RATIONAL HOMOTOPY TYPE AND COMPUTABILITY

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In memory of Edgar H. Brown, 1926–2021

Abstract. Given a simplicial pair (X, A), a simplicial complex Y, and a map $f : A \to Y$, does f have an extension to X? We show that for a fixed Y, this question is algorithmically decidable for all X, A, and f if Y has the rational homotopy type of an H-space. As a corollary, many questions related to bundle structures over a finite complex are likely decidable. Conversely, for all other Y, the question is at least as hard as certain special cases of Hilbert's tenth problem which are known or suspected to be undecidable.

1. Introduction

When can the set of homotopy classes of maps between spaces X and Y be computed? That is, when can this (possibly infinite) set be furnished with a finitely describable and computable structure? It is reasonable to restrict the question to the setting of finite complexes: otherwise one risks encountering spaces that themselves take an infinite amount of information to describe. Moreover, the question of whether this set has more than one element is undecidable for $X = S^1$, as shown by Novikov as early as 1955[1]. Therefore it is also reasonable to require the fundamental group not to play a role; in the present work, Y is always assumed to be simply connected.[2]

We answer this question with the following choice of quantifiers: for what Y and n can the set of homotopy classes $[X, Y]$ be computed for *every* n-dimensional X? Significant partial results in this direction were obtained by E. H. Brown [2] and much more recently by Čadek et al. [3], [5], [4] and Vokřínek [32]. The goal of the present work is to push their program closer to its logical limit.

To state the precise result, we need to sketch the notion of an H-space, which is defined precisely in §3. Essentially, an H-space is a space equipped with a binary operation which can be more or less "group-like"; if it has good enough properties, this allows us to equip sets of mapping classes to the H-space with a group structure.

The *cohomological dimension* $\mathrm{cd}(X, A)$ of a simplicial or CW pair (X, A) is the least integer d such that for all $n > d$ and every coefficient group π, $H^n(X, A; \pi) = 0$.

Theorem A. *Let Y be a simply connected simplicial complex of finite type and $d \geq 2$, and suppose*

(∗) *Y has the rational homotopy type of an H-space through dimension d. That is, there is a map from Y to an H-space (or, equivalently, to a product of Eilenberg–MacLane spaces) which induces isomorphisms on $\pi_n \otimes \mathbb{Q}$ for $n \leq d$.*

Then for any simplicial pair (X, A) of cohomological dimension $d+1$ and simplicial map $f : A \to Y$, the existence of a continuous extension of f to X is decidable.

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[1]This is the triviality problem for group presentations, translated into topological language. This work was extended by Adian and others to show that many other properties of nonabelian group presentations are likewise undecidable.

[2]The results can plausibly be extended to nilpotent spaces.

Moreover, there is an algorithm which, given a simply connected simplicial complex Y, a simplicial pair (X, A) of finite complexes of cohomological dimension d and a simplicial map $f : A \to Y$,

(1) Determines whether $(*)$ is satisfied;
(2) If it is, outputs the set of homotopy classes rel A of extensions $[X, Y]^f$ in the format of a (perhaps empty) set on which a finitely generated abelian group acts virtually freely and faithfully (that is, with a finite number of orbits each of which has finite stabilizer).

We give a few remarks about the statement. First of all, it is undecidable whether Y is simply connected; therefore, when given a non-simply connected input, the algorithm cannot detect this and returns nonsense, like previous algorithms of this type.

Secondly, we provide evidence for the conjectural converse to the first part of Theorem A: that if $(*)$ is not true, then the extension problem for pairs of cohomological dimension $d+1$ is undecidable. We prove this in a range of special cases, but the general case appears to be connected to deep unsolved problems in number theory. The best that can be said is that if the converse is false, it must be due to a strange number-theoretic coincidence.

Finally, the difference between $d + 1$ in the first part of the theorem and d in the second is significant: if $\mathrm{cd}(X, A) = d + 1$, then we can decide whether $[X, Y]^f$ is nonempty, but our method of describing this set breaks down. For example, a homotopy class of maps $S^1 \times S^2 \to S^2$ is determined by two numbers: the degree d of the map on the S^2 factor and the (relative) Hopf invariant h on the 3-cell. However, h is well-defined only up to multiples of $2d$, and so in a natural sense

$$[S^1 \times S^2, S^2] \cong \bigsqcup_{r \in \mathbb{Z}} \mathbb{Z}/2r\mathbb{Z}.$$

This structure does not fit into the framework we construct in this paper for describing $[X, Y]^f$. Other similar examples are described in [13, §3], and it would be interesting to give a general, perhaps computable, description for $[X, Y]^f$ (or even just $[X, S^{2n}]$) in this "critical" dimension.

1.1. **Examples.** The new computability result encompasses several previous results, as well as new important corollaries. Here are some examples of spaces which satisfy condition $(*)$ of Theorem A:

(a) Any simply connected space with finite homology groups (or, equivalently, finite homotopy groups) in every dimension is rationally equivalent to a point, which is an H-space. The computability of $[X, Y]$ when Y is of this form was already established by Brown [2].
(b) Any d-connected space is rationally an H-space through dimension $n = 2d$. Thus we recover the result of Čadek et al. [5] that $[X, Y]^f$ is computable whenever X is $2d$-dimensional and Y is d-connected. This implies that many "stable" homotopical objects are computable. One example is the group of oriented cobordism classes of n-manifolds, which is isomorphic to the set of maps from S^n to the Thom space of the tautological bundle over $\mathrm{Gr}_n(\mathbb{R}^{2n+1})$.
(c) The sphere S^n for n odd is rationally equivalent to the Eilenberg–MacLane space $K(\mathbb{Z}, n)$. Therefore $[X, S^n]^f$ is computable for any finite simplicial pair (X, A) and map $f : A \to S^n$; this is the main result of Vokřínek's paper [32].
(d) Any Lie group or simplicial group Y is an H-space, so if Y is simply connected then $[X, Y]^f$ is computable for any X, A, and f.
(e) Classifying spaces of connected Lie groups also have the rational homotopy type of an H-space [9, Prop. 15.15]. Therefore we have (somewhat aspirationally):

Corollary 1.1. *Let G be a connected Lie group, and suppose that the classifying space BG has a computable representation. Then:*

(i) Let X be a finite CW complex. Then the set of isomorphism classes of principal G-bundles over X is computable.

(ii) Let (X, A) be a finite CW pair. Then it is decidable whether a given principal G-bundle over A extends over X.

In particular, given a representation $G \to GL_n(\mathbb{R})$, we should be able to understand the set of vector bundles with a G-structure. This includes real oriented, complex, and symplectic bundles, as well as spin and metaplectic structures on bundles. However, doing this in practice requires paying attention to computational models of Lie groups, Grassmannians, bundles, and so forth.

(f) More generally, some classifying spaces of topological monoids have the rational homotopy type of an H-space. This includes the classifying space $BG_n = \mathrm{BAut}(S^n)$ for S^n-fibrations (see [16, Appendix 1] and [27]); therefore, the set of fibrations $S^n \to E \to X$ over a finite complex X up to fiberwise homotopy equivalence is computable.

Conversely, most sufficiently complicated simply connected spaces do not satisfy condition $*$. The main result of [4] shows that the extension problem is undecidable for even-dimensional spheres, which are the simplest example. Other examples include complex projective spaces and most Grassmannians and Stiefel manifolds.

1.2. **Proof ideas.** Suppose that Y has the rational homotopy type of an H-space through dimension d, but not through dimension $d+1$. To prove the main theorem, we must provide an algorithm which computes $[X, Y]^f$ if $\mathrm{cd}(X, A) \leq d$ and decides whether $[X, Y]^f$ is nonempty if $\mathrm{cd}(X, A) = d+1$. This builds on work of Čadek, Krčál, Matoušek, Vokřínek, and Wagner [5].

To provide an algorithm, we use the rational H-space structure of the dth Postnikov stage Y_d of Y. In this case, we can build an H-space H of finite type together with rational equivalences

$$H \to Y_d \to H$$

as well as an "H-space action" of H on Y_d, that is, a map $\mathrm{act} : H \times Y_d \to Y_d$ which satisfies various compatibility properties. These ensure that the set $[X/A, H]$ (where A is mapped to the basepoint) acts via composition with act on $[X, Y_d]^f$. In turn, $[X/A, H]$ is a product of cohomology groups and therefore easily computable, and this allows us to also compute $[X, Y_d]^f$. When $\mathrm{cd}(X, A) \leq d$, the obvious map $[X, Y]^f \to [X, Y_d]^f$ is a bijection; when $\mathrm{cd}(X, A) = d+1$, this map is a surjection. This gives the result.

In the last part of the paper, we study the extension problem in the case that Y is not a rational H-space through dimension d and connect it to Hilbert's tenth problem. Recall that Hilbert asked for an algorithm to determine whether a system of Diophantine equations has a solution. Work of Davis, Putnam, Robinson, and Matiyasevich showed that no such algorithm exists. It turns out that the problem is still undecidable for very restricted classes of systems of quadratic equations; this was used in [4] to show that the extension problem for maps to S^{2n} is undecidable. We generalize their work: extension problems for maps to a given Y are shown to encode systems of Diophantine equations in which terms are values on vectors of variables of a fixed bilinear map which depends on Y. We conjecture that Hilbert's tenth problem restricted to any such subtype is undecidable and prove this in certain special cases. However, the general case seems quite difficult; in particular, it would imply a long-standing conjecture on the undecidability of Hilbert's tenth problem over number rings.

1.3. **Computational complexity.** Unlike Čadek et al. [5], [6], whose algorithms are polynomial for fixed d, and like Vokřínek [32], we do not give any kind of complexity bound on the run time of the algorithm which computes $[X, Y]^f$. In fact, there are several steps in which the procedure is to iterate until we find a number that works, with no a priori bound on the size of the number, although it is likely possible to bound it in terms of dimension and other parameters such as the cardinality of the torsion subgroups in the homology of Y. There is much space to both optimize the algorithm and discover bounds on the run time.

1.4. **The fiberwise case.** In a paper of Čadek, Krčál, and Vokřínek [6], the results of [5] are extended to the *fiberwise* case, that is, to computing the set of homotopy classes of lifting-extensions completing the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow[g]{} & B, \end{array} \tag{1.2}$$

where X is $2d$-dimensional and the fiber of $Y \xrightarrow{p} B$ is d-connected. Vokřínek [32] also remarks that his results for odd-dimensional spheres extend to the fiberwise case. Is there a corresponding fiberwise generalization for the results of this paper? The naïve hypothesis would be that $[X, Y]^f_p$ is computable whenever the fiber of $Y \xrightarrow{p} B$ is a rational H-space through dimension n. This is false; as demonstrated by the example below, rational homotopy obstructions may still crop up in the interaction between base and fiber.

The correct fiberwise statement should relate to rational fiberwise H-spaces, as discussed for example in [12]. However, such a result presents technical difficulties which will require significant new ideas to overcome.

Example 1.3. Let $B = S^6 \times S^2$ and Y be the total space of the fibration

$$S^7 \to Y \xrightarrow{p_0} B \times (S^3)^2$$

whose Euler class (a.k.a. the k-invariant of the corresponding $K(\mathbb{Z}, 7)$-bundle) is

$$[S^6 \times S^2] + [(S^3)^2 \times S^2] \in H^8(B \times (S^3)^2).$$

Then the fiber of $p = \pi_1 \circ p_0 : Y \to B$ is the H-space $(S^3)^2 \times S^7$, but the intermediate k-invariant given above has a term which is quadratic in the previous part of the fiber.

