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CORRELATION DECAY AND PARTITION FUNCTION ZEROS: ALGORITHMS AND PHASE TRANSITIONS*

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Abstract. We explore connections between the phenomenon of correlation decay (more precisely, strong spatial mixing) and the location of Lee–Yang and Fisher zeros for various spin systems. In particular we show that, in many instances, proofs showing that weak spatial mixing on the Bethe lattice (infinite Δ-regular tree) implies that strong spatial mixing on all graphs of maximum degree Δ can be lifted to the complex plane, establishing the absence of zeros of the associated partition function in a complex neighborhood of the region in parameter space corresponding to strong spatial mixing. This allows us to give unified proofs of several recent results of this kind, including the resolution by Peters and Regts of the Sokal conjecture for the partition function of the hard-core lattice gas. It also allows us to prove new results on the location of Lee–Yang zeros of the antiferromagnetic Ising model. We show further that our methods extend to the case when weak spatial mixing on the Bethe lattice is not known to be equivalent to strong spatial mixing on all graphs. In particular, we show that results on strong spatial mixing in the antiferromagnetic Potts model can be lifted to the complex plane to give new zero-freeness results for the associated partition function, significantly sharpening previous results of Sokal and others. This new extension is also of independent algorithmic interest: it allows us to give the first polynomial time deterministic approximation algorithm (a fully polynomial time approximation scheme (FPTAS)) for counting the number of q-colorings of a graph of maximum degree Δ provided only that $q \geq 2\Delta$, a question that has been studied intensively. This matches the natural bound for randomized algorithms obtained by a straightforward application of Markov chain Monte Carlo. In the case when the graph is also triangle-free, we show that our algorithm applies under the weaker condition $q \geq \alpha\Delta + \beta$, where $\alpha \approx 1.764$ and $\beta = \beta(\alpha)$ are absolute constants.

Key words. approximate counting, graph coloring, Potts model, partition function, zeros of polynomials, derandomization

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1. Introduction.

1.1. Background and related work.

A standard approach to formalizing phase transitions in spin systems proceeds by characterizing when long-range correlations between spins appear in the system. More formally, one starts with an infinite graph, such as the Bethe lattice (infinite Δ-regular tree) or $\mathbb{Z}^d$, and then characterizes the regions in the space of parameters of the model in which, under the associated Gibbs distribution that assigns probabilities to configurations, correlations

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between spins decay exponentially with the distance between them. This correlation decay property is also known as "spatial mixing." This formalism can be extended to infinite families of finite graphs and has also been studied extensively due to its connections with the computational complexity of Markov chain Monte Carlo methods for sampling from the associated Gibbs distributions.

A related but different approach to studying phase transitions is via the so-called Yang–Lee theory [62]. Here, one views the infinite graph as a suitable limit of a sequence of finite graphs of growing size and studies the convergence of the free energy density (the logarithm of the partition function divided by the number of vertices) over this sequence. Yang and Lee showed that, under mild conditions, this limit exists and is analytic on a given subset S of the parameter space of the system, provided that none of the partition functions of the graphs appearing in the sequence have any roots in a *complex* neighborhood of S, uniformly over the graphs in the sequence. This classical approach has also recently found new algorithmic applications inspired by the work of Barvinok [2].

Classical work of Dobrushin and Shlosman [11, 12] establishes an equivalence between a strong version of spatial mixing (called "strong spatial mixing"[1]) and the Yang–Lee formalism of phase transitions in the special case of lattices $\mathbb{Z}^d$. However, their approach makes essential use of the amenability of the lattice (in the form that the size of a neighborhood of radius r grows only as a polynomial in r), and does not extend to non-amenable graph families. Until recently, few formal connections between them were known for the setting of general graphs. Sokal [57] conjectured that, for the hard-core lattice gas model, there is a complex neighborhood $\mathcal{N}$ of the interval $(0, \lambda_c(\Delta))$, where $\lambda_c(\Delta)$ is the critical activity of the model on the infinite regular tree (Bethe lattice) of degree Δ, such that the partition function of *any* finite graph of degree at most Δ does not vanish in $\mathcal{N}$. This conjecture was only recently resolved by Peters and Regts [47]. More recently, this correspondence between the two notions of phase transition for general bounded-degree graphs has been extended to the Ising model [40, 48, 51].

In this paper, we further explore this correspondence with a view to establishing it in more generality. Our first results show that previous arguments establishing an equivalence between weak spatial mixing on the infinite Δ-regular tree and strong spatial mixing of the Gibbs measure on all graphs of maximum degree Δ can be "lifted" to the complex plane in such a way as to also prove that the partition functions of all such graphs remain zero-free in a uniform complex neighborhood of the real parameter interval in which strong spatial mixing holds. This gives new and simpler proofs of Peters and Regts' resolution of the Sokal conjecture [47] described above, and of the results of the present authors on the Fisher zeros of the zero-field ferromagnetic Ising model [40]. In addition, our method allows us to prove new results on the Lee–Yang zeros of the *antiferromagnetic* Ising model, which we now describe.

Formally, the partition function of the Ising model on a graph $G = (V, E)$ can be written in terms of an edge activity (nearest-neighbor interaction) $\beta > 0$ and a vertex

[1] "(Weak) spatial mixing" simply refers to the decay of correlations property; "strong spatial mixing" is said to hold when correlations between spins decay with distance even in the presence of fixed spins (boundary conditions) close to the spins being measured. Strong spatial mixing is a crucial ingredient in the design of efficient algorithms, including Markov chain Monte Carlo and algorithms based on the self-avoiding walk tree.

activity (external field) $\lambda > 0$ as follows:

$$Z_G(\beta,\lambda) := \sum_{\sigma:V\to\{+,-\}} \beta^{d(\sigma)}\lambda^{p(\sigma)}, \tag{1}$$

where σ ranges over assignments of spins $\{+,-\}$ to vertices, $d(\sigma)$ is the number of edges $\{u,v\}$ for which $\sigma(u) \neq \sigma(v)$, and $p(\sigma)$ is the number of vertices for which $\sigma(v) = +$. As usual, the partition function implicitly defines the Gibbs distribution on configurations σ according to their weights in (1). When $\beta < 1$ the model is *ferromagnetic* and assigns larger weight to configurations with more aligned neighboring spins; conversely, when $\beta > 1$ the model is *antiferromagnetic*. For the infinite Δ-regular tree, it is known that weak spatial mixing holds when $\beta \in (1, \frac{\Delta}{\Delta-2})$ for all $\lambda > 0$, while for $\beta > \frac{\Delta}{\Delta-2}$ there exists a $\lambda_c(\beta,\Delta) > 0$ such that weak spatial mixing holds if $|\log\lambda| > \log\lambda_c(\beta,\Delta)$ [24, p. 255]; see also the remarks following Theorems 1 and 3 of [53]. We show that, in a complex neighborhood of this weak spatial mixing region, the model has no zeros in the parameter[2] λ.

THEOREM 1.1. *Fix $\Delta \geq 3$ and let $\beta > 1$ and $\lambda > 0$ be such that weak spatial mixing for the antiferromagnetic Ising model with edge activity β and vertex activity λ holds on the infinite Δ-regular tree. Then there exists $r_{\beta,\lambda,\Delta} > 0$ such that $Z_G(\beta,\lambda') \neq 0$ for any $\lambda' \in \mathbb{C}$ satisfying $|\lambda - \lambda'| \leq r_{\beta,\lambda,\Delta}$.*

Remark 1. Note that the width $r_{\beta,\lambda,\Delta}$ of the region depends on β and λ, and indeed tends to zero as the parameters approach their critical values. For any fixed *compact* subset of values of (β,λ) within the regime of weak spatial mixing, we get a fixed width r for the region.

Our second main result goes beyond the setting where a translation from weak spatial mixing on the infinite tree to strong spatial mixing on general graphs is known, and considers the antiferromagnetic Potts model. Even in this more general setting, we show that currently known arguments for proving strong spatial mixing for the model can be "lifted" to the complex plane to prove new zero-freeness results for the Potts model partition function. We now formally describe these results.

The partition function of the antiferromagnetic Potts model (at zero field) of a graph $G = (V,E)$ with a fixed number q of spins (which we often refer to as "colors") can be written as

$$Z_G(q;w) := \sum_{\sigma:V\to[q]} w^{m(\sigma)}. \tag{2}$$

Here σ ranges over arbitrary assignments of spins (colors) to vertices, and $m(\sigma)$ is the number of *monochromatic* edges, i.e., edges $\{u,v\}$ for which $\sigma(u) = \sigma(v)$. Note that the number of *proper* q-colorings of G (i.e., those with no monochromatic edges) is just $Z_G(q;0)$. The partition function again defines the Gibbs distribution on colorings σ of G, according to their weights in (2). We will often drop q from the notation, since it will be clear from the context, and write $Z_G(q;w)$ simply as $Z_G(w)$.

THEOREM 1.2. *Fix a positive integer Δ. Then there exists a $\tau_\Delta > 0$ such that the following is true. Let $\mathcal{D}_\Delta$ be a simply connected region in the complex plane obtained as the union of disks of radius τ_Δ centered at all points on the segment* $[0,1]$. *For any*

[2]Since $Z_G(\beta,\lambda)$ is a bivariate polynomial, one can investigate phase transitions in terms of either β or λ. Complex zeros of Z_G as a function of λ are known as *Lee–Yang zeros*, while zeros of Z_G as a function of β are known as *Fisher zeros*.

graph G of maximum degree at most $\Delta \geq 3$ and integer $q \geq 2\Delta$, we have $Z_G(q; w) \neq 0$ when $w \in \mathcal{D}_\Delta$.

Remark 2. The condition $q \geq 2\Delta$, as discussed in more detail below, corresponds to the Dobrushin uniqueness condition and the "path coupling" method for analyzing Gibbs samplers. The previous best result in this direction is due to Bencs et al. [6] and requires $q \geq e\Delta + 1$, which is much stronger than the Dobrushin bound.

As discussed later in section 1.3, our technique is also capable of directly harnessing tighter strong spatial mixing arguments used in the analysis of Markov chains for special classes of graphs. As an example, we can exploit such an argument of Gamarnik, Katz, and Misra [22] to improve the bound on q in Theorem 1.2 when the graph is triangle-free, for all but small values of Δ.

THEOREM 1.3. *Let $\alpha^\star \approx 1.7633$ be the unique positive solution of the equation $xe^{-1/x} = 1$. For every $\alpha > \alpha^\star$, there exists a $\beta = \beta(\alpha)$ such that for any integer $\Delta \geq 3$, there exists a $\tau_\Delta > 0$ for which the following is true. Let $\mathcal{D}_\Delta$ be a simply connected region in the complex plane obtained as the union of disks of radius τ_Δ centered at all points on the segment $[0, 1]$. For any* triangle-free *graph G of maximum degree at most Δ and integer $q \geq \alpha\Delta + \beta$, we have $Z_G(q; w) \neq 0$ when $w \in \mathcal{D}_\Delta$.*

Finally, for the special case of trees, the argument leading to the above theorems also leads to the following improved bound.

PROPOSITION 1.4. *Fix an integer $\Delta \geq 3$, and let $q \geq \Delta + 1$. Then, there exists a $\tau_\Delta > 0$ for which the following is true. Let $\mathcal{D}_\Delta$ be a simply connected region in the complex plane obtained as the union of disks of radius τ_Δ centered at all points on the segment $[0, 1]$. Then, for every tree T of maximum degree Δ, we have $Z_T(q; w) \neq 0$ when $w \in \mathcal{D}_\Delta$.*

We note that one can directly analyze the roots of the partition function of the Potts model on a tree, and they all lie at the points 0 and $1 - q$. Nevertheless, we include the above observation since the bound $q \geq \Delta + 1$ matches the optimal number of colors in the classical result of Jonasson [35] showing *weak* spatial mixing, and while proving *strong* spatial mixing for $q \geq \Delta + 1$ remains an important open problem, we find it interesting that one can obtain the above bound for the related property of zero-freeness using an argument that is based on spatial mixing. Further, the argument leading to Proposition 1.4 appears to be robust, and we expect that it may extend to more general settings (e.g., that of *list colorings* discussed later in this section).

For ease of later reference, we record the above three results in the following.

THEOREM 1.5. *Fix an integer $\Delta \geq 3$. Then there exists a $\tau_\Delta > 0$ such that for the simply connected region $\mathcal{D}_\Delta$ in the complex plane obtained as the union of disks of radius τ_Δ centered at all points on the segment $[0, 1]$, the following is true. For any graph G of maximum degree Δ and integer q satisfying the hypotheses of Theorem 1.2, Theorem 1.3, or Proposition 1.4, $Z_G(q; w) \neq 0$ when $w \in \mathcal{D}_\Delta$.*

Remark 3. Our proof of the above theorem actually holds under an abstract condition on coloring instances that we call *admissibility* (see Definition 4.3). We show in section 4 that the instances covered in Theorems 1.2 and 1.3 and Proposition 1.4 are all admissible. Proving admissibility for any larger class of instances would immediately extend Theorem 1.5 to such a class.

There has been extensive previous work on the complex zeros of the antiferromagnetic Potts model, which we now briefly summarize. Sokal [56, 58] proved (in the language of the Tutte polynomial) that the partition function has no zeros in the entire unit disk centered at $w = 0$, under the strong condition $q \geq 7.964\Delta$; the constant was later improved to 6.907 by Fernández and Procacci [16] (see also [32]). The results in these papers, since they are in terms of the Tutte polynomial, in fact extend to *complex* values of q—a setting which is not accessible in the Potts model formulation we use—and hence are not directly comparable to our result. However, for the most natural case of positive integer q, our result significantly improves upon them. Much more recently, the work of Bencs et al. [6] referred to above gave a zero-free region analogous to that in Theorem 1.2 above, but under the stronger condition $q \geq e\Delta + 1$. We note also that Barvinok and Soberón [4] (see also [2] for an improved version) established a zero-free region in a disk centered at $w = 1$ of radius significantly less than 1.

1.2. Algorithmic implications for the problem of counting colorings. The above theorems also allow us to make progress on a long-standing open problem in theoretical computer science: that of approximately counting proper colorings of a bounded degree graph using a deterministic algorithm. The problem of counting colorings is a benchmark problem in the theory of approximate counting, due both to its importance in combinatorics and statistical physics and to the fact that it has repeatedly challenged existing algorithmic techniques and stimulated the development of new ones. Below, we briefly summarize its history and current status.

Given a finite graph $G = (V, E)$ of maximum degree Δ, and a positive integer q, the goal is to count the number of (proper) vertex colorings of G with q colors. It is well known [7] that a greedy coloring exists if $q \geq \Delta + 1$. While counting colorings exactly is #P-complete, a long-standing conjecture asserts that approximately counting colorings is possible in polynomial time provided $q \geq \Delta + 1$. It is known that when $q < \Delta$, even approximate counting is NP-hard [20].

This question has led to numerous algorithmic developments over the past 25 years. The first approach was via Markov chain Monte Carlo (MCMC), based on the fact that approximate counting can be reduced to sampling a coloring (almost) uniformly at random. Sampling can be achieved by simulating a natural local Markov chain (or Glauber dynamics) that randomly flips colors on vertices: provided the chain is rapidly mixing, this leads to an efficient algorithm (a *fully polynomial randomized approximation scheme*, or *FPRAS*).

Jerrum's 1995 result [34] that the Glauber dynamics is rapidly mixing for $q \geq 2\Delta$ gave the first non-trivial randomized approximation algorithm for colorings and led to a plethora of follow-up work on MCMC (see, e.g., [13, 14, 17, 26, 28, 29, 30, 45, 59] and [18] for a survey), focusing on reducing the constant 2 in front of Δ. The best constant known for general graphs remains essentially $\frac{11}{6}$, obtained by Vigoda [59] using a more sophisticated Markov chain, though this was recently reduced to $\frac{11}{6} - \varepsilon$ for a very small ε by Chen et al. [9]. The constant can be substantially improved if additional restrictions are placed on the graph; e.g., Dyer et al. [14] achieve roughly $q \geq 1.49\Delta$ provided the girth is at least 6 and the degree is a large enough constant, while Hayes and Vigoda improve this to $q \geq (1 + \varepsilon)\Delta$ for girth at least 11 and degree $\Delta = \Omega(\log n)$, where n is the number of vertices.

A significant recent development in approximate counting is the emergence of *deterministic* approximation algorithms that in some cases match, or even improve upon, the best known MCMC algorithms.[3] Interestingly, these algorithms have made use of one of two main techniques, both of which are inspired by the two different notions of phase transitions in statistical physics described above. The first, based on *decay of correlations*, exploits the decreasing influence of the spins (colors) on distant vertices on the spin at a given vertex; while the second, based on *polynomial interpolation*, uses the absence of zeros of the partition function in a suitable region of the complex plane to perform a form of algorithmic analytic continuation. Early examples of the decay of correlations approach include [1, 5, 61], while for early examples of the polynomial interpolation method we refer the reader to the monograph of Barvinok [2] (see also [3, 15, 27, 31, 33, 41, 46] for more recent examples).

Unfortunately, however, in the case of colorings on general bounded degree graphs, these techniques have so far lagged well behind the MCMC algorithms mentioned above. One obstacle to getting correlation decay to work is the lack of a higher-dimensional analogue of Weitz's beautiful algorithmic framework [61], which allows correlation decay to be fully exploited via strong spatial mixing in the case of spin systems with just two spins (as opposed to the $q \geq 3$ colors present in coloring). For polynomial interpolation, the obstacle has been a lack of precise information about the location of the zeros of associated partition functions.

So far, the best algorithmic condition for colorings obtained via correlation decay is $q \geq 2.58\Delta+1$, due to Lu and Yin [43], and this remains the best available condition for any deterministic algorithm. This improved on an earlier bound of roughly $q \geq 2.78\Delta$ (proved only for triangle-free graphs), due to Gamarnik and Katz [21]. For the special case $\Delta = 3$, Lu et al. [42] give a correlation decay algorithm for counting 4-colorings. Furthermore, Gamarnik, Katz, and Misra [22] establish the related property of "strong spatial mixing" under the weaker condition $q \geq \alpha\Delta + \beta$ for any constant $\alpha > \alpha^\star$, where $\alpha^\star \approx 1.7633$ is the unique solution to $xe^{-1/x} = 1$ and β is a constant depending on α, and under the assumption that G is triangle-free (see also [23, 26] for similar results on restricted classes of graphs). However, as discussed in [22], this strong spatial mixing result unfortunately does not lead to a deterministic algorithm.[4]

The newer technique of polynomial interpolation, pioneered by Barvinok [2], has also recently been brought to bear on counting colorings. In a recent paper, Bencs et al. [6] use this technique to derive a fully polynomial time approximation scheme (FPTAS) for counting colorings provided $q \geq e\Delta + 1$. Although this result is weaker than those obtained via correlation deay, it is of independent interest because it uses a different algorithmic approach, and because it establishes a new zero-free region for the associated partition function in the complex plane (see below).

In this paper, we push the polynomial interpolation method further and obtain an FPTAS for counting colorings under the condition $q \geq 2\Delta$.

THEOREM 1.6. *Fix positive integers q and Δ such that $q \geq 2\Delta$. Then there exists an FPTAS for counting q-colorings in any graph of maximum degree Δ.*

This is the first deterministic algorithm (of *any* kind) that for all Δ matches the "natural" bound for MCMC, first obtained by Jerrum [34]. Indeed, $q \geq 2\Delta$ remains

[3]In this case, the notion of an FPRAS is replaced by that of a *fully polynomial time approximation scheme*, or *FPTAS*. An FPTAS for q-colorings of graphs of maximum degree at most Δ is an algorithm that, given as input the graph G and an error parameter $\varepsilon \in (0,1)$, produces a $(1 \pm \varepsilon)$-factor multiplicative approximation of the number of q-colorings of G in time poly $(|G|, 1/\varepsilon)$. (The degree of the polynomial is allowed to depend upon the constants q and Δ.)

[4]The strong spatial mixing condition does imply fast mixing of the Glauber dynamics, and hence an FPRAS, but only when the graph family being considered is "amenable," i.e., if the size of the ℓ-neighborhood of any vertex does not grow exponentially with ℓ. This restriction is satisfied by regular lattices, but fails, e.g., for random regular graphs.

the best bound known for rapid mixing of the basic Glauber dynamics that does not require either additional assumptions on the graph or a spectral comparison with another Markov chain: all the improvements mentioned above require either lower bounds on the girth and/or maximum degree, or (in the case of Vigoda's result [59]) analysis of a more sophisticated Markov chain. This is for good reason, since the bound $q \geq 2\Delta$ coincides with the closely related Dobrushin uniqueness condition from statistical physics [49], which in turn is closely related [60] to the path coupling method of Bubley and Dyer [8] that provides the simplest currently known proof of the $q \geq 2\Delta$ bound for the Glauber dynamics.

We therefore view our result also as a promising starting point for deterministic coloring algorithms to finally compete with their randomized counterparts. As pointed out above, our technique is capable of harnessing strong spatial mixing arguments used in the analysis of Markov chains for certain classes of graphs in order to relax the requirements on q. In particular, for the same reason as in Theorem 1.3 above, we can exploit such an argument of Gamarnik, Katz, and Misra [22] to improve the bound on q in Theorem 1.6 when the graph is triangle-free, for all but small values of Δ. (Recall that $\alpha^\star \approx 1.7633$ is the unique positive solution of the equation $xe^{-1/x} = 1$.)

THEOREM 1.7. *For every $\alpha > \alpha^\star$, there exists a $\beta = \beta(\alpha)$ such that the following is true. For all integers q and Δ such that $q \geq \alpha\Delta + \beta$, there exists an FPTAS for counting q-colorings in any triangle-free graph of maximum degree Δ.*

We mention also that our technique applies without further effort to the more general setting of *list* colorings, where each vertex has a list of allowed colors of size q, under the same conditions as above on q. Indeed, our proofs are written to handle this more general situation.

We now describe in more detail the connection between our results on the zeros of the Potts model and the above algorithmic results.

1.2.1. Our approach. Recall that the polynomial $Z_G(w)$ in eq. (2), being the partition function of the Potts model, implicitly defines a probability distribution on colorings σ according to their weights in eq. (2). The parameter w measures the strength of nearest-neighbor interactions. The value $w = 1$ corresponds to the trivial setting where there is no constraint on the colors of neighboring vertices, while $w = 0$ imposes the hard constraint that no neighboring vertices receive the same color. For intermediate values $w \in (0, 1)$, neighbors with the same color are penalized by a factor of w. We establish Theorems 1.6 and 1.7 as special cases of the following more general theorem.

THEOREM 1.8. *Suppose that the hypotheses of either Theorem* 1.6 *or Theorem* 1.7 *are satisfied, and fix $w \in [0, 1]$. Then there exists an FPTAS for the partition function $Z_G(w)$.*

Note that Theorems 1.6 and 1.7 follow immediately as the special case $w = 0$ of Theorem 1.8; however, the extension to other values of w is of independent interest as the computation of partition functions is a very active area of study in statistical physics and combinatorics.

Theorem 1.8 is obtained from our main result, Theorem 1.5, by appealing to a recent algorithmic paradigm of Barvinok [2]. The paradigm (see Lemma 2.2.3 of [2]) states that, for a partition function Z of degree m, if one can identify a simply connected, zero-free region $\mathcal{D}$ for Z in the complex plane that contains a τ-neighborhood of the interval $[0, 1]$, and a point in that interval where the evaluation of Z is easy (in our setting this is the point $w = 1$), then using the first $O\left(e^{\Theta(1/\tau)} \log\left(m/\varepsilon\right)\right)$ coefficients of Z one can obtain a $1 \pm \varepsilon$ multiplicative approximation of $Z(x)$ at any point $x \in \mathcal{D}$. Barvinok's framework is based on exploiting the fact that the zero-freeness of Z in $\mathcal{D}$ is equivalent to $\log Z$ being analytic in $\mathcal{D}$, and then using a carefully chosen transformation to deform $\mathcal{D}$ into a disk (with the easy point at the center) in order to obtain a convergent Taylor expansion. The coefficients of Z are used to compute the coefficients of this Taylor expansion.

Barvinok's framework in general leads to a *quasi-polynomial time* algorithm, because the computation of the $O\left(e^{\Theta(1/\tau)}\log(m/\varepsilon)\right)$ terms of the expansion may take time $O\big((m/\varepsilon)^{e^{\Theta(1/\tau)}\log m}\big)$ (which is only quasi-polynomial in m) for the partition functions considered here. However, additional insights provided by Patel and Regts [46] (see, e.g., the proof of Theorem 6.2 in [46]) show how to reduce this computation time to $O\big((m/\varepsilon)^{e^{\Theta(1/\tau)}\log \Delta}\big)$ for many models on graphs of degree at most Δ, including the Potts model with a bounded number of colors q at each vertex. Hence we obtain an FPTAS. This (by now standard) reduction is the same path as that followed by Bencs et al. [6, Corollary 1.2]; for completeness, we sketch some of the details in section 3.3. We note that, for each fixed Δ and q, the running time of our final algorithm is polynomial in n (the size of G) and ε^{-1}, as required for an FPTAS. However, as is typical of deterministic algorithms for approximate counting, the exponent in the polynomial depends on Δ (through the quantity τ_Δ in Theorem 1.5, which in the case where all lists are subsets of $[q]$ is inverse polynomial in q).

We conclude this introduction by sketching our approach to proving the zero-freeness results, which constitute the main technical contribution of the paper.

