Optimal Policies for Inventory Systems with Piecewise-Linear Concave Ordering Costs

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We study infinite-horizon stochastic inventory problems with general demand distributions and piecewise-linear concave ordering costs. Such costs arise in the important cases of quantity discounts or multiple suppliers. We consider the case of concave cost involving two linear segments. This corresponds to the case of one supplier with a fixed cost, a variable cost up to a given order quantity, and a quantity discount beyond that or, equivalently, the case of two suppliers, one with a low fixed cost along with a high variable cost and the other with a high fixed cost along with a low variable cost. We show that certain three and four parameter generalizations of the classical (s, S) policy are optimal. Our contributions consist of generalizing the demand, solving a functional Bellman equation for the value function that arises in the infinite-horizon framework, and providing an explicit solution in a special case of the exponential demand. We also give conditions under which our generalizations of the (s, S) policy reduce to the standard (s, S) policy. Finally and importantly, our method is also new in the sense that we construct explicitly the value function and we do not therefore need to utilize the notion of K-convexity used in the literature of inventory problems with fixed costs.

Key words: generalized (s, S) policy, inventory-production : stochastic, dynamic programming-optimal control : applications, functional Bellman equation, quantity discounts, contraction mapping.

1. Introduction

This paper studies the optimality of generalized (s, S) policies for infinite-horizon stochastic inventory control problems with quantity discounts or, equivalently, multiple suppliers. We choose to do this in the case of lost sales, although our method can be easily applied to the backlog case. The works most relevant to our study are Benjaafar et al. (2018) and Porteus (1971). The main differences between the problems we consider and the classical (s,S) framework of Scarf (1960) are that our cost structure, as in Porteus (1971) and Benjaafar et al. (2018), has a fixed ordering cost plus a cost that is concave in order quantity and, importantly, our horizon is infinite. The first difference gives rise to a generalized (s,S) policy and the second difference requires us to solve a functional Bellman equation.

Porteus (1971) was the first to study the problems with this kind of concave cost structure. He introduced the notion of a generalized (s,S) ordering policy (defined in Section 2) and proved its optimality, albeit under the restrictive assumption of only one-sided Polya density demands. Recently, Benjaafar et al. (2018) visited this inventory problem with a general demand. They specialized the cost function to be piecewise-linear concave and showed numerically its optimality under a long enough horizon, and conjectured the optimality of a generalized (s,S) policy provided the problem horizon is sufficiently long or infinite.

In this paper, we further specialize the piecewise-linear concave ordering costs treated in Benjaafar et al. (2018) by assuming only two linear segments for ease of exposition and prove analytically the optimality of a generalized (s,S) policy. This cost structure arises, for instance, when a supplier offers a buyer an incremental quantity discount for large orders. Specifically, this means that every order incurs a fixed cost (also known as a setup cost in the case of production) plus a per-unit cost which is identical for the first few items, and then a lower per-unit cost for the subsequent items beyond a given threshold. Sellers usually employ such a scheme to give buyers an incentive to order more. There is a considerable amount of literature on the practice of quantity discounts; see for example Sethi (1984), Chen and Robinson (2012), and references therein. The cost structure studied in this paper also results when ordering from two suppliers, one with a low fixed cost along with a high proportional cost and the other with a high fixed cost along with a low proportional cost. A typical case would be that of a local supplier with high labor cost of production and low shipping cost of a container and an overseas supplier with low labor cost of production and high cost of air-shipping a container. Moreover, a common assumption about production functions in economics is that of the economies of scale. The more a firm produces the same item, the lower the production cost will be. Our cost function structure is applicable to this case as well. Most economics platform such as Economicshelp (2017) discuss examples where such economies arise.

Specifically we address four different situations with regards to the cost structure. The first one is the classical case involving only one linear segment to illustrate our methodology in proving the optimality of the well known (s,S) policy shown in Figure 2. Here we offer a rigorous theory of solving stationary infinite-horizon optimal inventory control problems with general demands and cost parameters satisfying some technical assumptions usual in infinite-horizon settings. It is important to mention that our methodology does not require the notion of K-convexity, introduced in Scarf (1960), in proving our optimality results. This situation is treated in Section 4.

The second situation depicted in Figure 1(a) is the case of two suppliers, the first of whom charges a positive fixed ordering cost plus a proportional cost and the second one charges a higher fixed cost plus a lower proportional cost. Here we consider only an exponential demand for simplicity in exposition. In this case, we show that a generalized (s,S) policy, called a (σ,s,Σ,S) policy, where $\sigma < s \leq \Sigma < S$, is optimal. According to this policy if the beginning inventory is more than σ and less or equal to s, then order up to Σ; if it is less or equal to σ, then order up to S; and if it is more than s, then do not order. Moreover, we obtain these four parameters explicitly due to the simplicity of the exponential demand. This situation is treated in Section 7.

The third situation is an intermediate case that forms a bridge between the first and the third, and therefore it will be treated in Section 5. The idea is that if we introduce another supplier in the classical case of one supplier whose fixed cost is just slightly lower and/or proportional cost is slightly higher than the existing supplier, i.e., ϵ in Figure 1(a) is small, then we may still find a standard two parameter policy to remain optimal. What we find that it depends critically on the unit penalty cost of lost sales. Specifically, we show that for small penalty costs, in addition to some other conditions, two parameter policies remain optimal, whereas for large penalty costs, the optimal policies switch to generalized four parameter policies.

The fourth situation still considers two suppliers. However, now the fixed cost of the first supplier is zero as shown in Figure 1(b), and we allow the demand distribution to be general. We show that the four parameter policy of the second case reduces to a generalized three parameter policy, called a (σ, Σ, S) policy depicted in Figure 3, where $\sigma < s = \Sigma < S$. According to this policy, if the beginning inventory in any given period is more than σ and less or equal to Σ, then one should order up to Σ from the high proportional cost supplier; if it is less or equal to σ, then one should order up to S from the low proportional cost supplier; and if it is more than Σ, then do not order. Moreover, we provide a complete analytic solution in this case when the demand distribution is exponential. This situation is treated in Section 8 and Section 9.

In the process of obtaining our results, we also make the following contributions. We allow demand distributions which are more general than those considered by Porteus (1971). We also prove the conjecture of Benjaafar et al. (2018) about the infinite-horizon setting. Additionally we also make the following significant contributions:

- We offer a rigorous theory of the stationary infinite-horizon optimal inventory control problems with piecewise-linear concave ordering cost.
- The theory is constructive and does not rely on generalized K-convexity. We construct the value function explicitly by solving a functional Bellman equation in our infinite-horizon setting.
- We give conditions for optimality of the classical (s, S) policy and of the generalized (s, S) policy. This is achieved via an analysis conducted in several steps. First, we show that the generalized (s, S) policies reduce to standard two parameter policies under a cost assumption stated in equation (24) which necessarily requires that the unit penalty of lost sales is less than the unit purchase cost from the first supplier. Also required in addition are two necessary and sufficient condition that involves the fixed cost of the second supplier and possibly other problem parameters. Second, we prove that a four parameter generalization of the classical (s, S) policy is optimal under some precisely stated conditions that necessarily require the lost sales penalty to be larger than the proportional cost charged by the first supplier. We conduct this analysis only for the exponential demand case for

simplicity in the exposition, but more importantly for obtaining policy parameters explicitly. Third, we prove that the aforementioned four parameter policy reduces to a three parameter policy provided that the first supplier's fixed cost is zero. Moreover, we are able to obtain closed form expressions of the policy parameters in this case.

- We offer a complete analytical solution when the demand density is exponential.

1.1. Organization of the paper

The balance of the paper is organized as follows. In the next section, we briefly review the most relevant literature that would allow us to articulate the precise contributions of the paper. Section 3 provides a mathematical formulation of the problem under consideration in this paper. In Section 4 and Section 5, we introduce the Bellman equation and illustrate its use to demonstrate the optimality of the (s,S) policy for the classical inventory problem with a view toward extending it to the problems under study in this paper. The general problem is studied in Section 6. We construct a methodology and the road map in Section 6.1 and Section 6.2 to analyze the full problem. We use this methodology and a road map in Section 7 and prove in Theorem 2, the optimality of a generalized four parameter policy alluded to before. In Section 8 we consider general demand densities and perform a detailed analysis by using the road map provided in Section 6.2 and find a three parameter solution for the value function. In Section 9, we prove in Theorem 3 that this three parameter policy is optimal. Moreover, we provide the aforementioned analytic solution for the exponential demand case in Section 9.3. Figure 3 and Figure 4 in Section 9.2 give a pictorial representation of the type of results that we provide. Finally, Section 10 concludes the paper. The proofs of several preliminary technical results are relegated to the online companion of the paper.

2. Review of Literature

Arguably, the field of inventory management can be said to have begun in 1913 when Ford Whitman Harris came up with the Economic Order Quantity (EOQ) formula that balances the fixed ordering cost and inventory holding cost in the case of a constant demand rate and gives the optimal lot size or, equivalently, the optimal ordering frequency; see Erlenkotter (1990) for a historical account. It took many years before a decade long flurry of research activity, beginning with Arrow et al. (1951), which advanced the frontiers of inventory management by considering the realistic case of stochastic demand. The results on the optimality of the base stock and (s, S) policies were established during the 1950s, culminating in Arrow et al. (1958), long considered by many to be the Bible of Inventory Theory.

The classical paper Scarf (1960) proves the optimality of an (s, S) policy for a dynamic stochastic inventory problem where the costs structure consists of a fixed set-up or ordering cost, a unit cost, a linear holding cost, and a linear backlog cost. Whereas Scarf treated the case of backlog, Shreve (1976) extended it to the lost sales case. We should mention that the arguments employed by Shreve (1976) apply as well to many corresponding inventory problems with backlog. Following Scarf (1960), there has been a considerable amount of literature devoted to extending the optimality of (s, S) type policies to a variety of situations, which in some cases also necessitates the modification of the policy itself.

While the optimality of (s, S) policies has been explored in many different contexts, the direction that is most relevant to our paper are the works of Porteus (1971), Porteus (1972), Fox et al. (2006) and Benjaafar et al. (2018). A generalized K-convex function is used by Porteus (1971) to examine an inventory problem with an increasing concave ordering cost. He introduces a generalized (s, S) policy and proves its optimality when the demands follow one-sided Polya distributions. Porteus (1972) extends his earlier analysis to either uniform or convolutions of a finite number of uniform distributions. Fox et al. (2006) use a log-concave demand density distribution to show that a certain three parameter policy, which they call the (s, S_{HVC}, S_{LVC}) policy, is optimal by using the notions of K-convexity and quasi-convexity. Even though they are broader than the class of one-sided Polya densities, log-concave densities still only account for a limited range of distributions.

In practice, it frequently happens that an ordering cost or production cost has a number of break points, which results in a piecewise-linear concave cost function. Benjaafar et al. (2018) assumes this cost structure with arbitrary number n of break points along with a general demand. Under some limitations specified in Theorem 4 and Proposition 2 in their paper, they prove the optimality of a generalized (s,S) policy. This policy is characterized by a sequence of nested thresholds

$$s_n < s_{n-1} < \cdots < s_1 \leq S_1 < S_2 < \cdots < S_n, \tag{1}$$

such that when the beginning inventory is (i) $> s_1$, do not order; (ii) in $(s_i, s_{i-1}]$, order up to S_i, $i = 2, 3, \cdots n-1$; (iii) $\leq s_n$, order up to S_n. In the absence of these limitations, however, they show via a two-period counterexample that the policy may fail to be optimal outside an interval of beginning inventory levels. Moreover, their numerical experiments seem to suggest that the policy becomes optimal provided the problem horizon is sufficiently long. In their Theorem 6, they make a restrictive assumption on the cost and the demand density in order to establish the optimality of a generalized (s,S) policy. We can therefore take it as a conjecture that a generalized (s,S) policy will be optimal for an infinite-horizon problem. Moreover, we would like to establish this optimality without their restrictive assumption. While they consider the backlog case, they also mention that similar results can also be derived in the lost sales case. In this paper we take up the lost sales case and prove the optimality of a generalized (s,S) policy for an infinite-horizon problem. This is done with a general demand without the limitations and the restrictions imposed in Benjaafar et al. (2018). Before we describe the other contributions of this paper, we mention that our results easily extend to the backlog case as well.

Our method constructs the value function explicitly by solving the required functional Bellman equation in the stationary infinite-horizon setting. This setting causes the difficulty of having to solve a functional equation as opposed to solving a sequence of optimization problems in finite horizon cases. Thus, our method does not rely on the notion of K-convexity used in the literature for solving inventory problems with fixed ordering costs. In particular, the method used by Benjaafar et al. (2018) relies on the concept of c-convexity that generalizes the standard notion of K-convexity, and the resulting characterization of the policy parameter thresholds does not lead easily to constructive algorithms. Our method, on the other hand, provides an explicit formula for the value function in the general demand case. Moreover, in the case of exponential demand, we are able to give a closed-form solution for the value function as well as for the policy parameters. We consider only two linear segments in equation (2) in this paper for reasons of mitigating technical complexities, and leave the case of more than 2 segments as a topic of future research.

3. Problem Formulation

We study an infinite-horizon discrete-time stochastic inventory control problem where an order at the beginning of a period can be placed from one of two available sources, one charging a fix ordering cost of K_1 and a unit cost of c_1 and the other charging $K_2 > K_1 \geq 0$ and $0 \leq c_2 < c_1$, respectively. This cost structure is equivalent to ordering from a single source charging a fixed ordering cost of K_1 and a unit ordering cost of c_1 if an order does not exceed $(K_2 - K_1)/(c_1 - c_2) := \epsilon$, and if it does, then charging a unit ordering cost of c_2 for units in excess of ϵ; see Figure 1(a). The cost is incurred when the order is placed and its delivery is assumed to be instantaneous. After that, a random demand is realized and it is satisfied fully if less than or equal to the inventory on hand; if not then the demand in excess of the on-hand inventory is lost. A holding cost $h \geq 0$ is incurred at the end of the period, (or, equivalently at the beginning of the next period) for each unit carried to the next period or a penalty $p > c_2$ is incurred for each unit of unfilled demand, depending on the situation realized. The assumption $p > c_2$ is standard in the literature, without which it would always be optimal to order nothing and lose sale in each period. It is just by choice that we are choosing the lost sales case, while mentioning at the same time that the backlog case can be treated in a similar manner. Our purpose is to find an optimal ordering policy that minimizes the expected present value of the costs of ordering, holding and penalty over the infinite horizon. We shall define below the notation and then provide the mathematical formulation of the problem along with the functional Bellman equation that its value function satisfies.

Let us now introduce the following notation:

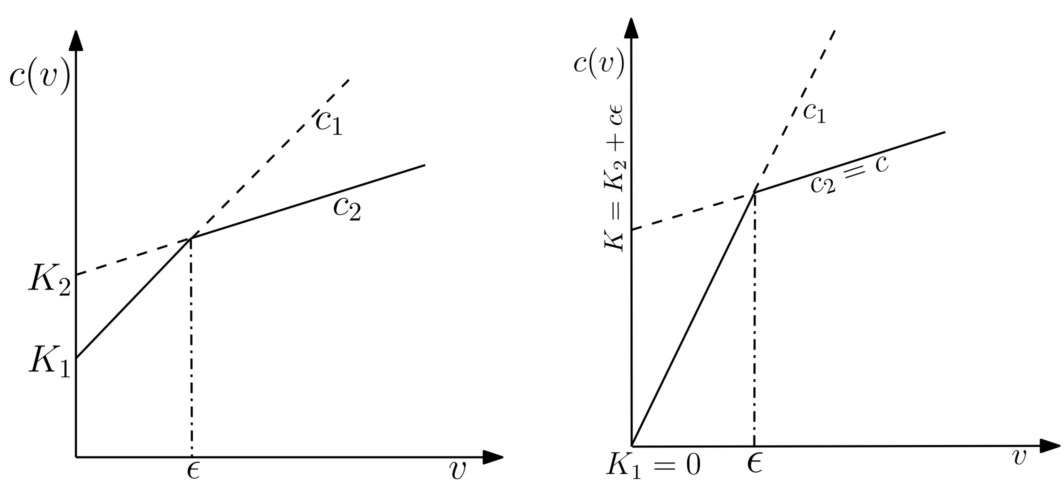
- K_1 - fixed cost to place an order from the first supplier; $K_1 \geq 0$
- K_2 - fixed cost to place an order from the second supplier; $K_2 > K_1$

- D - one period demand (random variable); $D \geq 0$ a.s.
- c_1 - per unit purchase cost from the first supplier
- c_2 - per unit purchase cost from the second supplier; $0 \leq c_2 < c_1$
- h - inventory holding cost per unit per period; $h \geq 0$
- p - penalty cost per unit incurred when the demand is not met and thus lost; $p > c_2$
- $f(x)$ - probability density function (pdf)
- $F(x)$ - cumulated distribution function (CDF); $F'(x) = f(x)$
- α - discount factor; $\alpha > 0$
- x - initial inventory level; $x \geq 0$
- v - total amount of order; $v \geq 0$
- y - inventory after receiving the order; $y = x + v$

We assume our cost function to be given by the formula

$$c(v) = \min_{i=1,2}(K_i + c_i v). \tag{2}$$

The problem we are considering with two linear segments in (2), arises in many practical situations. It not only arises in a problem of choosing amongst two suppliers but also represents the situation of rebate for large quantities in the case of a single supplier.



(a) Case $K_1 > 0$ (b) Case $K_1 = 0$

Figure 1 Figure (a) is the graph of cost function given in equation (2). Figure (b) is the specialization of Figure (a) to the case $K_1 = 0$. This defines the quantities c_2, K and c_1 in Section 8.1.

Indeed, set $\epsilon := \dfrac{K_2 - K_1}{c_1 - c_2}$, then it is easy to see that

$$c(v) = K_1 \mathbb{1}_{v>0} + c_1 v \mathbb{1}_{v<\epsilon} + (c_2(v-\epsilon) + c_1\epsilon)\mathbb{1}_{v\geq\epsilon}.$$

Here as usual $\mathbb{1}_v$ is the indicator function[1]. So the variable ordering cost per unit is reduced from c_1 to c_2, when the volume is larger than ϵ.