Given a system of s polynomial equations each of the form

$$\sum_{1 \le i < j \le r} a^{(k)}_{ij} (x_i y_j - x_j y_i) = b_k,$$

with variables $x_1, \ldots, x_r, y_1, \ldots, y_r$ and coefficients b_k and $a^{(k)}_{ij}$, we form a space X' by taking $\bigvee_r S^3$ and attaching s 6-cells, the kth one via an attaching map whose homotopy class is

$$\sum_{1 \le i < j \le r} a^{(k)}_{ij} [\mathrm{id}_i, \mathrm{id}_j],$$

where id_i is the inclusion map of the ith 3-sphere. We fix a map $f' : X' \to S^6$ which collapses the 3-cells and restricts to a map of degree $-b_k$ on the kth 6-cell. This induces a map $f = f' \times \mathrm{id}$ from $X = X' \times S^2$ to B.

A lift of f to $B \times (S^3)^2$ corresponds to an assignment of the variables x_i and y_i. The existence of a further lift to Y is then equivalent to whether this assignment is a solution to the system of equations above. Since the existence of such a solution is in general undecidable by [4, Lemma 2.1], so is the existence of a lift of f through p.

Remark 1.4. The role of S^2 in this example is to make the fiber into a rational H-space. If we let $B = S^6$ and Y be the total space of the fibration

$$S^5 \to Y \to B \times (S^3)^2$$

with Euler class $[S^6] + [(S^3)^2]$, then the fiber of $Y \to B$ is no longer a product $S^5 \times (S^3)^2$, even rationally, but rather has a nontrivial rational k-invariant in its Postnikov tower.

1.5. **Structure of the paper.** I have tried to make this paper readable to any topologist as well as anyone who is familiar with the work of Čadek et al. Thus §2 and 3 attempt to introduce all the necessary algebraic topology background which is not used in Čadek et al.'s papers: a bit of rational homotopy theory and some results about H-spaces. For the benefit of topologists, I have tried to separate the ideas that go into constructing a structure on mapping class sets from those required to compute this structure. The construction of the group and action in Theorem A is discussed in §4. In §5, we introduce previous results in computational homotopy theory from [5], [6], [10], and in §6 we use them to compute the structure we built earlier. Finally, in §7 and 8, we discuss Hilbert's tenth problem and its relation to undecidability of the extension problem.

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2. Rational homotopy theory

Rational homotopy theory is a powerful algebraicization of the topology of simply connected topological spaces first introduced by Quillen [21] and Sullivan [30]. The subject is well-developed, and the texts [11] and [9] are recommended as a comprehensive reference. This paper requires only a very small portion of the considerable machinery that has been developed, and this short introduction should suffice for the reader who is assumed to be familiar with Postnikov systems and other constructs of basic algebraic topology.

The key topological idea is the construction of rationalized spaces: to any simply connected CW complex X one can functorially (at least up to homotopy) associate a space $X_{(0)}$ whose homology (equivalently, homotopy) groups are $\mathbb{Q}$-vector spaces.[3] There are several ways of constructing such a rationalization, but the most relevant to us is by induction up the Postnikov tower: the rationalization of a point is a point, and then given a Postnikov stage

$$\begin{array}{ccccc} K(\pi_n(X),n) & \longrightarrow & X_n & \longrightarrow & E(\pi_n(X),n+1) \\ & & \downarrow & & \downarrow \\ & & X_{n-1} & \xrightarrow{k_n} & K(\pi_n(X),n+1), \end{array}$$

one replaces it with

$$\begin{array}{ccccc} K(\pi_n(X)\otimes\mathbb{Q},n) & \longrightarrow & X_{n(0)} & \longrightarrow & E(\pi_n(X)\otimes\mathbb{Q},n+1) \\ & & \downarrow & & \downarrow \\ & & X_{n-1(0)} & \xrightarrow{k_n\otimes\mathbb{Q}} & K(\pi_n(X)\otimes\mathbb{Q},n+1). \end{array}$$

This builds $X_{n(0)}$ given $X_{n-1(0)}$, and then $X_{(0)}$ is the homotopy type of the limit of this construction. We say two spaces are *rationally equivalent* if their rationalizations are homotopy equivalent.

The second key fact is that the homotopy category of rationalized spaces of *finite type* (that is, for which all homology groups, or equivalently all homotopy groups, are finite-dimensional vector spaces) is equivalent to several purely algebraic categories. The one most relevant for our purpose is the Sullivan DGA model.

[3] It's worth pointing out that this fits into a larger family of *localizations* of spaces, another of which is used in the proof of Lemma 3.3.

A *differential graded algebra* (DGA) over $\mathbb{Q}$ is a cochain complex of $\mathbb{Q}$-vector spaces equipped with a graded commutative multiplication which satisfies the (graded) Leibniz rule. A familiar example is the algebra of differential forms on a manifold. A key insight of Sullivan was to associate to every space X of finite type a *minimal* DGA $\mathcal{M}_X$ constructed by induction on degree as follows:

- $\mathcal{M}_X(1) = \mathbb{Q}$ with zero differential.
- For $n \geq 2$, the algebra structure is given by
$$\mathcal{M}_X(n) = \mathcal{M}_X(n+1) \otimes \Lambda \operatorname{Hom}(\pi_n(X); \mathbb{Q}),$$
where ΛV denotes the free graded commutative algebra generated by V.
- The differential is given on the elements of $\operatorname{Hom}(\pi_n(X); \mathbb{Q})$ (*indecomposables*) by the dual of the nth k-invariant of X,
$$\operatorname{Hom}(\pi_n(X); \mathbb{Q}) \xrightarrow{k_n^*} H^{n+1}(X; \mathbb{Q}),$$
and extends to the rest of the algebra by the Leibniz rule. Although it is only well-defined up to a coboundary, this definition makes sense because one can show by induction that $H^k(\mathcal{M}_X(n-1))$ is naturally isomorphic to $H^k(X_{n-1}; \mathbb{Q})$, independent of the choices made in defining the differential at previous steps.
  Note that from this definition, it follows that for an indecomposable y of degree n, dy is an element of degree $n+1$ which can be written as a polynomial in the indecomposables of degree $< n$. In particular, it has no linear terms.

The DGA $\mathcal{M}_X$ is the functorial image of $X_{(0)}$ under an equivalence of homotopy categories.

Many topological constructions can thus be translated into algebraic ones. This paper will use the following:

- The Eilenberg–MacLane space $K(\pi, n)$ corresponds to the DGA $\Lambda \operatorname{Hom}(\pi, \mathbb{Q})$ with generators concentrated in dimension n and zero differential.
- Product of spaces corresponds to tensor product of DGAs. In particular:

  Proposition 2.1. *The following are equivalent for a space X:*
  (a) X is rationally equivalent to a product of Eilenberg–MacLane spaces.
  (b) The minimal model of X has zero differential.
  (c) The rational Hurewicz map $\pi_*(X) \otimes \mathbb{Q} \to H_*(X; \mathbb{Q})$ is injective.

Finally, we note the following theorem of Sullivan:

Theorem 2.2 (Sullivan's finiteness theorem [30, Theorem 10.2(i)])**.** *Let X be a finite complex and Y a simply connected finite complex. Then the map $[X, Y] \to [X, Y_{(0)}]$ induced by the rationalization functor is finite-to-one.*

Note that this implies that if the map $Y \to Z$ between finite complexes induces a rational equivalence, then the induced map $[X, Y] \to [X, Z]$ is also finite-to-one.

3. H-spaces

A pointed space (H, o) is an H-space if it is equipped with a binary operation $\operatorname{add} : H \times H \to H$ satisfying $\operatorname{add}(x, o) = \operatorname{add}(o, x) = x$ (the basepoint acts as an identity). In addition, an H-space is *homotopy associative* if
$$\operatorname{add} \circ (\operatorname{add}, \operatorname{id}) \simeq \operatorname{add} \circ (\operatorname{id}, \operatorname{add})$$
and *homotopy commutative* if $\operatorname{add} \simeq \operatorname{add} \circ \tau$, where τ is the "twist" map sending $(x, y) \mapsto (y, x)$. We will interchangeably denote our H-space operations (most of which will be homotopy associative and commutative) by the usual binary operator $+$, as in $x + y = \operatorname{add}(x, y)$.

A classic result of Sugawara, see [29, Theorem 3.4], is that a homotopy associative H-space which is a connected CW complex automatically admits a *homotopy inverse* $x \mapsto -x$ with the expected property $\mathrm{add}(-x, x) = o = \mathrm{add}(x, -x)$.

Examples of H-spaces include topological groups and Eilenberg–MacLane spaces. If H is simply connected, then it is well-known that it has the rational homotopy type of a product of Eilenberg–MacLane spaces. Equivalently, from the Sullivan point of view, H has a minimal model $\mathcal{M}_H$ with zero differential; see [9, §12(a) Example 3] for a proof. On the other hand, a product of H-spaces is clearly an H-space. Therefore we can add "X is rationally equivalent to an H-space" to the list of equivalent conditions in Prop. 2.1. We will generally use the sloppy phrase "X is a rational H-space" to mean the same thing.

It is easy to see that an H-space operation plays nice with the addition on higher homotopy groups. That is:

Proposition 3.1. *Let (H, o, add) be an H-space. Given $f, g : (S^n, *) \to (H, o)$,*

$$[f] + [g] = [\mathrm{add} \circ (f, g)] \in \pi_n(H, o).$$

Another important and easily verified fact is the following:

Proposition 3.2. *If (H, o, add) is a homotopy associative H-space, then for any pointed space $(X, *)$, the set $[X, H]$ forms a group, with the operation given by $[\varphi] \cdot [\psi] = [\mathrm{add} \circ (\varphi, \psi)]$. If H is homotopy commutative, then this group is likewise commutative.*

Moreover, suppose that H is homotopy commutative, and let $A \to X$ be a cofibration (such as the inclusion of a CW subcomplex), and $f : A \to H$ a map with an extension $\tilde{f} : X \to H$. Then the set $[X, H]^f$ of extensions of f forms an abelian group with operation given by

$$[\varphi] + [\psi] = [\varphi + \psi - \tilde{f}].$$

Throughout the paper, we denote the "multiplication by r" map

$$\underbrace{\mathrm{id} + \cdots + \mathrm{id}}_{r \text{ times}} : H \to H$$

by χ_r. The significance of this map is in the following lemmas, which we will repeatedly apply to various obstruction classes:

Lemma 3.3. *Let H be an H-space of finite type, A be a finitely generated coefficient group, and let $\alpha \in H^n(H; A)$ be a cohomology class of finite order. Then there is an $r > 0$ such that $\chi_r^* \alpha = 0$.*

In other words, faced with a finite-order obstruction, we can always get rid of it by precomposing with a multiplication map. Before giving the proof, we develop a bit more of the theory:

Lemma 3.4. *Let H be a simply connected H-space of finite type. Then for every $r > 0$,*

$$\chi_r^*(H^*(H; \mathbb{Z})) \subseteq rH^*(H; \mathbb{Z}) + \textit{torsion}.$$

Proof. By Prop. 3.1, χ_r induces multiplication by r on $\pi_n(H)$. Therefore by Prop. 2.1(c), it induces multiplication by r on the indecomposables of the minimal model $\mathcal{M}_H$. Therefore it induces multiplication by some r^k on every class in $H^n(H; \mathbb{Q})$. □

Combining the two lemmas gives us a third:

Lemma 3.5. *Let H be a simply connected H-space of finite type and A a finitely generated coefficient group. Then for any $r > 0$ and any $n > 0$, there is an $s > 0$ such that*

$$\chi_s^*(H^n(H); A) \subseteq rH^n(H; A).$$

Proof of Lemma 3.3. I would like to thank Shmuel Weinberger for suggesting this proof.

Let q be the order of α. By Prop. 3.1, for $f : S^k \to H$, $(\chi_q)_*[f] = q[f]$.