1.3. Technical overview. We start with an outline of the proofs of our results for two-spin systems, including Theorem 1.1 and our simplified proofs of previous zero-freeness results. A standard observation in the area is that proving $Z_G(\beta, \lambda) \neq 0$ is equivalent to showing that the *occupation ratio* $R_{v,G}(\beta, \lambda)$ at a fixed vertex v, defined as the ratio of the sum of those terms in the partition function where the vertex v has spin $+$ to the sum of those terms in the partition function where the vertex v has spin $-$, is not equal to -1. In order to analyze this quantity, another standard step is to use an observation of Weitz [61],[5] which allows one to transfer the question from general graphs to trees. More precisely, for any fixed vertex v in the graph G, Weitz's theorem constructs a finite tree $T = T_{v,G}$ (with carefully chosen boundary conditions at the leaves), of maximum degree at most the maximum degree of G, such that if ρ is the root of T, then $R_{\rho,T}(\beta, \lambda) = R_{v,G}(\beta, \lambda)$ for all positive real β, λ. On a tree, one can easily write down a recurrence for the occupation ratio, and the problem then reduces to proving that, with initial conditions corresponding to the boundary conditions of T, the recurrence never reaches -1.

The convergence properties of such recurrences have been analyzed before, in the context of proving that *weak spatial mixing* (or uniqueness of the Gibbs measure) on the infinite Δ-regular tree implies *strong spatial mixing* on all graphs of maximum degree Δ, for the hard-core model [37, 52, 61] and the Ising model with and without field [53, 63]. These analyses, which are restricted to positive, real values of the parameters, often take the form of showing that the recurrence for an appropriate function $\phi(R)$ of the occupation ratio is a uniform contraction. Our main contribution is to show that the arguments in the above references are in fact robust enough that one can extend them to a *complex* neighborhood (independent of the size of the graph)

[5]The ideas behind Weitz's reduction first appeared in the work of Godsil [25], and also later in the work of Scott and Sokal [50].

of the real intervals on which they hold. Thus the behavior of the recurrence in this neighborhood remains close to what one sees for positive real parameters, and in particular the value of the occupation ratio remains away from -1, thus establishing zero-freeness.

The situation is more complicated for the case of the Potts model (where the number of spins is more than two), since neither the translation to trees, nor the tight recurrence analyses for tree recurrences, is known. The starting point for our proof of Theorem 1.5 is a simple geometric observation, versions of which have been used before for constructing inductive proofs of zero-freeness of partition functions (see, e.g., [2, 6]). Fix a vertex v in the graph G. Given $w \in \mathbb{C}$ and a color $k \in [q]$, let $Z_v^{(k)}(w)$ denote the *restricted partition function* in which one sums only over those colorings σ in which $\sigma(v) = k$. Then, since $Z_G(w) = \sum_{k\in[q]} Z_v^{(k)}(w)$, the zero-freeness of Z_G will follow if the angles between the complex numbers $Z_v^{(k)}(w)$, viewed as vectors in $\mathbb{R}^2$, are all small, and provided that at least one of the $Z_v^{(k)}$ is nonzero. (In fact, this condition on angles can be relaxed for those $Z_v^{(k)}(w)$ that are sufficiently small in magnitude, and this flexibility will be important for us when w is a complex number close to 0.) Therefore, one is naturally led to consider the so-called *marginal ratios*:

$$R_{G,v}^{(i,j)}(w) := \frac{Z_v^{(i)}(w)}{Z_v^{(j)}(w)}.$$

(In the q-coloring problem, this ratio is 1 by symmetry. However, in our recursive approach we have to handle the more general list-coloring problem, in which the ratio becomes non-trivial.) We then require that, for any two colors i, j for which $Z_v^{(k)}(w)$ is large enough in magnitude, the ratio $R_{G,v}^{(i,j)}(w)$ is a complex number with small argument. This is what we prove inductively in sections 5 and 6.

The broad contours of our approach as outlined so far are quite similar to some recent work [2, 6]. However, it is at the crucial step of how the marginal ratios are analyzed that we depart from these previous results. Instead of attacking the restricted partition functions or the marginal ratios directly for given $w \in \mathbb{C}$, as in these previous works, we crucially exploit the fact that for any real $\tilde{w} \in [0, 1]$ close to the given w, these quantities have natural probabilistic interpretations, and hence can be much better understood via probabilistic and combinatorial methods. For instance, when $\tilde{w} \in [0, 1]$, the marginal ratio $R_{G,v}^{(i,j)}(w)$ is in fact a ratio of the marginal probabilities $\Pr_{G,\tilde{w}}[\sigma(v) = i]$ and $\Pr_{G,\tilde{w}}[\sigma(v) = j]$, under the natural probability distribution on colorings σ. In fact, our analysis cleanly breaks into two separate parts:

1. First, understand the behavior of true marginal probabilities of the form $\Pr_{G,\tilde{w}}[\sigma(v) = i]$ for real $\tilde{w} \in [0, 1]$. This is carried out in section 4.
2. Second, argue that, for complex $w \approx \tilde{w}$, the ratios $R_{G,v}^{(i,j)}(w)$ remain well behaved. This is carried out separately for the two cases when w is close to 0 (in section 5) and when w is bounded away from 0 but still in the vicinity of $[0, 1]$ (in section 6).

A key technical point in our analysis is the notion of "niceness" of vertices, which stipulates that the marginal probability $\Pr_{G,\tilde{w}}[\sigma(v) = i] \le \frac{1}{\deg_G(v)+2}$, where $\deg_G(v)$ is the degree of v in G (see Definition 4.2). Note that this condition refers only to real nonnegative $\tilde{w}$, and hence is amenable to analysis via standard combinatorial tools. Indeed, our proofs that the conditions on q and Δ in Theorems 1.6 and 1.7 imply this niceness condition are similar to probabilistic arguments used by Gamarnik et al. [22] to establish strong spatial mixing (in the special case $\tilde{w} = 0$). We emphasize that this is the only place in our analysis where the lower bounds on q are used. One can therefore expect that combinatorial and probabilistic ideas used in the analysis of strong spatial mixing and the Glauber dynamics with a smaller number of colors in special classes of graphs can be combined with our analysis to obtain deterministic algorithms for those settings; indeed, our Theorem 1.7 demonstrates this point for triangle-free graphs, leveraging the strong spatial mixing argument of [22].

The above ideas are sufficient to understand the real-valued case (part 1 above). For the complex case in part 2, we start from a recurrence for the marginal ratios $R_{G,v}^{(i,j)}$ that is a generalization (to the case $w \neq 0$) of a similar recurrence used by Gamarnik, Katz, and Misra [22]; this recurrence is defined in Lemma 3.4. The inductive proofs in sections 5 and 6 use this recurrence to show that, if $\tilde{w} \in [0, 1]$ is close to $w \in \mathbb{C}$, then all the relevant $R_{G,v}^{(i,j)}(w)$ remain close to $R_{G,v}^{(i,j)}(\tilde{w})$ throughout. The actual induction, especially in the case when w is close to 0, requires a delicate choice of induction hypotheses (see Lemmas 5.2 and 6.3). The key technical idea is to use the "niceness" property of vertices established in part 1 to argue that the two recurrences (real and complex) remain close at every step of the induction. This in turn depends upon a careful application of the mean value theorem, *separately* to the real and imaginary parts (see Lemma 3.5), of a function f_κ that arises naturally in the analysis of the recurrence (see Lemma 3.6).

1.3.1. Comparison with correlation decay based algorithms. We conclude this overview with a brief discussion of how we are able to obtain a better bound on the number of colors than in correlation decay algorithms, such as those in [21, 43] cited earlier. In these algorithms, one first uses recurrences similar to the one mentioned above to *compute* the marginal probabilities, and then appeals to self-reducibility to compute the partition function. Of course, expanding the full tree of computations generated by the recurrence will in general give an exponential time (but exact) algorithm. The core of the analysis of these algorithms is to exploit the correlation decay property to show that, even if this tree of computations is only expanded to depth about $O(\log(n/\varepsilon))$, and the recurrence at that point is initialized with *arbitrary* values, the computation still converges to an ε-approximation of the true value. However, the requirement that the analysis be able to deal with arbitrary initializations implies that one cannot directly use properties of the actual probability distribution (e.g., the "niceness" property alluded to above); indeed, this issue is also pointed out by Gamarnik, Katz, and Misra [22]. In contrast, our analysis does not truncate the recurrence and thus only has to handle initializations that make sense in the context of the graph being considered. Moreover, the exponential size of the recursion tree is no longer a barrier for us since, in contrast to correlation decay algorithms, we are using the tree only as a tool to establish zero-freeness; the algorithm itself follows from Barvinok's polynomial interpolation paradigm. Our approach suggests that this paradigm can be viewed as a method for using (complex-valued generalizations of) strong spatial mixing results to obtain deterministic approximation algorithms.

2. Correlation decay implies absence of zeros. In this section, we present a sequence of results relating correlation decay and the absence of zeros for two-spin systems.[6] In addition to their intrinsic interest, these results will also serve as a "warm-up" to our results on the Potts model, which use similar methods in a more

[6]The results in this section were first reported in JL's Ph.D. thesis [38]. Subsequently, similar results, in a slightly more general context, have independently been obtained by Shao and Sun [51].

complex setting. We begin by re-proving the main result of [40] on the Fisher zeros of the Ising model (without external field). While the proof in [40] also implicitly used correlation decay, here we rewrite the argument as a special case of a more general method for "lifting" already known correlation decay results for various models to the complex plane. We go on to apply this generic method to prove new results on the Lee–Yang zeros of the antiferromagnetic Ising model (with field), and to give a new, simpler proof of the Sokal conjecture (first proved by Peters and Regts [47]) on the zeros of the hard core partition function. The ideas developed here will be extended to the Potts model in later sections of the paper.

2.1. Ising model. In this section we show that there are no *Fisher zeros* of the Ising model in a complex neighborhood around the correlation decay interval of the infinite Δ-regular tree (Bethe lattice). This gives a different proof of the main result of [40], making the role of the correlation decay arguments in the real domain more explicit.

Recall from eq. (1) that, given a graph G, an edge activity β, and a vertex activity λ, the Ising partition function is defined as $Z_G(\beta,\lambda) = \sum_\sigma \beta^{d(\sigma)}\lambda^{p(\sigma)}$, where $d(\sigma)$ is the number of edges between different spins, and $p(\sigma)$ is the number of vertices with spin $+$. Formally, we view this partition function as a polynomial in β for a fixed λ, and we study the complex zeros in β; these are known as *Fisher zeros*. In fact, in this section we fix $\lambda = 1$, and hence we will simply write $Z_G(\beta) := Z_G(\beta, 1)$ for the rest of the section. The correlation decay interval for the Ising model has been well studied: the Gibbs distribution of the Ising model on any graph of maximum degree Δ exhibits decay of correlations when β lies in the interval $(\frac{\Delta-2}{\Delta}, \frac{\Delta}{\Delta-2})$ [63], which corresponds exactly to the correlation decay interval for the Δ-regular tree [24]. The main result of this section will be Corollary 2.7, which says that there is a complex neighborhood of the correlation decay interval in which there are no Fisher zeros for the Ising model on any graph of maximum degree Δ. This provides a formal link between the "decay of correlations" and "analyticity of free energy density" views of phase transitions. Further, as discussed in more detail in [40], this zero-freeness result also implies the existence of efficient approximation algorithms for the partition function $Z_G(\beta)$ via Barvinok's paradigm discussed in section 3.3.

We recall some notation and definitions from [40]. Let G be any graph of maximum degree Δ. For any non-isolated vertex v of G, let $Z^+_{G,v}(\beta)$ (respectively, $Z^-_{G,v}(\beta)$) be the contribution to $Z_G(\beta)$ from configurations with $\sigma(v) = +$ (respectively, $\sigma(v) = -$), so that $Z_G(\beta) = Z^+_{G,v}(\beta) + Z^-_{G,v}(\beta)$. We also define the ratios $R_{G,v}(\beta) := \frac{Z^+_{G,v}(\beta)}{Z^-_{G,v}(\beta)}$. Note that $Z^+_{G,v}(\beta)$ and $Z^-_{G,v}(\beta)$ can be seen as Ising partition functions defined on the same graph G with the vertex v *pinned* to the appropriate spin. Without loss of generality, we assume that every pinned vertex has degree exactly one.[7] We will prove, inductively on the number of unpinned vertices, that neither $Z^+_{G,v}(\beta)$ nor $Z^-_{G,v}(\beta)$ vanishes. Under this induction hypothesis, the condition $Z_G(\beta) \neq 0$ is clearly equivalent to $R_{G,v}(\beta) \neq -1$. As we will see, for $\beta \in \mathbb{R}$, $R_{G,v}(\beta) > 0$. Thus it suffices to show that, for complex β sufficiently close to the correlation decay interval on the real line, $R_{G,v}(\beta) \approx R_{G,v}(\Re\beta)$.

As in [40], our development in this section is also based on the formal recurrences derived by Weitz [61] for computing ratios such as $R_{G,v}(\beta)$ in two-state spin

[7]Suppose that a vertex v of degree k is pinned in a graph G, and consider the graph G' obtained by replacing v with k copies of itself, each pinned to the same spin and connected to exactly one of the original neighbors of v. Then $Z_G(\beta) = Z_{G'}(\beta)$ for all β.

systems. However, instead of following [40], where Weitz's reduction to the so-called self-avoiding walk tree was used directly, we provide here a self-contained description in a form that is a simplification of the more complicated recurrences for the Potts model that we study in sections 3 and beyond.

We start with some notation and definitions. For a vertex u in a graph G, if u has s^+ neighbors pinned to spin $+$, and s^- neighbors pinned to spin $-$, then we say that u has $(s^- - s^+)$ *signed pinned neighbors*.

DEFINITION 2.1 (the graphs G_i). *Given a graph G and an unpinned vertex u in G, let $v_1, \ldots, v_k$ be the unpinned neighbors of u. We define G_i (the vertex u will be understood from the context) to be the graph obtained from G as follows:*

- *first, replace vertex u with $u_1, \ldots, u_k$, and connect u_1 to v_1, u_2 to v_2, and so on;*
- *next, pin vertices $u_1, \ldots, u_{i-1}$ to spin $+$, and vertices $u_{i+1}, \ldots, u_k$ to spin $-$;*
- *finally, remove vertex u_i.*

Note that the graph G_i has one fewer unpinned vertex than G. Moreover, the number of unpinned neighbors of v_i in G_i is at most $\Delta - 1$.

LEMMA 2.2. *Let ω be a formal variable. Given a graph G and an unpinned vertex u, let k be the number of unpinned neighbors of u, and s be the number of signed pinned neighbors of u. Defining $h_\omega(x) := \frac{\omega + x}{\omega x + 1}$, we have*

$$R_{G,u}(\omega) = \omega^s \prod_{i=1}^{k} h_\omega \left(R_{G_i, v_i}(\omega) \right).$$

Remark 4. (i) Note that the above formal equalities become valid numerical equalities when a numerical value $\beta \in \mathbb{C}$ is substituted for ω, provided that (a) $\beta x_i + 1 \neq 0$ for any $\boldsymbol{x}$ appearing in the computation, and (b) $Z^-_{G,u}(\beta) \neq 0$.

(ii) Moreover, since the number of unpinned neighbors of v_i in G_i is at most $\Delta - 1$, the tree recurrence will be applied with $k \leq \Delta - 1$ except possibly at the root, where k may be Δ.

Proof. Let $v_1, v_2, \ldots, v_k$ be the unpinned neighbors of u, and let $v_{k+1}, \ldots, v_{\deg_G(u)}$ be its pinned neighbors. For $0 \leq i \leq \deg_G(u)$, let H_i be the graph obtained from G as follows:

- replace vertex u with $u_1, \ldots, u_{\deg_G(u)}$, and connect u_1 to v_1, u_2 to v_2, and so on;
- pin vertices $u_1, \ldots, u_i$ to spin $+$, and vertices $u_{i+1}, \ldots, u_{\deg_G(u)}$ to spin $-$.

Note that H_i is the same as G_i, except that the last step of the construction of G_i is skipped, i.e., the vertex u_i is not removed, and, further, u_i is pinned to spin $+$. We can now write

$$R_{G,u}(\omega) = \frac{Z^+_{G,u}(\omega)}{Z^-_{G,u}(\omega)} = \frac{Z_{H_{\deg_G(u)}}(\omega)}{Z_{H_0}(\omega)} = \prod_{i=1}^{\deg_G(u)} \frac{Z_{H_i}(\omega)}{Z_{H_{i-1}}(\omega)} = \omega^s \cdot \prod_{i=1}^{k} \frac{Z_{H_i}(\omega)}{Z_{H_{i-1}}(\omega)}, \tag{3}$$

where k, s are the numbers of unpinned neighbors and signed pinned neighbors, respectively, of u. We observe that

$$Z_{H_i}(\omega) = Z^+_{G_i,v_i} + \omega \cdot Z^-_{G_i,v_i};$$
$$Z_{H_{i-1}}(\omega) = \omega \cdot Z^+_{G_i,v_i} + Z^-_{G_i,v_i}.$$

Substituting these expressions into eq. (3) gives

$$R_{G,u}(\omega) = \omega^s \cdot \prod_{i=1}^{k} \frac{Z^+_{G_i,v_i} + \omega \cdot Z^-_{G_i,v_i}}{\omega \cdot Z^+_{G_i,v_i} + Z^-_{G_i,v_i}} = \omega^s \cdot \prod_{i=1}^{k} \frac{\frac{Z^+_{G_i,v_i}}{Z^-_{G_i,v_i}} + \omega}{\omega \cdot \frac{Z^+_{G_i,v_i}}{Z^-_{G_i,v_i}} + 1} = \omega^s \prod_{i=1}^{k} h_\omega\left(R_{G_i,v_i}(\omega)\right).$$

This completes the proof. ∎

Lemma 2.2 leads to the following recurrence relation on the ratios:

$$F_{\beta,k,s}(\boldsymbol{x}) := \beta^s \prod_{i=1}^{k} h_\beta(x_i), \tag{4}$$

where as before $h_\beta(x) := \frac{\beta+x}{\beta x+1}$. As in several previous studies of this recurrence in the literature (see, e.g., [40, 44, 63]), it is useful to reparameterize it in terms of logarithms of likelihood ratios as follows. Let $\varphi(x) := \log x$, and define

$$F^\varphi_{\beta,k,s}(\boldsymbol{x}) := \left(\varphi \circ F_{\beta,k,s} \circ \varphi^{-1}\right)(\boldsymbol{x}) = s \log \beta + \sum_{i=1}^{k} \log h_\beta(e^{x_i}). \tag{5}$$

One may then derive the correlation decay property in the form of a convenient stepwise contraction [63]. The version here (and its proof, which we include for completeness) is taken from [40].

PROPOSITION 2.3. *Fix a degree $\Delta \geq 3$ and integers $k \geq 0$ and s. If $\frac{\Delta-2}{\Delta} < \beta < \frac{\Delta}{\Delta-2}$, then there exists $\eta > 0$ (depending upon β and Δ) such that $\|\nabla F^\varphi_{\beta,k,s}(\boldsymbol{x})\|_1 \leq \frac{k}{\Delta-1}(1-\eta)$ for every $\boldsymbol{x} \in \mathbb{R}^k$.*

Proof. A direct calculation of the derivative gives

$$\|\nabla F^\varphi_{\beta,k,s}(\boldsymbol{x})\|_1 = \sum_{i=1}^{k} \frac{\left|1-\beta^2\right|}{\beta^2 + 1 + \beta(e^{x_i} + e^{-x_i})}.$$

Since $e^x + e^{-x} \geq 2$ for every real x, the right-hand side is at most $k \times \frac{|1-\beta|}{1+\beta}$. The condition on β implies that $\frac{|1-\beta|}{1+\beta} \leq \frac{1-\eta}{\Delta-1}$ for some fixed $\eta > 0$. Therefore, we have $\|\nabla F^\varphi_{\beta,k,s}(\boldsymbol{x})\|_1 \leq k \times \frac{|1-\beta|}{1+\beta} \leq \frac{k}{\Delta-1}(1-\eta)$. ∎

We recall a few further computations from [40]. First, we bound $R_{G,u}(\beta)$ for real-valued β. From (4), for any integers $k \geq 0$ and s and a positive real β, we have $\beta^{k+|s|} \leq F_{\beta,k,s}(\boldsymbol{x}) \leq \frac{1}{\beta^{k+|s|}}$ when $\beta \leq 1$, and $\frac{1}{\beta^{k+|s|}} \leq F_{\beta,k,s}(\boldsymbol{x}) \leq \beta^{k+|s|}$ when $\beta \geq 1$, for all $\boldsymbol{x} \in \left(\mathbb{R}_+ \cup \{0,\infty\}\right)^k$. Noting that $k+|s| \leq \Delta$ and taking the logarithm of these bounds motivates the definition of the intervals $I_0(\beta,\Delta)$ as follows:

$$I_0 = I_0(\beta,\Delta) := \left[-\Delta\left|\log\beta\right|, \Delta\left|\log\beta\right|\right]. \tag{6}$$

Recalling Lemma 2.2, we see that the ratios $R_{G,u}(\beta)$ can be obtained by recursively applying the recurrence $F_{\beta,k,s}(\boldsymbol{x})$. Therefore, for $\beta \in \mathbb{R}_+$, any graph G, and unpinned vertex u, we have $\log R_{G,u}(\beta) \in I_0(\beta,\Delta)$. Our second point of departure from the strategy followed in [40] is the following corollary of Proposition 2.3 in the complex plane.

COROLLARY 2.4. *Fix a degree* $\Delta \geq 3$ *and integers* $k \geq 0$ *and* s. *If* $\frac{\Delta-2}{\Delta} < \beta < \frac{\Delta}{\Delta-2}$, *then there exist positive constants* $\eta, \varepsilon, \delta$ *(depending upon* β *and* Δ*) such that the following is true. Let* $D := D(\beta, \Delta)$ *be the closed and convex set of points within distance* ε *of* $I_0(\beta, \Delta)$ *in* $\mathbb{C}$. *Then* $\|\nabla F^{\varphi}_{\beta,k,s}(\boldsymbol{x})\|_1 \leq (1-\eta/2)$ *for all* k, s *satisfying* $k + |s| \leq \Delta - 1$ *and every* $\boldsymbol{x} \in D^k$. *Moreover, there is a finite constant* $M \geq 1$ *(depending upon* β *and* Δ*) such that*

$$\sup_{\boldsymbol{x}\in D^k,\ \beta'\in\mathbb{C}:|\beta'-\beta|<\delta} \left| F^{\varphi}_{\beta,k,s}(\boldsymbol{x}) - F^{\varphi}_{\beta',k,s}(\boldsymbol{x}) \right| \leq M\,|\beta - \beta'|\,; \tag{7}$$

$$\sup_{x,y:\varphi(x),\varphi(y)\in D} |\varphi(x) - \varphi(y)| \leq M\,|x-y|\,; \tag{8}$$

$$\sup_{x,y\in D} \left|\varphi^{-1}(x) - \varphi^{-1}(y)\right| \leq M\,|x-y|\,;\ \textit{and} \tag{9}$$

$$\sup_{\boldsymbol{x}\in D^k} \|\nabla F^{\varphi}_{\beta,k,s}(\boldsymbol{x})\|_1 \leq M \ \textit{when}\ k+|s| = \Delta. \tag{10}$$

Proof. Observe that $\|\nabla F^{\varphi}_{\beta,k,s}(\boldsymbol{x})\|_1 = \sum_{i=1}^{k} \frac{|1-\beta^2|}{\beta^2+1+\beta(e^{x_i}+e^{-x_i})}$ is a continuous function in x_i for every i. Since by Proposition 2.3 it is uniformly upper bounded by $\frac{k}{\Delta-1}(1-\eta)$ for all $\boldsymbol{x} \in I_0(\beta,\Delta)^k$, for small enough ε the expression can be bounded by $\frac{k}{\Delta-1}(1-\eta/2)$ for all $\boldsymbol{x} \in D^k$; this in turn is bounded by $(1-\eta/2)$ when $k+|s| \leq \Delta-1$.

Finally, the existence of M follows from the analyticity of $F^{\varphi}_{\beta,k,s}$ on D^k, of φ^{-1} on D, and of φ on $\varphi^{-1}(D)$, respectively. ∎

We will also need the following standard consequence of the mean value theorem for complex functions (also used, e.g., in [40]).

LEMMA 2.5. *Let* $K(\boldsymbol{x})$ *be a holomorphic function on a convex subset* D *of* $\mathbb{C}^k$. *For any* $\boldsymbol{x}, \boldsymbol{x'} \in D$, *we have*

$$|K(\boldsymbol{x}) - K(\boldsymbol{x'})| \leq \sup_{\boldsymbol{\xi}\in D^k} \|\nabla K(\boldsymbol{\xi})\|_1 \cdot \|\boldsymbol{x} - \boldsymbol{x'}\|_\infty.$$

Proof. Consider $g(t) := K(\boldsymbol{x} + t(\boldsymbol{x'} - \boldsymbol{x}))$ for $t \in [0,1]$. Since D is convex, $\boldsymbol{x} + t(\boldsymbol{x'} - \boldsymbol{x})$ lies in D for all $t \in [0,1]$. Now, observe that

$$g'(t) = \nabla K(\boldsymbol{x} + t(\boldsymbol{x'} - \boldsymbol{x}))^{\mathsf{T}} (\boldsymbol{x} - \boldsymbol{x'}).$$

Thus, for any $\boldsymbol{x}, \boldsymbol{x'} \in D$, we have

$$\begin{aligned} |K(\boldsymbol{x}) - K(\boldsymbol{x'})| = |g(1) - g(0)| = \left|\int_0^1 g'(t)dt\right| &\leq \sup_{t\in[0,1]} |g'(t)| \\ &\leq \sup_{\boldsymbol{\xi}\in D^k} \|\nabla K(\boldsymbol{\xi})\|_1 \cdot \|\boldsymbol{x} - \boldsymbol{x'}\|_\infty. \end{aligned}$$ ∎

Finally, we are ready to give a proof of the main result of this section (also the main result of [40]). The proof of the theorem below serves as a template for our arguments for establishing zero-free regions for the Potts model partition function in sections 5 and 6.