Let x be the initial inventory level and y be the inventory level after the order is received. The generic demand is denoted by D, which is a random variable with probability density $f(\xi)$ with its support on $[0,\infty)$. The CDF is denoted by $F(d) = \int_0^d f(\xi)d\xi$ and $\bar{F}(d) = 1 - F(d)$. Let $u(x)$ denote the expected value of the discounted costs for an inventory problem. Then, $u(x)$ satisfies the functional equation

$$\begin{aligned} u(x) = hx + \inf_{y\geq x} \Big\{ & K_1 \mathbb{1}_{y>x} + c_1(y-x)\mathbb{1}_{y-x<\epsilon} + (c_2(y-x) + (c_1-c_2)\epsilon)\mathbb{1}_{y-x\geq\epsilon} \\ & + pE(D-y)^+ + \alpha Eu((y-D)^+)\Big\},\ x>0, \end{aligned} \tag{3}$$

where h is the unit holding cost and $pE(D-y)^+$ represents the penalty cost. Equation (3) is a Bellman functional equation for the value function $u(x)$, for an initial inventory x. We can write equation (3) as

$$\begin{aligned} u(x) = (h-c_2)x + \inf_{y\geq x} \Big\{ & c_2 y + K_1 \mathbb{1}_{y>x} + (c_1-c_2)(y-x)\mathbb{1}_{y-x<\epsilon} + (c_1-c_2)\epsilon\mathbb{1}_{y-x\geq\epsilon} \\ & + pE(D-y)^+ + \alpha Eu((y-D)^+)\Big\},\ x>0. \end{aligned} \tag{4}$$

We make the transformation

$$u(x) = (h-c_2)x + H(x) + \rho, \tag{5}$$

where ρ is a constant to be chosen. Substitute equation (5) into equation (4). This defines H for which we get the expansion

$$H(x) + \rho(1-\alpha) = \inf_{y\geq x} \Big\{ K_1 \mathbb{1}_{y>x} + (c_1-c_2)(y-x)\mathbb{1}_{y-x\leq\epsilon} + (c_1-c_2)\epsilon\mathbb{1}_{y-x>\epsilon} + g(y) + \alpha EH((y-D)^+)\Big\}, \tag{6}$$

where

$$g(y) = c_2 y + pE(D-y)^+ + \alpha(h-c_2)E(y-D)^+. \tag{7}$$

The idea is to find an optimal policy that minimizes the expected total discounted cost, or in other words, that achieves the minimum of the right hand side of equation (6). We give conditions that guarantee that a standard (s,S) policy will remain optimal. With regard to the generalized (s,S) policies, we categorize our work in two different cases, namely, $K_1>0$ and $K_1=0$. In general, we develop a common methodology which is applicable to both cases in Section 6.1, and we lay out a road map for completing the analysis in Section 6.2. In the non-case-specific context, we explain all the steps, but in the $K_1>0$ case, we solve the problem completely only for an exponential demand.

For $K_1>0$, we show that a generalized (s,S) policy is optimal, which we call a $(\sigma_\epsilon, s_\epsilon, \Sigma_\epsilon, S_\epsilon)$ policy, where $\sigma_\epsilon < s_\epsilon \leq \Sigma_\epsilon < S_\epsilon$. This can be written as an (s_2, s_1, S_1, S_2) policy, which is a special case of (1) for two linear segments in the cost function. In the case $K_1=0$, we solve the problem for a general demand, because then $\Sigma_\epsilon = s_\epsilon$, and the policy simplifies from four parameters to three. In particular, this happens because equation (31) in Section 6.2, which is the most intricate step in Section 6.2, greatly simplifies. Similarly, when $K_1>0$ and the demand is exponential, equation (31) is again quite manageable. Therefore, we obtain a complete solution for the four policy parameters in this case. We are also able to obtain a closed-form solution when $K_1=0$ and the demand is exponential. In this case, all steps in the road map have explicit solutions.

For a graphical interpretation of our results, see Figure 3 and Figure 4. Figure 3 visualizes the results in the case where both suppliers impose a set up cost. We observe that, with a lower fixed cost from the first supplier, if the beginning inventory is more than σ_ϵ and less or equal to s_ϵ, then order up to Σ_ϵ; if it is less or equal to σ_ϵ, then order up to S_ϵ; and if it is more than s_ϵ, then do not order.

Figure 4 illustrates a situation in which the first supplier's fixed cost is zero. We see that if inventory level is between σ_ϵ and $s_\epsilon = \Sigma_\epsilon$, then ordering up to Σ_ϵ is optimal with zero or negligible fixed cost from the first supplier. In the case of a second supplier with a substantial amount of fixed cost and a relatively small variable cost, if the beginning inventory in any given period is more than σ_ϵ and less or equal to Σ_ϵ, then one should order up to Σ_ϵ from the high proportional cost supplier; if it is less or equal to σ_ϵ, then one should order up to S_ϵ from the low proportional cost supplier; and if it is more than Σ_ϵ, then do not order.

4. Preliminaries

In this section we briefly outline the classical case of the (s,S) policy in Section 4.1, obtained when there is only one supplier. We rewrite in Section 4.2 the Bellman equation, formulated in the previous section. Here we see that for a sufficiently small ϵ, the optimal policy remains a two parameter policy. We analyze this case as a bridge to solving the problem for any arbitrarily given positive ϵ.

4.1. Classical Case

The classical case involves just one supplier with fixed ordering cost K and per unit purchase cost c. It can be viewed as a limiting case of our problem formulated in Section 3 by considering $K_1 = K_2 = K$, $c_1 = c_2 = c$, and $\epsilon = 0$, so that equation (6) specializes to

$$H(x) + \rho(1-\alpha) = \inf_{y \geq x} \{K \mathbb{1}_{y>x} + g(y) + \alpha E H((y-D)^+)\}.$$

We already know in this case that an (s,S) policy is optimal. Nevertheless, we describe briefly how to use our methodology in this special case so as to foresee the steps needed for the analysis of equation (6) in general. This analysis begins with any $s > 0$, chooses $\rho(1-\alpha) = g(s)$, and then obtains $H_s(x)$ as the solution of

$$\begin{cases} H_s(x) = g(x) - g(s) + \alpha E H_s((x-D)^+), & \text{if } x > s, \\ H_s(x) = 0, & \text{if } x \leq s. \end{cases}$$

Since $g(x)$ is a convex function, $H_s(x)$ attains its minimum at a point (if there are several points we will take the smallest one). We define the minimum point to be S so that $H_s(S) = \inf_{y>s} H_s(y)$. Then define s so that

$$0 = K + \inf_{y \geq s} H_s(y).$$

A unique s can be defined, leading to a function $S(s)$. This is the well-known (s,S) policy, an optimal feedback law given by

$$v(x) = \begin{cases} S - x, & \text{if } x \leq s, \\ 0, & \text{if } x > s. \end{cases}$$

For more details, see Chapter 9 in Bensoussan (2011). We follow such a procedure to show that a two parameter policy remains optimal for ϵ sufficiently small.

4.2. Rewriting Bellman Equation

Returning to the case $\epsilon > 0$, one can expect that as ϵ is small, an (s_ϵ, S_ϵ) policy will be optimal. To show that, we first rewrite the Bellman functional equation defined in equation (6) as follows:

$$\begin{aligned} H(x) + \rho(1-\alpha) = \min \Big\{ & g(x) + \alpha E H((x-D)^+), \\ & K_1 + \inf_{x<y\le x+\epsilon} ((c_1 - c_2)(y-x) + g(y) + \alpha E H((y-D)^+)), \\ & K_2 + \inf_{y>x+\epsilon} (g(y) + \alpha E H((y-D)^+)) \Big\}. \end{aligned} \tag{8}$$

To reconcile this with a two parameter (s_ϵ, S_ϵ) policy, we show that in equation (8) the intermediate term in the right-hand side can be dispensed with to get

$$H(x) + \rho(1-\alpha) = \min \Big\{ g(x) + \alpha E H((x-D)^+), K_2 + \inf_{y>x+\epsilon} (g(y) + \alpha E H((y-D)^+)) \Big\}. \tag{9}$$

In other words, we have to show that the term $K_1 + \inf_{x<y\le x+\epsilon} ((c_1 - c_2)(y-x) + g(y) + \alpha E H((y-D)^+))$ of equation (8) is not relevant for the minimization in equation (8). In the next section, we provide the necessary conditions for the optimality of an (s_ϵ, S_ϵ) policy, where the pair (s_ϵ, S_ϵ) satisfies equation (9). Then we show that equation (5) is the value function (3), and hence the (s_ϵ, S_ϵ) policy is optimal.

5. Optimality of (s_ϵ, S_ϵ) Policy

In this section we show that there exists a threshold s_ϵ such that if $x \le s_\epsilon$ then it is optimal to order, and not to order otherwise. Further, there exists a smallest minimizer S_ϵ such that ordering up to that point is optimal. By analogy with the case $\epsilon = 0$ described above, for any $s > 0$, we take $\rho(1-\alpha) = g(s)$ and solve the problem

$$\begin{cases} H_s(x) = g(x) - g(s) + \alpha E H_s((x-D)^+), & \text{if } x > s, \\ H_s(x) = 0, & \text{if } x \le s. \end{cases} \tag{10}$$

This problem has a unique solution as follows. Differentiating in x, we get

$$\begin{cases} H_s'(x) = g'(x) + \alpha E H_s'(x-D), & \text{if } x > s, \\ H_s'(x) = 0, & \text{if } x \le s. \end{cases} \tag{11}$$

Since $g'(x)$ is bounded, this problem appears as a contraction problem on the space of bounded functions. Therefore, by the contraction mapping theorem, the function has a unique fixed point. So, the above problem has one and only one solution. Next from (7),

$$g'(x) = c_2 - p\bar{F}(x) + \alpha(h - c_2)F(x) = c_2 - p + (p + \alpha(h - c_2))F(x). \tag{12}$$

As $p > c_2$ by assumption, the function $g'(x)$ is increasing from $-(p - c_2)$ to $c_2(1-\alpha) + \alpha h$, and we have $-(p - c_2) \leq g'(x) \leq c_2(1-\alpha) + \alpha h$. Also by standard estimates from equation (11), we have

$$-\frac{p - c_2}{1-\alpha} \leq H_s'(x) \leq c_2 + \frac{\alpha h}{1-\alpha}, \tag{13}$$
$$H_s'(+\infty) = c_2 + \frac{\alpha h}{1-\alpha}.$$

Therefore, the function $H_s(x)$ is defined by

$$H_s(x) = \int_s^x H_s'(\xi)d\xi,\ x > s. \tag{14}$$

The function $H_s'(x)$ is not continuous at s, since $H_s'(x+0) = g'(s)$ and $H_s'(x-0) = 0$, unless $s = \bar{s}$, defined by

$$g'(\bar{s}) = 0. \tag{15}$$

This means from equation (12) we have

$$F(\bar{s}) = \frac{p - c_2}{p + \alpha(h - c_2)}. \tag{16}$$

Additionally, $H_s(x) \to \infty$ as $x \to \infty$, and thus as $H_s(x)$ is continuous, the function attains it's minimum on $[s, \infty)$. Therefore, we claim that there exists a unique $S(s)$ such that

$$H_s(S(s)) = \inf_{y \geq s} H_s(y). \tag{17}$$

Clearly, if $s \geq \bar{s}$, then $H_s(x)$ is increasing in x and for $s < \bar{s}$, $H_s(x)$ is decreasing on $[s, \bar{s})$. Necessarily,

$$S(s) \geq \bar{s}, \text{if}\, s < \bar{s},\ \text{ and } S(s) = s,\ \text{if}\, s \geq \bar{s}. \tag{18}$$

Now find an s_ϵ such that

$$0 = K_2 + \inf_{y \geq s_\epsilon + \epsilon} H_{s_\epsilon}(y). \tag{19}$$

Necessarily $s_\epsilon + \epsilon < \bar{s}$ (otherwise, (18) remains unsatisfied), which implies the assumption

$$\epsilon < \bar{s}, \tag{20}$$

where $\bar{s}$ is defined in (15). It is important to emphasize that when we say ϵ to be small or sufficiently small in this paper, we only mean that it is less than $\bar{s}$. We are not using it in the sense of a limit or it being close to zero.

Now from the inequality $s_\epsilon + \epsilon < \bar{s}$, the definition of $\bar{s}$, and equation (11), we can assert that $H'_{s_\epsilon}(y) < 0$, if $y \in [s_\epsilon, s_\epsilon + \epsilon]$. Therefore, $y \to H_{s_\epsilon}(y)$ is decreasing on $[s_\epsilon, s_\epsilon + \epsilon]$. Hence,

$$\inf_{y \in [s_\epsilon, s_\epsilon + \epsilon]} H_{s_\epsilon}(y) = H_{s_\epsilon}(s_\epsilon + \epsilon) \geq \inf_{y \geq s_\epsilon + \epsilon} H_{s_\epsilon}(y).$$

Therefore, to show equation (19) is satisfied, it is enough to show that

$$0 = K_2 + \inf_{y \geq s_\epsilon} H_{s_\epsilon}(y). \tag{21}$$

Furthermore, the function $s \mapsto \inf_{y > s} H_s(y) = H_s(S(s))$ is monotone increasing on $(0, \bar{s})$; see the online companion for its proof. So, to be able to find a unique solution s_ϵ of (19), it is necessary and sufficient to assume

$$\begin{cases} K_2 + \inf_{y > 0} H_0(y) < 0, \\ K_2 + \inf_{y > \bar{s} - \epsilon} H_{\bar{s} - \epsilon}(y) > 0. \end{cases} \tag{22}$$

Therefore, we obtain the following result.

PROPOSITION 1. *Assume $\epsilon < \bar{s}$ and (22). Then there exists a unique $s_\epsilon < \bar{s} - \epsilon$ that solves (19) with the function $H_s(x)$ given by (14). Moreover, $H_s(x)$ is the unique solution of (10).*

We want to check that the solution $H_{s_\epsilon}(x)$ is a solution of our original problem stated in equation (8) with

$$\rho(1 - \alpha) = g(s_\epsilon). \tag{23}$$

This is done next under an additional condition. Then with $S_\epsilon = S(s_\epsilon)$, the optimal (s_ϵ, S_ϵ) policy becomes defined.

THEOREM 1. *In addition to the assumptions of Proposition 1, let*

$$p \leq c_1(1-\alpha)+\alpha c_2. \tag{24}$$

Then the (s,S) *policy defined by* (17) *and* (23) *is optimal.*

Proof. It is given in the electronic companion Section EC.1.

REMARK 1. The purpose of assumptions (22) and (24) is to insure a standard (s,S) policy to be optimal. In particular, assumption (24) requires the penalty cost to be small, without which we can only prove that $H_{s_\epsilon}(x)$ is the solution of (9). But this is not enough to show the optimality of an (s,S) policy for the case under consideration. Assumption (24) ensures that $H_{s_\epsilon}(x)$ is the solution of equation (8). Taken together, Proposition 1 and Theorem 1 show that the (s_ϵ, S_ϵ) policy is optimal. On the other hand when $p > c_1$, which necessarily means that (24) is violated, we will show in Section 8 that a generalized (s,S) policy is optimal.

REMARK 2. Since (24) implies $p \leq c_1$, we have $c_2 < p \leq c_1$. This means that any purchase from supplier 1 to meet a demand will cost c_1 in addition to any fixed cost, whereas not meeting that demand will only cost p which is less than c_1, and thus only order that will be placed will be with the second supplier. Moreover, if we make ϵ small only by increasing K_1 and/or c_1, then the optimal policy will be exactly the (s,S) policy of the classical problem with only the second supplier, and that will hold for every ϵ as long as it is small and (24) continues to be satisfied. Of course, the optimal policy parameters will depend on ϵ, if making ϵ small includes changing K_2 or c_2 or both, according as the classical solution whose single supplier will be the second supplier with the changed values of fixed and variable costs.

REMARK 3. On the other hand, if (24) does not hold and if $p > c_1$, then it becomes economical to order also from the first supplier. Indeed we show in Section 7, Section 8, and Section 9 that for a large p along with some other conditions, a generalized (s,S) policy is optimal.

6. Generalized (s, S) Policy

In view of Remark 1, we now take up the case when (24) is not satisfied. In this case, we develop the methodology and indicate a road map to prove the optimality of generalized (s, S) policies. In Section 7 for $K_1 > 0$ and Section 8 for $K_1 = 0$, we will execute the steps in this road map under additional assumptions.

6.1. Methodology

At this stage, we look for a solution of equation (8) of the following form: there are two numbers σ and s, with $0 < \sigma < s$, and a function $H(x)$ that satisfy

$$\begin{cases} H(x) = H(\sigma), & \forall x \leq \sigma, \\ H(x) = H(s) + (c_1 - c_2)(s - x), & \sigma < x \leq s, \\ H(x) = g(x) - g(s) + \alpha E H((x-D)^+), & x > s. \end{cases} \tag{25}$$

The values of $H(s)$ and $H(\sigma)$ are given by the condition that $H(x)$ is continuous. We first have, from the first two segments of equation (25), that $H(\sigma) = H(s) + (c_1 - c_2)(s - \sigma)$. From the third, we get

$$H(s) = \alpha E H((s-D)^+). \tag{26}$$

Next, we integrate equation (26) and plug the expressions of the first two segments of (25) into (26), to obtain

$$H(s) = \frac{\alpha}{1-\alpha}(c_1 - c_2) \int_0^{s-\sigma} \bar{F}(\xi) d\xi.$$

So for $x > s$, the function $H(x)$ is a solution of the integral equation

$$\begin{aligned} H(x) = \alpha \int_0^{x-s} H(x-\xi) f(\xi) d\xi + g(x) - g(s) + \\ \alpha(c_1 - c_2)[\int_0^{s-\sigma} \bar{F}(x - s + \eta) d\eta + \frac{\alpha}{1-\alpha} \bar{F}(x-s) \int_0^{s-\sigma} \bar{F}(\xi) d\xi. \end{aligned} \tag{27}$$

Clearly, $H(x)$ is completely defined once the constants σ and s are fixed. To obtain them, we impose the following two conditions:

$$H(s) = K_1 + \inf_{s < y \leq s+\epsilon} ((c_1 - c_2)(y - s) + H(y)). \tag{28}$$

$$H(\sigma) = K_2 + \inf_{y>s} H(y). \tag{29}$$

The challenge then is to show that equations (24) and (25), with the unknowns σ and s, define them completely, and that the corresponding function $H(x)$ solves equation (8) with $\rho(1-\alpha) = g(s)$. We then associate to the pair (σ, s), a pair (Σ, S) which realizes the infimum on the right-hand sides of (28) and (29), respectively. The quartet $(\sigma_\epsilon, s_\epsilon, \Sigma_\epsilon, S_\epsilon)$, where $\sigma_\epsilon < s_\epsilon \leq \Sigma_\epsilon < S_\epsilon$, will define the optimal generalized (s,S) policy.