Let $H[1/q]$ be the universal cover of the mapping torus of χ_q; this should be thought of as an infinite mapping telescope. By the above, the homotopy groups of $H[1/q]$ are $\mathbb{Z}[1/q]$-modules (the telescope localizes them away from q). This implies, by [31, Thm. 2.1], that the reduced homology groups are also $\mathbb{Z}[1/q]$-modules. To understand the cohomology groups, we use the exact sequence

$$0 \to \operatorname{Ext}(H_{n-1}(H[1/q]), A) \to H^n(H[1/q]; A) \to \operatorname{Hom}(H_{n-1}(H[1/q]), A) \to 0$$

coming from the universal coefficient theorem. If M is a $\mathbb{Z}[1/q]$-module, then so is $\operatorname{Hom}(M, G)$ for any abelian group G: for any homomorphism h, we take $[h/q](m) = h(m/q)$. Since $\operatorname{Ext}(M, G)$ is the first homology of the chain complex

$$0 \to \operatorname{Hom}(M, I^0) \to \operatorname{Hom}(M, I^1) \to \cdots,$$

where I^* is an injective resolution of G, it is also a $\mathbb{Z}[1/q]$-module. It follows that $H^n(H[1/q]; A)$ is a $\mathbb{Z}[1/q]$-module. Now, by the Milnor exact sequence [15], the map

$$H^n(H[1/q]; A) \to \varprojlim\big(\cdots \xrightarrow{\chi_q^*} H^n(H; A) \xrightarrow{\chi_q^*} H^n(H; A)\big)$$

is surjective, and hence this inverse limit is also a $\mathbb{Z}[1/q]$-module.

Now we would like to show that for some t, $(\chi_q^*)^t\alpha = 0$, so that we can take $r = q^t$. Suppose not, so that $(\chi_q^*)^t\alpha$ is nonzero for every t. Clearly every element in the sequence

$$\alpha, \chi_q^*\alpha, (\chi_q^*)^2\alpha, \ldots$$

has order which divides q; moreover, since there are finitely many such elements, the sequence eventually cycles. Extrapolating this cycle backward gives us a nonzero element of the inverse limit above, which likewise has order dividing q. This contradicts the fact that this inverse limit is a $\mathbb{Z}[1/q]$-module. □

Note that this proof does not produce an effective bound on t. This prevents our algorithmic approach from yielding results that are as effective as those of Vokřínek in [32].

We will also require the similar but more involved fact.

Lemma 3.6. *Let (H, o, add) be a simply connected H-space of finite type, U another space of finite type, A a finitely generated coefficient group, and $n > 0$.*

(i) Suppose that $\alpha \in H^n(H \times U, o \times U; A)$ is torsion. Then there is an $r > 0$ such that $(\chi_r, \mathrm{id})^*\alpha = 0$.

(ii) Let $\alpha \in H^n(H \times U, o \times U; \mathbb{Z})$. Then for every $r > 0$,

$$(\chi_r, \mathrm{id})^*\alpha \in rH^n(H \times U, o \times U; \mathbb{Z}) + \textit{torsion}.$$

(iii) For every $r > 0$ there is an $s > 0$ such that

$$(\chi_s, \mathrm{id})^*H^n(H \times U, o \times U; A) \subseteq rH^n(H \times U, o \times U; A).$$

Proof. Let where $i_2 : U \to H \times U$ is the inclusion $u \mapsto (o, u)$. We first note that since the map i_2^* on cohomology is a surjection in every degree, $H^n(H \times U, o \times U; A) = \ker i_2^*$. Thus we can equivalently prove parts (i) and (ii) for an $\alpha \in H^n(H \times U; A)$ such that $i_2^*\alpha = 0$. We use several not-quite-standard algebraic topology facts which can be found in [28, §5.5].

We first consider $A = \mathbb{Z}$. For this we use the Künneth formula for cohomology, which gives a natural short exact sequence

$$(3.7) \qquad 0 \to \bigoplus_{k+\ell=n} H^k(H) \otimes H^\ell(U) \to H^n(H \times U) \to \bigoplus_{k+\ell=n+1} \operatorname{Tor}(H^k(H), H^\ell(U)) \to 0.$$

To demonstrate (i), we will first show that we can choose an r_0 such that $(\chi_{r_0}, \mathrm{id})^*\alpha$ is in the image of $\bigoplus_{k+\ell=n} H^k(H) \otimes H^\ell(U)$; in other words, such that the projection of $(\chi_{r_0}, \mathrm{id})^*\alpha$ to $\bigoplus_{k+\ell=n+1} \mathrm{Tor}(H^k(H), H^\ell(U))$ is zero. To see this, recall that for cyclic groups A and B, $\mathrm{Tor}(A, B)$ is trivial unless both A and B are finite, and that the Tor functor distributes over direct sum. Therefore $\mathrm{Tor}(H^k(H), H^\ell(U))$ is generated by elementary tensors $\eta \otimes \nu$ where $\eta \in H^k(H)$ and $\nu \in H^\ell(U)$ are torsion elements. By Lemma 3.3, for each such elementary tensor, we can pick $r(\eta)$ such that $\chi^*_{r(\eta)}\eta = 0$ and therefore

$$(\chi_{r(\eta)}, \mathrm{id})^*(\eta \otimes \nu) = 0 \in \mathrm{Tor}(H^k(H), H^\ell(U)).$$

We then choose r_0 to be the least common multiple of all the $r(\eta)$'s.

Now fix a decomposition of each $H^k(H)$ and $H^\ell(U)$ into cyclic factors to write $(\chi_{r_0}, \mathrm{id})^*\alpha$ as a sum of elementary tensors. Since $i_2^*\alpha = 0$, $(\chi_{r_0}, \mathrm{id})^*\alpha$ has no summands of the form $1 \otimes u$; moreover, each summand is itself torsion. For every other elementary tensor $h \otimes u$, we can use Lemma 3.3 (if h is torsion) or Lemma 3.5 (otherwise, since then u is torsion) to find an $s(h, u)$ such that $\chi^*_{s(h,u)} h \otimes u = 0$.

Finally, we can take r to be the product of r_0 with the least common multiple of the $s(h, u)$'s. This completes the proof of (i) for $A = \mathbb{Z}$.

To see (ii), we use the fact that the Künneth sequence (3.7) splits, albeit non-naturally. Therefore, since we are ignoring torsion, we may assume $\alpha \in \bigoplus_{k+\ell=n} H^k(H) \otimes H^\ell(U)$. Applying Lemma 3.4 to $H^k(H)$ for all $0 < k < n$, we get the result.

Finally, (iii) with integer coefficients follows from (i) and (ii).

Now we need to handle other coefficient groups. We can assume A is a finite cyclic group, since everything we need commutes with direct sums. For this case we use a version of the universal coefficient theorem which states that

$$0 \to H^n(H \times U) \otimes A \to H^n(H \times U; A) \to \mathrm{Tor}(H^{n+1}(H \times U), A) \to 0$$

is an exact sequence. Let $\alpha \in H^n(H \times U, o \times U; A)$ be torsion. We use the same outline as before to show that (i) holds. First we see that there is an r_0 such that $(\chi_{r_0}, \mathrm{id})^*\alpha$ is in the kernel of the map to $\mathrm{Tor}(H^{n+1}(H \times U), A)$; this follows from the integral case of (i) and the fact that $\mathrm{Tor}(G, A)$ contains only the A-torsion elements of G. Next we see that the preimage of $(\chi_{r_0}, \mathrm{id})^*\alpha$ in $H^n(H \times U) \otimes A$ is also annihilated by some $(\chi_{r_1}, \mathrm{id})^*$; this follows from the integral case of (iii). Then $(\chi_{r_0 r_1}, \mathrm{id})^*\alpha = 0$.

The general case of (iii) again follows from (i) and (ii). □

4. The algebraic structure of $[X, Y]^f$

We start by constructing the desired structure on $[X, Y]^f$ when Y is a rational H-space. From the previous section, such a Y is rationally equivalent to a product of Eilenberg–MacLane spaces. In particular, it is rationally equivalent to $H = \prod_{n=2}^{\infty} K(\pi_n(Y), n)$, which we give the product H-space structure. We will harness this to prove the following result.

Theorem 4.1. *Suppose that Y is a rational H-space through dimension d, denote by Y_d the dth Postnikov stage of Y, and let $H_d = \prod_{n=2}^{\infty} K(\pi_n(Y), n)$. Suppose (X, A) is a finite simplicial pair and $f : A \to Y$ a map. Then $[X, Y_d]^f$ admits a virtually free and faithful action by $[X, H_d]^f$ induced by a map $H_d \to Y_d$.*

The proof of this theorem occupies the rest of the section. Later, in §6, we give an algorithm for computing this action which closely mirrors this proof. Before beginning the proof of Theorem 4.1, we see how such an algorithm would also provide the algorithms whose existence is asserted in Theorem A.

If (X, A) has cohomological dimension $d+1$, then there is no obstruction to lifting an extension $X \to Y_d$ of f to Y, as the first obstruction lies in $H^{d+2}(X, A; \pi_{d+1}(Y)) \cong 0$. Therefore $[X, Y]^f$ is nonempty if and only if $[X, Y_d]^f$ is nonempty.

If (X, A) has cohomological dimension d, then in addition every such lift is unique: the first obstruction to homotoping two lifts lies in $H^{d+1}(X, A; \pi_{d+1}(Y)) \cong 0$. Therefore $[X, Y]^f \cong [X, Y_d]^f$.

4.1. **An H-space action on Y_n.** Denote the nth Postnikov stages of Y and H by Y_n and H_n, respectively, and the H-space zero and multiplication on H_n by o_n and by $+$ or $\mathrm{add}_n : H_n \times H_n \to H_n$. We will inductively construct the following additional data:

(i) Maps $H_n \xrightarrow{u_n} Y_n \xrightarrow{v_n} H_n$ inducing rational equivalences such that $v_n u_n$ is homotopic to the multiplication map χ_{r_n} for some integer r_n.
(ii) A map $\mathrm{act}_n : H_n \times Y_n \to Y_n$ defining an *H-space action*, (that is such that $\mathrm{act}_n(o_n, x) = x$ and the diagram

$$\begin{array}{ccc} H_n \times H_n \times Y_n & \xrightarrow{(\mathrm{add}_n, \mathrm{id})} & H_n \times Y_n \\ \downarrow{\scriptstyle (\mathrm{id}, \mathrm{act}_n)} & & \downarrow{\scriptstyle \mathrm{act}_n} \\ H_n \times Y_n & \xrightarrow{\mathrm{act}_n} & Y_n \end{array} \tag{4.2}$$

commutes up to homotopy) which is "induced by u_n" in the sense of the homotopy commutativity of

$$\begin{array}{ccccc} H_n \times H_n & \xrightarrow{(\mathrm{id}, u_n)} & H_n \times Y_n & \xrightarrow{(\chi_{r_n}, v_n)} & H_n \times H_n \\ \downarrow{\scriptstyle \mathrm{add}_n} & & \downarrow{\scriptstyle \mathrm{act}_n} & & \downarrow{\scriptstyle \mathrm{add}_n} \\ H_n & \xrightarrow{u_n} & Y_n & \xrightarrow{v_n} & H_n. \end{array} \tag{4.3}$$

Note that when we pass to rationalizations, the existence of such a structure is obvious: one takes $u_{n(0)}$ to be the identity, $\mathrm{act}_{n(0)} = \mathrm{add}_{n(0)}$, and $v_{n(0)}$ to be multiplication by r_n.

4.2. **The action of $[X/A, H_d]$ on $[X, Y_d]^f$.** Now suppose that we have constructed the above structure. Then add_d induces the structure of a finitely generated abelian group on the set $[X/A, H_d]$, which we identify with the set of homotopy classes of maps $X \to H_d$ sending A to $o_d \in H_d$. Moreover, this group acts on $[X, Y_d]^f$ via the action $[\varphi] \cdot [\psi] = [\mathrm{act}_d \circ (\varphi, \psi)]$.

