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Real Options Problem with Nonsmooth Obstacle*

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Abstract. We consider a real options problem, which is posed as a stochastic optimal control problem. The investment strategy, which plays the role of control, involves a one-time option to expand (invest) and a one-time option to abandon (terminate) the project. The timing and amount of the investment and the termination time are parameters to be optimized in order to maximize the expected value of the profit. This stochastic optimization problem amounts to solving a deterministic variational inequality in dimension one, with the associated obstacle problem. Because we consider both cessation and expansion options and fixed and variable costs of expansion, the obstacle is nonsmooth. Due to the lack of smoothness, we use the concept of a weak solution. However, such solutions may not lead to a straightforward investment strategy. Therefore, we further consider strong (C^1) solutions based on thresholds. We propose sufficient conditions for the existence of such solutions to the variational inequality with a nonsmooth obstacle in dimension one. When applied to the real options problem, these sufficient conditions yield a simple optimal investment strategy with the stopping times defined in terms of the thresholds.

Key words. stochastic optimal control, variational inequality, weak formulation, strong two-threshold solution, real options

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1. Introduction.

Stochastic control and optimization problems arise in various applications in finance. They are mainly related to investment problems where the control is usually given by an investment strategy [5]. A choice made available to the managers of a company concerning business investment opportunities is called a real option. In particular, common types of real options include the decision to expand, wait, or abandon a project [10, 18].

Real options in real estate investment would be an example worth considering. In particular, expansion options study the timing and the scale of a new construction development. By contrast, abandonment (exit) options focus on the optimal timing to close activities by selling the properties entirely. The cost of expansion can be divided into two parts: a variable cost and a fixed cost. The variable cost is a function of the capital outlay (e.g., the number of floors in the building). The fixed cost is independent of the capital outlay; it can include legal fees and construction permits.

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In this work, we study a real options problem with investment flexibility in time and capacity. The problem is formalized following the impulse control methodology [14], in which the optimal exit and expansion strategy is formulated in terms of the value function defined as the expected value of the future cumulative profit. Mathematically, the problem amounts to solving a variational inequality (V.I.) with a *nonsmooth* obstacle in dimension one.

It is now standard in investment theory and growth of firms theory that V.I.s obtained via the dynamic programming method [7] provide a methodology for solving the problem of finding the right time to invest, called a stopping time. The solution of the V.I. (or one solution if there is no uniqueness) gives the value function. One expects optimal stopping times to be expressed through thresholds, i.e., one stops when the state variable attains a threshold. When the obstacle is smooth (at least C^1), this is easily obtained. In our case, because we have two stopping times (investment and cessation of activities) and fixed and variable costs for the size of expansion, the obstacle cannot be C^1. It is C^0, and its derivative has the left and right limits. The value function may have just the same regularity as the obstacle, but it can also have a C^1 regularity. Because of this potential lack of smoothness of the value function, the rigorous theory of V.I. uses the concept of weak solutions. In that case, the optimal stopping can only be characterized as a time when the value function and the obstacle coincide. This characterization may not lead to a single threshold.

In what follows, we present a rigorous theory of the existence and uniqueness of a weak solution to the V.I. with a nonsmooth obstacle in dimension one. We further provide sufficient conditions for the solution to be C^1. In that case, the V.I. can be written in the standard form, called the strong formulation. We propose sufficient conditions for the existence of a strong two-threshold solution, which coincides with the obstacle below the lower threshold and above the upper threshold. These conditions simultaneously suggest a method for obtaining the solution numerically. An important condition postulates that the derivative of the obstacle has a positive increment at each discontinuity point, i.e., an assumption, which is naturally satisfied for the real options problem. In the context of this problem, the thresholds provide the stopping times for exercising the cessation option and the investment option of the optimal investment strategy. For the most general theory of weak solutions in any dimension we refer the reader to [7].

A rather general framework for solving optimal stopping and impulse control problems with a nonsmooth obstacle for Itô diffusion processes was developed in [1, 2, 3]. This approach uses the theory of r-excessive and r-harmonic mappings [9, 13] and involves showing the convexity of the minimal r-excessive mappings (fundamental solutions) for the diffusion process. In [1], an explicit solution of the optimal stopping problem with a terminal payoff is obtained. It is shown that the value function increases and the continuation region expands with higher volatility. These results are extended to impulse control problems with a controlled diffusion in [2, 3], where the associated optimal stopping problem can have a nonincreasing running payoff $\pi \in L_1(\mathbb{R}_+)$ in addition to the terminal payoff. In contrast to these results, in the present work a typical running payoff increases with the price and is unbounded, which is a natural setting for investment decision problems. The fundamental solutions for this paper are explicit convex power functions because we assume the geometric Brownian motion model for the price, but the obstacle is not necessarily convex or concave.

Moreover, our findings are also broadly related to the growing literature applying stochastic control to a firm's optimal decisions. Explicit closed form solutions for investment models with entry and exit decisions and investment decisions incurring a fixed cost for opening and closing a project are obtained in [11, 12, 19, 20] using the method of quasivariational inequalities (Q.V.I.s). In the case of a smooth concave obstacle, a singular control problem with discretionary stopping is solved for controlled geometric Brownian motion in [16], where the penalty method is used to analyze the degenerate V.I., show the concavity of the value function, and obtain the optimal control rule. An impulse control problem with discretionary stopping and multiple investment options incurring a proportional investment cost is solved in [5] for the geometric Brownian motion. The Q.V.I. of this problem also has a smooth obstacle. The comparative statics with respect to the fixed and proportional cost of investment presented in section 4.4.2 are in line with numerical findings in [17], where a lower total equity issuance cost delays liquidation and a higher fixed cost component of the equity issuance cost increases the amount of equity issued.

The real options problem with one-time investment flexibility and a cessation option was studied in [6] under the assumption that the investment cost is proportional to the investment amount and no fixed cost of investment is incurred. Also, a specific form of the profit function (which is a particular case of the profit function considered below in section 4.2) was assumed. In the setting ensuring the smoothness of the obstacle, the associated V.I. was shown to have a two-threshold strong solution with one continuation interval defining the optimal strategy. In this work, we consider the one-time investment problem with a cessation option in a more general setting. That is, we allow any investment cost function and profit function satisfying a set of simple natural assumptions. In particular, this includes investments incurring a fixed cost (in addition to the variable cost), which necessarily leads to the V.I. with a nonsmooth obstacle. By invoking the technique of weak solutions and an adapted form of the maximum principle, we obtain a set of conditions which ensure the existence of a weak solution and a strong two-threshold solution in the nonsmooth setting.

The paper is organized as follows. In the next section, the real options problem is discussed. In section 3, we consider the weak and strong formulations of V.I. with a nonsmooth obstacle of the general form in one dimension. Sufficient conditions for the existence of a weak solution and a strong two-threshold solution are presented. This analysis is applied to the real options problem in section 4. Proofs are presented in the appendix.

2. Optimal investment/exit problem.

In a continuous time setting, let us consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a Wiener process $w : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, with a filtration $\mathbb{F} = (\mathcal{F}_t; t \geq 0)$. Assume that a commodity price $X(t)$ follows the geometric Brownian motion process

$$dX = \mu X dt + \sigma X dw, \qquad X(0) = x. \tag{1}$$

Suppose that a firm has a capital $\delta > 0$, and this capital does not degrade, nor is maintenance needed. Let $\pi(X(t), \delta)$ denote the profit of the firm per unit time as a function of the current price and the firm's capital. If no investment is allowed, then the manager at any moment decides whether to continue the operation or go into cessation. This leads to the classical optimal stopping problem with the value function

$$\varphi(x, \delta) = \sup_{\tau} \mathbb{E}_{x,\delta} \int_0^{\tau} e^{-rt} \pi(X(t), \delta) dt, \tag{2}$$

where $\mathbb{E}_{x,\delta}$ denotes the expected value for fixed x, δ, and $r > 0$ is the discount rate.

Everywhere below we assume that the manager can make an investment prior to cessation, and at most one investment is allowed. Hence, admissible investment strategies are represented by the set U of triplets $\{\tau_0, \xi, \tau_1\}$, where $\tau_0 \geq 0$ is the time of investment, $\xi > 0$ is the amount of investment, and $\tau_1 \geq 0$ is the cessation time. In particular, the manager can choose not to exercise the cessation option ($\tau_1 = \infty$) or not to invest ($\tau_0 \geq \tau_1$). Formally, δ_t follows controlled dynamics,

$$\delta_t = \delta + \xi \mathbf{1}_{[\tau_0,\infty)}(t), \tag{3}$$

where the control consists of a stopping time τ_0 at which an impulse ξ (an $\mathcal{F}_{\tau_0}$-measurable random variable) is applied.

Further, the one-time investment ξ into the capital incurs a cost of $v(\xi)$, where we assume that $v : \mathbb{R}_+ \to \mathbb{R}_+$ increases, is continuous on $(0, \infty)$, and satisfies

$$v(0) = 0 < K_0 = \lim_{\xi\to 0} v(\xi), \qquad \lim_{\xi\to\infty} \frac{v(\xi)}{\xi} = k > 0. \tag{4}$$

The first assumption means that there is a fixed cost K_0 associated with the investment of any amount ξ, while $v(\xi) - K_0$ is the variable cost increasing with ξ. Further technical assumptions on the investment cost will be imposed later. The case $K_0 = 0$ is actually simpler and can be covered by our results, but the explicit assumption $K_0 > 0$ simplifies the formulations.

The manager's objective is to maximize the market value of equity. Assume that the value function of this optimization problem is given by

$$y(x, \delta) := \sup_{\{\tau_0,\xi,\tau_1\}\in U} \mathbb{E}_{x,\delta} \left[\int_0^{\tau_1} e^{-rt} \pi \left(X(t), \delta + \xi \mathbf{1}_{[\tau_0,\tau_1)}(t) \right) \mathrm{d}t - e^{-r\tau_0} v(\xi) \mathbf{1}_{[0,\tau_1)}(\tau_0) \right], \tag{5}$$

where $\mathbf{1}_A$ denotes the indicator function of a set A. On the other hand, the profit from investment equals

$$\Phi(x, \delta) = \sup_{\xi\geq 0} [\varphi(x, \delta + \xi) - v(\xi)] = \max \left\{ \varphi(x, \delta),\ \sup_{\xi>0} [\varphi(x, \delta + \xi) - v(\xi)] \right\}, \tag{6}$$

where φ is given by (2). With this notation, the value function (5) can be equivalently expressed using one stopping time $\theta = \tau_0 \wedge \tau_1$ (see also [5]) as

$$y(x, \delta) := \sup_{\theta} \mathbb{E}_{x,\delta} \left[\int_0^{\theta} e^{-rt} \pi (X(t), \delta) \, \mathrm{d}t + e^{-r\theta} \Phi(X(\theta), \delta) \right]. \tag{7}$$

Define the differential operator

$$\mathcal{L}y(x) := ry(x) - \mu x y'(x) - \frac{1}{2}\sigma^2 x^2 y''(x). \tag{8}$$

According to [7], the value function (5) should satisfy the V.I.

$$y(x, \delta) \geq \Phi(x, \delta) \quad \text{for all } x \in \mathbb{R}_+, \tag{9}$$

$$\mathcal{L}y(x, \delta) \geq \pi(x, \delta), \quad \text{a.e. } x \in \mathbb{R}_+, \tag{10}$$

$$[y(x, \delta) - \Phi(x, \delta)] \times [\mathcal{L}y(x, \delta) - \pi(x, \delta)] = 0, \quad \text{a.e. } x \in \mathbb{R}_+ \tag{11}$$

(which is equivalent to $\min\{y-\Phi;\mathcal{L}y-\pi\}=0$) and the boundary conditions

$$y(0,\delta)=\Phi(0,\delta)=0, \tag{12}$$

$$y(x,\delta)-\Phi(x,\delta)\to 0 \quad \text{as} \quad x\to\infty. \tag{13}$$

The continuous nonnegative function Φ is called the *obstacle*. Typically, the obstacle (6) has points of nonsmoothness. Examples will be considered in section 4.

In what follows, we consider a weak formulation of problem (9)–(13) and a strong solution to this problem, which possesses additional properties. The so-called *two-threshold* solution y is twice continuously differentiable in x a.e. and satisfies the condition

$$y(x,\delta)=\Phi(x,\delta), \quad x\in[0,X_0(\delta)]\cup[X_5(\delta),\infty) \tag{14}$$

with a priori unknown thresholds (free boundaries) $X_5(\delta)>X_0(\delta)>0$. We choose the notation X_5 for the upper threshold because the assumptions, results, and proofs presented later use a number of important intermediate points X_1,X_2,X_3,X_4 located between the thresholds X_0 and X_5. Condition (14) implies (12), (13).

3. Variational inequality with nonsmooth obstacle.

3.1. Setting of the problem. Motivated by problem (9)–(13), in this section we consider the existence of solutions to a V.I. with a nonsmooth obstacle in a rather general setting. We will then apply the results of this section to the real options problem considered above.

We work on $\mathbb{R}_+$. We consider the second order differential operator (8) with positive constants r,μ,σ. The obstacle $\Phi(x)$ is assumed to be a.e. differentiable and satisfy the conditions

$$\Phi(x)\ge 0, \quad \Phi(0)=0, \quad \int_0^{+\infty}\frac{\Phi^2(x)}{1+x^m}dx<+\infty, \quad \int_0^{+\infty}\frac{|\Phi'(x)|^2x^2}{1+x^m}dx<+\infty \tag{15}$$

with an $m\ge 3$. The Hilbert space of a.e. differentiable functions satisfying the last two conditions (15) with the scalar product

$$(\phi_1,\phi_2)_{\mathcal{H}_m}=\int_0^{+\infty}\frac{\phi_1(x)\phi_2(x)}{1+x^m}dx+\int_0^{+\infty}\frac{\phi_1'(x)\phi_2'(x)x^2}{1+x^m}dx$$

and the corresponding norm $\|\cdot\|_{\mathcal{H}_m}$ is denoted $\mathcal{H}_m$. Clearly, functions with bounded derivatives belong to $\mathcal{H}_m$. Assume that a function $\pi(x)$ (the running profit in the above economic interpretation) is continuous and satisfies

$$\int_0^{+\infty}\frac{\pi^2(x)}{1+x^m}dx<+\infty, \quad \inf_{x>0}\pi(x)<0. \tag{16}$$

We next consider the set of continuous functions $z(x)$ which satisfy

$$\mathcal{L}z(x)\ge\pi(x) \ \text{ in the sense of distributions on } \mathbb{R}_+; \tag{17}$$

$$z(x)\ge\Phi(x) \ \text{ for all } x\in\mathbb{R}_+; \quad z(0)=0, \quad z(x)-\Phi(x)\to 0 \quad \text{as} \quad x\to+\infty,$$

where $\mathcal{L}$ is the differential operator (8). We denote by $\mathcal{Z}$ the set of continuous functions $z(x)$ satisfying (17). We can define the value function as the smallest function in the set (17). The value function $y(x)$ thus satisfies

$$y(x) \in \mathcal{Z}; \quad y(x) \le z(x) \quad \text{for all} \quad x \in \mathbb{R}_+,\ z \in \mathcal{Z}. \tag{18}$$

This definition will be too loose. We will make it more precise in what follows.

3.2. Existence and uniqueness of a weak solution.

We give the definition of the V.I. in a weak sense. We look for a function $y(x)$ such that

$$y(x) \in \mathcal{H}_m,\ y(x) \ge \Phi(x) \quad \text{for all } x \in \mathbb{R}_+; \tag{19}$$

$$y(0) = 0, \quad y(x) - \Phi(x) \to 0 \quad \text{as} \quad x \to +\infty;$$

(20)
$$r\int_0^{+\infty} \frac{y(z-y)}{1+x^m}dx + \frac{\sigma^2}{2}\int_0^{+\infty} y' \frac{d}{dx}\left(\frac{(z-y)x^2}{1+x^m}\right)dx - \mu\int_0^{+\infty}\frac{xy'(z-y)}{1+x^m}dx \ge \int_0^{+\infty}\frac{\pi(x)(z-y)}{1+x^m}dx$$

for all $z \in \mathcal{H}_m,\ z(x) \ge \Phi(x)$ for all $x \in \mathbb{R}_+$ with $z(0) = 0, \quad z(x) - \Phi(x) \to 0$ as $x \to +\infty$.

Note that, since we are in dimension one, functions in $\mathcal{H}_m$ are continuous. Besides,

$$\frac{d}{dx}\left(\frac{(z-y)x^2}{1+x^m}\right) = \frac{(z'-y')x^2}{1+x^m} + (z-y)\left(\frac{2x}{1+x^m} - \frac{mx^{m+1}}{(1+x^m)^2}\right)$$

holds a.e., and since $m \ge 3$, the integral $\int_0^{+\infty} y' \frac{d}{dx}\left(\frac{(z-y)x^2}{1+x^m}\right)dx$ is well defined.

Theorem 3.1. *Let*

$$r_m := r - \frac{\mu}{2}(m-1) - \frac{\sigma^2}{4}(m-1)(m-2) > 0 \qquad \text{with} \qquad m \ge 3, \tag{21}$$

and let (15), (16) *hold. Then there is one and only one solution y of the V.I. (in a weak sense)* (19), (20).

A constructive intuitive proof using the penalty method is presented in the appendix.

The weak solution can be characterized as follows using a rigorous interpretation of (17). A function $z(x)$ in $\mathcal{H}_m$ will be called an upper solution of the V.I. (20) if it satisfies

$$r\int_0^{+\infty}\frac{z\zeta}{1+x^m}dx - \mu\int_0^{+\infty}\frac{xz'\zeta}{1+x^m}dx + \frac{\sigma^2}{2}\int_0^{+\infty} z'\frac{d}{dx}\left(\frac{x^2\zeta}{1+x^m}\right)dx \ge \int_0^{+\infty}\frac{\pi\zeta}{1+x^m}dx \tag{22}$$

for every $\zeta \in \mathcal{H}_m$ such that $\zeta(x) \ge 0,\ x \in \mathbb{R}_+; \quad \zeta(0) = 0, \quad \zeta(x) \to 0$ as $x \to +\infty$

and

$$z(x) \ge \Phi(x),\ x \in \mathbb{R}_+; \quad z(0) = 0, \quad z(x) - \Phi(x) \to 0 \quad \text{as} \quad x \to +\infty. \tag{23}$$

Proposition 3.2. *The solution y of the V.I.* (19), (20) *is the smallest upper solution.*

3.3. Existence of a strong solution with thresholds. Under the assumptions of Theorem 3.1 the V.I. (19), (20) has a unique solution y. The coincidence set is the set of points x such that $y(x) = \Phi(x)$, when the value function coincides with the obstacle. As discussed above (see the last paragraph of section 2), in applications, we would like this set to be the union of disjoint intervals. In particular, we are interested in the situation in which we can find two points, denoted X_0 and X_5 with $X_0 < X_5$, such that (cf. (14))

$$y(x) = \Phi(x) \quad \text{for} \quad x \in [0, X_0] \cup [X_5, +\infty), \tag{24}$$

whereas

$$\mathcal{L}y(x) = \pi(x), \quad x \in (X_0, X_5). \tag{25}$$

We also want $y(x)$ to be C^1, which leads to the smooth pasting conditions

$$\begin{aligned} y(X_0) = \Phi(X_0), \quad y(X_5) = \Phi(X_5), \\ y'(X_0) = \Phi'(X_0), \quad y'(X_5) = \Phi'(X_5). \end{aligned} \tag{26}$$

Clearly, the obstacle must be C^1 on the coincidence set. For a function $y(x)$ satisfying (24)–(26) to be a solution of the V.I. (9)–(13), we must also have

$$\begin{aligned} \mathcal{L}\Phi(x) \geq \pi(x) \quad \text{for a.e.} \quad x \in [0, X_0] \cup [X_5, +\infty); \\ y(x) \geq \Phi(x), \quad x \in (X_0, X_5). \end{aligned} \tag{27}$$

Below we present sufficient conditions for existence of a solution to problem (24)–(27). Let us consider the following assumptions about the obstacle.