THEOREM 2.6. *Fix a degree* $\Delta \geq 3$, *and let* $\beta \in (\frac{\Delta-2}{\Delta}, \frac{\Delta}{\Delta-2})$. *There exist positive constants* δ_0, τ *(both depending on* β *and* Δ*) such that, for any graph* G *of maximum degree* Δ, *any vertex* u *in* G, *and any complex* β' *with* $|\beta' - \beta| < \delta_0$, *the following are true:*

1. $|Z^{+}_{G,u}(\beta')| > 0, |Z^{-}_{G,u}(\beta')| > 0$.
2. $|\varphi(R_{G,u}(\beta)) - \varphi(R_{G,u}(\beta'))| < \tau$ *if* u *has degree at most* $\Delta - 1$ *in* G.
3. $Z_G(\beta') \neq 0$.

We will refer to the two items above as the "induction hypothesis." We remark that the assumption $\beta \in (\frac{\Delta-2}{\Delta}, \frac{\Delta}{\Delta-2})$ is needed only so that we may appeal to correlation decay (in the form of Corollary 2.4).

Proof. We use induction on the number of unpinned vertices in G. Without loss of generality, we assume that the graph G is connected.

For the base case, if u is the only unpinned vertex in G, with s^+ neighbors pinned to spin $+$ and s^- neighbors pinned to spin $-$, then $Z^{+}_{G,u}(\beta') = (\beta')^{s^-}, Z^{-}_{G,u}(\beta') = (\beta')^{s^+}$. Item 1 of the induction hypothesis is thus satisfied for all small enough positive δ_0. For item 2, we note that $R_{G,u}(\beta') = (\beta')^{s^+ - s^-}$, and, since $0 \leq s^+, s^- \leq \Delta$, also that $\varphi(R_{G,u}(\beta)) \in I_0(\beta, \Delta)$. Now, let $\eta, \varepsilon, \delta, M$ be the constants and D the closed convex set (depending on β and Δ) whose existence is guaranteed by Corollary 2.4. For all small enough positive δ_0, we then also have (i) $\Re(R_{G,u}(\beta')) > 0$ (since $\beta > (\Delta-2)/\Delta$ and $|\beta' - \beta| \leq \delta_0$); and (ii) $\varphi(R_{G,u}(\beta')) \in D$. Combined with eq. (8) in the statement of the corollary, inequality (ii) implies item 2, provided τ is chosen to be small enough (in terms of ε and M). Item 3 follows from item 1 and inequality (i), since $\Re(R_{G,u}(\beta')) = \Re\big(Z^{+}_{G,u}(\beta')/Z^{-}_{G,u}(\beta')\big) > 0$.

We now proceed to the inductive step. In this case, G has at least two unpinned vertices. We begin by deriving a useful consequence of the induction hypothesis. Let u be an arbitrary unpinned vertex in G, with s pinned and k unpinned neighbors. Let the k unpinned neighbors be $v_1, \ldots, v_k$. We denote $B_i(\beta) := \varphi(R_{G_i, v_i}(\beta))$, $\boldsymbol{B}(\beta) := \{B_1(\beta), B_2(\beta), \ldots, B_k(\beta)\}$, and $H_\beta(x_1, x_2, \ldots, x_k) := F^{\varphi}_{\beta,k,s}(x_1, x_2, \ldots, x_k)$, where the graphs G_i are as in Definition 2.1. Note that the above quantities are all well defined: this is because, by construction, each G_i has one fewer unpinned vertex than G, and also the degree of v_i in G_i is at most $\Delta - 1$ (since u, which is an unpinned neighbor of v_i in G, is not present in G_i), so that items 1 and 2 of the induction hypothesis apply at vertex v_i in G_i, and also (by item 3) $Z_{G_i}(\beta') \neq 0$. These items also imply that $B_i(\beta), B_i(\beta') \in D$, and further that $|B_i(\beta) - B_i(\beta')| < \tau$ (provided that $\tau \leq \varepsilon$ and δ_0 is small enough). The triangle inequality then gives (again, assuming $\delta_0 \leq \delta$)

$$\begin{aligned}
|H_\beta(\boldsymbol{B}(\beta)) - H_{\beta'}(\boldsymbol{B}(\beta'))| &\leq |H_\beta(\boldsymbol{B}(\beta)) - H_\beta(\boldsymbol{B}(\beta'))| + |H_\beta(\boldsymbol{B}(\beta')) - H_{\beta'}(\boldsymbol{B}(\beta'))| \\
&\leq \sup \|\nabla F^{\varphi}_{\beta,k,s}\|_1 \cdot \max_i |B_i(\beta) - B_i(\beta')| + M\,|\beta - \beta'|, \\
&\leq \tau \sup \|\nabla F^{\varphi}_{\beta,k,s}\|_1 + M\delta_0.
\end{aligned}$$

Here, in the second line, the first term comes from Lemma 2.5 (where the supremum is over all $\boldsymbol{x} \in D^k$), and the second term comes from eq. (7) of Corollary 2.4. Now let δ_0 be chosen so that it is also smaller than $\tau \cdot \min\{1, \eta/2M\}$. We have two cases.

Case 1: $k + |s| = \Delta$. In this case, we use eq. (10) of Corollary 2.4 to get

$$|H_\beta(\boldsymbol{B}(\beta)) - H_{\beta'}(\boldsymbol{B}(\beta'))| \leq M(\tau + \delta_0) < 2M\tau. \tag{11}$$

Case 2: $k + |s| \leq \Delta - 1$. In this case, we use the case $k + |s| \leq \Delta - 1$ of Corollary 2.4 to get

$$|H_\beta(\boldsymbol{B}(\beta)) - H_{\beta'}(\boldsymbol{B}(\beta'))| \leq (1 - \eta/2)\tau + M\delta_0 < \tau. \tag{12}$$

Armed with these consequences of the induction hypothesis, we now proceed to establish the inductive step. Before proceeding, we note that, by Lemma 2.2, $H_\beta(\boldsymbol{B}(\beta)) = \varphi(R_{G,u}(\beta))$, and, when $Z^+_{G,u}, Z^-_{G,u} \neq 0$, $H_{\beta'}(\boldsymbol{B}(\beta')) = \varphi(R_{G,u}(\beta'))$.

For item 1, we consider the graph G' where we pin vertex u to spin $+$. Note that by definition, $Z^+_{G,u}(\beta') = Z_{G'}(\beta')$. Let v be any unpinned vertex in G'. Since G' has one fewer unpinned vertex than G, by the induction hypothesis we have $|Z^-_{G',v}(\beta')| > 0$. Thus, $R_{G',v}(\beta')$ is well defined and is in D. The calculations leading to eqs. (11) and (12) applied to the vertex v in G' imply that $|\varphi(R_{G',v}(\beta)) - \varphi(R_{G',v}(\beta'))| < 2M\tau$ (in fact, the upper bound improves to τ in case v has degree at most $\Delta - 1$ in G'). Applying eq. (9) of Corollary 2.4 then shows that $|R_{G',v}(\beta) - R_{G',v}(\beta')| < 2M^2\tau$. Thus, since $R_{G',v}(\beta) > \min\{\beta^\Delta, 1/\beta^\Delta\}$, we have, for all small enough τ and δ_0,

$$\Re(R_{G',v}(\beta')) > 0. \tag{13}$$

We can therefore write

$$|Z_{G'}(\beta')| = |Z^+_{G',v}(\beta') + Z^-_{G',v}(\beta')| = |Z^-_{G',v}(\beta')| \cdot |1 + R_{G',v}(\beta')|.$$

But then, by eq. (13), $|1 + R_{G',v}(\beta')| \geq \Re(1 + R_{G',v}(\beta')) > 1$. Thus,

$$|Z^+_{G,u}(\beta')| = |Z_{G'}(\beta')| \geq |Z^-_{G',v}(\beta')| > 0.$$

An identical argument also proves that $|Z^-_{G,u}(\beta')| > 0$, completing the verification of item 1 of the induction hypothesis.

Now, since $Z^-_{G,u}(\beta')$ and $Z^+_{G,u}(\beta')$ have both been proved to be nonzero, it follows that $R_{G,u}(\beta')$ is well defined, and, by Lemma 2.2, is equal to $H_{\beta'}(\boldsymbol{B}(\beta'))$. Item 2 of the induction hypothesis then follows immediately from eq. (12). Finally, item 3 follows from item 1 and eq. (13). ∎

The main result of this section, establishing the absence of Fisher zeros in a complex region around the correlation decay interval, now follows immediately from item 3 of the previous theorem.

COROLLARY 2.7. *Fix a degree $\Delta \geq 3$, and let $\beta \in (\frac{\Delta-2}{\Delta}, \frac{\Delta}{\Delta-2})$. There exists a positive constant δ_0 (depending on β and Δ) such that, for any graph G of maximum degree Δ, and any complex β' with $|\beta' - \beta| < \delta_0$, we have $Z_G(\beta') \neq 0$.*

2.2. Antiferromagnetic Ising model. In this section we consider the antiferromagnetic Ising model. Recall from the introduction that, for the infinite Δ-regular tree, weak spatial mixing holds when $\beta \in (1, \frac{\Delta}{\Delta-2})$ for all $\lambda > 0$, while for $\beta > \frac{\Delta}{\Delta-2}$ there exists a $\lambda_c(\beta, \Delta) > 0$ such that weak spatial mixing holds if $|\log\lambda| > \log\lambda_c(\beta, \Delta)$ [24, 53]. We will refer to this as the *correlation decay region* for the antiferromagnetic Ising model. Fix any β, λ in the correlation decay region. As claimed in Theorem 1.1 of the introduction, we will show that there exists $\delta > 0$ such that, for any λ' with $|\lambda' - \lambda| < \delta$, the partition function $Z_G(\beta, \lambda') \neq 0$. This is apparently the first result precisely relating correlation decay to absence of Lee–Yang zeros for the antiferromagnetic Ising model on general graphs.

As before, for a fixed vertex v, we write $Z_G(\beta, \lambda) = Z^+_{G,v}(\beta, \lambda) + Z^-_{G,v}(\beta, \lambda)$ and let $R_{G,v}(\beta, \lambda) := \frac{Z^+_G(\beta,\lambda)}{Z^-_G(\beta,\lambda)}$. Then we can write a formal recurrence relation analogous to that in Lemma 2.2, as follows.

LEMMA 2.8. *Let $\omega_\beta, \omega_\lambda$ be formal variables. Given a graph G and an unpinned vertex u, let k be the number of unpinned neighbors of u, and let s be the number of signed pinned neighbors of u. Denoting $h_\omega(x) := \frac{\omega+x}{\omega x+1}$, we have*

$$R_{G,u}(\omega_\beta, \omega_\lambda) = \omega_\lambda \omega_\beta^s \prod_{i=1}^{k} h_{\omega_\beta}\left(R_{G_i,v_i}(\omega_\beta, \omega_\lambda)\right),$$

where the graphs G_i are defined as in Definition 2.1.

Given integers k and s, let $F_{\beta,\lambda,k,s}(\boldsymbol{x}) := \lambda\beta^s \prod_{i=1}^{k} h_\beta(x_i)$. This recurrence has been studied before in the literature [36, 53], and as in the case of the ferromagnetic Ising model, it has been found useful to reparameterize $F_{\beta,\lambda,k,s}$ with a "potential function" φ as follows: $F^{\varphi}_{\beta,\lambda,k,s} := \varphi \circ F_{\beta,\lambda,k,s} \circ \varphi^{-1}$. In [53] the function $\varphi(x) := \log \frac{x+D}{1-x+D}$ was used, where $D > 0$ is a constant depending on β and Δ (but not on λ). (This choice of φ is by no means unique: alternative choices can be found in, e.g., [36, 37].) For this choice of φ, the following stepwise correlation decay in the 1-norm is proved in [53].

THEOREM 2.9 ([53]). *Fix a degree $\Delta \geq 3$ and integers $k > 0, s$ such that $k + |s| \leq \Delta - 1$. If (β, λ) is in the correlation decay region of the infinite Δ-regular tree, then there exists an $\eta > 0$ (depending upon β, λ, and Δ) such that $\|\nabla F^{\varphi}_{\beta,\lambda,k,s}(\boldsymbol{x})\|_1 < 1-\eta$ for every $\boldsymbol{x} \in \mathbb{R}^k$.*[8]

We also note that an analogue of the calculation leading to eq. (6) gives the bound $\frac{\lambda}{\beta^\Delta} \leq R_{G,u}(\beta, \lambda) \leq \lambda\beta^\Delta$. Thus we define the analogous interval

$$I_0(\beta, \lambda, \Delta) := \left[\varphi\left(\tfrac{\lambda}{\beta^\Delta}\right), \varphi\left(\lambda\beta^\Delta\right)\right]. \tag{14}$$

The following corollary is analogous to Corollary 2.4 and is an immediate consequence of Theorem 2.9 and the analyticity of $F^{\varphi}_{\beta,\lambda,k,s}$ and φ^{-1} at points close to $I_0(\beta, \lambda, \Delta)$.

COROLLARY 2.10. *Fix a degree $\Delta \geq 3$ and integers $k \geq 0$ and s. If (β, λ) is in the correlation decay region of the infinite Δ-regular tree, then there exist positive constants $\eta, \varepsilon, \delta$ (depending upon β, λ, and Δ) such that the following is true. Let $D := D(\beta, \lambda, \Delta)$ be the set of points within distance ε of $I_0(\beta, \lambda, \Delta)$ in $\mathbb{C}$. Then $\|\nabla F^{\varphi}_{\beta,\lambda,k,s}(\boldsymbol{x})\|_1 < 1 - \eta/2$ for every $\boldsymbol{x} \in D^k$ whenever $k + |s| \leq \Delta - 1$. Moreover, there is a finite constant $M \geq 1$ (depending upon β, λ, and Δ) such that*

$$\sup_{\boldsymbol{x}\in D^k,\ \lambda'\in\mathbb{C}:|\lambda'-\lambda|<\delta} \left|F^{\varphi}_{\beta,\lambda,k,s}(\boldsymbol{x}) - F^{\varphi}_{\beta,\lambda',k,s}(\boldsymbol{x})\right| \leq M\,|\lambda' - \lambda|;$$

$$\sup_{x,y:\varphi(x),\varphi(y)\in D} |\varphi(x) - \varphi(y)| \leq M\,|x-y|;$$

$$\sup_{x,y\in D} \left|\varphi^{-1}(x) - \varphi^{-1}(y)\right| \leq M\,|x-y|;\ \text{and}$$

$$\sup_{\boldsymbol{x}\in D^k} \|\nabla F^{\varphi}_{\beta,\lambda,k,s}(\boldsymbol{x})\|_1 \leq M \text{ when } k + |s| = \Delta.$$

Finally, given Lemma 2.8 and Corollary 2.10, an identical argument to that in the proof of Theorem 2.6 establishes the following.

[8]Reference [53] uses a different convention for the Ising model, in which β corresponds to our $1/\beta$ (see eq. (1) of [53]).

THEOREM 2.11. *Fix a degree $\Delta \geq 3$, and let (β, λ) be in the correlation decay region for the infinite Δ-regular tree. There exist positive constants δ_0, τ (both depending on β, λ, and Δ) such that, for any graph G of maximum degree Δ, any unpinned vertex u in G, and any complex λ' with $|\lambda' - \lambda| < \delta_0$, the following are true:*

1. $|Z^+_{G,u}(\beta, \lambda')| > 0, |Z^-_{G,u}(\beta, \lambda')| > 0$.
2. $|\varphi(R_{G,u}(\beta, \lambda)) - \varphi(R_{G,u}(\beta, \lambda'))| < \tau$ *if u has degree at most $\Delta - 1$ in G.*
3. $Z_G(\beta, \lambda') \neq 0$.

The main result of this section, which is a restatement of Theorem 1.1 in the introduction, is a direct consequence of item 3 of the above theorem.

COROLLARY 2.12. *Fix a degree $\Delta \geq 3$, and let (β, λ) be in the correlation decay region for the antiferromagnetic Ising model on the infinite Δ-regular tree. Then, there exists a positive constant δ_0 (depending on β, λ, and Δ) such that, for any graph G of maximum degree Δ, and any complex λ' with $|\lambda' - \lambda| < \delta_0$, we have $Z_G(\beta, \lambda') \neq 0$.*

2.3. Hard-core model.

In this section we consider the *independence polynomial*, which is the partition function of the hard-core model. Formally, given a graph $G = (V, E)$ and a *vertex activity* $\lambda > 0$, we let $\mathcal{I}(G)$ be the set of independent sets of vertices in G. Then the independence polynomial is given by

$$Z_G(\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}.$$

The hard-core model is a simple model of the "excluded volume" phenomenon: vertices in the independent set I correspond to particles, each of which prevents neighboring sites from being occupied. The parameter λ controls the density of particles in the system.

It is known from seminal work of Weitz and Sly that there is a critical activity $\lambda_c(\Delta)$ such that, when $\lambda < \lambda_c(\Delta)$, the partition function for graphs of maximum degree Δ can be approximated efficiently [61], while for $\lambda > \lambda_c(\Delta)$ it is NP-hard to approximate the partition function [54] (see also [19, 55]). We will refer to $\lambda < \lambda_c(\Delta)$ as the *correlation decay interval* for the hard-core model. In this section, we view $Z_G(\lambda)$ as a polynomial in λ and study its complex zeros. The main result of this section will again be that there are no zeros in a complex neighborhood of the correlation decay interval $(0, \lambda_c(\Delta))$.

In similar fashion to the Ising model, for a fixed vertex v we write the partition function as $Z_G(\lambda) = Z_{G\setminus v}(\lambda) + \lambda \cdot Z_{G \setminus N_G[v]}(\lambda)$ and let $R_{G,v}(\lambda) := \frac{Z_{G\setminus N_G[v]}(\lambda)}{Z_{G\setminus v}(\lambda)}$. Note that $Z_{G\setminus v}(\lambda)$ corresponds to pinning v to be "unoccupied" (not in the independent set) in G, while $Z_{G\setminus N_G[v]}(\lambda)$ corresponds to pinning v to be "occupied" (in the independent set) in G. By analogy with Lemmas 2.2 and 2.8, we have the following formal recurrence relation for $R_{G,u}$ [61], which is easily verified.

LEMMA 2.13. *Let ω be a formal variable. Given a graph G and a vertex u in G, let k be the number of neighbors of u. We then have*

$$R_{G,u}(\omega) = \lambda \prod_{i=1}^{k} \frac{1}{1 + R_{G_i, v_i}(\omega)},$$

where the graphs $G_i := G \setminus \{u, v_1, \ldots, v_{i-1}\}$ are defined in an analogous fashion to Definition 2.1.

For a nonnegative integer k, let $F_{\lambda,k}(\boldsymbol{x}) := \lambda \prod_{i=1}^{k} \frac{1}{1+x_i}$. This recurrence has been studied before in the literature. As with the Ising model examples above, it has been found useful to reparameterize $F_{\lambda,k}$ using a "potential function" φ in the form $F^{\varphi}_{\lambda,k} := \varphi \circ F_{\lambda,k} \circ \varphi^{-1}$. As shown by Li, Lu, and Yin [37], using the function $\varphi(x) = 2\sinh^{-1}(\sqrt{x}\,)$ leads to the following stepwise correlation decay in the 1-norm.[9]

THEOREM 2.14 ([37]). *Fix a degree $\Delta \geq 3$, and let $k \leq \Delta - 1$ be a positive integer. If λ is in the correlation decay interval, then there exists an $\eta > 0$ (depending upon λ and Δ) such that $\|\nabla F^{\varphi}_{\lambda,k}(\boldsymbol{x})\|_1 < 1 - \eta$ for every $\boldsymbol{x} \in \mathbb{R}^k$.*

Again by analogy with the Ising model, we have $\lambda/(1+\lambda)^{\Delta} \leq R_{G,u}(\lambda) \leq \lambda$, leading us to define the following analogue of the interval in eq. (6):

$$I_0(\lambda, \Delta) := \left[\varphi\left(\lambda/(1+\lambda)^{\Delta}\right), \varphi(\lambda)\right]. \tag{15}$$

The following corollary is again a consequence of Theorem 2.14 and the smoothness properties of $F^{\varphi}_{\lambda,k}$ and φ^{-1} at points close to the set $I_0(\lambda, \Delta)$.

COROLLARY 2.15. *Fix a degree $\Delta \geq 3$, and let $k \geq 0$. If λ is in the correlation decay interval, then there exist positive constants $\eta, \varepsilon, \delta$ (depending on λ and Δ) such that the following is true. Let $D := D(\lambda, \Delta)$ be the set of points within distance ε of $I_0(\lambda, \Delta)$ in $\mathbb{C}$. Then, whenever $k \leq \Delta - 1$, $\|\nabla F^{\varphi}_{\lambda,k}(\boldsymbol{x})\|_1 < 1 - \eta/2$ for every $\boldsymbol{x} \in D^k$. Moreover, there is a finite constant $M \geq 1$ (depending on λ and Δ) such that*

$$\sup_{\boldsymbol{x} \in D^k,\ \lambda' \in \mathbb{C}: |\lambda' - \lambda| < \delta} \left| F^{\varphi}_{\lambda,k}(\boldsymbol{x}) - F^{\varphi}_{\lambda',k}(\boldsymbol{x}) \right| \leq M\,|\lambda - \lambda'|;$$

$$\sup_{x,y: \varphi(x), \varphi(y) \in D} |\varphi(x) - \varphi(y)| \leq M\,|x - y|;$$

$$\sup_{x,y \in D} \left|\varphi^{-1}(x) - \varphi^{-1}(y)\right| \leq M\,|x - y|; \text{ and}$$

$$\sup_{\boldsymbol{x} \in D^k} \|\nabla F^{\varphi}_{\lambda,k}(\boldsymbol{x})\|_1 \leq M \text{ when } k = \Delta.$$

Finally, given Lemma 2.13 and Corollary 2.15, an identical argument to that in the proof of Theorem 2.6 establishes the following.

THEOREM 2.16. *Fix a degree $\Delta \geq 3$, and let λ be in the correlation decay interval. Then there exist positive constants δ_0 and τ (depending on λ and Δ) such that, for any graph G of maximum degree Δ, any unpinned vertex u in G, and any λ' with $|\lambda' - \lambda| < \delta_0$, the following are true:*
1. *$|Z_G(\lambda')| > 0$.*
2. *$|\varphi(R_{G,u}(\lambda')) - \varphi(R_{G,u}(\lambda))| < \tau$.*

The main result of this section, establishing a zero-free region containing the correlation decay interval, now follows as an immediate corollary of the above theorem.

COROLLARY 2.17. *Fix a degree $\Delta \geq 3$, and let λ lie in the correlation decay interval for the hard-core model on the infinite Δ-regular tree. There exist positive constants δ, ε (both depending on λ and Δ) such that, for any graph G of maximum degree Δ, and any λ' with $|\lambda' - \lambda| < \delta$, we have $Z_G(\lambda') \neq 0$.*

[9] This is a special case of Lemma 4.4 of [37], taken in combination with item 5 of Lemma 3.1 of that paper, obtained by setting $\beta = 0$ and $\gamma = 1$ in their notation. Also note that in [37], only the derivative Φ of the message φ is explicitly mentioned (at the bottom of page 76, at the end of column 1). The function $\varphi(x) = 2\sinh^{-1}(\sqrt{x})$ is obtained by integrating $\Phi(x) = 1/\sqrt{x(1+x)}$.

The above result was conjectured by Sokal [57] and first proved (with more detailed information about the geometry of the zero-free region) by Peters and Regts [47] via a different argument involving a tailor-made "potential function." Our argument above describes a simpler route to the result starting from the previously known correlation decay properties for real parameters.

2.4. Related work and discussion. There are a few recent papers which use correlation decay-like arguments for proving absence of complex zeros: Peters and Regts [47] considered the case of the roots of the independence polynomial, while an earlier paper by the present authors [40] looked at the Fisher zeros of the zero-field Ising model. A recent paper of Peters and Regts [48] on the Lee–Yang zeros of the antiferromagnetic Ising model on graphs of maximum degree at most Δ for $\beta \in (\frac{\Delta-2}{\Delta}, \frac{\Delta}{\Delta-2})$ is also in a similar spirit. The main message of this section is that the somewhat different arguments used in these results can in fact be carried out in a unified framework which allows one to "lift" known analyses of Weitz recurrences for the corresponding models [37, 53, 61, 63] to show that, in each case, there is a zero-free region of constant width that contains the entire correlation decay interval. Thus, as mentioned earlier, this puts on a more formal footing the observation that Weitz's algorithm can be seen as a bridge between the "decay of correlations" and "analyticity of free energy density" formalisms of phase transitions. We also note in passing that, via Barvinok's general paradigm, the results in this section lead to polynomial time approximation algorithms for the model partition functions in the respective correlation decay intervals (and indeed in a complex neighborhood of those intervals). However, in all of the above cases, these algorithmic consequences (at least for real-valued parameters) can be derived directly from correlation decay [37, 53, 61, 63], so we do not pursue this direction here.

In the following section we turn to the Potts model, where such tight correlation decay results are not known. We show that, with a more careful analysis, less tight correlation decay arguments can also be lifted to the complex plane in a similar fashion to the results of this section. Further, in contrast to the two-spin systems considered in this section, the algorithmic consequences are also novel and resolve open questions; indeed, it is not yet known how to obtain them directly from correlation decay without passing to the complex plane.