It remains to show that this action is virtually free and faithful. Indeed, notice that pushing this action forward along v_d gives the action of of $[X/A, H_d]$ on $[X, H_d]^{v_d f}$ via $[\varphi] \cdot [\psi] = r_d[\varphi] + [\psi]$, which is clearly virtually free and faithful. This implies that the action on $[X, H_d]^{v_d f}$ is virtually free. Moreover, the map $v_d \circ : [X, Y_d]^f \to [X, H_d]^{v_d f}$ is finite-to-one by Sullivan's finiteness theorem. Thus the action on $[X, Y_d]^f$ is also virtually faithful.

4.3. **The Postnikov induction.** Now we construct the H-space action. For $n = 1$ all the spaces are points and all the maps are trivial. So suppose we have constructed the maps u_{n-1}, v_{n-1}, and act_{n-1}, and let $k_n : Y_{n-1} \to K(\pi_n(Y), n+1)$ be the nth k-invariant of Y. For the inductive step, it suffices to prove the following lemma:

Lemma 4.4. *There is an integer $q > 0$ such that we can define u_n to be a lift of $u_{n-1}\chi_q$, and construct v_n and a solution $\mathrm{act}_n : H_n \times Y_n \to Y_n$ to the homotopy lifting-extension problem*

$$\begin{array}{ccccccc} H_n \times H_n & & \xrightarrow{\mathrm{add}_n} & & H_n & \xrightarrow{u_n} & Y_n \\ \downarrow{\scriptstyle (\mathrm{id},u_n)} & & & \nearrow^{\mathrm{act}_n} & & & \downarrow \\ H_n \times Y_n & \xrightarrow[(\chi_q,\mathrm{id})]{} & H_n \times Y_n & \twoheadrightarrow & H_{n-1} \times Y_{n-1} & \xrightarrow{\mathrm{act}_{n-1}} & Y_{n-1} \end{array} \tag{4.5}$$

so that the desired conditions are satisfied.

Proof. First, since Y is rationally a product, k_n is of finite order, so by Lemma 3.3 there is some q_0 such that $k_n u_{n-1}\chi_{q_0} = 0$, and therefore

$$\begin{array}{ccc} H_n & \xrightarrow{\hat{u}} & Y_n \\ \downarrow & & \downarrow \\ H_{n-1} & \xrightarrow{u_{n-1}\chi_{q_0}} & Y_{n-1}; \end{array}$$

is a pullback square. We will define $u_n = \hat{u}\chi_{q_2 q_1}$, with q_1 and q_2 to be determined and $q = q_2 q_1 q_0$.

Now we construct act_n. Given a map f, we write $M(f)$ to mean its mapping cylinder, and let

$$\mathrm{Hact}_{n-1} : H_{n-1} \times M(u_{n-1}) \to Y_{n-1}$$

be a map which restricts to act_{n-1} on $H_{n-1} \times Y_{n-1}$ and add_{n-1} on $H_{n-1} \times H_{n-1}$. Such a map exists because (4.3) holds in degree $n-1$. We will construct a lifting-extension

$$\begin{array}{ccccccc} (H_n \times H_n) \cup (o_n \times M(\hat{u})) & & \xrightarrow{[\mathrm{add}_n \circ (\chi_{q_1},\mathrm{id})] \cup \mathrm{id}} & & M(\hat{u}) & \xrightarrow{\text{project}} & Y_n \\ \downarrow & & & \nearrow^{\widehat{\mathrm{Hact}}} & & & \downarrow \\ H_n \times M(\hat{u}) & \xrightarrow[(\chi_{q_1 q_0},\mathrm{id})]{} & H_n \times M(\hat{u}) & \twoheadrightarrow & H_{n-1} \times M(u_{n-1}) & \xrightarrow{\mathrm{Hact}_{n-1}} & Y_{n-1} \end{array}$$

It is easy to see that then for any $q_2 > 0$,

$$\mathrm{act}_n = (\widehat{\mathrm{Hact}}|_{H_n \times Y_n}) \circ (\chi_{q_2}, \mathrm{id})$$

satisfies (4.5). Moreover, then the desired identity $\mathrm{act}_n(o_n, x) = x$ is automatically satisfied.

Note that the outer rectangle commutes since we know (4.3) holds in degree $n-1$. Now, write

$$\begin{aligned} A &= H_n \times M(\hat{u}) \\ B &= (H_n \times H_n) \cup (o_n \times M(\hat{u})) \\ C &= o_n \times M(\hat{u}). \end{aligned}$$

Since $\hat{u}$ is a rational equivalence, so are the inclusions of $H_n \times H_n$ into A and B. Therefore, the obstruction $\mathcal{O} \in H^{n+1}(A, B; \pi_n(Y))$ to finding the lifting-extension is of finite order. We will show that when q_1 is large enough, this obstruction is zero.

The obstruction group fits into the exact sequence of the triple (A, B, C):

$$\cdots \to H^n(B, C; \pi_n(Y)) \xrightarrow{\delta} H^{n+1}(A, B; \pi_n(Y)) \xrightarrow{\mathrm{rel}^*} H^{n+1}(A, C; \pi_n(Y)) \to \cdots,$$

and so the image $\mathrm{rel}^* \mathcal{O}$ in $H^{n+1}(A, C; \pi_n(Y))$ is torsion. By Lemma 3.6(i), that means that $(\chi_s, \mathrm{id})^*(\mathrm{rel}^* \mathcal{O}) = 0$ for some $s > 0$.

Now we look at a preimage under δ of $(\chi_s, \mathrm{id})^*\mathcal{O}$, which we call $\alpha \in H^n(B, C; \pi_n(Y))$. By excision,

$$H^n(B, C; \pi_n(Y)) \cong H^n(H_n \times H_n, o_n \times H_n; \pi_n(Y)).$$

Applying Lemma 3.6(iii), we can find a t such that $\chi_t^*\alpha \in \ker\delta$ and therefore

$$\delta((\chi_t, \mathrm{id})^*\alpha) = (\chi_{st}, \mathrm{id})^*\mathcal{O} = 0.$$

Thus for $q_1 = st$, we can find a map $\widehat{\mathrm{Hact}}$ completing the diagram.

Now we ensure that (4.2) commutes by picking an appropriate q_2. Define $\widehat{\mathrm{act}} = \widehat{\mathrm{Hact}}|_{H_n \times Y_n}$; then the diagram

$$\begin{array}{ccc} H_n \times H_n \times Y_n & \xrightarrow{(\mathrm{add}_n, \mathrm{id})} & H_n \times Y_n \\ \big\downarrow{\scriptstyle (\mathrm{id}, \widehat{\mathrm{act}})} & & \big\downarrow{\scriptstyle \widehat{\mathrm{act}}} \\ H_n \times Y_n & \xrightarrow{\widehat{\mathrm{act}}} & Y_n \end{array}$$

commutes after rationalization. Since (4.2) commutes in degree $n-1$, the sole obstruction to homotopy commutativity is a torsion class in $H^n(H_n \times H_n \times Y; \pi_n(Y_n))$. Therefore we can again apply Lemma 3.6(i), this time with $H = H_n \times H_n$ and $U = Y_n$, to find a q_2 which makes the obstruction zero.

All that remains is to define v_n. But we know that u_n is rationally invertible, and so we can find some v_n such that $v_n u_n$ is multiplication by some r_n. Moreover, for any such v_n, the right square of (4.3) commutes up to finite order. Thus by increasing r_n (that is, replacing v_n by $\chi_{\hat{r}} v_n$ for some $\hat{r} > 0$) we can make it commute up to homotopy. □

5. Building blocks of homotopy-theoretic computation

We now turn to describing the algorithms for performing the computations outlined in the previous two sections. This relies heavily on machinery and results from [5], [6], and [10] as building blocks, which in turn rely on building blocks from the work of Rubio, Sergeraert, and others [25], [23], [24]. This section is dedicated to explaining these building blocks.

Our spaces are stored as simplicial sets *with effective homology*. Roughly speaking this means a computational black box equipped with:

- A way to refer to individual simplices and compute their face and degeneracy operators. This allows us to, for example, represent a function from a finite simplicial complex or simplicial set to a simplicial set with effective homology.
- A *fully effective* chain complex with a chain homotopy equivalence to this set. We do not need to make this completely precise, but for example it allows one to compute the homology and cohomology in any degree and with respect to any finitely generated coefficient group, and to know both their isomorphism type and (co)chains representing individual classes.

This is easy to construct for finite simplicial complexes. But effective homology is designed to work with simplicial sets that can be described algorithmically but are not necessarily finite; in our case, these are finite Postnikov stages of spaces of finite type. We refer to [23] for a more detailed overview.

Now we summarize the operations which are known to be computable from previous work.

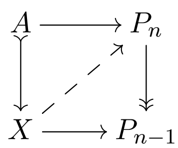
Theorem 5.1. *(a) Given a finitely generated abelian group π and $n \geq 2$, a model of the Eilenberg–MacLane space $K(\pi, n)$ can be represented as a simplicial set with effective homology and a computable simplicial group operation. Moreover, there are algorithms implementing a chain-level bijection between n-cochains in a finite simplicial complex or simplicial set X with coefficients in π and maps from X to $K(\pi, n)$ (the observation dates back to at least* [25]*, but see* [5, §3.7] *or* [24, §7.5] *for a detailed explanation).*

(b) Given a finite family of simplicial sets with effective homology, their product can be represented as a simplicial set with effective homology (see [24, §8.2] *or* [5, §3.1]*).*

(c) Given a simplicial map $f : X \to Y$ between simplicial sets with effective homology, there is a way of representing the mapping cylinder $M(f)$ as a simplicial set with effective homology. (In [6] *this is remarked to be "very similar to but easier than Prop. 5.11"; the related algebraic mapping cylinder construction is done explicitly in e.g.* [22, §3]*.)*

(d) Given a map $p : Y \to B$, we can compute the nth stage of the Moore–Postnikov tower for p, in the form of a sequence of Kan fibrations between simplicial sets with effective homology [6, Theorem 3.3] *(cf.* [5, Theorem 1.2] *for the non-relative version).*

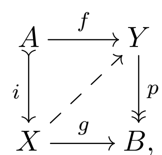
(e) Given a diagram



where $P_n \to P_{n-1}$ is a step in a (Moore–)Postnikov tower as above, there is an algorithm to decide whether a diagonal exists and, if it does, compute one [6, Prop. 3.7].

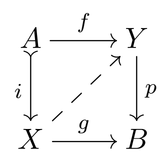
(f) Given a fibration $p : Y \to B$ of simply connected simplicial complexes and a map $f : X \to B$, we can compute any finite Moore–Postnikov stage of the pullback of p along f [6, Addendum 3.4].

(g) Given a diagram



where A is a subcomplex of a finite complex X and p is a fibration of simply connected complexes of finite type, we can compute whether two maps $u, v : X \to Y$ completing the diagram are homotopic relative to A and over B [10, see "Equivariant and Fiberwise Setup"].

(h) Given a diagram



where A is a subcomplex of a finite complex X, Y and B are simply connected, and p has finite homotopy groups, we can compute the (finite and perhaps empty) set $[X, Y]^f_p$ of homotopy classes of maps completing the diagram up to homotopy.

Proof. We prove only the part which is not given a citation in the statement.

Part (h). Let $d = \dim X$. One starts by computing the dth stage of the Moore–Postnikov tower of $p : Y \to B$ using (d). From there, we induct on dimension. At the kth step, we have computed the (finite) set of lifts to the kth stage P_k of the Moore–Postnikov tower. For each such lift, we use (e) to decide whether it lifts to the $(k+1)$st stage, and compute a lift $u : X \to P_{k+1}$ if it does. Then we compute all lifts by computing representatives of each element of $H^{k+1}(X, A; \pi_{k+1}(p))$ and modifying u by each of them. Finally, we use (g) to decide which of the maps we have obtained are duplicates and choose one representative for each homotopy class in $[X, P_{k+1}]^f_p$. We are done after step d since $[X, P_d]^f_p \cong [X, Y]^f_p$. □

6. Computing $[X, Y]^f$

We now explain how to compute the group and action described in §4. We work with a representation of (X, A) as a finite simplicial set and a Postnikov tower for Y, and perform the induction outlined in that section to compute $[X, Y_d]^f$ for a given dimension d. The algorithm verifies that Y is indeed a rational H-space through dimension d; however, it assumes that Y is simply connected and returns nonsense otherwise.