Assume that there exist a number X_1 such that

$$X_1 > 0; \quad \Phi(x) = 0 \quad \text{for all} \quad x \in [0, X_1]; \quad \Phi'(X_1) = 0; \tag{28}$$

$$\pi(x) < 0 \quad \text{for all} \quad x \in [0, X_1),$$

and a number X_M such that

$$X_M > X_1; \quad \Phi(x) \text{ is } C^1 \text{ for } x > X_M; \tag{29}$$

$$\Phi''(x) \text{ has left and right limits at every continuity point } x \in \mathbb{R}_+ \text{ of the function } \Phi'.$$

We set

$$g(x) = \pi(x) - \mathcal{L}\Phi(x); \tag{30}$$

this function has left and right limits at every continuity point $x \in \mathbb{R}_+$ of the function Φ' (in particular, on $(X_M, +\infty)$). Assume that there exists X_4 such that

$$X_4 > X_M; \quad g(x) = \pi(x) - \mathcal{L}\Phi(x) < 0 \quad \text{for all} \quad x > X_4; \tag{31}$$

$$g(x) < -b < 0 \quad \text{for all sufficiently large } x.$$

On $[X_1, X_M]$, $\Phi(x)$ may not be C^1, but $\Phi'(x)$ has left and right limits. We then assume that

$$\text{if } x \in (X_1, X_4) \text{ is a discontinuity point of } \Phi'(x), \text{then } \Phi'(x-0) < \Phi'(x+0); \tag{32}$$
$$\text{if } x \in (X_1, X_4) \text{ is a continuity point of } \Phi'(x), \text{then } g(x) > 0,$$

where again $g(x)$ must be interpreted as left and right limits.[1]

Next, define the polynomial

$$Q(\beta) = r - \left(\mu - \frac{\sigma^2}{2}\right)\beta - \frac{1}{2}\sigma^2\beta^2. \tag{33}$$

The roots of this polynomial satisfy $\beta_1 < 0 < \beta_2$. We further assume that

$$\Phi(X_4) < \frac{2}{\sigma^2(\beta_2 - \beta_1)} \int_{X_1}^{X_4} \frac{\pi(\xi)}{\xi} \left[\left(\frac{X_4}{\xi}\right)^{\beta_1} - \left(\frac{X_4}{\xi}\right)^{\beta_2}\right] d\xi. \tag{34}$$

Consider the function

$$L(x) = \frac{2}{\sigma^2(\beta_1 - \beta_2)} \int_{X_4}^{x} \frac{g(\xi)}{\xi} \left[\left(\frac{X_4}{\xi}\right)^{\beta_1} - \left(\frac{X_4}{\xi}\right)^{\beta_2}\right] d\xi, \quad x > X_4, \tag{35}$$

and its derivative

$$L'(x) = \frac{2}{\sigma^2(\beta_1 - \beta_2)} \frac{g(x)}{x} \left[\left(\frac{X_4}{x}\right)^{\beta_1} - \left(\frac{X_4}{x}\right)^{\beta_2}\right], \quad x > X_4.$$

From the assumption (31) we obtain $L'(x) > 0$. Also, (21) implies $r > \mu$, and hence $\beta_1 + 1 < 0$. Since $g(x) < -b$ for x sufficiently large, $L'(x) \to +\infty$ as $x \to +\infty$. Therefore, also $L(x) \to +\infty$ as $x \to +\infty$. Due to $L(X_4) = 0$, the function $L(x)$ is monotonically increasing on $(X_4, +\infty)$ from 0 to $+\infty$. Assumption (34) implies that there exists a single value $X_5 = X_5(X_1) > X_4$ such that

$$\begin{aligned} &\frac{2}{\sigma^2(\beta_2 - \beta_1)} \int_{X_1}^{X_4} \frac{\pi(\xi)}{\xi} \left[\left(\frac{X_4}{\xi}\right)^{\beta_1} - \left(\frac{X_4}{\xi}\right)^{\beta_2}\right] d\xi - \Phi(X_4) \\ &= \frac{2}{\sigma^2(\beta_1 - \beta_2)} \int_{X_4}^{X_5(X_1)} \frac{g(\xi)}{\xi} \left[\left(\frac{X_4}{\xi}\right)^{\beta_1} - \left(\frac{X_4}{\xi}\right)^{\beta_2}\right] d\xi. \end{aligned} \tag{36}$$

We can now state our last assumption,

$$\begin{aligned} \Phi'(X_4) > \frac{2}{\sigma^2(\beta_2 - \beta_1)} &\left(\int_{X_1}^{X_4} \frac{\pi(\xi)}{\xi^2} \left[\beta_1 \left(\frac{X_4}{\xi}\right)^{\beta_1 - 1} - \beta_2 \left(\frac{X_4}{\xi}\right)^{\beta_2 - 1}\right] d\xi \right. \\ &\left. + \int_{X_4}^{X_5(X_1)} \frac{g(\xi)}{\xi^2} \left[\beta_1 \left(\frac{X_4}{\xi}\right)^{\beta_1 - 1} - \beta_2 \left(\frac{X_4}{\xi}\right)^{\beta_2 - 1}\right] d\xi \right). \end{aligned} \tag{37}$$

[1] We have introduced X_1, X_3, X_4 but not X_2. This notation is reserved for a multiple of X_1 defined below in (87).

Theorem 3.3. *Assume that relations (28), (29), (31), (32), (34), (37) hold. Then there exists a single pair of thresholds X_0, X_5 satisfying $X_0 < X_1$, $X_5 > X_4$ such that system (24)–(27) has a unique solution $y(x)$. This function is a solution of the V.I. (19), (20).*

We now formulate a more restrictive sufficient condition for the existence of a two-threshold solution, which can be easier to verify in particular cases such as those considered in section 4. Let us consider a particular solution $y = p(x)$ of the equation $\mathcal{L}y(x) = \pi(x)$. Define the functions

$$C_1(x) = \frac{xp'(x) - \beta_2 p(x)}{(\beta_2 - \beta_1)x^{\beta_1}}, \qquad C_2(x) = \frac{-xp'(x) + \beta_1 p(x)}{(\beta_2 - \beta_1)x^{\beta_2}}, \quad x > 0; \tag{38}$$

$$D_1(x) = C_1(x) + \frac{-x\Phi'(x) + \beta_2\Phi(x)}{(\beta_2 - \beta_1)x^{\beta_1}}, \qquad D_2(x) = C_2(x) + \frac{x\Phi'(x) - \beta_1\Phi(x)}{(\beta_2 - \beta_1)x^{\beta_2}}, \quad x > X_M. \tag{39}$$

It is straightforward to check that

$$C_1'(x) = -\frac{2\pi(x)}{\sigma^2(\beta_2 - \beta_1)x^{\beta_1+1}}, \qquad C_2'(x) = \frac{2\pi(x)}{\sigma^2(\beta_2 - \beta_1)x^{\beta_2+1}};$$

$$D_1'(x) = -\frac{2g(x)}{\sigma^2(\beta_2 - \beta_1)x^{\beta_1+1}}, \qquad D_2'(x) = \frac{2g(x)}{\sigma^2(\beta_2 - \beta_1)x^{\beta_2+1}}.$$

Therefore conditions (28), (29), (31) imply

$$C_1'(x) > 0, \; C_2'(x) < 0 \text{ for } x \in (0, X_1); \qquad D_1'(x) > 0, \; D_2'(x) < 0 \text{ for } x \in (X_4, \infty). \tag{40}$$

Hence, each of the functions $C_1(x), C_2(x)$ converges to either a finite or infinite limit as $x \to 0$ and each of the functions $D_1(x), D_2(x)$ converges to either a finite or infinite limit as $x \to \infty$.

Theorem 3.4. *Assume that relations (28), (29), (31), (32) hold. Suppose that*

$$D_1(X_4) < C_1(0); \qquad \lim_{x\to\infty} D_1(x) > C_1(X_1). \tag{41}$$

$$\lim_{x\to\infty} D_2(x) \geq C_2(X_1); \qquad D_2(X_4) \leq \lim_{x\to 0} C_2(x). \tag{42}$$

Then there exists a pair X_0, X_5 such that system (24)–(27) has a solution $y(x)$. This function is a solution of the V.I. (19), (20).

Note that replacing a particular solution $p(x)$ in formulas (38) and (39) by another solution $\hat{p}(x)$ of $\mathcal{L}y(x) = \pi(x)$ results in shifting each of the functions $C_i(x)$, $D_i(x)$ by a constant c_i, $i = 1, 2$. Therefore, if conditions (41) and (42) are satisfied for a particular solution $p(x)$, they are also satisfied for any other particular solution $\hat{p}(x)$ of the equation $\mathcal{L}y(x) = \pi(x)$, i.e., conditions of Theorem 3.4 are independent of the choice of a particular solution $p(x)$.

Remark 3.5. The results of this section can be extended to optimization problem (5) with the general one-dimensional diffusion process $X(t)$. In this setting, the drift and volatility depend on X, and the differential operator (8) has variable coefficients. In this scenario, the sufficient conditions of Theorems 2 and 3 should be formulated in terms of the fundamental solutions of the equation $\mathcal{L}y(x) = 0$ and the solution $p(x)$ of the equation $\mathcal{L}y(x) = \pi(x)$. Important properties of these solutions and the associated optimal stopping problem are provided by the theory of r-excessive mappings [1].

4. Application to real options.

4.1. Verification theorem. In this section, we apply the results of section 3 to the real options problem introduced in section 2. We start with the verification theorem, which identifies the solution to the V.I. with the value function (7) and provides the optimal investment/exit rule.

Consider the payoff functional

$$J_x(\theta) = \mathbb{E}_x \left[\int_0^\theta e^{-rt} \pi \left(X(t) \right) \mathrm{d}t + e^{-r\theta} \Phi(X(\theta)) \right], \tag{43}$$

where $X(t)$ is the process (1) with the initial value $X(0) = x$ and θ is a stopping time adapted to the filtration $\mathbb{F} = (\mathcal{F}_t; t \geq 0)$. The value function of the optimization problem is given by

$$y(x) = \sup_\theta J_x(\theta) \tag{44}$$

(cf. (5)). Let us check that under the assumptions of Theorem 3.1,

$$\sup_\theta |J_x(\theta)| < +\infty \quad \text{for any} \quad x \geq 0. \tag{45}$$

For $x = 0$, the equation $X(t) = 0$ holds at all times, and hence

$$J_0(\theta) = \pi(0) \frac{1 - e^{-r\theta}}{r} + \Phi(0) e^{-r\theta}.$$

Therefore, the optimal stopping time $\theta = \hat{\theta}_0$ equals $\hat{\theta}_0 = 0$ and $y(0) = \Phi(0)$ if $\Phi(0) > \pi(0)/r$; on the other hand, $\hat{\theta}_0 = +\infty$ and $y(0) = \pi(0)/r$ if $\Phi(0) < \pi(0)/r$. For $x > 0$, we have

$$\left| \mathbb{E}_x \int_0^\theta \pi(X(t)) e^{-rt}\, dt \right| \leq \mathbb{E}_x \int_0^{+\infty} |\pi|(X(t)) e^{-rt}\, dt =: \tilde{\Gamma}(x), \tag{46}$$

where, due to Itô's lemma and the assumptions (16) and (21), the function $\tilde{\Gamma}(x)$ is a well-defined locally bounded solution of the equation $\mathcal{L}\tilde{\Gamma}(x) = |\pi(x)|$ satisfying $\tilde{\Gamma} \in \mathcal{H}_m$. Further, using (15) we can assert that

$$\frac{x\Phi^2(x)}{1 + x^m} \leq 2\|\Phi\|^2_{\mathcal{H}_m},$$

and hence

$$\mathbb{E}_x \big(\Phi(X(\theta)) e^{-r\theta} \big) \leq \sqrt{2}\, \|\Phi\|_{\mathcal{H}_m} \sqrt{\mathbb{E}_x \left(\left(\frac{1}{X(\theta)} + X^{m-1}(\theta) \right) e^{-2r\theta} \right)}. \tag{47}$$

Also, due to (21),

$$\mathbb{E}_x \left(\left(\frac{1}{X(\theta)} + X^{m-1}(\theta) \right) e^{-2r\theta} \right) \leq \frac{1}{x} + x^{m-1}. \tag{48}$$

Combining (46)–(48), we obtain (45).

Define

$$y_\infty(x) := \mathbb{E}_x \int_0^{+\infty} \pi(X(t)) e^{-rt}\, dt. \tag{49}$$

Theorem 4.1. *Under the assumptions of Theorem 3.1, the solution y of the V.I. (19), (20) coincides with the value function $y(x) = \sup_\theta J_x(\theta)$. If the function (49) satisfies*

$$y_\infty(x) \geq \Phi(x) \quad \textit{for all} \quad x \in \mathbb{R}_+, \tag{50}$$

then $y(x) = y_\infty(x)$ and the optimal stopping time is at infinity:

$$\sup_\theta J_x(\theta) = J_x(+\infty) \quad \textit{for all} \quad x \in \mathbb{R}_+. \tag{51}$$

If (50) is not true, then the optimal stopping time is defined by $\hat\theta_x = \inf\{t \geq 0 |\, y(X(t)) \leq \Phi(X(t))\}$.

4.2. Nonsmoothness of the obstacle. Everywhere below we assume a specific form for the running profit $\pi(x,\delta)$ in (2) and the form (6) for the obstacle. Namely, it is supposed that the profit of the firm per unit time is given by

$$\pi(X(t),\delta) = -\alpha_0 + \alpha_1 X(t)^\gamma \delta^\varepsilon \tag{52}$$

with parameters $\gamma > 0$, $\alpha_0 > 0, \alpha_1 > 0$, and $\varepsilon \in (0,1)$. The first term captures fixed operating costs incurred by the firm. The second term corresponds to the variable profits of the firm which features standard decreasing returns to scale with respect to δ, and allows for curvature with respect to X to capture, for example, market power. We additionally assume that $Q(\gamma) > 0$, where Q is the polynomial (33)—in other words $0 < \gamma < \beta_2$ (recall that $\beta_1 < 0$, $\beta_2 > 0$ are the roots of Q). For the running profit (52), the optimal stopping problem (2) has an explicit C^1 solution [15]:

$$\varphi(x,\delta) = \left[\eta_0 + \eta_\gamma x^\gamma \delta^\varepsilon + \frac{\eta_0 \gamma}{\beta_1 - \gamma}\left(\frac{\delta^\varepsilon}{\lambda}\right)^{\frac{\beta_1}{\gamma}} x^{\beta_1}\right] \mathbf{1}_{x \geq X_1(\delta)} \quad \text{with} \quad X_1(\delta) = \left(\frac{\lambda}{\delta^\varepsilon}\right)^{\frac{1}{\gamma}}, \tag{53}$$

where

$$\eta_0 = -\frac{\alpha_0}{r} < 0, \qquad \eta_\gamma = \frac{\alpha_1}{Q(\gamma)} > 0, \qquad \lambda = -\frac{\eta_0}{\eta_\gamma}\frac{\beta_1}{\beta_1 - \gamma} > 0;$$

see Figure 1.

The obstacle (6) with φ given by (53) has points of nonsmoothness where the left and right derivatives with respect to x are different. We make this statement precise in the following two lemmas.

Lemma 4.2. *There is a continuous function $X_3 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\Phi(x,\delta) = \begin{cases} \varphi(x,\delta), & x \leq X_3(\delta), \\ \sup_{\xi>0}\big[\varphi(x,\delta+\xi) - v(\xi)\big], & x > X_3(\delta), \end{cases} \tag{54}$$

and

$$\Phi(x,\delta) > \varphi(x,\delta) \quad \textit{for} \quad x > X_3(\delta). \tag{55}$$

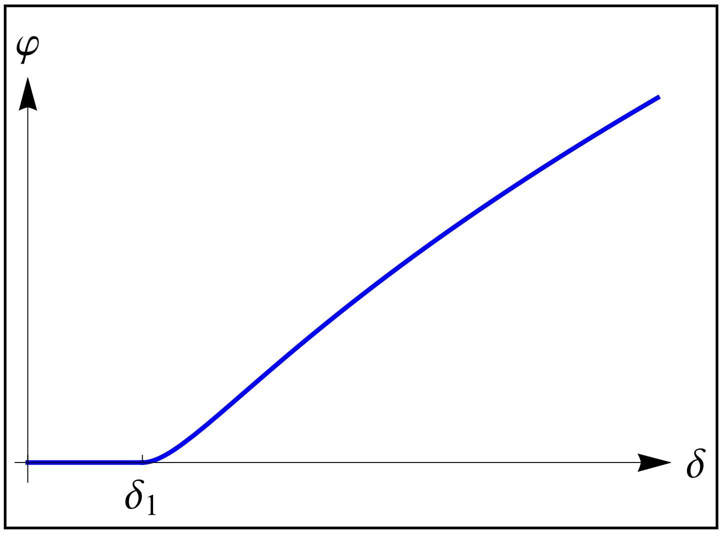

Figure 1. *The graph of the value function $\varphi(x,\cdot)$ with fixed $x = 10$ for the parameter values $\mu = 0.01$, $\sigma = 0.10$, $r = 0.12$, $\gamma = 1$, $\varepsilon = 0.5$, $\alpha_0 = 13.5$, $\alpha_1 = 1, \delta_1 = \left(\frac{\lambda}{x^\gamma}\right)^{\frac{1}{\epsilon}}$.*

For every $\delta > 0$, $x \geq X_3(\delta)$, the relations (54) and

$$\varphi(x,\delta) - v(0+) < \varphi(x,\delta) \leq \Phi(x,\delta); \quad \varphi(x,\delta+\xi) - v(\xi) \to -\infty \quad \text{as} \quad \xi \to +\infty \tag{56}$$

imply that there is a closed nonempty set $\Xi_{max}(x,\delta) \subset (0,\infty)$ of values of ξ such that

$$\Phi(x,\delta) = \varphi(x,\delta+\xi) - v(\xi) \quad \Leftrightarrow \quad \xi \in \Xi_{max}(x,\delta). \tag{57}$$

Define

$$\xi_M(x,\delta) = \sup \Xi_{max}(x,\delta), \qquad \xi_m(x,\delta) = \inf \Xi_{max}(x,\delta). \tag{58}$$

Let $D_x^+ f$ and $D_x^- f$ denote the right and left (partial) derivatives of the function f with respect to x.

Lemma 4.3. *For $x > X_3(\delta)$, the left and right derivatives of Φ with respect to x satisfy*

$$D_x^+\Phi(x,\delta) = \varphi_x\big(x,\delta+\xi_M(x,\delta)\big), \qquad D_x^-\Phi(x,\delta) = \varphi_x\big(x,\delta+\xi_m(x,\delta)\big), \tag{59}$$

and

$$D_x^+\Phi(x,\delta) - D_x^-\Phi(x,\delta) > 0 \quad \text{if} \quad \xi_M(x,\delta) > \xi_m(x,\delta). \tag{60}$$

If in addition

$$\xi_M(x,\delta) = \xi_m(x,\delta) \quad \text{and} \quad \varphi_{\delta\delta}\big(x,\delta+\xi_M(x,\delta)\big) - v''\big(\xi_M(x,\delta)\big) \neq 0, \tag{61}$$

then $\Phi_{xx}(x,\delta)$ is well-defined.

Further, the left and right derivatives of Φ *with respect to* x *on the line* $x = X_3(\delta)$ *satisfy*

$$D_x^+\Phi\big(X_3(\delta),\delta\big) = \varphi_x\big(X_3(\delta), \delta + \xi_M(X_3(\delta),\delta)\big), \qquad D_x^-\Phi\big(X_3(\delta),\delta\big) = \varphi_x\big(X_3(\delta),\delta\big),$$

and

$$D_x^+\Phi\big(X_3(\delta),\delta\big) - D_x^-\Phi\big(X_3(\delta),\delta\big) > 0. \tag{62}$$

According to (62), the derivative $\Phi_x(\cdot,\cdot)$ of the obstacle is discontinuous on the line $x = X_3(\delta)$. This discontinuity is induced by the nonzero fixed cost associated with any positive investment amount. Let us consider an example of a discontinuity of the type (60).

Assume that the variable cost with $v(0) = 0$ is piecewise linear:

$$v(\xi) = K_0 + k_1\xi + (k_2 - k_1)(\xi - \xi_0)\, H(\xi - \xi_0), \qquad \xi > 0, \tag{63}$$

where H is the Heaviside step function and $k_1 > k_2$. This cost function (63) captures important features of economic realism such as fixed expansion cost K_0 (e.g., the cost of obtaining a development license in the case of real estate or obtaining a license to drill in the case of oil and gas) and a decreasing marginal cost per additional unit. The latter reflects the savings incurred from bulk buying machinery or raw materials since typically sellers provide discounts for larger purchases which taper off at a minimum cost per additional unit. In (63), a discount applies when the investment amount exceeds ξ_0. The investment cost function (63) satisfies (4) with $k = k_2$.

Given a $\delta > 0$, let us show that if x and ξ_0 are large, then the function $\varphi(x, \delta + \cdot) - v(\cdot)$ can achieve the same global maximum value on the positive semiaxis $\xi \geq 0$ at two distinct points $\xi_m(x,\delta) \neq \xi_M(x,\delta)$. When this is the case, then according to Lemma 4.3, both $X_3(\delta)$ and x are discontinuity points of the function $\Phi_x(\cdot,\delta)$.

