 3.10 and Remark 4.6 we have:

$$\begin{aligned}
d_{\mathrm{I}}^{\mathbf{Simp}(Z)}(\varphi_X^*\mathbf{F}_X, \varphi_Y^*\mathbf{G}_Y) &= \max_{K \in \Delta(Z)} d_{\mathrm{H}}\left((\varphi_X^*\mathbf{F}_X)^{-1}(K^{\uparrow}), (\varphi_Y^*\mathbf{G}_Y)^{-1}(K^{\uparrow})\right) \\
&\overset{(*)}{=} \max_{\sigma \in \mathbf{pow}_{\geq 1}(Z)} d_{\mathrm{H}}\big([b_{\mathbf{F}_X}(\varphi_X(\sigma)), \infty), [b_{\mathbf{G}_Y}(\varphi_Y(\sigma)), \infty)\big) \\
&= \max_{\sigma \in \mathbf{pow}_{\geq 1}(Z)} \left|b_{\mathbf{F}_X}(\varphi_X(\sigma)) - b_{\mathbf{G}_Y}(\varphi_Y(\sigma))\right|,
\end{aligned}$$

where $(*)$ follows from the bijection $s : \mathbf{pow}_{\geq 1}(X) \to \Delta(X)$ (Definition 4.5 and Remark 4.6). The desired equality directly follows. □

In the rest of this section we fix a poset with a flow $(\mathcal{P}, \leq, \Omega)$. In light of Theorems 3.10 and 4.8, we obtain the following generalization of d_{GH} which we still denote by the same symbol:

Definition 4.9 Given any filtrations $\mathbf{F}_X : (\mathcal{P}, \leq) \to \mathbf{Simp}(X)$ and $\mathbf{G}_Y : (\mathcal{P}, \leq) \to \mathbf{Simp}(Y)$, the **generalized Gromov-Hausdorff distance** between them is defined by:

$$\begin{aligned}
d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{G}_Y) &:= \frac{1}{2} \min_R d_{\mathrm{I}}^{\mathbf{Simp}(Z)}\left(\varphi_X^*\mathbf{F}_X, \varphi_Y^*\mathbf{G}_Y\right) \\
&= \frac{1}{2} \min_R \max_{\sigma \in \mathbf{pow}_{\geq 1}(Z)} d_{\widehat{\Omega}}\left((\varphi_X^*\mathbf{F}_X)^{-1}(\sigma^{\uparrow}), (\varphi_Y^*\mathbf{G}_Y)^{-1}(\sigma^{\uparrow})\right),
\end{aligned}$$

where the minimum is taken over all tripods $R : X \xleftarrow{\varphi_X}\!\!\!\!\leftarrow Z \xrightarrow{\varphi_Y}\!\!\!\!\rightarrow Y$.

That d_{GH} above is an extended pseudometric is shown in the appendix (Proposition A7). We remark that (1) a sufficient condition for $d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{G}_Y)$ to be finite is that $\mathbf{F}_X$ and $\mathbf{G}_Y$ are full, and $d_{\widehat{\Omega}}$ is finite for every pair of upper sets in $\mathcal{P}$. (2) When $\mathcal{P} = \mathbb{R}^n$, $d_{\widehat{\Omega}}$ above boils down to the Hausdorff distance between upper sets in $\mathbb{R}^n$.

Remark 4.10 All instances of d_{GH} mentioned in this paper can be viewed as special instances of d_{GH} in Definition 4.9; see Proposition A13 in the appendix for the precise statement.

Universality property of d_{GH}

Let Met be the space of finite pseudometric spaces. It is known that the Gromov-Hausdorff distance $d_{\mathrm{GH}}^{\mathrm{Met}}$ between finite pseudometric spaces is the *largest* distance D satisfying the following two conditions: (i) If there is a surjection $\varphi : (Z, d_Z) \twoheadrightarrow (X, d_X)$ such that $d_Z(z, z') = d_X(\varphi(z), \varphi(z'))$ for all $z, z' \in Z$, then $D((X, d_X), (Z, d_Z)) = 0$. (ii) For any two metrics d_1, d_2 on a set X, we have:

$$2 \cdot D((X, d_1), (X, d_2)) \leq \max_{x, x' \in X} \left| d_1(x, x') - d_2(x, x') \right|.$$

(see [13, Theorem F, p.11] for a more general statement). We extend this universality property to:

Theorem 4.11 (Universality) *Let d_{GH} be given by Definition 4.9. Let D be any metric on $\mathcal{P}$-indexed finite simplicial filtrations with the following properties.*

(i) Given any $\mathbf{F}_X : (\mathcal{P}, \leq) \to \mathbf{Simp}(X)$ and a surjection $\varphi_X : Z \twoheadrightarrow X$, we have $D(\mathbf{F}_X, \varphi_X^* \mathbf{F}_X) = 0$.

(ii) Given any $\mathbf{F}_X, \mathbf{G}_X : (\mathcal{P}, \leq) \to \mathbf{Simp}(X)$, we have $2 \cdot D(\mathbf{F}_X, \mathbf{G}_X) \leq d_{\mathrm{I}}^{\mathbf{Simp}(X)}(\mathbf{F}_X, \mathbf{G}_X)$.

Then, $D \leq d_{\mathrm{GH}}$.

When restricting d_{GH} to *Vietoris-Rips filtrations* of finite pseudo metric spaces, this theorem boils down to the aforementioned universality property of $d_{\mathrm{GH}}^{\mathrm{Met}}$; this is a consequence of Proposition A13 (i) of the appendix.

Our proof of Theorem 4.11 is similar to the one in [24].

Proof of Theorem 4.11 Given any two filtrations $\mathbf{F}_X : (\mathcal{P}, \leq) \to \mathbf{Simp}(X)$ and $\mathbf{G}_Y : (\mathcal{P}, \leq) \to \mathbf{Simp}(Y)$, pick any tripod $R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$. Then, by invoking conditions (i) and (ii) in order, we have:

$$D(\mathbf{F}_X, \mathbf{G}_Y) = D(\varphi_X^* \mathbf{F}_X, \varphi_Y^* \mathbf{G}_Y) \leq d_{\mathrm{I}}^{\mathbf{Simp}(Z)}(\varphi_X^* \mathbf{F}_X, \varphi_Y^* \mathbf{G}_Y).$$

Since R is arbitrary, we have $D(\mathbf{F}_X, \mathbf{G}_Y) \leq 2 \cdot d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{G}_Y)$. □

Now we generalize the fact that $d_{\mathrm{GH}}^{\mathrm{Met}}$ is geodesic [21, 67], [13, Section 6.8]:

Theorem 4.12 *The generalized Gromov-Hausdorff distance is geodesic.*

Proof Let $\mathbf{F}_X : (\mathcal{P}, \leq) \to \mathbf{Simp}(X)$ and $\mathbf{F}_Y : (\mathcal{P}, \leq) \to \mathbf{Simp}(Y)$ be any two filtrations with $d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{F}_Y) =: \rho < \infty$. Let us take any tripod minimizer $R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ of $d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{F}_Y)$. The existence of R is guaranteed by the fact that X and Y are finite. Note that $d_{\mathrm{GH}}(\mathbf{F}_X, \varphi_X^* \mathbf{F}_X) = 0$ by taking the tripod $X \xleftarrow{\varphi_X} Z \xrightarrow{\mathbf{id}_Z} Z$. Similarly, $d_{\mathrm{GH}}(\mathbf{F}_Y, \varphi_Y^* \mathbf{F}_Y) = 0$. Hence, $2\rho = d_{\mathrm{GH}}(\varphi_X^* \mathbf{F}_X, \varphi_Y^* \mathbf{F}_Y)$ and it suffices to construct a geodesic path between $\varphi_X^* \mathbf{F}_X$ and $\varphi_Y^* \mathbf{F}_Y$ with respect to $d_{\mathrm{I}}^{\mathbf{Simp}(Z)}$. Invoking that $\mathbf{Simp}(Z)$ is a complete lattice, a geodesic path between $\varphi_X^* \mathbf{F}_X$ and $\varphi_Y^* \mathbf{F}_Y$ exists in $[(\mathcal{P}, \leq), \mathbf{Simp}(Z)]$, as desired. □

By **Simp** we denote the category of abstract simplicial complexes and simplicial maps. By $\mathbf{Simp}^{\mathcal{P}}$, we denote the category of $\mathcal{P}$-indexed simplicial filtrations (i.e. functors $\mathcal{P} \to \mathbf{Simp}$) and natural transformations between them.

Remark 4.13 (Generalization) The construction of a geodesic path in the proof above generalizes two constructions of geodesic paths existing in the literature: (i) A geodesic path $g : [0, 1] \to (\mathrm{Met}, d_{\mathrm{GH}})$ between any (X, d_X) and (Y, d_Y) that was constructed in [67], and (ii) the corresponding path $\mathrm{VR}(g) : [0, 1] \to (\mathbf{Simp}^{\mathbb{R}}, d_{\mathrm{GH}})$ $\mathrm{VR}(X, d_X)$ and $\mathrm{VR}(Y, d_Y)$ that was considered in [21].

Given an abstract simplicial complex S, let $|S|$ denote its geometric realization. Given a simplicial map $\xi : S \to L$ between abstract simplicial complexes, let $|\xi| : |S| \to |L|$ be the induced continuous map. The following proposition is useful to establish Theorem 4.15 below.

Proposition 4.14 *Given a surjective map $\varphi_X : Z \twoheadrightarrow X$, a simplicial filtration $\mathbf{F}_X : (\mathcal{P}, \leq) \to \mathbf{Simp}(X)$, and any $p \leq q$ in $\mathcal{P}$, the following diagram commutes*

$$\begin{tikzcd} \left|\varphi_X^* \mathbf{F}_X(p)\right| \arrow[r, "|\subseteq|"] \arrow[d, "|(\varphi_X)_p|"] & \left|\varphi_X^* \mathbf{F}_X(q)\right| \arrow[d, "|(\varphi_X)_q|"] \\ |\mathbf{F}_X(p)| \arrow[r, "|\subseteq|"'] & |\mathbf{F}_X(q)| \end{tikzcd}$$

where $(\varphi_X)_p$ denotes the restriction of φ_X to the vertex set of $\varphi_X^* \mathbf{F}_X(p)$. Furthermore, for each $p \in \mathcal{P}$, $\left|(\varphi_X)_p\right|$ is a homotopy equivalence.

To prove this, we recall the simplicial version of Quillen's Theorem A [21, Theorem 2.1], [69, Theorem 1.1] : *Let $\xi : S \to L$ be a simplicial map between two finite abstract simplicial complexes. Suppose that for each simplex $\sigma \in L$, $\left|\varphi_X^{-1}(\sigma)\right|$ is contractible. Then $|\xi|$ is a homotopy equivalence.*

Proof For $p \leq q$ in $\mathcal{P}$, we have

$$(\varphi_X)_q \circ (\varphi_X^* \mathbf{F}_X)(p \leq q) = \mathbf{F}_X(p \leq q) \circ (\varphi_X)_p$$

as set maps from the vertex set of $\varphi_X^* \mathbf{F}_X(p)$ to the vertex set of $\mathbf{F}_X(q)$. Therefore, the diagram above commutes.

Let $p \in \mathcal{P}$. For each simplex $\sigma \in \mathbf{F}_X(p)$, the preimage $\varphi_X^{-1}(\sigma)$ is a simplex in $\varphi_X^* \mathbf{F}_X(p)$ and thus $\left|\varphi_X^{-1}(\sigma)\right|$ contractible. By Quillen's Theorem A, $\left|(\varphi_X)_p\right| : |\varphi^* \mathbf{F}_X(p)| \to |\mathbf{F}_X(p)|$ is a homotopy equivalence. $\square$

For $k \in \mathbb{Z}_{\geq 0}$, H_k will denote the k-th simplicial homology functor with coefficients in a field $\mathbb{F}$ [68]. Recall that **vect** denotes the category of finite dimensional vector spaces over a field $\mathbb{F}$. Given an arbitrary category $\mathcal{C}$, the interleaving distance $d_{\mathrm{I}}^{\mathcal{C}}$ between two functors $(\mathcal{P}, \leq) \to \mathcal{C}$ is recalled in Definition A3 of the appendix.

Theorem 4.15 *Given any two filtrations $\mathbf{F}_X : (\mathcal{P}, \leq) \to \mathbf{Simp}(X)$ and $\mathbf{G}_Y : (\mathcal{P}, \leq) \to \mathbf{Simp}(Y)$, for any $k \in \mathbb{Z}_{\geq 0}$, we have:*

$$d_{\mathrm{I}}^{\mathbf{vect}}(\mathrm{H}_k(\mathbf{F}_X), \mathrm{H}_k(\mathbf{G}_Y)) \leq 2\, d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{G}_Y). \tag{12}$$

Proof Consider the metric $D(-,-) := \frac{1}{2} \cdot d_{\mathrm{I}}^{\mathbf{vect}}(\mathrm{H}_k(-), \mathrm{H}_k(-))$ on $\mathbf{Simp}^{\mathcal{P}}$. Given any surjection $\varphi_X : Z \twoheadrightarrow X$, by Proposition 4.14, we have the isomorphism $\mathrm{H}_k(\varphi_X^* \mathbf{F}_X) \cong \mathrm{H}_k(\mathbf{F}_X)$ and in turn that D satisfies condition (i) in Theorem 4.11. Functoriality of H_k guarantees condition (ii) in Theorem 4.11. Now the claim follows from Theorem 4.11. $\square$

Remark 4.16 (i) The theorem above subsumes [21, Theorem 4.2] which addresses the case $\mathcal{P} = \mathbb{R}$. Our proof relying on the universality of d_{GH} gives an alternative proof of [21, Theorem 4.2].