3. Potts model: Preliminaries.

3.1. Colorings and the Potts model. Throughout, we assume that the graphs that we consider are augmented with a list of colors for every vertex. Formally, a graph is a triple $G = (V, E, L)$, where V is the vertex set, E is the edge set, and $L : V \to 2^{\mathbb{N}}$ specifies a list of colors for every vertex. The partition function as defined in the introduction generalizes naturally to this setting: the sum is now over all those colorings σ which satisfy $\sigma(v) \in L(v)$.

We also allow graphs to contain *pinned* vertices: a vertex v is said to be *pinned* to a color c if only those colorings of G are allowed in which v has color c. Suppose that a vertex v of degree d_v in a graph G is pinned to a color c, and consider the graph G' obtained by replacing v with d_v copies of itself, each of which is pinned to c and connected to exactly one of the original neighbors of v in G. It is clear that $Z_{G'}(w) = Z_G(w)$ for all w. We will therefore assume that the operation of *pinning* a vertex comprises this operation as well; in particular, this means we can assume that all pinned vertices in our graphs have degree at most one. Further, if a pinned vertex u has another pinned vertex v as a neighbor, then u and v must form a connected component consisting of a single edge. The *size* of graph G is defined to be the number of unpinned vertices. Note that the above operation of duplicating pinned vertices does not change the size of the graph.

Let G be a graph and v an unpinned vertex in G. A color c in the list of v is said to be *good* for v if every pinned neighbor u of v is pinned to a color different from c. The set of good colors for a vertex v in graph G is denoted $\Gamma_{G,v}$. We sometimes omit the graph G and write Γ_v when G is clear from the context. A color c that is not in Γ_v is called *bad* for v. Further, given a graph G possibly with pinned vertices, we say that the graph is *unconflicted* if no two neighboring vertices in G are pinned to the same color. Note that since all pinned vertices have degree exactly one, any conflicted graph is the vertex-disjoint union of an unconflicted graph and a collection of disjoint, conflicted edges.

We will assume throughout that all unconflicted graphs G we consider have at least one proper coloring: this will be guaranteed in our applications since we will always have $|L(u)| \geq \deg_G(u) + 1$ for every unpinned vertex u in G.

Definition 3.1. *For a graph G, a vertex v, and a color $i \in L(v)$, the* restricted partition function *$Z^{(i)}_{G,v}(w)$ is the partition function restricted to colorings in which vertex v receives color i.*

Definition 3.2. *Let ω be a formal variable. For any G, a vertex v, and colors $i, j \in L(v)$, we define the* marginal ratio *of color i to color j as $R^{(i,j)}_{G,v}(\omega) := \frac{Z^{(i)}_{G,v}(\omega)}{Z^{(j)}_{G,v}(\omega)}$. Similarly we also define formally the corresponding* pseudo marginal probability *as $\mathcal{P}_{G,\omega}\left[c(v) = i\right] := \frac{Z^{(i)}_{G,v}(\omega)}{Z_G(\omega)}$.*

Remark 5. Note that when a numerical value $w \in \mathbb{C}$ is substituted in place of ω in the above formal definition, $R^{(i,j)}_{G,v}(w)$ is numerically well defined as long as $Z^{(j)}_{G,v}(w) \neq 0$, and $\mathcal{P}_{G,w}\left[c(v) = i\right]$ is numerically well defined as long as $Z_G(w) \neq 0$. In the proof of the main theorem in sections 5 and 6, we will ensure that the above definitions are numerically instantiated only in cases where the above conditions for such an instantiation to be well defined are satisfied. For instance, when $w \in [0, 1]$, this is the case for the first definition when either (i) $w \neq 0$; or (ii) $w = 0$, G is unconflicted, and $j \in \Gamma_{G,v}$. And for the second definition, this is the case when either (i) $w \neq 0$; or (ii) $w = 0$ and G is unconflicted.

Remark 6. Note also that when $w \in [0, 1]$, the pseudo probabilities, if well defined, are actual marginal probabilities. In this case, we will also write $\mathcal{P}_{G,w}\left[c(v) = i\right]$ as $\Pr_{G,w}\left[c(v) = i\right]$. For arbitrary complex w, this interpretation as probabilities is of course not valid (since $\mathcal{P}_{G,w}\left[c(v) = i\right]$ can be non-real), but provided that $Z_G(w) \neq 0$ it is still true that $\sum_{i \in L(v)} \mathcal{P}_{G,w}\left[c(v) = i\right] = \frac{1}{Z_G(w)} \sum_{i \in L(v)} Z^{(i)}_{G,v}(w) = \frac{Z_G(w)}{Z_G(w)} = 1$. We also note that if v is pinned to color k, then $\mathcal{P}_{G,w}\left[c(v) = i\right]$ is 1 when $k = i$ and 0 when $k \neq i$.

Notation. For the case $w = 0$ (proper colorings) we will sometimes shorten the notation $\mathcal{P}_{G,0}\left[c(v) = i\right]$ and $\Pr_{G,0}\left[c(v) = i\right]$ to $\mathcal{P}_G\left[c(v) = i\right]$ and $\Pr_G\left[c(v) = i\right]$, respectively.

Definition 3.3 (the graphs $G^{(i,j)}_k$). *Given a graph G and a vertex u in G, let $v_1, \ldots, v_{\deg_G(u)}$ be the neighbors of u. We define $G^{(i,j)}_k$ (the vertex u will be understood from the context) to be the graph obtained from G as follows:*

- *first we replace vertex u with $u_1, \ldots, u_{\deg_G(u)}$, and connect u_1 to v_1, u_2 to v_2,* *and so on;*

- *next we pin vertices $u_1, \dots, u_{k-1}$ to color i, and vertices $u_{k+1}, \dots, u_{\deg_G(u)}$ to color j;*
- *finally we remove the vertex u_k.*

Note that the graph $G_k^{(i,j)}$ has one fewer unpinned vertex than G. Moreover, the vertices $u_1, \dots, u_{\deg_G(u)}$ are of degree one, so this construction maintains the property that pinned vertices have degree one.

We now derive a recurrence relation between the marginal ratios of the graph G and pseudo marginal probabilities of the graphs $G_k^{(i,j)}$. This is an extension to the Potts model of a similar recurrence relation derived by Gamarnik, Katz, and Misra [22] for the special case of colorings (that is, $w = 0$).

LEMMA 3.4. *Let ω be a formal variable. For a graph G, a vertex u, and colors $i, j \in L(u)$, we have*

$$R_{G,u}^{(i,j)}(\omega) = \prod_{k=1}^{\deg_G(u)} \frac{1 - \gamma \cdot \mathcal{P}_{G_k^{(i,j)},\omega}[c(v_k) = i]}{1 - \gamma \cdot \mathcal{P}_{G_k^{(i,j)},\omega}[c(v_k) = j]},$$

where we define $\gamma := 1 - \omega$. In particular, when a numerical value $w \in \mathbb{C}$ is substituted in place of ω, the above recurrence is valid as long as the quantities $Z_{G_k^{(i,j)}}(w)$ and $1 - \gamma \cdot \mathcal{P}_{G_k^{(i,j)},w}[c(v_k) = j]$ for $1 \le k \le \deg_G(u)$ are all nonzero.

Proof. Let $t := \deg_G(u)$. For $0 \le k \le t$, let H_k be the graph obtained from G as follows:

- first we replace vertex u with $u_1, \dots, u_t$, and connect u_1 to v_1, u_2 to v_2, and so on;
- we then pin vertices $u_1, \dots, u_k$ to color i, and vertices $u_{k+1}, \dots, u_t$ to color j.

Note that H_k is the same as $G_k^{(i,j)}$, except that the last step of the construction of $G_k^{(i,j)}$ is skipped, i.e., the vertex u_k is not removed, and, further, u_k is pinned to color i. We can now write

$$R_{G,u}^{(i,j)}(\omega) = \frac{Z_{G,u}^{(i)}(\omega)}{Z_{G,u}^{(j)}(\omega)} = \frac{Z_{H_t}(\omega)}{Z_{H_0}(\omega)} = \prod_{k=1}^{t} \frac{Z_{H_k}(\omega)}{Z_{H_{k-1}}(\omega)}.$$

Next, for $1 \le k \le t$, let $Y_k := Z_{G_k^{(i,j)}}(\omega)$ and $Y_k^{(i)} := Z_{G_k^{(i,j)},v_k}^{(i)}(\omega)$. We observe that

$$\mathcal{P}_{G_k^{(i,j)},\omega}[c(v_k) = i] = \frac{Y_k^{(i)}}{Y_k};$$
$$Z_{H_k}(\omega) = Y_k - (1 - \omega) \cdot Y_k^{(i)};$$
$$Z_{H_{k-1}}(\omega) = Y_k - (1 - \omega) \cdot Y_k^{(j)}.$$

Therefore, we have

$$R_{G,u}^{(i,j)}(\omega) = \prod_{k=1}^{t} \frac{Y_k - (1 - \omega) \cdot Y_k^{(i)}}{Y_k - (1 - \omega) \cdot Y_k^{(j)}} = \prod_{k=1}^{t} \frac{1 - \gamma \cdot \mathcal{P}_{G_k^{(i,j)},\omega}[c(v_k) = i]}{1 - \gamma \cdot \mathcal{P}_{G_k^{(i,j)},\omega}[c(v_k) = j]},$$

where $\gamma = 1 - \omega$. The claim about the validity of the recurrence on numerical substitution then follows from the conditions outlined in Remark 5. ∎

3.2. Complex analysis. In this subsection we collect some tools and observations from complex analysis. Throughout this paper, we use ι to denote the imaginary unit $\sqrt{-1}$, in order to avoid confusion with the symbol "i" used for other purposes. For a complex number $z = a + \iota b$ with $a, b \in \mathbb{R}$, we denote its real part a as $\Re z$, its imaginary part b as $\Im z$, its *length* $\sqrt{a^2 + b^2}$ as $|z|$, and, when $z \neq 0$, its *argument* $\sin^{-1}(\frac{b}{|z|}) \in (-\pi, \pi]$ as $\arg z$. We also generalize the notation $[x, y]$ used for closed real intervals to the case when $x, y \in \mathbb{C}$, and use it to denote the closed straight line segment joining x and y.

We start with a consequence of the mean value theorem for complex functions, specifically tailored to our application. Let D be any domain in $\mathbb{C}$ with the following properties.

- For any $z \in D$, $\Re z \in D$.
- For any $z_1, z_2 \in D$, there exists a point $z_0 \in D$ such that one of the numbers $z_1 - z_0, z_2 - z_0$ has zero real part while the other has zero imaginary part.
- If $z_1, z_2 \in D$ are such that either $\Im z_1 = \Im z_2$ or $\Re z_1 = \Re z_2$, then the segment $[z_1, z_2]$ lies in D.

We remark that a rectangular region symmetric about the real axis will satisfy all of the above properties.

LEMMA 3.5 (mean value theorem for complex functions). *Let f be a holomorphic function on a domain D as above such that, for $z \in D$, $\Im f(z)$ has the same sign as $\Im z$. Suppose further that there exist positive constants ρ_I and ρ_R such that*

- *for all $z \in D$, $|\Im f'(z)| \leq \rho_I$;*
- *for all $z \in D$, $\Re f'(z) \in [0, \rho_R]$.*

Then for any $z_1, z_2 \in D$, there exists $C_{z_1,z_2} \in [0, \rho_R]$ such that

$$|\Re\left(f(z_1) - f(z_2)\right) - C_{z_1,z_2} \cdot \Re\left(z_1 - z_2\right)| \leq \rho_I \cdot |\Im\left(z_1 - z_2\right)|,$$

and, furthermore,

$$|\Im\left(f(z_1) - f(z_2)\right)| \leq \rho_R \cdot \begin{cases} |\Im(z_1 - z_2)| & \text{when } (\Im z_1) \cdot (\Im z_2) \leq 0; \\ \max\{|\Im z_1|, |\Im z_2|\} & \text{otherwise.} \end{cases}$$

Proof. We write $f = u + \iota v$, where $u, v : D \to \mathbb{R}$ are seen as differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$ satisfying the Cauchy–Riemann equations

$$u^{(1,0)} = v^{(0,1)} \quad \text{and} \quad u^{(0,1)} = -v^{(1,0)}.$$

This implies in particular that $\Re f'(z) = u^{(1,0)}(z) = v^{(0,1)}(z)$ and $\Im f'(z) = v^{(1,0)}(z) = -u^{(0,1)}(z)$.

Let z_0 be a point in D such that $\Re(z_2 - z_0) = 0$ and $\Im(z_1 - z_0) = 0$ (by the conditions imposed on D, such a z_0 exists, possibly after interchanging z_1 and z_2). Now we have

$$\begin{aligned} \Re\left(f(z_1) - f(z_2)\right) &= u(z_1) - u(z_0) + u(z_0) - u(z_2) \\ &= u^{(1,0)}(z') \cdot \Re(z_1 - z_0) + u(z_0) - u(z_2), \end{aligned}$$

where z' is a point lying on the segment $[z_0, z_1]$, obtained by applying the standard mean value theorem to the function u along this segment (note that the segment is parallel to the real axis). On the other hand, since the segment $[z_0, z_2]$ is parallel to the imaginary axis, we may apply the standard mean value theorem to the real-valued function u to get (after recalling that $\left|u^{(0,1)}(z)\right| = |\Im f'(z)| \leq \rho_I$ for all $z \in D$)

$$|u(z_0) - u(z_2)| \leq \rho_I \left|\Im\left(z_2 - z_0\right)\right| = \rho_I \left|\Im(z_2 - z_1)\right|.$$

This proves the first part, once we set $C_{z_1,z_2} = u^{(1,0)}(z') = \Re f'(z')$, which must lie in $[0, \rho_R]$ since $z' \in D$.

For the second part, we note that since $\Im f(z) = 0$ when $\Im z = 0$, we have, for $z \in D$,

$$\begin{aligned}\Im f(z) = \Im\left(f(z) - f(\Re z)\right) &= v(z) - v(\Re z)\\ &= v^{(0,1)}(z') \cdot \Im z,\end{aligned}$$

where z' is a point lying on the segment $[z, \Re z]$, obtained by applying the standard mean value theorem to the function v along this segment (note that the segment is parallel to the imaginary axis).

Since $v^{(0,1)}(z') = u^{(1,0)}(z') \in [0, \rho_R]$ for all $z' \in D$, there therefore exist $a, b \in [0, \rho_R]$ such that

$$|\Im(f(z_1) - f(z_2))| = |a\Im z_1 - b\Im z_2|,$$

so that we get

$$|\Im\ (f(z_1) - f(z_2))| = |a\Im z_1 - b\Im z_2| \leq \rho_R \cdot \begin{cases}|\Im(z_1 - z_2)| \text{ when } (\Im z_1)\cdot(\Im z_2) \leq 0;\\ \max\left\{|\Im z_1|, |\Im z_2|\right\} \text{ otherwise.}\end{cases}$$

This completes the proof. ∎

Later, we will apply the above lemma to the function

$$f_\kappa(x) := -\ln(1 - \kappa e^x), \tag{16}$$

which will play a central role in our proofs. (We note that here, and also later in the paper, we use ln to denote the principal branch of the complex logarithm; i.e., if $z = re^{\iota\theta}$ with $r > 0$ and $\theta \in (-\pi, \pi)$, then $\ln z = \ln r + \iota\theta$.) In the following lemma, we verify that, for real $\kappa \in [0, 1]$, f_κ indeed satisfies the hypotheses of Lemma 3.5 so that such an application is valid, and we also quantify the deviation in $f_\kappa(z)$ for complex z close to the real interval.

LEMMA 3.6. *Consider the domain D given by*

$$D := \{z \mid \Re z \in (-\infty, -\zeta) \text{ and } |\Im z| < \tau\},$$

where $\tau < 1/2$ and ζ are positive real numbers such that $\tau^2 + e^{-\zeta} < 1$. Suppose $\kappa \in [0, 1]$ and consider the function f_κ defined in eq. (16). Then the following hold:

1. *The function f_κ and the domain D satisfy the hypotheses of Lemma 3.5 if ρ_R and ρ_I in the statement of the theorem are taken to be $\frac{e^{-\zeta}}{1-e^{-\zeta}}$ and $\frac{\tau \cdot e^{-\zeta}}{(1-e^{-\zeta})^2}$, respectively.*
2. *If $\varepsilon > 0$ and $\kappa' \in \mathbb{C}$ are such that $|\kappa' - \kappa| < \varepsilon$ and $(1+\varepsilon) < e^\zeta$, then, for any $z \in D$,*

$$|f_{\kappa'}(z) - f_\kappa(z)| \leq \frac{\varepsilon}{e^\zeta - 1 - \varepsilon}.$$

Proof. Note first that the domain D is rectangular and symmetric about the real axis, so it satisfies the properties listed before Lemma 3.5. We also note that since $\kappa \leq 1$, $f_\kappa(z)$ is well defined when $\Re z < 0$, and maps real numbers in D to real numbers. Further, a direct calculation shows that $\Im f_\kappa(z) = -\arg(1 - \kappa e^z)$ has the same sign as $\sin(\Im z)$ when $\Re z < 0$ (since $\kappa \in [0, 1]$). Since $|\Im z| \leq \tau < \pi$, we see therefore that $\Im f_\kappa(z)$ has the same sign as $\Im z$, and hence f_κ satisfies the first hypothesis of Lemma 3.5.

Note that $f'_\kappa(z) = \frac{\kappa e^z}{1-\kappa e^z}$. A direct calculation shows that $\Re f'_\kappa(z) = \frac{\kappa \Re e^z - \kappa^2 |e^z|^2}{|1-\kappa e^z|^2}$ and $\Im f'_\kappa(z) = \frac{\kappa \Im e^z}{|1-\kappa e^z|^2}$. Now, for $z \in D$, $|\arg e^z| \leq \tau$, so that $\Re e^z \geq |e^z| \cos \arg e^z \geq |e^z| (1 - \tau^2)$. Thus, $\kappa \Re e^z - \kappa^2 |e^z|^2 \geq \kappa |e^z| \left(1 - \tau^2 - \kappa |e^z|\right) \geq \kappa |e^z| \left(1 - \tau^2 - \kappa e^{-\zeta}\right)$. Since $\kappa \in [0, 1]$ and $\tau^2 + e^{-\zeta} < 1$ by assumption, we therefore have $\Re f'_\kappa(z) \geq 0$. Further, $\Re f'_\kappa(z) \leq |f'_\kappa(z)| = \frac{\kappa |e^z|}{|1-\kappa e^z|} \leq \frac{\kappa |e^z|}{1-\kappa |e^z|} \leq \frac{\kappa e^{-\zeta}}{1-e^{-\zeta}}$, since $\kappa \in [0, 1]$. Together, these show that $\Re f'_\kappa(z) \in \left[0, \frac{e^{-\zeta}}{1-e^{-\zeta}}\right]$ for $z \in D$, so that the claimed choice of the parameter ρ_R in Lemma 3.5 is justified.

Similarly, for the imaginary part, we have $|\Im f'_\kappa(z)| = \frac{\kappa |\Im e^z|}{|1-\kappa e^z|^2}$, which in turn is at most $\frac{\kappa \cdot \tau \cdot e^{-\zeta}}{(1-\kappa e^{-\zeta})^2}$ for $z \in D$. Since $\kappa \in [0, 1]$, this justifies the choice of the parameter ρ_I and concludes the verification of item 1.

We now turn to item 2. The derivative of $f_x(z)$ with respect to x is $\frac{e^z}{1-xe^z}$, which for x within distance ε (satisfying $(1 + \varepsilon) < e^\zeta$) of κ and $z \in D$ has length at most $\frac{1}{e^\zeta - 1 - \varepsilon}$. Thus, the standard mean value theorem applied along the segment $[\kappa, \kappa']$ (which is of length at most ε) yields the claim. ∎

We will also need the following simple geometric lemma, versions of which have been used in the work of Barvinok [2] and also Bencs et al. [6].

LEMMA 3.7. *Let $z_1, z_2, \ldots, z_n$ be complex numbers such that the angle between any two nonzero z_i's is at most $\alpha \in [0, \pi/2)$. Then $|\sum_{i=1}^n z_i| \geq \cos(\alpha/2) \sum_{i=1}^n |z_i|$.*

Proof. Fix a nonzero z_i, and without loss of generality let z_1 and z_2 be the nonzero elements giving the maximum and minimum values, respectively, of the quantity $\arg(z_j/z_i)$, as z_j varies over all the nonzero elements (breaking ties arbitrarily). Consider the ray z bisecting the angle between z_1 and z_2. Then, by the assumption, the angle made by z and any of the nonzero z_i's is at most $\alpha/2$, so that the projection of z_i on z is of length at least $|z_i| \cos(\alpha/2)$ and is in the same direction as z. Thus, denoting by S' the projection of $S = \sum_{i=1}^n z_i$ on z, we have

$$|S| \geq |S'| \geq \sum_{i=1}^n |z_i| \cos(\alpha/2). \qquad ∎$$

3.3. Sketch of the algorithm. In this subsection we outline how to apply Barvinok's algorithmic paradigm to translate our zero-freeness result (Theorem 1.5) into the FPTAS claimed in Theorem 1.8. Let G be a graph with n vertices, m edges, and maximum degree Δ. Recall that our goal is to obtain a $1 \pm \varepsilon$ approximation of the Potts model partition function $Z_G(w)$ at any point $w \in [0, 1]$. Note that Z_G is a polynomial of degree m, and that computing Z_G at $w = 1$ is trivial since $Z_G(1) = q^n$. Recall also that Theorem 1.5 ensures that Z_G has no zeros in the region $\mathcal{D}_\Delta$ of width τ_Δ around the real interval $[0, 1]$. For technical convenience we will actually work with a slightly smaller zero-free region consisting of the rectangle

$$\mathcal{D}'_\Delta = \{w \in \mathbb{C} : -\tau'_\Delta \leq \Re w \leq 1 + \tau'_\Delta;\ |\Im w| \leq \tau'_\Delta\},$$

where $\tau'_\Delta = \tau_\Delta/\sqrt{2}$. Note that $\mathcal{D}'_\Delta \subset \mathcal{D}_\Delta$ so $\mathcal{D}'_\Delta$ is also zero-free. In the rest of this section, we drop the subscript Δ from these quantities.

Now let $f(z)$ be a complex polynomial of degree d for which $f(0)$ is easy to evaluate, and suppose we wish to approximate $f(1)$. Barvinok's basic paradigm [2, section 2.2] achieves this under the assumption that f has no zeros in the open disk $\mathcal{B}(0, 1+\delta)$ of radius $1+\delta$ centered at 0: the approximation simply consists of the first $k = O(\frac{1}{\delta}\log(\frac{d}{\varepsilon\delta}))$ terms of the Taylor expansion of $\log f$ around 0. (Note that this expansion is absolutely convergent within $\mathcal{B}(0, 1+\delta)$ by the zero-freeness of f.) These terms can in turn be expressed as linear combinations of the first k coefficients of f itself. We now sketch how to reduce our computation of $Z_G(w)$ to this situation.

First, for any fixed $w \in [0, 1]$, define the polynomial $g(z) := Z_G(z(w-1)+1)$. Note that $g(0) = Z_G(1)$ is trivial, while $g(1) = Z_G(w)$ is the value we are trying to compute. Moreover, plainly $g(z) \neq 0$ for all $z \in \mathcal{D}'$. Next, define a polynomial $\phi : \mathbb{C} \to \mathbb{C}$ that maps the disk $\mathcal{B}(0, 1+\delta)$ into the rectangle $\mathcal{D}'$, so that $\phi(0) = 0$ and $\phi(1) = 1$; Barvinok [2, Lemma 2.2.3] gives an explicit construction of such a polynomial, with degree $N = \exp(\Theta(\tau^{-1}))$ and with $\delta = \exp(-\Theta(\tau^{-1}))$. Now we have reduced the computation of $Z_G(w)$ to that of $f(1)$, where $f(z) := g(\phi(z))$ is a polynomial of degree $\deg(g)\cdot\deg(\phi) = mN$ that is nonzero on the disk $\mathcal{B}(0, 1+\delta)$, so the framework of the previous paragraph applies. Note that the number of terms required in the Taylor expansion of $\log f$ is $k = O(\frac{1}{\delta}\log(\frac{mN}{\varepsilon\delta})) = \exp(O(\tau^{-1}))\log(\frac{n\Delta}{\varepsilon})$.

Naive computation of these k terms requires time $n^{\Theta(k)}$, which yields only a quasi-polynomial algorithm since k contains a factor of $\log n$. This complexity comes from the need to enumerate all colorings of subgraphs induced by up to k edges. However, a technique of Patel and Regts [46], based on Newton's identities and an observation of Csikvári and Frenkel [10], can be used to reduce this computation to an enumeration over subgraphs induced by *connected* sets of edges (see [46, section 6] for details). Since G has bounded degree, this reduces the complexity to $\Delta^{O(k)} = (\frac{n\Delta}{\varepsilon})^{\log(\Delta)\exp(O(\tau^{-1}))}$. For any fixed Δ this is polynomial in (n/ε), thus satisfying the requirement of an FPTAS.

Note that the degree of the polynomial is exponential in τ^{-1}; since τ^{-1} in turn is exponential in Δ (see the discussion following the proof of Theorem 1.5), the degree of the polynomial is doubly exponential in Δ. The same discussion explains how this can be improved to singly exponential for the case of uniformly large list sizes.

4. Properties of the real-valued recurrence. In this section we prove some basic properties of the real-valued recurrence in Lemma 3.4, that is, in the case where $w \in [0, 1]$ is real (and hence $\gamma = 1 - w \in [0, 1]$).