Indeed, for large x, the function φ can be approximated by the function

$$\hat\varphi(x,\delta) = \eta_\gamma x^\gamma \delta^\varepsilon.$$

Now notice that, given $x, \delta > 0$, the system

$$\hat\varphi_\delta(x,\delta+\xi_1) = k_1, \quad \hat\varphi_\delta(x,\delta+\xi_2) = k_2, \quad \hat\varphi(x,\delta+\xi_1) - k_1\xi_1 = \hat\varphi(x,\delta+\xi_2) - k_2\xi_2 + (k_2-k_1)\xi_0 \tag{64}$$

has the solution

$$\xi_0 = \frac{k_2^{-\frac{\varepsilon}{1-\varepsilon}} - k_1^{-\frac{\varepsilon}{1-\varepsilon}}}{\frac{\varepsilon}{1-\varepsilon}(k_1 - k_2)} \left(\varepsilon\eta_\gamma x^\gamma\right)^{\frac{1}{1-\varepsilon}} - \delta,$$

$$\xi_i = k_i^{-\frac{1}{1-\varepsilon}} \left(\varepsilon\eta_\gamma x^\gamma\right)^{\frac{1}{1-\varepsilon}} - \delta, \qquad i = 1, 2.$$

Due to the convexity of the function $k^{-\frac{\varepsilon}{1-\varepsilon}}$, from $k_1 > k_2$ it follows that

$$-\frac{\varepsilon}{1-\varepsilon} k_1^{-\frac{1}{1-\varepsilon}} > \frac{k_1^{-\frac{\varepsilon}{1-\varepsilon}} - k_2^{-\frac{\varepsilon}{1-\varepsilon}}}{k_1 - k_2} > -\frac{\varepsilon}{1-\varepsilon} k_2^{-\frac{1}{1-\varepsilon}};$$

hence $\xi_1 < \xi_0 < \xi_2$, and therefore system (64) is equivalent to the system

$$\hat{\varphi}_\delta(x,\delta+\xi_1)-v'(\xi_1)=0, \quad \hat{\varphi}_\delta(x,\delta+\xi_2)-v'(\xi_2)=0, \quad \hat{\varphi}(x,\delta+\xi_1)-v(\xi_1)=\hat{\varphi}(x,\delta+\xi_2)-v(\xi_2).$$

In other words, ξ_1, ξ_2 are two maximum points of the function $\hat{\varphi}(x,\delta+\cdot)-v(\cdot)$, which achieves the same value at these points. But, for large values of x, the function φ can be considered as a small perturbation of the function $\hat{\varphi}$. Also, ξ_0 grows with x. Hence, we conclude that, given a $\delta > 0$, if the parameter ξ_0 of the price function (63) is sufficiently large, then the function $\varphi(x,\delta+\cdot)-v(\cdot)$ also achieves the global maximum at two distinct points ξ_i for at least one value of $x = x(\xi_0)$, which grows with ξ_0. Therefore, $\Phi_x(\delta,\cdot)$ has at least two discontinuity points for large ξ_0, one of them due to the fixed cost associated with any investment, the other due to the discount on investment amounts exceeding ξ_0.

It is worth noting that the situation is similar for smooth cost functions $v(\cdot)$, which are close to (63), i.e., the corresponding obstacle has multiple points of nonsmoothness.

4.3. Investment cost of general form.

Assume that the investment cost function $v(\xi)$ is C^2 for $\xi > 0$ and

$$\lim_{\xi\to\infty} \xi\, v''(\xi) = 0. \tag{65}$$

Proposition 4.4. *Let (21) hold with $m \geq 3$ and let*

$$\frac{\gamma}{1-\varepsilon} < \frac{m-1}{2}. \tag{66}$$

Let the investment cost satisfy (4) and (65). Then there is one and only one solution y of the V.I. (in a weak sense) (19), (20) with the running profit $\pi(x) = \pi(x,\delta)$ and the obstacle $\Phi(x) = \Phi(x,\delta)$ defined by (52), (53), (6).

We now turn to the existence of a strong two-threshold solution.

Given a $\delta > 0$ (firm's capital), let us recall the definition of the quantities $X_1 = X_1(\delta)$ and $X_3 = X_3(\delta)$; see (53) and (54), respectively. To be definite, assume that $X_1 < X_3$. In this case, formulas (53) and (54) imply that assumption (28) of Theorem 3.4 holds. The following statement asserts that assumption (29) is also satisfied.

Lemma 4.5. (i) *Suppose that (65) holds. Then, for each $\delta > 0$, there is an $X_M(\delta) \geq X_3(\delta)$ such that $\xi_M(x,\delta) = \xi_m(x,\delta)$ (cf. (57) and (58)), the functions $\xi_M(x,\delta)$ and (6) are twice continuously differentiable in x for $x > X_M(\delta)$, and*

$$\Phi_{xx}(x,\delta) = \varphi_{xx}(x,\delta+\xi_M(x,\delta)) - \frac{\left(\varphi_{x\delta}(x,\delta+\xi_M(x,\delta))\right)^2}{\varphi_{\delta\delta}(x,\delta+\xi_M(x,\delta)) - v''(\xi_M(x,\delta))}. \tag{67}$$

(ii) *If in addition*

$$(1-\varepsilon)\beta_2 - \gamma > 0, \tag{68}$$

then function (30) satisfies

$$g(x) \to -\infty \quad as \quad x \to +\infty. \tag{69}$$

Relation (69) ensures that condition (31) of Theorem 3.4 is satisfied if X_4 is sufficiently large.

Lemmas 4.3 and 4.5 imply that for $x > X_M = X_M(\delta)$,

$$\Phi(x,\delta) = \varphi\big(x,\delta+\xi_M(x,\delta)\big) - v\big(\xi_M(x,\delta)\big); \quad \Phi_x(x,\delta) = \varphi_x\big(x,\delta+\xi_M(x,\delta)\big). \tag{70}$$

Further, due to Lemmas 4.2 and 4.5, the function (30) is well-defined on the intervals $(0, X_3)$ and (X_M, ∞). Set

$$\mathcal{J} = \big\{x \in (X_3, X_M] : \ \xi_M(x,\delta) > \xi_m(x,\delta)\big\} \cup \{X_3\}. \tag{71}$$

We make an assumption that the following genericity condition is satisfied:

$$\varphi_{\delta\delta}\big(x,\delta+\xi_M(x,\delta)\big) - v''\big(\xi_M(x,\delta)\big) \neq 0, \qquad x \in (X_3, X_M] \setminus \mathcal{J}. \tag{72}$$

Then Lemma 4.3 implies that the function (30) is well-defined on the domain $x \in \mathbb{R}_+ \setminus \mathcal{J}$ and is not defined on $\mathcal{J}$. Therefore, conditions (31) and (32) of Theorem 3.4 are satisfied if there exists X_4 such that

$$X_4 > X_M; \quad g(x) = \pi(x) - \mathcal{L}\Phi(x) < 0 \quad \text{for all} \quad x > X_4; \tag{73}$$

$$g(x) > 0 \quad \text{for all} \quad (X_3, X_M] \setminus \mathcal{J}.$$

For the running profit (52), the equation $\mathcal{L}y(x) = \pi(x)$ has a particular solution

$$p(x) = \eta_0 + \eta_\gamma \delta^\varepsilon x^\gamma. \tag{74}$$

Therefore, functions (38) and (39) are given by

$$C_1(x) = -\frac{\eta_0\beta_2}{\beta_2-\beta_1}x^{-\beta_1} + \frac{\eta_\gamma(\gamma-\beta_2)\delta^\varepsilon}{\beta_2-\beta_1}x^{\gamma-\beta_1}; \tag{75}$$

$$C_2(x) = \frac{\eta_0\beta_1}{\beta_2-\beta_1}x^{-\beta_2} + \frac{\eta_\gamma(\beta_1-\gamma)\delta^\varepsilon}{\beta_2-\beta_1}x^{\gamma-\beta_2}; \tag{76}$$

$$D_1(x) = \frac{x^{-\beta_1}}{\beta_2-\beta_1}\big((\gamma-\beta_2)\eta_\gamma\delta^\varepsilon x^\gamma - \beta_2\eta_0 + \beta_2\Phi(x,\delta) - x\Phi_x(x,\delta)\big); \tag{77}$$

$$D_2(x) = \frac{x^{-\beta_2}}{\beta_2-\beta_1}\big((\beta_1-\gamma)\eta_\gamma\delta^\varepsilon x^\gamma + \beta_1\eta_0 - \beta_1\Phi(x,\delta) + x\Phi_x(x,\delta)\big). \tag{78}$$

Theorem 3.4 implies the following statement.

Proposition 4.6. *Let the running profit and the obstacle be defined by* (52), (53), (6). *Assume that relations* (4), (21), (65), (68), *and* (72) *hold. Given a* $\delta > 0$, *suppose that* $X_1 < X_3$, *where* $X_1 = X_1(\delta)$ *and* $X_3 = X_3(\delta)$ *are defined by* (53) *and* (54), *respectively. Suppose that* (29) *holds and there is an* $X_4 > X_M$ *such that function* (30) *satisfies* (73) *and function* (78) *satisfies* $D_1(X_4) < 0$. *Then there exists a pair of thresholds* $X_0 = X_0(\delta)$, $X_5 = X_5(\delta)$ *such that system* (24)–(27) *has a solution* $y(x) = y(x, \delta)$. *In particular, this function satisfies*

$$\begin{aligned} y(x) &= \Phi(x, \delta), \quad x \in [0, X_0] \cup [X_5, \infty), \\ y(x) &> \Phi(x, \delta), \quad x \in (X_0, X_5), \end{aligned}$$

and is a unique solution of the V.I. (9)–(13). *The value function* (5) *is equal to this solution, and the optimal stopping times are given by*

$$\hat{\tau}_1 = \inf\{t \geq 0 | \; X(t) \leq X_0\}, \qquad \hat{\tau}_0 = \inf\{t \geq 0 | \; X(t)) \geq X_5\} \tag{79}$$

with the optimal investment amount

$$\hat{\xi} = \text{argmax}_{\xi \in \mathbb{R}_+} \left(\varphi\big(\max\{X_5, x\}, \delta + \xi\big) - v(\xi)\right). \tag{80}$$

We assumed above that $X_3 > X_1$. A slight modification of the proof of Proposition 4.6 (we omit the details) allows us to obtain its counterpart for the complementary case $X_1 \geq X_3$.

Proposition 4.7. *Assume that relations* (4), (21), (65), (68), *and* (72) *hold, and* $X_1 = X_1(\delta)$, $X_3 = X_3(\delta)$ *satisfy* $X_1 \geq X_3$ *for a given* $\delta > 0$. *Let relation* (73) *hold with an* $X_4 > X_M$. *Suppose that there are* X', X'' *satisfying* $X'' > X' > X_4$ *such that*

$$D_1(X') = 0, \quad D_1(X'') = C_1(X_3), \quad D_2(X'') > C_2(X_3). \tag{81}$$

Then there exists a pair of thresholds $X_0 = X_0(\delta), X_5 = X_5(\delta)$ *such that system* (24)–(27) *has a solution* $y(x) = y(x, \delta)$. *The value function* (5) *is equal to this solution, and the optimal stopping times and investment amount are given by* (79) *and* (80).

Remark 4.8. According to Propositions 4.6 and 4.7, given an initial capital $\delta > 0$ and a two-threshold solution, the (a priori unknown) thresholds X_0 and X_5 determine the optimal strategy as follows. If the price (1) of the commodity initially belongs to the continuation (waiting) region (X_0, X_5) and exits this region through the left end, then it is wiser to exit without any additional investment at the first moment τ_1 when $X(\tau_1) = X_0$. On the other hand, if the price exits the continuation region from the right end, then it is best to apply the one-time investment at the first moment τ_0 when $X(\tau_0) = X_5$. The optimal amount $\hat{\xi}$ of this investment is given by (80). In this case, the exit time is determined as the first moment $\tau_1 > \tau_0$ when $\varphi(X(\tau_1), \delta + \hat{\xi}) = 0$. Naturally, if the initial price $x = X(0)$ does not exceed X_0, then it is best to exit immediately, while if $x \geq X_5$, then it is best to invest at the initial moment, and the optimal amount of the investment is given by (80).

4.4. Example: Investment with fixed cost and linear variable cost.

4.4.1. Existence of a two-threshold solution. In this section we consider a particular investment cost of the form

$$v(\xi) = k\xi + K_0\mathbf{1}_{\xi>0}, \tag{82}$$

where $k\xi$ is the variable cost proportional to the investment and $K_0 > 0$ is a fixed cost.[2] Let us discuss the implications of Propositions 4.6 and 4.7 for this case. Here, the obstacle (6) is given by

$$\Phi(x,\delta) = \sup_{\xi\geq 0}\left[\varphi(x,\xi+\delta) - k\xi - K_0\mathbf{1}_{\xi>0}\right], \tag{83}$$

where φ is the function (53). We will see that this obstacle has a unique point of non-smoothness, X_3, induced by the fixed cost of investment. Equivalently,

$$\Phi(x,\delta) = \max\left\{\varphi(x,\delta), \hat{\Phi}(x,\delta) - K_0\right\}, \tag{84}$$

where

$$\hat{\Phi}(x,\delta) = \sup_{\Delta\geq\delta}\left[\varphi(x,\Delta) - k\Delta\right] + k\delta \tag{85}$$

is the profit of the investment with the same variable cost and zero fixed cost.

First, we specify the result of Lemma 4.2 for the obstacle (83). Formulas (84) and (85) show that properties of the function $\varphi(x,\delta) - k\delta$ play an important role for the analysis of the obstacle. Define

$$\rho = \left(\frac{\gamma - \varepsilon\beta_1}{\gamma(1-\varepsilon)}\right)^{\frac{\gamma}{\gamma-\beta_1}} > 1$$

and

$$\delta_1(x) = \left(\frac{\lambda}{x^\gamma}\right)^{\frac{1}{\varepsilon}}, \qquad \delta_2(x) = \delta_1(x)\rho^{\frac{1}{\varepsilon}}, \qquad x > 0. \tag{86}$$

Note that $\delta_1(x)$ is the inverse of the function $X_1(\delta)$ defined in (53), the functions $\delta_1(x), \delta_2(x)$ strictly decrease, and their inverse functions $X_1(\delta)$, $X_2(\delta)$, respectively, satisfy

$$X_2(\delta) = X_1(\delta)\rho^{\frac{1}{\gamma}} > X_1(\delta). \tag{87}$$

Lemma 4.9. *Therefore, the equation $\varphi_\delta(x,\delta_2(x)) = k$ has a unique solution $x = x^*$. Moreover, if $x < x^*$, then $\varphi_\delta(x,\delta) < k$ for all $\delta \geq 0$. On the other hand, if $x > x^*$, then the equation $\varphi_\delta(x,\delta) = k$ has exactly two solutions $\delta_3'(x), \delta_3(x)$ satisfying*

$$\delta_1(x) < \delta_3'(x) < \delta_2(x) < \delta_3(x) \tag{88}$$

and

$$\varphi_\delta(x,\delta) < k, \quad \delta \in [0,\delta_3'(x)) \cup (\delta_3(x),\infty); \qquad \varphi_\delta(x,\delta) > k, \quad \delta \in (\delta_3'(x),\delta_3(x)). \tag{89}$$

[2]The case of $K_0 = 0$ was studied in [5].

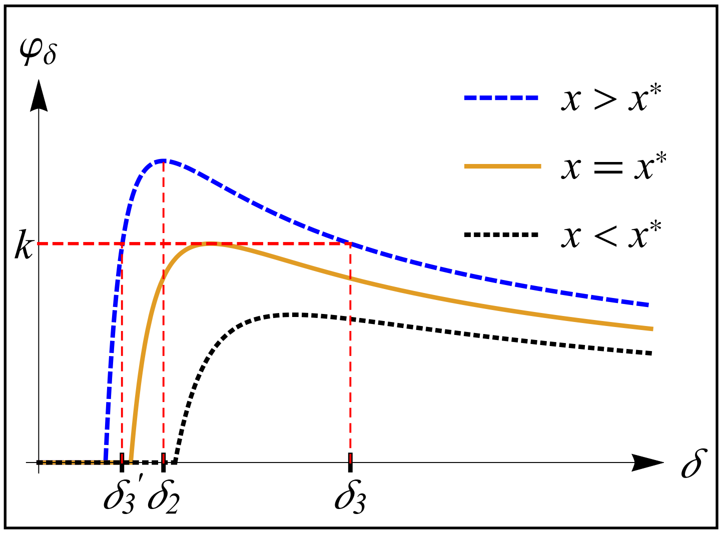

Figure 2. *The function $\varphi_\delta(x, \cdot)$ for $x = 7, 8.52, 10$. Parameters are the same as in Figure 1; $x^* = 8.52$. The horizontal dashed line is $\varphi_\delta(x, \cdot) = k$ for $k = 20$. The points $\delta_2 = \delta_2(x)$, $\delta_3 = \delta_3(x)$, $\delta_3' = \delta_3'(x)$ are shown for $x = 10$.*

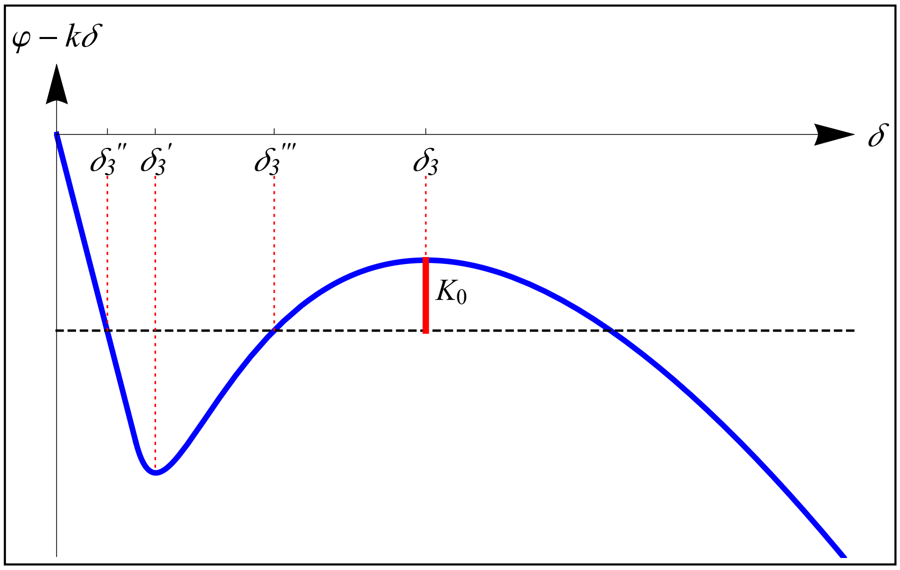

Figure 3. *The graph of $\varphi(x, \delta) - k\delta$ for fixed $x = 10 > x^*$. Other parameters are the same as in Figure 1. The horizontal dashed line shows the value of $\varphi(x, \delta_3(x)) - k\delta_3(x) - K_0$.*

According to Lemma 4.9, for any $x \le x^*$, the function $\varphi(x, \delta) - k\delta$ strictly decreases in δ on the whole domain $\delta \ge 0$, while for any $x > x^*$, the function $\varphi(x, \delta) - k\delta$ strictly decreases in δ on the intervals $[0, \delta_3'(x)]$ and $[\delta_3(x), \infty)$, strictly increases on the interval $\delta_3'(x) \le \delta \le \delta_3(x)$, and attains a unique local minimum at the point $\delta = \delta_3'(x)$ and a unique local maximum at the point $\delta = \delta_3(x)$; see Figures 2 and 3.

We need to consider a few additional properties of the function $\varphi(x,\delta)-k\delta$. Define

$$\tilde{\Delta}(x)=\varphi(x,\delta_3(x))-k\delta_3(x)-\big(\varphi(x,\delta_3'(x))-k\delta_3'(x)\big). \tag{90}$$

Lemma 4.10. *Equation $\tilde{\Delta}(x)=K_0$ has a unique solution $x^{***}>x^*$. Equation $\varphi(x,\delta_3(x))-k\delta_3(x)=K_0$ has a unique solution $x^{**}>x^{***}$. Further, for every $x>x^{***}$, the equation*

$$\varphi(x,\delta_3(x))-k\delta_3(x)-K_0=\varphi(x,\delta)-k\delta \tag{91}$$

has a unique solution $\delta=\delta_3'''(x)$ in the interval $\delta_3'(x)<\delta_3'''(x)<\delta_3(x)$. In addition, for every $x\in(x^{***},x^{**})$, (91) has a unique solution $\delta=\delta_3''(x)$ satisfying $0<\delta_3''(x)<\delta_3'(x)$ (see Figure 3). The function $\delta_3'''(x)$ strictly increases on its domain (x^{***},∞), the function $\delta_3''(x)$ strictly decreases on its domain $(x^{***},x^{**}]$, and these functions are smooth and $\delta_3''(x^{***})=\delta_3'''(x^{***})$, $\delta_3''(x^{**})=0$.