6.2. Road Map

For $0 < \sigma < s$ fixed, we can define a unique function $H_{\sigma,s}(x)$, $x > s$, as the solution of the integral equation (27). This is done by defining its derivative $H'_{\sigma,s}(x)$ as the unique solution of the equation

$$H'_{\sigma,s}(x) = \alpha \int_0^{x-s} H'_{\sigma,s}(x-\xi) f(\xi) d\xi + g'(x) + \alpha(c_1 - c_2)(\bar{F}(x-\sigma) - \bar{F}(x-s)),\ x > s. \tag{30}$$

Just as in equation (11), this equation also has a unique solution, because it is a fixed point of a contraction mapping on the set of bounded continuous functions on $[s, +\infty)$. Since

$$H_{\sigma,s}(s) = \frac{\alpha}{1-\alpha}(c_1 - c_2) \int_0^{s-\sigma} \bar{F}(\xi) d\xi,$$

the function $H_{\sigma,s}(x)$, $x > s$, is given by the integral

$$H_{\sigma,s}(x) = H_{\sigma,s}(s) + \int_s^x H'_{\sigma,s}(\xi) d\xi.$$

The next steps are at the core of the difficulty. For a fixed σ, we define $s(\sigma) > \sigma$ by solving the algebraic equation

$$\frac{\alpha}{1-\alpha}(c_1 - c_2) \int_0^{s-\sigma} \bar{F}(\xi) d\xi = K_1 + \inf_{s<y\leq s+\epsilon}((c_1 - c_2)(y-s) + H_{\sigma,s}(y)), \tag{31}$$

and finally we seek σ as the solution of the algebraic equation

$$(c_1 - c_2)(s(\sigma) - \sigma) + \frac{\alpha}{1-\alpha}(c_1 - c_2) \int_0^{s(\sigma)-\sigma} \bar{F}(\xi) d\xi = K_2 + \inf_{y>s(\sigma)} H_{\sigma,s(\sigma)}(y).$$

We shall carry out this program completely in the case $K_1 = 0$, because then the step corresponding to equation (31) simplifies greatly. In fact, the infimum in (31) will be attained at s (recall that $H_{\sigma,s}(x)$

is a continuous function). So $s(\sigma)$ will be defined by the much simpler problem $H'_{\sigma,s}(s)+c_1-c_2=0$, which in view of equation (30) reduces to

$$0=c_1-c_2+g'(s)-\alpha(c_1-c_2)F(s-\sigma). \tag{32}$$

The last equation does not involve the function $H'_{\sigma,s}$ directly, and this is the simplification mentioned above.

The case $K_1>0$ will be considered only for the exponential demand, namely for $f(\zeta)=\lambda e^{-\lambda\zeta}$. In this case, equation (30) has an explicit solution.

7. Case $K_1>0$

Assuming the demand density to be exponential in the case $K_1>0$, we will extend the methodology described in Section 6.1. We will show that a four parameter $(\sigma_\epsilon, s_\epsilon, \Sigma_\epsilon, S_\epsilon)$ policy is optimal. We follow the road map as outlined in Section 6.2 and find the four parameters $\sigma_\epsilon, s_\epsilon, \Sigma_\epsilon, S_\epsilon$ as described in the last paragraph of section Section 6.1.

7.1. Preliminary Calculations

We implement the road map for $f(\zeta)=\lambda e^{-\lambda\zeta}$. We first consider the integral equation (30) which now becomes

$$\begin{aligned} H'(x)=\lambda\alpha\int_0^{x-s} H'(x-\xi)e^{-\lambda\xi}\,d\xi+c_2+\alpha(h-c_2)+ \\ +e^{-\lambda x}[-(p+\alpha(h-c_2))+\alpha(c_1-c_2)(e^{\lambda\sigma}-e^{\lambda s})]. \end{aligned} \tag{33}$$

It has an explicit solution given by

$$\begin{aligned} H'(x)=c_2+\frac{\alpha h}{1-\alpha}+e^{-\lambda(1-\alpha)(x-s)}[\alpha(c_1-c_2)e^{-\lambda(s-\sigma)}- \\ -(p+\alpha(h-c_2))e^{-\lambda s}-\alpha(c_1+\frac{\alpha h}{1-\alpha})]. \end{aligned} \tag{34}$$

The next step is to find the function $s(\sigma)$ by solving equation (31). We expect that the infimum on the right-hand side of equation (31) is attained at a point Σ inside the interval $(s,s+\epsilon)$. Therefore, we have $H'(\Sigma)+c_1-c_2=0$, which along with equation (34) yields

$$e^{\lambda(1-\alpha)(\Sigma-s)}=\frac{(p+\alpha(h-c_2))e^{-\lambda s}-\alpha(c_1-c_2)e^{-\lambda(s-\sigma)}+\alpha(c_1+\frac{\alpha h}{1-\alpha})}{c_1+\frac{\alpha h}{1-\alpha}}. \tag{35}$$

Condition (31) specializes to $0 = K_1 + \int_s^\Sigma (c_1 - c_2 + H'(x))dx$, which yields

$$\lambda(1-\alpha)[K_1 + (c_1 + \frac{\alpha h}{1-\alpha})(\Sigma - s)] = \tag{36}$$

$$(1 - e^{-\lambda(1-\alpha)(\Sigma - s)})[(p + \alpha(h - c_2))e^{-\lambda s} - \alpha(c_1 - c_2)e^{-\lambda(s-\sigma)} + \alpha(c_1 + \frac{\alpha h}{1-\alpha})].$$

Equations (35) and (36) give s and Σ as functions of σ. It is convenient to define

$$X = X(\sigma, s) = \frac{(p + \alpha(h - c_2))e^{-\lambda s} - \alpha(c_1 - c_2)e^{-\lambda(s-\sigma)}}{c_1 + \frac{\alpha h}{1-\alpha}}. \tag{37}$$

Then, by combining (35) and (36), we see that X solves the algebraic equation

$$X = \log(X + \alpha) + (1-\alpha)(1 + \frac{K_1 \lambda}{c_1 + \frac{\alpha h}{1-\alpha}}). \tag{38}$$

It has a unique solution $X_0 > 1 - \alpha$, which is thus a fixed constant independent of σ. Therefore, $s(\sigma)$ is obtained by the formula

$$e^{\lambda s} = \frac{p + \alpha(h - c_2) - \alpha(c_1 - c_2)e^{\lambda \sigma}}{X_0(c_1 + \frac{\alpha h}{1-\alpha})}. \tag{39}$$

We naturally need σ sufficiently small. Specifically, we need $\sigma \le \bar{s}_\epsilon$ with

$$e^{\lambda \bar{s}_\epsilon} = \frac{p + \alpha(h - c_2)}{X_0(c_1 + \frac{\alpha h}{1-\alpha}) + \alpha(c_1 - c_2)}. \tag{40}$$

In order to get $\bar{s}_\epsilon > 0$, we need the condition

$$p + \alpha(h - c_1) > X_0(c_1 + \frac{\alpha h}{1-\alpha}) \tag{41}$$

and $s(\bar{s}_\epsilon) = \bar{s}_\epsilon$. We have also $s(0) = \bar{s}^*_\epsilon$ with

$$e^{\lambda \bar{s}^*_\epsilon} = \frac{p + \alpha(h - c_1)}{X_0(c_1 + \frac{\alpha h}{1-\alpha})}. \tag{42}$$

Assumption (41) implies immediately $\bar{s}_\epsilon < \bar{s}^*_\epsilon$. Recall the definition of $\bar{s}$, that is $g'(\bar{s}) = 0$, to obtain

$$e^{\lambda \bar{s}} = \frac{p + \alpha(h - c_2)}{c_2 + \alpha(h - c_2)}. \tag{43}$$

In view of $X_0 > 1 - \alpha$, comparing equations (42) and (43) yields $\bar{s}^*_\epsilon < \bar{s}$. Summarizing we have:

PROPOSITION 2. *Assume (41), i.e., $p+\alpha(h-c_1) > X_0(c_1+\dfrac{\alpha h}{1-\alpha})$. Then for $\sigma \in [0,\bar{s}_\epsilon]$, $s(\sigma)$ is uniquely defined by (39), i.e., $e^{\lambda s} = \dfrac{p+\alpha(h-c_2)-\alpha(c_1-c_2)e^{\lambda\sigma}}{X_0(c_1+\dfrac{\alpha h}{1-\alpha})}$, and it decreases from $s(0)=\bar{s}^*_\epsilon$ to $s(\bar{s}_\epsilon)=\bar{s}_\epsilon$. We further have $\bar{s}_\epsilon < \bar{s}^*_\epsilon < \bar{s}$.*

Next, $\Sigma(\sigma)$ is defined by (35), which reads as

$$\Sigma(\sigma)-s(\sigma)=\frac{\log(X_0+\alpha)}{\lambda(1-\alpha)}. \tag{44}$$

We can also define $S(\sigma)$ by the condition $H'(S)=0$. From (34) and (37), we have

$$H'(x)=c_2+\frac{\alpha h}{1-\alpha}-(X_0+\alpha)(c_1+\frac{\alpha h}{1-\alpha})e^{-\lambda(1-\alpha)(x-s)}. \tag{45}$$

Therefore,

$$S(\sigma)-s(\sigma)=\frac{\log\left[(X_0+\alpha)\dfrac{c_1+\dfrac{\alpha h}{1-\alpha}}{c_2+\dfrac{\alpha h}{1-\alpha}}\right]}{\lambda(1-\alpha)}. \tag{46}$$

Looking at the form of (33) and noting $\sigma<s$, we can assert

$$H'(x)\le \lambda\alpha\int_0^{x-s}H'(x-\xi)e^{-\lambda\xi}\,d\xi+c_2+\alpha(h-c_2)-e^{-\lambda x}(p+\alpha(h-c_2)). \tag{47}$$

For $s<x<\bar{s}$, we have $c_2+\alpha(h-c_2)-e^{-\lambda x}(p+\alpha(h-c_2))<0$. Hence,

$$H'(x)\le \lambda\alpha\int_0^{x-s}H'(x-\xi)e^{-\lambda\xi}\,d\xi,\ s<x<\bar{s}. \tag{48}$$

Also from equation (37),

$$H'(s+0)=c_2+\alpha(h-c_1)-X_0(c_1+\frac{\alpha h}{1-\alpha})<-(c_1-c_2). \tag{49}$$

From (48) and (49), we see that

$$H'(x)<0,\ s<x<\bar{s}, \tag{50}$$

which implies necessarily $S(\sigma)>\bar{s}$. Calling $\widetilde{H}'(x)=H(x)+c_1-c_2$, we can write the equation for $\widetilde{H}'(x)$ as

$$\widetilde{H}'(x)=\lambda\alpha\int_0^{x-s}\widetilde{H}'(x-\xi)e^{-\lambda\xi}\,d\xi+c_1+\alpha(h-c_1)+ \tag{51}$$

$$+e^{-\lambda x}[-(p+\alpha(h-c_2))+\alpha(c_1-c_2)e^{\lambda\sigma}].$$

Hence, for $x>s$, we have

$$\widetilde{H}'(x)\le\lambda\alpha\int_0^{x-s}\widetilde{H}'(x-\xi)e^{-\lambda\xi}\,d\xi+c_1+\alpha(h-c_2)-(p+\alpha(h-c_2))e^{-\lambda x}.$$

We introduce the number $\bar{s}^*$ defined by

$$e^{\lambda\bar{s}^*}=\frac{p+\alpha(h-c_2)}{c_1+\alpha(h-c_2)}. \tag{52}$$

We can show, just as we had showed $S(\sigma)>\bar{s}$, that $\Sigma(\sigma)>\bar{s}^*$.

7.2. Formulae for $(\sigma_\epsilon,s_\epsilon,\Sigma_\epsilon,S_\epsilon)$

Calling $H_\sigma(x)$, the function constructed in the previous section, we can see by the above results that

$$H_\sigma(\sigma)-H_\sigma(S(\sigma))=\frac{c_2+\frac{\alpha h}{1-\alpha}}{\lambda}\left(\frac{c_1-c_2}{c_2+\frac{\alpha h}{1-\alpha}}\log\frac{(p+\alpha(h-c_2))e^{-\lambda\sigma}-\alpha(c_1-c_2)}{X_0(c_1+\dfrac{\alpha h}{1-\alpha})}\right. \tag{53}$$
$$\left.+\frac{1}{1-\alpha}[-\log\frac{(X_0+\alpha)(c_1+\dfrac{\alpha h}{1-\alpha})}{c_2+\frac{\alpha h}{1-\alpha}}+\frac{(X_0+\alpha)(c_1+\dfrac{\alpha h}{1-\alpha})}{c_2+\frac{\alpha h}{1-\alpha}}-1]\right).$$

We find $\sigma=\sigma_\epsilon$ by the equation

$$H_\sigma(\sigma)-H_\sigma(S(\sigma))=K_2,$$

and therefore,

$$K_2=\frac{c_2+\frac{\alpha h}{1-\alpha}}{\lambda}\left(\frac{c_1-c_2}{c_2+\frac{\alpha h}{1-\alpha}}\log\frac{(p+\alpha(h-c_2))e^{-\lambda\sigma}-\alpha(c_1-c_2)}{X_0(c_1+\dfrac{\alpha h}{1-\alpha})}\right. \tag{54}$$
$$\left.+\frac{1}{1-\alpha}[-\log\frac{(X_0+\alpha)(c_1+\dfrac{\alpha h}{1-\alpha})}{c_2+\frac{\alpha h}{1-\alpha}}+\frac{(X_0+\alpha)(c_1+\dfrac{\alpha h}{1-\alpha})}{c_2+\frac{\alpha h}{1-\alpha}}-1]\right).$$

Since the function $H_\sigma(\sigma)-H_\sigma(S(\sigma))$ is decreasing in σ, a unique solution of (54) exists provided that K_2 satisfies the conditions

$$H_0(0)-H_0(S(0))>K_2>H_{\bar{s}_\epsilon}(\bar{s}_\epsilon)-H_{\bar{s}_\epsilon}(S(\bar{s}_\epsilon)). \tag{55}$$

This means that

$$\frac{c_2+\frac{\alpha h}{1-\alpha}}{\lambda}\left(\frac{c_1-c_2}{c_2+\frac{\alpha h}{1-\alpha}}\log\frac{p+\alpha(h-c_1)}{X_0(c_1+\dfrac{\alpha h}{1-\alpha})}\right.$$
$$\left.+\frac{1}{1-\alpha}\left[-\log\frac{(X_0+\alpha)(c_1+\dfrac{\alpha h}{1-\alpha})}{c_2+\frac{\alpha h}{1-\alpha}}+\frac{(X_0+\alpha)(c_1+\dfrac{\alpha h}{1-\alpha})}{c_2+\frac{\alpha h}{1-\alpha}}-1\right]\right)>K_2>$$
$$\frac{c_2+\frac{\alpha h}{1-\alpha}}{\lambda(1-\alpha)}\left[-\log\frac{(X_0+\alpha)(c_1+\dfrac{\alpha h}{1-\alpha})}{c_2+\frac{\alpha h}{1-\alpha}}+\frac{(X_0+\alpha)(c_1+\dfrac{\alpha h}{1-\alpha})}{c_2+\frac{\alpha h}{1-\alpha}}-1\right].$$

Once σ_ϵ is defined, we can solve (54) to obtain the quantity X_0 within it and then obtain s_ϵ, Σ_ϵ and S_ϵ explicitly. Specifically the next equation gives s_ϵ in terms of X_0 and σ_ϵ.

$$(c_1-c_2)(s_\epsilon-\sigma_\epsilon)=K_2-\frac{c_2+\frac{\alpha h}{1-\alpha}}{\lambda(1-\alpha)}\left[-\log\frac{(X_0+\alpha)(c_1+\dfrac{\alpha h}{1-\alpha})}{c_2+\frac{\alpha h}{1-\alpha}}+\frac{(X_0+\alpha)(c_1+\dfrac{\alpha h}{1-\alpha})}{c_2+\frac{\alpha h}{1-\alpha}}-1\right], \quad (56)$$

Then we use the next two equations to obtain S_ϵ and Σ_ϵ.

$$S_\epsilon-s_\epsilon=\frac{1}{\lambda(1-\alpha)}\log\frac{(X_0+\alpha)(c_1+\dfrac{\alpha h}{1-\alpha})}{c_2+\frac{\alpha h}{1-\alpha}}, \quad (57)$$

$$\Sigma_\epsilon-s_\epsilon=\frac{1}{\lambda(1-\alpha)}\log(X_0+\alpha). \quad (58)$$

Now the function $H_\epsilon(x)=H_{\sigma_\epsilon}(x)$ is defined by the formulae

$$H_\epsilon(x)=\begin{cases} H_\epsilon(\sigma_\epsilon), & \forall x\le\sigma_\epsilon,\\ H_\epsilon(s_\epsilon)-(c_1-c_2)(x-s_\epsilon), & \sigma<x<s_\epsilon,\\ H_\epsilon(s_\epsilon)+(c_2+\frac{\alpha h}{1-\alpha})(x-s_\epsilon)-\frac{(X_0+\alpha)\left(c_1+\dfrac{\alpha h}{1-\alpha}\right)}{\lambda(1-\alpha)}(1-e^{-\lambda(1-\alpha)(x-s_\epsilon)}), & \forall x\ge s_\epsilon,\end{cases} \quad (59)$$

with

$$H_\epsilon(s_\epsilon)=\frac{\alpha(c_1-c_2)}{\lambda(1-\alpha)}(1-e^{-\lambda(s_\epsilon-\sigma_\epsilon)}), \quad (60)$$

and

$$H_\epsilon(\sigma_\epsilon)=H_\epsilon(s_\epsilon)+(c_1-c_2)(s_\epsilon-\sigma_\epsilon). \quad (61)$$

Therefore, the four parameters $(\sigma_\epsilon,s_\epsilon,\Sigma_\epsilon,S_\epsilon)$ are defined explicitly. We next show the optimality of the $(\sigma_\epsilon,s_\epsilon,\Sigma_\epsilon,S_\epsilon)$ policy.

7.3. Optimality of $(\sigma_\epsilon, s_\epsilon, \Sigma_\epsilon, S_\epsilon)$ Policy

We begin by verifying the following proposition.

PROPOSITION 3. *We have the inequality*

$$\Sigma_\epsilon - \sigma_\epsilon < \epsilon. \tag{62}$$

Proof. From (54), we can write $H_\epsilon(\sigma_\epsilon) < K_2 + H_\epsilon(\Sigma_\epsilon)$. We use the last equation in (59) to compute

$$H_\epsilon(\Sigma_\epsilon) = H_\epsilon(s_\epsilon) + (c_2 + \frac{\alpha h}{1-\alpha})(\Sigma_\epsilon - s_\epsilon) - \frac{(X_0+\alpha)\left(c_1 + \frac{\alpha h}{1-\alpha}\right)}{\lambda(1-\alpha)}(1 - e^{-\lambda(1-\alpha)(\Sigma_\epsilon - s_\epsilon)}).$$

Using (58), and (38), we obtain

$$H_\epsilon(\Sigma_\epsilon) = H_\epsilon(s_\epsilon) + \frac{X_0+\alpha-1}{\lambda(1-\alpha)}(c_2 - c_1) - K_1 \frac{c_2 + \frac{\alpha h}{1-\alpha}}{c_1 + \frac{\alpha h}{1-\alpha}}.$$

Using (61) we can assert

$$(c_1 - c_2)(s_\epsilon - \sigma_\epsilon) < K_2 - K_1 + \frac{c_1 - c_2}{\lambda(1-\alpha)}\left((1-\alpha)(\frac{K_1\lambda}{c_1 + \frac{\alpha h}{1-\alpha}} + 1) - X_0\right)$$

$$K_2 - K_1 - \frac{c_1 - c_2}{\lambda(1-\alpha)}\log(X_0+\alpha).$$

Finally, by using the definition of ϵ and (58), we obtain the inequality (62). ■

Collecting all the assumptions one obtains the following result.

THEOREM 2. *Assume*

- (41), *i.e.,* $p + \alpha(h - c_1) > X_0(c_1 + \frac{\alpha h}{1-\alpha})$.
- (55), *i.e.,* $H_0(0) - H_0(S(0)) > K_2 > H_{\bar{s}_\epsilon}(\bar{s}_\epsilon) - H_{\bar{s}_\epsilon}(S(\bar{s}_\epsilon))$.
- $\bar{s}_\epsilon + \epsilon < \bar{s}$.
- $f(\zeta) = \lambda e^{-\lambda\zeta}$.