6.1. **Setup.** Let d be such that Y_d is a rational H-space. Since the homotopy groups of Y can be computed, we can use Theorem 5.1(a) and (b) to compute once and for all the space

$$H_d = \prod_{n=2}^{d} K(\pi_n(Y), n),$$

and the binary operation $\mathrm{add}_d : H_d \times H_d \to H_d$ is given by the product of the simplicial group operations on the individual $K(\pi_n(Y), n)$'s. The group of homotopy classes $[X/A, H_d]$ is naturally isomorphic to $\prod_{n=2}^{d} H^n(X, A; \pi_n(Y))$, making this also easy to compute. Finally, given an element of this group expressed as a word in the generators, we can compute a representative map $X \to H_d$, constant on A, by generating the corresponding cochains of each degree on (X, A) and using them to build maps to $K(\pi_n(Y), n)$.

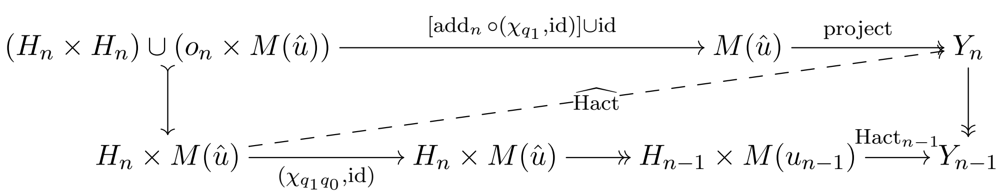
We then initialize the induction which will compute maps u_d, v_d, and act_d and an integer r_d satisfying the conditions of §4. Since $H_1 = Y_1$ is a point, we can set $r_1 = 1$ and u_1, v_1, and act_1 to be the trivial maps.

6.2. **Performing the Postnikov induction.** The induction is performed as outlined in §4.3, although we have to be careful to turn homotopy lifting and extension problems into genuine ones. Suppose that maps u_{n-1}, v_{n-1}, and act_{n-1} as desired have been constructed, along with a map

$$\mathrm{Hact}_{n-1} : H_n \times M(u_{n-1}) \to Y_{n-1}$$

which restricts to add_{n-1} on $H_{n-1} \times H_{n-1}$ and act_{n-1} on $H_{n-1} \times Y_{n-1}$. There are five steps to constructing the maps in the nth step:

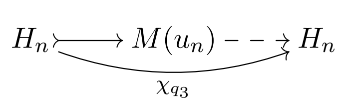
1. Find q_0 such that $u_{n-1}\chi_{q_0}$ lifts to a map $\hat{u} : H_n \to Y_n$, and fix such a map.
2. Find q_1 such that the diagram



has a lifting-extension $\widehat{\mathrm{Hact}}$ along the dotted arrow, and fix such a map.
3. Find q_2 such that $\widehat{\mathrm{Hact}}|_{H_n \times Y_n} \circ (\chi_{q_2}, \mathrm{id})$ makes the diagram (4.2) commute up to homotopy. Now we can define

$$u_n : H_n \to Y_n \quad \text{by} \quad u_n = \hat{u}\chi_{q_1 q_2};$$
$$\mathrm{Hact}_n : H_n \times M(u_n) \to Y_n \quad \text{by} \quad \mathrm{Hact}_n = \widehat{\mathrm{Hact}} \circ (\chi_{q_2}, \mathrm{id} \cup \chi_{q_1 q_2});$$
$$\mathrm{act}_n : H_n \times Y_n \to Y_n \quad \text{by} \quad \mathrm{act}_n = \mathrm{Hact}_n |_{H_n \times Y_n}.$$

4. Find q_3 so that the diagram



can be completed by some $\hat{v} : M(u_n) \to H_n$, and fix such a map.

5. Find q_4 so that setting

$$v_n = \hat{v}\chi_{q_4} \quad \text{and} \quad r_n = r_{n-1}q_0q_1q_2q_3q_4$$

makes the diagram (4.3) commute up to homotopy.

The first step is done by determining the order of the k-invariant $k_n \in H^{n+1}(Y_{n-1}; \pi_n(Y))$. If this order is infinite, then Y is not rationally a product of Eilenberg–MacLane spaces, and the algorithm returns failure. Otherwise q_0 is guaranteed to exist, and we can compute it by iterating over multiples of the order until we find one that works.

The rest of the steps are guaranteed to succeed for some value of q_i, and each of the conditions can be checked using the operations of Theorem 5.1, so this part can also be completed by iterating over all possible values until we find one that works.

6.3. **Computing the action.** Let $G = [X/A, H_d]$; we now explain how to compute $[X,Y]^f$ as a set with a virtually free and faithful action by G.

First we must decide whether there is a map $X \to H_d$ extending $v_d f : A \to H_d$. If the set $[X, Y_d]^f$ has an element e, then $v_d f$ has an extension $v_d e$, so if we find that there is no such extension, we return the empty set. Otherwise we compute such an extension ψ_0.

Lemma 6.1. *We can determine whether an extension $\psi_0 : X \to H_d$ of $v_d f$ exists, and compute one if it does.*

Proof. Recall that $H_d = \prod_{n=2}^d K(\pi_n(Y), n)$. Write proj_n for the projection to the $K(\pi_n(Y), n)$ factor. Then the extension we desire exists if and only if for each $n < d$, the cohomology class in $H^n(A; \pi_n(Y))$ represented by $\mathrm{proj}_n\, v_d f$ has a preimage in $H^n(X; \pi_n(Y))$ under the map i^*.

We look for an explicit cocycle $\sigma_n \in C^n(X; \pi_n(Y))$ whose restriction to A is $\mathrm{proj}_n\, v_d f$. We can compute cycles which generate $H^n(X; \pi_n(Y))$ (because X has effective homology) as well as generators for $\delta C^{n-1}(X; \pi_n(Y))$ (the coboundaries of individual $(n-1)$-simplices in X). Then finding σ_n or showing it does not exist is an integer linear programming problem with the coefficients of these chains as variables.

Now if σ_n exists, then it also determines a map $X \to K(\pi_n(Y), n)$. Taking the product of these maps for all $n \leq d$ gives us our ψ_0. □

We now compute a representative a_N for each coset N of $r_d G \subseteq G$. Since this is a finite-index subgroup of a fully effective abelian group, this can be done algorithmically, for example by trying all words of increasing length in a generating set until a representative of each coset are obtained. For each a_N, we compute a representative map $\varphi_N : X \to H_d$ which is constant on A. Then the finite set

$$S = \{\psi_N = \psi_0 + v_d u_d \varphi_N : N \in G/r_d G\}$$

contains representatives of the cosets of the action of $[X/A, H_d]$ on $[X, H_d]^{v_d f}$ obtained by pushing the action on $[X,Y]^f$ forward along v_d.

Now, for each element of S we apply Theorem 5.1(h) to the square

$$\begin{array}{ccc} A & \xrightarrow{f} & Y_d \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle v_d} \\ X & \xrightarrow{\psi_N} & H_d \end{array}$$

to compute the finite set of preimages under v_d in $[X, Y_d]^f$. To obtain a set of representatives of each coset for the action of $[X/A, H_d]$ on $[X, Y_d]^f$, we must then eliminate any preimages that are in the same coset. In other words, we must check whether two preimages $\tilde{\psi}$ and $\tilde{\psi}'$ of ψ_N differ by an element of $[X/A, H_d]$; any such element stabilizes $v_d\tilde{\psi}$, and so its order must divide r_d. Since there are finitely many elements whose order divides r_d, we can check for each such element φ in turn whether $[\varphi] \cdot [\tilde{\psi}] \simeq [\tilde{\psi}']$.

Finally, to finish computing $[X, Y_d]^f$ we must compute the finite stabilizer of each coset. This stabilizer is contained in the finite subgroup of $[X/A, H_d]$ of elements whose order divides r_d. Therefore we can again go through all elements of this subgroup and check whether they stabilize our representative.

6.4. **Summary.** We conclude this section with a formal summary of the algorithm.

Input:
- A simplicial pair (X, A).
- A simplicial complex Y, assumed to be simply connected.
- A simplicial map $f : A \to Y$.
- A positive integer d.

Output: If Y_d is not rationally an H-space, ALGORITHM NOT APPLICABLE. Otherwise:
- The dth Postnikov stage Y_d of Y, represented as a simplicial set with effective homology.
- A product of Eilenberg–MacLane spaces H_d, represented as a simplicial set with effective homology.
- The group $[X/A, H_d]$, represented as a fully effective abelian group.
- A finite (possibly empty) set $\mathcal{C}$ of maps $\tilde{f}_i : X \to Y_d$ representing cosets of the action of $[X/A, H_d]$ on $[X, Y_d]^f$.
- For each i, the stabilizer of $\tilde{f}_i$, represented as a finite subgroup $\Sigma_i \subseteq [X/A, H_d]$.

Main steps: Here is the outline of the algorithm:

A. Initialize the computation:
   - Compute the homotopy groups of Y through dimension d.
   - Construct the space $H_d = \prod_{n=2}^{d} K(\pi_n(Y), n)$, and compute the group $[X/A, H_d]$.
   - Set $r_1 = 1$, and u_1, v_1, act_1, and Hact_1 to be the unique maps between the relevant spaces (which are all points).

B. **for** $n = 2$ through d:
   - Compute the k-invariant $k_n \in H^{n+1}(Y_{n-1}; \pi_n(Y))$. If it is of infinite order, **return** ALGORITHM NOT APPLICABLE.
   - Otherwise, compute the action of H_n on Y_n and associated data as outlined in §6.2, namely the positive integer r_n and maps u_n, v_n, act_n, and Hact_n.

C. Using the algorithm of Lemma 6.1, determine whether there is a map $X \to H_d$ which extends $v_d f : A \to H_d$.
   - If there isn't, **return** $(Y_d, H_d, [X/A, H_d], \mathcal{C} = \emptyset, \emptyset)$.
   - If there is, compute such a map $\psi_0 : X \to H_d$.

D. **for** each $N \in [X/A, H_d]/r_d[X/A, H_d]$:
   - Choose a representative homotopy class in $[X/A, H_d]$, and a representative map $\varphi : (X, A) \to (H_d, o)$ in this homotopy class.
   - Compute the map $\psi = \psi_0 + v_d u_d \varphi : X \to H_d$.
   - For each homotopy class of maps completing the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & Y_d \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle v_d} \\ X & \xrightarrow{\psi} & H_d \end{array}$$

     up to homotopy, compute a representative $\tilde{f}_i : X \to Y_d$.

   Write $\mathcal{C}_0$ for the set of all the $\tilde{f}_i$.

E. Remove duplicates from $\mathcal{C}_0$, that is, take a subset $\mathcal{C} \subseteq \mathcal{C}_0$ which includes only one map from each orbit of the action of $[X/A, H_d]$ on $[X, Y_d]^f$.
F. For each $\tilde{f}_i \in \mathcal{C}$, compute the stabilizer as a subgroup of the torsion subgroup of $[X/A, H_d]$ and **return** $(Y_d, H_d, [X/A, H_d], \mathcal{C}, \text{stabilizers})$.

7. Variants of Hilbert's tenth problem

In [4], the authors show that the existence of an extension is undecidable by using the undecidability of the existence of solutions to systems of diophantine equations of particular shapes:

Lemma 7.1 (Lemma 2.1 of [4]). *The solvability in the integers of a system of equations of the form*

(Q-SYM) $$\sum_{1 \le i < j \le r} a_{ij}^{(q)} x_i x_j = b_q, \qquad q = 1, \ldots, s \quad \text{or}$$

(Q-SKEW) $$\sum_{1 \le i < j \le r} a_{ij}^{(q)} (x_i y_j - x_j y_i) = b_q, \qquad q = 1, \ldots, s$$

for unknowns x_i and (for (Q-SKEW)) y_i, $1 \le i \le r$, is undecidable.