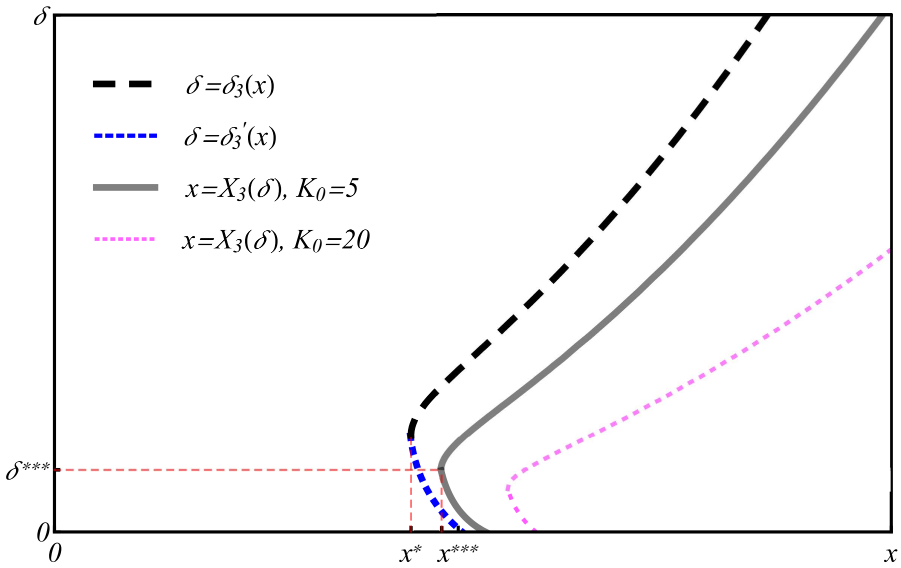

Figure 4. *The graphs of $X_3(\delta)$ for $K_0=5,20$. Other parameters are the same as in Figure 1.*

Having defined the functions

$$\delta_3''':[x^{***},\infty)\to[\delta^{***},\infty),\qquad \delta_3'':[x^{***},x^{**}]\to[0,\delta^{***}],$$

where $\delta^{***}=\delta_3'''(x^{***})=\delta_3''(x^{***})$, we can specify formula (54) for the obstacle (6) with the running profit (52) and the investment cost (82). By Lemma 4.10, these functions are invertible; see Figure 4. Denote by $(\delta_3''')^{-1}$, $(\delta_3'')^{-1}$ their inverse functions, respectively.

Lemma 4.11. *The obstacle function (83) satisfies*

$$\Phi(x,\delta)=\begin{cases}\varphi(x,\delta), & x\le X_3(\delta),\\ \varphi(x,\delta_3(x))-k\delta_3(x)+k\delta-K_0, & x>X_3(\delta),\end{cases} \tag{92}$$

and (55)*, where* $X_3(\delta)$ *is the continuous concatenation of the inverse functions* $(\delta_3''')^{-1}$, $(\delta_3'')^{-1}$ *(with* $\delta_3'''(x)$*,* $\delta_3''(x)$ *defined in Lemma* 4.10*):*

$$X_3(\delta) = \begin{cases} (\delta_3'')^{-1}(\delta), & 0 \le \delta \le \delta^{***}, \\ (\delta_3''')^{-1}(\delta), & \delta > \delta^{***}. \end{cases} \tag{93}$$

Remark 4.12. From relations (89) and the definition of the functions $\delta_3''(x)$, $\delta'''(x)$ (cf. (91)), it follows that the domain $x > X_3(\delta)$, where $\Phi > \varphi$ (see Figure 4), shrinks with increasing K_0.

Lemma 4.11 implies that the values (58) satisfy $\xi_m(x,\delta) = \xi_M(x,\delta) = \delta_3(x) - \delta$. Therefore $X_M = X_3$ (see Lemma 4.5) and the set (71) consists of one point for each $\delta > 0$,

$$\mathcal{J} = \{X_3\}.$$

In particular, (72) is trivially satisfied. Conditions (4) and (65) of Propositions 4.6 and 4.7 are also trivially satisfied for the investment cost (82).

Lemma 4.11 allows us to express function (30) explicitly in terms of the function $\delta_3(x)$:

$$\begin{aligned} g(x,\delta) &= \pi(x,\delta) - \mathcal{L}(\varphi(x,\delta_3(x)) - k\delta_3(x) + k\delta - K_0) \\ &= -\eta_\gamma Q(\gamma)(\delta_3(x)^\varepsilon - \delta^\varepsilon)x^\gamma + r(\delta_3(x)-\delta)\varepsilon\eta_\gamma\delta_3(x)^{\varepsilon-1}x^\gamma\left[1 - \left(\frac{\delta_1(x)}{\delta_3(x)}\right)^{\frac{\varepsilon(\gamma-\beta_1)}{\gamma}}\right] \\ &\quad - \frac{\sigma^2}{2}\varepsilon\eta_\gamma\delta_3(x)^\varepsilon x^\gamma \frac{\left[\gamma - \beta_1\left(\frac{\delta_1(x)}{\delta_3(x)}\right)^{\frac{\varepsilon(\gamma-\beta_1)}{\gamma}}\right]^2}{\varepsilon - 1 + \left(1 - \varepsilon\frac{\beta_1}{\gamma}\right)\left(\frac{\delta_1(x)}{\delta_3(x)}\right)^{\frac{\varepsilon(\gamma-\beta_1)}{\gamma}}} + rK_0, \quad x \in (X_3(\delta),\infty), \end{aligned} \tag{94}$$

where $\delta_3(x)$ is implicitly uniquely defined by the relations

$$\varphi_\delta(x,\delta_3(x)) = k, \qquad \delta_3(x) > \delta_2(x),$$

and $\delta_1(x)$, $\delta_2(x)$ are given by (86). Further, substituting formula (92) in (77) and (78) gives

$$D_1(x) = \frac{\eta_\gamma(\beta_2-\gamma)(\delta_3(x)^\varepsilon - \delta^\varepsilon)}{\beta_2-\beta_1}x^{\gamma-\beta_1} - \frac{k\beta_2(\delta_3(x)-\delta) + \beta_2 K_0}{\beta_2-\beta_1}x^{-\beta_1} + \frac{\eta_0\gamma}{(\beta_1-\gamma)}\left(\frac{\delta_3(x)^\varepsilon}{\lambda}\right)^{\frac{\beta_1}{\gamma}} \tag{95}$$

and

$$D_2(x) = \frac{\eta_\gamma(\gamma-\beta_1)(\delta_3(x)^\varepsilon - \delta^\varepsilon)}{\beta_2-\beta_1}x^{\gamma-\beta_2} + \frac{k\beta_1(\delta_3(x)-\delta)+\beta_1 K_0}{\beta_2-\beta_1}x^{-\beta_2} \tag{96}$$

for $x > X_3(\delta)$. Therefore, Proposition 4.6 implies the following statement.

Theorem 4.13. *Assume that the running profit and the investment cost are defined by* (52) *and* (82)*. Let* (21) *and* (68) *hold. Given a* $\delta > 0$*, let* $X_1 < X_3$*, where* $X_1 = X_1(\delta)$ *and*

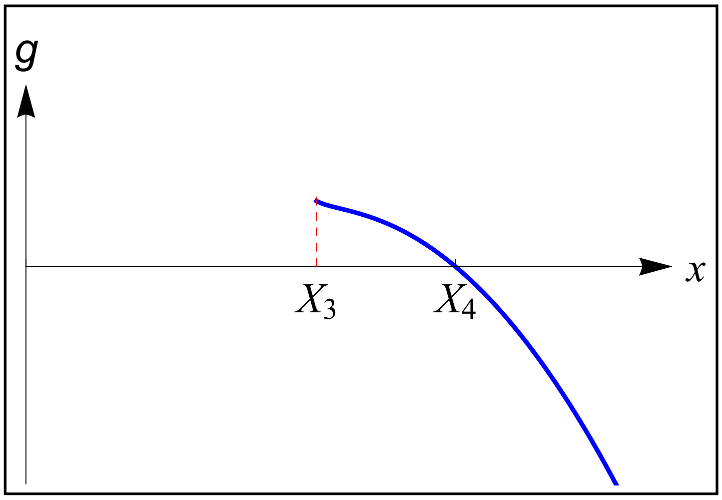

Figure 5. *The graph of the function $g(\cdot,\delta)$ with fixed $\delta = 3$ for the investment cost of the form $v(\xi) = k\xi + K_0\mathbf{1}_{\xi>0}$ with $k = 20, K_0 = 5$. Other parameters are the same as in Figure 1.*

$X_3 = X_3(\delta)$ are defined by (53) and (93), respectively. Suppose that there is an $X_4 \geq X_3$ such that function (94) satisfies

$$(97) \qquad g(x,\delta) > 0 \quad for \quad X_3 < x < X_4; \qquad g(x,\delta) < 0 \quad for \quad x > X_4$$

(see Figure 5), and function (95) satisfies $D_1(X_4) < 0$. Then there exists a pair of thresholds X_0, X_5 such that system (24)–(27) has a solution $y(x)$ (see Figure 6). The value function (5) is equal to this solution, and the optimal stopping times and investment amount are given by (79) and (80).

It follows from the proof of Theorem 3.4 that the thresholds have to satisfy $X_0 < X_1$, $X_5 > X_4$. In particular, condition (97) is satisfied if $g(x,\delta) < 0$ for all $x > X_3$, in which case $X_4 = X_3$. On the other hand, if $g(X_3,\delta) > 0$, then relation (69) ensures the existence of a zero X_4 of g such that $g(x,\delta) < 0$ for $x > X_4$, as required by the second part of condition (97). If X_4 is a unique zero of g, then also the first part of condition (97) is satisfied. This is the case for all the numerical simulations, and the corresponding parameter values, presented in the following section. However, g is not necessarily monotone with respect to x. The authors don't know whether condition (97) can be violated due to g having multiple zeros. The simple condition $D_1(X_4) < 0$ is quite strong and more restrictive than the assumptions of Theorem 3.3. However, this condition is useful because it is easy to check for a given parameter set. It holds true for realistic parameter values presented in the next section.

Proposition 4.7 can be adapted in a fashion similar to that in the case $X_1 \geq X_3$. The optimal control/exit rule is described in Remark 4.8.

4.4.2. Numerical solutions and comparative statics. Figure 7 presents comparative statics for the value function y with a fixed δ with respect to the parameters K_0, k, γ, r, and σ. Panel A shows the effect of the fixed cost K_0. Higher fixed cost reduces the benefit of investment, and as a result the firm delays investment. In panel B, the value gets smaller as the

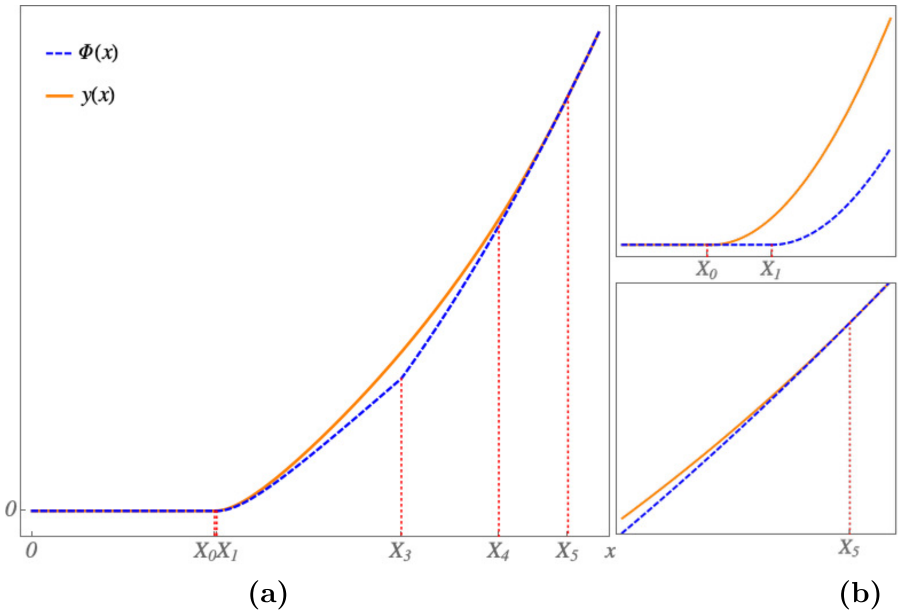

Figure 6. (a) *Solution y (solid line) of problem* (9)–(11) *and the obstacle Φ (dashed line). Φ has a discontinuity of the derivative at the point $x = X_3$.* (b) *Zoom-in of panel* (a) *near the point $x = X_0$ (above) and near the point $x = X_5$ (below). Parameters are the same as in Figure* 1.

variable cost k increases, and hence the firm exits earlier. It also delays investment, since the benefit of investment is lower. In panel C, as γ increases, the return on investment grows. Thus, the firm exits later and invests earlier. Panel D shows the comparison of two discount rates r. The higher the discount rate, the earlier the firm chooses to exit. Also it invests later. As shown in panel E, increasing σ delays both exit and investment decisions because higher volatility increases the option value of waiting for both options.

Figure 8 shows comparative statics for the parameters of the optimal strategy, i.e., the exit threshold X_0, the investment threshold X_5, and the investment lump $\delta_3(X_5) - \delta$. Both fixed cost (K_0) and proportional cost (k) have similar effects on the exit and investment decisions. As K_0 and k increase, the firm exits earlier (panels A, D) and invests later (panels B, E). As K_0 increases, the investment amount also increases (panel C), but it decreases (panel F) when k increases, since k is a proportional cost.

Higher γ implies a higher return to scale in X. Thus, as γ increases, the firm exits later (panel G) and invests more (panel H). Importantly, the timing of investment is U-shaped in γ. On the one hand, increasing γ increases the net present value of investment, thereby hastening investment. On the other hand, higher γ means that there is more complementarity between δ and X, rendering the option value of waiting for a larger value of X to be reached more valuable, thereby delaying investment. The first effect dominates for low γ, while the second effect dominates for large γ. We stress that the second effect arises in our model because of the additional flexibility to choose the scale of investment, since waiting for a higher X also increases the investment lump (panel I). Hence, when the size of investment is fixed, X_5 is monotonically decreasing in γ.

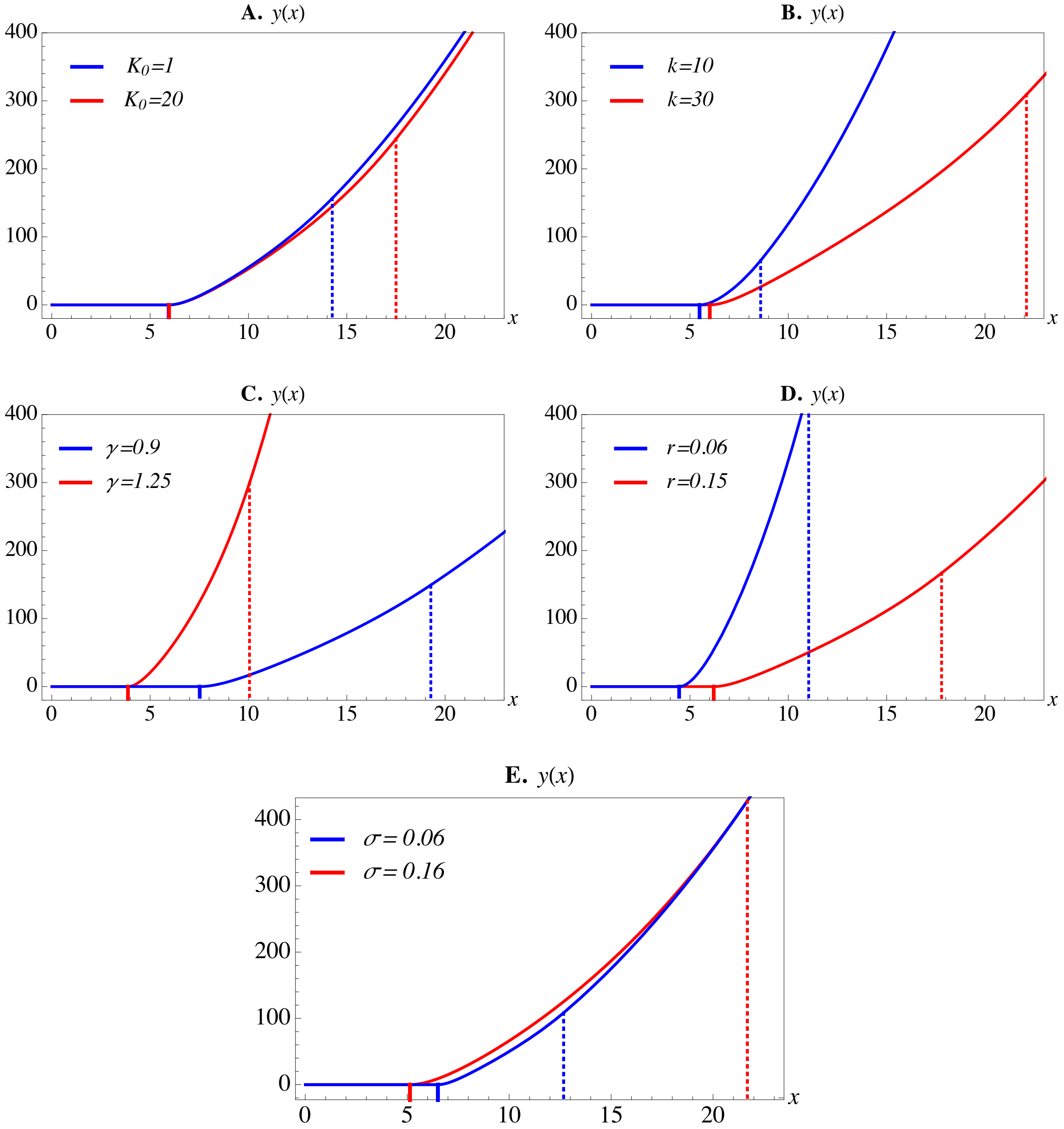

Figure 7. *Comparative statics for the value function $y(x, \delta)$ with respect to the parameters K_0, k, γ, r, and σ for a fixed δ. The solid vertical lines correspond to the exit threshold $X_0(\delta)$, the dashed vertical lines correspond to the investment thresholds $X_5(\delta)$. Baseline parameter values are $K_0 = 5$, $k = 20$, $r = 0.12$, $\sigma = 0.10$, $\mu = 0.01$, $\alpha_0 = 13.5$, $\alpha_1 = 1$, $\theta = 0.1$, $\epsilon = 0.5$, $\gamma = 1$, and $\delta = 3$.*

A larger discount rate r implies earlier exit (panel J) (since the option value of eventually recovering is discounted at a higher rate) and delayed investment (panel K) with lower investment lump (panel L) (since the benefits of investment are discounted at a higher rate).

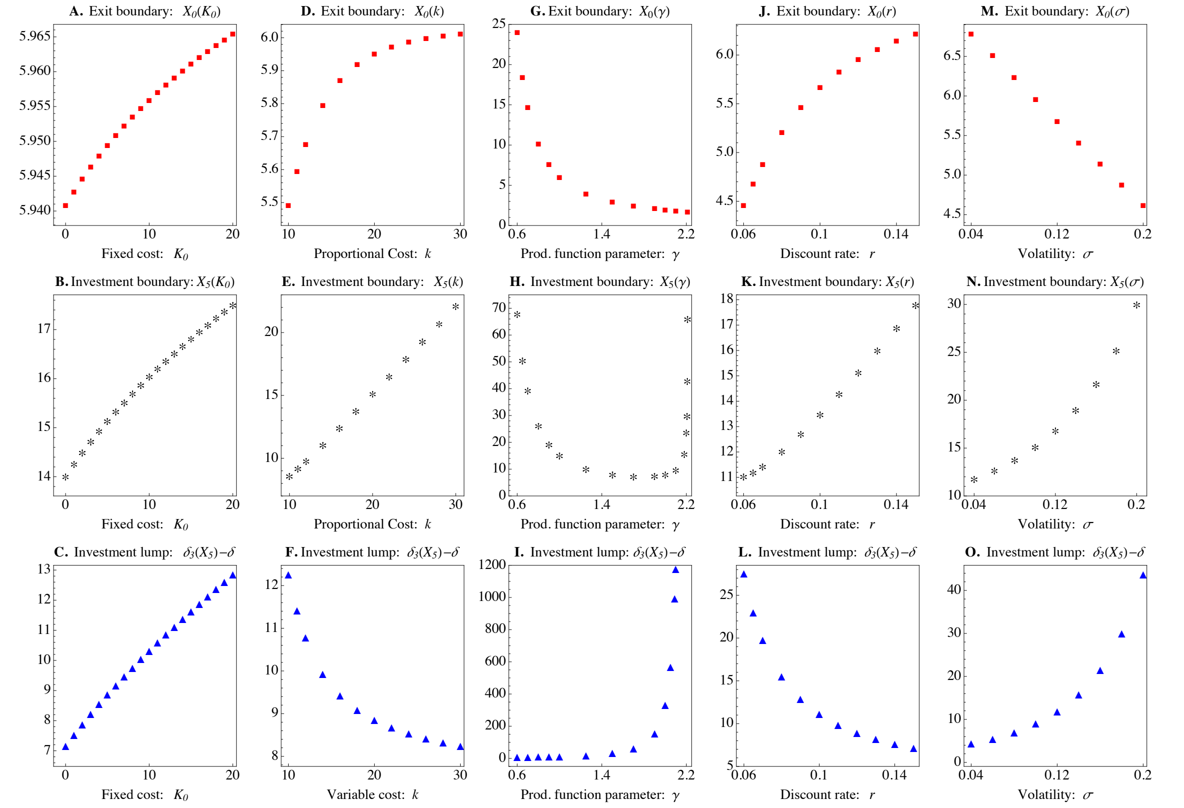

Figure 8. *Comparative statics of the exit boundary, investment boundary, and investment lump for different values of the parameters K_0, k, γ, r, and σ for a fixed δ. Baseline parameter values are the same as in Figure 7.*

A larger σ delays both exit and investment decisions (panels M, N), because the value of waiting increases in a volatile market. Also, the investment lump is larger if the volatility is higher (panel O).