(ii) When $\mathbf{F}_X$ and $\mathbf{G}_Y$ are the (spatiotemporal) Vietoris-Rips filtrations of (dynamic) metric spaces on X and Y, the inequality in Eq. 12 coincides with the (spatiotemporal) Vietories-Rips filtration stability theorems [49, 54, 55]. This is a corollary of Proposition A13 (i) and (ii).

(iii) The distance $d_{\mathrm{GH}}(-,-)$ is more discriminative than $\max_{k \in \mathbb{Z}_{\geq 0}} d_{\mathrm{I}}^{\mathbf{vect}}(\mathrm{H}_k(-), \mathrm{H}_k(-))$ as can be seen through the following example.

Example 4.17 Let $X = \{x_1\}$ and let $Y = \{y_1, y_2\}$. Define $\mathbf{F}_X : \mathbb{R} \to \mathbf{Simp}(X)$ and $\mathbf{F}_Y : \mathbb{R} \to \mathbf{Simp}(Y)$ by $\mathbf{F}_X(t) = \begin{cases} x_1, & t \in [0, \infty) \\ \emptyset, & \text{otherwise,} \end{cases}$ and $\mathbf{F}_Y(t) := \begin{cases} \{\{y_1\}\}, & t \in [0,1) \\ \{\{y_1\}, \{y_2\}, \{y_1, y_2\}\}, & t \in [1, \infty) \\ \emptyset, & \text{otherwise,} \end{cases}$

respectively. Note that $\max_{k \in \mathbb{Z}_{\geq 0}} d_{\mathrm{I}}^{\mathbf{vect}}(\mathrm{H}_k(\mathbf{F}_X), \mathrm{H}_k(\mathbf{F}_Y)) = 0$, but $d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{F}_Y) = 1$. Noting that $\mathbf{F}_X$ and $\mathbf{F}_Y$ are homotopy equivalent, this example shows that d_{GH} between homotopy equivalent filtrations can be strictly positive.

We further remark that d_{GH} between $\mathbb{R}$-indexed simplicial filtrations can be arbitrarily larger than the homotopy interleaving distance [23]. e.g. when comparing the Vietoris-Rips filtrations of metric spaces that are (almost) homotopy equivalent but are far from being isometric, such as a pair of a circle and a circle with long flares, cf. [22, Fig. 1].

In the appendix we provide other upper and lower bounds for d_{GH}; see Section D.

4.3 Hierarchical Clusterings over Posets

Let X be a nonempty finite set. A **$(\mathcal{P}, \leq)$-indexed hierarchical clustering** (over X) is any poset map $(\mathcal{P}, \leq) \to \mathbf{SubPart}(X)$. Standard examples include the case of $\mathcal{P} = \mathbb{R}^n$ with the product order, often referred to as *hierarchical clustering* (when $n = 1$) or *multiparameter hierarchical clustering* (when $n \geq 2$) [26–30, 49]. From Remark 2.5 (i), (ii), and Proposition 2.6 (i) recall that $\mathbf{SubPart}(X)$ is $\bigvee$-irreducibly generated and the join-irreducible elements of $\mathbf{SubPart}(X)$ are either singleton or doubleton blocks of X.

Recall that given any point p in a poset $\mathcal{P}$, $p^{\uparrow} := \{r \in \mathcal{P} : p \leq r\}$. Invoking Theorem 3.10, we have:

Theorem 4.18 *Let $(\mathcal{P}, \leq, \Omega)$ be a poset with a flow. For any $\mathbf{F}, \mathbf{G} : (\mathcal{P}, \leq, \Omega) \to \mathbf{SubPart}(X)$,*

$$d_{\mathrm{I}}(\mathbf{F}, \mathbf{G}) = \max_{x, x' \in X} d_{\widehat{\Omega}} \left(\mathbf{F}^{-1}\left(\left\{x, x'\right\}^{\uparrow} \right), \mathbf{G}^{-1}\left(\left\{x, x'\right\}^{\uparrow} \right) \right). \tag{13}$$

(Note: when $x = x'$, the set $\left\{x, x'\right\}$ is the singleton $\{x\}$).

If $\mathcal{P} = \mathbb{R}^n$ (resp. $\mathbf{Int}$), the equality above reduces computing the interleaving distance between multiparameter hierarchical clusterings (of a common space X) to computing the

Hausdorff distance between upper sets of $\mathbb{R}^n$ (resp. **Int**) a finite number of times. In the next section, we discuss the computational complexity of the RHS when $\mathcal{P} = \mathbf{Int}$ (Theorem 4.30). Also we will discuss the comparison of two $(\mathcal{P}, \leq)$-indexed hierarchical clusterings *over different underlying sets.*

4.4 Computing Distances between Formigrams

In this section we elucidate the structure of the two distances, d_{F} and d_{GH}, between formigrams introduced in Section 2.4 (Theorems 4.21 and 4.24). Thereby we find equivalences between several known metrics for comparing hierarchical clusterings (Remark 4.25 (i)). Also, we clarify the computational cost of d_{F}, d_{GH} and other related metrics (Remark 4.25 (ii) and Theorem 4.30). d_{GH} between formigrams will be extended to a distance between any poset-indexed hierarchical clusterings (Definition 4.26).

Formigrams can be viewed as poset maps

The poset **Int** in Example 2.13 is isomorphic to the poset of nonempty *closed* intervals of $\mathbb{R}$ whose partial order is inclusion. Each $(a, b) \in \mathbf{Int}$ will be identified with the closed interval $[a, b] \subset \mathbb{R}$. Let us fix a nonempty finite set X. A poset map $(\mathbf{Int}, \leq) \to (\mathbf{SubPart}(X), \leq)$ will be simply denoted by $\mathbf{Int} \to \mathbf{SubPart}(X)$. Any formigram θ over X induces a map $\mathbf{Int} \to \mathbf{SubPart}(X)$:

Definition 4.19 For $\theta \in \mathbf{Formi}(X)$, define $\widehat{\theta} : \mathbf{Int} \to \mathbf{SubPart}(X)$ as $I \mapsto \bigvee_{s\in I} \theta(s)$.

See Fig. 5 (A) and (B) for an illustrative example of Definition 4.19.

Recall the flow Ω on **Int** in Eq. 2. Definition 2.9 can be specialized as follows: Given $\alpha, \alpha' : \mathbf{Int} \to \mathbf{SubPart}(X)$

$$\widehat{d_{\mathrm{F}}}\left(\alpha, \alpha'\right) := \inf\left\{\varepsilon \in [0, \infty) : \alpha, \alpha' \text{ are } \varepsilon\text{-interleaved w.r.t. } - \cdot \Omega\right\}.$$

It is not difficult to check that $d_{\mathrm{F}}(-,-)$ in Definition 2.20 coincides with $\widehat{d_{\mathrm{F}}}(\hat{-}, \hat{-})$:

Proposition 4.20 ([45, Definition 4.11]) *For any $\theta, \theta' \in \mathbf{Formi}(X)$, we have:*

$$d_{\mathrm{F}}\left(\theta, \theta'\right) = \widehat{d_{\mathrm{F}}}\left(\widehat{\theta}, \widehat{\theta'}\right).$$

In view of Definition 4.19 and Proposition 4.20, for ease of notation in what follows, any formigram θ over X will be identified with the poset map $\widehat{\theta} : \mathbf{Int} \to \mathbf{SubPart}(X)$ and d_{F} will be identified with $\widehat{d_{\mathrm{F}}}$. Also for any $x, x' \in X$, we define

$$\theta^{-1}\left(\left\{x, x'\right\}^{\uparrow}\right) := \left\{I \in \mathbf{Int} : \{\{x, x'\}\} \leq \bigvee_{s\in I} \theta(s)\right\}.$$

d_{F} via interleaving by parts

Let d_{H} be the *Hausdorff distance* (Definition A4) in $(\mathbf{Int}, ||-||_\infty)$ (Example 2.13). As a corollary of Theorem 4.18, we have:

Theorem 4.21 *For any two formigrams θ and θ' over X,*

$$d_{\mathrm{F}}\left(\theta, \theta'\right) = \max_{x,x'\in X} d_{\mathrm{H}}\left(\theta^{-1}\left(\left\{x, x'\right\}^{\uparrow}\right), \theta'^{-1}\left(\left\{x, x'\right\}^{\uparrow}\right)\right). \tag{14}$$

We utilize Theorem 4.21 for elucidating the computational complexity of d_{F} (Proposition 4.29). Inspired by Theorem 4.21 we define:

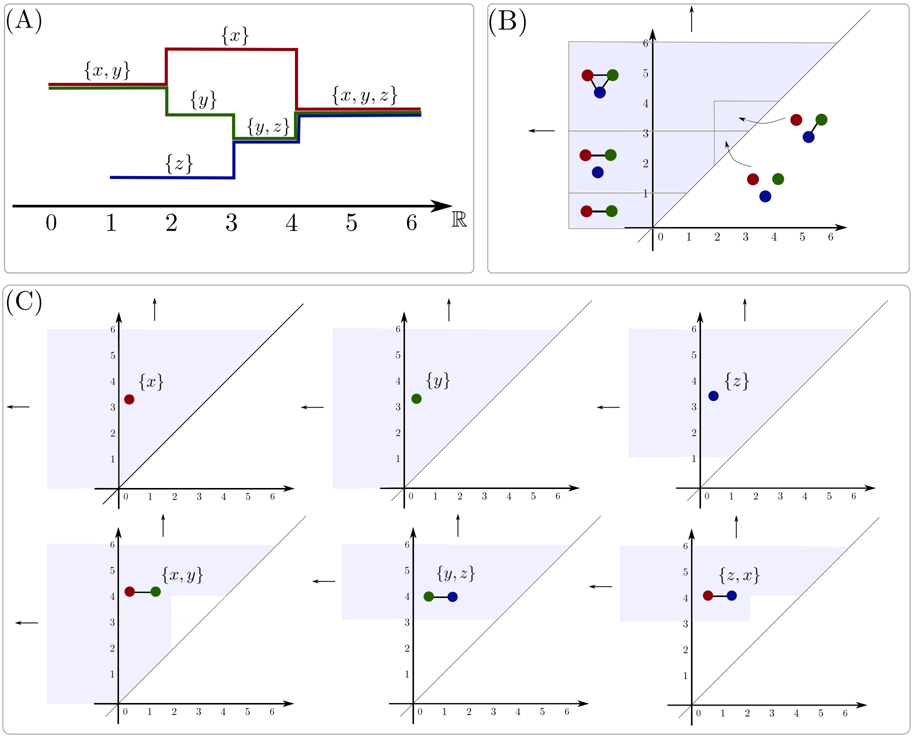

Fig. 5 (A) A formigram θ over $\{x, y, z\}$ such that $\theta(t) = \emptyset$ for $t \notin [0, 6]$. x, y, z are colored in red, green and blue, respectively. (B) The corresponding map $\widehat{\theta} : \mathbf{Int} \to \mathbf{SubPart}(X)$ (Definition 4.19). (C) The cosheaf-code of θ (Definition 4.22)

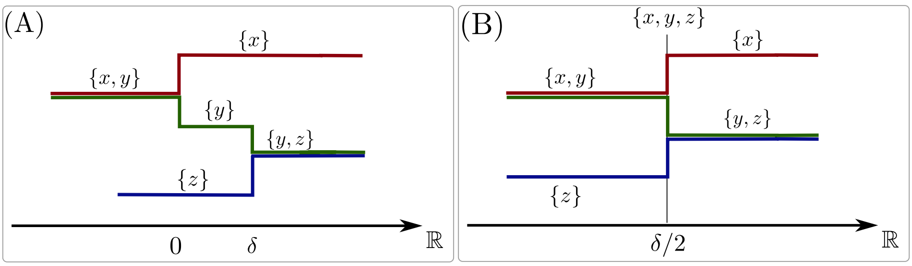

Fig. 6 (A) A formigram θ over $\{x, y, z\}$. x, y, z are colored in red, green and blue, respectively. (B) The formigram $\theta' := \mathbf{S}_{\delta/2}(\theta)$ (cf. Definition 2.18). The cosheaf-codes of θ and θ' are illustrated in Fig. 7

Definition 4.22 For $\theta \in \mathbf{Formi}(X)$, we call $B(\theta) := \left\{\theta^{-1}\left(\left\{x, x'\right\}^{\uparrow}\right)\right\}_{x,x' \in X}$ the **cosheaf-code** of θ (see Fig. 5 for an example).