We conjecture a very broad generalization of this result.

Conjecture 7.2. *For any nonzero bilinear map $\mathbf{B} : \mathbb{Z}^m \times \mathbb{Z}^n \to \mathbb{Z}^p$, the solvability in the integers of a system of equations of the form*

(Q-BLIN($\mathbf{B}$)) $$\sum_{i,j=1}^{r} a_{ij}^{(q)} \mathbf{B}(\mathbf{u}_i, \mathbf{v}_j) = \mathbf{c}_q, \qquad q = 1, \ldots, s$$

for unknowns $\mathbf{u}_i = (u_{i1}, \ldots, u_{im})$ and $\mathbf{v}_j = (v_{j1}, \ldots, v_{jn})$, $1 \le i, j \le r$, is undecidable.

We will show this conjecture in certain special cases, most notably the case $p = 1$. However, the general case would, for instance, imply the undecidability of Hilbert's tenth problem over the ring of integers of any number field, first conjectured by Denef and Lipshitz [8]. This narrower conjecture is still open in general, although Mazur and Rubin [14] show using work of Poonen [18] and Shlapentokh [26] that it is implied by the Shafarevich–Tate conjecture in number theory. On the other hand, undecidability is known unconditionally in many cases, for example for totally real number fields and their quadratic extensions. For a survey, see [19, Theorem 14.1].

Before discussing the relationship between these two problems, we give a precise definition:

Definition. Given a ring R and a subring S, *Hilbert's tenth problem over R with coefficients in S* is the decision problem: given a finite list of polynomials in $S[x_1, \ldots, x_n]$, do they have a simultaneous zero in R^n?

Proposition 7.3. *Let R be the ring of integers of a number field. Then Hilbert's tenth problem over R with coefficients in R and in $\mathbb{Z}$ are computationally equivalent.*

This is implicit in Poonen's survey [19]; I would like to thank Emil Jeřábek on MathOverflow for the following proof.

Proof. Given a system of polynomials $p_1, \ldots, p_m \in R[x_1, \ldots, x_n]$, we construct an equivalent system with coefficients in $\mathbb{Z}$. Let $\xi \in R$ be such that $R_{(0)} = \mathbb{Q}(\xi)$. We introduce a new variable z representing ξ, and replace the coefficients of each p_i with corresponding polynomials in z to obtain polynomials $q_i(x_1, \ldots, x_n, z)$ with rational coefficients. Finally we add the minimal polynomial $f_\xi(z)$ of ξ to our system. Then $q_1, \ldots, q_m, f_\xi$ has a solution over R if and only if $p_1, \ldots, p_m$ does, since for any ξ' such that $f_\xi(\xi') = 0$, there is an automorphism of R taking ξ to ξ'.

The polynomials $q_1, \dots, q_m, f_\xi$ have rational coefficients, and we can clear the denominators by multiplying by a sufficiently large integer. $\square$

Now we further reduce the problem to fit in our framework.

Lemma 7.4. *Let R be any ring. If Hilbert's tenth problem with coefficients in a ring $S \subseteq R$ is undecidable over R, then so is the solvability of a system of equations of the form*

$$\text{(Q-DIFF)} \qquad \sum_{i,j=1}^{r} a_{ij}^{(q)} x_i y_j = c_q, \qquad q = 1, \dots, s$$

in unknowns x_i and y_j, $1 \le i, j \le r$, and again with coefficients in S.

Proof. The proof exactly follows that of Lemma 2.1 of [4], but we give it for completeness. We reduce any system of equations over R to a system of the form (Q-DIFF). First, we note that any system of equations can be converted into a quadratic system by introducing new unknowns representing products and powers. Now to convert a general quadratic system in unknowns $z_1, \dots, z_r$ to a system of the form (Q-DIFF), we introduce variables $x_0, \dots, x_r$ and $y_0, \dots, y_r$, replace every quadratic term of the form $z_i z_j$ (where i and j are not necessarily distinct) with $x_i y_j$, every linear term z_i with $x_i y_0$, and introduce the following additional equations:

$$x_0 y_0 = 1; \qquad x_i y_0 - x_0 y_i = 0, \quad i = 1, \dots, r.$$

This forces x_0 and y_0 to be units and inverses of each other; moreover, if $x_0, \dots, x_r, y_0, \dots, y_r$ is a solution to the newly constructed system of the form (Q-DIFF), then $z_i = x_i y_0 = x_0 y_i$ is a solution to the original quadratic system. Conversely, given a solution $z_1, \dots, z_r$ to the original system, we can take $x_0 = y_0 = 1$ and $x_i = y_i = z_i$ for $i = 1, \dots, r$. $\square$

This immediately implies:

Proposition 7.5. *Let R be a ring which is finitely generated and free as a $\mathbb{Z}$-module (for example, the ring of integers of a number field, or the matrix ring $M_n(\mathbb{Z})$). Then Hilbert's tenth problem over R with coefficients in $\mathbb{Z}$ is undecidable if and only if (Q-BLIN($\mathbf{B}$)) is undecidable, where $\mathbf{B}$ describes the multiplication law in R in terms of some $\mathbb{Z}$-basis (or, possibly, three different $\mathbb{Z}$-bases for the left factor, the right factor, and the product).*

Example 7.6. The solvability in the integers of systems of the form (Q-BLIN($\mathbf{B}$)) is undecidable, where $\mathbf{B} : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}^2$ is given by

$$\mathbf{B}(\mathbf{u}, \mathbf{v}) = \left(\mathbf{u}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{v}, \mathbf{u}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{v} \right).$$

This bilinear map describes the multiplication law for $\mathbb{Z}[i]$ in the basis $\{1, i\}$; Hilbert's tenth problem over $\mathbb{Z}[i]$ and any other quadratic number ring is undecidable by [7].

We conclude the section with two more special cases of Conjecture 7.2.

Proposition 7.7. *Suppose that $\mathbf{B} : \mathbb{Z}^m \times \mathbb{Z}^n \to \mathbb{Z}^p$ is a bilinear map such that for some $L : \mathbb{Z}^p \to \mathbb{Z}$, $L \circ \mathbf{B}$ has rank 1. Then the solvability in the integers of systems of the form* (Q-BLIN($\mathbf{B}$)) *is undecidable.*

Proof. After changes of basis for $\mathbb{Z}^m$, $\mathbb{Z}^n$, and $\mathbb{Z}^p$, we can assume that $B_1(\mathbf{u}, \mathbf{v}) = c u_1 v_1$ for some $c \in \mathbb{Z}$, where B_1 is the first coordinate of $\mathbf{B}$. Now consider a general system of the form (Q-DIFF). We use it to build a corresponding system

$$\text{(7.8)} \qquad \sum_{i,j=1}^{r} a_{ij}^{(q)} \mathbf{B}(\mathbf{u}_i, \mathbf{v}_j) = c_q \mathbf{B}(\mathbf{e}_1, \mathbf{e}_1), \qquad q = 1, \dots, s,$$

where $\mathbf{e}_1 = (1, 0, \ldots, 0)$. We claim this system is equivalent.

Given a solution $x_1, \ldots, x_r, y_1, \ldots, y_r$ to (Q-DIFF), clearly $x_1\mathbf{e}_1, \ldots, x_r\mathbf{e}_r, y_1\mathbf{e}_1, \ldots, y_r\mathbf{e}_r$ is a solution to (7.8). Conversely, given a solution $\mathbf{u}_1, \ldots \mathbf{u}_r, \mathbf{v}_1, \ldots, \mathbf{v}_r$ to (7.8), $u_{11}, \ldots, u_{r1}, v_{11}, \ldots, v_{r1}$ is a solution to (Q-DIFF). □

Theorem 7.9. *The solvability in the integers of systems of the form* (Q-BLIN($\mathbf{B}$)) *is undecidable when $p = 1$, that is, when $\mathbf{B}(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T B\mathbf{v}$ for some $m \times n$ matrix B.*

Remark 7.10. One readily sees from the proof that this result admits various generalizations:

(1) The result holds with the integers replaced by any PID R in which Hilbert's tenth problem is undecidable, such as $R = \mathbb{Z}[i]$. When R is finite-dimensional and free as a $\mathbb{Z}$-module, a Diophantine system of this form over R with integer coefficients can be reinterpreted as an integral Diophantine system of the form (Q-BLIN($\mathbf{A} \otimes B$)), where $\mathbf{A} : \mathbb{Z}^d \otimes \mathbb{Z}^d \to \mathbb{Z}^d$ describes the multiplication law in R and $\mathbf{A} \otimes B$ is interpreted as a map

$$(\mathbb{Z}^d \otimes \mathbb{Z}^m) \otimes (\mathbb{Z}^d \otimes \mathbb{Z}^n) \to (\mathbb{Z}^d \otimes \mathbb{Z}).$$

Therefore, the solvability of systems of the form (Q-BLIN($\mathbf{A} \otimes B$)) is again undecidable.

(2) The result also holds for $p > 1$ if the following algebraic condition is satisfied: there are decompositions $\mathbb{Q}^m = L \oplus S$ and $\mathbb{Q}^n = L' \oplus S'$ such that L and L' are one-dimensional and the bilinear map $\mathbf{B} \otimes \mathbb{Q} : \mathbb{Q}^m \otimes \mathbb{Q}^n \to \mathbb{Q}^p$ restricts to zero on $L \otimes S'$ and $S \otimes L'$ and is nonzero on $L \otimes L'$.

Remark 7.11. Proposition 7.7 and Theorem 7.9 are in some sense opposite extremes: the more independent coordinates in the image of $\mathbf{B}$, the likelier one is to find a direction in which the rank is low. In between we have the case where $\mathbf{B} : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ has full rank in every direction; this includes multiplication laws of rings of integers of number fields and may be the most difficult situation.

Proof of Theorem 7.9. We show that a system of the form (Q-DIFF) can be simulated with one of the form Q-BLIN($\mathbf{B}$). By Lemma 7.4, this is sufficient to show that solvability of systems of the form Q-BLIN($\mathbf{B}$) is undecidable. The proof is again closely related to that of the undecidability of (Q-SYM) in [4].

We first show that we can replace B with a diagonal matrix.

Lemma 7.12. *Given an $m \times n$ matrix B, there is a square diagonal full-rank matrix B' such that for every choice of $\{a_{ij}\}$ and c_q, the system*

$$\sum_{i,j=1}^{r} a_{ij}^{(q)} \mathbf{u}_i^T B \mathbf{v}_j = c_q, \qquad q = 1, \ldots, s \tag{7.13}$$

has a solution if and only if the system

$$\sum_{i,j=1}^{r} a_{ij}^{(q)} (\mathbf{u}'_i)^T B' \mathbf{v}'_j = c_q, \qquad q = 1, \ldots, s \tag{7.14}$$

has a solution.

Proof. We can write $B = SAT$ where A is the Smith normal form and S and T are invertible $m \times m$ and $n \times n$ matrices, respectively. Then the vectors $(\mathbf{u}_i, \mathbf{v}_j)_{i,j=1,\ldots,r}$ are a solution to the system (7.13) if and only if $(S^T\mathbf{u}_i, T\mathbf{v}_j)_{i,j=1,\ldots,r}$ are a solution to the system

$$\sum_{i,j=1}^{r} a_{ij}^{(q)} \mathbf{u}_i^T A \mathbf{v}_j = c_q, \qquad q = 1, \ldots, s.$$

The matrix A consists of a full-rank diagonal submatrix B' in the top left corner and zeros everywhere else. After removing variables which don't appear in any terms with nonzero coefficients, we obtain the system (7.14) with this B'. □

Thus we may assume that $m = n$ and $B = (b_{k\ell})$ is a diagonal matrix of full rank.