The examples presented in Figure 8 satisfy all the conditions of Theorem 4.13.

In order to obtain the value function $y(x)$ and the threshold values X_0, X_5, we solved the system

$$C_1(X_0) = D_1(X_5), \qquad C_2(X_0) = D_2(X_5), \tag{98}$$

where C_i, D_i are defined by formulas (75), (76), (95), and (96). According to (40), the functions C_i are invertible on the interval $X_0 \in (0, X_1)$, and therefore system (98) is equivalent to the scalar equation

$$C_1^{-1}(D_1(X_5)) = C_2^{-1}(D_2(X_5)).$$

After finding the upper threshold X_5 from this equation, we obtained the lower threshold as $X_0 = C_1^{-1}(D_1(X_5))$. The value function is given by

$$y(x) = p(x) + C_1(X_0)x^{\beta_1} + C_2(X_0)x^{\beta_2} = p(x) + D_1(X_5)x^{\beta_1} + D_2(X_5)x^{\beta_2}$$

on the interval $x \in (X_0, X_5)$ with $p(x) = \eta_0 + \eta_\gamma \delta^\varepsilon x^\gamma$ (cf. (74)) and by $y(x) = \Phi(x)$ outside this interval (see section A.5 of the appendix for details).

5. Conclusions. We showed that the value function of a simple real options problem with an exit option and a one-time investment option with fixed and variable components of the investment cost satisfies a free boundary problem with a nonsmooth obstacle. Due to nonsmoothness, we invoked the concept of a weak solution. We showed how the existence and uniqueness of a solution for a weak formulation of the variational inequality with a nonsmooth obstacle in one dimension can be obtained using a constructive intuitive penalty method. However, the weak formulation does not lead to a straightforward control strategy, like a two threshold strategy. We further proposed sufficient conditions for the existence of a strong solution with the coincidence set defined by two thresholds. The crucial assumption is that the increments of the derivative of the obstacle at the discontinuity points of this derivative are all positive. This condition is naturally satisfied for the real options problem. The theorem, which ensures the existence of a strong two-threshold solution, results in a naturally simple optimal investment strategy with stopping times expressed through the thresholds.

It would be natural to consider real options problems with a wider class of controls which allow for a sequence of compound capacity expansion options, as, for example, in [4]. In this case, dynamic programming leads to a quasi-variational inequality [8]. This more complex setting is beyond the scope of the present paper and will be the subject of future work.

Appendix A. Proofs.

A.1. Proof of Lemmas 4.2 and 4.3.

A.1.1. Proof of Lemma 4.2. By definition of φ, its mixed second derivative satisfies $\varphi_{x\delta} = 0$ for $x < X_1(\delta)$ and

$$\varphi_{x\delta} = \varepsilon\gamma\eta_\gamma\delta^{\varepsilon-1}x^{\gamma-1} + \frac{\varepsilon\beta_1^2}{\gamma}\frac{\eta_0\gamma}{\beta_1-\gamma}\frac{\delta^{\frac{\varepsilon\beta_1}{\gamma}-1}}{\lambda^{\frac{\beta_1}{\gamma}}}x^{\beta_1-1} > 0 \quad \text{for} \quad x > X_1(\delta). \tag{99}$$

Define the function

$$\phi(x,\delta,\xi) = \varphi(x,\delta+\xi) - v(\xi) - \varphi(x,\delta), \quad \xi > 0.$$

This function increases in x because

$$\phi(x+\Delta x,\delta,\xi) - \phi(x,\delta,\xi) = \varphi(x+\Delta x,\delta+\xi) - \varphi(x+\Delta x,\delta) - \varphi(x,\delta+\xi) + \varphi(x,\delta)$$

$$= \int_0^\xi \varphi_\delta(x+\Delta x,\delta+\tau)\,d\tau - \int_0^\xi \varphi_\delta(x,\delta+\tau)\,d\tau = \int_0^\xi d\tau \int_0^{\Delta x} \varphi_{x\delta}(x+s,\delta+\tau)\,ds \tag{100}$$

and $\varphi_{x\delta} \geq 0$. Therefore, the continuous function

$$\psi(x,\delta) := \sup_{\xi>0} \phi(x,\delta,\xi)$$

also increases in x. Further,

$$\psi(0+,\delta) = \sup_{\xi>0}\bigl(-v(\xi)\bigr) = -v(0+) < 0,$$

$$\psi(+\infty,\delta)=\lim_{x\to+\infty}\sup_{\xi>0}\big[\eta_\gamma x^\gamma\big((\delta+\xi)^\varepsilon-\delta^\varepsilon\big)-v(\xi)\big]\geq\lim_{x\to+\infty}\big[\eta_\gamma x^\gamma\big((\delta+1)^\varepsilon-\delta^\varepsilon\big)-v(1)\big]=+\infty.$$

Hence, there is an interval $[x_-(\delta),x_+(\delta)]\subset(0,\infty)$ such that

$$\psi(x,\delta)<0,\quad x<x_-(\delta);\qquad \psi(x,\delta)=0,\quad x\in[x_-(\delta),x_+(\delta)];\qquad \psi(x,\delta)>0,\quad x>x_+(\delta).$$

Let us show that in fact $x_-(\delta)=x_+(\delta)$. Relations (53), (4) imply that $\phi(x,\delta,\xi)\to-\infty$ as $\xi\to\infty$. On the other hand, $\phi(x,\delta,0+)=-v(0+)<0$. Hence, the relationship $\psi(x_-(\delta),\delta)=0$ and the definition of ψ imply that there is a $\xi_-(\delta)>0$ such that $\phi(x_-(\delta),\delta,\xi_-(\delta))=0$. Since ϕ increases in x,

$$0=\phi(x_-(\delta),\delta,\xi_-(\delta))\leq\phi(x_+(\delta),\delta,\xi_-(\delta))\leq\psi(x_+(\delta),\delta)=0,\tag{101}$$

and hence relations (100) with $x=x_-(\delta)$, $x+\Delta x=x_+(\delta)$ and $\xi=\xi_-(\delta)$ imply that either $x_-(\delta)=x_+(\delta)$ or $\varphi_{x\delta}(x,\delta+\xi)=0$ for all $x\in[x_-(\delta),x_+(\delta)]$, $\xi\in(0,\xi_-(\delta)]$. The latter option is only possible if $\varphi(x,\delta+\xi)=0$ in the rectangle $x\in[x_-(\delta),x_+(\delta)]$, $\xi\in[0,\xi_-(\delta)]$. But in this case $\varphi(x_-(\delta),\delta+\xi_-(\delta))=\varphi(x_-(\delta),\delta)=0$ and further $\phi(x_-(\delta),\delta,\xi_-(\delta))=-v(\xi_-(\delta))<0$, which contradicts (101). Hence the only possible option is $x_-(\delta)=x_+(\delta)$.

Further, since the continuous function ψ increases in x and the equation $\psi(x,\delta)=0$ has a unique solution $x=x_-(\delta)$ for each $\delta>0$, the implicit function theorem implies that $x_-(\delta)$ is a continuous function. Hence, we set $X_3(\delta)=x_-(\delta)=x_+(\delta)$ and conclude from the definition of ψ and (6) that (54) holds. As $\psi(x,\delta)>0$ for $x>x_+(\delta)$, we also see that (55) holds. ■

A.1.2. Proof of Lemma 4.3. Let us fix $\delta>0$, $x\geq X_3(\delta)$ and, to shorten the notation, denote $\zeta(x,\delta,\xi)=\varphi(x,\delta+\xi)-v(\xi)$. Then relations (56) can be written as

$$\Phi(x,\delta)-\zeta(x,\delta,0+)\geq v(0+)=K_0>0,\qquad \zeta(x,\delta,\xi)=-\infty\quad\text{as}\quad\xi\to+\infty\tag{102}$$

and Lemma 4.2 implies that $\Phi(x,\delta)=\sup_{\xi>0}\zeta(x,\delta,\xi)$. Therefore,

$$\begin{aligned}\Phi(x+\Delta x,\delta)=\sup_{\xi>0}\zeta(x+\Delta x,\delta,\xi)&\geq\zeta\big(x+\Delta x,\delta,\xi_M(x,\delta)\big)\\&=\zeta\big(x,\delta,\xi_M(x,\delta)\big)+\Delta x\,\zeta_x\big(x,\delta,\xi_M(x,\delta)\big)+o(\Delta x)\\&=\Phi(x,\delta)+\Delta x\,\varphi_x\big(x,\delta+\xi_M(x,\delta)\big)+o(\Delta x),\end{aligned}\tag{103}$$

where $o(\Delta x)/\Delta x\to0$ as $\Delta x\to0+$. On the other hand, due to (58) and (102), given any $\varepsilon_0>0$ there is a $\varepsilon_1>0$ such that $\xi>\xi_M(x,\delta)+\varepsilon_0$ implies $\zeta(x,\delta,\xi)<\Phi(x,\delta)-\varepsilon_1$. Therefore, using (102) and the continuity of $\zeta(x,\delta,\xi)$,

$$\sup_{\xi>\xi_M(x,\delta)+\varepsilon_0}\zeta(x+\Delta x,\delta,\xi)<\Phi(x,\delta)-\varepsilon_1/2\tag{104}$$

if $\Delta x>0$ is sufficiently small. Further, if $\xi\leq\xi_M(x,\delta)+\varepsilon_0$, then

$$\begin{aligned}\zeta(x+\Delta x,\delta,\xi)&=\zeta(x,\delta,\xi)+\Delta x\,\zeta_x(x,\delta,\xi)+o(\Delta x)\leq\Phi(x,\delta)+\Delta x\,\zeta_x(x,\delta,\xi)+o(\Delta x)\\&=\Phi(x,\delta)+\Delta x\,\zeta_x\big(x,\delta,\xi_M(x,\delta)\big)+\Delta x\int_{\xi_M(x,\delta)}^{\xi}\zeta_{x\xi}(x,\delta,\eta)\,d\eta+o(\Delta x)\\&=\Phi(x,\delta)+\Delta x\,\varphi_x\big(x,\delta+\xi_M(x,\delta)\big)+\Delta x\int_{\xi_M(x,\delta)}^{\xi}\varphi_{x\delta}(x,\delta+\eta)\,d\eta+o(\Delta x).\end{aligned}$$

The integral in this expression is nonpositive for all $\xi \le \xi_M(x,\delta)$ because $\varphi_{x\delta}(x,\delta) \ge 0$; on the other hand, this integral has the order $O(\varepsilon_0)$ for $\xi_M(x,\delta) < \xi \le \xi_M(x,\delta)+\varepsilon_0$. Hence, for $\Delta x > 0$,

$$\sup_{\xi \le \xi_M(x,\delta)+\varepsilon_0} \zeta(x+\Delta x,\delta,\xi) \le \Phi(x,\delta) + \Delta x\, \varphi_x\big(x,\delta+\xi_M(x,\delta)\big) + \Delta x\, O(\varepsilon_0) + o(\Delta x).$$

Combining this with (104), we obtain

$$\Phi(x+\Delta x,\delta) \le \Phi(x,\delta) + \Delta x\, \varphi_x\big(x,\delta+\xi_M(x,\delta)\big) + \Delta x\, O(\varepsilon_0) + o(\Delta x)$$

for any sufficiently small $\Delta x > 0$. Since $\varepsilon_0 > 0$ is arbitrarily small, this relation together with (103) implies $D_x^+\Phi(x,\delta) = \varphi_x\big(x,\delta+\xi_M(x,\delta)\big)$, i.e., the first of relations (59). The second relation can be obtained similarly for $x > X_3(\delta)$.

Let us notice that

$$x > X_1(\delta+\xi_m(x,\delta)) \ge X_1(\delta+\xi_M(x,\delta)). \tag{105}$$

Indeed, according to (53), the opposite inequality $x \le X_1(\delta+\xi_m(x,\delta))$ implies $\varphi\big(x,\delta+\xi_m(x,\delta)\big) = 0$ and hence $\Phi(x,\delta) = -v\big(\xi_m(x,\delta)\big) < 0$, which contradicts the nonnegativity of Φ. Relations (99) and (105) imply

$$\varphi_{x\delta}(x,\delta+\xi) > 0 \quad \text{for} \quad \xi \ge \xi_m(x,\delta). \tag{106}$$

Therefore, from (59) it follows that if $\xi_M(x,\delta) > \xi_m(x,\delta)$, then

$$D_x^+\Phi(x,\delta) - D_x^-\Phi(x,\delta) = \int_{\xi_m(x,\delta)}^{\xi_M(x,\delta)} \varphi_{x\delta}(x,\delta+\eta)\, d\eta > 0,$$

which proves (60).

Now, suppose that conditions (61) hold, i.e., $\xi_M(x,\delta) = \xi_m(x,\delta)$ and $\zeta_{\xi\xi}\big(x,\delta,\xi_M(x,\delta)\big) \ne 0$. In this case, since $\Phi(x,\delta) = \sup_{\xi>0}\zeta(x,\delta,\xi) = \zeta\big(x,\delta,\xi_M(x,\delta)\big)$ implies $\zeta_\xi\big(x,\delta,\xi_M(x,\delta)\big) = 0$, the function $\zeta_\xi(\cdot,\delta,\cdot)$ of two variables with the fixed δ satisfies the conditions of the implicit function theorem in a neighborhood of the point $(x,\xi_M(x,\delta))$. Hence, there is a smooth function $\xi_*(\hat{x})$ such that $\zeta_\xi(\hat{x},\delta,\xi_*(\hat{x})) = 0$ in a neighborhood of the point x and $\xi_*(x) = \xi_M(x,\delta)$. Since $\xi = \xi_M(x,\delta) = \xi_M(x,\delta) = \xi_*(x)$ is a unique point of global maximum for the function $\zeta(x,\delta,\cdot)$ of one variable ξ (with x,δ fixed), and this point satisfies $\zeta_{\xi\xi}(x,\delta,\xi_*(x)) < 0$, we conclude by continuity that there are open intervals $\mathcal{O}_\xi \ni \xi_*(x)$ and $\mathcal{O}_x \ni x$ such that if $\hat{x} \in \mathcal{O}_x$, then the function $\zeta(\hat{x},\delta,\cdot)$ of one variable ξ has a unique critical point $\xi = \xi_*(\hat{x})$ within the interval $\mathcal{O}_\xi$, and this is a maximum point. Furthermore, relations (102) ensure that $\xi = \xi_*(\hat{x})$ is a unique point of global maximum for the function $\zeta(\hat{x},\delta,\cdot)$ whenever $\hat{x}$ is sufficiently close to x. Therefore, $\Phi(\hat{x},\delta) = \zeta(\hat{x},\delta,\xi_*(\hat{x}))$ in a neighborhood of the point x, and consequently $\Phi_{xx}(x,\delta)$ is well-defined.

Finally, for $x = X_3(\delta)$, Lemma 4.2 implies that $D_x^-\Phi\big(X_3(\delta),\delta\big) = \varphi_x\big(X_3(\delta),\delta\big)$. On the other hand, we have already shown that $D_x^+\Phi\big(x,\delta\big) = \varphi_x\big(x,\delta+\xi_M(x,\delta)\big)$ for $x = X_3(\delta)$. Hence,

$$D_x^+\Phi(X_3(\delta),\delta) - D_x^-\Phi(X_3(\delta),\delta) = \int_0^{\xi_M(X_3(\delta),\delta)} \varphi_{x\delta}(X_3(\delta),\delta+\eta)\, d\eta.$$

This quantity is positive because (105) and (106) hold for $x = X_3(\delta)$ and every ξ, which is sufficiently close to $\xi_M(X_3(\delta),\delta)$. Thus, relation (62) and the lemma are proved. ∎

A.2. Proof of Theorem 3.1.

A.2.1. Proof of uniqueness. We prove the uniqueness. We shall provide a constructive proof of existence in the next section. Suppose we have two solutions y^1 and y^2. By writing the inequality (20) for y^1 and y^2 and choosing $z = y^2$ and y^1, respectively, we obtain after adding the inequality

$$-r\int_0^{+\infty}\frac{(y^1-y^2)^2}{1+x^m}dx-\frac{\sigma^2}{2}\int_0^{+\infty}(y^1-y^2)'\frac{d}{dx}\left(\frac{(y^1-y^2)x^2}{1+x^m}\right)dx$$
$$+\mu\int_0^{+\infty}\frac{x(y^1-y^2)'(y^1-y^2)}{1+x^m}dx\geq 0.$$

Rearranging with one integration by parts using $y^1(0)-y^2(0)=0, y^1(x)-y^2(x)\to 0$ as $x\to+\infty$, we obtain easily

$$\int_0^{+\infty}\frac{(y^1-y^2)^2}{1+x^m}\left(r_m+\frac{\mu m}{2(1+x^m)}+\frac{m\sigma^2(3x^m(m-1)+m-3)}{4(1+x^m)^2}\right)dx+\frac{\sigma^2}{2}\int_0^{+\infty}\frac{[(y^1-y^2)']^2x^2}{1+x^m}dx\leq 0,$$

from which we obtain immediately $y^1-y^2=0$. ■

A.2.2. Constructive proof of existence. There are nonconstructive proofs. We present rather the penalty method, which is constructive and very intuitive. We consider the nonlinear second order differential equation

$$\mathcal{L}y^\epsilon=\pi(x)+\frac{(y^\epsilon-\Phi)^-}{\epsilon}; \tag{107}$$

$$y^\epsilon(0)=0,\quad y^\epsilon(x)-\Phi(x)\to 0\quad\text{as}\quad x\to+\infty,\quad y^\epsilon\in\mathcal{H}_m,$$

where we use the notation $f^-=(|f|-f)/2$ for the negative part of the function. We begin with the following statement.

Proposition A.1. *Under the assumptions of Theorem 3.1, there exists one and only one solution of problem (107) in the space $\mathcal{H}_m$. This solution satisfies $y^\epsilon\in C[0,\infty)\cap C^2(0,\infty)$.*

Proof. We begin with uniqueness. If we have two solutions y_1, y_2 (we delete ϵ in the notation), then setting $\widetilde{y}=y_1-y_2$ we can write

$$\mathcal{L}\widetilde{y}=\frac{(y_1-\Phi)^--(y_2-\Phi)^-}{\epsilon};\quad \widetilde{y}(0)=0,\quad \widetilde{y}(x)\to 0\quad\text{as}\quad x\to+\infty.$$

Suppose $\widetilde{y}$ is strictly positive at some point; then it has a strictly positive maximum at some $x^*>0$, i.e., $y_1(x^*)-y_2(x^*)>0$, and hence $(y_1(x^*)-\Phi(x^*))^--(y_2(x^*)-\Phi(x^*))^-\leq 0$. But the maximum principle implies $\widetilde{y}(x^*)\leq 0$, which is a contradiction. Therefore $\widetilde{y}(x)\leq 0$. By interchanging the roles of x_1, x_2 we can also state $\widetilde{y}(x)\geq 0$. Hence $\widetilde{y}(x)=0$. The uniqueness is proved.