Then the function $H_\epsilon(x)$ defined by (59), with σ_ϵ and s_ϵ defined by (53), (54) and (56), is the solution of the Bellman Equation (6) with $\rho(1-\alpha) = g(s_\epsilon)$.

Proof. See the online companion.

REMARK 4. It is interesting to mention that since $X_0 > 1-\alpha$, (41) implies $p > c_1$.

REMARK 5. (**Structure of the four parameter** $(\sigma_\epsilon, s_\epsilon, \Sigma_\epsilon, S_\epsilon)$ **policy**): The proof of Theorem 2 completes all the steps in the road map and thus the following ordering policy is optimal in the case $K_1 > 0$:

- If the beginning inventory level is less or equal to σ_ϵ, then order up to S_ϵ.
- If the beginning inventory level is less or equal to s_ϵ but more than σ_ϵ, then order up to Σ_ϵ.
- If the beginning inventory level is more than s_ϵ, then do not order.

8. Case $K_1 = 0$

Similar to the previous case, we will develop the methodology described in Section 6.1 for the case $K_1 = 0$, with reference to the cost structure in equation (2). We allow general demands. The main result in this section is to prove the optimality of a three parameter $(\sigma_\epsilon, \Sigma_\epsilon, S_\epsilon)$ policy in Theorem 3. This is made possible by showing that $s_\epsilon = \Sigma_\epsilon$ in this case. Figure 3 provides a pictorial illustration of the policy.

8.1. Problem Setting

We denote $c_2 = c$, $K = K_2 + c\epsilon$, then we have $c_1 = \dfrac{K}{\epsilon}$. Refer to Figure 1(b) to visualize c_2, K and c_1. We assume $\dfrac{K}{\epsilon} > c$. Then equation (32) becomes

$$\frac{K}{\epsilon} - p + (p + \alpha(h-c))F(s(\sigma)) = \alpha(\frac{K}{\epsilon} - c)F(s(\sigma) - \sigma). \tag{63}$$

Define the function $H_\sigma(x) = H_{\sigma, s(\sigma)}(x)$ as follows:

$$H_\sigma(x) = \begin{cases} \dfrac{\alpha}{1-\alpha}(\dfrac{K}{\epsilon} - c)\int_0^{s(\sigma)-\sigma} \bar{F}(\xi)d\xi + (\dfrac{K}{\epsilon} - c)(s(\sigma) - \sigma), & \text{if } x \leq \sigma, \\ \dfrac{\alpha}{1-\alpha}\left(\dfrac{K}{\epsilon} - c\right)\int_0^{s(\sigma)-\sigma} \bar{F}(\xi)d\xi + (\dfrac{K}{\epsilon} - c)(s(\sigma) - x), & \text{if } \sigma < x \leq s(\sigma), \\ \begin{cases} \alpha(\dfrac{K}{\epsilon} - c)[\int_0^{s(\sigma)-\sigma} \bar{F}(x - s(\sigma) + \eta)d\eta + \dfrac{\alpha}{1-\alpha}\bar{F}(x - s(\sigma))\int_0^{s(\sigma)-\sigma} \bar{F}(\xi)d\xi] \\ +g(x) - g(s(\sigma)) + \alpha\int_0^{x-s(\sigma)} H_\sigma(x-\xi)f(\xi)d\xi, \end{cases} & \text{if } x > s(\sigma). \end{cases} \tag{64}$$

We obtain σ by solving the equation

$$(\frac{K}{\epsilon}-c)(s(\sigma)-\sigma)+\frac{\alpha}{1-\alpha}(\frac{K}{\epsilon}-c)\int_0^{s(\sigma)-\sigma}\bar{F}(\xi)d\xi = K-c\epsilon+\inf_{y>s(\sigma)}H_\sigma(y). \tag{65}$$

We shall first study the definition of σ and $H_\sigma(x)$, and then check whether it satisfies equation (8) with $\rho(1-\alpha)=g(s(\sigma))$. Equation (8) becomes

$$\begin{aligned} H(x)=\min\Big\{ & g(x)-g(s(\sigma))+\alpha EH((x-D)^+), \\ & \inf_{x<y\le x+\epsilon}((\frac{K}{\epsilon}-c)(y-x)+g(y)+\alpha EH((y-D)^+)), \\ & K-c\epsilon+\inf_{y>x+\epsilon}(g(y)+\alpha EH((y-D)^+))\Big\}. \end{aligned} \tag{66}$$

8.2. Solution of Equation (63)

If equation (63) has a solution, then necessarily $\frac{K}{\epsilon}-p<\alpha(\frac{K}{\epsilon}-c)$, which means

$$\frac{K}{\epsilon}<\frac{p-\alpha c}{1-\alpha}. \tag{67}$$

Note that this is exactly the opposite of inequality (24) assumed in Theorem 1 for the classical (s,S) case. Moreover,

$$0\le\frac{K}{\epsilon}-p+(p+\alpha(h-c))F(s(\sigma))\le\alpha(\frac{K}{\epsilon}-c)F(s(\sigma)). \tag{68}$$

This leads us to define $\bar{s}_\epsilon$ as follows:

$$s(\sigma)\ge\bar{s}_\epsilon,\ F(\bar{s}_\epsilon)=\frac{(p-\frac{K}{\epsilon})^+}{p+\alpha(h-c)}. \tag{69}$$

Since $p<\frac{p-\alpha c}{1-\alpha}$ in view of $p>c_2=c$, we have two cases: either $p>\frac{K}{\epsilon}$ or

$$p\le\frac{K}{\epsilon}<\frac{p-\alpha c}{1-\alpha}. \tag{70}$$

In the second case, the second inequality in (68) implies necessarily that $p+\alpha(h-c)\le\alpha(\frac{K}{\epsilon}-c)$, for otherwise there will be no solution to (63). But then from (67), we have $p+\alpha(h-c)<\alpha\frac{p-c}{1-\alpha}$, which implies $p(1-\alpha)<\alpha p$. This requires $\alpha>1/2$. This is too restrictive an assumption as it would correspond economically to offering too large a discount. So we will not consider the case (70) and assume instead

$$p>c_1=\frac{K}{\epsilon}>c. \tag{71}$$

Therefore, looking at (69) we have

$$F(\bar{s}_\epsilon) = \frac{p - \dfrac{K}{\epsilon}}{p + \alpha(h-c)}. \tag{72}$$

If (71) holds, then $p + \alpha(h-c) > \alpha(\dfrac{K}{\epsilon} - c)$ and we define $\bar{s}^*_\epsilon$ by

$$s(\sigma) \le \bar{s}^*_\epsilon,\ F(\bar{s}^*_\epsilon) = \frac{p - \dfrac{K}{\epsilon}}{p - \alpha\dfrac{K}{\epsilon} + \alpha h}. \tag{73}$$

A solution of (63) necessarily belongs to the interval

$$\bar{s}_\epsilon < s(\sigma) < \bar{s}^*_\epsilon < \bar{s}, \tag{74}$$

and the last inequality comes from the definition of $\bar{s}$ (see equation (16)), since $\dfrac{p - \dfrac{K}{\epsilon}}{p - \alpha\dfrac{K}{\epsilon} + \alpha h} < \dfrac{p-c}{p - \alpha c + \alpha h}$, as is easily checked. We now define for $s \in [\bar{s}_\epsilon, \bar{s}^*_\epsilon]$, the map

$$s \mapsto T_\sigma(s) = \sigma + F^{-1}\left(\frac{\frac{K}{\epsilon} - p + (p + \alpha(h-c))F(s)}{\alpha(\frac{K}{\epsilon} - c)} \right).$$

We have $T_\sigma(\bar{s}_\epsilon) = \sigma$ and $T_\sigma(\bar{s}^*_\epsilon) = \sigma + \bar{s}^*_\epsilon$. Therefore, if $\sigma = 0$, we have $T_0((\bar{s}^*_\epsilon)) = (\bar{s}^*_\epsilon)$, i.e., $(\bar{s}^*_\epsilon)$ is a fixed point of $T_0(s)$. Similarly, if $\sigma = \bar{s}_\epsilon$, we have $T_{\bar{s}_\epsilon}(\bar{s}_\epsilon) = \bar{s}_\epsilon$, i.e., $\bar{s}_\epsilon$ is a fixed point of $T_{\bar{s}_\epsilon}(s)$. For $0 < \sigma < \bar{s}_\epsilon$, we have $T_\sigma(\bar{s}_\epsilon) < \bar{s}_\epsilon$ and $T_\sigma(\bar{s}^*_\epsilon) > \bar{s}^*_\epsilon$. Since the function $s \to T_\sigma(s)$ is continuous, there exists a fixed point of $T_\sigma(s)$. If there are several, we take the smallest one and have the following result:

Proposition 4. *We assume (71),i.e., $p > \dfrac{K}{\epsilon} > c$. Then for $\sigma \in [0, \bar{s}_\epsilon]$, there exists a unique smallest fixed point of the map $s \mapsto T_\sigma(s)$. This fixed point is denoted as $s(\sigma)$, and the inequality (74), i.e., $\bar{s}_\epsilon < s(\sigma) < \bar{s}^*_\epsilon < \bar{s}$, holds.*

8.3. Additional Properties

We first check that the infimum $\inf\limits_{y > s(\sigma)} H_\sigma(y)$ is attained. Indeed, from equation (30), as $x \to +\infty$, $H'_\sigma(x) \to \dfrac{\lim\limits_{x\to+\infty} g'(x)}{1-\alpha} = c + \dfrac{\alpha h}{1-\alpha}$. Therefore, $H_\sigma(x) \to +\infty$ as $x \to +\infty$. But thanks to equation (63), $H_\sigma(x)$ is continuously differentiable at $s(\sigma)$ with $H'_\sigma(s(\sigma)) = -(\dfrac{K}{\epsilon} - c)$. So the infimum of

$H_s(y)$ for $y > s(\sigma)$ is attained, and we denote the smallest minimum by $S(\sigma)$. Moreover, noting that $\bar{F}(x-\sigma) - \bar{F}(x - s(\sigma)) < 0$, we have

$$g'(x) + \alpha(\frac{K}{\epsilon} - c)(\bar{F}(x-\sigma) - \bar{F}(x - s(\sigma))) < 0, \forall x \leq \bar{s}$$

Therefore, necessarily, $S(\sigma) > \bar{s}$. This is because $g'(\bar{s}) = 0$ at $\bar{s}$, and consequently $H_s(y) < 0$ at $\bar{s}$. So the smallest minimum $S(\sigma)$ satisfies the inequality $S(\sigma) > \bar{s}$. Since we are expecting a threshold at $s(\sigma)$, it is necessary to have a downward slope for the function $\sigma \mapsto s(\sigma)$. Indeed, from equation (63), we can compute the derivative

$$s'(\sigma) = -\frac{\alpha(\frac{K}{\epsilon} - c) f(s(\sigma) - \sigma)}{(p + \alpha(h-c)) f(s(\sigma)) - \alpha(\frac{K}{\epsilon} - c) f(s(\sigma) - \sigma)}.$$

We know that $\bar{s}_\epsilon < s(\sigma) < \bar{s}^*_\epsilon$. So to ensure that $s'(\sigma) < 0$, it is sufficient to make the assumption

$$\inf_{\substack{\bar{s}_\epsilon < \xi < \bar{s}^*_\epsilon \\ \xi - \bar{s}_\epsilon < \eta < \xi}} \frac{f(\xi)}{f(\eta)} > \frac{\alpha(\frac{K}{\epsilon} - c)}{p + \alpha(h-c)}. \tag{75}$$

Note that this assumption is trivially satisfied in the exponential demand case, where $\inf_{\substack{\bar{s}_\epsilon < \xi < \bar{s}^*_\epsilon \\ \xi - \bar{s}_\epsilon < \eta < \xi}} \frac{f(\xi)}{f(\eta)} = e^{-\lambda \bar{s}_\epsilon} = \frac{\frac{K}{\epsilon} + \alpha(h-c)}{p + \alpha(h-c)} > \frac{\alpha(\frac{K}{\epsilon} - c)}{p + \alpha(h-c)}$. Based on the above discussion, we can state the following results.

Lemma 1. *Assume (75). Then the following holds:*

- *The function $\sigma \mapsto s(\sigma)$ is decreasing on $[0, \bar{s}_\epsilon]$ and $s(0) = \bar{s}^*_\epsilon$, $s(\bar{s}_\epsilon) = \bar{s}_\epsilon$.*
- *The function $\sigma \mapsto H_\sigma(x)$ is decreasing $\forall x$.*

Proposition 5. *The functions*

- *$\sigma \mapsto H'_\sigma(x)$ increases for $s(\sigma) < x$.*
- *$\sigma \mapsto S(\sigma)$ and $\sigma \mapsto H_\sigma(\sigma) - H_\sigma(S(\sigma))$ are decreasing.*

Proof. The proofs for Lemma 1 and Proposition 5 can be found in the online companion.

To demonstrate the main results, it remains to find a solution of equation (65) for σ.

8.4. Obtaining σ and Policy Structure

We can rewrite equation (65) as

$$H_\sigma(\sigma) - H_\sigma(S(\sigma)) = K - c\epsilon. \tag{76}$$

Since the function on the left hand side is monotone decreasing on $[0, \bar{s}_\epsilon]$, there exists a unique solution of equation (76) for σ if and only if,

$$(\frac{K}{\epsilon} - c)\bar{s}^*_\epsilon + \frac{\alpha}{1-\alpha}(\frac{K}{\epsilon} - c)\int_0^{\bar{s}^*_\epsilon} \bar{F}(\xi)d\xi - H_0(S(0)) > K - c\epsilon > -H_{\bar{s}_\epsilon}(S(\bar{s}_\epsilon)). \tag{77}$$

This leads to the following result.

PROPOSITION 6. *Assume*

- (71), *i.e.,* $p > \dfrac{K}{\epsilon} > c.$
- (75), *i.e.,* $\inf\limits_{\substack{\bar{s}_\epsilon < \xi < \bar{s}^*_\epsilon \\ \xi - \bar{s}_\epsilon < \eta < \xi}} \dfrac{f(\xi)}{f(\eta)} > \dfrac{\alpha(\frac{K}{\epsilon} - c)}{p + \alpha(h-c)}.$
- (77), *i.e.,* $(\dfrac{K}{\epsilon} - c)\bar{s}^*_\epsilon + \dfrac{\alpha}{1-\alpha}(\dfrac{K}{\epsilon} - c)\int_0^{\bar{s}^*_\epsilon} \bar{F}(\xi)d\xi - H_0(S(0)) > K - c\epsilon > -H_{\bar{s}_\epsilon}(S(\bar{s}_\epsilon)).$

Then there exists one and only one σ_ϵ solution of equation (65)*. Moreover, $\sigma_\epsilon \in [0, \bar{s}_\epsilon]$. If we set $s_\epsilon = s(\sigma_\epsilon) \in [\bar{s}_\epsilon, \bar{s}^*_\epsilon]$ and $S_\epsilon = S(\sigma_\epsilon)$, then the $(\sigma_\epsilon, s_\epsilon, S_\epsilon)$ policy is defined. We also have*

$$s_\epsilon - \sigma_\epsilon \leq \epsilon. \tag{78}$$

The proof follows from the previous results. Indeed, (65) implies

$$H_{\sigma_\epsilon}(\sigma_\epsilon) = (\frac{K}{\epsilon} - c)(s_\epsilon - \sigma_\epsilon) + H_{\sigma_\epsilon}(s_\epsilon),$$

$$(\frac{K}{\epsilon} - c)(s_\epsilon - \sigma_\epsilon) + H_{\sigma_\epsilon}(s_\epsilon) \leq K - c\epsilon + H_{\sigma_\epsilon}(s_\epsilon).$$

and we can therefore conclude inequality (78).

The next section is devoted to the proof of the optimality of the proposed three parameter policy.

9. Optimality of $(\sigma_\epsilon, \Sigma_\epsilon, S_\epsilon)$ Policy

In this section, we prove the optimality of the $(\sigma_\epsilon, \Sigma_\epsilon, S_\epsilon)$ policy stated in Proposition 6. We show in Theorem 3 that the function $H_\sigma(x)$ satisfies the Bellman equation and hence is the value function. A closed-form solution for the optimal $(\sigma_\epsilon, \Sigma_\epsilon, S_\epsilon)$ policy in the exponential demand case will be given in Section 9.3.

9.1. Problem Setting

We want to show that the function $H_{\sigma_\epsilon}(x)$ defined by (64), (65), and $\sigma=\sigma_\epsilon$, is the solution of (6) with $\rho(1-\alpha)=g(s_\epsilon)$. To simplify the notation, we suppress ϵ and use $H(x)$ for $H_{\sigma_\epsilon}(x)$, σ for σ_ϵ, and s for s_ϵ. We write (6) as

$$H(x)=\inf_{y\geq x}\{(\frac{K}{\epsilon}-c)(y-x)\mathbb{1}_{y-x\leq\epsilon}+(K-c\epsilon)\mathbb{1}_{y-x>\epsilon}+g(y)-g(s)+\alpha EH((y-D)^+)\}.$$

Taking in account the form of equation (64), we have to first verify the following three relations:

$$H(x)=\inf_{y\geq x}\{(\frac{K}{\epsilon}-c)(y-x)\mathbb{1}_{y\leq x+\epsilon}+(K-c\epsilon)\mathbb{1}_{y>x+\epsilon}+H(y)\},\text{ if }x\geq s, \tag{79}$$

$$\begin{aligned}H(\sigma)-(\frac{K}{\epsilon}-c)(x-\sigma)=\min\Big\{&\inf_{s\geq y\geq x}[(\frac{K}{\epsilon}-c)(y-x)\mathbb{1}_{y\leq x+\epsilon}+(K-c\epsilon)\mathbb{1}_{y>x+\epsilon}+g(y)-g(s)+\alpha H(\sigma)\\&-\alpha(\frac{K}{\epsilon}-c)\int_0^{y-\sigma}F(\xi)d\xi],\inf_{y>s}[(\frac{K}{\epsilon}-c)(y-x)\mathbb{1}_{y\leq x+\epsilon}+(K-c\epsilon)\mathbb{1}_{y>x+\epsilon}+H(y)]\Big\},\text{ if }s>x>\sigma,\end{aligned} \tag{80}$$

$$\begin{aligned}H(\sigma)=\min\Big\{&\inf_{\sigma\geq y\geq x}[(\frac{K}{\epsilon}-c)(y-x)\mathbb{1}_{y\leq x+\epsilon}+(K-c\epsilon)\mathbb{1}_{y>x+\epsilon}+g(y)-g(s)+\alpha H(\sigma)],\\&\inf_{s\geq y>\sigma}[(\frac{K}{\epsilon}-c)(y-x)\mathbb{1}_{y\leq x+\epsilon}+(K-c\epsilon)\mathbb{1}_{y>x+\epsilon}+g(y)-g(s)+\alpha H(\sigma)-\alpha(\frac{K}{\epsilon}-c)\int_0^{y-\sigma}F(\xi)d\xi],\\&\inf_{y>s}[(\frac{K}{\epsilon}-c)(y-x)\mathbb{1}_{y\leq x+\epsilon}+(K-c\epsilon)\mathbb{1}_{y>x+\epsilon}+H(y)]\Big\},\text{ if }x>s(\sigma).\end{aligned} \tag{81}$$

Then we show that the proposed $(\sigma_\epsilon,\Sigma_\epsilon,S_\epsilon)$ policy is optimal. For this, we need an additional assumption which extends the previous assumption (75).