See Figs. 6 and 7 for an illustrative example of an application of Theorem 4.21.

Remark 4.23 (i) Theorem 4.21 is analogous to the *isometry theorem for zigzag modules* [6, 70], which says that a certain interleaving distance between **vect**-valued zigzag modules is equal to the bottleneck distance between their *block barcodes*.

(ii) Recall the notions of dendrograms and ultrametrics induced by dendrograms (Definition 2.14). In Theorem 4.21, let us assume that θ and θ' are dendrograms over X. Then, for each $x, x' \in X$, $\theta^{-1}\left(\{x, x'\}^{\uparrow}\right) = \{(a, b) \in \mathbf{Int} : b \in [u_\theta(x, x'), \infty)\}$ and similarly for $\theta'^{-1}\left(\{x, x'\}^{\uparrow}\right)$. Therefore,

$$d_{\mathrm{H}}\left(\theta^{-1}(\{x, x'\}^{\uparrow}), \theta'^{-1}(\{x, x'\}^{\uparrow})\right) = |u_\theta(x, x') - u_{\theta'}(x, x')|$$

and in turn $d_{\mathrm{F}}(\theta, \theta') = \max_{x, x' \in X} |u_\theta(x, x') - u_{\theta'}(x, x')|$.

Structure of d_{GH} and related metrics

Consider a tripod $R:\ X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ between sets X and Y. For $x \in X$ and $y \in Y$, we write $(x, y) \in R$ when there exists $z \in Z$ such that $x = \varphi_X(z)$ and $y = \varphi_Y(z)$. We can reformulate the Gromov-Hausdorff distance between formigrams (Definition 2.22) via cosheaf-codes of formigrams:

Theorem 4.24 *Let θ_X,θ_Y be any two formigrams over X and Y, respectively. Then, we have:*

$$d_{\mathrm{GH}}(\theta_X, \theta_Y) = \frac{1}{2} \min_R \max_{\substack{(x,y)\in R \\ (x',y')\in R}} d_{\mathrm{H}}\left(\theta_X^{-1}(\{x, x'\}^{\uparrow}), \theta_Y^{-1}(\{y, y'\}^{\uparrow})\right), \tag{15}$$

where the minimum is taken over all tripods $R:\ X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ between X and Y.

Proof This follows from Theorem 4.21 and the equivalence in Eq. 6. Details are omitted. □

Remark 4.25 (i) Theorem 4.24 shows that d_{GH} in Eq. 15 has exactly the same structure as the distance d_Q introduced in [28, page 69], and the distance d_{CI} in [29, Definition 2.14]. The only difference is that d_{GH}, d_Q and d_{CI} compare respectively **Int**-indexed hierarchical clusterings, $\mathbb{R}^2$-indexed hierarchical clusterings, and $\mathbb{R}^n$-indexed hierarchical clusterings.

(ii) In Theorem 4.24, assume that θ_X and θ_Y are *dendrograms* over X and Y, respectively (Definition 2.14). Then, by Remark 4.23 (ii), we have that

$$d_{\mathrm{GH}}(\theta_X, \theta_Y) = \frac{1}{2} \min_R \max_{\substack{(x,y)\in R \\ (x',y')\in R}} |u_{\theta_X}(x, x') - u_{\theta_Y}(y', y)|.$$

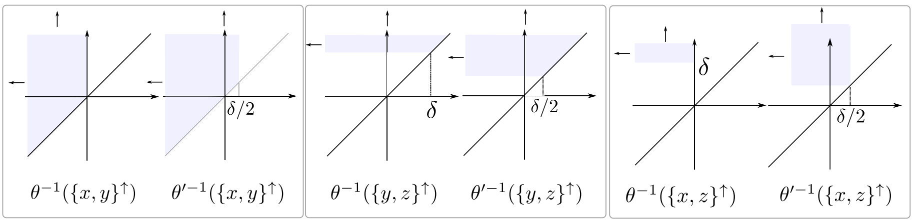

Fig. 7 Consider the formigrams θ and θ' in Fig. 6. Observe that $d_{\mathrm{H}}\left(\theta^{-1}(\{x, y\}^{\uparrow}), \theta'^{-1}(\{x, y\}^{\uparrow})\right) = d_{\mathrm{H}}\left(\theta^{-1}(\{y, z\}^{\uparrow}), \theta^{-1}(\{y, z\}^{\uparrow})\right) = d_{\mathrm{H}}\left(\theta^{-1}(\{x, z\}^{\uparrow}), \theta'^{-1}(\{x, z\}^{\uparrow})\right) = \delta/2$. Also, for any $w \in \{x, y, z\}$, $d_{\mathrm{H}}\left((\theta^{-1}(\{w\}^{\uparrow}), \theta'^{-1}(\{w\}^{\uparrow})\right) = d_{\mathrm{H}}(\mathbf{Int}, \mathbf{Int}) = 0$. By Theorem 4.21 we obtain $d_{\mathrm{F}}(\theta, \theta') = \delta/2$

Note that the RHS coincides with the Gromov-Hausdorff between (ultra)metric spaces (Definition A5), which is known to be NP-hard to compute [71]. Therefore, computing d_{GH} between formigrams is also NP-hard [45]. It is not difficult to see that the previous item implies that computing $d_{\mathcal{Q}}$ and d_{CI} is also NP-hard.

All distances mentioned in Remark 4.25 can now be seen as specializations of the following (cf. Theorem 4.18):

Definition 4.26 Let $(\mathcal{P}, \leq, \Omega)$ be a poset with a flow. Given any $\theta_X : (\mathcal{P}, \leq, \Omega) \to \mathbf{SubPart}(X)$ and $\theta_Y : (\mathcal{P}, \leq, \Omega) \to \mathbf{SubPart}(Y)$, the **Gromov-Hausdorff** distance between them is defined by:

$$\begin{aligned} d_{\mathrm{GH}}(\theta_X, \theta_Y) &= \frac{1}{2} \min_R d_{\mathrm{I}}(\varphi_X^* \theta_X, \varphi_Y^* \theta_Y) \\ &= \frac{1}{2} \min_R \max_{\substack{(x,y)\in R \\ (x',y')\in R}} d_{\mathrm{H}} \left(\theta_X^{-1}(\{x, x'\}^{\uparrow}), \theta_Y^{-1}(\{y, y'\}^{\uparrow}) \right), \end{aligned}$$

where the minimum is taken over all tripods $R: \ X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ between X and Y.

4.4.1 Computational Cost for the Calculation of d_{F}

In this section we elucidate the computational complexity of the formigram interleaving distance d_{F} (Theorem 4.30). We do this by clarifying the complexity of each preliminary step for the calculation of d_{F}. Many ideas in this section can be adapted to the case of the interleaving distance between multiparameter hierarchical clustering (cf. Eq. 13.

Let θ be a formigram over X (Definition 2.16). Let $n := |X|$ and $m := \mathbf{crit}(\theta)$.

Proposition 4.27 *Computing the corresponding poset map $\widehat{\theta} : \mathbf{Int} \to \mathbf{SubPart}(X)$ (Definition 4.19) requires at most time $O(n^2 m^2)$.*

Proposition 4.28 *Given $\widehat{\theta} : \mathbf{Int} \to \mathbf{SubPart}(X)$, computing the cosheaf-code of θ takes at most time $O(n^{\omega} m^2)$.*

Proposition 4.29 *Assume that the cosheaf-codes of two formigrams θ and θ' over X are given where $n := |X|$, $m := |\mathbf{crit}(\theta)|$ and $m' := |\mathbf{crit}(\theta')|$. Computing $d_{\mathrm{F}}(\theta, \theta')$ requires at most time $O(n^2(m + m'))$.*

In sum:

Theorem 4.30 *Given two formigrams θ and θ', computing $d_{\mathrm{F}}(\theta, \theta')$ requires at most $O(n^{\omega}\ell^2)$. where $\ell := \max(m, m')$.*

We can significantly reduce the complexity $O(n^{\omega}\ell^2)$ mentioned above to $O(n^2 \cdot \ell^{1.5} \log \ell)$ by restricting ourselves to "simple" formigrams; see Section E in the appendix. To prove this claim, we utilize a special relationship between the bottleneck distance and the Hausdorff distance on the real line which may be of independent interest (Theorem A14).

***Proof of Proposition* 4.27**. We begin with the following lemma:

Lemma 4.31 *Let P_1 and P_2 be any two subpartitions of X, with $n := |X|$. Computing $P_1 \vee P_2$ requires at most time $O(n^2)$.*

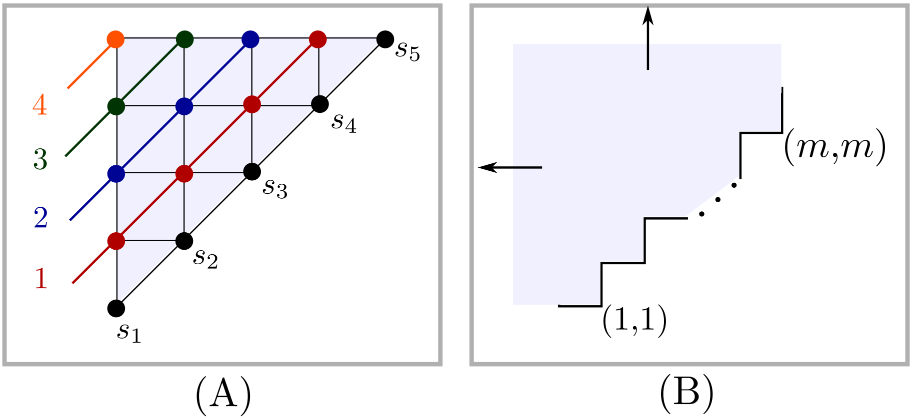

Fig. 8 (A) For a given formigram θ, let us assume that $\mathbf{crit}(\theta) = \{s_1 < s_2 < s_3 < s_4 < s_5\}$. The main diagonal line stands for the real line via the bijection $(r, r) \in \mathbb{R}^2 \leftrightarrow r \in \mathbb{R}$. We assign a subpartition of X to each colored point in the grid as follows: For each point v in the line 1, assign $\theta(s_i) \vee \theta(s_{i+1})$ where s_i and s_{i+1} are adjacent to v in the grid. For each point in line i, assign $P_1 \vee P_2$ where P_1 and P_2 are the subpartitions assigned to two points in line $i-1$ which are adjacent to v. Observe that, given any $I \in \mathbf{Int}$ with $I \cap \mathbf{crit}(\theta) \neq \emptyset$, $\widehat{\theta}(I)$ is equal to the subpartition assigned to the maximal point $v = (v_1, v_2)$ in the grid $(\subset \mathbb{R}^{\mathrm{op}} \times \mathbb{R})$ s.t. $v_1, v_2 \in I$. (B) An illustration of $\theta_X^{-1}(\{x, x'\}^{\uparrow})$ of which the number of its corner points is maximal, $2m-1$

Proof For $i = 1, 2$, let $X_i \subset X$ be the underlying set of P_i. Let us consider the undirected simple graph $G_i = (X_i, E_i)$ derived from P_i, where $\{x, y\} \in E_i$ if and only if x and y belong to the same block of X_i. Note that $P_1 \vee P_2$ is the partition of $X_1 \cup X_2$ where each block of $P_1 \vee P_2$ constitutes a connected component of the graph $G_1 \cup G_2 = (X_1 \cup X_2, E_1 \cup E_2)$. Therefore, computing $P_1 \vee P_2$ is equivalent to computing the connected components of $G_1 \cup G_2$. One needs $O(|X_1 \cup X_2| + |E_1 \cup E_2|)$ in time to partition $X_1 \cup X_2$ according to the components of $G_1 \cup G_2$ [72, Section 5]. Since $|X_1 \cup X_2| \leq |X| = n$ and $|E_1 \cup E_2| \leq \binom{n}{2}$, at worst time $O(n^2)$ will be necessary. □

Proof of Proposition 4.27 The claim directly follows from Lemma 4.31 and the observation that, in order to compute $\widehat{\theta}$, it suffices to compute $O(\binom{m}{2}) = O(m^2)$ different join operations between two (sub)partitions of X. See Fig. 8 (A) for an illustrative example.

Proof of Proposition 4.28 Let $X = \{x_1, \ldots, x_n\}$, and let $P = \{C_1, \ldots, C_k\}$ be a subpartition of X. This P can be encoded as the $n \times k$ *membership matrix* $M_P = (m_{ij})$ where the i-th row and the j-th column correspond to $x_i \in X$ and $C_j \in P$ respectively, and

$$m_{ij} := \begin{cases} 1, & \text{if } x_i \text{ belongs to } C_j \\ 0, & \text{otherwise.} \end{cases}$$

Note that the (i, j)-entry of $M_P(M_P)^t$ is 1 iff x_i and x_j belong to the same block, and 0 otherwise. Then, since k cannot exceed n, computing $M_P(M_P)^t$ takes $O(n^\omega)$ where $\omega \in [2, 2.373)$ is the exponent of matrix multiplication.