To prove the existence, we consider the sequence of two-point boundary value problems, indexed by R defined by

$$\mathcal{L}y_R^\epsilon = \pi + \frac{(y_R^\epsilon - \Phi)^-}{\epsilon}, 0 < x < R; \quad y_R^\epsilon(0) = 0, \quad y_R^\epsilon(R) = \Phi(R), \tag{108}$$

and we extend $y_R^\epsilon(x)$ beyond R by writing $y_R^\epsilon(x) = \Phi(x), x \geq R$. We can solve (108) by using a fixed point argument for the map

$$\mathcal{L}y = \pi + \frac{(z - \Phi)^-}{\epsilon}, 0 < x < R; \quad y(0) = 0,\ y(R) = \Phi(R). \tag{109}$$

We first notice that for a z bounded function on $[0, R]$ the solution of (109) is uniquely defined. We can find bounds on z which depend only on π and Φ, implying the same bounds on y. Indeed we first have

$$\mathcal{L}y \geq \pi; \quad y(0) = 0,\ y(R) \geq 0.$$

The maximum principle implies

$$y(x) \geq \frac{\inf_{x\in[0,R]} \pi(x)}{r}. \tag{110}$$

We now assume that $z(x) \geq \frac{\inf_{x\in[0,R]} \pi(x)}{r}$ and consider the maximum of $y(x)$ attained at some point x^*. We have $y(x^*) \geq \Phi(R)$. If $x^* = R$, we have $y(x^*) = \Phi(R)$. Otherwise, x^* is in the interior. The maximum principle implies

$$ry(x^*) \leq \pi(x^*) + \frac{(z(x^*) - \Phi(x^*))^-}{\epsilon}.$$

If $z(x^*) \geq \Phi(x^*)$, then $y(x^*) \leq \frac{\pi(x^*)}{r} \leq \frac{\sup_{0\leq x\leq R} \pi(x)}{r}$. If $z(x^*) < \Phi(x^*)$, then

$$ry(x^*) \leq \pi(x^*) + \frac{\Phi(x^*) - z(x^*)}{\epsilon} \leq \pi(x^*) + \frac{\Phi(x^*)}{\epsilon} - \frac{\inf_{x\in[0,R]} \pi(x)}{\epsilon r}.$$

Therefore, we can also assert that

$$y(x^*) \leq \frac{\sup_{0\leq x\leq R} \pi(x)}{r} + \frac{\sup_{0\leq x\leq R} \Phi(x)}{\epsilon r} - \frac{\inf_{x\in[0,R]} \pi(x)}{\epsilon r^2}.$$

To simplify the formulas, assume that $\epsilon r < 1$; then we get

$$\sup_{0\leq x\leq R} y(x) \leq \frac{\sup_{0\leq x\leq R} \pi(x)}{r} + \frac{\sup_{0\leq x\leq R} \Phi(x)}{\epsilon r} - \frac{\inf_{x\in[0,R]} \pi(x)}{\epsilon r^2}.$$

We may also assume, again to simplify the formulas, that $\epsilon r^2 < 1$. Then we obtain

$$\sup_{0\leq x\leq R} |y(x)| \leq M_0 = \frac{\sup_{0\leq x\leq R} \pi(x)}{r} + \frac{\sup_{0\leq x\leq R} \Phi(x)}{\epsilon r} - \frac{\inf_{x\in[0,R]} \pi(x)}{\epsilon r^2}. \tag{111}$$

Clearly, if $z(x)$ satisfies $\sup_{0\le x\le R}|z(x)|\le M_0$, then the same inequality holds for $y(x)$. But then, from (109), we also obtain fixed bounds for the derivatives:

$$\sup_{0\le x\le R}|y'(x)|\le M_1,\qquad \sup_{0\le x\le R}|y''(x)|\le M_2. \tag{112}$$

If z satisfies these bounds, y satisfies them also. It is then immediate to apply the Schauder fixed point theorem to see that the map $z\to y$ has a fixed point. So there exists a solution y_R^ϵ of (108).

The next step is to let $R\to+\infty$. We first obtain estimates on y_R^ϵ. We multiply (108) with $\frac{y_R^\epsilon-\Phi}{1+x^m}$ and integrate. We get, making use of the boundary conditions,

$$r\int_0^R \frac{y_R^\epsilon(y_R^\epsilon-\Phi)}{1+x^m}dx+\frac{\sigma^2}{2}\int_0^R (y_R^\epsilon)'\frac{d}{dx}\left(\frac{x^2(y_R^\epsilon-\Phi)}{1+x^m}\right)dx-\mu\int_0^R (y_R^\epsilon)'\frac{x(y_R^\epsilon-\Phi)}{1+x^m}dx$$
$$=\int_0^R\frac{\pi(y_R^\epsilon-\Phi)}{1+x^m}dx-\frac{1}{\epsilon}\int_0^R\frac{((y_R^\epsilon-\Phi)^-)^2}{(1+x^m)}dx.$$

After rearranging and making use of the fact that $y_R^\epsilon-\Phi=0$ for $x>R$, we obtain

$$r\int_0^{+\infty}\frac{y_R^\epsilon(y_R^\epsilon-\Phi)}{1+x^m}dx+\frac{\sigma^2}{2}\int_0^{+\infty}\frac{(y_R^\epsilon)'(y_R^\epsilon-\Phi)'x^2}{1+x^m}dx+\frac{1}{\epsilon}\int_0^{+\infty}\frac{((y_R^\epsilon-\Phi)^-)^2}{1+x^m}dx$$
$$+\int_0^{+\infty}(y_R^\epsilon)'(y_R^\epsilon-\Phi)\left[-\frac{\mu x}{1+x^m}+\sigma^2\frac{d}{dx}\left(\frac{x^2}{2(1+x^m)}\right)\right]dx=\int_0^{+\infty}\frac{\pi(y_R^\epsilon-\Phi)}{1+x^m}dx. \tag{113}$$

The first, second, and fourth integrals in the left-hand side of (113) can be equivalently written as

$$r\int_0^{+\infty}\frac{y_R^\epsilon(y_R^\epsilon-\Phi)}{1+x^m}dx=r\int_0^{+\infty}\left[\frac{(y_R^\epsilon-\Phi)^2}{1+x^m}+\frac{\Phi(y_R^\epsilon-\Phi)}{1+x^m}\right]dx,$$
$$\frac{\sigma^2}{2}\int_0^{+\infty}\frac{(y_R^\epsilon)'(y_R^\epsilon-\Phi)'x^2}{1+x^m}dx=\frac{\sigma^2}{2}\int_0^{+\infty}\left[\frac{\left[(y_R^\epsilon-\Phi)'\right]^2x^2}{1+x^m}+\frac{\Phi'(y_R^\epsilon-\Phi)'x^2}{1+x^m}\right]dx,$$

and

$$\int_0^{+\infty}(y_R^\epsilon)'\,(y_R^\epsilon-\Phi)\left[-\frac{\mu x}{1+x^m}+\sigma^2\left(\frac{2x+(2-m)x^{m+1}}{2(1+x^m)^2}\right)\right]dx$$
$$=\int_0^{+\infty}\left(\Phi'(y_R^\epsilon-\Phi)+\frac{1}{2}\frac{d}{dx}(y_R^\epsilon-\Phi)^2\right)\left[-\frac{\mu x}{1+x^m}+\sigma^2\left(\frac{2x+(2-m)x^{m+1}}{2(1+x^m)^2}\right)\right]dx.$$

Further, using integration by parts the last integral transforms into

$$\int_0^{+\infty}(y_R^\epsilon)'\,(y_R^\epsilon-\Phi)\left[-\frac{\mu x}{1+x^m}+\sigma^2\left(\frac{2x+(2-m)x^{m+1}}{2(1+x^m)^2}\right)\right]dx$$
$$=\int_0^{+\infty}(y_R^\epsilon-\Phi)^2\left[\frac{\mu(1+(1-m)x^m}{2(1+x^m)^2}-\sigma^2\frac{(m^2-3m+2)x^{2m}-(m^2+3m-4)x^m+2}{4(1+x^m)^3}\right]dx$$
$$-\int_0^{+\infty}\frac{x\Phi'(y_R^\epsilon-\Phi)}{1+x^m}\left[\mu+\frac{\sigma^2((m-2)x^m-2)}{2(1+x^m)}\right]dx.$$

Using these results in (113), we obtain the relation

$$r_m \int_0^{+\infty} \frac{(y_R^\epsilon - \Phi)^2}{1+x^m} dx + \frac{\sigma^2}{2} \int_0^{+\infty} \frac{[(y_R^\epsilon - \Phi)']^2 x^2}{1+x^m} dx + \frac{1}{\epsilon} \int_0^{+\infty} \frac{((y_R^\epsilon - \Phi)^-)^2}{1+x^m} dx$$
$$+ \int_0^{+\infty} \frac{(y_R^\epsilon - \Phi)^2}{1+x^m} \left[\frac{\mu m}{2(1+x^m)} + \frac{\sigma^2 m(m-3+3x^m(m-1))}{4(1+x^m)^2} \right] dx = \int_0^{+\infty} \frac{\pi(y_R^\epsilon - \Phi)}{1+x^m} dx$$
$$-r\int_0^{+\infty} \frac{\Phi(y_R^\epsilon - \Phi)}{1+x^m} dx - \frac{\sigma^2}{2}\int_0^{+\infty} \frac{\Phi'(y_R^\epsilon - \Phi)' x^2}{1+x^m} dx + \int_0^{+\infty} \frac{x\Phi'(y_R^\epsilon - \Phi)}{1+x^m}\left[\mu + \frac{\sigma^2((m-2)x^m - 2)}{2(1+x^m)}\right] dx,$$

where r_m is defined by (21). We deduce immediately

$$\|y_R^\epsilon - \Phi\|_{\mathcal{H}_m} \le C_{\Phi,\pi}; \quad \frac{1}{\epsilon} \int_0^{+\infty} \frac{((y_R^\epsilon - \Phi)^-)^2}{1+x^m} dx \le C_{\Phi,\pi}, \tag{114}$$

where the constant $C_{\Phi,\pi}$ does not depend on R or on ϵ. At this stage ϵ is fixed and we let $R \to +\infty$. From (114) we can extract a subsequence, still denoted by y_R^ϵ, which converges weakly in $\mathcal{H}_m$ to y^ϵ as $R \to +\infty$. It follows that $y_R^\epsilon(x) \to y^\epsilon(x)$ for all x. Since

$$\frac{(y_R^\epsilon)^2(x)\, x}{1+x^m} = 2\int_0^x \frac{y_R^\epsilon(\xi)(y_R^\epsilon)'(\xi)\,\xi}{1+\xi^m} d\xi + \int_0^x \frac{(y_R^\epsilon)^2(\xi)[1-(m-1)\xi^m]}{(1+\xi^m)^2} d\xi,$$

it follows that

$$\frac{(y_R^\epsilon)^2(x)\, x}{1+x^m} \le 2\|y_R^\epsilon\|^2_{\mathcal{H}_m} \le C_{\Phi,\pi} \quad \text{for all } x. \tag{115}$$

Now if φ is a smooth function with compact support in $\mathbb{R}_+$, we can assume that the support of φ is contained in $(0, R)$ for R sufficiently large. From the pointwise convergence and the estimate (115), we can assert that

$$\int_0^{+\infty} \frac{(y_R^\epsilon - \Phi)^-(x)}{\epsilon}\varphi(x)dx \to \int_0^{+\infty} \frac{(y^\epsilon - \Phi)^-(x)}{\epsilon}\varphi(x)dx.$$

Since $\int_0^{+\infty} \mathcal{L}y_R^\epsilon(x)\varphi(x)dx \to \int_0^{+\infty} \mathcal{L}y^\epsilon(x)\varphi(x)dx$, the function y^ϵ satisfies (107). It satisfies also the boundary conditions (107). This completes the proof of existence of a solution of problem (107) in $\mathcal{H}_m$. Finally, due to $y^\epsilon \in \mathcal{H}_m \subset C$, the right-hand side $z = \pi + (y^\epsilon - \Phi)^-/\epsilon$ of (107) is continuous, and hence $\mathcal{L}y^\epsilon = z$ implies $y^\epsilon \in C^2(0,\infty)$. ■

A.2.3. End of proof. From the uniform estimates (114) and (115), we get immediately

$$\|y^\epsilon - \Phi\|_{\mathcal{H}_m} \le C_{\Phi,\pi}; \quad \frac{1}{\epsilon}\int_0^{+\infty} \frac{((y^\epsilon - \Phi)^-)^2}{(1+x^m)} dx \le C_{\Phi,\pi}; \quad \frac{(y^\epsilon)^2(x)\, x}{1+x^m} \le C_{\Phi,\pi} \quad \text{for all } x. \tag{116}$$

As $\epsilon \to 0$, we can extract a subsequence, still denoted y^ϵ, which converges weakly in $\mathcal{H}_m$ to y. It follows that $y^\epsilon(x) \to y(x)$ pointwise. From the second inequality in (116) we get $(y^\epsilon - \Phi)^-(x) \to 0$ pointwise; therefore we obtain

$$y(x) - \Phi(x) \ge 0 \quad \text{for all } x. \tag{117}$$

Now let $z \in \mathcal{H}_m$, such that $z(x) - \Phi(x) \geq 0$ and $z(0) = 0, z(x) - \Phi(x) \to 0$ as $x \to +\infty$. We multiply (107) by $\left(\frac{z(x)-y^\epsilon(x)}{1+x^m}\right)$ and integrate on $\mathbb{R}_+$. We note that $(y^\epsilon - \Phi)^-(z - y^\epsilon) \geq 0$. Therefore

$$\int_0^{+\infty} \frac{\mathcal{L}y^\epsilon(x)(z(x) - y^\epsilon(x))}{1 + x^m} dx \geq \int_0^{+\infty} \frac{\pi(x)(z(x) - y^\epsilon(x))}{1 + x^m} dx.$$

We can perform integration by parts on the left-hand side, taking account of the boundary conditions. We obtain

$$r \int_0^{+\infty} \frac{y^\epsilon(z - y^\epsilon)}{1 + x^m} dx + \frac{\sigma^2}{2} \int_0^{+\infty} (y^\epsilon)' \frac{(z' - (y^\epsilon)')x^2}{1 + x^m} dx$$

$$+ \int_0^{+\infty} \frac{x(y^\epsilon)'(z - y^\epsilon)}{1 + x^m} \left[-\mu + \frac{\sigma^2(2 - (m-2)x^m)}{2(1 + x^m)}\right] dx \geq \int_0^{+\infty} \frac{\pi(z - y^\epsilon)}{1 + x^m} dx.$$

Rearranging and performing integration by parts we get

$$r \int_0^{+\infty} \frac{z(z - y^\epsilon)}{1 + x^m} dx + \frac{\sigma^2}{2} \int_0^{+\infty} z' \frac{(z' - (y^\epsilon)')x^2}{1 + x^m} dx$$

$$+ \int_0^{+\infty} \frac{xz'(z - y^\epsilon)}{1 + x^m} \left[-\mu + \frac{\sigma^2(2 - (m-2)x^m)}{2(1 + x^m)}\right] dx \geq \int_0^{+\infty} \frac{\pi(z - y^\epsilon)}{1 + x^m} dx$$

$$+ \frac{\sigma^2}{2} \int_0^{+\infty} \frac{(z' - (y^\epsilon)')^2 x^2}{1 + x^m} dx + \int_0^{+\infty} \frac{(z - y^\epsilon)^2}{1 + x^m} \left[r_m + \frac{\mu m}{2(1 + x^m)} + \frac{m\sigma^2(3x^m(m-1) + m - 3)}{4(1 + x^m)^2}\right] dx.$$

On the left-hand side, we pass to the limit by using the weak convergence in $\mathcal{H}_m$. On the right-hand side, we first note that we can modify the norm in $\mathcal{H}_m$ by using the equivalent norm

$$\frac{\sigma^2}{2} \int_0^{+\infty} \frac{(z')^2 x^2}{1 + x^m} dx + \int_0^{+\infty} \frac{z^2}{1 + x^m} \left[r_m + \frac{\mu m}{2(1 + x^m)} + \frac{m\sigma^2(3x^m(m-1) + m - 3)}{4(1 + x^m)^2}\right] dx,$$

and we use the weak lower semicontinuity of the norm, to finally get the inequality

$$r \int_0^{+\infty} \frac{z(z - y)}{1 + x^m} dx + \frac{\sigma^2}{2} \int_0^{+\infty} z' \frac{(z' - y')x^2}{1 + x^m} dx$$

$$+ \int_0^{+\infty} \frac{xz'(z - y)}{1 + x^m} \left[-\mu + \frac{\sigma^2(2 - (m-2)x^m)}{2(1 + x^m)}\right] dx \geq \int_0^{+\infty} \frac{\pi(z - y)}{1 + x^m} dx$$

$$+ \frac{\sigma^2}{2} \int_0^{+\infty} \frac{(z' - y')^2 x^2}{1 + x^m} dx + \int_0^{+\infty} \frac{(z - y)^2}{1 + x^m} \left[r_m + \frac{\mu m}{2(1 + x^m)} + \frac{m\sigma^2(3x^m(m-1) + m - 3)}{4(1 + x^m)^2}\right] dx.$$

By cancelling terms and performing integration by parts, we reduce it to (20). This completes the proof of existence of the solution of the V.I. (19), (20). The proof of Theorem 3.1 is complete. ■

A.3. Proof of Proposition 3.2. We are going to show that $y^\epsilon \leq z$. Going to the limit, we have $y \leq z$. But y is an upper solution, since in (20) we can take $z = y + \zeta$, where ζ satisfies the conditions in (22). So it is the smallest upper solution. Let us show that $y^\epsilon \leq z$. We have $y^\epsilon - \Phi \geq y^\epsilon - z$, and hence

$$\mathcal{L}y^\epsilon \leq \pi + \frac{1}{\epsilon}(y^\epsilon - z)^-.$$

We multiply this inequality by $\left(\frac{(y^\epsilon - z)^+}{1+x^m}\right)$, integrate over $\mathbb{R}_+$, and perform integration by parts to obtain the following, since $(y^\epsilon - z)^+(y^\epsilon - z)^- = 0$:

$$r\int_0^{+\infty} \frac{y^\epsilon(y^\epsilon - z)^+}{1+x^m}dx - \mu\int_0^{+\infty} \frac{x(y^\epsilon)'(y^\epsilon - z)^+}{1+x^m}dx + \frac{\sigma^2}{2}\int_0^{+\infty}(y^\epsilon)'\frac{d}{dx}\left(\frac{x^2(y^\epsilon - z)^+}{1+x^m}\right)dx$$
$$\leq \int_0^{+\infty}\frac{\pi(y^\epsilon - z)^+}{1+x^m}dx. \tag{118}$$

Since z is an upper solution, we also have

$$r\int_0^{+\infty} \frac{z(y^\epsilon - z)^+}{1+x^m}dx - \mu\int_0^{+\infty} \frac{xz'(y^\epsilon - z)^+}{1+x^m}dx + \frac{\sigma^2}{2}\int_0^{+\infty} z'\frac{d}{dx}\left(\frac{x^2(y^\epsilon - z)^+}{1+x^m}\right)dx \geq \int \frac{\pi(y^\epsilon - z)^+}{1+x^m}dx,$$

and thus

$$r\int_0^{+\infty} \frac{(y^\epsilon - z)(y^\epsilon - z)^+}{1+x^m}dx - \mu\int_0^{+\infty} \frac{x((y^\epsilon)' - z')(y^\epsilon - z)^+}{1+x^m}dx$$
$$+ \frac{\sigma^2}{2}\int_0^{+\infty}((y^\epsilon)' - z')\frac{d}{dx}\left(\frac{x^2(y^\epsilon - z)^+}{1+x^m}\right)dx \leq 0, \tag{119}$$

from which it follows, through calculations already made in section A.2.3, that $(y^\epsilon - z)^+ = 0$. ■

A.4. Proof of Theorem 3.3. We begin by proving that there exists a unique pair X_0, X_5 such that the system (24), (25), (26) has a unique solution $y(x)$. Solving (24), (25), (26) is equivalent to the following problem. We fix $X_0 < X_1$ and $X_5 > X_4$ and consider two initial-value problems,

$$\mathcal{L}y = \pi, \quad y(X_0) = 0, \quad y'(X_0) = 0, \quad x > X_0, \tag{120}$$

$$\mathcal{L}z = g, \quad z(X_5) = 0, \quad z'(X_5) = 0, \quad x < X_5, \tag{121}$$

where $z = y - \Phi$. The first problem is solved on the interval $[X_0, X_4]$; the second problem is solved on the interval $[X_4, X_5]$. If the solutions $y : [X_0, X_4] \to \mathbb{R}$ and $z : [X_4, X_5] \to \mathbb{R}$ of (120) and (121), respectively, satisfy the matching conditions

$$y(X_4) = \Phi(X_4) + z(X_4), \tag{122}$$

$$y'(X_4) = \Phi'(X_4) + z'(X_4) \tag{123}$$

at the point X_4, then the concatenation of the functions $y : [X_0, X_4] \to \mathbb{R}$, $z+\Phi : [X_4, X_5] \to \mathbb{R}$, $\Phi : [X_5, \infty) \to \mathbb{R}$ and the zero function on $[0, X_0]$ is a solution of problem (24), (25), (26) (recall that $\Phi = 0$ on $[0, X_1]$). Hence, it suffices to prove the existence of a unique pair of numbers X_0, X_5 satisfying (120), (121), (122), (123).