9.2. Optimal Policy

We are going to assume

$$\inf_{0<\eta<\xi<\eta+\bar{s}_\epsilon}\frac{f(\xi)}{f(\eta)}>\frac{\alpha(\frac{K}{\epsilon}-c)}{p+\alpha(h-c)}. \tag{82}$$

Note that in the exponential demand case, this condition holds trivially since

$$\frac{f(\xi)}{f(\eta)}=e^{-\lambda(\xi-\eta)}>e^{-\lambda\bar{s}_\epsilon}=\frac{\frac{K}{\epsilon}+\alpha(h-c)}{p+\alpha(h-c)}>\frac{\alpha(\frac{K}{\epsilon}-c)}{p+\alpha(h-c)}.$$

We need also to assume that $\bar{s}_\epsilon < \bar{s} - \epsilon$. In the exponential demand case, this, in view of the definition of $\bar{s}_\epsilon$ and $\bar{s}$ in (72) and (16), means

$$\frac{K}{\epsilon} - c \leq (e^{\lambda\epsilon} - 1)(c(1-\alpha) + \alpha h). \tag{83}$$

This is an assumption about the smallness of the difference $\frac{K}{\epsilon} - c$. We then claim the following result.

THEOREM 3. *Assume*

- (71), *i.e.,* $p > \frac{K}{\epsilon} > c$.
- (77), *i.e.,* $(\frac{K}{\epsilon} - c)\bar{s}^*_\epsilon + \frac{\alpha}{1-\alpha}(\frac{K}{\epsilon} - c)\int_0^{\bar{s}^*_\epsilon} \bar{F}(\xi)d\xi - H_0(S(0)) > K - c\epsilon > -H_{\bar{s}_\epsilon}(S(\bar{s}_\epsilon))$.
- (82), *i.e.,* $\inf_{0<\eta<\xi<\eta+\bar{s}_\epsilon} \frac{f(\xi)}{f(\eta)} > \frac{\alpha(\frac{K}{\epsilon} - c)}{p + \alpha(h-c)}$.
- $\bar{s}_\epsilon < \bar{s} - \epsilon$.

Then the function $H(x) = H_{\sigma_\epsilon}(x)$, *obtained by formulae* (64) *and* (65), *satisfies the relations* (79), (80), *and* (81), *and therefore is the value function by way of being the solution of equation* (6) *with* $\rho(1-\alpha) = g(s_\epsilon)$.

Proof: See the online companion.

REMARK 6. (**Structure of the three parameter** $(\sigma_\epsilon, \Sigma_\epsilon, S_\epsilon)$ **policy**)

- If the beginning inventory level is less or equal to σ_ϵ, then order up to S_ϵ.
- If the beginning inventory level is less or equal to Σ_ϵ but more than σ_ϵ, then order up to Σ_ϵ.
- If the beginning inventory level is more than Σ_ϵ, then do not order.

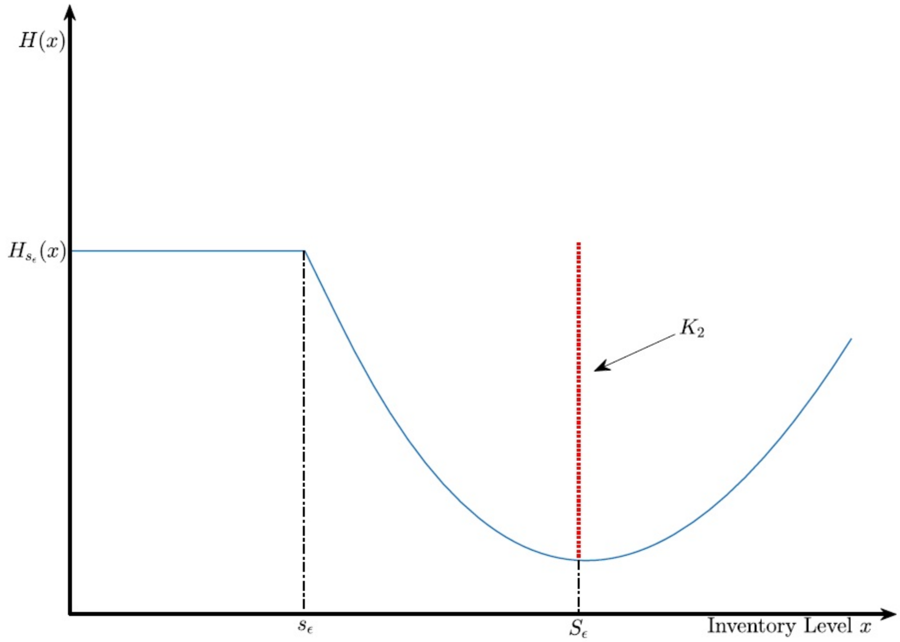


Figure 2 The graph of the standard (s, S) policy.

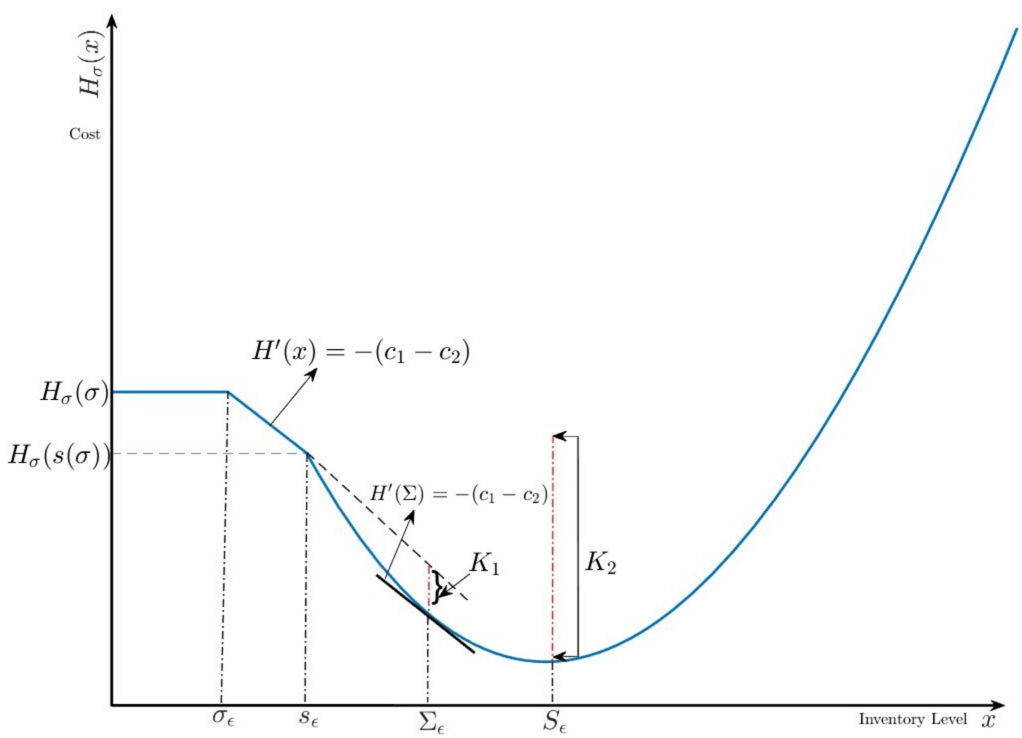


Figure 3 The graph of $H_\sigma(x)$ when $K_1 > 0$. The four parameter $(\sigma_\epsilon, s_\epsilon, \Sigma_\epsilon, S_\epsilon)$ policy is optimal. The solid line represents the optimal cost function and σ_ϵ and s_ϵ are the ordering points.

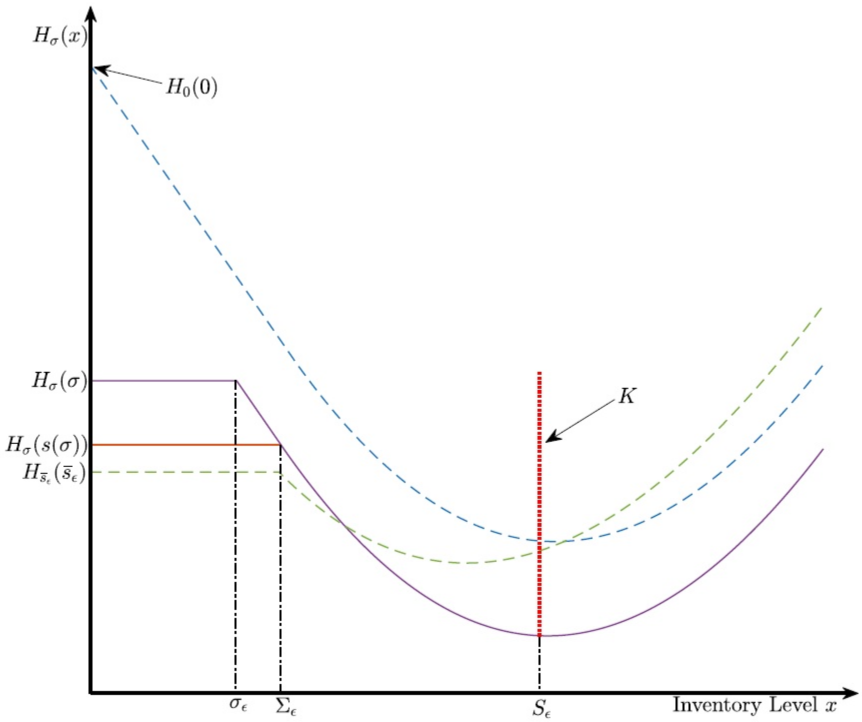


Figure 4 The graph of $H_\sigma(x)$ when $K_1 = 0$. The three parameter $(\sigma_\epsilon, \Sigma_\epsilon, S_\epsilon)$ policy is optimal.

9.3. Explicit Solution of $(\sigma_\epsilon, \Sigma_\epsilon, S_\epsilon)$ Policy for Exponential Demand

Let the demand density be $f(\zeta) = \lambda e^{-\lambda\zeta}$. Assume (71) and (83) and consider equations (16), (72), (74). Then there is a σ_ϵ that solves

$$\frac{1}{1-\alpha}\left[\frac{\frac{K}{\varepsilon}-c}{c+\frac{\alpha h}{1-\alpha}} - \log(1+\frac{\frac{K}{\varepsilon}-c}{c+\frac{\alpha h}{1-\alpha}})\right]+$$
$$+\frac{\frac{K}{\varepsilon}-c}{c+\frac{\alpha h}{1-\alpha}}\log\frac{\frac{(p+\alpha(h-c))e^{-\lambda\sigma_\epsilon}}{c+\frac{\alpha h}{1-\alpha}} - \alpha\frac{\frac{K}{\varepsilon}-c}{c+\frac{\alpha h}{1-\alpha}}}{(1-\alpha)(1+\frac{\frac{K}{\varepsilon}-c}{c+\frac{\alpha h}{1-\alpha}})} = \lambda\epsilon\frac{\frac{K}{\varepsilon}-c}{c+\frac{\alpha h}{1-\alpha}}. \quad (84)$$

With the help of equation (63), we then set

$$s_\epsilon = \sigma_\epsilon + \frac{1}{\lambda}\log\frac{(p+\alpha(h-c))e^{-\lambda\sigma_\epsilon} - \alpha(\frac{K}{\epsilon}-c)}{\frac{K}{\epsilon}(1-\alpha)+\alpha h}, \quad (85)$$

$$S_\epsilon = s_\epsilon + \frac{1}{\lambda(1-\alpha)}\log\frac{\frac{K}{\epsilon}+\frac{\alpha h}{1-\alpha}}{c+\frac{\alpha h}{1-\alpha}}. \quad (86)$$

This gives a complete analytic solution for the $(\sigma_\epsilon, s_\epsilon = \Sigma_\epsilon, S_\epsilon)$ policy. The details of this explicit solution are in the online companion Section EC.8.

10. Conclusion

In this paper, we study an infinite-horizon stochastic inventory model when the ordering cost is piecewise-linear concave resulting from quantity discounts or multiple suppliers. The most related recent work to our paper is Benjaafar et al. (2018) who consider a finite-horizon problem with a number of suppliers, and show the optimality of a generalized (s, S) policy under some assumptions relating to demand and cost. Without these limitations, however, their numerical examples show that the policy fails to be optimal when the problem horizon is not long enough. We take up their problem with only two suppliers for ease of technical exposition and show a generalized (s, S) policy is optimal without the assumptions imposed in Benjaafar et al. (2018). Of course, in our case, the generalized (s, S) policy can be characterized by four parameters with two representing ordering points and the other two representing order-up-to thresholds. In the special case when one of the suppliers has no fixed ordering cost, the policy has only three parameters, as the larger ordering point and the smaller order-up-to level equals in this situation. Moreover, when the demand distribution is exponential, we are able to obtain the optimal policy in a closed form.

In addition to the proof of optimality, we make some important contributions. Unlike in finite-horizon cases, an infinite-horizon problem requires a functional Bellman equation for the value function that is more difficult to solve than a sequence of optimization problems occurring in finite horizon problems. Our method constructs the value function explicitly by solving the functional Equation, and thus it does not rely on the notions related to K-convexity used in the literature for solving inventory problems with our cost structure.

Although we have worked out only the two linear segment case in equation (2), our method is general. In our future work, we plan first to look into the case of three suppliers and then extend it to the general case of an arbitrary number of suppliers. Finally, let us mention that while we have chosen to study the lost sales case in this paper, our methodology applies equally well to the backlog case.

Endnotes

1. $\mathbb{1}_v$ comes with value 1 if v satisfies the indicated condition otherwise 0.

Acknowledgments

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EC.1. Proof of Theorem 1 of Section 5.

Without assumption (24) we will prove that $H_{s_\epsilon}(x)$ is solution of (9), i.e., we will show that

$$H_{s_\epsilon}(x) = \min\left(g(x) - g(s_\epsilon) + \alpha E H_{s_\epsilon}((x-D)^+), K_2 + \inf_{y > x+\epsilon}(g(y) - g(s_\epsilon) + \alpha E H_{s_\epsilon}((y-D)^+))\right). \tag{EC.1}$$

Assume first $x \le s_\epsilon$. Then $H_{s_\epsilon}(x) = 0$ and $g(x) - g(s_\epsilon) + \alpha E H_{s_\epsilon}((x-D)^+) = g(x) - g(s_\epsilon) > 0$, since $g(x)$ decreases for $x < \bar{s}$. Next If $x + \epsilon < s_\epsilon$,

$$\begin{aligned}\inf_{y > x+\epsilon}(g(y) - g(s_\epsilon) + \alpha E H_{s_\epsilon}((y-D)^+)) &= \min\Big(\inf_{x+\epsilon < y < s_\epsilon}(g(y) - g(s_\epsilon), \\ &\qquad \inf_{s_\epsilon \le y}(g(y) - g(s_\epsilon) + \alpha E H_{s_\epsilon}((y-D)^+))\Big) \\ &= \min(\inf_{x+\epsilon < y < s_\epsilon}(g(y) - g(s_\epsilon), \inf_{s_\epsilon \le y} H_{s_\epsilon(y)}) = -K_2,\end{aligned}$$

since $\inf_{x+\epsilon < y < s_\epsilon}(g(y) - g(s_\epsilon)) > 0$. Hence the right hand side of (EC.1) is 0, which is indeed the value of $H_{s_\epsilon}(x)$, when $x \le s_\epsilon$. Hence $H_{s_\epsilon}(x)$ is solution of (EC.1) when $x + \epsilon \le s_\epsilon$. When $s_\epsilon \ge x > s_\epsilon - \epsilon$,

$$\inf_{y > x+\epsilon}(g(y) - g(s_\epsilon) + \alpha E H_{s_\epsilon}((y-D)^+)) = \inf_{y > x+\epsilon} H_{s_\epsilon}(y) = H_{s_\epsilon}(S_\epsilon) = \inf_{y \ge s_\epsilon} H_{s_\epsilon}(y) = -K_2.$$

Hence $H_{s_\epsilon}(x)$ is solution of (EC.1) when $x \le s_\epsilon$. If now $x > s_\epsilon$, then (EC.1) means

$$H_{s_\epsilon}(x) = \min(H_{s_\epsilon}(x), K_2 + \inf_{y > x+\epsilon} H_{s_\epsilon}(y)). \tag{EC.2}$$

To check (EC.2) we have to prove that

$$H_{s_\epsilon}(x) \le K_2 + H_{s_\epsilon}(y),\ \forall y > x + \epsilon \tag{EC.3}$$

But, in fact we have

$$H_{s_\epsilon}(x) \le K_2 + H_{s_\epsilon}(y),\ \forall y > x, \tag{EC.4}$$

which is a classical result. However, for convenience of the reader, a proof of (EC.4) is provided following the proof of this theorem.

So we have proven that $H_{s_\epsilon}(x)$ is the solution of (EC.1), and thus of (9) with $\rho(1-\alpha) = g(s_\epsilon)$. Assumption (24) is not needed for that. However, under assumption (24), to prove that $H_{s_\epsilon}(x)$ is the solution of (8) it suffices to show

$$H_{s_\epsilon}(x) \le K_1 + \inf_{x<y\le x+\epsilon}((c_1-c_2)(y-x) + g(y) - g(s_\epsilon) + \alpha E H_{s_\epsilon}((y-D)^+)). \tag{EC.5}$$

If $x+\epsilon \le s_\epsilon$, then (EC.5) becomes

$$0 \le K_1 + \inf_{x<y\le x+\epsilon}((c_1-c_2)(y-x) + g(y) - g(s_\epsilon)).$$

This is, of course, true because $g(y) - g(s_\epsilon) > 0$ for $y \le s_\epsilon$. So we may assume $s_\epsilon < x+\epsilon$. If $x \ge s_\epsilon$, then (EC.5) becomes

$$\begin{aligned} H_{s_\epsilon}(x) &\le K_1 + \inf_{x<y\le x+\epsilon}((c_1-c_2)(y-x) + H_{s_\epsilon}(y)) \\ &= K_1 + H_{s_\epsilon}(x). \end{aligned}$$

Next we see why we need condition (24). For this let us first observe that the left inequality in (13) can be strict when $x>0$. By adding (c_1-c_2) to both sides of the inequality, we get $H_s'(x) + (c_1-c_2) > c_1(1-\alpha) + c_2\alpha - p$ for $x>0$. Then we see that under condition (24), the function $(c_1-c_2)y + H_{s_\epsilon}(y)$ is monotone increasing when $y > s_\epsilon$. Therefore, the assertion (EC.5) is true if $x \le s_\epsilon - \epsilon$ or $x \ge s_\epsilon$. Finally if $s_\epsilon - \epsilon < x < s_\epsilon$, (EC.5) reduces to

$$0 \le K_1 + \min\left(\inf_{x<y\le s_\epsilon}((c_1-c_2)(y-x) + g(y) - g(s_\epsilon)),\ \inf_{s_\epsilon<y\le x+\epsilon}((c_1-c_2)(y-x) + H_{s_\epsilon}(y))\right).$$

The last inequality is correct by previously used arguments. The proof has been completed.■

EC.2. Proof of Equation (EC.4).