Since $m := |\mathbf{crit}(\theta)|$, for any $i, j \in \{1, \ldots, n\}$, the set $\widehat{\theta}^{-1}\left(\{x_i, x_j\}^{\uparrow}\right)$ is directly obtained by checking the (i, j)-components of $\binom{m}{2}$ matrices of dimension $n \times n$, which takes time $O\left(\binom{m}{2} \cdot 1\right) = O(m^2)$. As there are $n + \binom{n}{2}$ pairs of $i, j \in \{1, \ldots, n\}$ with $i \leq j$, the cosheaf-code of θ contains $n + \binom{n}{2}$ elements. Therefore, computing the cosheaf-code of θ takes in time $O(n^\omega m^2 + (n + \binom{n}{2})m^2) = O(n^\omega m^2)$.

***Proof of Proposition* 4.29**. Recall from Theorem 4.21 that computing d_{F} reduces to the calculation of the Hausdorff distance between upper sets of $(\mathbf{Int}, ||-||_\infty)$. The lemma below provides an insight into computing d_{F}:

Lemma 4.32 *If A, B are upper sets of $(\mathbf{Int}, ||-||_\infty)$, then their Hausdorff distance is given by the formula*

$$d_{\mathrm{H}}(A, B) = \sup_{\ell} d_{\mathrm{H}}(A \cap \ell, B \cap \ell),$$

where ℓ ranges over all lines of slope -1 in $\mathbb{R}^2$.

Proof ($\geq$) Pick any line ℓ of slope -1. Let $\varepsilon := d_{\mathrm{H}}(A, B)$. Let $x = (x_1, x_2) \in A \cap \ell$. Then, there exists $y \in B$ such that $||x - y||_\infty \leq \varepsilon$. Since B is an upper set, $y \leq (x_1 - \varepsilon, x_2 + \varepsilon) \in B$. Since ℓ is of slope -1, $(x_1 - \varepsilon, x_2 + \varepsilon)$ lies on the line ℓ, and thus $(x_1 - \varepsilon, x_2 + \varepsilon) \in B \cap \ell$. In the same way, one can prove that for any $x \in B \cap \ell$, there exists $y \in A \cap \ell$ such that $||x - y||_\infty \leq \varepsilon$.

($\leq$) Assume the RHS is less or equal to ε. Pick $a \in A$ and a line ℓ of slope -1 which passes through a. By assumptions, there is $b \in B \cap \ell$ such that $||a - b||_\infty \leq \varepsilon$. This b also belongs to B. By symmetry, for all $b \in B$, there exists $a \in A$ such that $||a - b||_\infty \leq \varepsilon$. □

***Proof of Proposition* 4.29** For $x, x' \in X$, the upper sets $A_{\{x,x'\}} := \theta_X^{-1}(\{x, x'\}^\uparrow)$ and $B_{\{x,x'\}} := (\theta'_X)^{-1}(\{x, x'\}^\uparrow)$ have at most $2m - 1$ and $2m' - 1$ corner points, respectively (see Fig. 8 (B)). By Theorem 4.21, computing $d_{\mathrm{F}}(\theta, \theta')$ reduces to computing $d_{\mathrm{H}}\left(A_{\{x,x'\}}, B_{\{x,x'\}}\right)$ for all $x, x' \in X$. By Lemma 4.32, computing $d_{\mathrm{H}}\left(A_{\{x,x'\}}, B_{\{x,x'\}}\right)$ requires at most $O((2m-1) + (2m'-1)) = O(m + m')$ computations of $d_{\mathrm{H}}\left(A_{\{x,x'\}} \cap \ell, B_{\{x,x'\}} \cap \ell\right)$ where ℓ are (-1)-slope-lines passing through at least one of $O(m + m')$ corner points of $A_{\{x,x'\}}$ or $B_{\{x,x'\}}$. For any ℓ, let $p, q \in \mathbb{R}^{\mathrm{op}} \times \mathbb{R}$ be the unique minima of $A_{\{x,x'\}} \cap \ell$ and $B_{\{x,x'\}} \cap \ell$, respectively. Then, it is not difficult to check that $d_{\mathrm{H}}\left(A_{\{x,x'\}} \cap \ell, B_{\{x,x'\}} \cap \ell\right) = ||p - q||_\infty$. Hence the claim follows. □

5 Discussion

One open question is the following: *What is the relationship between d_{GH} in Definition 4.9 and the edit distance d_{edit} in [73] between lattice-indexed simplicial filtrations*? Both distances are a generalization or a rendition of certain distances that satisfy *universality*; For d_{GH}, see Theorem 4.11 (also [13, Proposition 6.2.21]). d_{edit} is a rendition of the edit distance on Reeb graphs which satisfies another universal property [24] (and is itself inspired on d_{GH}). Currently we know that d_{GH} and d_{edit} cannot be strongly equivalent; it is not difficult to find a pair of simplicial filtrations such that d_{GH} vanishes, but d_{edit} does not.

Appendix

Persistence Modules and Rank Functions

Let **vect** be the category of finite dimensional vector spaces and linear maps over a field $\mathbb{F}$.

Definition A1 Any functor $M : (\mathbb{R}, \leq) \to \mathbf{vect}$ is said to be a (standard) **persistence module**, i.e. each $r \in \mathbb{R}$ is sent to a vector space $M(r)$ and each pair $r \leq s$ in $\mathbb{R}$ is sent to a linear map $M(r \leq s)$. In particular, for all $r \leq s \leq t$, we have:

$$M(s \leq t) \circ M(r \leq s) = M(r \leq t).$$

M is called **constructible** if there exists a finite set $\{c_1, \ldots, c_n\} \subset \mathbb{R}$ such that (i) for $i = 1, \ldots, n$, whenever $r, s \in [c_i, c_{i+1})$ with $r \leq s$, $M(r \leq s)$ is the identity map (let $c_{n+1} := \infty$), (ii) for $r \in (-\infty, a_1)$, $M(r) = 0$.

By replacing the indexing poset $(\mathbb{R}, \leq)$ by $(\mathbb{R}^d, \leq)$ for $d \geq 2$, we obtain a **multiparameter persistence module**.

Let $M : (\mathbb{R}, \leq) \to \mathbf{vect}$. For any $r \leq r' \leq s' \leq s$ in $\mathbb{R}$, we have

$$M(r \leq s) = M(s' \leq s) \circ M(r' \leq s') \circ M(r \leq r'),$$

which implies $\operatorname{rank}(M(r \leq s)) \leq \operatorname{rank}(M(r' \leq s'))$.

Definition A2 Let $M : (\mathbb{R}, \leq) \to \mathbf{vect}$. The **rank function** $\operatorname{rk}(M) : (\mathbf{Int}, \subset) \to \mathbb{Z}_{\geq 0}^{\mathrm{op}}$ of M is defined as the map $[a, b] \mapsto \operatorname{rank}\big(M(a \leq b)\big)$.

Interleaving Distances in General

We review the general notion of interleaving distance in the language of [11] (Definition 2.9 is a special instance of the definition below). Consult [74] for general definitions related to category theory.

Let $(\mathcal{P}, \leq, \Omega)$ be a poset with a flow. If $\mathcal{P}$ is viewed as a category, then, for each $\varepsilon \in [0, \infty)$, Ω_ε is an endofunctor on $\mathcal{P}$ and we have $\mathbf{I}_{\mathcal{P}} \leq \Omega_\varepsilon$. We view this inequality as a natural transformation $\eta_\varepsilon : \mathbf{I}_{\mathcal{P}} \to \Omega_\varepsilon$. Let $\mathcal{C}$ be any category and let $M : \mathcal{P} \to \mathcal{C}$ be any functor. Then, we have a natural transformation $M\eta_\varepsilon : M \to M\Omega_\varepsilon$.

Definition A3 Let $(\mathcal{P}, \leq, \Omega)$ be a poset with a flow. Any two functors $M, N : \mathcal{P} \to \mathcal{C}$ are said to be Ω_ε**-interleaved** if there exists a pair of natural transformations $\varphi : M \to N\Omega_\varepsilon$ and $\psi : N \to M\Omega_\varepsilon$ such that the diagram below commutes.

$$\begin{tikzcd}
M \arrow[r, "M\eta_\varepsilon"] \arrow[dr, "\varphi"] & M\Omega_\varepsilon \arrow[r, "M\eta_\varepsilon\Omega_\varepsilon"] \arrow[dr, "\varphi\Omega_\varepsilon"] & M\Omega_\varepsilon\Omega_\varepsilon \\
N \arrow[r, "N\eta_\varepsilon"'] \arrow[ur, "\psi"] & N\Omega_\varepsilon \arrow[r, "N\eta_\varepsilon\Omega_\varepsilon"'] \arrow[ur, "\psi\Omega_\varepsilon"] & N\Omega_\varepsilon\Omega_\varepsilon
\end{tikzcd}$$

The interleaving distance (with respect to Ω) is:

$$d_{\mathrm{I}}^{\mathcal{C}}(M, N) = \inf\{\varepsilon \geq 0 : \ M, N \text{ are } \Omega_\varepsilon - \text{interleaved}\}$$

where $d_{\mathrm{I}}^{\mathcal{C}}(M, N) := \infty$ if there is no ε-interleaving for any $\varepsilon \geq 0$.

When $\mathcal{C}$ is a poset $\mathcal{Q}$, this definition reduces to Definition 2.9. When $\mathcal{P} = \mathbb{R}^n$ with the flow in Eq. 1 and $\mathcal{C} = \mathbf{vect}$, the distance $d_{\mathrm{I}}^{\mathcal{C}}$ is the standard interleaving distance [5, 12].

(Gromov-)Hausdorff and Bottleneck Distances.

We recall the definitions of the **Hausdorff distance**, **Gromov-Hausdorff distance** [66, Section 7.3.3] and the **Bottleneck distance** [75] in that order.

Definition A4 (Hausdorff distance) Let A and B be closed subsets of a metric space (M, d). The Hausdorff distance between A and B is defined as $d_{\mathrm{H}}(A, B) = \inf\{\varepsilon \in [0, \infty) : A \subset B^{\varepsilon}$ and $B \subset A^{\varepsilon}\}$, where $A^{\varepsilon} := \{m \in M : \exists a \in A,\ d(a, m) \leq \varepsilon\}$.

Let (X, d_X) and (Y, d_Y) be any two metric spaces and let $R : X \xtwoheadleftarrow{\varphi_X} Z \xtwoheadrightarrow{\varphi_Y} Y$ be a tripod (i.e. a pair of surjective maps from a common space) between X and Y. Then, the *distortion* of R is defined as

$$\operatorname{dis}(R) := \sup_{z,z' \in Z} \left| d_X\left(\varphi_X(z), \varphi_X(z')\right) - d_Y\left(\varphi_Y(z), \varphi_Y(z')\right) \right|.$$

The Gromov-Hausdorff distance d_{GH} measures how far two metric spaces are from being isometric.

Definition A5 (Gromov-Hausdorff distance) The Gromov-Hausdorff distance between compact metric spaces (X, d_X) and (Y, d_Y) is defined as

$$d_{\mathrm{GH}}\left((X, d_X), (Y, d_Y)\right) := \frac{1}{2} \inf_R \operatorname{dis}(R),$$

where the infimum is taken over all tripods R between X and Y.

Bottleneck distance
The bottleneck distance is an extensively studied metric. We adopt notation in [62] to describe it. Partial bijections are referred to as *matchings*. Given two nonempty sets A and B, we use $\sigma : A \nrightarrow B$ to denote a matching $\sigma \subset A \times B$. The canonical projections of σ onto A and B are denoted by $\operatorname{coim}(\sigma)$ and $\operatorname{im}(\sigma)$, respectively. By $\langle a, b\rangle$ for $a < b$ in $\mathbb{R}$, we denote one of the real intervals (a, b), $(a, b]$, $[a, b)$, and $[a, b]$.

Letting $\mathcal{A}$ be a multiset of intervals in $\mathbb{R}$ and $\varepsilon \geq 0$,

$$\mathcal{A}^{\varepsilon} := \{\langle b, d\rangle \in \mathcal{A} : b + \varepsilon < d\} = \{I \in \mathcal{A} : [t, t + \varepsilon] \subset I \text{ for some } t \in \mathbb{R}\}.$$

Note that $\mathcal{A}^0 = \mathcal{A}$.