By standard methods, one can check that the solution of (120) is given by

$$y(x) = \frac{2}{\sigma^2(\beta_2 - \beta_1)} \int_{X_0}^{x} \frac{\pi(\xi)}{\xi} \left[\left(\frac{x}{\xi}\right)^{\beta_1} - \left(\frac{x}{\xi}\right)^{\beta_2} \right] d\xi, \tag{124}$$

$$y'(x) = \frac{2}{\sigma^2(\beta_2 - \beta_1)} \int_{X_0}^{x} \frac{\pi(\xi)}{\xi^2} \left[\beta_1 \left(\frac{x}{\xi}\right)^{\beta_1 - 1} - \beta_2 \left(\frac{x}{\xi}\right)^{\beta_2 - 1} \right] d\xi,$$

and the solution of (121) is given by

$$z(x) = \frac{2}{\sigma^2(\beta_2 - \beta_1)} \int_{x}^{X_5} \frac{g(\xi)}{\xi} \left[\left(\frac{x}{\xi}\right)^{\beta_2} - \left(\frac{x}{\xi}\right)^{\beta_1} \right] d\xi, \tag{125}$$

$$z'(x) = \frac{2}{\sigma^2(\beta_2 - \beta_1)} \int_{x}^{X_5} \frac{g(\xi)}{\xi^2} \left[\beta_2 \left(\frac{x}{\xi}\right)^{\beta_2 - 1} - \beta_1 \left(\frac{x}{\xi}\right)^{\beta_1 - 1} \right] d\xi.$$

We can then express conditions (122) and (123) as follows:

$$\begin{aligned} &\frac{2}{\sigma^2(\beta_2 - \beta_1)} \int_{X_0}^{X_4} \frac{\pi(\xi)}{\xi} \left[\left(\frac{X_4}{\xi}\right)^{\beta_1} - \left(\frac{X_4}{\xi}\right)^{\beta_2} \right] d\xi - \Phi(X_4) \\ &\quad = \frac{2}{\sigma^2(\beta_2 - \beta_1)} \int_{X_4}^{X_5} \frac{g(\xi)}{\xi} \left[\left(\frac{X_4}{\xi}\right)^{\beta_2} - \left(\frac{X_4}{\xi}\right)^{\beta_1} \right] d\xi, \end{aligned} \tag{126}$$

$$\begin{aligned} &\frac{2}{\sigma^2(\beta_2 - \beta_1)} \int_{X_0}^{X_4} \frac{\pi(\xi)}{\xi^2} \left[\beta_1 \left(\frac{X_4}{\xi}\right)^{\beta_1 - 1} - \beta_2 \left(\frac{X_4}{\xi}\right)^{\beta_2 - 1} \right] d\xi - \Phi'(X_4) \\ &\quad = \frac{2}{\sigma^2(\beta_2 - \beta_1)} \int_{X_4}^{X_5} \frac{g(\xi)}{\xi^2} \left[\beta_2 \left(\frac{X_4}{\xi}\right)^{\beta_2 - 1} - \beta_1 \left(\frac{X_4}{\xi}\right)^{\beta_1 - 1} \right] d\xi. \end{aligned} \tag{127}$$

We first consider the function

$$x \to \Lambda(x) = \frac{2}{\sigma^2(\beta_2 - \beta_1)} \int_{x}^{X_4} \frac{\pi(\xi)}{\xi} \left[\left(\frac{X_4}{\xi}\right)^{\beta_1} - \left(\frac{X_4}{\xi}\right)^{\beta_2} \right] d\xi - \Phi(X_4)$$

for $x \in (0, X_1)$. From the assumption (34), we have $\Lambda(X_1) > 0$. We have also

$$\Lambda'(x) = \frac{-2}{\sigma^2(\beta_2 - \beta_1)} \frac{\pi(x)}{x} \left[\left(\frac{X_4}{x}\right)^{\beta_1} - \left(\frac{X_4}{x}\right)^{\beta_2} \right] < 0 \tag{128}$$

using the fact that $\pi(x) < 0$ on $(0, X_1)$. Moreover $\Lambda(0) = +\infty$. So if we fix $X_0 \in (0, X_1)$, the left-hand side of (126) is positive. Now, the right-hand side of (126) is $L(X_5)$. We have seen earlier that $L(x)$ is monotone increasing on $(X_4, +\infty)$ from 0 to $+\infty$. Therefore for X_0 fixed on $(0, X_1)$, equation (126) is satisfied for a unique value $X_5(X_0)$ on $(X_4, +\infty)$ with $X_5(0) = +\infty$. Because of (128) and $L'(x) > 0$, we can assert that $X_5'(X_0) < 0$. We can then reduce the problem to one equation for X_0 on $(0, X_1)$ given by

$$\frac{2}{\sigma^2(\beta_2-\beta_1)}\int_{X_0}^{X_4}\frac{\pi(\xi)}{\xi^2}\left[\beta_1\left(\frac{X_4}{\xi}\right)^{\beta_1-1}-\beta_2\left(\frac{X_4}{\xi}\right)^{\beta_2-1}\right]d\xi \tag{129}$$

$$-\frac{2}{\sigma^2(\beta_2-\beta_1)}\int_{X_4}^{X_5(X_0)}\frac{g(\xi)}{\xi^2}\left[\beta_2\left(\frac{X_4}{\xi}\right)^{\beta_2-1}-\beta_1\left(\frac{X_4}{\xi}\right)^{\beta_1-1}\right]d\xi = \Phi'(X_4).$$

For $x \in (0, X_1)$, set

$$\Gamma(x) = \frac{2}{\sigma^2(\beta_2-\beta_1)}\int_{x}^{X_4}\frac{\pi(\xi)}{\xi^2}\left[\beta_1\left(\frac{X_4}{\xi}\right)^{\beta_1-1}-\beta_2\left(\frac{X_4}{\xi}\right)^{\beta_2-1}\right]d\xi$$

$$-\frac{2}{\sigma^2(\beta_2-\beta_1)}\int_{X_4}^{X_5(x)}\frac{g(\xi)}{\xi^2}\left[\beta_2\left(\frac{X_4}{\xi}\right)^{\beta_2-1}-\beta_1\left(\frac{X_4}{\xi}\right)^{\beta_1-1}\right]d\xi.$$

Assumption (37) reads $\Phi'(X_4) > \Gamma(X_1)$. We next have

$$\Gamma'(x) = \frac{-2\pi(x)}{\sigma^2(\beta_2-\beta_1)x^2}\left[\beta_1\left(\frac{X_4}{x}\right)^{\beta_1-1}-\beta_2\left(\frac{X_4}{x}\right)^{\beta_2-1}\right]$$

$$-\frac{2X_5'(X_0)g(X_5(x))}{\sigma^2(\beta_2-\beta_1)X_5(x)^2}\left[\beta_2\left(\frac{X_4}{X_5(x)}\right)^{\beta_2-1}-\beta_1\left(\frac{X_4}{X_5(x)}\right)^{\beta_1-1}\right].$$

Using $\pi(x) < 0, g(X_5(x)) < 0, X_5'(X_0) < 0$, we see immediately that $\Gamma'(x) < 0$. Now $\Gamma(0) = +\infty$. Therefore (129) has one and only one solution X_0 in $(0, X_1)$. So the system (126), (127) in X_0, X_5 has one and only one solution in $(0, X_1)$ and $(X_4, +\infty)$. Therefore system (24), (25), (26) has a unique solution $y(x)$.

To show that $y(x)$ is a solution of the V.I., it remains to show (27). The first part is immediate since on $(0, X_1)$ we have $\Phi(x) = 0$ and $\pi(x) < 0$ and on $(X_4, +\infty)$ we have $g(x) = \pi(x) - \mathcal{L}\Phi(x) < 0$. So, the only thing to show is $y(x) \geq \Phi(x)$ on (X_0, X_5). We split this interval into $(X_0, X_1], (X_1, X_4), [X_4, X_5)$. On (X_4, X_5) the function $z(x) = y(x) - \Phi(x)$ satisfies

$$\mathcal{L}z(x) = g(x); \quad z(X_5) = 0,\ z'(X_5) = 0. \tag{130}$$

We note that $z''(X_5 - 0) = -\frac{2g(X_5)}{\sigma^2 X_5^2} > 0$. Therefore, $z(x) > 0$ for $x < X_5$ close to X_5. But z cannot have a strictly positive maximum on (X_4, X_5), which would contradict the maximum principle, since $g(x) < 0$. Therefore $z(x)$ is decreasing on $[X_4, X_5)$ and therefore positive. So we have $z(x) = y(x) - \Phi(x) > 0$ on $[X_4, X_5)$. In particular $y(X_4) > \Phi(X_4)$. Similarly, on (X_0, X_1), the function $y(x)$ satisfies

$$\mathcal{L}y(x) = \pi(x); \quad y(X_0) = 0,\ y'(X_0) = 0. \tag{131}$$

Since $\pi(x) < 0$ on (X_0, X_1), a reasoning similar to that for $z(x)$ shows that $y(x)$ is positive in a right neighborhood of the point X_0 and increases on $(X_0, X_1]$, and hence y is positive on the whole interval $(X_0, X_1]$. Since the obstacle is zero on $(X_0, X_1]$, we have $y(x) - \Phi(x) > 0$ on $(X_0, X_1]$. In particular, $y(X_1) - \Phi(X_1) > 0$. Consider finally the interval (X_1, X_4). If $y(x) - \Phi(x)$ has a strictly negative minimum in this interval, it is necessarily in the interior. Let x_0 be such a point. Then x_0 cannot be a point of discontinuity of $\Phi'(x)$, because in this case $y'(x_0) - \Phi'(x_0 - 0) < 0 < y'(x_0) - \Phi'(x_0 + 0)$, which implies $\Phi'(x_0 + 0) < \Phi'(x_0 - 0)$. But this contradicts the first part of assumption (32). On the other hand, if x_0 is a point of continuity of $\Phi'(x)$, then the left and right limits of $\mathcal{L}(y - \Phi)(x_0)$ are negative, but this contradicts the second part of assumption (32). So we must have $y(x) - \Phi(x) \geq 0$ on (X_1, X_4). This concludes the proof that $y(x)$ is a solution of the V.I.

A.5. Proof of Theorem 3.4. From relations (40), it follows that $C_1(0) < C_1(X_1)$. Therefore, assumption (41) implies the existence of an interval $[a, b] \subset (X_4, \infty)$ such that

$$D_1(a) = C_1(0); \quad D_1(b) = C_1(X_1); \quad C_1(0) < D_1(x) < C_1(X_1) \quad \text{for} \quad x \in (a, b). \tag{132}$$

Since $a > X_4$ and, according to (40), $D_2(x)$ decreases on (X_4, ∞), assumption (42) implies that

$$C_2(X_1) < D_2(x) < C_2(0) \quad \text{for} \quad x \in [a, b], \tag{133}$$

where $C_2(0) := \lim_{x \to 0} C_2(x)$ is either finite or infinite.

Consider the restriction of the functions C_1, C_2 to the intervals $[0, X_1]$ and $(0, X_1)$, respectively. Relations (40) ensure that the inverse functions

$$C_1^{-1} : [C_1(0), C_1(X_1)] \to [0, X_1], \qquad C_2^{-1} : (C_2(X_1), C_2(0)) \to (0, X_1)$$

are well-defined. Now, let us consider the functions

$$f_1(x) = C_1^{-1}(D_1(x)), \quad f_2 = C_2^{-1}(D_2(x))$$

on the interval $[a, b]$. They are well-defined due to (132) and (133). Relations (132) imply that

$$f_1(a) = 0, \quad f_1(b) = X_1,$$

and hence

$$f_1(a) - f_2(a) < 0, \quad f_1(b) - f_2(b) > 0$$

because the range of f_2 belongs to $(0, X_1)$. We conclude that there is a point $X_5 \in (a, b) \subset (X_4, \infty)$ such that $f_1(X_5) = f_2(X_5)$. We set $X_0 = f_1(X_5) = f_2(X_5) \in (0, X_1)$. By definition of f_1, f_2,

$$C_1(X_0) = D_1(X_5), \quad C_2(X_0) = D_2(X_5).$$

Consider the solution

$$y(x) = p(x) + C_1(X_0)x^{\beta_1} + C_2(X_0)x^{\beta_2} = p(x) + D_1(X_5)x^{\beta_1} + D_2(X_5)x^{\beta_2}$$

of the equation $\mathcal{L}y(x) = \pi(x)$. The definition (38), (39) of the functions C_i, D_i combined with the assumption (28) ensures that this solution satisfies the smooth pasting conditions (26). It remains to establish 27, which can be done exactly as in the last part of the proof of Theorem 3.3 presented above.

A.6. Proof of Theorem 4.1. The assumptions (16) and (21) ensure that the function (49) is a unique locally bounded solution of $\mathcal{L}\, y_\infty(x) = \pi(x)$ satisfying $y_\infty \in \mathcal{H}_m$. Further, by the Markov property of $X(t)$,

$$\begin{aligned} J_x(\theta) &= y_\infty(x) - \mathbb{E}_x \int_\theta^{+\infty} \pi(X(t))e^{-rt}\, dt + \mathbb{E}_x\big(\Phi(X(\theta))e^{-r\theta}\big) \\ &= y_\infty(x) - \mathbb{E}_x\big(y_\infty(X(\theta))e^{-r\theta}\big) + \mathbb{E}_x\big(\Phi(X(\theta))e^{-r\theta}\big). \end{aligned}$$

Therefore, if (50) holds, then $J_x(\theta) \leq y_\infty(x)$. Hence, in this simple case, (49) is the value function, and the optimal stopping is at infinity, i.e., we have (51). Also, due to (50), the function (49) is a solution of the V.I.

Now, assume that (50) is not true. We begin the analysis of this main case with an important property of the penalized problem.

Lemma A.2. *The function $\epsilon \to y^\epsilon(x)$ is decreasing.*

Proof. Consider $y^\epsilon(x)$ and $y^{\epsilon'}(x)$, with $\epsilon < \epsilon'$. We set $\widetilde{y}(x) = y^\epsilon(x) - y^{\epsilon'}(x)$. From (107), it follows that

$$\mathcal{L}\widetilde{y}(x) = \frac{(y^\epsilon - \Phi)^- - (y^{\epsilon'} - \Phi)^-}{\epsilon} + (y^{\epsilon'} - \Phi)^- \left(\frac{1}{\epsilon} - \frac{1}{\epsilon'}\right)$$

and $\widetilde{y}(0)=0$, $\widetilde{y}(x) \to 0$ as $x \to +\infty$. We see that

$$\int_0^{+\infty} \mathcal{L}\widetilde{y}(x) \frac{\widetilde{y}^-(x)}{1 + x^m} dx \geq 0.$$

Performing integration by parts, we check easily that $\widetilde{y}^-(x) = 0$, and hence $y^\epsilon(x) \geq y^{\epsilon'}(x)$, which proves the property. ∎

We also conclude that

$$y^\epsilon(x) \uparrow y(x) \quad \text{as} \quad \epsilon \downarrow 0.$$

Now, we interpret $y^\epsilon(x)$. Since $y^\epsilon(x)$ is C^1, it is standard. Problem (107) is the Bellman equation of a continuous stochastic control problem. Indeed, it can be written as

$$\mathcal{L}y^\epsilon(x) = \pi(x) + \frac{1}{\epsilon} \sup_{0 \leq v \leq 1} \big((\Phi(x) - y^\epsilon(x))v\big).$$

Therefore, considering controls as a stochastic processes $v(t)$ adapted to the filtration $\mathbb{F}$ with values in $(0, 1)$, we introduce the functional

$$J_x^\epsilon(v(\cdot)) = \mathbb{E}_x \int_0^{+\infty} \left(\pi(X(t)) + \frac{v(t)}{\epsilon}\, \Phi(X(t))\right) e^{-rt - \frac{1}{\epsilon}\int_0^t v(s)\, ds}\, dt$$

with $x = X(0)$. Then we have

$$y^\epsilon(x) = \sup_{v(\cdot)} J_x(v(\cdot)).$$

We observe that the control does not affect the state dynamics $X(t)$, and the optimal control is given by

$$\hat{v}^\epsilon(t) = \begin{cases} 1 & \text{if } y^\epsilon(X(t)) \leq \Phi(X(t)), \\ 0 & \text{if } y^\epsilon(X(t)) > \Phi(X(t)). \end{cases}$$

Next, we introduce the stopping time

$$\widehat{\theta}_x^\epsilon = \inf\{t > 0 \,|\, y^\epsilon(X(t)) \leq \Phi(X(t))\},$$

which satisfies

$$\begin{aligned} &\widehat{\theta}_x^\epsilon > 0 \quad \text{if} \quad y^\epsilon(x) > \Phi(x), \\ &\widehat{\theta}_x^\epsilon = 0 \quad \text{if} \quad y^\epsilon(x) \leq \Phi(x). \end{aligned} \tag{134}$$

We can also see that

$$y^\epsilon(x) = J_x(\widehat{\theta}_x^\epsilon) \quad \text{if} \quad y^\epsilon(x) > \Phi(x). \tag{135}$$

Indeed, we have $v^\epsilon(t) = 0$ for $t < \widehat{\theta}_x^\epsilon$. Therefore, using Itô's formula,

$$y^\epsilon(x) = \mathbb{E}_x \int_0^{\widehat{\theta}_x^\epsilon} \pi(X(t))e^{-rt}\,dt + \mathbb{E}_x\big(y^\epsilon(X(\widehat{\theta}_x^\epsilon))e^{-r\widehat{\theta}_x^\epsilon}\big)$$

and since $y^\epsilon(X_x(\widehat{\theta}_x^\epsilon)) = \Phi(X_x(\widehat{\theta}_x^\epsilon))$ due to the definition of $\widehat{\theta}_x^\epsilon$, we obtain (135).

Now, take θ to be an $\mathbb{F}$ stopping time, and R large enough so that $x < R$. Let $\tau_R = \inf\{t > 0 \,|\, X(t) > R\}$. Using Itô's calculus, we can write

$$y^\epsilon(x) \geq \mathbb{E}_x \int_0^{\theta\wedge\tau_R} \pi(X(t))e^{-rt}\,dt + \mathbb{E}_x\big(y^\epsilon(X(\theta\wedge\tau_R))e^{-r(\theta\wedge\tau_R)}\big).$$

Then, as $\epsilon \downarrow 0$, we obtain

$$y(x) \geq \mathbb{E}_x \int_0^{\theta\wedge\tau_R} \pi(X(t))e^{-rt}\,dt + \mathbb{E}_x\big(y(X(\theta\wedge\tau_R))e^{-r(\theta\wedge\tau_R)}\big).$$

Since $y(x) \geq \Phi(x)$, it follows that

$$y(x) \geq \mathbb{E}_x \int_0^{\theta\wedge\tau_R} \pi(X(t))e^{-rt}\,dt + \mathbb{E}_x\big(\Phi(X(\theta\wedge\tau_R))e^{-r(\theta\wedge\tau_R)}\big).$$

By letting $R \uparrow +\infty$, we obtain $y(x) \geq J_x(\theta)$, and since θ is arbitrary,

$$y(x) \geq \sup_\theta J_x(\theta). \tag{136}$$

Now, if $y(x) = \Phi(x)$, it means that $y(x) = J_x(0)$. Therefore, if $y(x) = \Phi(x)$, then $\hat{\theta}_x = 0$ is the optimal stopping time and (44) holds. So, we can assume that $y(x) > \Phi(x)$. Hence, also $\hat{\theta}_x > 0$. Since $y^\epsilon(x) \uparrow y(x)$, we can assume that $y^\epsilon(x) > \Phi(x)$; then from (134) and (135) it follows that $\widehat{\theta}^\epsilon_x > 0$ and $y^\epsilon(x) = J_x(\widehat{\theta}^\epsilon_x)$. Also, by definition of the stopping time,

$$y^\epsilon(X(\widehat{\theta}^\epsilon_x)) = \Phi(X(\widehat{\theta}^\epsilon_x)). \tag{137}$$

Let us show that the sequence $\epsilon \to \widehat{\theta}^\epsilon_x$ is decreasing. Indeed, we want to show that

$$\widehat{\theta}^\epsilon_x \geq \widehat{\theta}^{\epsilon'}_x \quad \text{if} \quad \epsilon \leq \epsilon'. \tag{138}$$

It is enough to show it when $\widehat{\theta}^{\epsilon'}_x > 0$. In this case,

$$0 \leq s < \widehat{\theta}^{\epsilon'}_x \quad \Rightarrow \quad y^{\epsilon'}(X(s)) > \Phi(X(s)) \quad \Rightarrow \quad y^\epsilon(X(s)) > \Phi(X(s))$$

by Lemma A.2. This implies immediately $\widehat{\theta}^\epsilon_x \geq \widehat{\theta}^{\epsilon'}_x$, hence the property (138) holds.

Since $y^\epsilon(x) \uparrow y(x)$ as $\epsilon \downarrow 0$, the same argument shows that

$$\hat{\theta}_x \geq \widehat{\theta}^\epsilon_x. \tag{139}$$

Finally, let us show that

$$\widehat{\theta}^\epsilon_x \uparrow \hat{\theta}_x \quad \text{as} \quad \epsilon \downarrow 0. \tag{140}$$

This property follows from (139) if $\hat{\theta}_x = 0$. If $\hat{\theta}_x > 0$, let $0 < \delta < \hat{\theta}_x$; then for $0 \leq s \leq \hat{\theta}_x - \delta$, we have $y(X(s)) > \Phi(X(s))$. Therefore, there exists an $\epsilon_0 = \epsilon_0(x, \omega, \delta) > 0$ such that for $\epsilon \leq \epsilon_0(x, \omega, \delta)$, we have $y^\epsilon(X(s)) > \Phi(X(s))$ for all $s \in [0, \hat{\theta}_x - \delta]$. This implies $\widehat{\theta}^\epsilon_x \geq \hat{\theta}_x - \delta$. Since δ is arbitrarily small, this argument proves (140).