Equation (EC.4) is true if $H_\epsilon(x) \le 0$ since for $x \ge s_\epsilon$

$$K_2 + \inf_{y\ge x} H_\epsilon(y) \ge K_2 + \inf_{y\ge s_\epsilon} H_\epsilon(y) = 0 \ge H_\epsilon(x).$$

Let x_ϵ be the first point such that $H_\epsilon(x_\epsilon) = 0$. Set $S_\epsilon = S(s_\epsilon)$. Then $x_\epsilon > S_\epsilon$ and $H_\epsilon(x) < 0$ for $x < x_\epsilon$. So we need to prove equation (EC.4) only for $x > x_\epsilon$. Define

$$B_\epsilon(x) = H_\epsilon(x) - \inf_{y\ge x} H_\epsilon(y).$$

Thus we need to prove

$$B_\epsilon(x) \le K_2,\, \forall x > x_\epsilon. \tag{EC.6}$$

But $B_\epsilon(x) < K_2$, $\forall x < x_\epsilon$. Let x'_ϵ be the smallest number satisfying $x'_\epsilon > x_\epsilon$ and $B_\epsilon(x'_\epsilon) = K_2$. Let us consider now x_2^ϵ such that

$$H_\epsilon(x_2^\epsilon) = \inf_{\eta \ge x'_\epsilon} H_\epsilon(\eta).$$

The point x_2^ϵ is well defined since $H_\epsilon(\eta) \to \infty$ as $\eta \to \infty$. Thus $H_\epsilon(\eta)$ has a minimum and we take x_2^ϵ to be the smallest minimum. Further, $x_2^\epsilon \ne x'_\epsilon$ since

$$B_\epsilon(x'_\epsilon) = H_\epsilon(x'_\epsilon) - H_\epsilon(x_2^\epsilon) = K_2 \ne 0.$$

Next, consider x_1^ϵ such that

$$H_\epsilon(x_1^\epsilon) = \sup_{s^\epsilon \le \eta \ge x_2^\epsilon} H_\epsilon(\eta).$$

We claim that $x_1^\epsilon \ge x'_\epsilon$. Indeed if $x_1^\epsilon < x'_\epsilon$ then $\inf_{\eta \ge x_1^\epsilon} H_\epsilon(\eta) \le \inf_{\eta \ge x'_\epsilon} H_\epsilon(\eta)$ and

$$B_\epsilon(x_1^\epsilon) = H_\epsilon(x_1^\epsilon) - \inf_{\eta \ge x_1^\epsilon} H_\epsilon(\eta) \ge H_\epsilon(x'_\epsilon) - \inf_{\eta \ge x'_\epsilon} H_\epsilon(\eta) = B_\epsilon(x'_\epsilon) = K_2.$$

recalling that $x'_\epsilon < x_2^\epsilon$. This contradicts the definition of x'_ϵ. Therefore, $x_1^\epsilon > x'_\epsilon$. Further, $x_1^\epsilon \ne x_2^\epsilon$ since

$$H_\epsilon(x_1^\epsilon) - H_\epsilon(x_2^\epsilon) \ge H_\epsilon(x'_\epsilon) - H_\epsilon(x_2^\epsilon) = B_\epsilon(x'_\epsilon) = K_2 > 0.$$

Therefore we have

$$x_\epsilon < x'_\epsilon \le x_1^\epsilon < x_2^\epsilon. \tag{EC.7}$$

We next check the property that $\forall\, x \le y \le x_2^\epsilon$, we have

$$H_\epsilon(x) - H_\epsilon(y) \le H_\epsilon(x_1^\epsilon) - H_\epsilon(x_2^\epsilon). \tag{EC.8}$$

Indeed if $x'_\epsilon \le x \le y \le x_2^\epsilon$ then $H_\epsilon(y) \ge H_\epsilon(x_2^\epsilon)$ and $H_\epsilon(x) \le H_\epsilon(x_1^\epsilon)$, which proves (EC.8) in this case. Now, if $x < x'_\epsilon$ then

$$H_\epsilon(x) - H_\epsilon(y) \le H_\epsilon(x) - \inf_{y \ge x} H_\epsilon(y) = B_\epsilon(x) < K_2 \le H_\epsilon(x_1^\epsilon) - H_\epsilon(x_2^\epsilon),$$

which completes the proof of (EC.8). Next we write

$$H_\epsilon(x_1^\epsilon) = g(x_1^\epsilon) - g(s^\epsilon) + \alpha E H_\epsilon((x_1^\epsilon - D)^+),$$

$$H_\epsilon(x_2^\epsilon) = g(x_2^\epsilon) - g(s^\epsilon) + \alpha E H_\epsilon((x_2^\epsilon - D)^+).$$

So,

$$H_\epsilon(x_1^\epsilon) - H_\epsilon(x_2^\epsilon) = g(x_1^\epsilon) - g(x_2^\epsilon) + \alpha E H_\epsilon(x_1^\epsilon - D)^+) - \alpha E H_\epsilon(x_2^\epsilon - D)^+) \leq g(x_1^\epsilon) - g(x_2^\epsilon) + \alpha(H_\epsilon(x_1^\epsilon) - H_\epsilon(x_2^\epsilon)),$$

$$H_\epsilon(x_1^\epsilon) - H_\epsilon(x_2^\epsilon) \leq \frac{g(x_1^\epsilon) - g(x_2^\epsilon)}{1-\alpha} < 0.$$

This is because g is increasing on $(\bar{s}, \infty)$. This is a contradiction since $H_\epsilon(x_1^\epsilon) > H_\epsilon(x_2^\epsilon)$. Therefore the proof of equation (EC.4) is complete. ■

Note: We recommend to the reader to consult Bensoussan (2011) for a similar proof for the case $\epsilon = 0$.

EC.3. Proof of Monotonicity Property at page 15.

Indeed, we first write (11) as

$$H_s(x) = g(x) - g(s) + \alpha \int_0^{x-s} H_s(x-\xi) f(\xi) d\xi, \; x > s,$$

$\dfrac{\partial H_s(x)}{\partial s}$ is, for $x > s$, the solution of the equation

$$\frac{\partial H_s(x)}{\partial s} = -g'(s) + \alpha \int_0^{x-s} \frac{\partial H_s(x-\xi)}{\partial s} f(\xi) d\xi, \; x > s.$$

For $s < \bar{s}$, we have $g'(s) < 0$, therefore $\dfrac{\partial H_s(x)}{\partial s} > 0$, $x > s$. Next, $\inf_{y>s} H_s(y) = H_s(S(s))$ and

$$\frac{d}{ds} H_s(S(s)) = \frac{\partial H_s(S(s))}{\partial s},$$

in view of the optimality of $S(s)$. Therefore $\frac{d}{ds} H_s(S(s)) > 0$, when $s \in (0, \bar{s})$, which proves the property.

EC.4. Proof of Theorem 2 of Section 7.3

To simplify notation, we drop the index ϵ and denote $H(x), \sigma, s, \Sigma, S$. We must check that

$$H(x) = \inf_{x \leq y} \left\{ K_1 1\!\!1_{y>x} + (c_1 - c_2)(y - x) 1\!\!1_{y-x\leq\epsilon} + (c_1 - c_2)\epsilon 1\!\!1_{y-x>\epsilon} + g(y) - g(s) + \alpha E H((y - D)^+) \right\}.$$

There are 3 intervals $(0, \sigma), (\sigma, s), (s, +\infty)$. The verification amounts to ascertaining that:

- (a)

$$H(\sigma) = \min\left\{ \inf_{x\leq y\leq\sigma} [K_1 1\!\!1_{y>x} + (c_1 - c_2)(y - x) 1\!\!1_{y-x\leq\epsilon} + (c_1 - c_2)\epsilon 1\!\!1_{y-x>\epsilon} + g(y)] - g(s) + \alpha H(\sigma), \right.$$
$$K_1 + \inf_{\sigma<y\leq s} [(c_1 - c_2)(y - x) 1\!\!1_{y-x\leq\epsilon} + (c_1 - c_2)\epsilon 1\!\!1_{y-x>\epsilon} + g(y) - g(s) + \alpha H(\sigma) - \alpha(c_1 - c_2) \int_0^{y-\sigma} F(\xi) d\xi],$$
$$\left. K_1 + \inf_{s<y} [(c_1 - c_2)(y - x) 1\!\!1_{y-x\leq\epsilon} + (c_1 - c_2)\epsilon 1\!\!1_{y-x>\epsilon} + H(y)] \right\}, \text{ if } 0 \leq x \leq \sigma, \quad \text{(EC.9)}$$

- (b)

$$H(\sigma) - (c_1 - c_2)(x - \sigma) = \min\left\{ \inf_{x\leq y<s} [K_1 1\!\!1_{y>x} + (c_1 - c_2)(y - x) 1\!\!1_{y-x\leq\epsilon} + (c_1 - c_2)\epsilon 1\!\!1_{y-x>\epsilon} + g(y) - g(s) \right.$$
$$\left. + \alpha H(\sigma) - \alpha(c_1 - c_2) \int_0^{y-\sigma} F(\xi) d\xi], K_1 + \inf_{s\leq y} [(c_1 - c_2)(y - x) 1\!\!1_{y-x\leq\epsilon} + (c_1 - c_2)\epsilon 1\!\!1_{y-x>\epsilon} + H(y)] \right\},$$
$$\text{if } \sigma < x < s, \quad \text{(EC.10)}$$

- (c)

$$H(x) = \inf_{x\leq y} \left\{ K_1 1\!\!1_{y>x} + (c_1 - c_2)(y - x) 1\!\!1_{y-x\leq\epsilon} + (c_1 - c_2)\epsilon 1\!\!1_{y-x>\epsilon} + H(y) \right\}, \text{ if } s \leq x. \quad \text{(EC.11)}$$

We begin with (EC.11). It reduces to

$$H(x) = \min\left[H(x), K_1 + \inf_{x<y\leq x+\epsilon} ((c_1 - c_2)(y - x) + H(y)), K_2 + \inf_{y>x+\epsilon} H(y) \right]. \quad \text{(EC.12)}$$

In view of the formula (45) the function $H'(x)$ is increasing on $(s. + +\infty)$, and has a unique 0 at $S > s$. So $H(y)$ increases on $(S, +\infty)$ and decreases on (s, S). Clearly

$$H(x) < H(y) < K_2 + H(y), \; \forall x \geq S, \; y > x$$

$$H(x) < H(s) < H(\sigma) = K_2 + H(S)$$

$$\leq K_2 + H(y),\, \forall y \geq s,\, \forall s \leq x \leq S.$$

Therefore

$$H(x) < K_2 + \inf_{y > x+\epsilon} H(y),\, \forall x \geq s. \tag{EC.13}$$

Similarly $H'(x) + c_1 - c_2$ is increasing on $(s, +\infty)$, and has a unique 0 at $\Sigma > s$. Therefore $H(y) + (c_1 - c_2)y$ increases for $y > \Sigma$ and decreases on (s, Σ). So $H(x) + (c_1 - c_2)x < H(y) + (c_1 - c_2)y < K_1 + H(y) + (c_1 - c_2)y$, $\forall x \geq \Sigma$, $y > x$ and

$$H(x) + (c_1 - c_2)x \leq H(s) + (c_1 - c_2)s = K_1 + H(\Sigma) + (c_1 - c_2)\Sigma$$

$$\leq K_1 + H(y) + (c_1 - c_2)y,\, \forall y \geq s,\, \forall s \leq x \leq \Sigma.$$

Therefore,

$$H(x) < K_1 + \inf_{x < y \leq x+\epsilon} ((c_1 - c_2)(y - x) + H(y)),\, \forall x \geq s. \tag{EC.14}$$

From (EC.13) and (EC.14) we obtain immediately (EC.12).Therefore equation (EC.12) has been proved.

We next prove (EC.10). We first consider the term

$$U = K_1 + \inf_{s \leq y} \left\{ (c_1 - c_2)(y - x)\mathbb{1}_{y - x \leq \epsilon} + (c_1 - c_2)\epsilon \mathbb{1}_{y - x > \epsilon} + H(y) \right\}.$$

Since $\sigma < x < s < \Sigma < \sigma + \epsilon < x + \epsilon$, we can write

$$U = \min \left(K_1 + \inf_{s \leq y \leq x+\epsilon} [(c_1 - c_2)(y - x) + H(y)],\, K_2 + \inf_{y > x+\epsilon} H(y) \right).$$

But, since $\Sigma \in [s, x + \epsilon]$, we have

$$K_1 + \inf_{s \leq y \leq x+\epsilon} [(c_1 - c_2)(y - x) + H(y)] = K_1 + (c_1 - c_2)(\Sigma - x) + H(\Sigma) =$$

$$H(s) + (c_1 - c_2)(s - x) = H(\sigma) - (c_1 - c_2)(x - \sigma).$$

Now, since $x + \epsilon > s$, we have $\inf_{y > x+\epsilon} H(y) > H(S)$. Hence

$$K_2 + \inf_{y > x+\epsilon} H(y) > K_2 + H(S) = H(\sigma) > H(\sigma) - (c_1 - c_2)(x - \sigma).$$

Therefore, collecting the above results, we can write

$$U = H(\sigma) - (c_1 - c_2)(x - \sigma). \tag{EC.15}$$

Consider next

$$V = \inf_{x \le y < s} \Big\{ K_1 \mathbb{1}_{y>x} + (c_1 - c_2)(y - x)\mathbb{1}_{y-x\le\epsilon} + (c_1 - c_2)\epsilon \mathbb{1}_{y-x>\epsilon} + g(y)\} - g(s)$$
$$+ \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{y-\sigma} F(\xi)d\xi \Big\}.$$

Since $y < s < x + \epsilon$

$$V = \inf_{x \le y < s} \Big\{ K_1 \mathbb{1}_{y>x} + (c_1 - c_2)(y - x) + g(y) - g(s) + \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{y-\sigma} F(\xi)d\xi \Big\}.$$

Define for $y < s$,

$$\Phi(y) = (c_1 - c_2)y + g(y) - \alpha(c_1 - c_2)\int_0^{y-\sigma} F(\xi)d\xi.$$

We thus have

$$\Phi''(y) = -\alpha(c_1 - c_2)f(y - \sigma) + (p + \alpha(h - c_2))f(y)$$
$$= \lambda e^{-\lambda y}[(p + \alpha(h - c_2)) - \alpha(c_1 - c_2)e^{\lambda\sigma}].$$

Since $\sigma < \bar{s}_\epsilon$, it follows that

$$\Phi''(y) \ge \lambda e^{-\lambda y}[(p + \alpha(h - c_2)) - \alpha(c_1 - c_2)e^{\lambda \bar{s}_\epsilon}]$$
$$= \frac{\lambda e^{-\lambda y}(p + \alpha(h - c_2))X_0(c_1 + \frac{\alpha h}{1-\alpha})}{X_0(c_1 + \frac{\alpha h}{1-\alpha}) + \alpha(c_1 - c_2)} > 0.$$

So $\Phi'(y) = c_1 + \alpha(h - c_1) + \alpha(c_1 - c_2)\bar{F}(y - \sigma) - (p + \alpha(h - c_2))\bar{F}(y)$ is increasing for $y < s$. But

$$\Phi'(s) = c_1 + \alpha(h - c_1) + \alpha(c_1 - c_2)\bar{F}(s - \sigma) - (p + \alpha(h - c_2))\bar{F}(s)$$
$$= c_1 + \alpha(h - c_1) + e^{-\lambda s}[-(p + \alpha(h - c_2)) + \alpha(c_1 - c_2)e^{\lambda\sigma}].$$

Therefore from (37), we have $\Phi'(s) = (c_1 + \alpha(h - c_1))(1 - \dfrac{X_0}{1-\alpha}) < 0$. Hence, $\Phi'(y) < 0$, for $y < s$, and the function $\Phi(y)$ decreases for $y < s$. So it follows that

$$V > \inf_{x \le y < s} \Big\{ (c_1 - c_2)(y - x) + g(y) - g(s) + \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{y-\sigma} F(\xi)d\xi \Big\}$$
$$= (c_1 - c_2)(s - x) + \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{s-\sigma} F(\xi)d\xi$$
$$= -(c_1 - c_2)(x - \sigma) + (c_1 - c_2)(1 - \alpha)(s - \sigma) + \alpha(c_1 - c_2)\int_0^{s-\sigma} \bar{F}(\xi)d\xi + \alpha H(\sigma)$$
$$= -(c_1 - c_2)(x - \sigma) + H(\sigma) = U, \quad (\text{from}(6.2))$$

and (EC.10) has therefore been proved. We finally turn to (EC.9). We first consider

$$J = K_1 + \inf_{s<y}\left\{(c_1-c_2)(y-x)\mathbb{1}_{y-x\leq\epsilon} + (c_1-c_2)\epsilon\mathbb{1}_{y-x>\epsilon} + H(y)\right\}.$$

If $x+\epsilon < s$, then

$$J = K_2 + \inf_{y>s} H(y) = H(\sigma)$$

If $x+\epsilon > s$, then

$$J = \min\left\{K_1 + \inf_{y<s\leq x+\epsilon}((c_1-c_2)(y-x)+H(y)), K_2 + \inf_{y>x+\epsilon} H(y)\right\}.$$

Now

$$K_1 + \inf_{y<s\leq x+\epsilon}\left\{(c_1-c_2)(y-x)+H(y)\right\} = \begin{cases} K_1+(c_1-c_2)(\Sigma-x)+H(\Sigma), & \text{if } x+\epsilon\geq\Sigma, \\ K_2+H(x+\epsilon), & \text{if } x+\epsilon<\Sigma, \end{cases}$$

and

$$K_1+(c_1-c_2)(\Sigma-x)+H(\Sigma) = H(\sigma)+(c_1-c_2)(\sigma-x) > H(\sigma), \text{ if } x+\epsilon\geq\Sigma$$

$$K_2+H(x+\epsilon) > K_2+H(S) = H(\sigma), \text{ if } x+\epsilon<\Sigma.$$

Therefore,

$$K_1 + \inf_{y<s\leq x+\epsilon}\left\{(c_1-c_2)(y-x)+H(y)\right\} > H(\sigma), \text{ if } x+\epsilon>s.$$

On the other hand, since $x<\sigma$, and $\bar{s}_\epsilon+\epsilon<\bar{s}$ we have that $x+\epsilon<\sigma+\epsilon<\bar{s}_\epsilon+\epsilon<\bar{s}$. Therefore $x+\epsilon<S$, hence

$$K_2 + \inf_{y>x+\epsilon} H(y) = K_2 + H(S) = H(\sigma).$$

Collecting results we have checked that $J=H(\sigma)$. We next consider

$$X = K_1 + \inf_{\sigma<y\leq s}\left\{(c_1-c_2)(y-x)\mathbb{1}_{y-x\leq\epsilon} + (c_1-c_2)\epsilon\mathbb{1}_{y-x>\epsilon} + g(y) - g(s) + \alpha H(\sigma) - \alpha(c_1-c_2)\int_0^{y-\sigma} F(\xi)d\xi\right\}.$$