Definition A6 (Bottleneck distance) Let $\mathcal{A}$ and $\mathcal{B}$ be multisets of intervals in $\mathbb{R}$. We define a δ-matching between $\mathcal{A}$ and $\mathcal{B}$ to be a matching $\sigma : \mathcal{A} \nrightarrow \mathcal{B}$ such that $\mathcal{A}^{2\delta} \subset \operatorname{coim}(\sigma)$, $\mathcal{B}^{2\delta} \subset \operatorname{im}(\sigma)$, and if $\sigma\langle b, d\rangle = \langle b', d'\rangle$, then

$$\langle b, d\rangle \subset \langle b' - \delta, d' + \delta\rangle, \qquad \langle b', d'\rangle \subset \langle b - \delta, d + \delta\rangle.$$

with the convention $+\infty + \delta = +\infty$ and $-\infty - \delta = -\infty$. We define the bottleneck distance d_{B} by

$$d_{\mathrm{B}}(\mathcal{A}, \mathcal{B}) := \inf\{\delta \in [0, \infty) : \exists \delta\text{-matching between } \mathcal{A} \text{ and } \mathcal{B}\}.$$

We declare $d_{\mathrm{B}}(\mathcal{A}, \mathcal{B}) = +\infty$ when there is no δ-matching between $\mathcal{A}$ and $\mathcal{B}$ for any $\delta \in [0, \infty)$.

On the (generalized) Gromov-Hausdorff Distance

The goal of this section is to establish basic properties of d_{GH} in Definition 4.9; Propositions A7, A8, A11, A12, and A13. Throughout this section, X, Y, Z, and W will denote nonempty finite sets.

Proposition A7 *d_{GH} in Definition 4.9 is an extended pseudometric.*

Proof Symmetry and non-negativity are clear. We prove the triangle inequality. Consider any three filtrations $\mathbf{F}_X : (\mathcal{P}, \leq, \Omega) \to \mathbf{Simp}(X)$, $\mathbf{G}_Y : (\mathcal{P}, \leq, \Omega) \to \mathbf{Simp}(Y)$, and $\mathbf{H}_W : (\mathcal{P}, \leq, \Omega) \to \mathbf{Simp}(W)$. Assume that, for $\eta_1, \eta_2 > 0$, we have:

$$d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{G}_Y) < \eta_1 \text{ and } d_{\mathrm{GH}}(\mathbf{G}_Y, \mathbf{H}_W) < \eta_2.$$

Then, there exist tripods $R_1 : X \xleftarrow{\varphi_X} Z_1 \xrightarrow{\varphi_Y} Y$ and $R_2 : Y \xleftarrow{\psi_Y} Z_2 \xrightarrow{\psi_W} W$ such that

$$d_{\mathrm{I}}^{\mathbf{Simp}(Z_1)}(\varphi_X^* \mathbf{F}_X, \varphi_Y^* \mathbf{G}_Y) < \eta_1 \text{ and } d_{\mathrm{I}}^{\mathbf{Simp}(Z_2)}(\psi_Y^* \mathbf{G}_Y, \psi_Y^* \mathbf{G}_Y) < \eta_2.$$

Consider the set $Z := \{(z_1, z_2) \in Z_1 \times Z_2 : \varphi_Y(z_1) = \psi_Y(z_2)\}$ and let $\pi_1 : Z \to Z_1$ and $\pi_2 : Z \to Z_2$ be the canonical projections to the first and the second coordinate, respectively. We define the composite tripod $R_2 \circ R_1$ as follows:

$$R_2 \circ R_1 : X \xleftarrow{\omega_X} Z \xrightarrow{\omega_W} W, \text{ where } \omega_X := \varphi_X \circ \pi_1, \quad \omega_W := \psi_W \circ \pi_2. \tag{16}$$

$$\begin{tikzcd}
 & & Z \arrow[dl, two heads, "\pi_1"'] \arrow[dr, two heads, "\pi_2"] & & \\
 & Z_1 \arrow[dl, two heads, "\varphi_X"'] \arrow[dr, two heads, "\varphi_Y"] & & Z_2 \arrow[dl, two heads, "\psi_Y"'] \arrow[dr, two heads, "\psi_W"] & \\
X & & Y & & W
\end{tikzcd}$$

Also, let $\omega_Y := \varphi_Y \circ \pi_1 = \psi_Y \circ \pi_2$. By taking the composite tripod, we have:

$$\begin{aligned}
d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{H}_W) &\leq d_{\mathrm{I}}^{\mathbf{Simp}(Z)}(\omega_X^* \mathbf{F}_X, \omega_Z^* \mathbf{H}_Z) \\
&\leq d_{\mathrm{I}}^{\mathbf{Simp}(Z)}(\omega_X^* \mathbf{F}_X, \omega_Y^* \mathbf{G}_Y) + d_{\mathrm{I}}^{\mathbf{Simp}(Z)}(\omega_Y^* \mathbf{G}_Y, \omega_Z^* \mathbf{H}_Z) \\
&= d_{\mathrm{I}}^{\mathbf{Simp}(Z_1)}(\varphi_X^* \mathbf{F}_X, \varphi_Y^* \mathbf{G}_Y) + d_{\mathrm{I}}^{\mathbf{Simp}(Z_2)}(\psi_Y^* \mathbf{G}_Y, \psi_Z^* \mathbf{H}_Z) \\
&< \eta_1 + \eta_2.
\end{aligned}$$

The desired inequality follows by letting $\eta_1 \searrow d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{G}_Y)$ and $\eta_2 \searrow d_{\mathrm{GH}}(\mathbf{G}_Y, \mathbf{H}_W)$. □

Next we show that vanishing d_{GH} implies the existence of a functorial homotopy equivalence between filtrations.

Proposition A8 *Let $\mathbf{F}_X : (\mathcal{P}, \leq, \Omega) \to \mathbf{Simp}(X)$ and $\mathbf{F}_Y : (\mathcal{P}, \leq, \Omega) \to \mathbf{Simp}(Y)$ be any two filtrations. Suppose that $d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{F}_Y) = 0$. Then there exists a family $(f_p : |\mathbf{F}_X(p)| \to |\mathbf{F}_Y(p)|)_{p \in \mathcal{P}}$ of homotopy equivalences such that the following diagram commutes*

$$\begin{tikzcd}
{|\mathbf{F}_X(p)|} \arrow[r, "|\subseteq|"] \arrow[d, "f_p", "\cong"'] & {|\mathbf{F}_X(q)|} \arrow[d, "f_q", "\cong"'] \\
{|\mathbf{F}_Y(p)|} \arrow[r, "|\subseteq|"'] & {|\mathbf{F}_Y(q)|}
\end{tikzcd}$$

(the converse does not hold; see Example 4.17). [12]

This proposition can be proved in a similar way to [21, Corollary 2.1].

Proof Let $R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ be a minimizer for d_{GH} (cf. the first footnote in Definition 2.22). Then, $d_{\mathrm{I}}(\varphi_X^* \mathbf{F}_X, \varphi_Y^* \mathbf{F}_Y) = 0$. By Definition 4.9, we obtain

$$\max_{\sigma \in \mathbf{pow}_{\geq 1}(Z)} d_{\widehat{\Omega}} \left((\varphi_X^* \mathbf{F}_X)^{-1}(\sigma^{\uparrow}), (\varphi_Y^* \mathbf{G}_Y)^{-1}(\sigma^{\uparrow}) \right) = 0.$$

This implies that $\varphi_X^* \mathbf{F}_X = \varphi_Y^* \mathbf{F}_Y$. By Proposition 4.14, we have the required functorial homotopy equivalence between $\mathbf{F}_X$ and $\varphi_X^* \mathbf{F}_X$ and between $\mathbf{F}_Y$ and $\varphi_Y^* \mathbf{F}_Y$, thus completing the proof. □

Upper bound for d_{GH}
We aim at obtaining a coarse upper bound for d_{GH} (Proposition A11).

Lemma A9 (Weak join-preserving property of $d_{\widehat{\Omega}}$) *Let* $(\mathcal{P}, \leq, \Omega)$ *be a poset with a flow. Let* (A_j) *be a family of upper sets in* $\mathcal{P}$ *and let* B *be another upper set in* $\mathcal{P}$*. Then,*

$$d_{\widehat{\Omega}} \left(\bigcup_j A_j, B \right) \leq \sup_{j \in J} d_{\widehat{\Omega}} \left(A_j, B \right) \tag{17}$$

It is not difficult to find an example that shows the inequality in Eq. 17 can be strict.

Proof Let $\varepsilon > 0$ such that $d_{\widehat{\Omega}} \left(A_j, B \right) < \varepsilon$ for all $j \in J$. Then, for each j we have $A_j \subset \widehat{\Omega}_\varepsilon(B)$ and thus $\bigcup_j A_j \subset \widehat{\Omega}_\varepsilon(B)$. Also, for every j, we have $B \subset \widehat{\Omega}(A_j)$, which implies $B \subset \bigcup_j \widehat{\Omega}(A_j)$. By Remark 2.12 (ii), we have $B \subset \widehat{\Omega}_\varepsilon \left(\bigcup_j A_j \right)$. Therefore, the left-hand side of inequality Eq. 17 is at most ε. □

For the singleton set $\{*\}$, note that $\mathbf{Simp}(\{*\})$ includes only the two simplicial complexes; the empty complex and $\{\{*\}\}$. Definition 4.9 directly implies:

Lemma A10 *Given any* $\mathbf{F}_X : (\mathcal{P}, \leq, \Omega) \to \mathbf{Simp}(X)$ *and* $\mathbf{H}_{\{*\}} : (\mathcal{P}, \leq, \Omega) \to \mathbf{Simp}(\{*\})$*, we have*

$$d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{H}_{\{*\}}) = \max_{\substack{\sigma \subset X \\ \sigma \neq \emptyset}} d_{\widehat{\Omega}} \left(\mathbf{F}_X^{-1}(\sigma^{\uparrow}), \mathbf{H}^{-1}(*^{\uparrow})) \right).$$

By invoking the triangle inequality of d_{GH} and the lemma above, we have:

Proposition A11 (Coarse upper bound for d_{GH}) *Given any filtrations* $\mathbf{F}_X : (\mathcal{P}, \leq, \Omega) \to \mathbf{Simp}(X)$ *and* $\mathbf{G}_Y : (\mathcal{P}, \leq, \Omega) \to \mathbf{Simp}(Y)$*, we have*

$$d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{G}_Y) \leq \min_{A \in U(\mathcal{P})} \left(\max_{\substack{\sigma \subset X \\ \sigma \neq \emptyset}} d_{\widehat{\Omega}} \left(\mathbf{F}_X^{-1}(\sigma^{\uparrow}), A \right) + \max_{\substack{\tau \subset Y \\ \tau \neq \emptyset}} d_{\widehat{\Omega}} \left(\mathbf{G}_Y^{-1}(\tau^{\uparrow}), A \right) \right).$$

[12] When $\mathcal{P} = \mathbb{R}$, a more general statement can be found in [13, Remark 6.8.9] in connection with the *homotopy interleaving distance* [23].

Lower bound for d_{GH}
Let $K \in \mathbf{Simp}(X)$ and let $K^{(0)}$ be the vertex set of K. For $x, x' \in K^{(0)}$, we write $x \sim_{\pi_0(K)} x'$ if there exists a sequence of 1-simplices in K connecting x and x', i.e. there exist $x_1, \ldots, x_n \in K^{(0)}$ with $\{x, x_1\}, \{x_1, x_2\}, \ldots, \{x_n, x'\} \in K$ (if the sequence is empty, then $x = x'$). This defines an equivalence relation on $K^{(0)}$ and thus $\pi_0(K) := K^{(0)}/\sim_{\pi_0(K)} \in \mathbf{SubPart}(X)$.

Then note that π_0 serves as a poset map $\mathbf{Simp}(X) \to \mathbf{SubPart}(X)$. Recall d_{GH} in Definition 4.26.

Proposition A12 *For any* $\mathbf{F}_X : (\mathcal{P}, \leq, \Omega) \to \mathbf{Simp}(X)$ *and* $\mathbf{G}_Y : (\mathcal{P}, \leq, \Omega) \to \mathbf{Simp}(Y)$,

$$d_{\mathrm{GH}}(\pi_0 \circ \mathbf{F}_X, \pi_0 \circ \mathbf{G}_Y) \leq d_{\mathrm{GH}}(\mathbf{F}_X, \mathbf{G}_Y).$$

We remark that the LHS is a better lower bound for d_{GH} than the quantity $d_{\mathrm{I}}(\mathrm{H}_0(\mathbf{F}_X), \mathrm{H}_0(\mathbf{G}_Y))$ from Theorem 4.15. For example, if $\mathbf{F}_X$ and $\mathbf{G}_Y$ are the respective Rips filtrations of metric spaces (X, d_X) and (Y, d_Y), then the inequality above coincides with the inequality in Eq. 4. The LHS of Eq. 4 is known to be a better lower bound than the interleaving distance between the zeroth homology of the Rips filtrations of (X, d_X) and (Y, d_Y) [49, Remark D.6].

Ubiquity of d_{GH}
The definitions in the following three items are relevant to the three statements in Proposition A13 respectively in order.