We remark that in all graphs G appearing in our analysis, we will be able to assume that for any unpinned vertex u in G, $|L(u)| \geq \deg_G(u) + 1$. Thus, $Z_G(w) \neq 0$ whenever either (i) $w \in (0, 1]$; or (ii) $w = 0$ and G is unconflicted. As discussed in the previous section, this implies that the marginal ratios and the pseudo marginal probabilities are well defined, and, further, the latter are actual probabilities. Note also that G is not connected, and G' is the connected component containing u; then we have $R^{(i,j)}_{G,u}(w) = R^{(i,j)}_{G',u}(w)$ and $\mathcal{P}_{G,w}[c(u) = i] = \mathcal{P}_{G',w}[c(u) = i]$.

As noted in the introduction, we will prove our main theorem about zero-freeness under a certain abstract condition on list-coloring instances which we call *admissibility*. In this section, we define admissibility and then show that all three classes of instances referred to in Theorems 1.2 and 1.3 and Proposition 1.4 are admissible. The last two sections of the paper will be devoted to proving zero-freeness for all admissible instances.

To define admissibility, we augment our list-coloring instances by *marking* certain unpinned vertices; we call the resulting instances *marked* instances.

The first key property of admissible instances is that they are "hereditary" in the following sense.

DEFINITION 4.1 (hereditary). *A condition on marked list-coloring instances is* hereditary *if it is preserved under each of the following operations:*

1. *Remove a pinned vertex from the graph without changing the set of marked vertices.*
2. *Pin a marked vertex u to any color in its list $L(u)$, and mark (if they are not already marked) all unpinned neighbors of u. (Note that u itself is no longer marked since it is now a pinned vertex, while all other marked vertices, if any, remain marked.)*
3. *Given a graph G and a marked unpinned vertex u in G, take any neighbor v_k of u and colors i, j in the list $L(u)$, and construct the instance $G_k^{(i,j)}$ as in Definition 3.3, with the set of marked vertices in $G_k^{(i,j)}$ consisting of all unpinned neighbors of u in G and all other marked vertices v' in G.*
4. *Take a connected component H of G, with the set of marked vertices in H being those vertices of H that were marked in G.*

The second key property of admissible instances is that the marginal distributions of colors on certain vertices have "large" min-entropy. This "niceness" property is spelled out in the following definition. We emphasize that establishing niceness is the only place in our analysis where the lower bounds on the list sizes are used.

DEFINITION 4.2 (niceness). *Given a graph G and an unpinned vertex u in G, let d be the number of unpinned neighbors of u. We say the vertex u is* nice *in G if for any real $w \in [0,1]$ and any color $i \in L(u)$, $\Pr_{G,w}[c(u) = i] \leq \frac{1}{d+2}$.*

We are now in a position to define admissible instances, as previously advertised.

DEFINITION 4.3 (admissibility). *A condition $\mathcal{L}$ on marked list-coloring instances is an* admissible *list condition if it satisfies all of the following properties:*

(i) *$\mathcal{L}$ is hereditary;*
(ii) *if a list-coloring instance G satisfies $\mathcal{L}$, then for every unpinned vertex u in G, $|L(u)| \geq \deg_G(u) + 1$;*
(iii) *if a list-coloring instance G satisfies $\mathcal{L}$, and G has at least one unpinned vertex, then G also has at least one marked unpinned vertex;*
(iv) *if a list-coloring instance G satisfies $\mathcal{L}$, then for any marked unpinned vertex u in G, and any unpinned neighbor v_k of u, v_k is nice in $G_k^{(i,j)}$.*

We now recall the three conditions on coloring instances from the introduction, appropriately generalized to include list-coloring and marking.

CONDITION A. $|L(v)| \geq \max\{2, 2 \cdot \deg_G(v)\}$ *for every unpinned vertex v in G, and all unpinned vertices are marked.*

CONDITION B. *The graph G is triangle-free, and further, for every unpinned vertex v of G,*

$$|L(v)| \geq \alpha \cdot \deg_G(v) + \beta,$$

where α is any fixed constant larger than the unique positive solution $\alpha^\star$ of the equation $xe^{-\frac{1}{x}} = 1$ and $\beta = \beta(\alpha) \geq 2\alpha$ is a constant chosen so that $\alpha \cdot e^{-\frac{1}{\alpha}(1+\frac{1}{\beta})} \geq 1$. We note that $\alpha^\star$ lies in the interval $[1.763, 1.764]$, and β as chosen above is at least $7/2$. Further, all unpinned vertices are marked.

CONDITION C. *G is a forest of maximum degree Δ, with the same list of $q \geq \Delta+1$ colors for every unpinned vertex. Further, each connected component of the forest that does not consist entirely of pinned vertices has exactly one marked unpinned vertex, and all unpinned vertices with pinned vertices as neighbors are marked.*

Remark 7. Note that the condition $|L(v)| \geq 2$ imposed in Condition A above is without loss of generality, since any vertex with $|L(v)| = 1$ can be removed from G after removing the unique color in its list from the lists of its neighbors without changing the number of colorings of G.

Remark 8. Condition B is essentially identical to Assumption 1 of Gamarnik, Katz, and Misra [22]. Indeed, an important technical calculation for us, which appears in Lemma 4.7, is essentially identical to a similar calculation in [22]. The differences between Condition B and Assumption 1 of [22] are of a technical nature and are driven by the form of the upper bound we require in Lemma 4.7. In particular, Assumption 1 of [22] puts a somewhat weaker restriction on β ($\beta \geq 2 + \sqrt{2}$), but then requires the stronger condition $(1 - 1/\beta) \cdot \alpha \cdot \exp(-1/\alpha \cdot (1 + 1/\beta) > 1$ on α and β together.

Our goal in the remainder of this section is to prove that all three of the above conditions are admissible.

LEMMA 4.4. *Conditions* A*,* B*, and* C *above are all admissible.*

To prove this lemma, we first verify the easy fact that all three list conditions are hereditary.

PROPOSITION 4.5. *Conditions* A*,* B*, and* C *above are all hereditary.*

Proof. Recall that hereditary conditions must be preserved under the four operations listed in Definition 4.1.

For the first operation, observe that removing any number of pinned vertices does not increase the degree or change the lists at any unpinned vertices. Further, if the graph is triangle-free, it remains so after such a removal. Finally, this operation does not change which vertices are marked. Hence the first operation preserves all three conditions.

For the second operation, we note that pinning a vertex does not change the degree or the list at any unpinned vertex. Further, if the graph is either triangle-free or a tree, it remains so after the operation of pinning a vertex. This already establishes that the second operation preserves Conditions A and B, as all unpinned vertices remain marked. For Condition C, we note that on pinning a marked vertex u in the forest, the component in which u lies breaks into connected components (trees) indexed by the neighbors of u, none of which are marked in G (since, by the hypothesis, u is the unique marked vertex in its connected component). Further, the components indexed by the pinned neighbors of u are just single edges with both endpoints pinned, while those indexed by an unpinned neighbor v of u get v as their unique marked vertex. Thus, Condition C is also preserved under the second operation.

We now turn to the third operation. Again, as in the second operation, none of the lists at the unpinned vertices change, while the degree of v_k drops by one. As all unpinned vertices remain marked, this already establishes that this operation preserves Conditions A and B. For the case when G is a forest (Condition C), we note that in $G_k^{(i,j)}$, the component of G containing u breaks into connected components (trees) indexed by the neighbors of u in G. Further, since only u was marked in its connected component in G, and only the unpinned neighbors of u get marked in the new connected components created in $G_k^{(i,j)}$, the condition that each connected component not consisting entirely of pinned vertices must have exactly one marked vertex is satisfied. Finally, we observe that the only pinned vertices in the newly created connected components in $G_k^{(i,j)}$ must correspond to either (i) pinned neighbors of u in G; or (ii) pinned copies of u that are now neighbors of (marked) vertices that were the unpinned neighbors of u in G. All these pinned vertices have either a pinned vertex or a marked vertex as their (unique) neighbor. This establishes that Condition C is also preserved by the third operation.

Finally, the fourth operation of passing to a connected component trivially maintains all three conditions. ∎

Continuing with our proof of Lemma 4.4, we note next that property (ii) is trivially true for all three of Conditions A, B, and C, while property (iii) is also easily verified in all three cases. To conclude the proof, it therefore remains only to prove the niceness property (iv). We do this separately for each of the three conditions in the following subsections.

Remark 9. In the remainder of this section, we adopt the convention that if G is a conflicted graph (so that it has no proper colorings) and $w = 0$, then $\Pr_{G,w}[c(u) = i] = 0$ for every color i and every unpinned vertex u in G. This is just to simplify the presentation in this section by avoiding the need to explicitly exclude this case from the lemmas below. In the proof of our main result in sections 5 and 6, we will never consider conflicted graphs in a situation where w could be 0, so that this convention will then be rendered moot.

4.1. Analysis for Condition A.

LEMMA 4.6. *Let G be a graph that satisfies Condition* A. *Then for any unpinned vertex u in G, and any unpinned neighbor v_k of u, we have that v_k is nice in $G_k^{(i,j)}$.*

Proof. For ease of notation, we denote $G_k^{(i,j)}$ by H and v_k by v. Since G satisfies Condition A, and $\deg_H(v) = \deg_G(v_k) - 1$ (since the neighbor u of v_k in G is dropped in the construction of $H = G_k^{(i,j)}$), we have $|L_H(v)| = |L_G(v_k)| \geq 2\deg_G(v_k) \geq 2 \cdot \deg_H(v) + 2$.

Consider any valid coloring[10] σ' of the neighbors of v in H. For $k \in L_H(v)$, let n_k denote the number of neighbors of v that are colored k in σ'. Then for any $w \in [0, 1]$ and $i \in L_H(v)$,

$$\Pr_{H,w}[c(v) = i|\sigma'] = \frac{w^{n_i}}{\sum_{j \in L_H(v)} w^{n_j}} \leq \frac{1}{|L_H(v)| - \deg_H(v)},$$

since at most $\deg_H(v)$ of the n_j can be positive. Note in particular that if i is not a good color for v in H, then the probability is 0. Since this holds for any coloring σ', we have $\Pr_{H,w}[c(v) = i] \leq \frac{1}{|L_H(v)| - \deg_H(v)}$. Now, let d be the number of unpinned neighbors of v in H. Noting that $\deg_H(v) \geq d$, and recalling the observation above that $|L_H(v)| \geq 2\deg_H(v) + 2$, we thus have

$$\Pr_{G_k^{(i,j)},w}[c(v_k) = i] = \Pr_{H,w}[c(v) = i] \leq \frac{1}{|L_H(v)| - \deg_H(v)} \leq \frac{1}{d+2}.$$

Thus v_k is nice in $G_k^{(i,j)}$. ∎

[10]Here, we say that a coloring σ is *valid* if the color σ assigned to any vertex v is from $L(v)$, and further, in case $w = 0$, no two neighbors are assigned the same color by σ.

4.2. Analysis for Condition B. Note that, as established in Proposition 4.5, if G satisfies Condition B, then so does $G_k^{(i,j)}$. Thus in order to show that v_k is nice in $G_k^{(i,j)}$, it suffices to show the following more general fact.

LEMMA 4.7. *Let G be any graph that satisfies Condition* B*, and let u be any unpinned vertex in G. Then u is nice in G.*

The proof of this lemma is almost identical to arguments that appear in the work of Gamarnik, Katz, and Misra [22] on strong spatial mixing; we include a proof here for completeness.

Proof. We show first that $\Pr_{G,w}[c(u) = i] \le \frac{1}{\beta}$ whenever $L_G(u) \ge \deg_G(u) + \beta$; this will be required later in the proof. To do so, we repeat the arguments in the proof of Lemma 4.6 to see that $\Pr_{G,w}[c(u) = i] \le \frac{1}{|L(u)|-\deg_G(u)}$. The claimed bound then follows since $|L(u)| - \deg_G(u) \ge \beta$.

Next we show that the upper bound of $\frac{1}{d+2}$, where d is the number of unpinned neighbors of u in G, holds conditioned on every coloring of the neighbors of the (unpinned) neighbors of u, by following a similar path as in [22]. Consider any valid coloring σ' (defined as in the proof of the previous lemma) of the vertices at distance *two* from u. Since G is triangle-free, we claim there is a tree T of depth two rooted at u, with all the leaves pinned according to σ', such that

$$\Pr_{G,w}[c(u) = i|\sigma'] = \Pr_{T,w}[c(u) = i]. \tag{17}$$

To see this, notice that once we condition on the coloring of the vertices at distance 2 from u, the distribution of the color at u becomes independent of the distribution of colors of vertices at distance 3 or more. Further, because of triangle freeness, no two neighbors of u have an edge between them, and hence any cycle in the distance-2 neighborhood, if one exists, must go through at least one pinned vertex. We then observe that such a cycle can be broken by replacing any pinned vertex v' in it with $\deg(v')$ copies, one for each of its neighbors: as discussed earlier, this operation cannot change the partition function or probabilities. This operation therefore ensures that every pinned vertex in the resulting graph is now a leaf of a tree T of depth 2 rooted at u. Further, in T, the root u has d unpinned children, and all vertices at depth 2 are pinned according to σ'.

Let $v_1, \dots, v_d$ be the d unpinned neighbors of u in T, and let $T_1, \dots, T_d$ be the subtrees rooted at $v_1, \dots, v_d$, respectively. For each $k \in L_G(u)$, let n_k be the number of neighbors of u that are pinned to color k. Then by Lemma 3.4,

$$R_{T,u}^{(j,i)}(w) = \frac{w^{n_j} \cdot \prod_{k=1}^{d} (1 - \gamma \cdot \mathcal{P}_{T_k,w}[c(v_k) = j])}{w^{n_i} \cdot \prod_{k=1}^{d} (1 - \gamma \cdot \mathcal{P}_{T_k,w}[c(v_k) = i])}.$$

Define $t_{kj} := \gamma \cdot \Pr_{T_k,w}[c(v_k) = j]$, and note that from the calculation at the beginning of the proof, we have $0 \le t_{kj} \le \frac{\gamma}{\beta} \le \frac{1}{\beta} \le 1/2$. Note also that $t_{kj} = 0$ if $j \notin L(v_k)$. Thus, we have

$$\sum_{j \in \Gamma_u} t_{kj} = \gamma \sum_{j \in \Gamma_u \cap L(v_k)} \Pr_{T_k,w}[c(v_k) = j] \le \gamma \le 1. \tag{18}$$

Therefore,

$$\Pr_{T,w}[c(u)=i] = \frac{1}{\sum_{j\in L(v)} R_{T,v}^{(j,i)}(w)}$$
$$= \frac{w^{n_i}\cdot\prod_{k=1}^{d}(1-t_{ki})}{\sum_{j\in L(u)} w^{n_j}\prod_{k=1}^{d}(1-t_{kj})}$$
$$\le \frac{1}{\sum_{j\in\Gamma_u}\prod_{k=1}^{d}(1-t_{kj})}, \tag{19}$$

where, in the last inequality we use that $n_j = 0$ when j is good for u in G, and also that $w \in [0,1]$.

Since $\Pr_{G,w}[c(u)=i|\sigma'] = \Pr_{T,w}[c(u)=i]$, it remains to lower bound the denominator term $\sum_{j\in\Gamma_u}\prod_{k=1}^{d}(1-t_{kj})$. We begin by recalling the following standard consequence of the Taylor expansion of $\ln(1-x)$ around 0: when $0 \le x \le \frac{1}{\beta} < 1$, and β is such that $(1-1/\beta)^2 \ge 1/2$,

$$\ln(1-x) \ge -x - \frac{x^2}{2(1-1/\beta)^2} \ge -x - x^2 \ge -\left(1+\frac{1}{\beta}\right)x. \tag{20}$$

Note that the condition required of β is satisfied since $\beta \ge 2\alpha \ge 7/2$, as stipulated in Condition B. Since $0 \le t_{kj} \le 1/\beta$, we therefore obtain, for every $j \in \Gamma_u$,

$$\prod_{k=1}^{d}(1-t_{kj}) \ge \prod_{k=1}^{d}\exp\left(-\left(1+\frac{1}{\beta}\right)t_{kj}\right) = \exp\left(-\left(1+\frac{1}{\beta}\right)\sum_{k=1}^{d}t_{kj}\right). \tag{21}$$

For convenience of notation, we denote $|\Gamma_u|$ by q_u. Note that since $|L(u)| \ge \alpha\deg(u)+\beta$, and u has $\deg(u)-d$ pinned neighbors, we have

$$q_u \ge |L(u)| - (\deg(u)-d) \ge |L(u)| - \alpha(\deg(u)-d) \ge \alpha d + \beta, \tag{22}$$

where in the second inequality we use $\alpha \ge 1$. Now, by the AM-GM inequality, we get

$$\begin{aligned}
\sum_{j\in\Gamma_u}\prod_{k=1}^{d}(1-t_{kj}) &\ge q_u\left(\prod_{j\in\Gamma_u}\prod_{k=1}^{d}(1-t_{kj})\right)^{\frac{1}{q_u}} \\
&\ge q_u\exp\left(-\frac{1+1/\beta}{q_u}\cdot\sum_{k=1}^{d}\sum_{j\in\Gamma_u}t_{kj}\right) && \text{using eq. (21)} \\
&\ge (\alpha d+\beta)\exp\left(-\frac{d(1+1/\beta)}{\alpha d+\beta}\right) && \text{by eqs. (18) and (22)} \\
&\ge (d+2)\alpha\cdot\exp\left(-\frac{(1+1/\beta)}{\alpha}\right) && \text{using } \beta \ge 2\alpha \\
&\ge (d+2),
\end{aligned}$$

where the last line uses the stipulation in Condition B that α and β satisfy $\alpha\cdot\exp\left(-\frac{(1+1/\beta)}{\alpha}\right) \ge 1$. From eqs. (17) and (19) we therefore get

$$\Pr_{G,w}[c(u)=i|\sigma'] \le \frac{1}{d+2}.$$

Since this holds for any conditioning σ' of the colors of the neighbors of the neighbors of u in G, we then have

$$\mathrm{Pr}_{G,w}\left[c(u)=i\right] \leq \frac{1}{d+2},$$

which concludes the proof. ∎

4.3. Analysis for Condition C.

LEMMA 4.8. *Let G be a list-coloring instance that satisfies Condition* C *(in particular, G is a forest), and let u be a marked unpinned vertex in G. Then any unpinned neighbor v_k of u is nice in $G_k^{(i,j)}$.*

Proof. Since G is a forest, and all pinned vertices in the connected component of u in G must be neighbors of u (since u is, by Condition C, the unique marked vertex in its component), we see that the connected component of v_k in $G_k^{(i,j)}$ contains no pinned vertices. Since all unpinned vertices in G have the same list, which is of size $q \geq \Delta+1$ (where Δ is the maximum degree of G), it follows by symmetry that the marginal distribution of the color of v_k is uniform. Further, since the neighbor u of v_k in G is not present in $G_k^{(i,j)}$, we know that v_k has $d \leq \Delta - 1$ unpinned neighbors in $G_k^{(i,j)}$. Thus, for each $i \in L(v_k)$,

$$\mathrm{Pr}_{G_k^{(i,j)},w}\left[c(v_k)=i\right] = \frac{1}{q} \leq \frac{1}{\Delta+1} \leq \frac{1}{d+2},$$

which establishes that v_k is nice in $G_k^{(i,j)}$. ∎

Proof of Lemma 4.4. The proof of Lemma 4.4 now follows by combining Proposition 4.5 and Lemmas 4.6, 4.7, and 4.8, along with the simple observations about properties (ii) and (iii) preceding Remark 9. ∎

We conclude this section by noting that the niceness condition can be strengthened in the case when all the list sizes are uniformly large (e.g., as in the case of standard q-colorings).

Remark 10. In Conditions A and B, if we replace the degree of a vertex by the maximum degree Δ (i.e., in Condition A, if we assume $|L(v)| \geq 2\Delta$, and in Condition B, if we assume $|L(v)| \geq \alpha\Delta+\beta$, for each v), then for every vertex v in the graph G we also have $\mathrm{Pr}_{G,w}\left[c(v)=i\right] < \min\left\{\frac{4}{3\Delta}, 1\right\}$.

To see this, notice that the same calculation as in the proof of Lemma 4.6 above gives

$$\mathrm{Pr}_{G,w}\left[c(v)=i\right] \leq \frac{1}{|L(v)|-\Delta} < \frac{4}{3\Delta},$$

under the maximum degree versions of both Conditions A and B. We will refer to this stronger condition on list sizes as the *uniformly large list size* condition. Note that the maximum degree versions of the conditions are also admissible by the same arguments as those for Conditions A and B.

5. Zero-free region for small $|w|$. As explained in the introduction, all our algorithmic results follow from Theorem 1.5, which establishes a zero-free region for the partition function $Z_G(w)$ around the interval $[0,1]$ in the complex plane. We split the proof of Theorem 1.5 into two parts: in this section, we establish the existence of a zero-free disk around the endpoint $w=0$ (see Theorem 5.1): this is the most delicate case because $w = 0$ corresponds to proper colorings. Then in section 6 (see Theorem 6.1) we derive a zero-free region around the remainder of the interval, using a similar but less delicate approach. Taken together, Theorems 5.1 and 6.1 immediately imply Theorem 1.5, so this will conclude our analysis.

THEOREM 5.1. *Fix a positive integer Δ, and let $\mathcal{L}$ be an admissible list condition. There exists a $\nu_w = \nu_w(\Delta)$ such that the following is true. Let G be a graph of maximum degree Δ satisfying the admissible list condition $\mathcal{L}$, and having no pinned vertices. Then, $Z_G(w) \neq 0$ for any w satisfying $|w| \le \nu_w$.*

In the proof, we will encounter several constants which we now fix. Given the degree bound $\Delta \ge 1$, we define

$$\varepsilon_R := \frac{0.01}{\Delta^2}; \quad \varepsilon_I := \varepsilon_R \cdot \frac{0.01}{\Delta^2}; \quad \text{and } \varepsilon_w := \varepsilon_I \cdot \frac{0.01}{\Delta^3}. \tag{23}$$

We will then see that the quantity ν_w in the statement of the theorem can be chosen to be $0.2\varepsilon_w/2^{\Delta}$. (In fact, we will show that if one has the slightly stronger assumption of uniformly large list sizes, as considered in Remark 10, then ν_w can be chosen to be $\varepsilon_w/(300\Delta)$.)

Throughout the rest of this section, we fix Δ to be the maximum degree of the graphs, and let $\varepsilon_w, \varepsilon_I, \varepsilon_R$ be as above.

We now briefly outline our strategy for the proof. Recall that, for a vertex u and colors i, j, the marginal ratio is given by $R^{(i,j)}_{G,u}(w) = \frac{Z^{(i)}_{G,u}(w)}{Z^{(j)}_{G,u}(w)}$. When G is an unconflicted graph, $R^{(i,j)}_{G,u}(0)$ is always a well-defined nonnegative real number. Intuitively, we would like to show that $R^{(i,j)}_{G,u}(w) \approx R^{(i,j)}_{G,u}(0)$, independent of the size of G, when $w \in \mathbb{C}$ is close to 0. Given such an approximation, one can use a simple geometric argument (see Consequence 5.3) to conclude that the partition function does not vanish for such w. In order to prove the above approximate equality inductively for a given graph G, we take an approach that exploits the properties of the "real" case (i.e., of $R^{(i,j)}_{G,u}(0)$) and then uses the notion of "niceness" of certain vertices described earlier to control the accumulation of errors. To this end, we will prove the following lemma via induction on the number of unpinned vertices in G. Theorem 5.1 will follow almost immediately from the lemma; see the end of this section for the details. Throughout the section, we fix an admissible list condition $\mathcal{L}$, and a $w \in \mathbb{C}$ satisfying $|w| \le \nu_w$ (as in the statement of Theorem 5.1).

LEMMA 5.2. *Let G be an unconflicted graph of maximum degree Δ satisfying an admissible list condition $\mathcal{L}$, and let u be any marked unpinned vertex in G. Then, the following are true (with ε_w, ε_I, and ε_R as defined in eq.* (23)*):*

1. *For $i \in \Gamma_u$, $\left|Z^{(i)}_{G,u}(w)\right| > 0$.*
2. *For $i, j \in \Gamma_u$, if u has all neighbors pinned, then $R^{(i,j)}_{G,u}(w) = R^{(i,j)}_{G,u}(0) = 1$.*
3. *For $i, j \in \Gamma_u$, if u has $d \ge 1$ unpinned neighbors, then*
$$\frac{1}{d}\left|\Re \ln R^{(i,j)}_{G,u}(w) - \Re \ln R^{(i,j)}_{G,u}(0)\right| < \varepsilon_R.$$
4. *For $i, j \in \Gamma_u$, if u has $d \ge 1$ unpinned neighbors, then $\frac{1}{d}\left|\Im \ln R^{(i,j)}_{G,u}(w)\right| < \varepsilon_I$.*
5. *For $i \notin \Gamma_u, j \in \Gamma_u$, we have $\left|R^{(i,j)}_{G,u}(w)\right| \le \varepsilon_w$.*

We will refer to items 1 to 5 as "items of the induction hypothesis." The rest of this section is devoted to the proof of this lemma via induction on the number of unpinned vertices in G.

We begin by verifying that the induction hypothesis holds in the base case when u is the only unpinned vertex in an unconflicted graph G. In this case, items 3 and 4 are vacuously true since u has no unpinned neighbors. Since all neighbors of u in G are pinned, the fact that all pinned vertices have degree at most one implies that G can be decomposed into two disjoint components G_1 and G_2, where G_1 consists of u and its pinned neighbors, while G_2 consists of a disjoint union of unconflicted edges (since G is unconflicted). Now, since G_1 and G_2 are disjoint components, we have $Z^{(i)}_{G,u}(w) = Z_{G_2}(w) = 1$ for all $i \in \Gamma_{G,u}$ and all $w \in \mathbb{C}$. This proves items 1 and 2. Similarly, when $i \notin \Gamma_{G,u}$, we have $Z^{(i)}_{G,u}(w) = w^{n_i}$, where $n_i \geq 1$ is the number of neighbors of u pinned to color i. This gives

$$\left| R^{(i,j)}_{G,u}(w) \right| \leq |w|^{n_i} \leq \varepsilon_w,$$

since $|w| \leq \varepsilon_w \leq 1$, and proves item 5.