If $x+\epsilon > s$, then

$$X = K_1 + \inf_{\sigma<y\leq s}\left\{(c_1-c_2)(y-x) + g(y) - g(s) + \alpha H(\sigma) - \alpha(c_1-c_2)\int_0^{y-\sigma} F(\xi)d\xi\right\}.$$

Since $\Phi(y) = (c_1 - c_2)y + g(y) - \alpha(c_1 - c_2)\int_0^{y-\sigma} F(\xi)d\xi$ is decreasing for $y \le s$, we have

$$X = K_1 + (c_1 - c_2)(s - x) + \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{s-\sigma} F(\xi)d\xi$$
$$= K_1 + (c_1 - c_2)(\sigma - x) + H(\sigma) > H(\sigma).$$

If $x + \epsilon < s$, then

$$X = \min\left\{K_1 + \inf_{\sigma < y \le x+\epsilon}\left[(c_1 - c_2)(y - x) + g(y) - g(s) + \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{y-\sigma} F(\xi)d\xi\right],\right.$$
$$\left. K_2 + \inf_{x+\epsilon < y \le s}\left[g(y) - g(s) + \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{y-\sigma} F(\xi)d\xi\right]\right\}.$$

Using the decreasing property of $\Phi(y)$, we have

$$K_1 + \inf_{\sigma < y \le x+\epsilon}\left\{(c_1 - c_2)(y - x) + g(y) - g(s) + \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{y-\sigma} F(\xi)d\xi\right\} =$$
$$K_2 + g(x+\epsilon) - g(s) + \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{x+\epsilon-\sigma} F(\xi)d\xi.$$

Thus, if $x + \epsilon < s$, then

$$X = K_2 + \inf_{x+\epsilon < y \le s}\left\{g(y) - g(s) + \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{y-\sigma} F(\xi)d\xi\right\}$$
$$> K_2 + \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{s-\sigma} F(\xi)d\xi$$
$$> (c_1 - c_2)(s - \sigma) + \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{s-\sigma} F(\xi)d\xi = H(\sigma),$$

where we have used the fact that $s - \sigma < \epsilon$. So, we have proved that $X > H(\sigma)$. Finally, we consider

$$Y = \inf_{x \le y \le \sigma}\left\{K_1 \mathbb{1}_{y>x} + (c_1 - c_2)(y - x)\mathbb{1}_{y-x\le\epsilon} + (c_1 - c_2)\epsilon\mathbb{1}_{y-x>\epsilon} + g(y)\right\} - g(s) + \alpha H(\sigma).$$

If $x + \epsilon > \sigma$, then

$$Y = \inf_{x \le y \le \sigma}\left\{K_1 \mathbb{1}_{y>x} + (c_1 - c_2)(y - x) + g(y) - g(s) + \alpha H(\sigma)\right\}$$
$$> \inf_{x \le y \le s}\left\{(c_1 - c_2)(y - x) + g(y) - g(s) + \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{y-\sigma} F(\xi)d\xi\right\}$$
$$> (c_1 - c_2)(s - \sigma) + \alpha H(\sigma) - \alpha(c_1 - c_2)\int_0^{s-\sigma} F(\xi)d\xi = H(\sigma).$$

If $x + \epsilon < \sigma$, we have

$$Y = \min\left\{\inf_{x \le y \le x+\epsilon}\left[K_1 \mathbb{1}_{y>x} + (c_1 - c_2)(y - x) + g(y) - g(s) + \alpha H(\sigma)\right], K_2 + \inf_{x+\varepsilon < y \le \sigma}\Big(g(y) - g(s)\Big) + \alpha H(\sigma)\right\}.$$

But

$$K_2 + \inf_{x+\varepsilon<y\leq\sigma}(g(y)-g(s)) + \alpha H(\sigma) > K_2 + \alpha H(\sigma) > H(\sigma),$$

and

$$\begin{aligned}
&\inf_{x\leq y\leq x+\epsilon}\left\{K_1 1\!\!1_{y>x} + (c_1-c_2)(y-x) + g(y) - g(s) + \alpha H(\sigma)\right\}\\
&> \inf_{x\leq y\leq s}\left\{(c_1-c_2)(y-x) + g(y) - g(s) + \alpha H(\sigma)\right\}\\
&> \inf_{x\leq y\leq s}\left\{(c_1-c_2)(y-x) + g(y) - g(s) + \alpha H(\sigma) - \alpha(c_1-c_2)\int_0^{y-\sigma} F(\xi)d\xi\right\}\\
&> (c_1-c_2)(s-\sigma) + \alpha H(\sigma) - \alpha(c_1-c_2)\int_0^{s-\sigma} F(\xi)d\xi = H(\sigma).
\end{aligned}$$

Therefore we have obtained

$$Y > H(\sigma). \tag{EC.16}$$

With previous estimates this implies (EC.9) and thus the proof of the theorem is complete. ■

EC.5. Proof of Lemma 1 of Section 8.3.

The proof of the first part of the lemma follows from the consideration of the derivative $s'(\sigma)$ and assumption (75). For the second part, we first recall that

$$H_\sigma(s(\sigma)) = \frac{\alpha}{1-\alpha}(\frac{K}{\epsilon} - c)\int_0^{s(\sigma)-\sigma} \bar{F}(\xi)d\xi,$$

$$H_\sigma(\sigma) = \frac{\alpha}{1-\alpha}(\frac{K}{\epsilon} - c)\int_0^{s(\sigma)-\sigma} \bar{F}(\xi)d\xi + (\frac{K}{\epsilon} - c)(s(\sigma)-\sigma),$$

hence

$$\frac{d}{d\sigma}H_\sigma(s(\sigma)) = (s'(\sigma)-1)\frac{\alpha}{1-\alpha}(\frac{K}{\epsilon} - c)\bar{F}(s(\sigma)-\sigma) < 0,$$
$$\frac{d}{d\sigma}H_\sigma(\sigma) = (s'(\sigma)-1)(\frac{K}{\epsilon} - c)[\frac{\alpha}{1-\alpha}\bar{F}(s(\sigma)-\sigma) + 1] < 0,$$

thanks to this lemma's part (a). Next we take $\sigma < \sigma'$, then we have $\sigma < \sigma' < s(\sigma') < s(\sigma)$. For $x < \sigma$, $H_\sigma(x) = H_\sigma(\sigma)$ and $H_{\sigma'}(x) = H_{\sigma'}(\sigma')$, therefore $H_\sigma(\sigma) > H_{\sigma'}(\sigma')$ and hence

$$H_\sigma(x) > H_{\sigma'}(x). \tag{EC.17}$$

Take next $\sigma < x < \sigma'$, then

$$H_\sigma(x) = H_\sigma(s(\sigma)) + (\frac{K}{\epsilon} - c)(s(\sigma) - x),$$

$$H_{\sigma'}(x) = H_{\sigma'}(\sigma') = H_{\sigma'}(s(\sigma')) + (\frac{K}{\epsilon} - c)(s(\sigma') - \sigma').$$

But $H_\sigma(s(\sigma)) > H_{\sigma'}(s(\sigma'))$ and $s(\sigma) - x > s(\sigma') - x > s(\sigma') - \sigma'$. So (EC.17) holds again. We next take $\sigma' < x < s(\sigma')$, then

$$H_\sigma(x) = H_\sigma(s(\sigma)) + (\frac{K}{\epsilon} - c)(s(\sigma) - x),$$

$$H_{\sigma'}(x) = H_{\sigma'}(s(\sigma')) + (\frac{K}{\epsilon} - c)(s(\sigma') - x).$$

Hence

$$H_\sigma(x) - H_{\sigma'}(x) = H_\sigma(s(\sigma)) - H_{\sigma'}(s(\sigma')) + (\frac{K}{\epsilon} - c)(s(\sigma) - s(\sigma')).$$

So (EC.17) holds again.

We next take $s(\sigma') < x < s(\sigma)$. Then we have $H_{\sigma'}(x) < H_{\sigma'}(s(\sigma'))$ because $x \to H_{\sigma'}(x)$ is decreasing on $[0, \bar{s}]$. Then

$$H_{\sigma'}(x) < H_\sigma(s(\sigma)) = H_\sigma(x) - (\frac{K}{\epsilon} - c)(x - s(\sigma)) < H_\sigma(x).$$

Thus (EC.17) is satisfied once more. So (EC.17) holds for $x \in [0, s(\sigma)]$. We may then assume $x > s(\sigma) > s(\sigma')$. We can write from (64)

$$H_\sigma(x) = g(x) - g(s(\sigma)) + \alpha E H_\sigma((x - D)^+),$$

$$H_{\sigma'}(x) = g(x) - g(s(\sigma')) + \alpha E H_{\sigma'}((x - D)^+).$$

Set $H_{\sigma,\sigma'}(x) = H_\sigma(x) - H_{\sigma'}(x)$. Then $H_{\sigma,\sigma'}(x)$, for $x < s(\sigma)$, and

$$H_{\sigma,\sigma'}(x) = g(s(\sigma')) - g(s(\sigma)) + \alpha E H_{\sigma,\sigma'}((x - D)^+),\ x > s(\sigma).$$

But $g(s(\sigma')) - g(s(\sigma)) > 0$, since $s(\sigma') < s(\sigma) < \bar{s}$. Collecting the above observations together we see that $H_{\sigma,\sigma'}(x) > 0$, and the proof has been completed. ■

EC.6. Proof of Proposition 5 of Section 8.3.

The first part is a routine calculation. Indeed when $s(\sigma) < x$, we have, from (30)

$$H'_\sigma(x) = g'(x) + \alpha(\frac{K}{\epsilon} - c)(\bar{F}(x-\sigma) - \bar{F}(x - s(\sigma))) + \alpha \int_0^{x-s(\sigma)} H'_\sigma(x-\xi) f(\xi) d\xi. \tag{EC.18}$$

Differentiating with respect to σ, we can write

$$\frac{\partial}{\partial \sigma} H'_\sigma(x) = \alpha(\frac{K}{\epsilon} - c) f(x-\sigma) + \alpha \int_0^{x-s(\sigma)} \frac{\partial}{\partial \sigma} H'_\sigma(x-\xi) f(\xi) d\xi,$$

which implies $\frac{\partial}{\partial \sigma} H'_\sigma(x) > 0$. This yields the result. ■

Now for the second part, using $H'_\sigma(S(\sigma)) = 0$, we obtain by differentiation with respect to σ

$$\frac{\partial}{\partial \sigma} H'_\sigma(S(\sigma)) + H''_\sigma(S(\sigma)) S'(\sigma) = 0.$$

From the first part of this lemma we have $\frac{\partial}{\partial \sigma} H'_\sigma(S(\sigma)) > 0$. Since $S(\sigma)$ is a minimum, $H''_\sigma(S(\sigma)) > 0$. It follows that $S'(\sigma) < 0$, which is the desired result. Next

$$H_\sigma(\sigma) - H_\sigma(S(\sigma)) = (\frac{K}{\epsilon} - c)(s(\sigma) - \sigma) - \int_{s(\sigma)}^{S(\sigma)} H'_\sigma(\xi) d\xi.$$

Therefore,

$$\begin{aligned}\frac{d}{d\sigma}(H_\sigma(\sigma) - H_\sigma(S(\sigma))) &= (\frac{K}{\epsilon} - c)(s'(\sigma) - 1) + s'(\sigma) H'_\sigma(\xi) - \int_{s(\sigma)}^{S(\sigma)} \frac{\partial}{\partial \sigma} H'_\sigma(\xi) d\xi \\ &= -(\frac{K}{\epsilon} - c) - \int_{s(\sigma)}^{S(\sigma)} \frac{\partial}{\partial \sigma} H'_\sigma(\xi) d\xi < 0,\end{aligned}$$

and the proof has been completed. ■

EC.7. Proof of Theorem 3 of Section 9.2

We first obtain two preliminary properties, which are consequences of the assumptions (75),(82). The first one is

$$x \mapsto H'_\sigma(x) \text{ is monotone increasing for } x > \sigma, \tag{EC.19}$$

where $H_\sigma(x)$ is defined by (64). Indeed, $x \to H'_\sigma(x)$ is C^1 on $(\sigma, +\infty)$ and $H''_\sigma(x) = 0$, on $(\sigma, s(\sigma))$. On $(s(\sigma), +\infty)$, one has from (EC.18)

$$\begin{aligned}H''_\sigma(x) &= g''(x) + \alpha(\frac{K}{\epsilon} - c)(-f(x-\sigma) + f(x - s(\sigma)) + \alpha H'_\sigma(s(\sigma)) + \alpha \int_0^{x-s(\sigma)} H''_\sigma(x-\xi) f(\xi) d\xi \\ &= g''(x) - \alpha(\frac{K}{\epsilon} - c) f(x-\sigma) + \alpha \int_0^{x-s(\sigma)} H''_\sigma(x-\xi) f(\xi) d\xi.\end{aligned}$$

Now

$$g''(x)-\alpha(\frac{K}{\epsilon}-c)f(x-\sigma)=(p+\alpha(h-c))f(x)-\alpha(\frac{K}{\epsilon}-c)f(x-\sigma)$$
$$=f(x-\sigma)(p+\alpha(h-c))\left[\frac{f(x)}{f(x-\sigma)}-\frac{\alpha(\frac{K}{\epsilon}-c)}{p+\alpha(h-c)}\right].$$

Recalling that $x>s(\sigma)>\bar{s}_\epsilon$ and $x-\sigma>x-\bar{s}_\epsilon$, and using assumption (75)), we that $g''(x)-\alpha(\frac{K}{\epsilon}-c)f(x-\sigma)>0$. This implies $H''_\sigma(x)>0$, for $x>s(\sigma)$. This proves (EC.19). Next as,

$$H'_\sigma(x)=-(\frac{K}{\epsilon}-c),\text{ for }\sigma<x\leq s(\sigma),$$

the monotonicity property (EC.19) implies immediately that

$$H'_\sigma(x)+\frac{K}{\epsilon}-c\geq 0,\text{ for }\sigma<x. \tag{EC.20}$$

Hence, it follows that

$$\inf_{x\leq y\leq x+\epsilon}[(\frac{K}{\epsilon}-c)y+H_\sigma(y)]=H_\sigma(x). \tag{EC.21}$$

We now turn to proving that the function $H(x)=H_{\sigma_\epsilon}(x)$ satisfies (79), (80), (81). For (79), this amounts to showing

$$H(x)=\min[H(x),K-c\epsilon+\inf_{y>x+\epsilon}H(y)].$$

So we have to prove

$$H(x)\leq K-c\epsilon+H(y),\ \forall y>x+\epsilon.$$

But this is a consequence of

$$H(x)\leq K-c\epsilon+H(y),\ \forall y>x>s,$$

which is same as equation (EC.4) which has already been proved. We next prove (80). Since $s>x>\sigma$ and $s-\sigma<\epsilon$, we have $s<x+\epsilon$. Therefore,

$$\inf_{y>s}\left\{(\frac{K}{\epsilon}-c)(y-x)\mathbb{1}_{y\leq x+\epsilon}+(K-c\epsilon)\mathbb{1}_{y>x+\epsilon}+H(y)\right\}=$$
$$\min\left\{\inf_{s<y<x+\epsilon}[(\frac{K}{\epsilon}-c)(y-x)+H(y)],\ K-c\epsilon+\inf_{y>x+\epsilon}H(y)\right\}.$$

So from (EC.20) again, it follows that

$$\inf_{y>s}\left\{(\frac{K}{\epsilon}-c)(y-x)\mathbb{1}_{y\leq x+\epsilon}+(K-c\epsilon)\mathbb{1}_{y>x+\epsilon}+H(y)\right\} \tag{EC.22}$$

$$=\min\left\{(\frac{K}{\epsilon}-c)(s-x)+H(s),\, K-c\epsilon+\inf_{y>x+\epsilon}H(y)\right\}.$$

But

$$\begin{aligned}(\frac{K}{\epsilon}-c)(s-x)+H(s) &= (\frac{K}{\epsilon}-c)(s-\sigma)-(\frac{K}{\epsilon}-c)(x-\sigma)+H(s)\\ &= H(\sigma)-(\frac{K}{\epsilon}-c)(x-\sigma)\\ &< H(\sigma)=K-c\epsilon+\inf_{y>s}H(y)\\ &< K-c\epsilon+\inf_{y>x+\epsilon}H(y).\end{aligned}$$

Therefore, we have

$$\inf_{y>s}\left\{(\frac{K}{\epsilon}-c)(y-x)\mathbb{1}_{y\leq x+\epsilon}+(K-c\epsilon)\mathbb{1}_{y>x+\epsilon}+H(y)\right\}=H(\sigma)-(\frac{K}{\epsilon}-c)(x-\sigma). \tag{EC.23}$$

Next, since $s<x+\epsilon$

$$\inf_{s\geq y\geq x}\left\{(\frac{K}{\epsilon}-c)(y-x)\mathbb{1}_{y\leq x+\epsilon}+(K-c\epsilon)\mathbb{1}_{y>x+\epsilon}+g(y)-g(s)+\alpha H(\sigma)-\alpha(\frac{K}{\epsilon}-c)\int_0^{y-\sigma}F(\xi)d\xi\right\}=$$

$$\inf_{s\geq y\geq x}\left\{(\frac{K}{\epsilon}-c)(y-x)+g(y)-g(s)+\alpha H(\sigma)-\alpha(\frac{K}{\epsilon}-c)\int_0^{y-\sigma}F(\xi)d\xi\right\}.$$

Now, we use the assumption (82) to check that the function

$$\Phi(y)=(\frac{K}{\epsilon}-c)y+g(y)-\alpha(\frac{K}{\epsilon}-c)\int_0^{y-\sigma}F(\xi)d\xi,$$

is decreasing on $y\in(\sigma,s)$. Indeed,

$$\Phi'(y)=\frac{K}{\epsilon}-c+g'(y)-\alpha(\frac{K}{\epsilon}-c)F(y-\sigma)=$$

$$\frac{K}{\epsilon}-p+(p+\alpha(h-c))F(y)-\alpha(\frac{K}{\epsilon}-c)F(y-\sigma),$$

and

$$\Phi''(y)=(p+\alpha(h-c))f(y)-\alpha(\frac{K}{\epsilon}-c)f(y-\sigma)>0,$$

thanks to the assumption (82). Therefore $\Phi'(y)$ is increasing , and since $\Phi'(s) = 0$, we obtain $\Phi'(y) < 0$, on $y \in (\sigma, s)$. Hence $\Phi(y)$ is decreasing on $y \in (\sigma, s)$. Therefore, it follows that

$$\begin{aligned}
&\inf_{s \geq y \geq x} \left\{ (\frac{K}{\epsilon} - c)(y-x) + g(y) - g(s) + \alpha H(\sigma) - \alpha(\frac{K}{\epsilon} - c) \int_0^{y-\sigma} F(\xi) d\xi \right\} \\
&= (\frac{K}{\epsilon} - c)(s-x) + \alpha H(\sigma) - \alpha(\frac{K}{\epsilon} - c) \int_0^{s-\sigma} F(\xi) d\xi \\
&= -(\frac{K}{\epsilon} - c)(x-\sigma) + (\frac{K}{\epsilon} - c)(s-\sigma)(1-\alpha) + \alpha(\frac{K}{\epsilon} - c) \int_0^{s-\sigma} \bar{F}(\xi) d\xi + \alpha H(\sigma) \\
&= -(\frac{K}{\epsilon} - c)(x-\sigma) + H(\sigma),
\end{aligned}$$

which, combined with (EC.23) implies the relation (80). We turn finally to (81). We note that $\sigma \leq \bar{s}_\epsilon < \bar{s} - \epsilon$. If $x \leq \sigma$, then $x + \epsilon < \bar{s}$. Let us first check that

$$\inf_{y > s} \left\{ (\frac{K}{\epsilon} - c)(y-x) 1\!\!1_{y \leq x+\epsilon} + (K - c\epsilon) 1\!\!1_{y > x+\epsilon} + H(y) \right\} = H(\sigma). \tag{EC.24}$$