(i) Given a pseudometric space (X, d_X), let $\mathrm{VR}(X, d_X) : \mathbb{R} \to \mathbf{Simp}(X)$ be the **Vietoris-Rips filtration** of (X, d_X), i.e. each $r \in \mathbb{R}$ is sent to:

$$\mathrm{VR}(X, d_X)(r) = \{\sigma \subset X : |\sigma| < \infty \text{ and } d_X(x, x') \leq r \text{ for all } x, x' \in X\}.$$

(ii) Let us recall the simplexization map s from Definition 4.5. For any nonempty finite set X and any $P \in \mathbf{SubPart}(X)$, let $s(P) := \{s(B) : B \in P\} \in \mathbf{Simp}(X)$. Then s serves as a poset map $\mathbf{SubPart}(X) \to \mathbf{Simp}(X)$.

(iii) Let $\lambda > 0$. The λ-flow on the product poset $\mathbf{Int} \times \mathbb{R}_{\geq 0}$ is defined by

$$\Omega^\lambda := \{\Omega^\lambda_\varepsilon : ((a, b), r) \mapsto (a - \varepsilon, b + \varepsilon), r + \lambda\varepsilon\}_{\varepsilon \geq 0}.$$

Proposition A13 (Ubiquity of $d_{\mathrm{GH}}^{\mathrm{Simp}}$) *Let* $d_{\mathrm{GH}}^{\mathrm{Met}}$, $d_{\mathrm{GH}}^{\mathrm{HC}}$, *and* $d_{\mathrm{GH}}^{\mathrm{Simp}}$ *be the Gromov-Hausdorff distances between metric spaces, between hierarchical clusterings, and between simplicial filtrations, respectively (Definitions A5, 4.26, and 4.7). We have:*

(i) [22, Proposition 2.8] For any finite pseudometric spaces (X, d_X) *and* (Y, d_Y),

$$d_{\mathrm{GH}}^{\mathrm{Met}}((X, d_X), (Y, d_Y)) = d_{\mathrm{GH}}^{\mathrm{Simp}}(\mathrm{VR}(X, d_X), \mathrm{VR}(Y, d_Y)).$$

(ii) For any $(\mathcal{P}, \leq)$*-indexed hierarchical clusterings* θ_X *and* θ_Y *over* X *and* Y *respectively,*

$$d_{\mathrm{GH}}^{\mathrm{HC}}(\theta_X, \theta_Y) = d_{\mathrm{GH}}^{\mathrm{Simp}}(s \circ \theta_X, s \circ \theta_Y).$$

(iii) Let $\lambda > 0$. *For any dynamic metric spaces* γ_X *and* γ_Y,

$$d_{\mathrm{GH}}^{\lambda}(\gamma_X, \gamma_Y) = d_{\mathrm{GH}}^{\mathrm{Simp}}(\mathrm{VR}^\lambda(\gamma_X), \mathrm{VR}^\lambda(\gamma_Y)),$$

where the LHS is the λ*-slack interleaving distance [49, Definition 2.10].*

Since the respective proofs of (ii) and (iii) directly follow from the definitions of the involved metrics, we omit them.

Specialization of Theorem 4.30

We specialize Theorem 4.30 by restricting ourselves to *simple* formigrams (defined below); Theorem A16. This theorem is a rather direct consequence of Theorem A14 and Corollary A15 below.

Recall the Hausdorff and bottleneck distances in Section C. Recall that, by $\langle a, b\rangle$ for $a < b$ in $\mathbb{R}$, we denote one of the real intervals (a, b), $(a, b]$, $[a, b)$, and $[a, b]$. Given any real intervals $\langle a, b\rangle$ and $\langle c, d\rangle$, we write $\langle a, b\rangle < \langle c, d\rangle$ if $b < c$.

Theorem A14 *Let $\mathcal{A} := \{I_i := \langle a_i, b_i\rangle : 1 \le i \le m\}$ and $\mathcal{B} := \left\{J_j := \langle c_j, d_j\rangle : 1 \le j \le n\right\}$. Assume that $I_i < I_{i+1}$ for $i = 1, \ldots, m-1$ and $J_j < J_{j+1}$ for $j = 1, \ldots, n-1$. We have:*

$$d_{\mathrm{H}}\big(C(\mathcal{A}), C(\mathcal{B})\big) = d_{\mathrm{B}}(\mathcal{A}, \mathcal{B}),$$

where $C(\mathcal{A}) := \mathbb{R} \setminus \bigcup \mathcal{A}$.

We prove this theorem at the end of the section. By Theorem A14 and [76, Theorem 3.1], we obtain:

Corollary A15 *Let $\ell := \max(m, n)$. Computing $d_{\mathrm{H}}\big(C(\mathcal{A}), C(\mathcal{B})\big)$ requires at most time $O(\ell^{1.5}\log \ell)$.*

We call a formigram θ over X **simple** if the following condition holds: for any $x, x' \in X$ and for any interval $I \subset \mathbb{R}$, if x, x' belong to the same block of $\bigvee_I \theta$, then there exists $t \in I$ such that x, x' belong to the same block in $\theta(t)$. Dendrograms on X are examples of simple formigrams.

Theorem A16 *Assume that $\theta, \theta' \in$ **Formi**(X) are simple. Then, computing $d_{\mathrm{F}}(\theta, \theta')$ requires at most time $O(n^2\ell^{1.5}\log \ell)$, where $n := |X|$ and*

$$\ell := \max\left(|\mathbf{crit}(\theta_X)|, |\mathbf{crit}(\theta')|\right).$$

Proof Fix $x, x' \in X$ and let

$$\theta_{\{x,x'\}} := \left\{t \in \mathbb{R} : x \text{ and } x' \text{ belong to the same block in } \theta(t)\right\}.$$

We claim that $d_{\mathrm{H}}^{\mathbf{Int}}\left(\theta^{-1}(\{x, x'\}^{\uparrow}), \theta'^{-1}(\{x, x'\}^{\uparrow})\right)$ in Eq. 14 is equal to $d_{\mathrm{H}}^{\mathbb{R}}\left(\theta_{\{x,x'\}}, \theta'_{\{x,x'\}}\right)$. To show this, it suffices to show that for any $\eta > 0$ one distance is upper bounded by η implies the other is too. Indeed, the assumption that θ, θ' are simple implies that, for any $\eta > 0$, both distances are upper bounded by η if and only if the following holds; if x and x' belong to $\theta(t)$ (resp. $\theta'(t)$), then there exists $s \in [t-\varepsilon, t+\varepsilon]$ such that x and x' belong to $\theta'(s)$ (resp. $\theta(s)$).

By Corollary A15, computing $d_{\mathrm{H}}^{\mathbb{R}}\left(\theta_{\{x,x'\}}, \theta'_{\{x,x'\}}\right)$ requires at most time $O(\ell^{1.5}\log \ell)$. Since there are n^2 singleton or time doubleton subsets of X, by the equality in Eq. 14, we can compute $d_{\mathrm{F}}(\theta, \theta')$ in $O(n^2 \cdot \ell^{1.5}\log \ell)$ time. □

Proof of Theorem A14. We prove Theorem A14. To avoid trivialities, assume that $\mathcal{A} \neq \mathcal{B}$. Let $E_{\mathcal{A}}$ be the collection of I_i's endpoints, i.e. $\mathcal{A} = \{a_i\}_{i=1}^m \cup \{b_i\}_{i=1}^m$. Letting $b_0 = -\infty$ and $a_{m+1} = \infty$, we have $C(\mathcal{A}) = \mathbb{R} \setminus \bigcup \mathcal{A} = \bigsqcup_{i=0}^m \langle b_i, a_{i+1}\rangle$. Similarly, we define $E_{\mathcal{B}}$ and $C(\mathcal{B})$.

Lemma A17

$$d_{\mathrm{H}}(C(\mathcal{A}), C(\mathcal{B})) = \max\left\{ \max_{a\in C(\mathcal{A})\cap(\bigcup\mathcal{B})} \min_{b\in E_{\mathcal{B}}} |a-b|\,, \max_{b\in C(\mathcal{B})\cap(\bigcup\mathcal{A})} \min_{a\in E_{\mathcal{A}}} |a-b| \right\}.$$

Proof Since

$$d_{\mathrm{H}}(C(\mathcal{A}), C(\mathcal{B})) = \max\left\{ \max_{a\in C(\mathcal{A})} \min_{b\in C(\mathcal{B})} |a-b|\,, \max_{b\in C(\mathcal{B})} \min_{a\in C(\mathcal{A})} |a-b| \right\},$$

by symmetry, it suffices to prove that

$$\max_{a\in C(\mathcal{A})} \min_{b\in C(\mathcal{B})} |a-b| = \max_{a\in C(\mathcal{A})\cap(\bigcup\mathcal{B})} \min_{b\in E_{\mathcal{B}}} |a-b|\,. \tag{18}$$

For $a \in C(\mathcal{A})$, let $\varphi(a) := \min_{b\in C(\mathcal{B})} |a-b|$. If $a \in C(\mathcal{B})$, then clearly $\varphi(a) = 0$. Hence, restricting the domain $C(\mathcal{A})$ of φ to the intersection of $C(\mathcal{A})$ and $\mathbb{R}\setminus C(\mathcal{B}) = \bigcup\mathcal{B}$ does not affect the maximum of φ, which implies:

$$\max_{a\in C(\mathcal{A})} \min_{b\in C(\mathcal{B})} |a-b| = \max_{a\in C(\mathcal{A})\cap(\bigcup\mathcal{B})} \min_{b\in C(\mathcal{B})} |a-b|\,. \text{[13]} \tag{19}$$

Next, fix an arbitrary $a \in \bigcup\mathcal{B}$. Then the closest point in $\mathbb{R}\setminus\bigcup\mathcal{B} = C(\mathcal{B})$ to a is obviously located on the boundary of $C(\mathcal{B})$, the set of endpoints $E_{\mathcal{B}}$, which implies $\min_{b\in C(\mathcal{B})} |a-b| = \min_{b\in E_{\mathcal{B}}} |a-b|$. Therefore, the RHS of Eq. 18 coincides with the RHS of Eq. 19. □

Proof of $d_{\mathrm{H}}\big(C(\mathcal{A}), C(\mathcal{B})\big) \le d_{\mathrm{B}}(\mathcal{A}, \mathcal{B})$

Let $\sigma : \mathcal{A} \nrightarrow \mathcal{B}$ be a δ-matching. By Lemma A17 and symmetry, it suffices to prove that

$$\max_{a\in C(\mathcal{A})\cap(\bigcup\mathcal{B})} \min_{b\in E_{\mathcal{B}}} |a-b| \le \delta.$$

Fix any $a \in C(\mathcal{A}) \cap (\bigcup\mathcal{B})$. Then there are $0 \le i \le m$ and $1 \le j \le n$ such that

$$a \in \langle b_i, a_{i+1}\rangle \cap \langle c_j, d_j\rangle. \tag{20}$$

Case 1. Assume that length$\langle c_j, d_j\rangle \le 2\delta$. Since $a \in \langle c_j, d_j\rangle$, we have:

$$\min_{b\in E_{\mathcal{B}}} |a-b| \le \min\{\left|a-c_j\right|, \left|a-d_j\right|\} \le \delta.$$

Case 2. Assume that length$\langle c_j, d_j\rangle > 2\delta$. Then there exists $1 \le k \le m$ such that $\langle a_k, b_k\rangle$ is matched with $\langle c_j, d_j\rangle$ via the matching σ. Note that the intersection in Eq. 20 can possibly be expressed as follows:

$$\langle b_i, a_{i+1}\rangle \cap \langle c_j, d_j\rangle = \begin{cases} \langle c_j, d_j\rangle & \text{Case (a)} \\ \langle b_i, a_{i+1}\rangle & \text{Case (b)} \\ \langle b_i, d_j\rangle & \text{Case (c)} \\ \langle c_j, a_{i+1}\rangle & \text{Case (d)} \end{cases}$$

Assume Case (a), i.e. $J_j = \langle c_j, d_j\rangle \subset \langle b_i, a_{i+1}\rangle$ (See Fig. 9). Given any intervals $\langle a, b\rangle$ and $\langle c, d\rangle$, let $||\langle a, b\rangle - \langle c, d\rangle||_\infty := \max\{|a-c|, |b-d|\}$. Note that the closest intervals to $\langle c_j, d_j\rangle$ in $\mathcal{A}$ in the metric $||\cdot||_\infty$ are $I_i = \langle a_i, b_i\rangle$ and $I_{i+1} = \langle a_{i+1}, b_{i+1}\rangle$. However, both $||I_i - J_j||_\infty$ and $||I_{i+1} - J_j||_\infty$ are greater than δ because $2\delta \le |d_j - c_j| \le \left|b_i - d_j\right| \le$

Fig. 9 An illustration for Case 2, (a)

$||I_i - J_j||_\infty$ and $2\delta \leq |d_j - c_j| = \left|a_{i+1} - c_j\right| \leq ||I_{i+1} - J_j||_\infty$. This contradicts the fact that σ is a δ-matching. Therefore, Case (a) cannot happen.