We now derive some consequences of the above induction hypothesis that will be helpful in carrying out the induction. Throughout, we assume that G is an unconflicted graph satisfying an admissible list condition $\mathcal{L}$, and u is a marked unpinned vertex in G.

CONSEQUENCE 5.3. $|Z_G(w)| \geq 0.9 \min_{i\in\Gamma_u} \left| Z^{(i)}_{G,v}(w) \right| > 0$.

Proof. Note that $Z_G(w) = \sum_{i\in L(u)} Z^{(i)}_{G,u}(w)$. Recall also that since u is an unpinned vertex in G and G satisfies an admissible list condition $\mathcal{L}$, we have

$$|L(u)| \geq \deg_G(u) + 1.$$

Now, from item 4, we see that the angle between the complex numbers $Z^{(i)}_{G,u}(w)$ and $Z^{(j)}_{G,u}(w)$, when $i, j \in \Gamma_u$, is at most $d\varepsilon_I$. Applying Lemma 3.7 to the terms corresponding to the good colors and item 5 to the terms corresponding to the bad colors, we then have

$$\begin{aligned} \left| \sum_{i\in L(u)} Z^{(i)}_{G,u}(w) \right| &\geq \left(|\Gamma_u| \cos\frac{d\varepsilon_I}{2} - |L(u) \setminus \Gamma_u|\, \varepsilon_w \right) \min_{i\in\Gamma_u} \left| Z^{(i)}_{G,u}(w) \right| \\ &\geq \left(\cos\frac{d\varepsilon_I}{2} - \deg_G(u)\cdot\varepsilon_w \right) \min_{i\in\Gamma_u} \left| Z^{(i)}_{G,u}(w) \right|, \end{aligned}$$

where we use the fact that $|L(u) \setminus \Gamma_u| \leq \deg_G(u)$ and $|L(u)| \geq \deg_G(u) + 1$ in the last inequality. Since $d\varepsilon_I \leq 0.01$ and $\varepsilon_w \leq 0.01/\Delta$, we then have $\left| \sum_{i\in L(u)} Z^{(i)}_{G,u}(w) \right| \geq 0.9 \min_{i\in\Gamma_u} \left| Z^{(i)}_{G,v}(w) \right|$, which in turn is positive from item 1. ∎

CONSEQUENCE 5.4. *The pseudo probabilities approximate the real probabilities in the following sense:*

1. *for any* $i \notin \Gamma_u$, $|\mathcal{P}_{G,w}[c(u) = i]| \leq 1.2\varepsilon_w$;
2. *for any* $j \in \Gamma_u$,

$$\left| \Im \ln \frac{\mathcal{P}_{G,w}[c(u) = j]}{\mathcal{P}_G[c(u) = j]} \right| = |\Im \ln \mathcal{P}_{G,w}[c(u) = j]| \leq d\varepsilon_I + 2\Delta\varepsilon_w,$$

and

$$\left|\Re \ln \frac{\mathcal{P}_{G,w}\left[c(u)=j\right]}{\mathcal{P}_{G}\left[c(u)=j\right]}\right| \leq d\varepsilon_R + d\varepsilon_I + 2\Delta\varepsilon_w,$$

where d is the number of unpinned neighbors of u in G.

Proof. For part 1, by Consequence 5.3 we have

$$\begin{aligned}\left|\mathcal{P}_{G,w}\left[c(u)=i\right]\right| &= \frac{\left|Z^{(i)}_{G,u}(w)\right|}{\left|Z_G(w)\right|}\\ &\leq \frac{\left|Z^{(i)}_{G,u}(w)\right|}{0.9\min_{j\in\Gamma_u}\left|Z^{(j)}_{G,u}(w)\right|} \leq 1.2\varepsilon_w,\end{aligned}$$

where the last inequality follows from induction hypothesis item 5.

For part 2, by items 2 to 4 of the induction hypothesis, there exist complex numbers ξ_i (for all $i \in \Gamma_u$) satisfying $|\Re\xi_i| \leq d\varepsilon_R$ and $|\Im\xi_i| \leq d\varepsilon_I$ such that

$$\begin{aligned}\frac{1}{\mathcal{P}_{G,w}\left[c(u)=j\right]} &= \sum_{i\in L(u)} \frac{Z^{(i)}_{G,u}(w)}{Z^{(j)}_{G,u}(w)}\\ &= \underbrace{\sum_{i\in\Gamma_u} \frac{Z^{(i)}_{G,u}(0)}{Z^{(j)}_{G,u}(0)} e^{\xi_i}}_{:=A} + \underbrace{\sum_{i\in L(u)\setminus\Gamma_u} \frac{Z^{(i)}_{G,u}(w)}{Z^{(j)}_{G,u}(w)}}_{:=B}.\end{aligned}$$

Next we show that $A \approx \frac{1}{\mathcal{P}_G[c(u)=j]}$ and B is negligible. From item 5 of the induction hypothesis we have

$$\mathcal{P}_G\left[c(u)=j\right] \cdot |B| \leq \Delta\varepsilon_w. \tag{24}$$

Now, note that $\sum_{i\in\Gamma_u} \frac{Z^{(i)}_{G,u}(0)}{Z^{(j)}_{G,u}(0)} = \frac{1}{\mathcal{P}_G[c(u)=j]}$. Further, when $\varepsilon_I \leq 0.1/\Delta$, we also have[11]

$$\Re e^{\xi_i} \in \left(e^{-d\varepsilon_R} - d^2\varepsilon_I^2,\ e^{d\varepsilon_R}\right), \text{ and } |\arg e^{\xi_i}| \leq d\varepsilon_I. \tag{25}$$

The above will therefore be true also for any convex combination of the e^{ξ_i}. Noting that $\mathcal{P}_G\left[c(u)=j\right] \cdot A$ is just such a convex combination (as the coefficients of the e^{ξ_i} are non-negative reals summing to 1), we have

$$\mathcal{P}_G\left[c(u)=j\right] \cdot \Re A \in (e^{-d\varepsilon_R} - d^2\varepsilon_I^2,\ e^{d\varepsilon_R}), \tag{26}$$

$$|\arg\left(\mathcal{P}_G\left[c(u)=j\right] \cdot A\right)| \leq d\varepsilon_I. \tag{27}$$

Together, eqs. (24), (26), and (27) imply that if $C := \frac{\mathcal{P}_G[c(u)=j]}{\mathcal{P}_{G,w}[c(u)=j]}$, then (using the values of $\varepsilon_R, \varepsilon_I$, and ε_w)[12]

$$\begin{aligned}&\Re C \in \left(e^{-d\varepsilon_R} - d^2\varepsilon_I^2 - \Delta\varepsilon_w,\ e^{d\varepsilon_R} + \Delta\varepsilon_w\right), \text{ and}\\ &\arg C \in \left(-d\varepsilon_I - 2\Delta\varepsilon_w, d\varepsilon_I + 2\Delta\varepsilon_w\right).\end{aligned}$$

[11] Here, we also use the elementary facts that if z is a complex number satisfying $\Re z = r$ and $|\Im z| = \theta \leq 0.1$, then $|\arg e^z| = |\Im z| = \theta$, and $e^r \geq \Re e^z = e^r\cos\theta = \exp(r + \ln\cos\theta) \geq \exp(r - \theta^2) \geq e^r - e^r\theta^2$. Hence if $r < 0$, we have $\Re e^z \geq e^r - \theta^2$.

[12] Here, for the second inclusion, we use the following elementary computation. Let z, s be complex numbers such that $\Re z = r \in [0.9, 1.1]$, $|\arg z| = \theta \leq 0.1$, and $|s| \leq 0.1$. Then, we have $\Re(z+s) \geq r - |s|$ and $|\Im(z+s)| \leq r\theta + |s|$. Thus, $|\arg(z+s)| \leq \frac{|\Im(z+s)|}{|\Re(z+s)|} \leq \frac{r\theta+|s|}{r-|s|} = \theta + |s| \cdot \frac{1+\theta}{r-|s|} \leq \theta + 2|s|$.

Thus, since $\varepsilon_I, \varepsilon_R$ are small enough and $\varepsilon_w \leq 0.01 \min\{\varepsilon_I, \varepsilon_R\}$, we have

$$|\Re \ln C| \leq d\varepsilon_R + d\varepsilon_I + 2\Delta\varepsilon_w, \text{ and}$$
$$|\Im \ln C| \leq d\varepsilon_I + 2\Delta\varepsilon_w.$$

Here we use the elementary fact that for $z \in \mathbb{C}$, $\Re \ln z = \ln|z|$ and $\Im \ln z = \arg z$. Further, for z satisfying $\Re z = r \in [0.9, 1.1]$ and $|\arg z| = \theta \leq 0.1$, we also have $\ln r \leq \Re \ln z \leq \ln r + \ln \sec \theta \leq \ln r + \theta^2$. ∎

In the next consequence, we show that the error contracts during the induction. We first set up some notation. For a graph G, a vertex u, and a color $i \in \Gamma_u$, we let $a^{(i)}_{G,u}(w) = \ln \mathcal{P}_{G,w}[c(u) = i]$. We also recall that $\gamma := 1 - w$, and the definition of the function $f_\gamma(x) := -\ln(1 - \gamma e^x)$ from eq. (16).

CONSEQUENCE 5.5. *There exists a positive constant $\eta \in [0.9, 1)$ so that the following is true. Let d be the number of unpinned neighbors of u. Assume further that u is nice in G. Then, for any colors $i, j \in \Gamma_u$, there exists a real number $C = C_{G,u,i} \in [0, \frac{1}{d+\eta}]$ such that*

(28)
$$\left|\Re f_\gamma(a^{(i)}_{G,u}(w)) - f_1(a^{(i)}_{G,u}(0)) - C \cdot \Re(a^{(i)}_{G,u}(w) - a^{(i)}_{G,u}(0))\right| \leq \varepsilon_I + \varepsilon_w;$$

$$\left|\Im f_\gamma(a^{(i)}_{G,u}(w)) - \Im f_\gamma(a^{(j)}_{G,u}(w))\right| \leq \frac{1}{d+\eta} \cdot (d\varepsilon_I + 4\Delta\varepsilon_w) + 2\varepsilon_w; \tag{29}$$

$$\left|\Im f_\gamma(a^{(i)}_{G,u}(w))\right| \leq \frac{1}{d+\eta} \cdot (d\varepsilon_I + 4\Delta\varepsilon_w) + \varepsilon_w. \tag{30}$$

Proof. Since u is nice in G, the bound $\mathcal{P}_{G,0}[c(u) = k] \leq \frac{1}{d+2}$ (for any $k \in \Gamma_{G,u}$) applies. Combining them with Consequence 5.4 we see that $a^{(i)}_{G,u}(w), a^{(i)}_{G,u}(0), a^{(j)}_{G,u}(w)$, $a^{(j)}_{G,u}(0)$ lie in a domain D as described in Lemma 3.6 (with the parameter κ therein set to 1), with the parameters ζ and τ in that observation chosen as

$$\begin{aligned} \zeta &= \ln(d+2) - d\varepsilon_R - d\varepsilon_I - 2\Delta\varepsilon_w \text{, and} \\ \tau &= d\varepsilon_I + 2\Delta\varepsilon_w. \end{aligned} \tag{31}$$

Here, for the bound on ζ, we use the fact that for $j \in \Gamma_{G,u}$, $\mathcal{P}_G[c(u) = j] \leq \frac{1}{d+2}$, which is due to u being nice in G.

The bounds on ε_w, ε_I, and ε_R now imply $e^\zeta \geq (d+2)\left(1 - \frac{0.02}{\Delta}\right) \geq d + 1.94$, and also that $\tau \leq 0.02/\Delta$. Thus, the conditions required on ζ and τ in Lemma 3.6 (i.e., that $\tau < 1/2$ and $\tau^2 + e^{-\zeta} < 1$) are satisfied. Further, ρ_R and ρ_I as set in the observation satisfy $\rho_R \leq \frac{1}{d+\eta}$, where η can be taken to be 0.94, and $\rho_I < 3\varepsilon_I$.

Using Lemma 3.5 followed by the value of ε_w, and noting that $a^{(i)}_{G,u}(0)$ is a real number, we then have

$$\begin{aligned} \left|\Re f_1(a^{(i)}_{G,u}(w)) - f_1(a^{(i)}_{G,u}(0)) - C \cdot \Re\left(a^{(i)}_{G,u}(w) - a^{(i)}_{G,u}(0)\right)\right| &\leq \rho_I \cdot \left|\Im\left(a^{(i)}_{G,u}(w) - a^{(i)}_{G,u}(0)\right)\right| \\ &\leq 3\varepsilon_I(d\varepsilon_I + 2\Delta\varepsilon_w) \leq 4d\varepsilon_I^2 \leq \varepsilon_I \end{aligned} \tag{32}$$

for an appropriate nonnegative $C \leq 1/(d+\eta)$. This is almost eq. (28); the difference will be handled later.

Similarly, applying Lemma 3.5 to the imaginary part we have

$$\begin{aligned} &\left|\Im f_1(a^{(i)}_{G,u}(w)) - \Im f_1(a^{(j)}_{G,u}(w))\right| \\ &\quad \leq \rho_R \cdot \max\left\{\left|\Im\left(a^{(i)}_{G,u}(w) - a^{(j)}_{G,u}(w)\right)\right|,\ \left|\Im a^{(i)}_{G,u}(w)\right|, \left|\Im a^{(j)}_{G,u}(w)\right|\right\}, \end{aligned} \tag{33}$$

where, as noted above, $\rho_R \le \frac{1}{d+\eta}$. Now, note that the first term in the above maximum is less than $d\varepsilon_I$ by item 4 of the induction hypothesis, while the other two terms are at most $d\varepsilon_I + 2\Delta\varepsilon_w$ from item 2 of Consequence 5.4. This is almost the bound in eq. (29); again, the difference will be handled later.

To prove the bound in eq. (30), we first apply the imaginary part of Lemma 3.5 along with the fact that $\Im a^{(i)}_{G,u}(0) = 0$ to get

$$\begin{aligned}\left|\Im f_1(a^{(i)}_{G,u}(w))\right| &= \left|\Im f_1(a^{(i)}_{G,u}(w)) - f_1(a^{(i)}_{G,u}(0))\right| \\ &\le \rho_R \cdot \left|\Im\left(a^{(i)}_{G,u}(w)\right)\right| \\ &\le \frac{1}{d+\eta}(d\varepsilon_I + \Delta\varepsilon_w). \qquad (34)\end{aligned}$$

Finally, we use item 2 of Lemma 3.6 (with the parameter κ' therein set to γ) to conclude the proofs of eqs. (28) to (30). To this end, we note that γ satisfies $|\gamma - 1| \le \varepsilon_w$, so that the condition $(1+\varepsilon_w) < e^{\zeta}$ required for item 2 to apply is satisfied. Thus we see that for any $z \in D$,

$$|f_\gamma(z) - f_1(z)| \le \varepsilon_w,$$

so that the quantities $|\Re f_\gamma(a^{(i)}_{G,u}(w)) - \Re f_1(a^{(i)}_{G,u}(w))|$, $|\Im f_\gamma(a^{(i)}_{G,u}(w)) - \Im f_1(a^{(i)}_{G,u}(w))|$, $|\Im f_\gamma(a^{(j)}_{G,u}(w)) - \Im f_1(a^{(j)}_{G,u}(w))|$, and $|\Im f_\gamma(a^{(j)}_{G,u}(w)) - \Im f_1(a^{(j)}_{G,u}(w))|$ are all at most ε_w. The desired bounds of eqs. (28) to (30) now follow from the triangle inequality and the bounds in eqs. (32) to (34). ∎

We set up some further notation for the next consequence. For a color $i \in L(u)\backslash\Gamma_u$ we let $b^{(i)}_{G,u}(w) = \mathcal{P}_{G,w}\left[c(u) = i\right]$. We then consider the function $g_\gamma(x) := -\ln(1-\gamma x)$.

CONSEQUENCE 5.6. *For every color $i \notin \Gamma_u$, $\left|g_\gamma(b^{(i)}_{G,u}(w))\right| \le 2\varepsilon_w$.*

Proof. Item 1 of Consequence 5.4 implies that $\left|b^{(i)}_{G,u}(w)\right| \le 1.2\varepsilon_w$. Thus, recalling that $|\gamma - 1| \le \varepsilon_w$, we get that for all $\varepsilon_w < 0.01$, $\left|g_\gamma(b^{(i)}_{G,u}(w))\right| = \left|\ln(1 - \gamma b^{(i)}_{G,u}(w))\right| \le 2\varepsilon_w$. ∎

Inductive proof of Lemma 5.2. We are now ready to see the induction step in the proof of Lemma 5.2; recall that the base case (when u is the only unpinned vertex in G) was already established immediately following the statement of the lemma. Let G be any unconflicted graph which satisfies the admissible list condition $\mathcal{L}$ and has at least two unpinned vertices. We first prove induction item 1 for any marked unpinned vertex u in G. Consider the graph G' obtained from G by pinning vertex u to color i. Note that by the definition of the pinning operation, $Z^{(i)}_{G,u}(w) = Z_{G'}(w)$. When $i \in \Gamma_{G,u}$, the graph G' is also unconflicted and, further, since $\mathcal{L}$ is hereditary (because it is admissible), satisfies the admissible list condition $\mathcal{L}$. Also, G' has one fewer unpinned vertex than G. Thus, from Consequence 5.3 of the induction hypothesis applied to G', we have that $\left|Z^{(i)}_{G,u}(w)\right| = |Z_{G'}(w)| > 0$.

We now consider item 2. When all neighbors of u in G are pinned, the fact that all pinned vertices have degree at most one implies that G can be decomposed into two disjoint components G_1 and G_2, where G_1 consists of u and its pinned neighbors, while G_2 is also unconflicted (when G is unconflicted) and has one fewer unpinned vertex than G. Note also that G_2, being a connected component of G, also satisfies the admissible list condition $\mathcal{L}$ (since $\mathcal{L}$ is hereditary). Thus, from Consequence 5.3 of the induction hypothesis applied to G_2, we get that $Z_{G_2}(w)$ and $Z_{G_2}(0)$ are both nonzero. Now, since G_1 and G_2 are disjoint components, we have $Z^{(k)}_{G,u}(x) = Z_{G_2}(x)$ for all $k \in \Gamma_{G,u}$ and all $x \in \mathbb{C}$. It therefore follows that when $i, j \in \Gamma_{G,u}$, $R^{(i,j)}_{G,u}(w) = R^{(i,j)}_{G,u}(0) = 1$.

We now consider items 3 and 4. Recall that by Lemma 3.4, we have

$$R^{(i,j)}_{G,u}(w) = \prod_{k=1}^{\deg_G(u)} \frac{1 - \gamma \mathcal{P}_{G^{(i,j)}_k, w}\left[c(v_k) = i\right]}{1 - \gamma \mathcal{P}_{G^{(i,j)}_k, w}\left[c(v_k) = j\right]}. \tag{35}$$

For simplicity we write $G_k := G^{(i,j)}_k$. Note that when $i, j \in \Gamma_{G,u}$, and G is unconflicted, so are the G_k. Note also that when $i, j \in \Gamma_{G,u}$, we can restrict the product above to the d unpinned neighbors of u, since for such i, j, the contribution of the factor corresponding to a pinned neighbor is 1, irrespective of the value of w. Without loss of generality, we relabel these unpinned neighbors as $v_1, v_2, \ldots, v_d$.

Since $\mathcal{L}$ is hereditary, G_k also satisfies $\mathcal{L}$, and the vertex v_k is marked in G_k (since u was marked in G). Further, each G_k has exactly one fewer unpinned vertex than G, so that the induction hypothesis applies to each G_k at the vertex v_k.

Now, as before, for $s \in \Gamma_{G_k, v_k}$ we define $a^{(s)}_{G_k, v_k}(w) := \ln \mathcal{P}_{G_k, w}\left[c(v_k) = s\right]$; while for $t \in L(v_k) \setminus \Gamma_{G_k, v_k}$ we let $b^{(t)}_{G_k, v_k}(w) := \mathcal{P}_{G_k, w}\left[c(v_k) = t\right]$. For a graph G, a vertex u, and a color s, we let $B_{G,u}(s)$ be the set of those neighbors of u for which s is a bad color in $G \setminus \{u\}$. For simplicity we will also write $B(s) := B_{G,u}(s)$ when it is clear from the context. As before, we have $\gamma = 1 - w$, $f_\gamma(x) = -\ln(1 - \gamma e^x)$, $g_\gamma(x) = -\ln(1 - \gamma x)$. From the above recurrence, we then have

$$\begin{aligned} -\ln R^{(i,j)}_{G,u}(w) = &\sum_{v_k \in \overline{B(i)} \cap \overline{B(j)}} \left(f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right)\right) \\ &+ \sum_{v_k \in \overline{B(i)} \cap B(j)} f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - \sum_{v_k \in B(i) \cap \overline{B(j)}} f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) \\ &- \sum_{v_k \in \overline{B(i)} \cap B(j)} g_\gamma\left(b^{(j)}_{G_k,v_k}(w)\right) + \sum_{v_k \in B(i) \cap \overline{B(j)}} g_\gamma\left(b^{(i)}_{G_k,v_k}(w)\right) \\ &+ \sum_{v_k \in B(i) \cap B(j)} \left(g_\gamma\left(b^{(i)}_{G_k,v_k}(w)\right) - g_\gamma\left(b^{(j)}_{G_k,v_k}(w)\right)\right). \end{aligned} \tag{36}$$

Note that the same recurrence also applies when w is replaced by 0 (and hence γ by 1), except in that case the last three sums are 0 (as, when i is bad for v_k in G_k, we have $b^{(i)}_{G_k,v_k}(0) := \Pr_{G_k}\left[c(v_k) = i\right] = 0$):

$$\begin{aligned} -\ln R^{(i,j)}_{G,u}(0) = &\sum_{v_k \in \overline{B(i)} \cap \overline{B(j)}} \left(f_1\left(a^{(i)}_{G_k,v_k}(0)\right) - f_1\left(a^{(j)}_{G_k,v_k}(0)\right)\right) \\ &+ \sum_{v_k \in \overline{B(i)} \cap B(j)} f_1\left(a^{(i)}_{G_k,v_k}(0)\right) - \sum_{v_k \in B(i) \cap \overline{B(j)}} f_1\left(a^{(j)}_{G_k,v_k}(0)\right). \end{aligned} \tag{37}$$

Further, by Consequence 5.6 of the induction hypothesis applied to the graph G_k at a vertex $v_k \in B(i)$ (respectively, $v_k \in B(j)$) we see that $\left|g_\gamma\left(b^{(i)}_{G_k,v_k}(w)\right)\right| \leq 2\varepsilon_w$

(respectively, $\left|g_\gamma\left(b^{(j)}_{G_k,v_k}(w)\right)\right| \le 2\varepsilon_w$). Thus, applying the triangle inequality to the real part of the difference of the two recurrences, we get

$$\begin{aligned}
&\frac{1}{d}\left|\Re \ln R^{(i,j)}_{G,u}(0) - \ln R^{(i,j)}_{G,u}(w)\right| \le 2\Delta\varepsilon_w \\
&\quad + \max\Bigg\{ \max_{v_k \in \overline{B(i)}\cap\overline{B(j)}} \Big\{\Big|\left(\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_1\left(a^{(i)}_{G_k,v_k}(0)\right)\right) \\
&\qquad\qquad - \left(\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_1\left(a^{(j)}_{G_k,v_k}(0)\right)\right)\Big|\Big\}, \\
&\qquad \max_{v_k \in \overline{B(i)}\cap B(j)} \left\{\left|\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_1\left(a^{(i)}_{G_k,v_k}(0)\right)\right|\right\}, \\
&\qquad \max_{v_k \in \overline{B(j)}\cap B(i)} \left\{\left|\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_1\left(a^{(j)}_{G_k,v_k}(0)\right)\right|\right\}\Bigg\}. \qquad (38)
\end{aligned}$$

In what follows, we let v_k be the vertex that maximizes the above expression and d_k be the number of unpinned neighbors of v_k in G_k. Before proceeding with the analysis, we recall the observation above that the graphs G_k are unconflicted and satisfy the admissible list condition $\mathcal{L}$. Further, we note that v_k is (i) marked in G_k (this follows from the fact that $\mathcal{L}$ is hereditary); and (ii) nice in G_k (this last fact follows from Lemma 4.4 and the fact that G satisfies the admissible list condition $\mathcal{L}$). Thus, the preconditions of Consequence 5.5 apply to the vertex v_k in graph G_k. We now proceed with the analysis.