Indeed, suppose first that $s < x + \epsilon < \bar{s}$. Then

$$\begin{aligned}
&\inf_{y > s} \left\{ (\frac{K}{\epsilon} - c)(y-x) 1\!\!1_{y \leq x+\epsilon} + (K - c\epsilon) 1\!\!1_{y > x+\epsilon} + H(y) \right\} \\
&= \min \left\{ \inf_{s < y < x+\epsilon} ((\frac{K}{\epsilon} - c)(y-x) + H(y)),\, K - c\epsilon + \inf_{y > x+\epsilon} H(y) \right\}.
\end{aligned}$$

Thanks to the assumption (75) and to the fact that $x + \varepsilon < \bar{s}$, this is

$$\begin{aligned}
&\inf_{y > s} \left\{ (\frac{K}{\epsilon} - c)(y-x) 1\!\!1_{y \leq x+\epsilon} + (K - c\epsilon) 1\!\!1_{y > x+\epsilon} + H(y) \right\} \\
&= \min \left\{ \inf_{s < y < x+\epsilon} ((\frac{K}{\epsilon} - c)(y-x) + H(y)),\, K - c\epsilon + \inf_{y > x+\epsilon} H(y) \right\} \\
&= \min \left\{ (\frac{K}{\epsilon} - c)(s-x) + H(s), H(\sigma) \right\},
\end{aligned}$$

and $(\frac{K}{\epsilon} - c)(s-x) + H(s) > (\frac{K}{\epsilon} - c)(s-\sigma) + H(s) = H(\sigma)$, therefore (EC.24) is satisfied. In the case $x + \epsilon < s$, we have simply

$$\inf_{y > s} \left\{ (\frac{K}{\epsilon} - c)(y-x) 1\!\!1_{y \leq x+\epsilon} + (K - c\epsilon) 1\!\!1_{y > x+\epsilon} + H(y) \right\} = K - c\epsilon + \inf_{y > s} H(y) = H(\sigma),$$

and (EC.24) is again satisfied. Consider next

$$X = \inf_{s \geq y > \sigma} \left\{ (\frac{K}{\epsilon} - c)(y-x) 1\!\!1_{y \leq x+\epsilon} + (K - c\epsilon) 1\!\!1_{y > x+\epsilon} + g(y) - g(s) + \alpha H(\sigma) - \alpha(\frac{K}{\epsilon} - c) \int_0^{y-\sigma} F(\xi) d\xi \right\}.$$

Suppose again that the situation $x+\epsilon>s$. Then

$$X=\inf_{s\geq y>\sigma}\left\{(\frac{K}{\epsilon}-c)(y-x)+g(y)-g(s)+\alpha H(\sigma)-\alpha(\frac{K}{\epsilon}-c)\int_0^{y-\sigma}F(\xi)d\xi\right\}.$$

But, as already seen above, thanks to assumption (82) we have

$$\begin{aligned}X&=(\frac{K}{\epsilon}-c)(s-x)+\alpha H(\sigma)-\alpha(\frac{K}{\epsilon}-c)\int_0^{s-\sigma}F(\xi)d\xi\\&=-(\frac{K}{\epsilon}-c)(x-\sigma)+H(\sigma)>H(\sigma).\end{aligned}$$

If, on the other hand $x+\epsilon<s$, we can write

$$\begin{aligned}X&=\min\left\{\inf_{x+\epsilon\geq y>\sigma}[(\frac{K}{\epsilon}-c)(y-x)+g(y)-g(s)+\alpha H(\sigma)-\alpha(\frac{K}{\epsilon}-c)\int_0^{y-\sigma}F(\xi)d\xi],\right.\\&\qquad\left.K-c\epsilon+\inf_{s\geq y\geq x+\epsilon}[g(y)-g(s)+\alpha H(\sigma)-\alpha(\frac{K}{\epsilon}-c)\int_0^{y-\sigma}F(\xi)d\xi]\right\}\\&=K-c\epsilon+\min\left\{g(x+\epsilon)-g(s)+\alpha H(\sigma)-\alpha(\frac{K}{\epsilon}-c)\int_0^{x+\epsilon-\sigma}F(\xi)d\xi,\right.\\&\qquad\left.\alpha H(\sigma)-\alpha(\frac{K}{\epsilon}-c)\int_0^{s-\sigma}F(\xi)d\xi\right\}\\&>K-c\epsilon+\alpha H(\sigma)-\alpha(\frac{K}{\epsilon}-c)\int_0^{s-\sigma}F(\xi)d\xi\\&\geq(\frac{K}{\epsilon}-c)(s-\sigma)+\alpha H(\sigma)-\alpha(\frac{K}{\epsilon}-c)\int_0^{s-\sigma}F(\xi)d\xi\\&=(\frac{K}{\epsilon}-c)(s-\sigma)(1-\alpha)+\alpha H(\sigma)+\alpha(\frac{K}{\epsilon}-c)\int_0^{s-\sigma}\bar{F}(\xi)d\xi=H(\sigma).\end{aligned}$$

So $X>H(\sigma)$. It remains to consider

$$Y=\inf_{\sigma\geq y\geq x}\left\{(\frac{K}{\epsilon}-c)(y-x)\mathbb{1}_{y\leq x+\epsilon}+(K-c\epsilon)\mathbb{1}_{y>x+\epsilon}+g(y)-g(s)+\alpha H(\sigma)\right\}.$$

If $x+\epsilon>\sigma$, then

$$Y=\inf_{\sigma\geq y\geq x}\left\{(\frac{K}{\epsilon}-c)(y-x)+g(y)-g(s)+\alpha H(\sigma)\right\}.$$

Since $\sigma<\bar{s}_\epsilon$, the function $y\rightarrow(\frac{K}{\epsilon}-c)y+g(y)$ decreases for $y\leq\sigma$, hence

$$Y=(\frac{K}{\epsilon}-c)(\sigma-x)+g(\sigma)-g(s)+\alpha H(\sigma)>X>H(\sigma)$$

If $x+\epsilon<\sigma$, we have

$$
\begin{aligned}
Y = \min\Big\{ &\inf_{x+\epsilon\geq y\geq x}[(\frac{K}{\epsilon}-c)(y-x)+g(y)-g(s)+\alpha H(\sigma)], \\
&K - c\epsilon + \inf_{\sigma\geq y> x+\epsilon}(g(y)-g(s)+\alpha H(\sigma))\Big\} \\
&= K - c\epsilon + \min\Big\{ g(x+\epsilon)-g(s)+\alpha H(\sigma),\, g(\sigma)-g(s)+\alpha H(\sigma)\Big\} \\
&= K - c\epsilon + g(\sigma)-g(s)+\alpha H(\sigma) \\
&> (\frac{K}{\epsilon}-c)(s-\sigma)+g(\sigma)-g(s)+\alpha H(\sigma) \\
&> (\frac{K}{\epsilon}-c)(s-\sigma)+\alpha H(\sigma)-\alpha(\frac{K}{\epsilon}-c)\int_0^{s-\sigma}F(\xi)d\xi = H(\sigma).
\end{aligned}
$$

Thus $Y>H(\sigma)$. Therefore the proof of Theorem 3 is complete. ■

EC.8. Explicit Solution for $(\sigma_\epsilon, s_\epsilon, S_\epsilon)$ Policy for Exponential Demand.

EC.8.1. Preliminary Assumptions

We let $f(x)=\lambda e^{-\lambda x}$. and obtain the promised closed form solution. We begin by summarizing the assumptions of Theorem 3. We have

$$p > \frac{K}{\epsilon} > c. \tag{EC.25}$$

Next $\bar{s}_\epsilon$ is defined by ((72), hence

$$\bar{s}_\epsilon = \frac{1}{\lambda}\log\frac{p+\alpha(h-c)}{\frac{K}{\epsilon}+\alpha(h-c)}, \tag{EC.26}$$

and $\bar{s}^*_\epsilon$ is defined by (73)

$$\bar{s}^*_\epsilon = \frac{1}{\lambda}\log\frac{p-\alpha\frac{K}{\epsilon}+\alpha h}{\frac{K}{\epsilon}-\alpha\frac{K}{\epsilon}+\alpha h}. \tag{EC.27}$$

Now recall that $\bar{s}$ is given by (16), we get

$$\bar{s} = \frac{1}{\lambda}\log\frac{p+\alpha(h-c)}{c+\alpha(h-c)}. \tag{EC.28}$$

Therefore, assumption (83) becomes

$$\frac{K}{\epsilon}-c < (c+\alpha(h-c))(e^{\lambda\epsilon}-1). \tag{EC.29}$$

Next, assumption (82) becomes

$$e^{-\lambda \bar{s}_\epsilon} > \frac{\alpha(\frac{K}{\epsilon} - c)}{p + \alpha(h - c)},$$

which is automatically verified thanks to (EC.26). Next (82) reads

$$\frac{\frac{K}{\epsilon} - \alpha\frac{K}{\epsilon} + \alpha h}{p - \alpha\frac{K}{\epsilon} + \alpha h} > \frac{\alpha(\frac{K}{\epsilon} - c)}{p + \alpha(h - c)}. \tag{EC.30}$$

which is, like (EC.29), a smallness condition on $\dfrac{K}{\epsilon} - c$.

EC.8.2. Solution in the Exponential Case

To apply Theorem 3, we have to verify (77). Let us first define the function $s(\sigma)$. From Proposition 4, after some easy calculations, we obtain that

$$s(\sigma) = \sigma + \frac{1}{\lambda}\log\frac{(p + \alpha(h - c))e^{-\lambda\sigma} - \alpha(\frac{K}{\epsilon} - c)}{\frac{K}{\epsilon}(1 - \alpha) + \alpha h}. \tag{EC.31}$$

$s(\sigma)$ is defined for $\sigma < \bar{s}_\epsilon$. Moreover $s(\sigma) \leq s(0) = \bar{s}^*_\epsilon$. Next (EC.18) becomes

$$H'_\sigma(x) = c + \alpha(h - c) - (p + \alpha(h - c))e^{-\lambda x} - \tag{EC.32}$$

$$-\alpha(\frac{K}{\epsilon} - c)e^{-\lambda x}(e^{\lambda s(\sigma)} - e^{\lambda\sigma}) + \alpha\lambda\int_0^{x - s(\sigma)} H'_\sigma(x - \xi)e^{-\lambda\xi}\, d\xi,\ x > s(\sigma).$$

Combining with (EC.31) this reduces to

$$H'_\sigma(x) = c + \alpha(h - c) - e^{-\lambda x}\frac{\frac{K}{\epsilon} + \alpha(h - c)}{\frac{K}{\epsilon}(1 - \alpha) + \alpha h}(p + \alpha(h - c) - \alpha(\frac{K}{\epsilon} - c)e^{\lambda\sigma}) + \tag{EC.33}$$

$$+\lambda\alpha e^{-\lambda x}\int_{s(\sigma)}^{x} H'_\sigma(\theta)e^{\lambda\theta}d\theta,\ x > s(\sigma).$$

This integral equation has an analytic solution, given by

$$H'_\sigma(x) = c + \frac{\alpha h}{1 - \alpha} - (\frac{K}{\epsilon} + \frac{\alpha h}{1 - \alpha})e^{-\lambda(1 - \alpha)(x - s(\sigma))},\ x > s(\sigma). \tag{EC.34}$$

Since $S(\sigma)$ satisfies $H'_\sigma(S(\sigma)) = 0$, we get

$$e^{\lambda(1 - \alpha)(S(\sigma) - s(\sigma))} = \frac{\frac{K}{\epsilon} + \frac{\alpha h}{1 - \alpha}}{c + \frac{\alpha h}{1 - \alpha}}. \tag{EC.35}$$

By integration

$$H_\sigma(x) = H_\sigma(s(\sigma)) + (c + \frac{\alpha h}{1-\alpha})(x - s(\sigma)) \tag{EC.36}$$
$$-(\frac{K}{\epsilon} + \frac{\alpha h}{1-\alpha})\frac{1 - e^{-\lambda(1-\alpha)(x-s(\sigma))}}{\lambda(1-\alpha)},\ x \geq s(\sigma).$$

We also can write

$$H_\sigma(s(\sigma)) = \frac{\alpha}{\lambda(1-\alpha)}(\frac{K}{\epsilon} - c)(1 - e^{-\lambda(s(\sigma)-\sigma)}). \tag{EC.37}$$

Then using (EC.31)

$$H_\sigma(s(\sigma)) = \frac{\alpha}{\lambda(1-\alpha)}(\frac{K}{\epsilon} - c)\frac{(p + \alpha(h-c))e^{-\lambda\sigma} + \alpha(c-h) - \frac{K}{\epsilon}}{(p + \alpha(h-c))e^{-\lambda\sigma} - \alpha(\frac{K}{\epsilon} - c)}, \tag{EC.38}$$

$$H_\sigma(\sigma) = H_\sigma(s(\sigma)) + (\frac{K}{\epsilon} - c)(s(\sigma) - \sigma), \tag{EC.39}$$

$$H_\sigma(S(\sigma)) = H_\sigma(s(\sigma)) + (c + \frac{\alpha h}{1-\alpha})(S(\sigma) - s(\sigma)) - \frac{\frac{K}{\epsilon} - c}{\lambda(1-\alpha)}. \tag{EC.40}$$

Finally

$$H_\sigma(\sigma) - H_\sigma(S(\sigma)) = (\frac{K}{\epsilon} - c)(s(\sigma) - \sigma) + \frac{\frac{K}{\epsilon} - c}{\lambda(1-\alpha)} - (c + \frac{\alpha h}{1-\alpha})(S(\sigma) - s(\sigma)). \tag{EC.41}$$

Using (EC.31) and (EC.35) we obtain the expression

$$H_\sigma(\sigma) - H_\sigma(S(\sigma)) = \frac{c + \frac{\alpha h}{1-\alpha}}{\lambda}\left\{\frac{1}{1-\alpha}\left[\frac{\frac{K}{\varepsilon} - c}{c + \frac{\alpha h}{1-\alpha}} - \log(1 + \frac{\frac{K}{\varepsilon} - c}{c + \frac{\alpha h}{1-\alpha}})\right] + \right. \tag{EC.42}$$
$$\left. + \frac{\frac{K}{\varepsilon} - c}{c + \frac{\alpha h}{1-\alpha}}\log\frac{\frac{(p + \alpha(h-c))e^{-\lambda\sigma}}{c + \frac{\alpha h}{1-\alpha}} - \alpha\frac{\frac{K}{\varepsilon} - c}{c + \frac{\alpha h}{1-\alpha}}}{(1-\alpha)(1 + \frac{\frac{K}{\varepsilon} - c}{c + \frac{\alpha h}{1-\alpha}})}\right\},$$

which is clearly a decreasing function of σ. We can then interpret condition (77) as

$$\frac{1}{1-\alpha}\left[\frac{\frac{K}{\varepsilon} - c}{c + \frac{\alpha h}{1-\alpha}} - \log(1 + \frac{\frac{K}{\varepsilon} - c}{c + \frac{\alpha h}{1-\alpha}})\right] + \tag{EC.43}$$
$$+ \frac{\frac{K}{\varepsilon} - c}{c + \frac{\alpha h}{1-\alpha}}\log\frac{p + \alpha h - \alpha\frac{K}{\epsilon}}{\frac{K}{\epsilon}(1-\alpha) + \alpha h} > \lambda\epsilon\frac{\frac{K}{\varepsilon} - c}{c + \frac{\alpha h}{1-\alpha}},$$

and

$$\lambda\epsilon\frac{\frac{K}{\varepsilon} - c}{c + \frac{\alpha h}{1-\alpha}} > \frac{1}{1-\alpha}\left[\frac{\frac{K}{\varepsilon} - c}{c + \frac{\alpha h}{1-\alpha}} - \log(1 + \frac{\frac{K}{\varepsilon} - c}{c + \frac{\alpha h}{1-\alpha}})\right]. \tag{EC.44}$$

Then there exists a single value σ_ϵ which is the solution of the equation

$$\frac{1}{1-\alpha}\left[\frac{\frac{K}{\varepsilon}-c}{c+\frac{\alpha h}{1-\alpha}}-\log(1+\frac{\frac{K}{\varepsilon}-c}{c+\frac{\alpha h}{1-\alpha}})\right]+ \tag{EC.45}$$

$$+\frac{\frac{K}{\varepsilon}-c}{c+\frac{\alpha h}{1-\alpha}}\log\frac{\frac{(p+\alpha(h-c))e^{-\lambda\sigma_\epsilon}}{c+\frac{\alpha h}{1-\alpha}}-\alpha\frac{\frac{K}{\varepsilon}-c}{c+\frac{\alpha h}{1-\alpha}}}{(1-\alpha)(1+\frac{\frac{K}{\varepsilon}-c}{c+\frac{\alpha h}{1-\alpha}})}=\lambda\epsilon\frac{\frac{K}{\varepsilon}-c}{c+\frac{\alpha h}{1-\alpha}}.$$

Let us now set

$$s_\epsilon=\sigma_\epsilon+\frac{1}{\lambda}\log\frac{(p+\alpha(h-c))e^{-\lambda\sigma_\epsilon}-\alpha(\frac{K}{\epsilon}-c)}{\frac{K}{\epsilon}(1-\alpha)+\alpha h}, \tag{EC.46}$$

$$S_\epsilon=s_\epsilon+\frac{1}{\lambda(1-\alpha)}\log\frac{\frac{K}{\epsilon}+\frac{\alpha h}{1-\alpha}}{c+\frac{\alpha h}{1-\alpha}}. \tag{EC.47}$$

We then have the following proposition:

PROPOSITION EC.1. *We assume* (EC.25)*,* (EC.29)*,* (EC.30)*,* (EC.43)*, and* (EC.44)*. Then the triple* $(\sigma_\epsilon, s_\epsilon=\Sigma_\epsilon, S_\epsilon)$ *given by* (EC.45)*,* (EC.46)*,* (EC.47) *defines a* $(\sigma_\epsilon,\Sigma_\epsilon,S_\epsilon)$ *policy for the Bellman equation* (6)*.*

References

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