Assume Case (b). Also, assume that $J_j = \langle c_j, d_j \rangle$ is matched with I_k for $k \leq i$ via σ. Then since $b_k \leq b_i \leq a \leq d_j$,

$$\left|a - d_j\right| \leq \left|b_k - d_j\right| \leq ||I_k - J_j||_\infty \leq \delta.$$

Now, suppose that $\langle c_j, d_j \rangle$ is matched with I_k for $k > i$. Then since $c_j \leq a \leq a_{i+1} \leq a_k$,

$$\left|a - c_j\right| \leq \left|a_k - c_j\right| \leq ||I_k - J_j||_\infty \leq \delta.$$

Therefore, we have

$$\min_{b \in E_{\mathcal{B}}} |a - b| \leq \min\{\left|a - c_j\right|, \left|a - d_j\right|\} \leq \delta$$

as desired.

Assume Case (c), i.e., $c_j \leq b_i \leq d_j \leq a_{i+1}$. Note that I_k cannot be matched with J_j for $k > i$ via σ because $c_j < d_j \leq a_{i+1}$ and in turn

$$\delta < 2\delta < d_j - c_j \leq a_{i+1} - c_j \leq a_k - c_j \leq ||I_k - J_j||_\infty.$$

Hence, J_j must be matched with I_k for some $k \leq i$. Take $k \leq i$ such that I_k is matched with J_j via σ. Since $b_k \leq b_i \leq a < d_j$, we have

$$\left|a - d_j\right| \leq \left|d_j - b_k\right| \leq ||J_j - I_k||_\infty \leq \delta.$$

Therefore, we have

$$\min_{b \in E_{\mathcal{B}}} |a - b| \leq \left|a - d_j\right| \leq \delta.$$

Assume Case (d). By a similar argument to Case (c), J_j must be matched with I_k for some $k > i$ and this in turn implies $\left|a - c_j\right| \leq \delta$. Hence again

$$\min_{b \in E_{\mathcal{B}}} |a - b| \leq \left|a - c_j\right| \leq \delta.$$

We have shown that $\min_{b \in E_{\mathcal{B}}} |a - b| \leq \delta$ for all $a \in C(\mathcal{A}) \cap (\bigcup \mathcal{B})$ as desired. □

Proof of $d_{\mathrm{H}}\big(C(\mathcal{A}), C(\mathcal{B})\big) \geq d_{\mathrm{B}}(\mathcal{A}, \mathcal{B})$

Let $\eta > 0$. Define $\mathcal{A}^\eta$ to be the collection of intervals in $\mathcal{A}$ whose length is at least η. Also, given any interval $I = \langle a, b \rangle$, let

$$I^{-\eta} := \begin{cases} \emptyset & \text{if } b - a \leq 2\eta \\ \langle a + \eta, b - \eta \rangle & \text{otherwise.} \end{cases}$$

Let $\big(C(\mathcal{A})\big)^\eta$ be the η-thickening of $C(\mathcal{A})$, i.e. $\{r \in \mathbb{R} : \exists p \in C(\mathcal{A}),\ |p - r| \leq \eta\}$. We have:

[13] Since we assumed $\mathcal{A} \neq \mathcal{B}$, the set $C(\mathcal{A}) \cap (\bigcup \mathcal{B})$ cannot be empty.

Lemma A18 $\mathbb{R} \setminus \big(C(\mathcal{A})\big)^{\eta} = \bigcup_{I \in \mathcal{A}^{2\eta}} I^{-\eta}$ *(the proof is elementary but rather tedious so we omit it).*

Suppose that $d_{\mathrm{H}}\big(C(\mathcal{A}), C(\mathcal{B})\big) \leq \eta$ for some $\eta > 0$. We wish to construct an η-matching $\sigma : \mathcal{A} \nrightarrow \mathcal{B}$. Note that $C(\mathcal{B}) \subset \big(C(\mathcal{A})\big)^{\eta}$ by assumption and thus $\bigcup_{j=1}^{n} J_j = \mathbb{R} \setminus C(\mathcal{B}) \supset \mathbb{R}\setminus\big(C(\mathcal{A})\big)^{\eta} = \bigcup_{I_i \in \mathcal{A}^{2\eta}} I_i^{-\eta}$. This implies that there exists j such that $I_i^{-\eta} \subset J_j$, equivalently $I_i \subset J_j^{\eta}$, for each $I_i \in \mathcal{A}^{2\eta}$ since the union $\bigcup_{j=1}^{n} J_j$ is disjoint. We already have shown the following proposition.

Proposition A19 *Assume that $\eta \geq d_{\mathrm{H}}\big(C(\mathcal{A}), C(\mathcal{B})\big)$ for some $\eta > 0$. Then, there exist functions $f : A^{2\eta} \to B$ and $g : B^{2\eta} \to A$ such that*

$$I_i \subseteq (J_{f(i)})^{\eta} \text{ for all } i \in A^{2\eta} \text{ and } J_j \subseteq (I_{g(j)})^{\eta} \text{ for all } j \in B^{2\eta}$$

where $A^{2\eta} = \big\{1 \leq i \leq m : I_i \in \mathcal{A}^{2\eta}\big\}$ and $B^{2\eta} = \big\{1 \leq j \leq n : J_j \in \mathcal{B}^{2\eta}\big\}$.

Let f, g be as in the proposition. We construct an η-matching between $\mathcal{A}$ and $\mathcal{B}$. We write $A^{2\eta} = A_0^{2\eta} \sqcup A_*^{2\eta}$ where $A_0^{2\eta} := \big\{i \in A^{2\eta} : f(i) \notin B^{2\eta}\big\}$ and $A_*^{2\eta} := \big\{i \in A^{2\eta} : f(i) \in B^{2\eta}\big\}$. Similarly, we write $B^{2\eta} = B_0^{2\eta} \sqcup B_*^{2\eta}$ using the function g.

Proposition A20 $g \circ f|_{A_*^{2\eta}} = \mathrm{id}_{A_*^{2\eta}}$ *and* $f \circ g|_{B_*^{2\eta}} = \mathrm{id}_{B_*^{2\eta}}$.

Proof We only prove the first equality. Take any $i \in A_*^{2\eta}$. We know that

$$I_i \subseteq \big(J_{f(i)}\big)^{\eta} \subseteq \left(\big(I_{g(f(i))}\big)^{\eta}\right)^{\eta} = \big(I_{g(f(i))}\big)^{2\eta}.$$

Let $j = g(f(i))$. The above equation means that $\langle a_i, b_i\rangle \subseteq \langle a_j - 2\eta, b_j + 2\eta\rangle$. However, since length$\langle a_i, b_i\rangle \geq 2\eta$, this is impossible unless either $\langle a_i, b_i\rangle$ and $\langle a_j, b_j\rangle$ share one of their endpoints or have nonempty intersection. Since the intervals in $\mathcal{A}$ are disjoint and do not share their endpoints, we have $i = j$. $\square$

Notice two important implications of the above claim: The first is that $f(A_*^{2\eta}) \subseteq B_*^{2\eta}$ and $g(B_*^{2\eta}) \subseteq A_*^{2\eta}$. The second is that both $f|_{A_*^{2\eta}}$ and $g|_{B_*^{2\eta}}$ are injective. Now we are going to show that f and g are injective on $A_0^{2\eta}$ and $B_0^{2\eta}$ respectively as well.

Claim A21 The functions $f|_{A_0^{2\eta}}$ and $g|_{B_0^{2\eta}}$ are injective.

Proof We prove the claim for f. Assume that $i, j \in A_0^{2\eta}$, and $f(i) = f(j) = k$, which means $(J_k)^{\eta} \supseteq I_i$ and $(J_k)^{\eta} \supseteq I_j$ and hence $(J_k)^{\eta} \supseteq I_i \cup I_j$. Therefore,

$$\begin{aligned} 4\eta &\geq 2\eta + \mathrm{length} J_k && \because i, j \in A_0^{2\eta} \\ &= \mathrm{length}(J_k)^{\eta} \\ &\geq \mathrm{length} I_i \cup I_j \end{aligned}$$

This forces I_i and I_j to have nonempty intersection since each of them has the length$\geq 2\eta$. Thus $i = j$ because the intervals in $\mathcal{A}$ are disjoint. $\square$

We are now ready to define an η-matching $\sigma : \mathcal{A} \nrightarrow \mathcal{B}$. For the sake of simplicity, we will regard σ as a matching between index sets A and B of $\mathcal{A}$ and $\mathcal{B}$ respectively by identifying elements in $\mathcal{A}$ and $\mathcal{B}$ to their indexes. First, we define

$$\mathrm{coim}(\sigma) = A_0^{2\eta} \sqcup A_*^{2\eta} \sqcup g(B_0^{2\eta}), \quad \text{and} \quad \mathrm{im}(\sigma) = f(A_0^{2\eta}) \sqcup B_*^{2\eta} \sqcup B_0^{2\eta}.$$

Then, $\mathrm{coim}(\sigma) \supseteq A^{2\eta}$ and $\mathrm{im}(\sigma) \supseteq B^{2\eta}$. Now, define $\sigma : A \nrightarrow B$ as follows:

$$\sigma(i) := \begin{cases} f(i) & \text{if } i \in A^{2\eta} = A_*^{2\eta} \sqcup A_0^{2\eta} \\ g^{-1}(i) & \text{if } i \in g(B_0^{2\eta}). \end{cases}$$

By Claim A21, σ is well-defined . The following diagram depicts the construction of the matching:

$$\begin{tikzcd} A_0^{2\eta} \arrow[d, "f"'] & \sqcup & A_*^{2\eta} \arrow[d, shift right, "f"'] & \sqcup & g(B_0^{2\eta}) \\ f(A_0^{2\eta}) & \sqcup & B_*^{2\eta} \arrow[u, shift right, "g"'] & \sqcup & B_0^{2\eta} \arrow[u, "g"'] \end{tikzcd}$$

It remains to show that $||I_i - J_{\sigma(i)}||_\infty \leq \eta$, i.e., $I_i \subseteq (J_{\sigma(i)})^\eta$ and $J_{\sigma(i)} \subseteq (I_i)^\eta$ for all $i \in \mathrm{coim}(\sigma)$. Recall that $I_i = \langle a_i, b_i \rangle$ and $J_j = \langle c_j, d_j \rangle$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Case 1. Pick $i \in A_0^{2\eta}$ and let $\sigma(i) = f(i) = j$ so that $\mathrm{length} I_i \geq 2\eta$ whereas $\mathrm{length} J_j < 2\eta$. We wish to verify that $I_i \subseteq (J_j)^\eta$ and $J_j \subseteq (I_i)^\eta$. But, the first inclusion follows automatically from the definition of f and this implies that (1) $a_i \geq c_j - \eta$ and (2) $b_i \leq d_j + \eta$. So we are going to prove $J_j \subseteq (I_i)^\eta$ only, which amounts to show that (3) $c_j \geq a_i - \eta$ and (4) $d_j \leq b_i + \eta$. Suppose that (3) is false, i.e., $c_j < a_i - \eta$. Then we have

$$\begin{aligned} d_j &= c_j + \mathrm{length} J_j \\ &< c_j + 2\eta \\ &< a_i + \eta && \because c_j < a_i - \eta \\ &\leq b_i - \eta && \because a_i = b_i - \mathrm{length} I_i \leq b_i - 2\eta. \end{aligned}$$

This contradicts (2) and thus (3) must hold. Similarly, the negation of (4) deduce the contradiction to the inequality (1) and thus both (3) and (4) should hold as desired. This strategy works for the case of $i \in g(B_0^{2\eta})$ as well since g has the same property as f.

Case 2. Pick $i \in A_*^{2\eta}$ and let $\sigma(i) = f(i) = j$. Again by the definition of f, we know $I_i \subset (J_j)^\eta$. Further, $J_j \subseteq (I_{g(j)})^\eta$ by the definition of g but recalling $g(j) = g(f(i)) = i$ by Claim A20, we have $||I_i - J_{\sigma(i)}||_\infty \leq \eta$.

Assuming $\eta \geq d_{\mathrm{H}}\big(C(\mathcal{A}), C(\mathcal{B})\big)$ for some $\eta > 0$, we have constructed η-matching between $\mathcal{A}$ and $\mathcal{B}$. Therefore, we have inequality $d_{\mathrm{H}}\big(C(\mathcal{A}), C(\mathcal{B})\big) \geq d_{\mathrm{B}}(\mathcal{A}, \mathcal{B})$ as desired.