We first consider $v_k \in \overline{B(i)} \cap B(j)$. Note that this implies that $i \in \Gamma_{G_k,v_k}$. Thus, the conditions of Consequence 5.5 of the induction hypothesis instantiated on G_k apply to v_k with color i, and we thus have from eq. (28) that

$$\left|\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_1\left(a^{(i)}_{G_k,v_k}(0)\right)\right| \le \frac{1}{d_k+\eta}\left|\Re a^{(i)}_{G_k,v_k}(w) - a^{(i)}_{G_k,v_k}(0)\right| + \varepsilon_I + \varepsilon_w,$$

where d_k is the number of unpinned neighbors of v_k and $\eta \in [0.9, 1)$ is as in the statement of Consequence 5.5. Applying item 2 of Consequence 5.4 (which, again, is applicable because $i \in \Gamma_{G_k,v_k}$), we then have $\left|\Re a^{(i)}_{G_k,v_k}(w) - a^{(i)}_{G_k,v_k}(0)\right| \le d_k(\varepsilon_R + \varepsilon_I) + 2\Delta\varepsilon_w$, so that (recalling $\Delta \ge 3$ and $\eta \ge 0.9$, notably for the case $d_k = 0$)

$$\left|\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_1\left(a^{(i)}_{G_k,v_k}(0)\right)\right| \le \frac{d_k}{d_k+\eta}\varepsilon_R + 2\varepsilon_I + 3\Delta\varepsilon_w. \qquad (39)$$

By interchanging the roles of i and j in the above argument, we see that, for $v_k \in \overline{B(j)} \cap B(i)$,

$$\left|\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_1\left(a^{(j)}_{G_k,v_k}(0)\right)\right| \le \frac{d_k}{d_k+\eta}\varepsilon_R + 2\varepsilon_I + 3\Delta\varepsilon_w. \qquad (40)$$

We now consider $v_k \in \overline{B(i)} \cap \overline{B(j)}$. Note that both i and j are good for v_k in G_k, so that

$$\begin{aligned}
&\left|\left(\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_1\left(a^{(i)}_{G_k,v_k}(0)\right)\right) - \left(\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_1\left(a^{(j)}_{G_k,v_k}(0)\right)\right)\right| \\
&\le \max_{i',j' \in \Gamma_{G_k,v_k}} \left|\left(\Re f_\gamma\left(a^{(i')}_{G_k,v_k}(w)\right) - f_1\left(a^{(i')}_{G_k,v_k}(0)\right)\right) - \left(\Re f_\gamma\left(a^{(j')}_{G_k,v_k}(w)\right) - f_1\left(a^{(j')}_{G_k,v_k}(0)\right)\right)\right|.
\end{aligned}$$

Now, for any color $s \in \Gamma_{G_k,v_k}$, Consequence 5.5 of the induction hypothesis instantiated on G_k and applied to v_k and s shows that there exists a $C_s = C_{s,v_k,G_k} \in [0, 1/(d_k+\eta)]$ such that

$$\left|\Re f_\gamma\left(a^{(s)}_{G_k,v_k}(w)\right) - f_1\left(a^{(s)}_{G_k,v_k}(0)\right) - C_s\left(\Re a^{(s)}_{G_k,v_k}(w) - a^{(s)}_{G_k,v_k}(0)\right)\right| \le \varepsilon_I + \varepsilon_w. \tag{41}$$

Substituting this into the previous display shows that

$$\begin{aligned}
&\left|\left(\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_1\left(a^{(i)}_{G_k,v_k}(0)\right)\right) - \left(\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_1\left(a^{(j)}_{G_k,v_k}(0)\right)\right)\right| \\
&\quad\le \max_{i',j' \in \Gamma_{G_k,v_k}} \left|C_{i'}(\Re a^{(i')}_{G_k,v_k}(w) - a^{(i')}_{G_k,v_k}(0)) - C_{j'}(\Re a^{(j')}_{G_k,v_k}(w) - a^{(j')}_{G_k,v_k}(0))\right| + 2\varepsilon_I + 2\varepsilon_w \\
&\quad= 2\varepsilon_I + 2\varepsilon_w + \max_{i',j' \in \Gamma_{G_k,v_k}} |C_{i'}\Re\xi_{i'} - C_{j'}\Re\xi_{j'}| \\
&\quad= 2\varepsilon_I + 2\varepsilon_w + C_s\Re\xi_s - C_t\Re\xi_t,
\end{aligned} \tag{42}$$

where $\xi_l := a^{(l)}_{G_k,v_k}(w) - a^{(l)}_{G_k,v_k}(0)$ for $l \in \Gamma_{G_k,v_k}$, and s and t are given by

$$s := \arg\max_{i' \in \Gamma_{G_k,v_k}} C_{i'}\Re\xi_{i'} \quad \text{and} \quad t := \arg\min_{i' \in \Gamma_{G_k,v_k}} C_{i'}\Re\xi_{i'}.$$

We now have the following two cases.

Case 1: $(\Re\xi_s)\cdot(\Re\xi_t) \le 0$. Recall that C_s, C_t are nonnegative and lie in $[0, 1/(d_k + \eta)]$. Thus, in this case, we must have $\Re\xi_s \ge 0$ and $\Re\xi_t \le 0$, so that

$$C_s\Re\xi_s - C_t\Re\xi_t = C_s\Re\xi_s + C_t\left|\Re\xi_t\right| \le \frac{\Re\xi_s + |\Re\xi_t|}{d_k+\eta} = \frac{|\Re\xi_s - \Re\xi_t|}{d_k+\eta}. \tag{43}$$

Now, note that

$$\begin{aligned}
\Re\xi_s - \Re\xi_t &= \Re\ln\frac{\mathcal{P}_{G_k,w}\left[c(v_k)=s\right]}{\mathcal{P}_{G_k}\left[c(v_k)=s\right]} - \Re\ln\frac{\mathcal{P}_{G_k,w}\left[c(v_k)=t\right]}{\mathcal{P}_{G_k}\left[c(v_k)=t\right]} \\
&= \Re\ln\frac{\mathcal{P}_{G_k,w}\left[c(v_k)=s\right]}{\mathcal{P}_{G_k,w}\left[c(v_k)=t\right]} - \Re\ln\frac{\mathcal{P}_{G_k}\left[c(v_k)=s\right]}{\mathcal{P}_{G_k}\left[c(v_k)=t\right]} \\
&= \Re\ln R^{(s,t)}_{G_k,v_k}(w) - \ln R^{(s,t)}_{G_k,v_k}(0).
\end{aligned}$$

Note that all the logarithms in the above are well defined from Consequence 5.4 of the induction hypothesis applied to G_k and v_k (as $s, t \in \Gamma_{G_k,v_k}$). Further, from items 2 and 3 of the induction hypothesis, the last term is at most $d_k\varepsilon_R$ in absolute value. Substituting this into eq. (43), we get

$$C_s\Re\xi_s - C_t\Re\xi_t \le \frac{d_k}{d_k+\eta}\varepsilon_R. \tag{44}$$

This concludes the analysis of Case 1.

Case 2: *$\Re\xi_{i'}$ for $i' \in \Gamma_{G_k,v_k}$ all have the same sign.* Suppose first that $\Re\xi_{i'} \ge 0$ for all $i' \in \Gamma_{G_k,v_k}$. Then, we have

$$0 \le C_s\Re\xi_s - C_t\Re\xi_t \le \frac{\Re\xi_s}{d_k+\eta} \le \frac{d_k\cdot\varepsilon_R}{d_k+\eta} + \varepsilon_I + 4\Delta\varepsilon_w, \tag{45}$$

where the last inequality follows from item 2 of Consequence 5.4 of the induction hypothesis applied to G_k at vertex v_k with color s, which states that $|\Re\xi_s| \le d_k(\varepsilon_R + \varepsilon_I) + 4\Delta\varepsilon_w$. Similarly, when $\Re\xi_{i'} \le 0$ for all $i' \in \Gamma_{G_k,v_k}$, we have

$$\begin{aligned} 0 \le C_s\Re\xi_s - C_t\Re\xi_t &= C_t|\Re\xi_t| - C_s|\Re\xi_s| \\ &\le \frac{|\Re\xi_t|}{d_k+\eta} \\ &\le \frac{d_k\cdot\varepsilon_R}{d_k+\eta} + \varepsilon_I + 4\Delta\varepsilon_w, \end{aligned} \tag{46}$$

where the last inequality follows from item 2 of Consequence 5.4 of the induction hypothesis applied to G_k at vertex v_k with color t, which states that $|\Re\xi_t| \le d_k(\varepsilon_R + \varepsilon_I) + 4\Delta\varepsilon_w$. This concludes the analysis of Case 2.

Now, substituting eqs. (44) to (46) into eq. (42), we get

$$\begin{aligned} &\left|\left(\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_1\left(a^{(i)}_{G_k,v_k}(0)\right)\right) - \left(\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_1\left(a^{(j)}_{G_k,v_k}(0)\right)\right)\right| \\ &\qquad\qquad \le \frac{d_k}{d_k+\eta}\varepsilon_R + 3\varepsilon_I + 5\Delta\varepsilon_w. \end{aligned} \tag{47}$$

Substituting eqs. (39), (40), and (47) into eq. (38), we get

$$\frac{1}{d}\left|\Re\ln R^{(i,j)}_{G,u}(w) - \ln R^{(i,j)}_{G,u}(0)\right| \le \frac{d_k\cdot\varepsilon_R}{d_k+\eta} + 3\varepsilon_I + 7\Delta\varepsilon_w < \varepsilon_R,$$

where the last inequality follows since $\eta\varepsilon_R > (\Delta+1)(3\varepsilon_I + 7\Delta\varepsilon_w)$ (recalling that $0 \le d_k \le \Delta$ and $\eta \in [0.9, 1)$). This verifies item 3 of the induction hypothesis.

For item 4, we consider the imaginary part of eq. (36). As in the derivation of eq. (38), we use the fact that the induction hypothesis applied to the graph G_k at the vertex $v_k \in B(i)$ (respectively, $v_k \in B(j)$) implies that $\left|g_\gamma\left(b^{(i)}_{G_k,v_k}(w)\right)\right| \le 2\varepsilon_w$ (respectively, $\left|g_\gamma\left(b^{(j)}_{G_k,v_k}(w)\right)\right| \le 2\varepsilon_w$). This yields

$$\begin{aligned} \frac{1}{d}\left|\Im\ln R^{(i,j)}_{G,u}(w)\right| &\le 2\Delta\varepsilon_w \\ &+ \max\Bigg\{\max_{v_k\in\overline{B(i)}\cap\overline{B(j)}}\left|\Im f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - \Im f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right)\right|, \\ &\qquad \max_{v_k\in\overline{B(i)}\cap B(j)}\left|\Im f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right)\right|, \max_{v_k\in\overline{B(j)}\cap B(i)}\left|\Im f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right)\right|\Bigg\}. \end{aligned} \tag{48}$$

Again, let v_k be the vertex that maximizes the above expression and d_k be the number of unpinned neighbors of v_k in G_k. We first consider $v_k \in \overline{B(i)} \cap \overline{B(j)}$. Applying eq. (29) of Consequence 5.5 of the induction hypothesis to the graph G_k at vertex v_k with colors $i, j \in \Gamma_{G_k,v_k}$ gives

$$\left|\Im f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - \Im f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right)\right| \le \frac{d_k}{d_k+\eta}\varepsilon_I + 6\Delta\varepsilon_w. \tag{49}$$

Now consider $v_k \in \overline{B(i)} \cap B(j)$. For this case, eq. (30) of Consequence 5.5 of the induction hypothesis applied to G_k at vertex v_k with color $i \in \Gamma_{G_k,v_k}$ gives

$$\left|\Im f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right)\right| \le \frac{d_k}{d_k+\eta}\varepsilon_I + 5\Delta\varepsilon_w. \tag{50}$$

Similarly, for $v_k \in \overline{B(j)} \cap B(i)$, eq. (30) of Consequence 5.5 of the induction hypothesis applied to G_k at vertex v_k with color $j \in \Gamma_{G_k,v_k}$ gives

$$\left|\Im f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right)\right| \le \frac{d_k}{d_k+\eta}\varepsilon_I + 5\Delta\varepsilon_w. \tag{51}$$

Substituting eqs. (49) to (51) into eq. (48), we have

$$\frac{1}{d}\left|\Im \ln R^{(i,j)}_{G,u}(w)\right| \le \frac{d_k}{d_k+\eta}\varepsilon_I + 8\Delta\varepsilon_w < \varepsilon_I,$$

where the last inequality holds since $\eta\varepsilon_I > 8(\Delta+1)\Delta\varepsilon_w$ (recalling that $0 \le d_k \le \Delta$ and $\eta \in [0.9, 1)$). This completes the proof of item 4 of the induction hypothesis.

Finally, we prove item 5. Since $i \notin \Gamma_u$, there exist $n_i > 0$ neighbors of u that are pinned to color i. Let H be the graph obtained by removing these neighbors of u from G. Then, H is an unconflicted graph with the *same* number of unpinned vertices as G, which also satisfies the admissible list condition $\mathcal{L}$ (since $\mathcal{L}$ is hereditary). Further, u remains marked in H, and H further satisfies $i, j \in \Gamma_{H,u}$. We can therefore apply the already proved items 1 to 3 to H to conclude that

$$\left|R^{(i,j)}_H(w)\right| \le \left|R^{(i,j)}_H(0)\right| e^{d\varepsilon_R}. \tag{52}$$

Now, since $i, j \in \Gamma_{H,u}$, we can apply the recurrence of Lemma 3.4 in the same way as in the derivation of eq. (35) above to get

$$R^{(i,j)}_{H,u}(w) = \prod_{k=1}^{\deg_H(u)} \frac{1 - \mathcal{P}_{H_k^{(i,j)},w}\left[c(v_k)=i\right]}{1 - \mathcal{P}_{H_k^{(i,j)},w}\left[c(v_k)=j\right]}, \tag{53}$$

where, for the reasons described in the discussion following eq. (35), the product can be restricted to unpinned neighbors of u in H. Renaming these unpinned neighbors as $v_1, v_2, \ldots, v_d$, we then have

$$0 \le R^{(i,j)}_H(0) = \prod_{k=1}^{d} \frac{(1 - \mathcal{P}_{H_k}\left[c(v_k)=i\right])}{(1 - \mathcal{P}_{H_k}\left[c(v_k)=j\right])}, \tag{54}$$

where, as before, $H_k := H_k^{(i,j)}$. Now, as observed above, H satisfies the admissible list condition $\mathcal{L}$. Thus, for $1 \le k \le d$, v_k is nice in H_k (Lemma 4.4), and hence, $\mathcal{P}_{H_k}\left[c(v_k)=j\right] \le \frac{1}{d_k+2}$ for $1 \le k \le d$, where $d_k \ge 0$ is the number of unpinned neighbors of v_k in H_k. We then have

$$0 \le R^{(i,j)}_H(0) = \prod_{k=1}^{d} \frac{(1 - \mathcal{P}_{H_k}\left[c(v_k)=i\right])}{(1 - \mathcal{P}_{H_k}\left[c(v_k)=j\right])} \le \prod_{k=1}^{d} \frac{1}{1 - \frac{1}{d_k+2}} = \prod_{k=1}^{d} \frac{d_k+2}{d_k+1} \le 2^\Delta.$$

(As an aside, we note that one could get a better bound under the slightly stronger assumption of uniformly large list sizes considered in Remark 10. Under the conditions of that remark, we have $\mathcal{P}_{H_k}\left[c(v_k)=j\right] < \min\left\{\frac{4}{3\Delta}, 1\right\}$, so that the above upper bound can be improved to $R^{(i,j)}_H(0) \le e^4$ for $\Delta > 1$.)

Combining the estimate with eq. (52), we get $\left|R^{(i,j)}_H(w)\right| \le 5 \cdot 2^\Delta$ since $d\varepsilon_R \le 1/2$. Now note that since $j \in \Gamma_{G,u}$,

$$Z^{(i)}_{G,u}(w) = w^{n_i} Z^{(i)}_{H,u}(w), \quad \text{and} \quad Z^{(j)}_{G,u}(w) = Z^{(j)}_{H,u}(w),$$

so that $\left|R_{G,u}^{(i,j)}(w)\right| = |w|^{n_i}\left|R_{H,u}^{(i,j)}(w)\right| \leq 5 \cdot 2^{\Delta} \cdot |w|^{n_i}$. The latter is at most ε_w whenever $|w| \leq 0.2\varepsilon_w/2^{\Delta}$. This proves item 5 and also completes the inductive proof of Lemma 5.2. (Note also that using the stronger upper bound above under the condition of uniformly large list sizes, we can in fact relax the requirement further to $|w| \leq \varepsilon_w/(300\Delta)$.)

We conclude this section by using Lemma 5.2 to prove Theorem 5.1.

Proof of Theorem 5.1. Let G be a graph of maximum degree Δ satisfying the admissible list condition $\mathcal{L}$. Since G has no pinned vertices, G is unconflicted. Let u be an unpinned vertex that is marked in G. By Consequence 5.3 of the induction hypothesis (which we proved in Lemma 5.2), we then have $Z_w(G) \neq 0$ provided $\nu_w \leq 0.2\varepsilon_w/2^{\Delta}$.

Furthermore, as discussed above, under a slightly stronger assumption of uniformly large list sizes considered in Remark 10, ν_w can be chosen to be $\varepsilon_w/(300\Delta)$.∎

6. Zero-free region around the interval $(0, 1]$. In this section, we consider the case of w close to $[0, 1]$ but bounded away from 0. In particular, we prove the following theorem, which complements Theorem 5.1.

THEOREM 6.1. *Fix a positive integer Δ and an admissible list condition $\mathcal{L}$. Let $\nu_w = \nu_w(\Delta)$ be as in Theorem* 5.1. *Then, for any w satisfying*

$$\Re w \in [\nu_w/2, 1 + \nu_w^2/8] \qquad \text{and} \qquad |\Im w| \leq \nu_w^2/8, \tag{55}$$

and any graph G of maximum degree Δ which satisfies $\mathcal{L}$, we have $Z_G(w) \neq 0$.

(Here, we recall that, as described in the discussion following Theorem 5.1, ν_w can be chosen to be $\varepsilon_w/(300\Delta)$ when the uniformly large list size condition of Remark 10 is satisfied. However, as in that theorem, in the case of general list coloring, one chooses $\nu_w = 0.2\varepsilon_w/2^{\Delta}$.)

For w as in eq. (55), we define $\tilde{w}$ to be the point on the interval $[0, 1]$ which is closest to w. Thus

$$\tilde{w} := \begin{cases} \Re w & \text{when } \Re w \in [\nu_w/2, 1]; \\ 1 & \text{when } \Re w \in (1, 1 + \nu_w^2/8]. \end{cases}$$

We also define, in analogy with the last section, $\gamma := 1 - w$ and $\tilde{\gamma} := 1 - \tilde{w}$. We record a few properties of these quantities in the following observation.

OBSERVATION 6.2. *With $w, \gamma, \tilde{w}$, and $\tilde{\gamma}$ as above, we have*
1. $0 \leq \tilde{\gamma}, |\gamma| < 1$*;*
2. $|\ln w - \ln \tilde{w}| \leq \nu_w$.

Proof. We have $\tilde{\gamma} \in [0, 1 - \nu_w/2]$, $\Re\gamma \in [-\nu_w^2/8, 1 - \nu_w/2]$, and $|\Im\gamma| \leq \nu_w^2/8$. Since $\nu_w \leq 0.01$, these bounds taken together imply item 1. We also have $0 \leq \tilde{w} \leq |w| \leq \tilde{w} + \nu_w^2/4$ and $\tilde{w} \geq \nu_w/2$. Thus

$$0 \leq \Re(\ln w - \ln \tilde{w}) = \ln \frac{|w|}{\tilde{w}} \leq \ln\left(1 + \frac{\nu_w^2}{4\tilde{w}}\right) \leq \frac{\nu_w}{2}.$$

Similarly, $\Im(\ln w - \ln \tilde{w}) = \Im \ln w = \arg w$, so that

$$|\Im(\ln w - \ln \tilde{w})| \leq |\arg w| \leq \frac{|\Im w|}{\Re w} \leq \frac{\nu_w}{4}.$$

Together, the above two bounds imply item 2. ∎

In analogous fashion to the proof of Theorem 5.1, we would like to show that $R^{(i,j)}_{G,u}(w) \approx R^{(i,j)}_{G,u}(\tilde{w})$ independent of the size of G. (Note that for positive $\tilde{w}$, $R^{(i,j)}_{G,u}(\tilde{w})$ is a well-defined positive real number for any graph.) To this end, we will prove the following analogue of Lemma 5.2 via an induction on the number of unpinned vertices in G. The induction is very similar in structure to that used in the proof of Lemma 5.2, except that the fact that w has strictly positive real part allows us to simplify several aspects of the proof. In particular, we do not need to consider good and bad colors separately and do not require the underlying graphs to be unconflicted.

As in the previous section, we assume that all graphs in this section have maximum degree at most $\Delta \geq 1$, and we define the quantities $\varepsilon_w, \varepsilon_R, \varepsilon_I$ in terms of Δ using eq. (23). We again fix an admissible list condition $\mathcal{L}$ throughout this section.

LEMMA 6.3. *Let G be a graph of maximum degree Δ satisfying the admissible list condition $\mathcal{L}$, and let u be any marked unpinned vertex in G. Then, the following are true (here, $\varepsilon_w, \varepsilon_I, \varepsilon_R$ are as defined in eq. (23)):*

1. *For $i \in L(u)$, $\left|Z^{(i)}_{G,u}(w)\right| > 0$.*
2. *For $i, j \in L(u)$, if u has all neighbors pinned, then*

$$|\ln R^{(i,j)}_{G,u}(w) - \ln R^{(i,j)}_{G,u}(\tilde{w})| < \varepsilon_w.$$

3. *For $i, j \in L(u)$, if u has $d \geq 1$ unpinned neighbors, then*

$$\frac{1}{d}\left|\Re \ln R^{(i,j)}_{G,u}(w) - \Re \ln R^{(i,j)}_{G,u}(\tilde{w})\right| < \varepsilon_R.$$

4. *For $i, j \in L(u)$, if u has $d \geq 1$ unpinned neighbors, then*

$$\frac{1}{d}\left|\Im \ln R^{(i,j)}_{G,u}(w)\right| < \varepsilon_I.$$

We will refer to items 1 to 4 as "items of the induction hypothesis." The rest of this section is devoted to the proof of this lemma via an induction on the number of unpinned vertices in G.

We begin by verifying that the induction hypothesis holds in the base case when u is the only unpinned vertex in a graph G. In this case, items 3 and 4 are vacuously true since u has no unpinned neighbors. Since all neighbors of u in G are pinned, the fact that all pinned vertices have degree at most one implies that G can be decomposed into two disjoint components G_1 and G_2, where G_1 consists of u and its pinned neighbors, while G_2 consists of a disjoint union of edges with pinned endpoints. Let m be the number of conflicted edges on G_2, and let n_k denote the number of neighbors of u pinned to color k. We then have $Z^{(k)}_{G,u}(x) = x^{n_k} Z_{G_2}(x) = x^{n_k+m}$ for all $x \in \mathbb{C}$. This already proves item 1 since $w, \tilde{w} \neq 0$. Item 2 follows via the following computation (which uses item 2 of Observation 6.2):

$$|\ln R^{(i,j)}_{G,u}(w) - \ln R^{(i,j)}_{G,u}(\tilde{w})| = |n_i - n_j| \cdot |\ln w - \ln \tilde{w}| \leq \Delta \nu_w < \varepsilon_w.$$

We now derive some consequences of the above induction hypothesis that will be helpful in carrying out the induction. Throughout, we fix the graph G and the vertex u as in the statement of Lemma 6.3.

CONSEQUENCE 6.4. *If $|L(u)| \geq 1$, then $|Z_G(w)| > 0$.*

Proof. Note that $Z_G(w) = \sum_{i\in L(u)} Z^{(i)}_{G,u}(w)$. From item 4, we see that the angle between the complex numbers $Z^{(i)}_{G,u}(w)$ and $Z^{(j)}_{G,u}(w)$, for all $i, j \in L(u)$, is at most $d\varepsilon_I$. Applying Lemma 3.7, we then have

$$\Big|\sum_{i\in L(u)} Z^{(i)}_{G,u}(w)\Big| \geq |L(u)| \cos\frac{d\varepsilon_I}{2} \cdot \min_{i\in\Gamma_u} \Big|Z^{(i)}_{G,u}(w)\Big| \geq 0.9 \min_{i\in\Gamma_u} \Big|Z^{(i)}_{G,u}(w)\Big|$$

when $|L(u)| \geq 1$ and $d\varepsilon_I \leq 0.01$. This last quantity is positive from item 1. ∎

CONSEQUENCE 6.5. *For all $\varepsilon_R, \varepsilon_I, \varepsilon_w$ small enough such that $\varepsilon_I \leq \varepsilon_R$ and $\varepsilon_w \leq 0.01\varepsilon_I$, the pseudo probabilities approximate the real probabilities in the following sense: for any $j \in L(u)$,*

$$\left|\Im \ln \frac{\mathcal{P}_{G,w}[c(u)=j]}{\mathcal{P}_{G,\tilde{w}}[c(u)=j]}\right| = |\Im \ln \mathcal{P}_{G,w}[c(u)=j]| \leq d\varepsilon_I + 2\Delta\varepsilon_w;$$
$$\left|\Re \ln \frac{\mathcal{P}_{G,w}[c(u)=j]}{\mathcal{P}_{G,\tilde{w}}[c(u)=j]}\right| \leq d\varepsilon_R + d\varepsilon_I + 2\Delta\varepsilon_w,$$

where d is the number of unpinned neighbors of u in G.