Now consider a general system of the form (Q-DIFF). We use it to build a system of the form (Q-BLIN($\mathbf{B}$)) with variables

$$\begin{aligned} &u_{i1}, \ldots, u_{in} \text{ and } v_{j1}, \ldots, v_{jn}, && 1 \leq i \leq r, \\ &z_{k\ell} \text{ and } w_{k\ell}, && 1 \leq k, \ell \leq n. \end{aligned}$$

Define $n \times n$ matrices $Z = (z_{k\ell})$ and $W = (w_{k\ell})$. Then the equations of our new system are

$$\begin{cases} \displaystyle\sum_{i,j=1}^{r} a_{ij}^{(q)} \mathbf{u}_i^T B \mathbf{v}_j = b_{11} c_q, \quad q = 1, \ldots, s, \\ Z^T B W = B, \\ (\mathbf{u}_i^T B W)_\ell = 0, \qquad i = 1, \ldots, r, \quad \ell = 2, \ldots, n, \\ (Z^T B \mathbf{v}_j)_k = 0, \qquad j = 1, \ldots, r, \quad k = 2, \ldots, n. \end{cases} \tag{7.15}$$

To complete the proof, we must show that the system (7.15) has a solution if and only if (Q-DIFF) does. It is easy to see that $\{x_i, y_j\}_{1 \leq i,j \leq r}$ is a solution to (Q-DIFF) if and only if

$$Z = W = I_n, \qquad \mathbf{u}_i = x_i \mathbf{e}_1, \qquad \mathbf{v}_j = y_j \mathbf{e}_1,$$

where $\mathbf{e}_1$ is the basis vector $(1, 0, \ldots, 0)$, is a solution to (7.15). In particular, if (Q-DIFF) has a solution, then so does (7.15). Conversely, suppose that we have a solution for (7.15). Since they are integer matrices and B has nonzero determinant, Z and W must both have determinant ± 1 and are invertible over $\mathbb{Z}$. Then (7.15) also has the solution

$$\mathbf{u}_i' = Z^{-1} \mathbf{u}_i, \qquad \mathbf{v}_j' = W^{-1} \mathbf{v}_j, \qquad Z' = W' = I_n,$$

and $x_i = u_{i1}', y_j = v_{j1}'$ is a solution for (Q-DIFF). □

8. Undecidability of extension problems

Theorem 8.1. *Let Y be a simply connected finite complex which is not a rational H-space. Then the problem of deciding, for a finite simplicial pair (X, A) and a map $\varphi : A \to Y$, whether an extension to X exists is at least as hard as deciding solvability for systems of equations of the form* (Q-BLIN($\mathbf{B}$))*, for a bilinear map $\mathbf{B}$ depending on Y. Moreover, it is enough to consider pairs satisfying* $\operatorname{cd}(X, A) = d + 1$*, where d is the smallest degree such that Y_d is not a rational H-space.*

Examples of target spaces Y for which this gives us a proof of undecidability include $\mathbb{C}P^n$ for any n, $\mathbb{C}P^2 \# \mathbb{C}P^2$, punctured products of odd-dimensional spheres, Grassmannians, and any Y such that $\pi_d(Y)$ has rank 1. In general, one should be able to prove undecidability of the extension problem for a wide range of target spaces after computing their Sullivan minimal model.

Before proving the theorem in full generality, we review the proof in [4] of the case $Y = S^2$, where undecidability is shown by reduction from Hilbert's tenth problem for systems of equations of the type (Q-SYM). How do the authors encode equations in an extension problem? There are three ingredients, all encoded into cells of the pair (X, A):

- Variable cells: copies S_i^2 of S^2 in X which are not in A, and hence can be mapped to Y with arbitrary degree.
- 3-spheres encoding constant terms of equations: copies S_q^3 of S^3 in A, which are mapped to Y with a fixed Hopf invariant b_q by the map φ.

- 4-cells encoding equations. The qth 4-cell is attached to the rest of X by the map
$$-2b_q \operatorname{id}_{S^3_q} + \sum_{1\leq i<j\leq r} a_{ij}^{(q)}[\operatorname{id}_{S^2_i}, \operatorname{id}_{S^2_j}],$$
where $[\alpha,\beta]$ denotes the *Whitehead product* of α and β: the composition
$$S^3 \xrightarrow{\text{attaching map of the top cell of } S^2\times S^2} S^2 \vee S^2 \xrightarrow{\alpha\vee\beta} X^{(3)}.$$

In summary, A is a wedge of 3-spheres and X consists of a wedge of 2- and 3-spheres with 4-cells attached.

The homotopy class of a map $S^3 \to S^2$ is determined by its Hopf invariant, an integer. The Whitehead product $[\operatorname{id}_{S^2}, \operatorname{id}_{S^2}] : S^3 \to S^2$ has Hopf invariant 2, and the Whitehead product is bilinear in the two variables. Therefore, the 4-cells force the degrees x_i on S^2_i of an extension of φ to X to satisfy the equations (Q-SYM).

The minimal model of S^2 is
$$\big(\textstyle\bigwedge(a^2, b^3), da = 0, db = a^2\big).$$
The Hopf invariant can be thought of as the result of pairing with b. There is therefore a relationship between the differential and the Whitehead product:
$$\langle b, [f,g]\rangle = 2\langle a, f\rangle\langle a, g\rangle.$$
Such a relationship holds more generally.

In the general case, we use a similar tactic, but with higher-order Whitehead products, originally defined by Porter [20]. Given spheres $S^{n_1}, \ldots, S^{n_\kappa}$, their product can be given a cell structure with one cell for each subset of $\{1,\ldots,\kappa\}$. Define their *fat wedge* $\mathbb{V}_{i=1}^{\kappa} S^{n_i}$ to be this cell structure without the top face. Let $N = -1 + \sum_{i=1}^{\kappa} n_i$, and let $\tau : S^N \to \mathbb{V}_{i=1}^{\kappa} S^{n_i}$ be the attaching map of the missing face. By definition, $\alpha \in \pi_N(Y)$ is contained in the *κth-order Whitehead product* $[\alpha_1,\ldots,\alpha_\kappa]$, where $\alpha_i \in \pi_{n_i}(Y)$, if it has a representative which factors through a map
$$S^N \xrightarrow{\tau} \mathbb{V}_{i=1}^{\kappa} S^{n_i} \xrightarrow{f_\alpha} Y$$
such that $[f_\alpha|_{S^{n_i}}] = \alpha_i$. Note that there are many potential indeterminacies in how higher-dimensional cells are mapped, so $[\alpha_1,\ldots,\alpha_\kappa]$ is a set of homotopy classes rather than a unique class. This set may be empty: for example, if the ordinary Whitehead product $[\alpha,\beta]$ is nonzero, then $[\alpha,\beta,\gamma]$ is empty for any γ because there is no way to extend the map $\alpha\vee\beta$ to the product cell. However, this is not the case in our situation:

Lemma 8.2. *Suppose that Y is a rational H-space through degree $d-1$. Then every d-dimensional higher-order Whitehead product in $Y_{(0)}$ is nonempty.*

Proof. Let $\alpha_i : S^{n_i} \to Y_{(0)}$, for $i = 1,\ldots,\kappa$ be homotopy classes of maps, and suppose $\sum_i n_i = d+1$. Since $Y_{d-1(0)}$ is an H-space, $\bigvee_i \alpha_i : \bigvee_i S^{n_i} \to Y_{d-1(0)}$ extends via the H-space operation to a map $F : \prod_i S^{n_i} \to Y_{d-1(0)}$. The obstruction to lifting F to $Y_{(0)}$ lies in $(d+1)$-dimensional cohomology, and therefore the restriction of F to the fat wedge lifts to $Y_{(0)}$. □

Importantly, higher-order Whitehead products are graded symmetric and multilinear in a weak sense. It is easy to see that the factors commute or anticommute as determined by the grading. For multilinearity, notice that if two maps $f, g : S^n \times X \to Y$ agree on $* \times X$, where $* \in S^n$ is some base point, then there is a well-defined map "$f+g$" given by

$$S^n \times X \xrightarrow{\text{“pinch the waist”}\times\mathrm{id}} (S^n \vee S^n)\times X \xrightarrow{f\cup_{*\times X} g} Y, \tag{8.3}$$

which induces addition in $\pi_n(Y, f(*,x))$ on every fiber $S^n \times \{x\}$. Likewise, if $f, g : \mathbb{V}_{i=1}^{\kappa} S^{n_i} \to Y$, where $f\circ\tau$ and $g\circ\tau$ represent elements of $[\alpha_1,\ldots,\alpha_\kappa]$ and $[\alpha_1', \alpha_2, \ldots, \alpha_\kappa]$ respectively, agree on

$\prod_{i=2}^{\kappa} S^{n_i}$, then a similar operation yields a well-defined element $[f+g] \in [\alpha_1 + \alpha_1', \alpha_2, \ldots, \alpha_\kappa]$. In particular, taking $f = g$ and performing the operation repeatedly, we get that

$$[c\alpha_1, \ldots, \alpha_\kappa] \supseteq c[\alpha_1, \ldots, \alpha_\kappa]$$

(see e.g. [20, Theorem 2.13]). Note, however, that there is no more general notion of additivity.

We can use this weak multilinearity to relate Lemma 8.2 to Y itself:

Lemma 8.4. *For $i = 1, \ldots, \kappa$, let $\alpha_i \in \pi_{n_i}(Y)$, and denote the image in $\pi_{n_i}(Y_{(0)})$ by $\alpha_{i(0)}$. If $[\alpha_{1(0)}, \ldots, \alpha_{\kappa(0)}] \subseteq \pi_N(Y_{(0)})$ is nonempty, then there are positive integers $r_1, \ldots, r_\kappa$ such that $[r_1\alpha_1, \ldots, r_\kappa\alpha_\kappa]$ is nonempty in $\pi_N(Y)$. Moreover, if Y is of finite type, then the set of r_i can be chosen to depend only on the set of n_i.*

Proof. We prove a stronger statement: if $\beta \in \pi_N(Y_{(0)})$ is an element of $[\alpha_{1(0)}, \ldots, \alpha_{\kappa(0)}]$, then for some $R = r_1 \cdots r_\kappa$, $R\beta$ lifts to an element of $[r_1\alpha_1, \ldots, r_\kappa\alpha_\kappa] \subseteq \pi_N(Y)$.

Let $f : \mathbb{V}_{i=1}^{\kappa} S^{n_i} \to Y_{(0)}$ be a map such that $f \circ \tau$ is a representative of β. Denote by $(r_1, \ldots, r_\kappa)f : \mathbb{V}_{i=1}^{\kappa} S^{n_i} \to Y_{(0)}$ the corresponding map in which the ith factor is multiplied by r_i in the sense of (8.3). We construct a map $\mathbb{V}_{i=1}^{\kappa} S^{n_i} \to Y$, one cell at a time. Suppose that Z is a subcomplex of $\mathbb{V}_{i=1}^{\kappa} S^{n_i}$ including the boundary of a particular cell e (WLOG, the top cell of $\prod_{i=1}^{s} S^{n_i}$ for some $s < \kappa$), and $F : Z \to Y$ is a lift of $(r_1, \ldots, r_\kappa)f$ for some factors $r_1, \ldots, r_\kappa$. In particular, if $\tau_s : S^M \to Z$ is the attaching map of this cell, then the obstruction to extending $F \circ \tau_s$ to a lift of $(r_1, \ldots, r_\kappa)f|_e$ is a finite order element of $\pi_{M+1}(Y)$, and there is an $r_{\text{new}} > 0$ such that $r_{\text{new}}[F \circ \tau_s] = 0$. (In the finite type case, we can choose r_{new} to be the cardinality of the torsion subgroup of $\pi_{M+1}(Y)$.) Using an (8.3)-type construction to multiply the first factor of F by r_{new}, we get a lift of $(r_{\text{new}}r_1, r_2, \ldots, r_\kappa)f|_{Z\cup e}$ to a map $Z \cup e \to Y$.