From (140), it follows that

$$y^\epsilon(x) = J_x(\widehat{\theta}^\epsilon_x) \to J_x(\hat{\theta}_x) \quad \text{as} \quad \epsilon \downarrow 0 \quad \text{if} \quad \hat{\theta}_x > 0. \tag{141}$$

Since $y^\epsilon(x) \uparrow y(x)$ as $\epsilon \downarrow 0$, we obtain $y(x) = J_x(\hat{\theta}_x)$ if $\hat{\theta}_x > 0$, i.e., if $y(x) > \Phi(x)$. Moreover, due to (136), the optimal stopping time is $\hat{\theta}_x$. We have already seen that if $y(x) = \Phi(x)$, then the optimal stopping time is zero. This completes the proof of the verification theorem.

A.7. Proof of Lemma 4.5 and Proposition 4.4. We start with the proof of Lemma 4.5. Lemma 4.2 implies that at the points ξ of the set $\Xi_{max}(x, \delta)$ one has

$$\varphi_\delta(x, \delta + \xi) - v'(\xi) = 0 \tag{142}$$

(cf. (57)). According to (53), for $x > X_3$ this is equivalent to

$$\varepsilon \eta_\gamma x^\gamma (\delta + \xi)^{\varepsilon - 1} + \frac{\varepsilon \beta_1 \eta_0 x^{\beta_1}}{\beta_1 - \gamma} \left(\frac{1}{\lambda}\right)^{\frac{\beta_1}{\gamma}} (\delta + \xi)^{\frac{\varepsilon \beta_1}{\gamma} - 1} - v'(\xi) = 0. \tag{143}$$

Due to (4), for large x this leads to the asymptotics

$$\xi \sim \left(\frac{\varepsilon \eta_\gamma}{k}\right)^{\frac{1}{1-\varepsilon}} x^{\frac{\gamma}{1-\varepsilon}} \to \infty \quad \text{as} \quad x \to +\infty, \qquad \xi \in \Xi_{max}(x, \delta), \tag{144}$$

where we use the notation $f_1 \sim f_2$ if $f_1/f_2 \to 1$ as $x \to +\infty$. Differentiating the left-hand side of (143) with respect to ξ and using (65) and (144) gives

$$\varphi_{\delta\delta}(x, \delta+\xi) - v''(\xi) \sim \varepsilon(\varepsilon-1)\left(\frac{\varepsilon\eta_\gamma}{k}\right)^{\frac{\varepsilon-2}{1-\varepsilon}} x^{\frac{\gamma}{\varepsilon-1}} < 0, \qquad \xi \in \Xi_{max}(x,\delta). \tag{145}$$

Hence, by the implicit function theorem, (142) has a unique solution $\xi = \xi_m(x,\delta) = \xi_M(x,\delta)$ for sufficiently large x (i.e., $\Xi_{max}(x,\delta) = \{\xi_M(x,\delta)\}$), and this solution is a smooth function of x. Further, due to (145), Lemma 4.3 implies that $\Phi(x)$ is smooth and formulas (70) hold. Differentiating the equality $\varphi_\delta(x, \delta+\xi_M(x,\delta)) - v'(\xi_M(x,\delta)) = 0$ with respect to x, we obtain

$$\frac{\partial}{\partial x}\xi_M(x,\delta) = -\frac{\varphi_{x\delta}(x, \delta+\xi_M(x,\delta))}{\varphi_{\delta\delta}(x, \delta+\xi_M(x,\delta)) - v''(\xi_M(x,\delta))}.$$

Therefore, differentiating the second equation in (70) gives (67). This proves statement (i) of Lemma 4.5.

Using the asymptotic expressions (144) and $v(\xi) \sim k\xi$ and formulas (70), one obtains from (53) by direct calculation that

$$\Phi(x,\delta) \sim \eta_\gamma(1-\varepsilon)\left(\frac{\varepsilon\eta_\gamma}{k}\right)^{\frac{\varepsilon}{1-\varepsilon}} x^{\frac{\gamma}{1-\varepsilon}}, \qquad x\,\Phi_x(x,\delta) \sim \eta_\gamma\gamma\left(\frac{\varepsilon\eta_\gamma}{k}\right)^{\frac{\varepsilon}{1-\varepsilon}} x^{\frac{\gamma}{1-\varepsilon}}. \tag{146}$$

Computing the second derivatives φ_{xx}, $\varphi_{x\delta}$ of the function (53), substituting the asymptotic formula (144) in the expressions for these derivatives, and using relation (145), from (67) one gets

$$x^2\,\Phi_{xx}(x,\delta) \sim \eta_\gamma\gamma\left(\frac{\varepsilon\eta_\gamma}{k}\right)^{\frac{\varepsilon}{1-\varepsilon}}\left(\gamma - 1 + \frac{\gamma\varepsilon}{1-\varepsilon}\right) x^{\frac{\gamma}{1-\varepsilon}}. \tag{147}$$

The first of relations (146) implies that $\pi(x,\delta)/\Phi(x,\delta) \to 0$ as $x \to +\infty$, and hence $g(x,\delta) \sim -\mathcal{L}\Phi(x,\delta)$. Therefore, using the definition of the differential operator $\mathcal{L}$ and formulas (146) and (147), one obtains

$$g(x,\delta) \sim \frac{\eta_\gamma}{1-\varepsilon}\left(\frac{\varepsilon\eta_\gamma}{k}\right)^{\frac{\varepsilon}{1-\varepsilon}}\left(-r + \left(\mu - \frac{\sigma^2}{2}\right)\frac{\gamma}{1-\varepsilon} + \frac{\sigma^2\gamma^2}{2(1-\varepsilon)^2}\right) x^{\frac{\gamma}{1-\varepsilon}}$$

$$= -Q\left(\frac{\gamma}{1-\varepsilon}\right)\frac{\eta_\gamma}{1-\varepsilon}\left(\frac{\varepsilon\eta_\gamma}{k}\right)^{\frac{\varepsilon}{1-\varepsilon}} x^{\frac{\gamma}{1-\varepsilon}}. \tag{148}$$

Relation (68) implies that $\beta_1 < 0 < \frac{\gamma}{1-\varepsilon} < \beta_2$; hence $Q\big(\frac{\gamma}{1-\varepsilon}\big) > 0$ and therefore (148) implies (69). This completes the proof of Lemma 4.5.

Relations (66) and (146) imply that all the integrals in (15) and (16) converge. Also $\Phi(0) = 0$, $\Phi(x) \geq 0$, $x \in \mathbb{R}_+$, and $\inf_{x>0}\pi(x) < 0$ due to formulas (53), (54), and (52) where $\alpha_0 < 0$. Hence, all the conditions of Theorem 3.1 are satisfied and the proof of Proposition 4.4 is complete.

A.8. Proof of Proposition 4.6. Formulas (75) and (76) imply that

$$C_1(0) = 0, \qquad C_2(x) \to \infty \quad \text{as} \quad x \to 0. \tag{149}$$

Also, substituting $X_1 = X_1(\delta) = \left(\frac{\lambda}{\delta^\varepsilon}\right)^{\frac{1}{\gamma}}$ into (76) gives $C_2(X_1) = 0$ due to the definition of λ. On the other hand, substituting asymptotic formulas (146) into (77) and (78), one obtains

$$D_1(x) \sim \frac{(1-\varepsilon)\beta_2 - \gamma}{\beta_2 - \beta_1} \left(\frac{\varepsilon\eta_\gamma}{k}\right)^{\frac{\varepsilon}{1-\varepsilon}} \eta_\gamma \, x^{\frac{\gamma}{1-\varepsilon}-\beta_1}, \quad D_2(x) \sim \frac{-(1-\varepsilon)\beta_1 + \gamma}{\beta_2 - \beta_1} \left(\frac{\varepsilon\eta_\gamma}{k}\right)^{\frac{\varepsilon}{1-\varepsilon}} \eta_\gamma \, x^{\frac{\gamma}{1-\varepsilon}-\beta_2},$$

and hence (68) implies

$$D_1(x) \to \infty, \qquad D_2(x) \to 0 \quad \text{as} \quad x \to \infty. \tag{150}$$

Combining (149), (150), and $C_2(X_1) = 0$ with the assumption $D_1(X_4) < 0$, we see that conditions (41) and (42) of Theorem 3.4 are satisfied. As we saw in section 4, all the other conditions (28), (29), (31), (32) of this theorem are also satisfied, and hence (24)–(27) has a solution $y(x)$. The other conclusions of Proposition 4.6 follow by invoking the verification Theorem 4.1.

A.9. Proof of Lemma 4.9 . By direct calculation, formula (53) implies

$$\varphi_\delta(x,\delta) = \varepsilon\eta_\gamma\delta^{\varepsilon-1}x^\gamma \left[1 - \left(\frac{\delta_1(x)}{\delta}\right)^{\varepsilon\frac{\gamma-\beta_1}{\gamma}}\right] \mathbf{1}_{\delta \geq \delta_1(x)},$$

$$\varphi_{\delta\delta}(x,\delta) = \varepsilon(\varepsilon-1)\eta_\gamma\delta^{\varepsilon-2}x^\gamma \left[1 - \left(\frac{\delta_2(x)}{\delta}\right)^{\varepsilon\frac{\gamma-\beta_1}{\gamma}}\right] \mathbf{1}_{\delta \geq \delta_1(x)}.$$

We see that the function φ_δ is continuous for all positive x, δ and

$$\varphi_{\delta\delta}(x, \delta_2(x)) = 0; \quad \varphi_{\delta\delta}(x,\delta) > 0 \text{ for } \delta_1(x) < \delta < \delta_2(x); \quad \varphi_{\delta\delta}(x,\delta) < 0 \text{ for } \delta > \delta_2(x).$$

Hence, $\varphi_\delta(x,\delta)$ strictly increases in δ on the interval $\delta_1(x) \leq \delta \leq \delta_2(x)$, strictly decreases for $\delta \geq \delta_2(x)$, and has a unique (global) maximum at $\delta = \delta_2(x)$. Moreover, from the expression for φ_δ, one obtains

$$\varphi_\delta \geq 0 \quad \text{for all} \quad x, \delta > 0; \quad \varphi_\delta(x, 0) = 0 \quad \text{for} \quad \delta \in (0, \delta_1(x)); \quad \varphi_\delta(x,\delta) \to 0 \quad \text{as} \quad \delta \to \infty,$$

and, furthermore, the quantity

$$\max\{\varphi_\delta(x,\delta) : \delta \geq 0\} = \varphi_\delta(x, \delta_2(x)) = \frac{\varepsilon\eta_\gamma\lambda\rho(1 - \rho^{\frac{\beta_1-\gamma}{\gamma}})}{\delta_2(x)} = \varepsilon\eta_\gamma(\lambda\rho)^{\frac{\varepsilon-1}{\varepsilon}}\left(1 - \rho^{\frac{\beta_1-\gamma}{\gamma}}\right) x^{\frac{\gamma}{\varepsilon}}$$

strictly increases with x and satisfies

$$\varphi_\delta(0, \delta_2(0)) = 0; \qquad \varphi_\delta(x, \delta_2(x)) \to \infty \quad \text{as} \quad x \to \infty,$$

which implies the statement of the lemma.

A.10. Proof of Lemma 4.10. The following lemma was proved in [5].

Lemma A.3. *The function $\varphi(x,\delta_3(x))-k\delta_3(x)$ strictly increases on the interval $[x^*,\infty)$ and tends to infinity as $x\to\infty$. The function $\delta_3(x)$ strictly increases on $[x^*,\infty)$. The function $\delta_3'(x)$ strictly decreases on $[x^*,\infty)$. These functions are smooth on the interval (x^*,∞) and satisfy $\delta_3'(x^*)=\delta_3(x^*)$.*

Let us consider the derivative of the function $\varphi(x,\delta_3(x))-k\delta_3(x)$. The relations

$$\frac{d}{dx}\big(\varphi(x,\delta_3(x))-k\delta_3(x)\big)=\varphi_x(x,\delta_3(x))+(\varphi_\delta(x,\delta_3(x))-k)\,\frac{d\delta_3}{dx}(x)$$

and $k=\varphi_\delta(x,\delta_3(x))$ imply that

$$\frac{d}{dx}\big(\varphi(x,\delta_3(x))-k\delta(x)\big)=\varphi_x(x,\delta_3(x)). \tag{151}$$

Further, from (53) it follows that

$$k=\varphi_\delta(x,\delta_3(x))=\varepsilon\eta_\gamma\delta_3(x)^{\varepsilon-1}x^\gamma+\frac{\varepsilon\beta_1}{\gamma}\frac{\eta_0\gamma}{\beta_1-\gamma}\frac{\delta_3(x)^{\frac{\varepsilon\beta_1}{\gamma}-1}}{\lambda^{\frac{\beta_1}{\gamma}}}x^{\beta_1}.$$

Hence

$$\frac{\eta_0\gamma}{\beta_1-\gamma}\frac{\delta_3(x)^{\frac{\varepsilon\beta_1}{\gamma}}}{\lambda^{\frac{\beta_1}{\gamma}}}x^{\beta_1}=\frac{\gamma\delta}{\varepsilon\beta_1}\left(k-\varepsilon\eta_\gamma\delta_3(x)^{\varepsilon-1}x^\gamma\right),$$

and therefore

$$\begin{aligned}\varphi_x(x,\delta_3(x))&=\gamma\eta_\gamma\delta_3(x)^\varepsilon x^{\gamma-1}+\frac{\eta_0\gamma}{\beta_1-\gamma}\left(\frac{\delta_3(x)^\varepsilon}{\lambda}\right)^{\frac{\beta_1}{\gamma}}\beta_1x^{\beta_1-1}\\&=\gamma\eta_\gamma\delta_3(x)^\varepsilon x^{\gamma-1}+\frac{\gamma\delta_3(x)}{\varepsilon\beta_1}\left(k-\varepsilon\eta_\gamma\delta_3(x)^{\varepsilon-1}x^\gamma\right)\frac{\beta_1}{x}=\frac{\gamma k\delta_3(x)}{\varepsilon x}.\end{aligned}$$

Using (151), we conclude that

$$\frac{d}{dx}\left(\varphi(x,\delta_3(x))-k\delta_3(x)\right)=\frac{\gamma k\delta_3(x)}{\varepsilon x}.$$

A similar argument shows that

$$\frac{d}{dx}\big(\varphi(x,\delta_3'(x))-k\delta_3'(x)\big)=\frac{\gamma k\delta_3'(x)}{\varepsilon x}.$$

Combining these two equations, we see that function (90) satisfies

$$\frac{d}{dx}\tilde\Delta(x)=\frac{\gamma k}{\varepsilon x}\big(\delta_3(x)-\delta_3'(x)\big)>0,\qquad x>x^*.$$

Further, according to Lemma A.3 the difference $\delta_3(x)-\delta_3'(x)$ increases; therefore for all $x\ge x_0>x^*$,

$$\frac{d\tilde\Delta}{dx}(x)\ge\frac{\gamma k}{\varepsilon x}\big(\delta_3(x_0)-\delta_3'(x_0)\big),$$

and integrating, one obtains

$$\tilde{\Delta}(x) - \tilde{\Delta}(x_0) \geq \frac{\gamma k}{\varepsilon}\big(\delta_3(x_0) - \delta_3'(x_0)\big) \ln \frac{x}{x_0} .$$

We conclude that $\tilde{\Delta}(x)$ strictly increases on its domain $x \geq x^*$ and tends to infinity as $x \to \infty$. Also $\tilde{\Delta}(x^*) = 0$ because $\delta_3(x^*) = \delta_3'(x^*)$. Therefore, given any $K_0 > 0$, the equation $\tilde{\Delta}(x) = K_0$ has a unique solution $x^{***} > x^*$ and, furthermore, $\tilde{\Delta}(x) > K_0$ for $x > x^{***}$; see Figure 4.

From $\varphi(x, \delta_3'(x)) - k\delta_3'(x) < \varphi(x, 0) = 0$, it follows that $\hat{\Delta}(x) > \varphi(x, \delta_3(x)) - k\delta_3(x)$. But the function $\varphi(x, \delta_3(x)) - k\delta_3(x)$ strictly increases and tends to infinity as $x \to \infty$ (see Lemma A.3); hence from $K_0 = \tilde{\Delta}(x^{***}) > \varphi(x^{***}, \delta_3(x^{***})) - k\delta_3(x^{***})$ it follows that the equation $\varphi(x, \delta_3(x)) - k\delta_3(x) = K_0$ has a unique solution $x^{**} > x^{***}$.

Also, if $x > x^{***}$, then from $\tilde{\Delta}(x) > K_0$ it follows that (91) has a unique solution $\delta = \delta_3'''(x)$ in the interval $\delta_3'(x) < \delta < \delta_3(x)$ because the function $\varphi(x, \delta) - k\delta$ strictly increases in δ on this interval. On the other hand, if $x^{***} < x < x^{**}$, then the relations $\tilde{\Delta}(x) > K_0 > \varphi(x, \delta_3(x)) - k\delta_3(x)$ and $\varphi(x, 0) = 0$ imply that (91) has another solution $\delta = \delta_3''(x)$, which belongs to the interval $0 < \delta < \delta_3'(x)$ and is unique in this interval (because the function $\varphi(x, \delta) - k\delta$ strictly decreases in δ on $(0, \delta_3'(x))$).

It remains to show that

$$\frac{d}{dx}\delta_3''(x) < 0 < \frac{d}{dx}\delta_3'''(x). \tag{152}$$

To this end, recall that each of the functions $\delta = \delta_3'''(x), \delta_3''(x)$ is implicitly defined by (91). Substituting either of these functions in (91) and differentiating with respect to x gives

$$\varphi_x(x, \delta_3(x)) + \varphi_\delta(x, \delta_3(x))\frac{d}{dx}\delta_3(x) - k\frac{d}{dx}\delta_3(x) = \varphi_x(x, \delta(x)) + \varphi_\delta(x, \delta(x))\frac{d}{dx}\delta(x) - k\frac{d}{dx}\delta(x),$$

where $\delta = \delta_3'''$ or $\delta = \delta_3''$. Due to $\varphi_\delta(x, \delta_3(x)) = k$, this is equivalent to

$$\varphi_x(x, \delta_3(x)) = \varphi_x(x, \delta(x)) + \frac{d}{dx}\delta(x)\left(\varphi_\delta(x, \delta(x)) - k\right).$$

Consequently,

$$\frac{d}{dx}\delta(x) = \frac{\varphi_x(x, \delta_3(x)) - \varphi_x(x, \delta(x))}{\varphi_\delta(x, \delta(x)) - k}, \qquad \delta(x) = \delta_3'''(x),\ \delta_3''(x). \tag{153}$$

Now recall that $\varphi_{x\delta} > 0$ whenever $\delta > \delta_1(x)$ due to (99), while $\varphi_{\delta x} = 0$ for $\delta < \delta_1(x)$. Therefore, the numerator on the right-hand side of (153) is positive as $\delta_1(x), \delta_3'''(x), \delta_3''(x) < \delta_3(x)$. Finally, the denominator on the right-hand side of (153) satisfies

$$\varphi_\delta(x, \delta_3'''(x)) - k > 0 > \varphi_\delta(x, \delta_3''(x)) - k$$

because $\delta_3''(x) < \delta_3'(x) < \delta_3'''(x)$. This proves (152) and completes the proof of the lemma.

A.11. Proof of Lemma 4.11. From (84) and (85) it follows that

$$\Phi(x,\delta) = \max\left\{\sup_{\Delta\geq\delta}\left[\varphi(x,\Delta) - k\Delta\right] - K_0,\, \varphi(x,\delta) - k\delta\right\} + k\delta.$$

Therefore, the definition of $\delta_3'''(x), \delta_3''(x)$ and relations (89) imply that

$$\Phi(x,\delta) = \begin{cases} \varphi(x,\delta) & \text{if } \; 0 \leq \delta \leq \delta_3''(x) \; \text{ or } \; \delta \geq \delta_3'''(x), \\ \varphi(x,\delta_3(x)) - k\delta_3(x) + k\delta - K_0 & \text{if } \; \max\{0, \delta_3''(x)\} < \delta < \delta_3'''(x) \end{cases}$$

and

$$\varphi(x,\delta_3(x)) - k\delta_3(x) + k\delta - K_0 > \varphi(x,\delta) \quad \text{ if } \; \max\{0,\delta_3''(x)\} < \delta < \delta_3'''(x).$$

Combining these relations with (54) and (55) of Lemma 4.2 proves (92).

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