Proof. Using items 2 to 4 of the induction hypothesis, there exist complex numbers ξ_i (for all $i \in \Gamma_u$) satisfying $|\Re\xi_i| \leq d\varepsilon_R + \varepsilon_w$ and $|\Im\xi_i| \leq d\varepsilon_I + \varepsilon_w$ such that

(56)
$$\frac{\mathcal{P}_{G,\tilde{w}}[c(u)=j]}{\mathcal{P}_{G,w}[c(u)=j]} = \mathcal{P}_{G,\tilde{w}}[c(u)=j] \sum_{i\in L(u)} \frac{Z^{(i)}_{G,u}(w)}{Z^{(j)}_{G,u}(w)} = \mathcal{P}_{G,\tilde{w}}[c(u)=j] \sum_{i\in L(u)} \frac{Z^{(i)}_{G,u}(\tilde{w})}{Z^{(j)}_{G,u}(\tilde{w})} e^{\xi_i}.$$

Now, note that $\sum_{i\in L(u)} \frac{Z^{(i)}_{G,u}(\tilde{w})}{Z^{(j)}_{G,u}(\tilde{w})} = \frac{1}{\mathcal{P}_{G,\tilde{w}}[c(u)=j]}$, so that the sum above is a convex combination of the $\exp(\xi_i)$. From the bounds on the real and imaginary parts of the ξ_i quoted above, by a calculation similar to that in eq. (25), we also have (when $\varepsilon_I, \varepsilon_w \leq 0.01/\Delta$)

$$\Re e^{\xi_i} \in (e^{-d\varepsilon_R - \varepsilon_w} - (d\varepsilon_I + \varepsilon_w)^2,\ e^{d\varepsilon_R + \varepsilon_w}), \text{ and } |\arg e^{\xi_i}| \leq d\varepsilon_I + \varepsilon_w.$$

The above will therefore be true also for any convex combination of the e^{ξ_i}, in particular the one in eq. (56). We therefore have, for $C := \frac{\mathcal{P}_{G,\tilde{w}}[c(u)=j]}{\mathcal{P}_{G,w}[c(u)=j]}$,

$$\Re C \in \left(e^{-d\varepsilon_R - \varepsilon_w} - (d\varepsilon_I + \varepsilon_w)^2,\ e^{d\varepsilon_R + \varepsilon_w}\right),$$
$$|\arg C| \leq d\varepsilon_I + \varepsilon_w.$$

Now recall that for $|\theta| \leq \pi/4$, we have $-\theta^2 \leq \ln\cos\theta \leq -\theta^2/2$. Thus, using the values of ε_w, ε_I, and ε_R, we have

$$|\Re \ln C| \leq d\varepsilon_R + d\varepsilon_I + 2\Delta\varepsilon_w, \text{ and}$$
$$|\Im \ln C| \leq d\varepsilon_I + \varepsilon_w.$$ ∎

As before, we define $a^{(i)}_{G,u}(w) = \ln \mathcal{P}_{G,w}[c(u)=i]$ and recall the definition of the function $f_\gamma(x) := -\ln(1-\gamma e^x)$.

CONSEQUENCE 6.6. *There exists a positive constant* $\eta \in [0.9, 1)$ *so that the following is true. Let d be the number of unpinned neighbors of u. Assume further that the vertex u is nice in G. Then, for any colors* $i, j \in L(u)$*, there exists a real number* $C = C_{G,u,i} \in [0, \frac{1}{d+\eta}]$ *such that*

$$\left| \Re f_{\gamma}(a_{G,u}^{(i)}(w)) - f_{\tilde{\gamma}}(a_{G,u}^{(i)}(\tilde{w})) - C \cdot \Re\left(a_{G,u}^{(i)}(w) - a_{G,u}^{(i)}(\tilde{w})\right) \right| \leq \varepsilon_I + \varepsilon_w; \tag{57}$$

$$\left| \Im f_{\gamma}(a_{G,u}^{(i)}(w)) - \Im f_{\gamma}(a_{G,u}^{(j)}(w)) \right| \leq \frac{1}{d+\eta} \cdot (d\varepsilon_I + 4\Delta\varepsilon_w) + 2\varepsilon_w. \tag{58}$$

Proof. Since u is nice in G, the bound $\mathcal{P}_{G,\tilde{w}}\left[c(u) = k\right] \leq \frac{1}{d+2}$ (for any $k \in L(u)$) applies. Combining them with Consequence 6.5, we see that $a_{G,u}^{(i)}(w), a_{G,u}^{(i)}(\tilde{w}), a_{G,u}^{(j)}(w)$, $a_{G,u}^{(j)}(\tilde{w})$ lie in a domain D as described in Lemma 3.6, with the parameters ζ and τ in that lemma chosen as

$$\zeta = \ln(d+2) - d\varepsilon_R - d\varepsilon_I - 2\Delta\varepsilon_w \text{ , and}$$
$$\tau = d\varepsilon_I + 2\Delta\varepsilon_w.$$

Here, for the bound on ζ, we use the fact that for $k \in L(u)$, $\mathcal{P}_{G,\tilde{w}}\left[c(u) = k\right] \leq \frac{1}{d+2}$, since u is nice in G. As in the proof of Consequence 5.5, we use the values of $\varepsilon_w, \varepsilon_I, \varepsilon_R$ to verify that the condition $\tau < 1/2$ and $\tau^2 + e^{-\zeta} < 1$ are satisfied, so that item 1 of Lemma 3.6 applies (with the parameter κ therein set to $\tilde{\gamma}$) and further that ρ_R and ρ_I as set there satisfy $\rho_R \leq \frac{1}{d+\eta}$ and $\rho_I < 3\varepsilon_I$, with $\eta = 0.94$. Using Lemma 3.5 followed by the bound on ε_w, we then have

$$\begin{aligned} &\left| \Re f_{\tilde{\gamma}}(a_{G,u}^{(i)}(w)) - f_{\tilde{\gamma}}(a_{G,u}^{(i)}(\tilde{w})) - C \cdot \Re\left(a_{G,u}^{(i)}(w) - a_{G,u}^{(i)}(\tilde{w})\right) \right| \leq 3\varepsilon_I(d\varepsilon_I + 2\Delta\varepsilon_w) \\ &\quad \leq 4d\varepsilon_I^2 \leq \varepsilon_I, \end{aligned} \tag{59}$$

for an appropriate nonnegative $C \leq 1/(d+\eta)$. This is almost eq. (57), whose difference will be handled later.

Similarly, applying Lemma 3.5 to the imaginary part, we have

$$\begin{aligned} &\left| \Im\Big(f_{\tilde{\gamma}}(a_{G,u}^{(i)}(w)) - f_{\tilde{\gamma}}(a_{G,u}^{(j)}(w)) \Big) \right| \\ &\qquad \leq \rho_R \cdot \max\left\{ \left| \Im\left(a_{G,u}^{(i)}(w) - a_{G,u}^{(j)}(w)\right) \right|,\ \left| \Im a_{G,u}^{(i)}(w) \right|, \left| \Im a_{G,u}^{(j)}(w) \right| \right\}, \end{aligned} \tag{60}$$

where, as noted above, $\rho_R \leq \frac{1}{d+\eta}$. Now, note that the first term in the above maximum is less than $d\varepsilon_I + \varepsilon_w$ by items 2 and 4 of the induction hypothesis, while the other two are at most $d\varepsilon_I + 2\Delta\varepsilon_w$ from item 2 of Consequence 6.5.

Finally, we use item 2 of Lemma 3.6 with the parameter κ' therein set to γ. To this end, we note that $|\gamma - \tilde{\gamma}| \leq \varepsilon_w$, and that with the fixed values of ε_w, ε_R, and ε_I, the condition $(1 + \varepsilon_w) < e^{\zeta}$ is satisfied, so that the item applies. Using the item, we then see that for any $z \in D$,

$$|f_{\gamma}(z) - f_{\tilde{\gamma}}(z)| \leq \varepsilon_w.$$

Thus, the quantities $|\Re f_{\gamma}(a_{G,u}^{(i)}(w)) - \Re f_{\tilde{\gamma}}(a_{G,u}^{(i)}(w))|$, $|\Im f_{\gamma}(a_{G,u}^{(i)}(w)) - \Im f_{\tilde{\gamma}}(a_{G,u}^{(i)}(w))|$, $|\Im f_{\gamma}(a_{G,u}^{(j)}(w)) - \Im f_{\tilde{\gamma}}(a_{G,u}^{(j)}(w))|$, and $|\Im f_{\gamma}(a_{G,u}^{(j)}(w)) - \Im f_{\tilde{\gamma}}(a_{G,u}^{(j)}(w))|$ are all at most ε_w. The desired bounds now follow from the triangle inequality and the bounds in eqs. (59) and (60). ∎

Inductive proof of Lemma 6.3. We are now ready to see the inductive proof of Lemma 6.3; recall that the base case was already established immediately following the statement of the lemma. Let G be any graph which satisfies the admissible list condition $\mathcal{L}$ and has at least two unpinned vertices. We first prove induction item 1 for any marked unpinned vertex u in G. Consider the graph G' obtained from G by pinning vertex u to color i. Note that by the definition of the pinning operation, $Z^i_{G,u}(w) = Z_{G'}(w)$. Further, since $\mathcal{L}$ is hereditary (because it is admissible), the graph G' also satisfies $\mathcal{L}$ and has one fewer unpinned vertex than G. Thus, from Consequence 6.4 of the induction hypothesis applied to G', we have that $\left|Z^{(i)}_{G,u}(w)\right| = |Z_{G'}(w)| > 0$.

We now consider item 2. When all neighbors of u in G are pinned, the fact that all pinned vertices have degree at most one implies that G can be decomposed into two disjoint components G_1 and G_2, where G_1 consists of u and its pinned neighbors, while G_2 has one fewer unpinned vertex than G. Note also that G_2, being a connected component of G, also satisfies the admissible list condition $\mathcal{L}$ (since $\mathcal{L}$ is hereditary). Thus, from Consequence 6.4 of the induction hypothesis applied to G_2, we have that $Z_{G_2}(w)$ and $Z_{G_2}(\tilde{w})$ are both nonzero. Let n_k be the number of neighbors of u pinned to color k. Now, since G_1 and G_2 are disjoint components, we get $Z^{(k)}_{G,u}(x) = x^{n_k} Z_{G_2}(x)$ for all $k \in L(u)$ and all $x \in \mathbb{C}$. It therefore follows that

$$|\ln R^{(i,j)}_{G,u}(w) - \ln R^{(i,j)}_{G,u}(\tilde{w})| = |n_i - n_j| \cdot |\ln w - \ln \tilde{w}| \le \Delta\nu_w < \varepsilon_w.$$

We now consider items 3 and 4. Recall that by Lemma 3.4, we have

$$R^{(i,j)}_{G,u}(w) = \prod_{k=1}^{\deg_G(u)} \frac{\left(1 - \gamma \mathcal{P}_{G^{(i,j)}_k, w}\left[c(v_k) = i\right]\right)}{\left(1 - \gamma \mathcal{P}_{G^{(i,j)}_k, w}\left[c(v_k) = j\right]\right)}.$$

Without loss of generality, we relabel the unpinned neighbors of u as $v_1, v_2, \dots, v_d$. As before, for simplicity we write $G_k := G^{(i,j)}_k$. Note that each G_k has exactly one fewer unpinned vertex than G and satisfies $\mathcal{L}$ (since $\mathcal{L}$ is hereditary). Further, the vertex v_k is marked in G_k (as u was marked in G). Thus, the induction hypothesis applies to each G_k at the vertex v_k. Now, let n_k be the number of neighbors of u pinned to color k. Recalling that $1 - \gamma = w$, we can then simplify the above recurrence to

$$R^{(i,j)}_{G,u}(w) = w^{n_i - n_j} \prod_{k=1}^{d} \frac{\left(1 - \gamma \mathcal{P}_{G^{(i,j)}_k, w}\left[c(v_k) = i\right]\right)}{\left(1 - \gamma \mathcal{P}_{G^{(i,j)}_k, w}\left[c(v_k) = j\right]\right)}.$$

Now, as before, for $s \in L(v_k)$ we define $a^{(s)}_{G_k, v_k}(w) := \ln \mathcal{P}_{G_k, w}\left[c(v_k) = s\right]$. From the above recurrence, we then have

$$-\ln R^{(i,j)}_{G,u}(w) = (n_i - n_j)\ln w + \sum_{k=1}^{d} \left(f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right)\right). \tag{61}$$

Note that the same recurrence also applies when w is replaced by $\tilde{w}$ (and hence γ by $\tilde{\gamma}$):

$$-\ln R^{(i,j)}_{G,u}(\tilde{w}) = (n_i - n_j)\ln \tilde{w} + \sum_{k=1}^{d} \left(f_{\tilde{\gamma}}\left(a^{(i)}_{G_k,v_k}(\tilde{w})\right) - f_{\tilde{\gamma}}\left(a^{(j)}_{G_k,v_k}(\tilde{w})\right)\right).$$

(Recall that since $\Re w, \tilde{w} > 0$, $\ln w$ and $\ln \tilde{w}$ are well defined.)

Using item 2 of Observation 6.2, $|n_i - n_j| \leq \Delta$, and the fact that $\Delta \nu_w \leq \varepsilon_w$, we have

$$|n_i - n_j| \cdot |\ln w - \ln \tilde{w}| \leq \varepsilon_w.$$

Applying the triangle inequality to the real part of the difference of the two recurrences, we therefore get

$$\begin{aligned} &\frac{1}{d}\left|\Re \ln R^{(i,j)}_{G,u}(w) - \ln R^{(i,j)}_{G,u}(\tilde{w})\right| \\ &\leq \varepsilon_w + \max_{1\leq k\leq d}\left\{\left|\left(\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(i)}_{G_k,v_k}(\tilde{w})\right)\right) - \left(\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(j)}_{G_k,v_k}(\tilde{w})\right)\right)\right|\right\}. \end{aligned} \tag{62}$$

In what follows, we let v_k be the vertex that maximizes the above expression and d_k be the number of unpinned neighbors of v_k in G_k. Before proceeding with the analysis, we recall the observation above that the graphs G_k satisfy the admissible list condition $\mathcal{L}$. Further, we note that v_k is (i) marked in G_k (this follows from the fact that $\mathcal{L}$ is hereditary) and (ii) nice in G_k (this last fact follows from Lemma 4.4 and the fact that G satisfies the admissible list condition $\mathcal{L}$). Thus, the preconditions of Consequence 6.6 apply to the vertex v_k in graph G_k. We now proceed with the analysis.

We begin by noting that

$$\begin{aligned} &\left|\left(\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(i)}_{G_k,v_k}(\tilde{w})\right)\right) - \left(\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(j)}_{G_k,v_k}(\tilde{w})\right)\right)\right| \\ &\leq \max_{i',j'\in L(v_k)}\left|\left(\Re f_\gamma\left(a^{(i')}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(i')}_{G_k,v_k}(\tilde{w})\right)\right) - \left(\Re f_\gamma\left(a^{(j')}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(j')}_{G_k,v_k}(\tilde{w})\right)\right)\right|. \end{aligned}$$

On the other hand, for any color $s \in L(v_k)$, Consequence 6.6 of the induction hypothesis instantiated on G_k and applied to v_k and s shows that there exists a $C_s = C_{s,v_k,G_k} \in [0, 1/(d_k + \eta)]$ such that

$$\left|\Re f_\gamma\left(a^{(s)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(s)}_{G_k,v_k}(\tilde{w})\right) - C_s(\Re a^{(s)}_{G_k,v_k}(w) - a^{(s)}_{G_k,v_k}(\tilde{w}))\right| \leq \varepsilon_I + \varepsilon_w.$$

Substituting this into the previous display shows that

$$\begin{aligned} &\left|\left(\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(i)}_{G_k,v_k}(\tilde{w})\right)\right) - \left(\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(j)}_{G_k,v_k}(\tilde{w})\right)\right)\right| \\ &\leq \max_{i',j'\in L(v_k)}\left|C_{i'}(\Re a^{(i')}_{G_k,v_k}(w) - a^{(i')}_{G_k,v_k}(\tilde{w})) - C_{j'}(\Re a^{(j')}_{G_k,v_k}(w) - a^{(j')}_{G_k,v_k}(\tilde{w}))\right| + 2\varepsilon_I + 2\varepsilon_w \\ &= 2\varepsilon_I + 2\varepsilon_w + \max_{i',j'\in L(v_k)}|C_{i'}\Re\xi_{i'} - C_{j'}\Re\xi_{j'}| \\ &= 2\varepsilon_I + 2\varepsilon_w + C_s\Re\xi_s - C_t\Re\xi_t, \end{aligned} \tag{63}$$

where $\xi_l := a^{(l)}_{G_k,v_k}(w) - a^{(l)}_{G_k,v_k}(\tilde{w})$ for $l \in \Gamma_{G_k,v_k}$, and s and t are given by

$$s := \arg\max_{i'\in L(v_k)} C_{i'}\Re\xi_{i'} \quad \text{and} \quad t := \arg\min_{i'\in L(v_k)} C_{i'}\Re\xi_{i'}.$$

We now have the following two cases.

Case 1: $(\Re\xi_s)\cdot(\Re\xi_t) \leq 0$. Recall that C_s, C_t are nonnegative and lie in $[0, 1/(d_k + \eta)]$. Thus, in this case, we must have $\Re\xi_s \geq 0$ and $\Re\xi_t \leq 0$, so that

$$C_s\Re\xi_s - C_t\Re\xi_t = C_s\Re\xi_s + C_t|\Re\xi_t| \leq \frac{\Re\xi_s + |\Re\xi_t|}{d_k + \eta} = \frac{|\Re\xi_s - \Re\xi_t|}{d_k + \eta}. \tag{64}$$

Now, note that

$$\begin{aligned}\Re\xi_s - \Re\xi_t &= \Re\ln\frac{\mathcal{P}_{G_k,w}\left[c(v_k)=s\right]}{\mathcal{P}_{G_k,\tilde{w}}\left[c(v_k)=s\right]} - \Re\ln\frac{\mathcal{P}_{G_k,w}\left[c(v_k)=t\right]}{\mathcal{P}_{G_k,\tilde{w}}\left[c(v_k)=t\right]}\\ &= \Re\ln\frac{\mathcal{P}_{G_k,w}\left[c(v_k)=s\right]}{\mathcal{P}_{G_k,w}\left[c(v_k)=t\right]} - \Re\ln\frac{\mathcal{P}_{G_k,\tilde{w}}\left[c(v_k)=s\right]}{\mathcal{P}_{G_k,\tilde{w}}\left[c(v_k)=t\right]}\\ &= \Re\ln R^{(s,t)}_{G_k,v_k}(w) - \ln R^{(s,t)}_{G_k,v_k}(\tilde{w}).\end{aligned}$$

Note that all the logarithms in the above are well defined from Consequence 6.5 of the induction hypothesis applied to G_k and v_k. Further, from items 2 and 3 of the induction hypothesis, the last term is at most $d_k\varepsilon_R+\varepsilon_w$ in absolute value. Substituting this into eq. (64), we get

$$C_s\Re\xi_s - C_t\Re\xi_t \le \frac{d_k}{d_k+\eta}\varepsilon_R + \varepsilon_w. \tag{65}$$

This concludes the analysis of Case 1.

Case 2: $\Re\xi_{i'}$ *for* $i' \in L(v_k)$ *all have the same sign.* Suppose first that $\Re\xi_{i'} \ge 0$ for all $i' \in L(v_k)$. Then, we have

$$0 \le C_s\Re\xi_s - C_t\Re\xi_t \le \frac{\Re\xi_s}{d_k+\eta} \le \frac{d_k\cdot\varepsilon_R}{d_k+\eta} + \varepsilon_I + 4\Delta\varepsilon_w, \tag{66}$$

where the last inequality follows from the second inequality in Consequence 6.5 of the induction hypothesis applied to G_k at vertex v_k with color s, which states that $|\Re\xi_s| \le d_k(\varepsilon_R+\varepsilon_I)+4\Delta\varepsilon_w$. Similarly, when $\Re\xi_{i'} \le 0$ for all $i' \in \Gamma_{G_k,v_k}$, we have

$$\begin{aligned}0 \le C_s\Re\xi_s - C_t\Re\xi_t &= C_t|\Re\xi_t| - C_s|\Re\xi_s|\\ &\le \frac{1}{d_k+\eta}\left|\Re\xi_t\right| \le \frac{d_k}{d_k+\eta}\varepsilon_R + \varepsilon_I + 4\Delta\varepsilon_w,\end{aligned} \tag{67}$$

where the last inequality follows from the second inequality in Consequence 6.5 of the induction hypothesis applied to G_k at vertex v_k with color t, which states that $|\Re\xi_t| \le d_k(\varepsilon_R+\varepsilon_I)+4\Delta\varepsilon_w$. This concludes the analysis of Case 2.

Now, substituting eqs. (65) to (67) into eq. (63), we get

$$\begin{aligned}&\left|\left(\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(i)}_{G_k,v_k}(\tilde{w})\right)\right) - \left(\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(j)}_{G_k,v_k}(\tilde{w})\right)\right)\right|\\ &\qquad\le \frac{d_k}{d_k+\eta}\varepsilon_R + 3\varepsilon_I + 5\Delta\varepsilon_w.\end{aligned} \tag{68}$$

Substituting eq. (68) into eq. (62), we get

$$\frac{1}{d}\left|\Re\ln R^{(i,j)}_{G,u}(w) - \ln R^{(i,j)}_{G,u}(\tilde{w})\right| \le \frac{d_k}{d_k+\eta}\varepsilon_R + 3\varepsilon_I + 7\Delta\varepsilon_w < \varepsilon_R, \tag{69}$$

where the last inequality holds since $\eta\varepsilon_R > (\Delta+1)(3\varepsilon_I+7\Delta\varepsilon_w)$ (recalling that $0 \le d_k \le \Delta$ and $\eta \in [0.9,1)$). This verifies item 3 of the induction hypothesis.

Finally, to prove item 4, we consider the imaginary part of eq. (61). We first note that

$$|n_i - n_j| \cdot |\Im \ln w| \le \Delta \left|\ln w - \ln \tilde{w}\right| \le \Delta \nu_w \le \varepsilon_w.$$

We then have

$$\frac{1}{d}\left|\Im \ln R_{G,u}^{(i,j)}(w)\right| \le \varepsilon_w + \max_{1\le k\le d}\left|\Im f_\gamma\left(a_{G_k,v_k}^{(i)}(w)\right) - \Im f_\gamma\left(a_{G_k,v_k}^{(j)}(w)\right)\right|. \tag{70}$$

Again, let v_k be the vertex that maximizes the above expression and d_k be the number of unpinned neighbors of v_k in G_k. Applying eq. (58) of Consequence 6.6 of the induction hypothesis to the graph G_k at vertex v_k with colors $i, j \in L(v_k)$ gives

$$\left|\Im f_\gamma\left(a_{G_k,v_k}^{(i)}(w)\right) - \Im f_\gamma\left(a_{G_k,v_k}^{(j)}(w)\right)\right| \le \frac{d_k}{d_k+\eta}\varepsilon_I + 6\Delta\varepsilon_w. \tag{71}$$

Substituting eq. (71) into eq. (70), we then have

$$\frac{1}{d}\left|\Im \ln R_{G,u}^{(i,j)}(w)\right| \le \frac{d_k}{d_k+\eta}\varepsilon_I + 8\Delta\varepsilon_w < \varepsilon_I,$$

where the last inequality holds since $\eta\varepsilon_I > 8(\Delta+1)\Delta\varepsilon_w$ (recalling that $0 \le d_k \le \Delta$ and $\eta \in [0.9, 1)$). This proves item 4 and also completes the inductive proof of Lemma 6.3.

We now use Lemma 6.3 to prove Theorem 6.1.

Proof of Theorem 6.1. Let G be any graph of maximum degree Δ satisfying the admissible list condition $\mathcal{L}$. If G has no unpinned vertices, then $Z_G(w) = 1$, and there is nothing to prove. Otherwise, let u be an unpinned vertex that is marked in G. By Consequence 6.4 of the induction hypothesis (which we proved in Lemma 6.3), we then have $Z_w(G) \neq 0$ for w as in the statement of the theorem. ∎

The proof of Theorem 1.5 is now immediate.

Proof of Theorem 1.5. Let the quantity $\nu_w = \nu_w(\Delta)$ be as in the statements of Theorems 5.1 and 6.1. Fix the maximum degree Δ, and suppose that w satisfies

$$-\nu_w^2/8 \le \Re w \le 1 + \nu_w^2/8 \text{ and } |\Im w| \le \nu_w^2/8. \tag{72}$$

Now, if G satisfies the hypotheses of Theorem 1.2 (respectively, Theorem 1.3), we mark all its vertices so that the resulting instance satisfies Condition A (respectively, Condition B); whereas if G is a tree satisfying the hypotheses of Proposition 1.4, we root G at an arbitrary vertex and mark the root, so that the resulting instance satisfies Condition C.

By Lemma 4.4, the list coloring instance for G so generated then satisfies an admissible list condition. When w satisfying eq. (72) is such that $\Re w \le \nu_w/2$, we have $|w| \le \nu_w$, so that $Z_G(w) \neq 0$ by Theorem 5.1, while when such a w satisfies $\Re w \ge \nu_w/2$, we have $Z_G(w) \neq 0$ from Theorem 6.1. It therefore follows that $Z_G(w) \neq 0$ for all w satisfying eq. (72), and thus the quantity τ_Δ in the statement of Theorem 1.5 can be taken to be $\nu_w^2/8$. ∎

We conclude with a brief discussion of the dependence of τ_Δ on Δ. We saw above that τ_Δ can be taken to be $\nu_w(\Delta)^2/8$, so it is sufficient to consider the dependence of $\nu_w = \nu_w(\Delta)$ on Δ. Let $c = 10^{-6}$. As stated in the discussion following eq. (23), ν_w

can be chosen to be $0.2c/(2^{\Delta}\Delta^7)$ for the case of general list colorings, or $c/(300\Delta^8)$ with the assumption of uniformly large list sizes (which, we recall from Remark 10, is satisfied in the case of uniform q-colorings). We have not tried to optimize these bounds, and it is conceivable that a more careful accounting of constants in our proofs can improve the value of the constant c by a few orders of magnitude.

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