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Declarations

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References

1. Carlsson, G.: Topology and data. Bull. Am. Math. Soc. **46**(2), 255–308 (2009)
2. Edelsbrunner, H., Harer, J.: Persistent homology-a survey. Contemp. Math. **453**, 257–282 (2008)
3. Ghrist, R.: Barcodes: the persistent topology of data. Bull. Am. Math. Soc. **45**(1), 61–75 (2008)
4. Bubenik, P., Scott, J.A.: Categorification of persistent homology. Discrete Comput. Geom. **51**(3), 600–627 (2014)
5. Chazal, F., Cohen-Steiner, D., Glisse, M., Guibas, L.J., Oudot, S.Y.: Proximity of persistence modules and their diagrams. In: Proceedings of the Twenty-fifth Annual Symposium on Computational Geometry, pp. 237–246 (2009). ACM
6. Botnan, M., Lesnick, M.: Algebraic stability of zigzag persistence modules. Algebr. Geom. Topol. **18**(6), 3133–3204 (2018)
7. Botnan, M., Curry, J., Munch, E.: A relative theory of interleavings. arXiv preprint arXiv:2004.14286 (2020)
8. Bubenik, P., De Silva, V., Scott, J.: Metrics for generalized persistence modules. Found. Comput. Math. **15**(6), 1501–1531 (2015)
9. Curry, J.: Sheaves, Cosheaves and Applications. PhD thesis, University of Pennsylvania (2013)
10. De Silva, V., Munch, E., Patel, A.: Categorified Reeb graphs. Discrete Comp. Geom. **55**(4), 854–906 (2016)
11. de Silva, V., Munch, E., Stefanou, A.: Theory of interleavings on categories with a flow. Theory Appl. Categ. **33**(21), 583–607 (2018)
12. Lesnick, M.: The theory of the interleaving distance on multidimensional persistence modules. Found. Comput. Math. **15**(3), 613–650 (2015)
13. Scoccola, L.N.: Locally persistent categories and metric properties of interleaving distances. PhD thesis, The University of Western Ontario (2020)
14. Bjerkevik, H.B., Botnan, M.B., Kerber, M.: Computing the interleaving distance is NP-hard. Found. Comput. Math. 1–35 (2019)
15. Bjerkevik, H.B., Kerber, M.: Asymptotic improvements on the exact matching distance for 2-parameter persistence. arXiv preprint arXiv:2111.10303 (2021)
16. Cerri, A., Fabio, B.D., Ferri, M., Frosini, P., Landi, C.: Betti numbers in multidimensional persistent homology are stable functions. Math. Methods Appl. Sci. **36**(12), 1543–1557 (2013)
17. Kerber, M., Lesnick, M., Oudot, S.: Exact Computation of the Matching Distance on 2-Parameter Persistence Modules. In: 35th International Symposium on Computational Geometry (SoCG 2019). Leibniz International Proceedings in Informatics (LIPIcs), vol. 129, pp. 46–14615. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany (2019). https://doi.org/10.4230/LIPIcs.SoCG.2019.46. http://drops.dagstuhl.de/opus/volltexte/2019/10450
18. Landi, C.: The rank invariant stability via interleavings. In: Research in Computational Topology, pp. 1–10. Springer, Switzerland (2018)
19. Patel, A.: Generalized persistence diagrams. J. Appl. Comput. Topol. **1**(3–4), 397–419 (2018)
20. Kozlov, D.: Combinatorial Algebraic Topology vol. 21. Springer, (2008)
21. Mémoli, F.: A distance between filtered spaces via tripods. arXiv preprint arXiv:1704.03965 (2017)
22. Mémoli, F., Okutan, O.B.: Quantitative simplification of filtered simplicial complexes. Discrete Comput. Geom. **65**, 554–583 (2021)
23. Blumberg, A.J., Lesnick, M.: Universality of the homotopy interleaving distance, Transaction of the American Mathematical Society, https://doi.org/10.1090/tran/8738
24. Bauer, U., Landi, C., Mémoli, F.: The Reeb graph edit distance is universal. Foundations of Computational Mathematics (2020)
25. Kim, W., Mémoli, F., Smith, Z.: Analysis of dynamic graphs and dynamic metric spaces via zigzag persistence. In: Topological Data Analysis, pp. 371–389. Springer, ??? (2020)
26. Carlsson, G., Mémoli, F.: Characterization, stability and convergence of hierarchical clustering methods. J. Mach. Learn. Res. **1**1(Apr), 1425–1470 (2010)

27. Cai, C., Kim, W., Mémoli, F., Wang, Y.: Elder-rule staircodes for augmented metric spaces. In: Proceedings of the Thirty-sixth International Symposium on Computational Geometry (SoCG 2020) (2020)
28. Carlsson, G.,Mémoli, F.: Multiparameter hierarchical clustering methods. In: Classification as a Tool for Research, pp. 63–70. Springer, ??? (2010)
29. Rolle, A., Scoccola, L.: Stable and consistent density-based clustering. arXiv preprint arXiv:2005.09048 (2020)
30. Smith, Z., Chowdhury, S., Mémoli, F.: Hierarchical representations of network data with optimal distortion bounds. In: 2016 50th Asilomar Conference on Signals, Systems and Computers, pp. 1834–1838 (2016). IEEE
31. Buchin, K., Buchin, M., van Kreveld, M., Speckmann, B., Staals, F.: Trajectory grouping structure. In: Workshop on Algorithms and Data Structures, pp. 219–230 (2013). Springer
32. Kim, W., Mémoli, F.: Formigrams: Clustering summaries of dynamic data. In: Proceedings of the Thirtieth Canadian Conference on Computational Geometry, pp. 180–188 (2018)
33. Billera, L.J., Holmes, S.P., Vogtmann, K.: Geometry of the space of phylogenetic trees. Adv. Appl. Math. **27**(4), 733–767 (2001)
34. Sokal, R.R., Rohlf, F.J.: The comparison of dendrograms by objective methods. Taxon, 33–40 (1962)
35. Woese, C.R., Kandler, O., Wheelis, M.L.: Towards a natural system of organisms: proposal for the domains archaea, bacteria, and eucarya. Proc. Nat. Acad. Sci. **87**(12), 4576–4579 (1990)
36. Griffiths, R.C., Marjoram, P.: An ancestral recombination graph. In: Donnelly, P. and Tavaré, S. (Eds.), Progress in Population Genetics and Human Evolution, IMA Volumes in Mathematics and Its Applications vol. 87, pp. 257–270. Springer, Berlin (1997)
37. Huson, D.H., Rupp, R., Scornavacca, C.: Phylogenetic Networks: Concepts. Algorithms and Applications. Cambridge University Press, Cambridge, UK (2010)
38. Martin, S.H., Dasmahapatra, K.K., Nadeau, N.J., Salazar, C., Walters, J.R., Simpson, F., Blaxter, M., Manica, A., Mallet, J., Jiggins, C.D.: Genome-wide evidence for speciation with gene flow in heliconius butterflies. Genome Res. **23**(11), 1817–1828 (2013)
39. Parida, L., Utro, F., Yorukoglu, D., Carrieri, A.P., Kuhn, D., Basu, S.: Topological signatures for population admixture. In: International Conference on Research in Computational Molecular Biology, pp. 261–275 (2015). Springer
40. Gasparovic, E., Munch, E., Oudot, S., Turner, K., Wang, B., Wang, Y.: Intrinsic interleaving distance for merge trees. arXiv preprint arXiv:1908.00063 (2019)
41. Morozov, D., Beketayev, K., Weber, G.: Interleaving distance between merge trees. In: Proceedings of Topology-Based Methods in Visualization (2013)
42. Bauer, U., Munch, E., Wang, Y.: Strong Equivalence of the Interleaving and Functional Distortion Metrics for Reeb Graphs. In: Arge, L., Pach, J. (eds.) 31st International Symposium on Computational Geometry (SoCG 2015). Leibniz International Proceedings in Informatics (LIPIcs), vol. 34, pp. 461–475. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany (2015). https://doi.org/10.4230/LIPIcs.SOCG.2015.461
43. Chambers, E.W., Munch, E., Ophelders, T.: A Family of Metrics from the Truncated Smoothing of Reeb Graphs. In: 37th International Symposium on Computational Geometry (SoCG 2021). Leibniz International Proceedings in Informatics (LIPIcs), vol. 189, pp. 22–12217. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany (2021). https://doi.org/10.4230/LIPIcs.SoCG.2021.22. https://drops.dagstuhl.de/opus/volltexte/2021/13821
44. Stefanou, A.: Tree decomposition of Reeb graphs, parametrized complexity, and applications to phylogenetics. J. Appl. Comput. Topol. (2020). https://doi.org/10.1007/s41468-020-00051-1
45. Kim, W., Mémoli, F.: Extracting persistent clusters in dynamic data via Möbius inversion. arXiv preprint arXiv:1712.04064v5 (2022), to appear in Discrete Comput. Geom.
46. Weibel, C.A.: The K-book: An Introduction to Algebraic K-theory, vol. 145. American Mathematical Society, Providence, RI (2013)
47. Kim, W., Mémoli, F.: Generalized persistence diagrams for persistence modules over posets. J. Appl. Comput. Topol. **5**(4), 533–581 (2021)
48. Puuska, V.: Erosion distance for generalized persistence modules. Homol. Homotopy Appl. **22**(1), 233–254 (2020)
49. Kim, W., Mémoli, F.: Spatiotemporal persistent homology for dynamic metric spaces. Discrete Comput. Geom. **66**(3), 831–875 (2021)
50. Clause, N., Kim, W.: Spatiotemporal Persistent Homology Computation Tool. https://github.com/ndag/PHoDMSs (2020)
51. Bubenik, P., De Silva, V., Scott, J.: Interleaving and Gromov-Hausdorff distance. arXiv preprint arXiv:1707.06288 (2017)

52. Betthauser, L., Bubenik, P., Edwards, P.B.: Graded persistence diagrams and persistence landscapes. Discrete Comput. Geom. **67**(1), 203–230 (2022)
53. Bubenik, P., et al.: Statistical topological data analysis using persistence landscapes. J. Mach. Learn. Res. **16**(1), 77–102 (2015)
54. Chazal, F., Cohen-Steiner, D., Guibas, L.J., Mémoli, F., Oudot, S.Y.: Gromov-Hausdorff stable signatures for shapes using persistence. Comput. Graph. Forum **2**8(5), 1393–1403 (2009). Wiley Online Library
55. Chazal, F., De Silva, V., Oudot, S.: Persistence stability for geometric complexes. Geom. Dedicata **173**(1), 193–214 (2014)
56. Erné, M., Šešelja, B., Tepavčević, A.: Posets generated by irreducible elements. Order **20**(1), 79–89 (2003)
57. Roman, S.: Lattices and Ordered Sets. Springer, New York (2008)
58. Erné, M.: Compact generation in partially ordered sets. J. Aust. Math. Soc. **42**(1), 69–83 (1987)
59. Curry, J., Patel, A.: Classification of constructible cosheaves. Theory Appl. Categ. **35**(27), 1012–1047 (2020)
60. Cardona, G., Mir, A., Rosselló, F., Rotger, L., Sánchez, D.: Cophenetic metrics for phylogenetic trees, after Sokal and Rohlf. BMC Bioinform. **14**(1), 3 (2013)
61. Munch, E., Stefanou, A.: The ℓ^∞-cophenetic metric for phylogenetic trees as an interleaving distance. In: Research in Data Science, pp. 109–127. Springer, Switzerland (2019)
62. Bauer, U., Lesnick, M.: Induced matchings and the algebraic stability of persistence barcodes. J. Comput. Geom. **6**(2), 162–191 (2015)
63. Serra, J.: Hausdorff distances and interpolations. Comput. Imaging Vis. **12**, 107–114 (1998)
64. Chowdhury, S.: Geodesics in persistence diagram space. arXiv preprint arXiv:1905.10820 (2019)
65. Chazal, F., De Silva, V., Glisse, M., Oudot, S.: The Structure and Stability of Persistence Modules, vol. 10. Springer, Switzerland (2016)
66. Burago, D., Burago, I.D., Burago, Y., Ivanov, S.A., Ivanov, S.: A Course in Metric Geometry, vol. 33. American Mathematical Soc, Providence, Rhode Island (2001)
67. Chowdhury, S., Mémoli, F.: Explicit geodesics in Gromov-Hausdorff space. Electron. Res. Announc. **25**, 48–59 (2018)
68. Munkres, J.R.: Elements of Algebraic Topology. CRC Press, Boca Raton, FL, USA (2018)
69. Barmak, J.A.: On quillen's theorem a for posets. J. Comb. Theory Ser. A **118**(8), 2445–2453 (2011)
70. Bakke Bjerkevik, H.: On the stability of interval decomposable persistence modules. Discrete Comput. Geom. **66**(1), 92–121 (2021)
71. Schmiedl, F.: Computational aspects of the gromov-hausdorff distance and its application in non-rigid shape matching. Discrete Comput. Geom. **57**(4), 854–880 (2017)
72. Erickson, J.: Algorithms. Independent Publish, Urbana-Champaign, IL (2019)
73. McCleary, A., Patel, A.: Edit distance and persistence diagrams over lattices. SIAM J. Appl. Algebra Geom. **6**(2), 134–155 (2022)
74. Mac Lane, S.: Categories for the Working Mathematician, vol. 5. Springer, New York (2013)
75. Lawler, E.L.: Combinatorial Optimization: Networks and Matroids. Courier Corporation, Mineola, NY (2001)
76. Kerber, M., Morozov, D., Nigmetov, A.: Geometry helps to compare persistence diagrams. J. Exp. Algorithmics **22**, 1–20 (2017)