Starting with $S^{n_1} \vee \cdots \vee S^{n_\kappa}$ and performing the operation for every higher cell, we obtain a lift of $R\beta$ for some R. □

Finally, in the setting of the theorem, certain d-dimensional higher-order Whitehead products don't only exist but are virtually unique:

Lemma 8.5. *Suppose that Y is a rational H-space through degree $d-1$. Then for some $\kappa \geq 2$, there is a nonempty κth-order Whitehead product in $\pi_d(Y)$ containing no torsion elements. Moreover, for the smallest such κ, all κth-order Whitehead products in $\pi_d(Y)$ are unique up to torsion.*

Proof. Fix a minimal model $\mathcal{M}_Y$ for Y and a basis of generators for the indecomposables V_n in each degree n which is dual to a basis for $\pi_n(Y)/\text{torsion}$. Since Y is not a rational H-space, there is some least d such that the differential in the minimal model $\mathcal{M}_Y$ is nontrivial. Recall that for a minimal model, each nonzero term in the differential is at least quadratic. For each of the generators η of V_d, $d\eta$ is a polynomial in the lower-degree generators. Denote by *P-degree* the degree of an element of the minimal model as a polynomial in these generators, as opposed to the degree imposed by the grading. Let κ be the minimal P-degree of any monomial in any $d\eta$.

To prove the lemma, we use the connection, first investigated in [1], between the differential in the minimal model and higher-order Whitehead products. The main theorem of [1], Theorem 5.4, gives a formula for the pairing between an indecomposable $\eta \in V_n$ and any element of an ith-order Whitehead product set in π_n, assuming that every term of $d\eta$ has P-degree at least i. This formula is somewhat complicated, but is i-linear in the pairings between factors of the terms of $d\eta$ and factors of the Whitehead product.

It follows that, given an element of a κth-order Whitehead product set in $\pi_d(Y)$, its pairings with each of the generators η are given uniquely by this formula. Since V_d is dual to $\pi_d(Y) \otimes \mathbb{Q}$, all elements of the κth-order Whitehead product set are in the same rational homotopy class.

Consider a particular η and a particular term μ of $d\eta$ whose P-degree is κ. Let $\alpha_1, \ldots, \alpha_\kappa$ be elements of $\pi_*(Y)$ dual to the variables in this term. By Lemmas 8.2 and 8.4, some $[r_1\alpha_1, \ldots, r_\kappa\alpha_\kappa]$ is nonempty. Every element of this set pairs nontrivially with $d\eta$, since there is a nonzero contribution from μ and a zero contribution from all other terms. Therefore the set does not contain a torsion element. □

Proof of Theorem 8.1. We reduce from the problem (Q-BLIN($\mathbf{B}$)), for an appropriate bilinear map $\mathbf{B}$. For each instance of this problem, we construct a pair (X, A) and map $f : A \to Y$ such that an extension exists if and the instance has a solution.

Fix a minimal model $\mathcal{M}_Y$ for Y and a basis of generators $\eta_1, \ldots, \eta_p$ for the indecomposables V_d in degree d. By Lemma 8.5, there is a nontrivial and rationally unique higher-order Whitehead product $[\alpha_1, \ldots, \alpha_\kappa] \subset \pi_d(Y)$, where $\alpha_i \in \pi_{d_i}(Y)$. Moreover, by Lemma 8.4, we may choose $\alpha_3, \ldots, \alpha_\kappa$ and positive integers ρ_1 and ρ_2 so that for every choice of $\beta \in \rho_1\pi_{d_1}(Y)$ and $\gamma \in \rho_2\pi_{d_2}(Y)$, $[\beta, \gamma, \alpha_3, \ldots, \alpha_\kappa]$ is nonempty.

Now we fix $\alpha_3, \ldots, \alpha_\kappa$ and vary β and γ. For each η_k, the pairing $\langle \eta_k, [\beta, \gamma, \alpha_3, \ldots, \alpha_\kappa]\rangle$ is bilinear in β and γ. In particular, after fixing $\mathbb{Z}$-bases for $\rho_1\pi_{d_1}(Y) \cong \mathbb{Z}^m$ and $\rho_2\pi_{d_2}(Y) \cong \mathbb{Z}^n$, we get a bilinear map $\mathbf{B} : \mathbb{Z}^m \times \mathbb{Z}^n \to \mathbb{Z}^p$.

Now given a system of the form (Q-BLIN($\mathbf{B}$)), we will build a $(d+1)$-dimensional pair (X, A) and a map $f : A \to Y$ such that the extension problem has a solution if and only if the system does. We define

$$A = \bigvee_{q=1}^{s} S^d_q \vee \bigvee_{i=3}^{\kappa} S^{d_i},$$

and let $f : A \to Y$ send

- S^{d_i} to Y via a representative of α_i;
- S^d_q to Y via an element of $\pi_d(Y)$ whose pairing with η_k, for each k, is c_{kq}.

Finally, we build X from $A' = A \vee \bigvee_{i=1}^{r} S^{d_1}_i \vee \bigvee_{j=1}^{r} S^{d_2}_j$ as follows:

- Add on cells so that for every i and j, X includes the fat wedge $\mathbb{V}(S^{d_1}_i, S^{d_2}_j, S^{d_3}, \ldots, S^{d_\kappa})$, and these fat wedges only intersect in A'. Let $\varphi_{ij} : S^d \to X$ be the attaching map of the missing $(d+1)$-cell for the (i,j)th fat wedge.
- Add on spheres $S^{d_1}_i{}'$ together with the mapping cylinder of a map $S^{d_1}_i \to S^{d_1}_i{}'$ of degree ρ_1, and spheres $S^{d_2}_j{}'$ together with the mapping cylinder of a map $S^{d_2}_j \to S^{d_2}_j{}'$ of degree ρ_2.
- Then, for each q, add a $(d+1)$-cell attached along a representative of $\rho([S^d_q] - \sum_{i,j=1}^{r} a^{(q)}_{ij}[\varphi_{ij}])$, where ρ is the exponent of the torsion part of $\pi_d(Y)$.

It is easy to see that $H_n(X, A) = 0$ for $n > d$.

We claim that (X, A) and f pose the desired extension problem. Indeed, any extension of f to $\tilde{f} : X \to Y$ sends each $S^{d_1}_i$ to an element $\beta_i \in \rho_1\pi_{d_1}(Y)$ and each $S^{d_2}_j$ to an element $\gamma_j \in \rho_2\pi_{d_2}(Y)$, as constrained by the mapping cylinders. Then $\tilde{f} \circ \varphi_{ij} : S^d \to Y$ represents an element of $[\beta_i, \gamma_j, \alpha_3, \ldots, \alpha_\kappa]$. Then the $(d+1)$-cells force the equations of (Q-BLIN($\mathbf{B}$)) to hold.

Conversely, given a satisfying assignment for (Q-BLIN($\mathbf{B}$)), there is an extension $\tilde{f} : X \to Y$. To see this, note that such a satisfying assignment gives us values for β_i and γ_j up to torsion, and by construction there is an extension to the fat wedges and the mapping cylinders. Moreover, under any such extension, $f_*[S^d_q]$ and $\sum_{i,j=1}^{r} a^{(q)}_{ij}\tilde{f}_*[\varphi_{ij}] \in \pi_d(Y)$ are rationally equivalent; thus when multiplied by ρ they are equal, and the map extends to the $(d+1)$-cells of X. □

Examples. We conclude by discussing some instances of Y for which the extension problem is undecidable. For example, if there are $\alpha_3, \ldots, \alpha_\kappa$ such that there exists a rationally nontrivial

Whitehead product $[\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_\kappa] \in \pi_d(Y)$ and the image of the map

$$\pi_{d_1}(Y) \otimes \pi_{d_2}(Y) \to \pi_d(Y)$$
$$\beta \otimes \gamma \mapsto [\beta, \gamma, \alpha_3, \dots, \alpha_\kappa]$$

is one dimensional, then the extension problem is undecidable by Theorem 7.9. This situation includes the cases where Y is a fat wedge of odd spheres (in this case the one-dimensional subspace is generated by the universal Whitehead product) and $\mathbb{C}P^n$ (in this case the one-dimensional subspace is generated by $\underbrace{[\alpha, \dots, \alpha]}_{n+1 \text{ times}}$), as well as any Y such that $\pi_d(Y)$ has rank 1.

A different case is that of $\mathbb{C}P^2 \# \mathbb{C}P^2$. In this case, we give explicit generators for the low-dimensional part of the minimal model:

$$\Big(\bigwedge(a_1^2, a_2^2, b^3, c^3, \dots), da_i = 0, db = \frac{1}{2}a_1^2 - \frac{1}{2}a_2^2, dc = a_1 a_2, \dots\Big).$$

Here the a_i are dual to the two-dimensional classes α_i representing the spheres in the two copies of $\mathbb{C}P^2$. The 3-dimensional generators are governed by the 4-cell of $\mathbb{C}P^2 \# \mathbb{C}P^2$, which ensures that $[\alpha_1, \alpha_1] + [\alpha_2, \alpha_2] = 0$. Then the pairing of b and c with Whitehead products of linear combinations of α_1 and α_2 is described by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

from Example 7.6. Thus the extension problem for maps from 4-complexes to $\mathbb{C}P^2 \# \mathbb{C}P^2$ is equivalent to Hilbert's tenth problem for $\mathbb{Z}[i]$, and hence undecidable.

One can construct similar (though perhaps less natural) examples for other number rings.

Finally, as a demonstration of the wide range of natural examples that can be covered with our techniques, we show that the extension problem is undecidable for $Y = \widetilde{\mathrm{Gr}}_k(\mathbb{R}^n)$, the Grassmannian of oriented k-planes in $\mathbb{R}^n$, when $2 \leq k \leq n-2$. The minimal models of these spaces are computed explicitly in [17]. Letting d be the least dimension for which the differential is nontrivial, this computation tells us:

- Whenever $k \neq n/2$, or when $k = n/2$ is odd, $\pi_d(\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n))$ has rank 1, and therefore the extension problem for maps into $\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n)$ is undecidable.
- When $k = n/2$ and k is even, write $t = k/2$. In this case, $d = 4t - 1$, $\pi_d(\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n))$ has rank 2, and the two generators have differentials

$$dv_0 = h_t - \tau^2, \qquad du_0 = \sigma\tau.$$

Here σ and τ are generators of degree k, and h_t is a polynomial in generators $p_1, \dots, p_{t-1}$, where p_i has degree $4i$, and $p_t = \sigma^2$. Namely, h_t is such that

$$(1 + p_1 + \dots + p_t)(1 + h_1 + h_2 + \cdots) = 1.$$

From this formula one sees that the terms of dv_0 of multi-degree (k, k), which are "seen" by pairing with Whitehead products between elements of π_k, are

$$-\sigma^2 - \tau^2, \quad \text{when } t \text{ is odd}; \qquad p_{t/2}^2 - \sigma^2 - \tau^2, \quad \text{when } t \text{ is even.}$$

This lets us write down the bilinear maps

$$\pi_k(\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n)) \otimes \mathbb{Q} \times \pi_k(\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n)) \otimes \mathbb{Q} \to \pi_d(\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n)) \otimes \mathbb{Q}$$

induced by the Whitehead product. When t is odd, the bilinear form given by pairing with $2u_0 - v_0$ has rank 1, and we can apply Proposition 7.7. When t is even, we are in the situation of Remark 7.10(2). (Both these claims only rely on the rational structure and are independent of integral information.) In both cases, we know that the relevant version of

Hilbert's tenth problem is undecidable, and therefore so is the extension problem for maps into $\widetilde{\mathrm{Gr}}_k(\mathbb{R}^n)$.

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